J. Math. Anal. Appl. 515 (2022) 126427

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Regular Articles

A direct proof of the Brunn-Minkowski inequality in nilpotent Lie groups

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A R T I C L E   I N F O

Article history:
Received 15 November 2021
Available online 13 June 2022
Submitted by P. Koskela

Keywords:
Brunn-Minkowski inequality
Nilpotent Lie groups
Nilpotent Lie algebras

A B S T R A C T

The purpose of this work is to give a direct proof of the multiplicative Brunn-Minkowski inequality in nilpotent Lie groups based on Hadwiger-Ohmann's one of the classical Brunn-Minkowski inequality in Euclidean space.

1. Introduction

The classical Brunn-Minkowski inequality in Euclidean space asserts that, given $A, B \subset \mathbb{R}^d$ measurable sets such that $A + B$ is also measurable, we have

$$|A + B|^{1/d} \geq |A|^{1/d} + |B|^{1/d},$$

where $|\cdot|$ indicates the volume of a set, and $A+B = \{a+b : a \in A, b \in B\}$ is the classical Minkowski addition of sets. Taking $\lambda \in [0, 1]$, and replacing A by λA and B by $(1 - \lambda)B$, we get the equivalent inequality

$$|\lambda A + (1 - \lambda)B|^{1/d} \geq \lambda|A|^{1/d} + (1 - \lambda)|B|^{1/d}.$$

First connected to the isoperimetric theorem, this inequality is a cornerstone in convex geometry [21,9]. Through the equivalent functional formulation of the Brunn-Minkowski inequality, the Prékopa-Leindler inequality, we can see some of the implications in the preservation of logarithmic concavity under convolutions noticed by Brascamp and Lieb [5], as well as in the work of Bobkov and Ledoux [4] where it is derived the concentration of measure of Gaussian-like measures, Brascamp-Lieb and logarithmic Sobolev inequalities.

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[1] The author has been supported by Horizon 2020 project GHAIA ref. 777882, MEC-Feder grants MTM2017-84851-C2-1-P and PID2020-118180GB-I00, and Junta de Andalucía grants A-FQM-441-UGR18 and P20-00164.

[2] The author would like to thank his Ph.D. advisor Manuel Ritoré for sugesting the problem and his help.

https://doi.org/10.1016/j.jmaa.2022.126427

There are several ways of generalizing the Brunn-Minkowski inequality. In Lie groups we can define the Minkowski addition of sets using the group product and take as volume the Haar measure of the group. The Brunn-Minkowski inequality obtained this way is called the *multiplicative* Brunn-Minkowski inequality. In general metric measure spaces the notion of s-intermediate points can be used to replace the convex combination of points in Euclidean space, see [20]. This leads to the *geodesic* Brunn-Minkowski inequality.

A large number of proofs for the Brunn-Minkowski inequality in Euclidean space are known, some of them can be found in [9,10,13]. Ritoré and Yepes [20] proved the geodesic Brunn-Minkowski inequality for products of metric measures spaces. For Riemannian manifolds with a lower bound on the Ricci curvature this inequality is proven in [6] employing techniques of optimal transport. These techniques were latter applied to prove this inequality for CD spaces (see [8]). While Juillet [11] proved that no CD condition holds in sub-Riemannian Heisenberg groups $\mathbb{H}^n$, the optimal transport approach was followed by Balogh, Kristály and Sipos [1] and by Barilary and Rizzi [2] to prove geodesic Brunn-Minkowski inequalities in the sub-Riemannian setting (see also [18]).

In 2003, Monti [19] observed that the multiplicative Brunn-Minkowski inequality in $\mathbb{H}^n$ cannot hold with exponent $(2n+2)^{-1}$, corresponding to the homogeneous dimension of $\mathbb{H}^n$, since otherwise Carnot–Carathéodory balls would be isoperimetric sets.

Leonardi and Masnou [16] proved in 2005 that this inequality holds with exponent $(2n+1)^{-1}$, corresponding to the topological dimension of $\mathbb{H}^n$. Their proof was based on Hadwiger-Ohmann's proof of the classical Brunn-Minkowski inequality given in [10].

Later on, Tao [22,23] posted an entry in his blog in 2011 explaining how to produce a Prékopa-Leindler inequality in any nilpotent Lie group of topological dimension d, which provides a natural way to prove the multiplicative Brunn-Minkowski inequality with exponent d^{-1}.

Juillet [11] gave examples of sets for which the multiplicative Brunn-Minkowski inequality in $\mathbb{H}^n$ does not hold with exponent smaller than $(2n+1)^{-1}$.

In this article we prove a generalization of the Brunn-Minkowski inequality in Euclidean space where the Minkowski addition of sets is replaced by any product $*:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}^d$ of the form

$$z*w=z+w+(F_1,F_2(z,w),\dots,F_d(z,w))=z+w+F(z,w), \tag{*}$$

where F_1 is a constant and F_i are continuous functions that depend only on $z_1,\dots,z_{i-1},w_1,\dots,w_{i-1}$ $\forall i=2,\dots,d$. By a product here we mean a binary operation without assuming any further properties such as associativity.

Theorem 1.1 *(Brunn-Minkowski inequality for (*) products). Let $*:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}^d$ be a product of the form (*) and let $A,B\subset\mathbb{R}^d$ be measurable sets such that $A*B$ is measurable. Then we have*

$$|A*B|^{1/d}\geq|A|^{1/d}+|B|^{1/d}. \tag{1.1}$$

Any nilpotent Lie group verifies the hypothesis of Theorem 1.1 because of the expression of the group product in exponential coordinates of the first kind. This theorem is an extension of the result obtained by Leonardi and Masnou [16] in Heisenberg groups. While the proof of Leonardi and Masnou only works in Heisenberg groups, this argument can be seen as the first step of an induction argument developed in this paper (see Remark 3.5 for more details). In this paper, we shall consider a product $*$ of the form (*), that not necessarily comes from a group product, and change $*$ for another one $*_{z_1,w_1}$ of the form (*), depending on the sets A and B, that allows us to compare the volume of the Minkowski addition of sets for the products $*$ and $*_{z_1,w_1}$, as a consequence of Lemma 3.1. When the product $*$ comes from a nilpotent group it is not true that $*_{z_1,w_1}$ can define a group product. Then, by an induction argument, we will compare the volume of the Minkowski addition of sets A and B with the volume of the Euclidean Minkowski addition of A and B, and establish in Proposition 3.6 a sufficient condition in $\mathbb{H}^1$ for the strict inequality in (1.1).

