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Innovative Applications of O.R.

Hidden markov models in reliability and maintenance

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Although the hidden Markov models (HMM) are very popular in many applied areas their use in reliability engineering is limited. Problems such as the selection of the HMM model by choosing the appropriate number of states, or problems of prediction of failures have not been widely covered in the literature. This paper is concerned with the use of HMMs where the state of the system is not directly observable and instead certain indicators of the true situation are provided via a control system. A hidden model can provide key information about the system dependability such as the failed component of the system, the reliability of the system and related measures. A maximum-likelihood estimator of the system reliability is obtained and its asymptotic properties are studied. Finally, the maintenance of the system is considered in this context and new preventive maintenance strategies are defined and their efficiency is measured in terms of expected cost. To prove the finite sample performance of the methodology, an extensive simulation study is developed.

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1. Introduction

A stochastic model can describe the evolution-in-time of a stochastic system. The estimation of its local characteristics is derived from observation data of its evolution in a given interval of time, based on one or several trajectories. In general, data can be provided by sensors that can be interpreted in order to predict the real state of the system. All modern systems include sensors aimed to describe internal or environmental changes of the system functional conditions that influence to their performance level. For example, a car has sensors of pressure, temperature, etc., in order to describe the functional conditions of its engine, brakes, etc. The important challenge for engineers is to provide methods and devices to define the state of the engine, for example, of the car given the values of these indicators. This is a typical inverse problem where several methods can be applied. Some of these methods are the so called hidden Markov models (HMM), which are based on a coupled process (e.g. Markov chain), say (X, Y), where X is an unobserved random sequence, describing the state of the system (i.e., engine), and Y is an observable random sequence, giving the values of the parameters of some indicators (i.e., pressure, temperature, etc.), whose law depends on the value of the corresponding unobserved sequence X. In order to be able to handle the above coupled process, we have to assume some particular probabilistic structure. For example, for X we can suppose that it is an i.i.d sequence or a Markov or semi-Markov chain; while for Y it can be thought as conditionally independent on X sequence, with its law depending on the corresponding value of X.

We have a number of different situations where this model can be used in the real data case. For example in Rex data (field data), where we have a system (a device) with lifetime data and maintenance data matching together. If for example, in this case we have several identical systems and we estimate their lifetime distributions, they are not the same. That means we have some additional random factor. This random factor could be a Markov chain as a random media of our main system. The situation of experimental data is the same. This is the case of the Virkler's data that we present here (see Section 6). The same idea is possible considering expert's opinion to describe the lifetime of the device. The random factor here could be the different experience of each expert. A different situation is when we describe a system, i.e, our car etc.

It is a common practice in Statistics, in order to be able to control the results, to produce simulated data. For example, we produce trajectories (data) for Markov chain, from a given transition probability matrix, and then we use these trajectories as data entries to our estimator of the transition probability matrix without

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any reference to the given matrix. Finally, we can compare the given matrix with the estimated one. Nevertheless, as the ultimate goal is to estimate parameters and functions using real data, we have also to try using real data, even if these data is a little difficult to obtain (see Section 6).

Another important aspect for engineers is keeping the system in an appropriate working state. This is achieved by implementing good maintenance strategies for each particular system. Maintenance involves planned (preventive) and unplanned (corrective) actions carried out to retain a system in or restore it to an acceptable operating condition (Pham & Wang, 1996). Preventive maintenance (PM) actions attempt to retain the system in an acceptable operating condition preventing its failure. Corrective maintenance refers to all the actions that occur when the system has already failed. PM is essential to reduce operating costs and the risk of a catastrophic failure (Shey-Huei, Chin-Chih, Yen-Luan, & Zhe, 2015).

After a maintenance action takes place the system is in a new state, that depends not only on the performed maintenance action but also how well it was performed. Maintenance actions can be perfect, restoring the system to an as good as new condition (AGAN), minimal, keeping the system in the same state as before the maintenance action took place, or something in between (Labeau & Segovia, 2011). In most situations the system is not back to an AGAN condition, but rather to a state previous to the moment the maintenance action took place. This is known as Imperfect Maintenance. Estimating this state has been widely considered in the literature. For example, Kijima I and Kijima II (Kijima, 1989) are two well-known classical models. The first model considers that maintenance actions can only remove damage on the system since previous maintenance intervention, the second model assumes that maintenance can remove part of the cumulated damage up to the moment the intervention takes place. Other well established models are Arithmetic Reduction of the Intensity (ARI) and Arithmetic Reduction of the Age (ARA) models (Doyen & Gaudoin, 2004). ARI models consider that the failure intensity of the system is reduced after maintenance. ARA models focuses on reducing the age of the system after the intervention. These are just some examples but there are many others.

In this paper we will suppose that $X = \{X_n; n \geq 0\}$ is a Markov chain taking values in a set $E = \{e_1, e_2, \dots, e_d\}$ where transitions between states are given by an unknown matrix $\mathbf{P}$; and, that $Y = \{Y_n; n \geq 0\}$ is a random sequence conditionally independent and stationary such that $P(Y_n \in B | X_n = e_j) = M(e_j, B)$, with $j = 1, 2, \dots, d$, $B \subset \mathbb{R}^k$, for $k \geq 1$ in general, and $\mathbf{M}$ is an unknown function. We consider the case where Y takes values in a finite set $A = \{a_1, a_2, \dots, a_s\}$. The $d \times s$-matrix $\mathbf{M}$ with (i, j) element $M(e_i, a_j)$ is called the emission matrix. This model is described as HMM M1-M0, that is, a Markov chain of order one for X, and a Markov chain of order zero for Y. They are dynamical stochastic models and this is the main reason of their usefulness to model real systems.

The problem here is to estimate the transition matrix of the Markov chain X and the emission probability for Y as above. As it is always the case for missed data, we cannot obtain a closed form solution for the maximum likelihood estimator (MLE), but we have to use approximation numerical methods as the EM-algorithm. The main application of the proposed model concerns reliability and maintenance of complex systems, see Landon, Ozekici, & Soyer (2013), or Vrignat, Avila, Duculty, & Kratz (2015), among others, and also Wang (2002), and Jonge & Scarf (2020) for an up to date review on maintenance theory and applications.

Several studies in the literature concern this kind of models from theoretical point of view (Baum & Petrie, 1966; Trevezas & Limnios, 2009) and practical applications in modelling and analysing biological sequences, as DNA (Barbu & Limnios, 2008), in environmental sustainability problems analysis (Jiang & Liu, 2015), and recently in reliability analysis (Durand & Gaudoin, 2005; Fort, Mugnaini, & Vignoli, 2015; Simoes, Viegas, Torres Farinha, & Fonseca, 2017; Votsi, Limnios, Tsaklidis, & Papadimitriou, 2013; Zhou, Hu, Xu, Chen, & Zhou, 2010).

The present paper is organized as follows. In Section 2 general considerations about hidden Markov models are treated. In Section 3 a new approach of reliability analysis based on HMM is presented. Section 4 is devoted to maintenance issues. Numerical applications are developed in Section 5 and Section 6 where we discuss simulated data as well as a real dataset. Finally, Section 7 gives the conclusions and suggests future research lines.

2. Hidden Markov models

In a hidden modelling context, there are three basic problems that must be solved for the model to be useful in real-world applications (Rabiner, 1989). The most difficult one is the training problem, which consists of optimally estimate the parameters of the model from observed data. We usually call the available observations the *training dataset*. Once the model is constructed (trained), we need to evaluate the model, that is we address the evaluation problem. This involves the calculation of several probabilities associated to the model estimated parameters. Among others we want to score how well the estimated model fit the data, which is useful to choose between competing candidate models. Finally, we are mostly interested in uncovering the hidden part of the model, then we have a decoding problem. In short it means that we want to determine the “optimal” sequence of hidden states that originated the output sequence we actually observed. To do it, we first need to decide which optimality criterion best fits our purposes. In the particular case of reliability applications, solving these problems allow us to get key information about the system performance. More specifically: knowing an estimator $\widehat{\theta}$ of θ we can estimate by the usual plug in estimation procedure the reliability, availability, mean times, etc. (*evaluation*) and find the way the system fails (*decoding*), e.g. via a Viterbi algorithm.

2.1. Preliminary

Let us consider two finite sets, say $E = \{e_1, \dots, e_d\}$ and $A = \{a_1, \dots, a_s\}$, and a sequence of coupled r.v. $(X_n, Y_n)_{n \geq 0}$, where (X_n) is a Markov chain of order 1 (CM1), with values in E, and transition matrix $\mathbf{P}$, and initial law α, and (Y_n) a sequence of r.v. with values in A whose law depends on values of (X_n), in the following way:

$$M(i, l) = \mathbb{P}(Y_n = l \mid X_n = i), \quad i \in E, l \in A, \tag{1}$$

for all $n \in \mathbb{N}$. The matrix $\mathbf{M}$ is called an emission matrix. As usual we call the elements of E the *states* of the system while the elements of A are referred to as *signals*.

We can write then

$$\begin{aligned} &\mathbb{P}(X_n = j, Y_n = l \mid X_0 = i_0, Y_0 = l_0, \dots, X_{n-1} = i, Y_{n-1} = l_{n-1}) \\ &\quad = \mathbb{P}(X_n = j, Y_n = l \mid X_{n-1} = i) \\ &\quad = P(i, j) M(j, l). \end{aligned}$$

This model is denoted by $M1 - M0$, where $M1$ is referred to the Markov chain of order 1, X, and $M0$ to the chain Y of order zero with respect to itself. A more general case is the model $M1 - Mk$ with $k \geq 1$. In the last case, the conditional law of Y is

$$\begin{aligned} &\mathbb{P}(Y_n = l \mid X_0 = i_0, Y_0 = l_0, \dots, X_{n-1} = i_{n-1}, Y_{n-1} = l_{n-1}) \\ &\quad = \mathbb{P}(Y_n = l \mid X_{n-1} = i_{n-1}, Y_{n-1} = l_{n-1}, \dots, Y_{n-k} = l_{n-k+1}) \end{aligned}$$

It is obvious that the Markov chain X can be considered of order m, which is the model $Mm - Mk$.

In this paper we limit ourselves to the case *HMM (M1-M0)*. The problem here is to estimate the parameters of $\mathbf{P}$ and $\mathbf{M}$.

Let us denote the independent parameters of the model by $\boldsymbol{\theta} = (P(i,j)_{i\neq j; i,j\in E}; M(i,l)_{i\in E; l\in A\ \{a_s\}})$.

The log-likelihood can be written, by neglecting the term $\log P(X_0)$, as

$$\log p_\theta(Y) = \sum_X \log f(X, Y \mid \theta), \tag{2}$$

where $Y = (Y_0, \dots, Y_n)$ and $X = (X_0, \dots, X_n)$ and it is included in (3)

$$\log f(X, Y \mid \theta) = \sum_{k=1}^{n} \log P(X_{k-1}, X_k) + \sum_{k=0}^{n} \log M(X_k, Y_k). \tag{3}$$

2.2. The EM-algorithm

In order to estimate $\boldsymbol{\theta}$ we will apply E-M algorithm as follows. The function $Q(\theta \mid \theta^{(m)})$ will give us by successive iterations an approximation of the estimate of θ.

$$\begin{aligned} Q\left(\theta \mid \theta^{(m)}\right) &= \mathbb{E}_{\theta^{(m)}}[\log f(X, Y \mid \theta)] \\ &= \sum_{k=1}^{n} \mathbb{E}_{\theta^{(m)}}[\log P(X_{k-1}, X_k) \mid Y] \\ &\quad + \sum_{k=0}^{n} \mathbb{E}_{\theta^{(m)}}[\log M(X_k, Y_k) \mid Y]. \end{aligned}$$

Finally,

$$\begin{aligned} Q\left(\theta \mid \theta^{(m)}\right) &= \sum_{k=1}^{n} \sum_{i,j\in E} \mathbb{P}_{\theta^{(m)}}(X_{k-1} = i, X_k = j \mid Y) \log P(i,j) \\ &\quad + \sum_{k=0}^{n} \sum_{i\in E} \sum_{a\in A} \mathbb{P}_{\theta^{(m)}}(X_k = i \mid Y) \mathbf{1}_{\{Y_k = a\}} \log M(i,a). \end{aligned} \tag{4}$$

The calculus of this expectation needs the calculus of the probabilities $\mathbb{P}_{\theta(m)}(X_{k-1} = i, X_k = j \mid Y)$.

In order to pass from $\theta^{(m)}$ to $\theta^{(m+1)}$,

$$\theta^{(m+1)} = \arg\max_\theta Q\left(\theta \mid \theta^{(m)}\right), \qquad m = 0, 1, 2, \dots \tag{5}$$

the following two-steps algorithm is used. This is the well known EM algorithm.

EM-Algorithm

Step E (Expectation):

For given $\theta^{(m)}$, compute the probabilities:

$$\mathbb{P}_{\theta^{(m)}}(X_{k-1} = i, X_k = j \mid Y), \quad k = 1, 2, \dots, n; \quad i, j \in E$$

Step M (Maximization):

Update $\theta^{(m)}$ to $\theta^{(m+1)}$ via (5)

The maximization step M, is realized directly by the following formulas:

$$\hat{P}^{(m+1)}(i,j) = \frac{\sum_{k=1}^{n} \mathbb{P}_{\theta^{(m)}}(X_{k-1} = i, X_k = j \mid Y)}{\sum_{k=1}^{n} \mathbb{P}_{\theta^{(m)}}(X_{k-1} = i \mid Y)}, \tag{6}$$

and

$$\hat{M}^{(m+1)}(i,a) = \frac{\sum_{k=0}^{n} \mathbb{P}_{\theta^{(m)}}(X_k = i \mid Y) \mathbf{1}_{\{Y_k = a\}}}{\sum_{k=0}^{n} \mathbb{P}_{\theta^{(m)}}(X_k = i \mid Y)}. \tag{7}$$

2.3. The E-Step: Forward-backward equations

The probabilities that need to be computed in the E-step can be obtained by means of the "forward-backward" procedure as explained in the following.

For given $\theta^{(m)}$, compute the probabilities:

$$\mathbb{P}_{\theta^{(m)}}(X_{k-1} = i, X_k = j \mid Y), \quad k = 1, 2, \dots, n; \quad i, j \in E;$$

and

$$\mathbb{P}_{\theta^{(m)}}(X_k = i \mid Y), \quad k = 1, 2, \dots, n; \quad i \in E.$$

To do it a "forward-backward" procedure is used.

Define the forward probability function $F_k(i)$, for $k = 1, \dots, n$ and $i \in E$ as

$$F_k^{(m)}(i) = \mathbb{P}_{\theta^{(m)}}(Y_0^k; X_k = i), \tag{8}$$

and the backward probability function $B_k(i)$, as

$$B_k^{(m)}(i) = \mathbb{P}_{\theta^{(m)}}(Y_{k+1}^n \mid X_k = i), \tag{9}$$

for $k = 1, \dots, n$ and $i \in E$.

These functions meet, respectively, the following recurrence equations

$$F_k^{(m)}(i) = \sum_{j\in E} F_{k-1}^{(m)}(j) P^{(m)}(j,i) M^{(m)}(i, Y_k),$$

for all $k = 1, 2, \dots, n$, with $F_0^{(m)}(i) = \alpha^{(m)}(i)$, for $i \in E$; and,

$$B_k^{(m)}(i) = \sum_{j\in E} P^{(m)}(i,j) M^{(m)}(j, Y_{k+1}) B_{k+1}^{(m)}(j),$$

for $k = n-1, n-2, \dots, 1$, taking $B_n^{(m)}(i) = 1$ for all $i \in E$.

Also we have that $\mathbb{P}_{\theta^{(m)}}(Y) = \sum_{i\in E} F_n^{(m)}(i)$ and $\mathbb{P}_{\theta^{(m)}}(Y) = \sum_{i\in E} B_0^{(m)}(i)$.

Then it can be written that

$$\mathbb{P}_{\theta^{(m)}}(X_k = i \mid Y) = \frac{F_k^{(m)}(i) B_k^{(m)}(i)}{\mathbb{P}_{\theta^{(m)}}(Y)},$$

and

$$\mathbb{P}_{\theta^{(m)}}(X_k = i, X_{k+1} = j \mid Y) = \frac{F_k^{(m)}(i) P^{(m)}(i,j) M^{(m)}(j, Y_{k+1}) B_{k+1}^{(m)}(j)}{\mathbb{P}_{\theta^{(m)}}(Y)},$$

3. Reliability in HMM

We suppose here that the system structure is described by the hidden Markov chain X and that the state-space is split into two subsets $U := \{1, \dots, r\}$, the working states, and $D := \{r+1, \dots, d\}$, the down states. For simplicity, and without loss of generality, this notation is used for the states of the system.

Additionally, the system up states can be defined not only by $U \subset E$ but also by some subset of A. In some situations, the information we get about the system functioning can be categorized into two groups of signals. On the one hand, we have a group of signals indicating a good performance, the subset $A_1 \subset A$; and, on the other hand, there is a group of $s_1 < s$ signals for warning of some serious problem in the system A_2 that involves the operation interruption thus causing the system failure, that is, the subset $A_2 \subset A$. Then we have also the partition $A = A_1 \cup A_2$.