At the end of the paper, we state several classical variations of inequality (1.1) in the case of Carnot groups, where dilations can be defined.

After this article was completed, the author was informed that Theorem 1.1 was also proven by Bobkov [3] in 2011, where he used Knothe's map to get the Brunn-Minkowski inequality for convex sets and obtained the general result after proving the equivalent analytic version of the theorem, the Prékopa-Leindler inequality.

2. Preliminaries

For the convenience of the reader, we introduce some notation on Lie groups following the one in [25]. Given a Lie group G we shall denote 0 and l_σ the neutral element and the left-translation respectively. Its tangent plane at 0 is the Lie algebra $\mathfrak{g}$ and we write $[\cdot,\cdot]$ for the Lie bracket of vector fields. The exponential map of left-invariant vector field X will be denoted by $\exp(X)$, writing $\exp_G$ if specifying the group is needed. A left-invariant or Haar measure in G will be denoted by μ. In $\mathbb{R}^n$ it is the Lebesgue measure $|\cdot|_n$. We will drop the subscript when n is the topological dimension of G.

We recall some results on nilpotent and stratifiable groups. For a quite complete description of nilpotent Lie groups the reader is referred to [12], and to [14] for stratifiable and Carnot groups.

Let G be a Lie group with Lie algebra $\mathfrak{g}$. We define recursively $\mathfrak{g}_0 = \mathfrak{g}$, $\mathfrak{g}_{i+1} = [\mathfrak{g}, \mathfrak{g}_i] = \text{span}\{[X,Y] : X \in \mathfrak{g}, Y \in \mathfrak{g}_i\}$. The decreasing series

$$\mathfrak{g} = \mathfrak{g}_0 \supseteq \mathfrak{g}_1 \supseteq \mathfrak{g}_2 \supseteq \cdots$$

is called the *lower central series of* $\mathfrak{g}$. If $\mathfrak{g}_r = 0$ and $\mathfrak{g}_{r-1} \neq 0$ for some r, we say that $\mathfrak{g}$ is *nilpotent*, and the number r is called the *step of* $\mathfrak{g}$. A connected Lie group is said to be *nilpotent* if its Lie algebra is nilpotent.

Notice that each $\mathfrak{g}_i$ is an ideal in $\mathfrak{g}$. We shall write n_i for the dimension of $\mathfrak{g}_i$.

Lemma 2.1. *Let $\mathfrak{g}$ be a nilpotent Lie algebra. Then there exists a basis $\{X_1, \dots, X_d\}$ of $\mathfrak{g}$ such that*

i) for each $1 \leq n \leq d$, $\mathfrak{h}_n = \text{span}\{X_{d-n+1}, \dots, X_d\}$ is an ideal of $\mathfrak{g}$,
ii) for each $0 \leq i \leq r-1$, $\mathfrak{h}_{n_i} = \mathfrak{g}_i$,

where n_i denotes the dimension of $\mathfrak{g}_i$.

A basis verifying this is called a *strong Malcev basis*. This construction is adapted from [7].

Fixed a strong Malcev basis in a simply connected nilpotent group, the exponential is a diffeomorphism between $\mathbb{R}^d$ and G, and is given by the map

$$x = (x_1, \dots, x_d) \mapsto \exp(x_1 X_1 + \dots + x_d X_d).$$

This result can be found as Theorem 1.127 in [12]. By abuse of notation we shall denote $\exp(x_1 X_1 + \dots + x_d X_d) = \exp(x)$. The inverse of this map provides coordinates called *canonical coordinates of the first kind*, and we denote it as $\log : G \to \mathbb{R}^d$.

We define a multiplication map associated to the exponential in a simply connected nilpotent group by

$$z * w = \log(\exp(z) \cdot \exp(w)).$$

The structure of this product is given by the following theorem. It was first proved by Malcev in 1949 [17], and a proof can be found as Theorem 4.1 in [24], or with some modification as Proposition 1.2.7 in [7].

Theorem 2.2. *Let G be a simply connected nilpotent group. Then the multiplication map takes the following form:*

$$z * w = z + w + (P_1(z,w), \dots, P_d(z,w)), \tag{2.1}$$

where $z = (z_1, \dots, z_d)$, $w = (w_1, \dots, w_d)$, P_1 is a constant and P_i is a polynomial in the variables $z_1, \dots, z_{i-1}, w_1, \dots, w_{i-1}$ $\forall\ i = d - n_1 + 1, \dots, d$.

In the next result we show that, slightly refining Theorem 2.2, the multiplication map acts as a sum in the coordinates corresponding to the complement of $\mathfrak{g}_1$. This argument can be seen also in [15], Proposition 6.0.16.

Theorem 2.3. *Let G be a simply connected nilpotent group. Then the multiplication map takes the following form:*

$$z * w = z + w + (0, \dots, 0, P_{d-n_1+1}(z,w), \dots, P_d(z,w))$$

where $z = (z_1, \dots, z_d)$, $w = (w_1, \dots, w_d)$ and P_i is a polynomial in the variables $z_1, \dots, z_{i-1}, w_1, \dots, w_{i-1}$ $\forall\ i = d - n_1 + 1, \dots, d$.

Proof. Let $Z = \sum_{i=1}^d z_i X_i$, $W = \sum_{i=1}^d w_i X_i$. Since $\mathfrak{g}_1$ is an ideal in $\mathfrak{g}$, there is a normal Lie subgroup $G_1 \subseteq G$ whose Lie algebra is $\mathfrak{g}_1$. Let $\pi : G \to G/G_1$ denote the projection over the quotient, $\tilde{z} = \pi(z)$, $\tilde{w} = \pi(w)$, $\tilde{Z} = (d\pi)_0(Z)$, $\tilde{W} = (d\pi)_0(W)$. Notice that $\ker(d\pi)_0 = \mathfrak{h}_{n_1}$ and $\mathfrak{g}/\mathfrak{g}_1$ is a trivial Lie algebra with the induced product. Therefore, by the Baker-Campbell-Hausdorff formula,

$$\tilde{z} * \tilde{w} = \tilde{z} + \tilde{w}. \tag{2.2}$$

On the other hand, by Theorem 2.2 it holds that

$$\begin{aligned} &\exp_{G/G_1}(\tilde{Z}) \exp_{G/G_1}(\tilde{W}) = \pi(\exp_G(Z) \exp_G(W)) = \\ &\pi\Big(\exp_G\big(Z + W + \sum_{i=1}^d P_i(z,w) X_i\big)\Big) = \exp_{G/G_1}\big(\tilde{Z} + \tilde{W} + \sum_{i=1}^{d-n_1} P_i(z,w) X_i\big). \end{aligned} \tag{2.3}$$

Taking $\log_{G/G_1}$ in (2.3), we obtain

$$\tilde{z} * \tilde{w} = \tilde{z} + \tilde{w} + (P_1(z,w), \dots, P_{d-n_1}(z,w), 0, \dots, 0). \tag{2.4}$$

From (2.2) and (2.4), we obtain that $P_i = 0$ $\forall i = 1, \dots, d - n_1$. □

From Theorem 2.3 it can be proved that right translations are maps whose Jacobian determinant is equal to 1 at any point, and the change of variables gives us the following theorem. The interested reader can find the details as Theorem 1.2.9 and Theorem 1.2.10 in [7].