3.1. Definition

Let us denote T the first time the system visits the set of down states D, i.e. the hitting time of set D. Let us consider $\widetilde{\mathcal{U}} = U \times A_1$ and $\widetilde{\mathcal{D}} = \widetilde{\mathcal{E}}\ \widetilde{\mathcal{U}}$, being $\widetilde{\mathcal{E} = \mathcal{E}} \times A$. Then $T = \min\{n \geq 0 : \tilde{\mathcal{X}} = (X_n, Y_n) \in \widetilde{\mathcal{D}}\}$. Therefore the reliability of the system can be defined as $R(n) = \mathbb{P}(T > n)$, for $n = 1, 2, \dots$. Conditioning on $X_0 = i$, for $i \in U$, we write

$$\begin{aligned} R_i(n) &= \mathbb{P}_i(T > n) \\ &= \sum_{i_1, i_2, \dots, i_n \in U} \sum_{l_0, l_1, \dots, l_n \in A_1} \mathbb{P}_i(X_1^n = i_1^n) \mathbb{P}_i(Y_0^n = l_0^n \mid X_1^n = i_1^n) \\ &= \sum_{i_1^n \in U^n} \sum_{l_0^n \in A_1^{n+1}} P(i, i_1) P(i_1, i_2) \cdots P(i_{n-1}, i_n) \end{aligned}$$

$$\times M(i,l_0)M(i_1,l_1)\cdots M(i_n,l_n),$$

where we use the notation $i_1^n=(i_1,i_2,\ldots,i_n)\in U^n$ and $l_0^n=(l_0,l_1,\ldots,l_n)\in A_1^{n+1}$.

Finally

$$R(n)=\sum_{i\in U}\alpha_i R_i(n) \tag{10}$$

where $\alpha_i=\mathbb{P}(X_0=i)$, for $i\in U$, denotes the initial law.

As we know, the two-dimensional process $\widetilde{\mathcal{X}}=\{(X_n,Y_n);n\geq 0\}$ is a two-dimensional Markov chain of order 1 (CM1) with state-space $\widetilde{\mathcal{E}}$ of size $d\cdot s$ and transition probability matrix $\widetilde{\mathbf{P}}$, with elements

$$\widetilde{P}((i_1,l_1),(i_2,l_2))=P(i_1,i_2)\cdot M(i_2,l_2) \tag{11}$$

for all pairs $(i_1,l_1),(i_2,l_2)\in\widetilde{\mathcal{E}}$. Also, the initial distribution of chain (X,Y) is $\widetilde{\alpha}_{(i,l)}=\alpha_i M(i,l)$.

As can be seen, the matrix $\widetilde{\mathbf{P}}$ has dimension $d\cdot s\times d\cdot s$.

For convenience in the calculations, the states are organized in lexicographical order as follows

$$\widetilde{E}=\{(1,a_1),(2,a_1),\ldots,(d,a_1),(1,a_2),\ldots,\\(d,a_2),\ldots,(1,a_s),\ldots,(d,a_s)\}.$$

The matrix $\widetilde{\mathbf{P}}$ consists then of s blocks of sub-matrices $\mathbf{B}_1,\ldots,\mathbf{B}_s$, with dimension $d\times s\,d$ each. All blocks are identical, $\mathbf{B}_j=\mathbf{B}$, for all $j=1,\ldots,s$, and can be expressed as

$$\mathbf{B}=\left(\mathbf{P}\cdot\mathbf{D}_{\{M_{1,1},\ldots,M_{d,1}\}},\ldots,\mathbf{P}\cdot\mathbf{D}_{\{M_{1,j},\ldots,M_{d,j}\}},\ldots,\mathbf{P}\cdot\mathbf{D}_{\{M_{1,s},\ldots,M_{d,s}\}}\right)$$

where $\mathbf{D}_{\{m_1,\ldots,m_d\}}$ is a d-dimensional diagonal matrix with elements $m_1,\ldots,m_d$. Finally, $\widetilde{\mathbf{P}}=\left(\mathbf{B}'_1,\ldots,\mathbf{B}'_s\right)'$, where we denote $\mathbf{B}'$ the transpose of matrix $\mathbf{B}$.

With this in mind, we can write the reliability function just defined as

$$R(n)=\sum_{(i,l)_0^n\in\widetilde{U}}\widetilde{\alpha}_{(i_0,l_0)}\widetilde{P}((i_0,l_0),(i_1,l_1))\widetilde{P}((i_1,l_1),(i_2,l_2))\cdots\\ \widetilde{P}((i_{n-1},l_{n-1}),(i_n,l_n)), \tag{12}$$

where it is denoted $(i,l)_0^n=\{(i_0,l_0),\ldots,(i_n,l_n)\}$. That is, the system fails whenever the subset D is reached or a signal of subset A_2 is emitted. The summation in Eq. (12) is expanded for all the elements in $\widetilde{U}=U\times A_1$, then, using matrix notation, it can be written

$$R(n)=\widetilde{\alpha}_0\widetilde{\mathbf{P}}_0^n\mathbf{1}_{\widetilde{r}},$$

for $n\geq 1$, with $\mathbf{1}_{\widetilde{r}}$ a unitary vector of size $\widetilde{r}=r\cdot s_1$, being r the total number of up states and s_1 the size of the set A_1.

We define the following estimator of the reliability as

$$\widehat{R}(n;N)=\widehat{\widetilde{\alpha}}_0\widehat{\widetilde{\mathbf{P}}}_0^n\mathbf{1}_{\widetilde{r}},$$

for $n\geq 0$ and N the sample size.

3.2. Asymptotic properties of the reliability estimator

Let us derive the consistency and asymptotic normality of the reliability estimator defined above.

For $\{Y_0,\ldots,Y_N\}$ a sample path of observations, the log-likelihood function for an observation of the hidden Markov chain $\log f(X,Y|\boldsymbol{\theta})$, is given in Eq. (3). The vector of parameters of the model $\boldsymbol{\theta}$, can be written after removing the independent parameters as $\boldsymbol{\theta}=(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2)$, with

$$\boldsymbol{\theta}_1=(P(2,1),\ldots,P(d,1),P(1,2),P(3,2),\ldots,\\P(d,2),\ldots,P(1,d),\ldots,P(d,d-1)),$$

that is, the elements of the matrix $\mathbf{P}$ taken column-wise without the diagonal. The number of parameters to be estimated in this sub-vector is $b_1=d\cdot(d-1)$. On the other hand,

$$\boldsymbol{\theta}_2=(M(1,a_1),\ldots,M(1,a_{s-1}),M(2,a_1),\ldots,M(2,a_{s-1}),\ldots,\\(d,a_1),\ldots,M(d,a_{s-1})),$$

that is the elements of the matrix $\mathbf{M}$ taken column-wise. The size of the sub-vector $\boldsymbol{\theta}_2$ is $b_2=d\cdot(s-1)$. Then, the total number of independent parameters to be estimated is $b_1+b_2=d\cdot(d+s-2)$. So $\boldsymbol{\theta}\in\Theta\subset[0,1]^{d^2+ds-2d}$.

We need the following assumptions.

Assumptions Barbu & Limnios (2008):

A1 The Markov chain X is ergodic, i.e., irreducible and aperiodic; and stationary;

A2 There exists an integer $n\in\mathbb{N}$ such that the Fisher information matrix $I_n(\boldsymbol{\theta}^0)=-E_{\boldsymbol{\theta}^0}\left(\frac{\partial^2\log p_{\boldsymbol{\theta}}(Y_0^n)}{\partial\theta_i\partial\theta_j}\Big|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0}\right)_{i,j}$ is nonsingular, where $\log p_{\boldsymbol{\theta}}(Y_0^n)$ is the log-likelihood function defined in Eq. (2).

Let us denote $\boldsymbol{\theta}^0=\left(\boldsymbol{\theta}_1^0,\boldsymbol{\theta}_2^0\right)$ the true value of the parameter. The following theorem is deduced as a particular case of Theorem 6.1 and Theorem 6.4 in Barbu & Limnios (2008). See also Baum & Petrie (1966) and Bickel, Ritov, & Rydén (1998).

Theorem 1. *Under assumptions A1 - A2, given a sample of observations Y_0^N, the maximum-likelihood estimator $\widehat{\boldsymbol{\theta}}_N=\left(\widehat{\boldsymbol{\theta}}_1,\widehat{\boldsymbol{\theta}}_2\right)_N$ of $\boldsymbol{\theta}=(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2)$ is strongly consistent as N tends to infinity. Moreover, the random vector*

$$\sqrt{N}\left[\left(\widehat{\boldsymbol{\theta}}_1,\widehat{\boldsymbol{\theta}}_2\right)_N-\left(\boldsymbol{\theta}_1^0,\boldsymbol{\theta}_2^0\right)\right]\\=\sqrt{N}\left[\left(\left(\widehat{P}_N(i,j)_{1\leq i,j<d;(i\neq j)}\right),\left(\widehat{M}_N(i,a_l)_{1\leq i\leq d;1\leq l<s}\right)\right)\\-\left(\left(P^0(i,j)_{1\leq i,j<d;(i\neq j)}\right),\left(M^0(i,a_l)_{1\leq i\leq d;1\leq l<s}\right)\right)\right]$$

is asymptotically Normal, as $N\to+\infty$, with zero mean and covariance matrix the inverse of the asymptotic Fisher information matrix $I(\boldsymbol{\theta}^0)$.