Proposition 2.4. *Let G be a simply connected nilpotent group. Then, after having chosen a strong Malcev basis on $\mathfrak{g}$, the exponential takes the Lebesgue measure on $\mathbb{R}^d$ to a Haar measure μ on G, that is, for any $A \subset G$ measurable and any $f : G \to \mathbb{R}$ integrable, one has*

$$\mu(A) = |\log(A)| \quad \textit{and} \quad \int_G f d\mu = \int_{\mathbb{R}^d} (f \circ \exp)(x) dx.$$

We refer the reader to [14] for the details on the rest of this section.

A *stratification* of a Lie algebra $\mathfrak{g}$ is a direct-sum decomposition

$$\mathfrak{g} = V_1 \oplus \ldots \oplus V_r,$$

for some integer $r \geq 1$, where $V_r \neq \{0\}$, $[V_1, V_i] = V_{i+1}$ for all $i \in \{1, \ldots, r\}$ and $V_{r+1} = \{0\}$. We say that a Lie algebra is *stratifiable* if there exists a stratification on it. We say that a Lie algebra is *stratified* when it is stratifiable and endowed with a fixed stratification. We say that a Lie group is *stratifiable* if it is connected and simply connected and its Lie algebra is stratifiable.

The following lemma assures that any stratifiable group is a nilpotent group.

Lemma 2.5. *Let $\mathfrak{g} = V_1 \oplus \ldots \oplus V_r$ be a stratified Lie algebra. Then*

$$\mathfrak{g}_{k-1} = V_k \oplus \ldots \oplus V_r.$$

In particular, $\mathfrak{g}$ is a nilpotent Lie algebra of step r, and $\mathfrak{g} = V_1 \oplus \mathfrak{g}_1$.

It is worth checking that Theorem 2.3 manifests that the multiplication map acts as a sum in the coordinates corresponding to V_1. The reader can find the following proposition and an example of a nilpotent group which is not stratifiable in [14].

Proposition 2.6. *Let $\mathfrak{g}$ be a stratifiable Lie algebra with stratifications*

$$\mathfrak{g} = V_1 \oplus \ldots \oplus V_r = W_1 \oplus \ldots \oplus W_s.$$

Then $r = s$ and there exists a Lie algebra automorphism $A : \mathfrak{g} \to \mathfrak{g}$ such that $A(V_i) = W_i$ for $i = 1, \ldots, r$.

Proposition 2.6 guarantees that for a stratifiable group G, the natural number

$$Q = \sum_{i=1}^{r} i \dim(V_i),$$

does not depend on the particular stratification. Q is called the *homogeneous dimension of G*.

For $\lambda > 0$ we define the *dilation on $\mathfrak{g}$ of factor λ* as the unique linear map $\delta_\lambda : \mathfrak{g} \to \mathfrak{g}$ such that

$$\delta_\lambda(X) = \lambda^t X \quad \forall X \in V_t \ \forall t \in \{1, \ldots, r\}.$$

Remark 2.7. Dilations $\delta_\lambda : \mathfrak{g} \to \mathfrak{g}$ are Lie algebra isomorphisms.

The fact that G is simply connected certifies that there exists a unique Lie groups automorphism $\delta_\lambda : G \to G$ (denoted as the dilation on the Lie algebra) whose differential at 0 is the dilation on $\mathfrak{g}$ of factor λ. This automorphism is called *dilation on G of factor λ*.

Proposition 2.8. *Let G be a stratified group with Haar measure μ and let $\lambda > 0$. Then*

$$\int_G f d\mu = \lambda^Q \int_G (f \circ \delta_\lambda) d\mu,$$

where Q is the homogeneous dimension of G.

Let G be a stratified group, with the stratification $\mathfrak{g} = V_1 \oplus V_2 \oplus \ldots \oplus V_r$, and fix a norm $\|\cdot\|$ on V_1. We can construct a distance d homogeneous with respect to δ_λ, that is,

$$d(\delta_\lambda(p), \delta_\lambda(q)) = \lambda d(p,q) \qquad \forall \lambda > 0 \ \ \forall p,q \in G.$$

First we extend V_1 and $\|\cdot\|$ to a left-invariant subbundle Δ of the tangent bundle and a left-invariant norm on Δ by left translations:

$$\begin{cases} \Delta_\sigma = (dl_\sigma)_0 V_1 \ \ \forall \sigma \in G \\ \|(dl_\sigma)_0(v)\| = \|v\| \ \ \forall v \in V_1. \end{cases}$$

Now we define the *Carnot-Carathéodory distance* or *CC-distance* associated with Δ and $\|\cdot\|$ via piecewise smooth paths $\gamma \in C^\infty_{pw}([0,1], G)$ as

$$d(p,q) = \inf \left\{ \int_0^1 \|\gamma'(t)\| dt : \gamma \in C^\infty_{pw}([0,1],G),\ \gamma(0) = p,\ \gamma(1) = q,\ \gamma'(t) \in \Delta \right\}.$$

We call the data $(G, \delta_\lambda, \Delta, \|\cdot\|, d)$ a *Carnot group* or, more explicitly, *subFinsler Carnot group*. Usually, the term Carnot group is used when the norm comes from a scalar product, but in this paper we shall make no distinction.

3. The Brunn-Minkowski inequality

We have seen that any simply connected nilpotent group is isomorphic to $\mathbb{R}^d$ with a product of the form (2.1). Now we prove the Brunn-Minkowski inequality for any product $* : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ of the form (*). This product does not necessarily define a group structure in $\mathbb{R}^d$. Given such a map F and $z_1', w_1' \in \mathbb{R}$, we can define another product $*_{z_1', w_1'} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, by

$$z *_{z_1',w_1'} w = z + w + F((z_1', \tilde{z}), (w_1', \tilde{w})),$$

where $\tilde{z} = (z_2, \ldots, z_d)$, $\tilde{w} = (w_2, \ldots, w_d)$. We define the map $F_{(z_1',w_1')} : \mathbb{R}^{d-1} \times \mathbb{R}^{d-1} \to \mathbb{R}^{d-1}$ by

$$F_{(z_1',w_1')}(\tilde{z}, \tilde{w}) := (F_2, \ldots, F_d)((z_1', \tilde{z}), (w_1', \tilde{w})). \tag{3.1}$$

Notice that $F_i((z_1', \tilde{z}), (w_1', \tilde{w}))$ only depends on the first $i-2$ variables of $\tilde{z}$ and $\tilde{w}$ and so $F_2((z_1', \tilde{z}), (w_1', \tilde{w}))$ is constant. Thus the product $\tilde{*} : \mathbb{R}^{d-1} \times \mathbb{R}^{d-1} \to \mathbb{R}^{d-1}$ given by

$$\tilde{z} \,\tilde{*}\, \tilde{w} = \tilde{z} + \tilde{w} + F_{(z_1',w_1')}(\tilde{z}, \tilde{w}), \tag{3.2}$$

has the form (*). Notice that the product $\tilde{*}$ depends on the choice of z_1', w_1'.

Lemma 3.1. *Let $* : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ be a product of the form (*) and let $A, B \subset \mathbb{R}^d$ be $A = I \times \tilde{A}$, and $B = J \times \tilde{B}$, where I, J are compact intervals in $\mathbb{R}$ and $\tilde{A}, \tilde{B} \subset \mathbb{R}^{d-1}$ are compact. Then*

$$|A * B| \geq |I + J|_1 |\tilde{A} \,\tilde{*}\, \tilde{B}|_{d-1}, \tag{3.3}$$

where $\tilde{*}$ is the product described in (3.2) for certain $z_1' \in I$ and $w_1' \in J$. Moreover, if F does not depend on z_1, w_1, then equality holds in (3.3).

Proof. Notice that $A*B$ and $\tilde{A}\tilde{*}\tilde{B}$ are compact, and so measurable. Let $I=[a,b]$, $J=[a',b']$ and $l=b-a$, $l'=b'-a'$. The product is

$$A*B=\left\{z+w+F(z,w): z_1\in I,\ w_1\in J,\ \tilde{z}\in\tilde{A},\ \tilde{w}\in\tilde{B}\right\}.$$

We define a diffeomorphism $\phi:\mathbb{R}^2\to\mathbb{R}^2$ by $(s,z)\mapsto(z,s-z)$. The inverse $\phi^{-1}(z,w)=(z+w,z)$ is a diffeomorphism between the sets $I\times J$ and $\{(s_1,z_1): s_1\in I+J,\ z_1\in I\cap(s_1-J)=K(s_1)\}$. Hence, we clearly have

$$A*B=\left\{(s_1,\tilde{z}+\tilde{w})+(F_1,F_{\phi(s_1,z_1)})(\tilde{z},\tilde{w}): s_1\in I+J,\tilde{z}\in\tilde{A},z_1\in K(s_1),\tilde{w}\in\tilde{B}\right\}.$$

Now we use Fubini's theorem and we obtain

$$|A*B|=\int_{I+J}h(s_1)ds_1, \tag{3.4}$$

where $h:I+J\to\mathbb{R}_0^+$ is the function

$$h(s_1)=\left|\{\tilde{p}\in\mathbb{R}^{d-1}:(s_1+F_1,\tilde{p})\in A*B\}\right|_{d-1}=\left|\bigcup_{z_1\in K(s_1)}D_{(s_1,z_1)}\right|_{d-1}, \tag{3.5}$$

and

$$D_{(s_1,z_1)}=\{\tilde{z}+\tilde{w}+F_{\phi(s_1,z_1)}(\tilde{z},\tilde{w}):\ \tilde{z}\in\tilde{A},\ \tilde{w}\in\tilde{B}\}. \tag{3.6}$$

Now we compare $h(s_1)$ with the measure of $D_{(s_1,z_1)}$ for some z_1. Let $z_1:I+J\to\mathbb{R}$ be the function

$$z_1(s_1)=tl+a,$$

where $t=\frac{s_1-(a+a')}{l+l'}$. It is clear that $0\le t\le 1$, hence $z_1(s_1)\in I$. Moreover,

$$tl+a=s_1-tl'-a',$$

and therefore $z_1(s_1)\in s_1-J$. Then $z_1(s_1)\in K(s_1)$.

Let $f:I+J\to\mathbb{R}_0^+$ be the map given by $f(s_1)=\left|D_{(s_1,z_1(s_1))}\right|_{d-1}$. It is easy to check that f is continuous, and hence f reaches its minimum at a certain value s_1'. Thus, we get

$$\int_{I+J}h(s_1)ds_1\ge\int_{I+J}f(s_1)ds_1\ge\int_{I+J}f(s_1')ds_1=|I+J|_1 f(s_1'). \tag{3.7}$$

Denoting by $z_1':=z_1(s_1')$ and $w_1':=s_1'-z_1'$, we can write $F_{\phi(s_1',z_1')}=F_{(z_1',w_1')}$. Hence we have that $D_{(s_1',z_1')}=\tilde{A}\tilde{*}\tilde{B}$ and

$$f(s_1')=|\tilde{A}\tilde{*}\tilde{B}|_{d-1}. \tag{3.8}$$

From (3.4), (3.7) and (3.8) we obtain (3.3).

Suppose that F does not depend on z_1,w_1, let us prove that equality holds in (3.3). It is enough to prove equality in (3.7). For any $s_1\in I+J$ and $z_1\in K(s_1)$, we have that

$$F_{\phi(s_1',z_1')}=F_{z_1',w_1'}=F_{z_1,w_1}=F_{\phi(s_1,z_1)},$$

where $w_1 = s_1 - z_1$. Therefore

$$D_{(s_1',z_1')} = D_{(s_1,z_1)} = \bigcup_{z_1 \in K(s_1)} D_{(s_1,z_1)}. \tag{3.9}$$

Hence, from (3.5) and (3.9) we get that $f(s_1') = h(s_1)$ for all $s_1 \in I + J$. Thus equality holds in (3.7) and the result follows. □

Remark 3.2. The product $*_{z_1',w_1'}$ does not depend on z_1, w_1 and Lemma 3.1 guarantees

$$|A * B| \geq |I + J|_1 |\tilde{A} \tilde{*} \tilde{B}|_{d-1} = |A *_{z_1',w_1'} B|. \tag{3.10}$$

Recall that $*_{z_1',w_1'}$ acts as a sum in the first two coordinates, and someway (3.10) allows us to compare the measure of $A * B$ with the measure of a set more similar to the Euclidean Minkowski addition of A and B.