The asymptotic Fisher information matrix is given by

$$I(\boldsymbol{\theta}^0)=-E_{\boldsymbol{\theta}^0}\left(\frac{\partial^2\log\mathbb{P}_{\boldsymbol{\theta}}(Y_0|Y_{-1},Y_{-2},\ldots)}{\partial\theta_i\partial\theta_j}\Big|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0}\right)_{i,j},$$

see Baum & Petrie (1966), and in Douc (2005) it is shown that $I(\boldsymbol{\theta}^0)$ is nonsingular under Assumption A2.

From Theorem 1 we immediately obtain the consistency and the asymptotic normality of the reliability estimator $\widehat{R}$. Previously we need the following lemmas. First we consider the following partition of the matrix $\mathbf{P}$

$$\mathbf{P}=\left(\begin{array}{c|c}\mathbf{P}_{\mathbf{UU}}&\mathbf{P}_{UD}\\\mathbf{P}_{DU}&\mathbf{P}_{DD}\end{array}\right)$$

and denote $\boldsymbol{\theta}_{\widetilde{U}}=\left(\boldsymbol{\theta}_{1,U},\boldsymbol{\theta}_{2,1}\right)$ where in the sub-vector denoted $\boldsymbol{\theta}_{1,U}$ we keep only the elements of sub-matrix P_{UU}, which contains all transitions between the up-states; and, in the sub-vector denoted $\boldsymbol{\theta}_{2,1}$ we keep only the elements of sub-matrix M_{UA_1}, which contains all emission probabilities from up-states to good signals.

Accordingly, we can write matrix $\widetilde{\mathbf{P}}$ by blocks as follows

$$\widetilde{\mathbf{P}}=\left(\begin{array}{c|c}\widetilde{\mathbf{P}}_{\widetilde{\mathbf{U}}\widetilde{\mathbf{U}}}&\widetilde{\mathbf{P}}_{\widetilde{U}\widetilde{D}}\\\widetilde{\mathbf{P}}_{\widetilde{D}\widetilde{U}}&\widetilde{\mathbf{P}}_{\widetilde{D}\widetilde{D}}\end{array}\right),$$

where we have considered a similar partition of the state space $\widetilde{E}=\widetilde{U}\cup\widetilde{D}$, where we denote $\widetilde{D}=(U\times A_2)\cup(D\times A)$.

Consistency

Lemma 1. *Under the Assumptions A1 – A2, given a sample of observations Y_0^N, the maximum likelihood estimator of $\left(\widetilde{P}((i,l),(j,h))\right)_{(i,l),(j,h)\in\widetilde{U}}$, that is $\left(\widehat{\widetilde{P}}_N((i,l),(j,h))\right)_{(i,l),(j,h)\in\widetilde{U}}$, is strongly consistent as N tends to infinity.*

Proof. The transition probabilities for the two-dimensional process (X,Y) are obtained as $\widetilde{P}((i,l),(j,h)) = P(i,j)M(j,h)$ for all $i,j\in E$ and $l,h\in A$. Considering the vector of parameters $\boldsymbol{\theta} = \left(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2\right)$ we define the following function

$\Phi \ : \ [0,1]^{d^2+ds-2d} \longrightarrow [0,1]^{r^2\cdot s_1}$

such that $\Phi = \left(\Phi_m; m=1,2,\dots,r^2\cdot s_1\right)$.

For fixed $k=1,\dots,s_1$, let us consider values $j=1,2,\dots,r$. For fixed k and j, we can write $m=(k-1)r^2+(j-1)r+i$, for $1\le m\le r^2 s_1$, and, for $i=1,2,\dots,r$, consider two cases:

- for $i\neq j$, define

$$\Phi_m(\boldsymbol{\theta}) = \Phi_{(k-1)r^2+(j-1)r+i}(\boldsymbol{\theta}) = P(i,j)\cdot M(j,a_k);\ \text{and,} \quad (13)$$

- for $i=j$, define

$$\begin{aligned}\Phi_m(\boldsymbol{\theta}) &= \Phi_{(k-1)r^2+(j-1)r+j}(\boldsymbol{\theta}) \\ &= \left(1-\sum_{j'\in E;\,j'\neq j}(1-P(j,j'))\right)\cdot M(j,a_k). \end{aligned} \quad (14)$$

Then $\Phi = \left(\Phi_{(k-1)r^2+(j-1)r+i}\right)_{k=1,\dots,s_1;j=1,\dots,r;i=1,\dots,r} \in [0,1]^{r^2\cdot s_1}$.

This function returns a vector whose components are the elements of matrix $\widetilde{\mathbf{P}}_{\widetilde{U}\widetilde{U}}$ taken column-wise. Then, using the consistency of the estimator $\widehat{\boldsymbol{\theta}}$, which is deduced from Theorem 6.1 in Barbu & Limnios (2008), recalled by Theorem 1 above, and applying the continuous mapping theorem to the function Φ defined in (13)-(14), we obtain the desired result. □

Lemma 2. *We have that*

$$\max_{0\le n\le N}\ \max_{(i,l),(j,h)\in U\times A}\left|\widehat{\widetilde{P}}{}^n_N((i,l),(j,h)) - \widetilde{P}^n((i,l),(j,h))\right| \xrightarrow{a.s.} 0, \quad (N\to+\infty)$$

Proof. The proof is easily obtained from Lemma 1 given above that gives the proof for $n=1$, then, mathematical induction similarly to Lemma 1 of Sadek & Limnios (2002) can be applied to get the result for all $n\ge 2$. □

Proposition 1. *The estimator $\widehat{R}(n;N)$ is strongly consistent, as $N\to\infty$, for any $n\ge 1$, i.e.,*

$$\widehat{R}(n;N) \xrightarrow{a.s.} R(n)$$

Proof. The proof is similar to Theorem 3 in Sadek & Limnios (2002) and can be deduced straightforwardly from Lemma 1 and Lemma 2 above. □

Asymptotic normality

Lemma 3. *Under the Assumptions A1 – A2, given a sample of observations Y_0^N, the random vector $\mathbf{F}_N = \left(F_{(i,l),(j,h)};_{(i,l),(j,h)\in\widetilde{U}}\right)$ such that*

$$\begin{aligned}F_{(i,l),(j,h)} = \sqrt{N}\Bigg[&\left(\widehat{\widetilde{P}}_N((i,l),(j,h))\right)_{(i,l),(j,h)\in\widetilde{U}} \\ &-\left(\widetilde{P}((i,l),(j,h))\right)_{(i,l),(j,h)\in\widetilde{U}}\Bigg]\end{aligned}$$

is asymptotically Normal, as $N\to+\infty$ with 0 mean and covariance matrix $\Sigma_{\widetilde{P}} = \Phi'\cdot\Sigma_{\boldsymbol{\theta}}\cdot\Phi'^{\top}$, where $\Sigma_{\boldsymbol{\theta}}$ is the covariance matrix of the random vector $\widehat{\boldsymbol{\theta}} = \left(\widehat{\boldsymbol{\theta}}_1,\widehat{\boldsymbol{\theta}}_2\right)$ and Φ is the function defined in (13)-(14) whose partial derivative matrix Φ' has elements given in (15)-(18).

Proof. Theorem 6.4 in Barbu & Limnios (2008) gives the asymptotic normality of the maximum likelihood estimator $\widehat{\boldsymbol{\theta}} = (\widehat{\boldsymbol{\theta}}_1,\widehat{\boldsymbol{\theta}}_2)$, then we can apply the Delta method considering the function Φ defined in (13)-(14) and the derivative matrix is detailed as follows:

- For each $k=1,2,\dots,s_1$, each $j=1,2,\dots,r$, and $i=j$. For all $i',j'\in\{1,2,\dots,d\}$, we have that

$$\frac{\partial\Phi_{(k-1)r^2+(j-1)r+j}}{\partial P(i',j')} = \begin{cases}0, & \text{if } i'\neq j;\\ -M(j,a_k), & \text{if } i'=j.\end{cases} \quad (15)$$

For all $j'\in\{1,\dots,d\}$ and all $k'\in\{1,\dots,s\}$,

$$\frac{\partial\Phi_{(k-1)r^2+(j-1)r+j}}{\partial M(j',a_{k'})} = \begin{cases}0, & \text{if } j'\neq j \text{ or } k'\neq k;\\ 1-\sum_{h\in E;h\neq j}P(j,h), & \text{if } j'=j \text{ and } k'=k;\end{cases} \quad (16)$$