Proof of Theorem 1.1. The proof is divided into three steps.

Step 1. We first claim that (1.1) holds for a pair of d-rectangles A and B, that is,

$$A = I_1 \times \cdots \times I_d$$
$$B = J_1 \times \cdots \times J_d,$$

where I_i, J_j are compact intervals $\forall\ 1 \leq i, j \leq d$. We shall see that

$$|A * B| \geq |I_1 + J_1|_1 \ldots |I_d + J_d|_1 = |A + B|, \tag{3.11}$$

and the classical Brunn-Minkowski inequality in $\mathbb{R}^d$ would imply (1.1).

In order to prove (3.11), we use Lemma 3.1 to obtain

$$|A * B| \geq |I_1 + J_1|_1 |\tilde{A} \tilde{*} \tilde{B}|_{d-1},$$

but now $\tilde{A} = I_2 \times (I_3 \times \ldots \times I_d), \tilde{B} = J_2 \times (J_3 \times \ldots \times J_d)$ and $\tilde{*}$ has the form (*), and so we can apply Lemma 3.1 to the sets $\tilde{A}$ and $\tilde{B}$. Iterating this process, we get (3.11).

Step 2. Now we consider the case where A and B are finite unions of dyadic d-rectangles, that is, $A = A_1 \cup \ldots \cup A_n$, $B = B_1 \cup \ldots \cup B_m$ where $A_i = I_1^i \times \ldots \times I_d^i$, $B_j = J_1^j \times \ldots \times J_d^j$ and, for any $k = 1, \ldots, d$ and $r \neq s$ $(p \neq q)$, it is satisfied that either $\operatorname{int}(I_k^r) \cap \operatorname{int}(I_k^s) = \emptyset$ or $I_k^r = I_k^s$ (either $\operatorname{int}(J_k^p) \cap \operatorname{int}(J_k^q) = \emptyset$ or $J_k^p = J_k^q$), where $\operatorname{int}(I)$ denotes the interior of I.

We proceed by induction on the total number $n + m$ of d-rectangles. If $n + m = 2$, then A and B are d-rectangles and we can apply step 1. Suppose that the theorem holds for $n + m - 1$, where $n + m \geq 3$. Then we can find a hyperplane $P : \{z_i = a_i\}$ such that some $A_r \subset \{z_i \geq a_i\}$ and some $A_s \subset \{z_i \leq a_i\}$.

If the hyperplane has as equation $P : \{z_1 = a_1\}$, the proof is the same as the classical proof of Hadwiger and Ohmann for the addition of sets in $\mathbb{R}^d$. We include it for the sake of completeness. The sets

$$A^+ = A \cap \{z_1 \geq a_1\}, \qquad A^- = A \cap \{z_1 \leq a_1\}$$

are unions of d-rectangles whose sum is strictly less than n. We choose a parallel hyperplane $Q : \{z_1 = b_1\}$ verifying that

$$\frac{|B^\pm|}{|B|} = \frac{|A^\pm|}{|A|}, \tag{3.12}$$

where B^+ and B^- are the sets given by

$$B^+ = B \cap \{z_1 \geq b_1\}, \qquad B^- = B \cap \{z_1 \leq b_1\}.$$

Moreover, B^+ and B^- are disjoint unions of d-rectangles whose sum is at most m. We apply the induction hypothesis to the pairs A^+, B^+ and A^-, B^-, and we obtain

$$\begin{aligned} |A^+ * B^+| &\geq (|A^+|^{1/d} + |B^+|^{1/d})^d \\ |A^- * B^-| &\geq (|A^-|^{1/d} + |B^-|^{1/d})^d. \end{aligned} \tag{3.13}$$

On the other hand, $P * Q$ is contained in another vertical plane $\{z_1 = a_1 + b_1\} \subset \mathbb{R}^d$, $A^+ * B^+ \subset (P * Q)^+$, and $A^- * B^- \subset (P * Q)^-$. Therefore $A^+ * B^+$ and $A^- * B^-$ are disjoint sets (up to a null set) in $A * B$. Combining this with (3.12) and (3.13) we get the inequality

$$\begin{aligned} |A * B| &\geq |A^+ * B^+| + |A^- * B^-| \\ &\geq (|A^+|^{1/d} + |B^+|^{1/d})^d + (|A^-|^{1/d} + |B^-|^{1/d})^d \\ &= (|A^+| + |A^-|) \left[1 + \left(\frac{|B|}{|A|} \right)^{1/d} \right]^d \\ &= (|A|^{1/d} + |B|^{1/d})^d, \end{aligned}$$

and the theorem is proved for such A and B.

If there is no such hyperplane with equation $P : \{z_1 = a_1\}$ but with equation $P : \{z_2 = a_2\}$, then for any u, v, p, q, $I_1^u = I_1^v = I_1$, $J_1^p = J_1^q = J_1$ and for some $r \neq s$, $\operatorname{int}(I_2^r) \cap \operatorname{int}(I_2^s) = \emptyset$, and we can write

$$A = \bigcup_i I_1 \times I_2^i \times \ldots \times I_d^i = I_1 \times \left(\bigcup_i I_2^i \times \ldots \times I_d^i \right) = I_1 \times \tilde{A}$$

$$B = \bigcup_j J_1 \times J_2^j \times \ldots \times J_d^j = J_1 \times \left(\bigcup_j J_2^j \times \ldots \times J_d^j \right) = J_1 \times \tilde{B}.$$

We have seen in (3.10) that

$$|A * B| \geq |A *_{z_1', w_1'} B|.$$

Now we repeat the above argument, where now we apply the induction hypothesis to the product $*_{z_1', w_1'}$, thus the sets $A^+ *_{z_1', w_1'} B^+$ and $A^- *_{z_1', w_1'} B^-$ are disjoint (up to a null set). Hence, by (3.10) we obtain

$$|A * B| \geq |A *_{z_1', w_1'} B| \geq |A^+ *_{z_1', w_1'} B^+| + |A^- *_{z_1', w_1'} B^-| \geq (|A|^{1/d} + |B|^{1/d}|)^{1/d}$$

and the result is proved.