- For each $k=1,2,\dots,s_1$, each $j=1,2,\dots,r$, and each $i=1,2,\dots,r$, $i\neq j$. For all $i',j'\in\{1,2,\dots,d\}$

$$\frac{\partial\Phi_{(k-1)r^2+(j-1)r+i}}{\partial P(i',j')} = \begin{cases}0, & \text{if } i'\neq j, \text{ or } j'\neq j;\\ M(j,a_k), & \text{if } i'=i \text{ and } j'=j.\end{cases} \quad (17)$$

Finally, for all $k'\in\{1,\dots,s\}$, and all $j'\in\{1,\dots,d\}$,

$$\frac{\partial\Phi_{(k-1)r^2+(j-1)r+i}}{\partial M(j',a_{k'})} = \begin{cases}0, & \text{if } k'\neq k, \text{ or } j'\neq j\\ P(i,j), & \text{if } j'=j \text{ and } k'=k;\end{cases} \quad (18)$$

□

Lemma 4. *Let $F^n_{(i,l),(j,h)} = \sqrt{N}\left(\widehat{\widetilde{P}}{}^n_N((i,l),(j,h)) - \widetilde{P}^n((i,l),(j,h))\right)$, for all $n\ge 1$ and $(i,l),(j,h)\in\widetilde{E}$. Then, the random vector $\mathbf{F}^n_N = (F^n_{(i,l),(j,h)})$ converges, as $N\to+\infty$, to a Normal distribution with mean 0, and covariance matrix $\Sigma_{\mathbf{F}^n} = \Gamma_n\Sigma_{\mathbf{F}}\Gamma_n^{\top}$, where Γ_n is a constant matrix and $\Sigma_{\mathbf{F}}$ is the covariance matrix of $\mathbf{F}_N$.*

Proof. This result is directly deduced from the result in Lemma 3 given above which shows the result for $n=1$. Then we can follow similar steps as in Theorem 4 of Sadek & Limnios (2002) and apply mathematical induction to get the result for all $n\ge 2$. Specifically, for $n=2$ we have

$$\begin{aligned}F^2_{(i,l),(j,h)} &= \\ &= \sqrt{N}\left(\widehat{\widetilde{P}}{}^2((i,l),(j,h)) - \widetilde{P}^2((i,l),(j,h))\right) \\ &= \sqrt{N}\sum_{(i_1,l_1)}\Big(\widehat{\widetilde{P}}((i,l),(i_1,l_1))\widehat{\widetilde{P}}((i_1,l_1),(j,h)) \\ &\quad -\widetilde{P}((i,l),(i_1,l_1)\widetilde{P}((i_1,l_1),(j,h))\Big) \\ &= \sqrt{N}\sum_{(i_1,l_1)}\Big\{\widehat{\widetilde{P}}((i,l),(i_1,l_1))\Big[\widehat{\widetilde{P}}((i_1,l_1),(j,h)) \\ &\quad -\widetilde{P}((i_1,l_1),(j,h))\Big]+ \\ &\quad +\Big[\widehat{\widetilde{P}}((i,l),(i_1,l_1))-\widetilde{P}((i,l),(i_1,l_1))\Big]\widetilde{P}((i_1,l_1),(j,h))\Big\},\end{aligned}$$

which can also be written as

$$\begin{aligned}F^2_{(i,l),(j,h)} &= \\ &= \sqrt{N}\sum_{(i_1,l_1)}\Big[\widehat{\widetilde{P}}((i,l),(i_1,l_1)) - \widetilde{P}((i,l),(i_1,l_1))\Big]\end{aligned}$$

$$\left[\widehat{\widetilde{P}}((i_1,l_1),(j,h)) - \widetilde{P}((i_1,l_1),(j,h))\right]$$
$$+ \sqrt{N}\sum_{(i_1,l_1)} \widetilde{P}_{((i,l),(i_1,l_1))}\left[\widehat{\widetilde{P}}((i_1,l_1),(j,h)) - \widetilde{P}((i_1,l_1),(j,h))\right]$$
$$+ \sqrt{N}\sum_{(i_1,l_1)} \left[\widehat{\widetilde{P}}((i,l),(i_1,l_1)) - \widetilde{P}((i,l),(i_1,l_1))\right] \widetilde{P}_{((i_1,l_1),(j,h))}. \tag{19}$$

Following similar arguments as in Sadek & Limnios (2002), by Slutsky's theorem, the first term of the sum in 19 is of lower order so it can be ignored for the limit expression. Then, we have

$$F^2_{(i,l),(j,h)} = \sum_{(i_1,l_1)} \left[\widetilde{P}_{((i,l),(i_1,l_1))} F_{((i_1,l_1),(j,h))} + F_{((i,l),(i_1,l_1))}\widetilde{P}_{((i_1,l_1),(j,h))}\right]$$

So we can express $F^2_{(i,l),(j,h)}$ as a linear transformation of the vector

$$\left(F_{((i,l),(j,h))};\ (i,l),\ \ (j,h)\in\widetilde{E}\right).$$

Using Lemma 3, the vector $\mathbf{F}^2_N = \left(F^2_{(i,l),(j,h)};\ (i,l),(j,h)\in\widetilde{E}\right)$ has a centered Normal distribution. Moreover the covariance matrix can be obtained from the following

$$\begin{aligned}\mathrm{Cov}&\left(F^2_{((i_1,l_1),(j_1,h_1))}, F^2_{((i_2,l_2),(j_2,h_2))}\right)\\ &= \sum_{(i,l),(j,h)\in\widetilde{E}} \widetilde{P}_{(i_1,l_1),(i,l)}\widetilde{P}_{(i_2,l_2),(j,h)}\mathrm{Cov}\left(F_{(i,l),(j_1,h_1)}, F_{(j,h),(j_2,h_2)}\right)\\ &+ \sum_{(i,l),(j,h)\in\widetilde{E}} \widetilde{P}_{(i_1,l_1),(i,l)}\widetilde{P}_{(j,h),(j_2,l_2)}\mathrm{Cov}\left(F_{(i,l),(j_1,h_1)}, F_{(i_2,l_2),(j,h)}\right)\\ &+ \sum_{(i,l),(j,h)\in\widetilde{E}} \widetilde{P}_{(i,l),(j_1,h_1)}\widetilde{P}_{(i_2,l_2),(j,h)}\mathrm{Cov}\left(F_{(i_1,l_1),(i,l)}, F_{(j,h),(i_2,l_2)}\right)\\ &+ \sum_{(i,l),(j,h)\in\widetilde{E}} \widetilde{P}_{(i,l),(j_1,h_1)}\widetilde{P}_{(j,h),(j_2,h_2)}\mathrm{Cov}(F_{(i_1,l_1),(i,l)}, F_{(i_2,l_2),(j,h)}),\end{aligned}$$

which can be written, after conveniently defining the matrix Γ_2, as $\Sigma_{\mathbf{F}^2} = \Gamma_2\Sigma_{\mathbf{F}}\Gamma_2^{\top}$.

By the same reasoning as in Eq. (19), for any $n \geq 2$, the expression $F^{n+1}_{(i,l),(j,h)}$ has the same limit as the expression that follows

$$\sqrt{N}\sum_{(i_n,l_n)} \left[\widetilde{P}_{((i,l),(i_n,l_n))}F^n_{((i_1,l_1),(j,h))} + F_{((i,l),(i_n,l_n))}\widetilde{P}^n_{((i_n,l_n),(j,h))}\right].$$

Then, using mathematical induction, the following recurrence relation can be shown, for any $n \geq 2$,

$$\begin{aligned}F^{n+1}_{((i,l),(j,h))} &\equiv \sum_{(i_1,l_1)\in\widetilde{E}} \widetilde{P}_{(i,l),(i_1,l_1)}F_{((i_1,l_1),(j,h))}\\ &+\sum_{k=1}^{n-1}\left[\sum_{(i_k,l_k)\in\widetilde{E}}\ \sum_{(i_{k+1},l_{k+1})\in\widetilde{E}} \widetilde{P}^{n-k}_{(i,l),(i_{k+1},l_{k+1})}F_{(i_{k+1},l_{k+1}),(i_k,l_k)}\widetilde{P}^k_{(i_k,l_k),(j,h)}\right]\\ &+\sum_{(i_n,l_n)\in\widetilde{E}} F_{(i,l),(i_n,l_n)}\widetilde{P}^n_{((i_n,l_n),(j,h))},\end{aligned}$$

where $\equiv$ means equivalence in distribution. Then, we get that vector $\mathbf{F}^{n+1}_N = \left(F^{n+1}_{(i,l),(j,h)};\ (i,l),(j,h)\in\widetilde{E}\right)$ is a linear transformation of vector $\mathbf{F}_N$ and then it has Normal distribution with mean 0. Reasoning similar to the case $n=2$ we can write the covariance matrix conveniently defining the transformation matrix Γ_{n+1}. □

Proposition 2. *The estimator $\widehat{R}(n;N)$ is asymptotically Normal, as $N\to\infty$, for any $n\geq 1$, i.e.,*

$$\sqrt{N}(\widehat{R}(n;N) - R(n)) \xrightarrow{d} N(0, \Sigma_{R,n})$$

Proof. The proof is similar to Theorem 6 in Sadek & Limnios (2002) and can be deduced straightforwardly from Lemma 3 and Lemma 4 above. □

Remark 1. Conditional reliability for stationary Markov chains

When the Markov chain X is stationary the initial distribution $\boldsymbol{\alpha}$ coincides with the stationary distribution $\boldsymbol{\pi}$, then we have that $R(0) < 1$ because in general $\boldsymbol{\pi}(D) = \sum_{i\in D}\pi_i > 0$, where $E = U\cup D$, being U the subset of up-states U, and D the subset of down-states. To overcome this issue we introduce the concept of conditional reliability R_C, defined as

$$R_C(n) = \frac{1}{\boldsymbol{\pi}(U)}\sum_{i\in U}\pi_i R_i(n),$$

for all $n\geq 0$, where $\boldsymbol{\pi}(U) = \sum_{i\in U}\pi_i$. Taking $\alpha_i = \frac{\pi_i}{\boldsymbol{\pi}(U)}$ in Eq. (10) we can define the corresponding estimator and deduce its properties.

4. Maintenance in HMM

In this paper, we propose a maintenance policy for our HMM where the system states represent degradation levels. The policy restores the system to a previous, not necessarily AGAN, condition with certain probability. Similarly, Boussemart, Bickard, & Limnios (2001) considered a Markov chain that governs the system degradation, maintenance actions bring the system to a new state with certain probability, the new system state depends on the performed action. More details in this subject can be read in Section 1.

4.1. Maintenance strategy based on critical values

Let us consider a system that degrades with time. Every state of the system represents a degradation level, $E = \{1,\ldots,d\}$. State 1 indicates that the system is new and state d indicates the failure of the system. The system can only progress to a higher degradation level, not necessarily the following one.

The system is inspected at regular intervals to detect any problem and intervene if necessary. Two different maintenance policies are proposed for this system: the first one is based on the estimated probability that the system is visiting a certain subset of states at the nth inspection. The second one considers the observed signals assuming that when the signals associated to failure are observed, the system has failed.

The cost of the intervention depends on the degradation level reached by the system, being the corrective maintenance the most expensive one.

Case 1: Preventive maintenance based on critical state probability criterion (CSPC)

In our first approach we consider preventive maintenance criteria based on critical states probability criterion (CSPC). Roughly speaking, a preventive maintenance action is carried out once the system enters a subset of operational states that are considered critical in some sense. To get a better picture of the situation let us illustrate it with the following example.

Consider a system with two units that is working as long as at least one unit is operative (i.e. parallel structure). Let us assume that the two units are identical and the system evolution is modelled by a Markov chain with state space defined in terms of the number of down units, $E = \{0,1,2\}$; the set of up states is $U = \{0,1\}$ and the down-state set is then $D = \{2\}$. State 1 can be seen as critical in comparison with state 0. In general, we denote $U = \{1,2,\ldots,r\}$, the set of up states and let us assume that $U = U_1\cup U_2$ where the set of up states can be split into two subsets such states in U_2 are critical to the system performance. Let us assume that $card(U_2) = c$, for a $c < r$.

The preventive maintenance action is undertaken as soon as the subset U_2 is reached with pre-specified probability. More specifically, let us denote T_c the first time the system hits subset U_2 directly from subset U_1, that is, without visiting any state of subset D. The probability distribution of this time is

$$f_c(n) = \mathbb{P}(X_n \in U_2; X_k \in U_1, k = 1, \dots, n-1) = \alpha_{U_1}(\mathbf{P}_{U_1,U_1})^{n-1}\mathbf{P}_{U_1,U_2}\mathbf{1}_{U_2},$$

for $n \geq 1$. As long as transitions from U_1 to D are allowed there are non-zero elements in sub-matrix $P_{U_1,D}$, and then $\sum_{n\geq 1} f_c(n) < 1$. Let $f_c^*(n) = \frac{f_c(n)}{\sum_{k\geq 1} f_c(k)}$, for all $n = 1, 2\dots$ and let $F_c^*(n) = \sum_{k\leq n} f_c^*(k)$ denote the corresponding distribution function. A preventive maintenance action is carried out at time

$$N_c(q) := \min\{n \geq 0 : \ F_c^*(n) \geq q\}, \tag{20}$$

with q a critical probability value, $0 < q < 1$; that is, the quantile of order q of the distribution F_c^*.

Once the action is finished, the system is restored to a non-critical state. Then, by the memoryless property, a new preventive maintenance action will be scheduled following the rule just defined. Note that with this rule we decide when to do the preventive action, but it is still to be decided how the system is maintained.

Case 2: Preventive maintenance based on warning signals probability criterion (WSPC)

As explained in Section 3, the system performance is described not only in terms of the states of the set E, but also in terms of the set of observed signals, i.e. $A = A_1 \cup A_2$, where we distinguish between good signals A_1, and bad signals A_2. Then we can decide to undertake a preventive action as soon as a *warning* signal is observed. Notice that when a signal in the subset A_2 is emitted the system is in a failed state with probability 1.

Let as assume that the subset of good signals A_1 can in turn be split into two subsets such that $A_1 = A_{11} \cup A_{12}$, being $a_w \in A_{12}$ a signal that alerts of some non desirable behaviour in the system, that is, a_w is a warning signal. Let us also define τ_w the first time a warning signal is observed without having previously received a signal indicating the system failure, that is a signal of subset A_2. In other words, $\tau_w = n$ if and only if $Y_n \in A_{12}$ and $Y_k \in A_{11}$ for all $k = 1, 2, \dots, n-1$. The system has only emitted good signals until time n, when an alert is detected for the first time. To obtain the distribution probability of τ_w we can consider two different situations.

1. $\{Y_n; n = 1, \dots, N\}$ are independent identically distributed (i.i.d.)

$$g_w(n) = \mathbb{P}(Y_n \in A_{12}; Y_k \in A_{11}, k = 1, \dots, n-1) = \mathbb{P}(Y_n \in A_{12})\mathbb{P}(Y_{n-1} \in A_{11})\cdots\mathbb{P}(Y_1 \in A_{11}) \tag{21}$$

Let A_{1l}, for $l = 1, 2$ the corresponding subset of signals, then it can be written

$$P(Y_k \in A_{1l}) = \sum_{i\in E} P(X_k = i)M(i, A_{1l})$$

with $M(i, A_{1l}) = \sum_{a\in A_{1l}} M(i, a)$. Then,

$$g_w(n) = \mathbb{P}(Y_n \in A_{12}; Y_k \in A_{11}, k = 1, \dots, n-1) = \left(\prod_{k=1}^{n-1}\left\{\sum_{i\in E} P(X_k = i)M(i, A_{11})\right\}\right)\left(\sum_{j\in E} P(X_n = j)M(j, A_{12})\right)$$

2. $\{Y_n; n = 1, \dots, N\}$ are independent conditionally on $\{X_n; n = 1, \dots, N\}$

$$g_w(n) = \mathbb{P}(Y_n \in A_{12}; Y_k \in A_{11}, k = 1, \dots, n-1) = \sum_{i_1,i_2,\dots,i_n\in E} \alpha_{i_1} M(i_1, A_{11})P(i_1, i_2)M(i_2, A_{11})\cdots P(i_{n-1}, i_n)M(i_n, A_{12})$$

Then, for a pre-specified probability q, we can decide to undertake preventive maintenance actions at times

$$N_w(q) = \min\{n : G_w^*(n) > q\}, \tag{22}$$

for $n = 1, 2, \dots$, and $0 < q < 1$, a critical probability value, and G_w^* represents the distribution function of the corresponding normalized distribution.

4.2. Maintenance strategy expected cost

Let us consider again the state space $E = \{1, \dots, d\}$ as a set of degradation states of the system in the sense that 1 indicates the system is new and d the failure of the system. As above, let us assume that $U = U_1 \cup U_2$, that is $r = r_1 + r_2 < d$, with U_2 the subset of size r_2 containing the critical states of the system. D is the subset of failed states.