Repeating this reasoning we have covered the general case where $P : \{z_i = a_i\}$.

Step 3. Let us prove (1.1) for A and B are measurable sets such that $A * B$ is measurable. We can suppose that A, B and $A * B$ have finite measure, since otherwise the inequality is trivial. Fix $\varepsilon > 0$ and take an open set O such that $A * B \subset O$ and $|O \setminus A * B| < \varepsilon$. Take open sets $O_A \supset A$ and $O_B \supset B$ such that $|O_A \setminus A| < \varepsilon$ and $|O_B \setminus B| < \varepsilon$. Since $*$ is continuous, we can assume also that $O_A * O_B \subset O$. Now we approximate the open sets O_A and O_B from inside by dyadic d-rectangles, D_A and D_B so that $|O_A \setminus D_A| < \varepsilon$, $|O_B \setminus D_B| < \varepsilon$. Using step 2 for D_A and D_B, we obtain

$$(|A * B| + \varepsilon)^{1/d} \geq |O|^{1/d} \geq |O_A * O_B|^{1/d} \geq |D_A * D_B|^{1/d}$$
$$\geq |D_A|^{1/d} + |D_B|^{1/d} \geq (|A| - 2\varepsilon)^{1/d} + (|B| - 2\varepsilon)^{1/d}.$$

Taking $\varepsilon \to 0$ we obtain (1.1). □

As a particular case, we have the Brunn-Minkowski inequality in nilpotent groups.

Theorem 3.3 *(Brunn-Minkowski inequality in nilpotent groups). Let G be a simply connected nilpotent group of topological dimension d with Haar measure μ and let $A, B \subset G$ be measurable sets such that $A \cdot B$ is measurable. Then we have*

$$\mu(A \cdot B)^{1/d} \geq \mu(A)^{1/d} + \mu(B)^{1/d}. \tag{3.14}$$

Proof. We denote $\mathfrak{a} = \log(A)$, $\mathfrak{b} = \log(B)$. Using Proposition 2.4 and Theorem 1.1, we have

$$\mu(A \cdot B) = |\log(A \cdot B)| = |\log(\exp(\mathfrak{a}) \cdot \exp(\mathfrak{b}))| = |\mathfrak{a} * \mathfrak{b}| \geq (|\mathfrak{a}|^{1/d} + |\mathfrak{b}|^{1/d})^d$$
$$= (\mu(A)^{1/d} + \mu(B)^{1/d})^d. \quad \square$$

Remark 3.4. Since the right-hand side of (3.14) is symmetric in A and B, it follows

$$\min\{\mu(A \cdot B), \mu(B \cdot A)\}^{1/d} \geq \mu(A)^{1/d} + \mu(B)^{1/d}.$$

An example where $\mu(A \cdot B)$ and $\mu(B \cdot A)$ are different can be found in [16].

Remark 3.5. The arguments used by Leonardi and Masnou [16] can not be applied to this setting. They prove the theorem first for the case where A and B are cubes in $\mathbb{R}^{2n+1}$ of the form $A_1 \times A_2$ where A_1 is a dyadic cube in $\mathbb{R}^{2n}$ and A_2 is a measurable set in $\mathbb{R}$, then when A and B are unions of a finite number of cubes, using then an approximation argument. This has the crucial property that either exists a vertical hyperplane that separates cubes or the union is a cube itself. We call a hyperplane vertical when is also a hyperplane after left multiplication. Then we can consider only vertical hyperplanes to separate cubes. In $\mathbb{R}^d$ with a product of the form (*) this property is not true, since the union of the cubes takes the form

$$\bigcup_i I_1 \times \ldots \times I_{n_1} \times I^i_{n_1+1} \times \ldots \times I^i_d = I_1 \times \ldots \times I_{n_1} \times \left(\bigcup_i I^i_{n_1+1} \times \ldots \times I^i_d \right).$$

This set is not of the form $A_1 \times A_2$ and the argument fails.

3.1. A sufficient condition for strict inequality in the Heisenberg group

A set A in the Heisenberg group $\mathbb{H}^1$ of the form $A = A_1 \times A_2$, where A_1 is a measurable set in $\mathbb{R}^2$ and A_2 is a measurable set in $\mathbb{R}$ is called a *generalized cylinder*.

In this subsection we prove in Proposition 3.6 that the Brunn-Minkowski inequality (3.14) is strict in $\mathbb{H}^1$ for a pair of generalized cylinders A and B such that the volumes of A_1 and B_1 are positive.

Recall that a point a in $\mathbb{R}^d$ is a *density point of A* if

$$\lim_{r \to 0^+} \frac{|A \cap B(a,r)|}{|B(a,r)|} = 1,$$

where $B(a, r)$ is the Euclidean ball of center a and radius r. The set of density points of a set A will be denoted as A^o. We can always normalize a set by including its density points in the set. The existence of a density point in A implies that the volume of A is positive.

Proposition 3.6. *Let $A, B \subset \mathbb{H}^1$ be generalized cylinders such that $A \cdot B$ and $A + B$ are measurable. Suppose that $|A_1| > 0$ and $|B_1| > 0$. Then*

$$|A \cdot B| > |A + B|. \tag{3.15}$$

Proof. By Fubini's theorem, we have

$$|A \cdot B| = \int_{A_1+B_1} h(s_1) ds_1,$$

where $h(s_1) = |\{t + t' + \mathrm{Im}(z\overline{(s_1 - z)}) : t \in A_2,\ t' \in B_2,\ z \in K(s_1)\}|_1$ and $K(s_1) = I \cap (s_1 - J)$. Denoting $s_1 = (s_x, s_y)$, we can see that $\mathrm{Im}(z\overline{(s_1 - z)}) = \mathrm{Im}(z\overline{s_1}) = ys_x - xs_y$. We write

$$I_{s_1} = \{ys_x - xs_y : (x, y) \in K(s_1)\}.$$

By the Brunn-Minkowski inequality in $\mathbb{R}$,

$$\begin{aligned} h(s_1) &= |\{s_2 + \mathrm{Im}(z\overline{s_1}) : s_2 \in A_2 + B_2,\ z \in K(s_1)\}|_1 \\ &= |\{s_2 + a : s_2 \in A_2 + B_2,\ a \in I_{s_1}\}|_1 \\ &\geq |A_2 + B_2|_1 + |I_{s_1}|_1. \end{aligned}$$