Here a maintenance cost depending on the state of the system as well as the observed signal is considered. A system failure is followed by a corrective maintenance action which involves a cost vector of $\mathbf{C}_{CM}$, of dimension $d - r$. On the other hand, a preventive maintenance action is done at times $N_c(q)$ given in (20) when the maintenance is carried out following rule CSPC (see Case 1 above), and at times $N_w(q)$ given in (22), when the maintenance is carried out following rule WSPC (as described in Case 2). The associated cost depends on the hidden state that is being visited at the moment of the inspection. We then define a vector of costs as follows. The cost is 0 for the states 1 to r_1, and, on the other hand, for the critical states there is an associated PM cost given by c_j with $j = r_1 + 1, \dots, r_1 + r_2 = r$ where $c_{j_1} \leq c_{j_2}$, for $j_1 < j_2$. Let $\mathbf{C}_{PM} = (\mathbf{0}_{r_1}, c_{r_1+1}, \dots, c_r)'$ be a column vector of dimension r with $\mathbf{0}_{r_1}$, the first r_1 components equal to 0. Additionally, it is supposed that $\min \mathbf{C}_{CM} \geq c_r$. For a probability q, a PM inspection is carried out at a particular time n only if $N_\star(q) = n$. Let $\mathbf{C}_{q,\star}(n)$, the total cost associated with a potential maintenance action at a time n, with $N_\star = N_c$ for PM based on CSPC; or $N_\star = N_w$ in case PM is adopted according to WSPC. The expected cost at time n can be obtained as

$$E[\mathbf{C}_{q,\star}(n)] = \mathbf{1}_{\{n=N_\star\}}\sum_{i=1}^{r} P(X_n = i)C_{PM}(i) + \sum_{i=r+1}^{d} P(X_n = i)C_{CM}(i) \tag{23}$$

Let us assume that the system is allowed to operate for a pre-specified period of time, that is N_0. Each time a PM action is carried out, the system is returned to a functioning state in the subset U_1 chosen with a probability given by the vector α_1, that is the initial law restricted to the elements of U_1. After that, the conditions of the system are the same as they were at time 0, which means that the following state after PM is chosen according to the initial law, α and the following transitions are governed by the matrix $\mathbf{P}$. From that moment, new PM action will be carried out $N_\star$ times later. This behaviour continues until time N_0 is reached. The total number of PM actions developed is equal to $n_{0,\star}$, where it can be written $N_0 = n_{0,\star}\ N_\star + r_{0,\star}$. The total expected cost involved in the interval $(0, N_0]$ is then

$$E[C_\star(N_0)] =$$

$$= n_{0,\star}\left(\sum_{n=1}^{N_\star}\left\{\sum_{i=1}^{r}P(X_n=i)C_{PM}(i)\right\} + \sum_{i=r+1}^{d}P(X_{N_\star}=i)C_{CM}(i)\right) + \sum_{n=1}^{r_{0,\star}}\sum_{n=1}^{N_\star}\left\{\sum_{i=1}^{r}P(X_n=i)C_{PM}(i)\right\} \quad (24)$$

where as before, the subscript $\star$ indicates the type of PM maintenance, based on critical states or critical signals.

Remark 2. Controlling for false positives

Maintenance strategy based on observed signals can lead to assess the state of the system wrongly, and consequently an unnecessary maintenance cost will be involved. If an alarm signal is observed the system will be sent for repair. Maintenance crew will then assess the true system state that might not agree with the estimated one. There is a cost associated to it that should be minimized. This aspect of maintenance will be treated in a future research.

5. Numerical application

5.1. Example 1. A G: 1-out-of-n system.

Let us consider a system with d_0 identical units that operate independently. The system is operative while at least one unit is operative and fails as soon as all units are down. The state of the system (wear out level) is measured in terms of the number of units failed. After failure, the units are not repaired, and when there are no operative units, the system is replaced by a new identical one.

Let us assume that the units are exponentially distributed with equal failure rate λ.

Model description. Information on the system performance is collected periodically in such a way that only partial information is obtained regarding the system deterioration. More specifically, at regular instants of time one has access to some parameters or indicators (*signals*) related somehow to the level of wear out of the system. For simplicity, let us denote $A=\{1,2,\dots,s\}$, and consider that when signal $a=1$ is emitted, it means that the system is operating in optimal conditions. On the opposite, an observation $a=s$ indicates that a fatal failure has occurred in the system.

At any moment, the true state of the systems is unobservable. The state space is represented by $E=\{1,2,\dots,d\}$, where $i=1$ is the optimal functioning state, that is, the system is new with no unit failed. On the other hand, $i=d$ means that all units are down and then the system is in the failure state, with $d=d_0+1$.

Let us denote $X_0,X_1,\dots,X_n$ the successive (unobserved) states of the system, taking values in the set E; and, $Y_0,Y_1,\dots,Y_n$ the successive observed indicators, which are assumed to range in the set A. We consider that inspections are carried out at times $k=0,\Delta,2\cdot\Delta,\dots\dots$, for simplicity we take $\Delta=1$.

At time $k=0$ we assume that the system is new so that the initial state is $X_0=1$ and the transition probabilities $p_{ij}=\mathbb{P}(X_k=j\mid X_{k-1}=i)$, for $i=1,\dots,d-1$, and $k\geq 1$, are given by

$$p_{ij}=\frac{(d-i)!}{(j-i)!(d-j)!}(1-e^{-\lambda})^{j-i}e^{-(d-j)\lambda},$$

for and $i\leq j$, and $p_{ij}=0$ for $i>j$. Finally, $p_{d,1}=1$.

Successively, an output symbol is produced according to a probability distribution, which depends on the current state. This probability distribution is held fixed for the state regardless of when and how the state is entered. Specifically, for a given state of the system, $i\in E$, we denote $\mathbb{P}(Y_k=a\mid X_k=i)=M(i,a)$, for any $a\in A$. We have that $\sum_{a\in A}M(i,a)=1$, for all $i\in E$. Let $\mathbf{M}$ denote the matrix of dimension $d\times s$, whose (i,a) element is $M(i,a)$, for all $i\in E$, and $a\in A$. In particular we have that $M(1,1)=1$, and $M(d,s)=1$. In addition, it is quite realistic assumption that $M(i,1)=M(i,s)=0$ for $1<i<s$. Rows 2 to $d-1$ of matrix $\mathbf{M}$ are taken as the corresponding probability distribution of a Binomial law with size $s-2$ and probability p_i which is assumed to decrease with the value of i. Then we have that the signal emitted stochastically increases as the system deteriorates.

Then the parameters to be estimated are $\boldsymbol{\theta}:=(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2)$, with $\boldsymbol{\theta}_1=\lambda$ and $\boldsymbol{\theta}_2=(M(i,a);i=2,\dots,d-1;a=1,\dots,s-1)$. Eq. (4) can be approximated as

$$Q(\theta\mid\theta^{(m)})\approx\sum_{k=1}^{n}\sum_{i=1}^{d-1}\sum_{j=i}^{d}\mathbb{P}_{\theta^{(m)}}(X_{k-1}=i,X_k=j\mid Y)\big((j-i)\log(1-e^{-\lambda})-(d-j)\lambda\big)+\sum_{k=0}^{n}\sum_{i=2}^{d-1}\sum_{a=2}^{s-1}\mathbb{P}_{\theta^{(m)}}(X_k=i\mid Y)\mathbf{1}_{\{Y_k=a\}}\log M(i,a),$$

where all terms that do not depend on the unknown parameters have been omitted.

Using the EM algorithm, the maximization step M leads us to the following

$$\widehat{\lambda}^{(m+1)}=-\log\frac{\sum_{k=1}^{n}\sum_{i=1}^{d-1}\sum_{j=i}^{d-1}(d-j)\cdot\mathbb{P}_{\theta^{(m)}}(X_{k-1}=i,X_k=j\mid Y)}{\sum_{k=1}^{n}\sum_{i=1}^{d-1}\sum_{j=i}^{d}(d-i)\cdot\mathbb{P}_{\theta^{(m)}}(X_{k-1}=i,X_k=j\mid Y)}.$$

Changing the order of summation we get

$$\widehat{\lambda}^{(m+1)}=-\log\frac{\sum_{k=1}^{n}\sum_{j=1}^{d-1}(d-j)\cdot\mathbb{P}_{\theta^{(m)}}(X_k=j\mid Y)}{\sum_{k=1}^{n}\sum_{i=1}^{d-1}(d-i)\cdot\mathbb{P}_{\theta^{(m)}}(X_{k-1}=i\mid Y)}=-\log\frac{\sum_{k=1}^{n}\sum_{i=1}^{d-1}(d-i)\cdot\mathbb{P}_{\theta^{(m)}}(X_k=i\mid Y)}{\sum_{k=0}^{n-1}\sum_{i=1}^{d-1}(d-i)\cdot\mathbb{P}_{\theta^{(m)}}(X_k=i\mid Y)},$$

where we have used that $\sum_{i=1}^{j}\mathbb{P}_{\theta^{(m)}}(X_{k-1}=i,X_k=j\mid Y)=\mathbb{P}_{\theta^{(m)}}(X_k=j\mid Y)$; and $\sum_{j=i}^{d}\mathbb{P}_{\theta^{(m)}}(X_{k-1}=i,X_k=j\mid Y)=\mathbb{P}_{\theta^{(m)}}(X_{k-1}=i\mid Y)$. On the other hand, $\widehat{M}^{(m+1)}(i,a)$ as in Eq. (7).