We assert that if $|K(s_1)|_2 > 0$, then $|I_{s_1}|_1 > 0$. To see that, we can take the diffeomorphism $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ given by $(x, y) \mapsto \left(ys_x - xs_y, \frac{x}{2s_x} - \frac{y}{2s_y}\right)$. Then $|\mathrm{Jac}(\phi)| = 1$ and applying the change of variables formula to ϕ^{-1}, we have

$$0 < |K(s_1)|_2 = \int_{\mathbb{R}^2} \chi_{K(s_1)}(z) dz = \int_{\mathbb{R}^2} \chi_{\phi(K(s_1))}(z) dz = |\phi(K(s_1))|_2.$$

Now we use that, for any set $O \subseteq \mathbb{R}^2$ with $|O|_2 > 0$, it holds that $|\pi_1(O)|_1 > 0$ where $\pi_1(x, y) = x$, since $|\pi_1(O)|_1 = 0$ implies $|O|_2 \leq |\pi_1(O) \times \mathbb{R}|_2 = 0$. Hence

$$|I_{s_1}|_1 = |\pi_1(\phi(K(s_1)))|_1 > 0.$$

To complete the proof it remains to show that $\{s_1 \in A_1 + B_1 : |K(s_1)|_2 > 0\}$ has positive measure. Let $a \in A_1^o$, $b \in B_1^o$ and $s_1 = a + b \in A_1^o + B_1^o$. Then $a = s_1 - b$ is a density point in $s_1 - B_1$ and therefore a is a density point in $A_1 \cap (s_1 - B_1) = K(s_1)$ which implies that $|K(s_1)|_2 > 0$. Finally $A_1^o + B_1^o \subseteq A_1 + B_1$ has positive measure since $|A_1^o + B_1^o|_2 \geq |A_1^o|_2 = |A_1|_2 > 0$, and

$$|\{s_1 \in A_1 + B_1 : |K(s_1)|_2 > 0\}|_2 \geq |\{s_1 \in A_1^o + B_1^o : |K(s_1)|_2 > 0\}|_2 > 0. \quad \square$$

Remark 3.7. In order to characterize the equality in (3.14) for generalized cylinders, we can distinguish several cases. If A and B lie in parallel vertical hyperplanes, then $|A \cdot B| = 0$ and we have equality in (3.14). If A and B are convex and homothetic then either $|A_1|_2 > 0$ and B_1 is a point and the equality holds, or $|A_1|_2 > 0$ and $|B_1|_2 > 0$, and therefore, by Proposition 3.6 jointly with the (Euclidean) Brunn-Minkowski inequality, equality does not hold in (3.14). The same argument works if A and B lie in horizontal hyperplanes with $|A_1|_2 > 0$ and $|B_1|_2 > 0$. The case in which A and B lie in horizontal hyperplanes with $|A_1|_2 = 0$ is not known in general.

4. Consequences

Another equivalent version of the Brunn-Minkowski inequality in Euclidean space is the Prékopa-Leindler inequality. Now we show how the proof of the Prékopa-Leindler inequality from the Brunn-Minkowski inequality can be adapted to the case of nilpotent groups.

Theorem 4.1 *(Prékopa-Leindler inequality in nilpotent groups). Let G be a simply connected nilpotent group of topological dimension d with Haar measure μ. Let $f, g, h : G \to \mathbb{R}_0^+$ be measurable functions and $0 < \alpha < 1$ verifying*

$$h(a \cdot b) \geq f(a)^{1-\alpha} g(b)^{\alpha} \quad \forall a, b \in G. \tag{4.1}$$

Then

$$\int_G h d\mu \geq \frac{1}{(1-\alpha)^{d(1-\alpha)}\alpha^{d\alpha}} \left(\int_G f d\mu \right)^{1-\alpha} \left(\int_G g d\mu \right)^{\alpha}. \tag{4.2}$$

Proof. We proceed by induction on d.

Let $d = 1$ and $a \cdot b \in \{f > \lambda\} \cdot \{g > \lambda\}$. Then we have $h(a \cdot b) \geq f(a)^{1-\alpha} g(b)^{\alpha} > \lambda$, and as a consequence

$$\{h > \lambda\} \supset \{f > \lambda\} \cdot \{g > \lambda\}.$$

Now we can apply Theorem 3.3 to get

$$\mu(\{h > \lambda\}) \geq \mu(\{f > \lambda\}) + \mu(\{g > \lambda\}).$$

Integrating in λ and using Cavalieri's Principle,

$$\int_G h d\mu = \int_0^\infty \mu(\{h > \lambda\}) d\lambda \geq \int_0^\infty \big(\mu(\{f > \lambda\}) + \mu(\{g > \lambda\})\big) d\lambda = \int_G f d\mu + \int_G g d\mu. \tag{4.3}$$

Now we use the weighted inequality between the geometric and arithmetic means,

$$\int_G f d\mu + \int_G g d\mu \geq \left(\frac{\int_G f d\mu}{1-\alpha} \right)^{1-\alpha} \left(\frac{\int_G g d\mu}{\alpha} \right)^{\alpha}. \tag{4.4}$$

From (4.3) and (4.4) we have (4.2).

Suppose that Theorem 4.1 holds for $d-1$. We shall prove (4.4) for the functions f, g, h composed with exp and use Proposition 2.4. Let $z' = (z_1, \dots, z_{d-1})$, $w' = (w_1, \dots, w_{d-1}) \in \mathbb{R}^{d-1}$. By (2.1), we can write $(z', z_d) * (w', w_d) = (z' *' w', z_d + w_d + P_d(z', w'))$. Recall that $\mathbb{R}^d$ is isomorphic to $\mathfrak{g}$ once we fix the strong Malcev basis $\{X_1, \dots, X_d\}$, and X_d spans an ideal $\mathfrak{h}_1$ in $\mathfrak{g}$. Thus $\mathfrak{g}/\mathfrak{h}_1 \cong (\mathbb{R}^{d-1}, *')$ is a nilpotent group. Now we define the functions $\tilde{f}, \tilde{g}, \tilde{h} : \mathbb{R} \to \mathbb{R}_0^+$ by

$$\begin{aligned} \tilde{f}(z_d) &= (f \circ \exp)(z', z_d), \\ \tilde{g}(w_d) &= (g \circ \exp)(w', w_d), \\ \tilde{h}(t) &= (h \circ \exp)(z' *' w', t + P_d(z', w')). \end{aligned}$$

Let us see that these functions verify (4.1):

$$\tilde{h}(z_d + w_d) = (h \circ \exp)((z', z_d) * (w', w_d)) = h(\exp(z', z_d) \cdot \exp(w', w_d))$$
$$\geq (f \circ \exp)^{1-\alpha}(z', z_d)(g \circ \exp)^{\alpha}(w', w_d) = \tilde{f}^{1-\alpha}(z_d)\tilde{g}^{\alpha}(w_d). \quad (4.5)$$

By induction hypothesis,

$$\int_{\mathbb{R}} \tilde{h}(t)dt \geq \frac{1}{(1-\alpha)^{(1-\alpha)}\alpha^{\alpha}} \left(\int_{\mathbb{R}} \tilde{f}(z_d)dz_d \right)^{1-\alpha} \left(\int_{\mathbb{R}} \tilde{g}(w_d)dw_d \right)^{\alpha}. \quad (4.6)$$

By the invariance of the 1-dimensional Lebesgue measure by translations we get

$$\int_{\mathbb{R}} (h \circ \exp)(z' *' w', t)dt = \int_{\mathbb{R}} \tilde{h}(t)dt. \quad (4.7)$$

Inequality (4.5) is valid for any $z', w' \in \mathbb{R}^{d-1}$, and we can define the functions $F, G, H : \mathbb{R}^{d-1} \to \mathbb{R}_0^+$ given by

$$\begin{aligned} F(z') &= \frac{1}{(1-\alpha)} \int_{\mathbb{R}} \tilde{f}(z_d)dz_d \\ G(w') &= \frac{1}{\alpha} \int_{\mathbb{R}} \tilde{g}(w_d)dw_d \\ H(z') &= \int_{\mathbb{R}} (h \circ \exp)(z', t)dt. \end{aligned} \quad (4.8)$$

Applying (4.7) we can rewrite (4.6) as

$$H(z' *' w') = \int_{\mathbb{R}} \tilde{h}(t)dt \geq F(z')^{1-\alpha}G(w')^{\alpha} \quad \forall z', w' \in \mathbb{R}^{d-1},$$

and again by the induction hypothesis, we get

$$\int_{\mathbb{R}^{d-1}} H(z')dz' \geq \frac{1}{(1-\alpha)^{(d-1)(1-\alpha)}\alpha^{(d-1)\alpha}} \left(\int_{\mathbb{R}^{d-1}} F(z')dz' \right)^{1-\alpha} \left(\int_{\mathbb{R}^{d-1}} G(w')dw' \right)^{\alpha}.$$

The result now follows from Fubini's theorem. □

The Prékopa-Leindler inequality in $\mathbb{R}^d$ is usually stated using $h((1-\alpha)x + \alpha y)$ instead of $h(x+y)$ in order to eliminate the factor $((1-\alpha)^{d(1-\alpha)}\alpha^{d\alpha})^{-1}$. This can be done when dilations are defined, and in this case, this inequality takes a more pleasant expression.

Corollary 4.2. *Let G be a stratifiable group of topological dimension d with Haar measure μ and homogeneous dimension Q. Let $f, g, h : G \to \mathbb{R}_0^+$ be measurable functions, and $0 < \alpha < 1$ verifying*

$$h(\delta_{(1-\alpha)}a \cdot \delta_{\alpha}b) \geq f(a)^{1-\alpha}g(b)^{\alpha} \quad \forall a, b \in G.$$

Then

$$\int_G h d\mu \geq (1-\alpha)^{(Q-d)(1-\alpha)} \alpha^{(Q-d)\alpha} \left(\int_G f d\mu \right)^{1-\alpha} \left(\int_G g d\mu \right)^{\alpha}.$$

Proof. For the sake of simplicity, $\delta_\lambda(a)$ will be just written as λa for any $\lambda > 0$ and $a \in G$. We denote $a' = (1-\alpha)a$, $b' = \alpha b$, $f_{1-\alpha}(a) = f(\frac{a}{1-\alpha})$ and $g_\alpha(a) = g(\frac{a}{\alpha})$. Then we have

$$h(a' \cdot b') \geq f(a)^{1-\alpha} g(b)^\alpha = f\left(\frac{a'}{1-\alpha}\right)^{1-\alpha} g\left(\frac{b'}{\alpha}\right)^{\alpha} = f_{1-\alpha}(a')^{1-\alpha} g_\alpha(b')^\alpha.$$

By Theorem 4.1, we have

$$\int_G h d\mu \geq \frac{1}{(1-\alpha)^{d(1-\alpha)} \alpha^{d\alpha}} \left(\int_G f_{1-\alpha} d\mu \right)^{1-\alpha} \left(\int_G g_\alpha d\mu \right)^{\alpha}.$$

Using now Proposition 2.8,

$$\int_G f_{1-\alpha}(a) d\mu(a) = \int_G f\left(\frac{a}{1-\alpha}\right) d\mu(a) = (1-\alpha)^Q \int_G f(a') d\mu(a'),$$

and after using also Proposition 2.8 for the integral of g_α, we obtain

$$\int_G h d\mu \geq (1-\alpha)^{(Q-d)(1-\alpha)} \alpha^{(Q-d)\alpha} \left(\int_G f d\mu \right)^{1-\alpha} \left(\int_G g d\mu \right)^{\alpha}. \quad \square$$

As we can find in [21], there are several equivalent statements for the Brunn-Minkowski inequality in Euclidean space. Similarly, we have the following result.

Corollary 4.3 *(Multiplicative Brunn-Minkowski inequalities in Carnot groups). Let G be a Carnot group of topological dimension d with Haar measure μ and homogeneous dimension Q. Let $A, B \subset G$ be measurable sets such that $A \cdot B$ is measurable, and $0 < \alpha < 1$. Then*

$$\mu(\delta_{(1-\alpha)} A \cdot \delta_\alpha B)^{1/d} \geq (1-\alpha)^{Q/d} \mu(A)^{1/d} + \alpha^{Q/d} \mu(B)^{1/d}.$$
$$\mu(\delta_{(1-\alpha)} A \cdot \delta_\alpha B) \geq (1-\alpha)^{(Q-d)(1-\alpha)} \alpha^{(Q-d)\alpha} \mu(A)^{1-\alpha} \mu(B)^\alpha.$$

Proof. We use Theorem 3.3 with the sets $\delta_{(1-\alpha)} A$ and $\delta_\alpha B$, and from Proposition 2.8 we get the first inequality.

For the second one, we take $f = \chi_A$, $g = \chi_B$ and $h = \chi_{\delta_{(1-\alpha)} A \cdot \delta_\alpha B}$ and apply Corollary 4.2, obtaining the result. $\square$

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