Using the forward-backward probabilities in Eq. (8) and the backward probabilities given in (9), we calculate,

$$\widehat{\lambda}^{(m+1)}=-\log\frac{\sum_{k=1}^{n}\sum_{i=1}^{d-1}(d-i)F_k^{(m)}(i)B_k^{(m)}(i)}{\sum_{k=0}^{n-1}\sum_{i=1}^{d-1}(d-i)F_k^{(m)}(i)B_k^{(m)}(i)}. \quad (25)$$

and,

$$\widehat{M}^{(m+1)}(i,a)=\frac{\sum_{k=0}^{n}F_k^{(m)}(i)B_k^{(m)}(i)\mathbf{1}_{\{Y_k=a\}}}{\sum_{k=0}^{n}F_k^{(m)}(i)B_k^{(m)}(i)}. \quad (26)$$

Numerical results. In this example we are specially interested in evaluating the role of the system size. Then we consider different specifications for the number of units included in the system, specifically we take $d_0=3,5,10$. Besides, we consider $\lambda=0.1$. Figure 1 displays the true reliability functions corresponding to 3 systems with size: 3 (*solid line*), 5 (*dashed line*), 10 (*dotted line*), respectively.

For each system we have simulated markovian sample paths of size $n=150$ using the corresponding true model $(\alpha,\mathbf{P})$. Then, from the theoretical emission matrix $\mathbf{M}$, a sample of simulated outputs has been obtained. To avoid wrong conclusions due to the randomness in the simulation process the experiment has been repeated a total of 500 times for each system. The estimation results are represented in Fig. 2. The true reliability is given by the black curve. For each sample we have estimated the reliability function based on the HMM model. The results have been summarized through

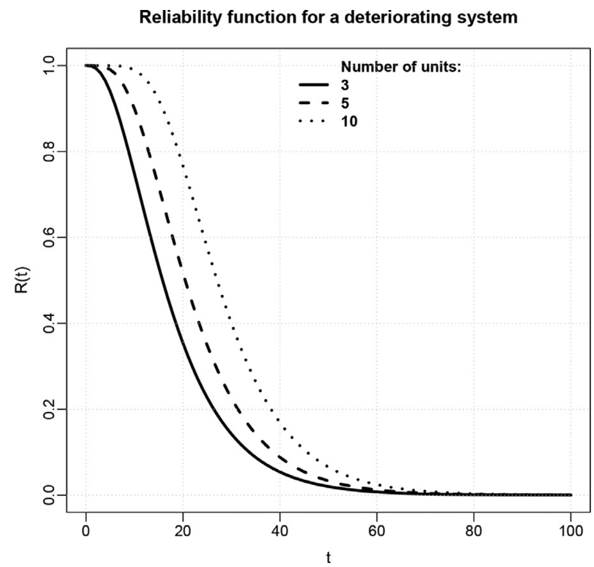

Fig. 1. A G: 1-out-of-n system.

averaging. That is, we consider the following

$$\widehat{R}_{av}(t) = \frac{1}{500}\sum_{r=1}^{500}\widehat{R}^{(r)}(n),$$

where $\widehat{R}^{(r)}$ is the estimated reliability function based on the rth sample, for $r = 1, \ldots, 500$.

The red curve represents the average of the estimated curves along the 500 replications for each case $d_0 = 3, 5, 10$ (from left to right). As expected, the accuracy of the estimator decreases with the complexity of the system. The bias increases with the number of components. It is worth noticing that in an acceptable reliability level, i.e. $[0, 20]$ time interval, for all figures we have a good accuracy.

5.2. Example 2. A progressively deteriorating system with shocks

Let us consider a system that receives shocks with time. The state of the system varies in the set $E = \{1, 2, \ldots, d\}$, from perfect functioning represented by state 1 to complete failure represented by state d. Each time a shock occurs, the state of the system changes from the current state i to $i+1$ with probability p or the system remains in the same state i with probability $1-p$, for $i = 1, 2, \ldots, d-1$. It is assumed that the system is designed such that it can only stand a maximum number of shocks after which it is replaced by a new and identical one. Equivalently when the system reaches level d it is restored to state 1 of perfect functioning. That is, the system is designed to stand a maximum number of chocks, i.e. d. In this case the hidden Markov chain is given by a random walk with state space E, with a reflecting barrier at d, whose probability transition matrix is given by

$$\mathbf{P} = \begin{pmatrix} 1-p & p & 0 & 0 \\ 0 & 1-p & p & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

As in the previous case, we do not observe the true state of the system but an output symbol wich is produced depending on the current state with a particular probability distribution. Again, for a given state of the system, $i \in E$, we denote $\mathbb{P}(Y_k = a \mid X_k = i) = M(i, a)$, for any $a \in A$. Let $\mathbf{M}$ be a $d \times s$-matrix whose (i, a) element is $M(i, a)$, for all $i \in E$, and $a \in A$. In this case we only assume that $M(d, s) = 1$ and for rows 1 to $d-1$ we consider the corresponding probability distribution of a Binomial law with size $s-1$ and probability p_i which is assumed to decrease with the value of i.

The parameters to be estimated are $\boldsymbol{\theta} := (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$, with $\boldsymbol{\theta}_1 = p$ and $\boldsymbol{\theta}_2 = (M(i, a); i = 2, \ldots, d-1; a = 1, \ldots, s-1)$. Eq. (4), can be written as

$$\begin{aligned} Q(\theta \mid \theta^{(m)}) \approx \sum_{k=1}^{n}\sum_{i<d}\{&\mathbb{P}_{\theta^{(m)}}(X_{k-1} = i, X_k = i+1 \mid Y)\log p \\ &+ \mathbb{P}_{\theta^{(m)}}(X_{k-1} = i, X_k = i \mid Y)\log(1-p)\} \\ &+ \sum_{k=0}^{n}\sum_{i=1}^{d-1}\sum_{a=1}^{s-1}\mathbb{P}_{\theta^{(m)}}(X_k = i \mid Y)\mathbf{1}_{\{Y_k = a\}}\log M(i, a). \end{aligned}$$

Using the EM, the maximization step M leads us to the following expression

$$\widehat{p}^{(m+1)} = \frac{\sum_{k=1}^{n}\sum_{i<d}\mathbb{P}_{\theta^{(m)}}(X_{k-1} = i, X_k = i+1 \mid Y)}{\sum_{k=1}^{n}\sum_{i<d}\mathbb{P}_{\theta^{(m)}}(X_{k-1} = i \mid Y)}.$$

Let us consider the following particular model: $d = 10$; $s = 20$; $p = 0.6$. From this model we generate a total of 500 samples of size $n = 150$ and, as in the previous example, estimate for each case the reliability function. The estimation results are presented in Fig. 3. The true reliability is given by the black curve. For each sample we have estimated the reliability function based on the HMM model. The solid red curve represents the average of the estimated curves along the 500 replications, the two dotted red lines represent the corresponding bootstrap confidence intervals at a confidence level of 95%, calculated at each estimation point, that is, $n = 1, 2, \ldots$.

5.3. Example 3: A repairable system with two failure states

Let us consider now a system with four possible levels of performance, that is $E = \{u_1, u_2, d_1, d_2\}$, where $u_{\cdot}$ denotes a function-

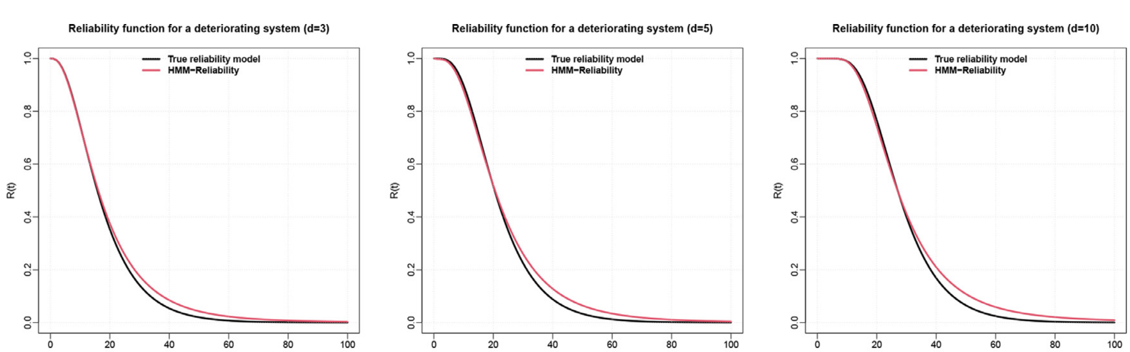

Fig. 2. Deteriorating system with identical Exponential components. From left to right. Panel 1: $d_0 = 3$ and $s = 5$. Panel 2: $d_0 = 5$ and $s = 7$. Panel 3: $d_0 = 10$ and $s = 12$. For all graphs, the black line is true reliability curve; the red line is the averaged reliability estimations from 500 samples. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

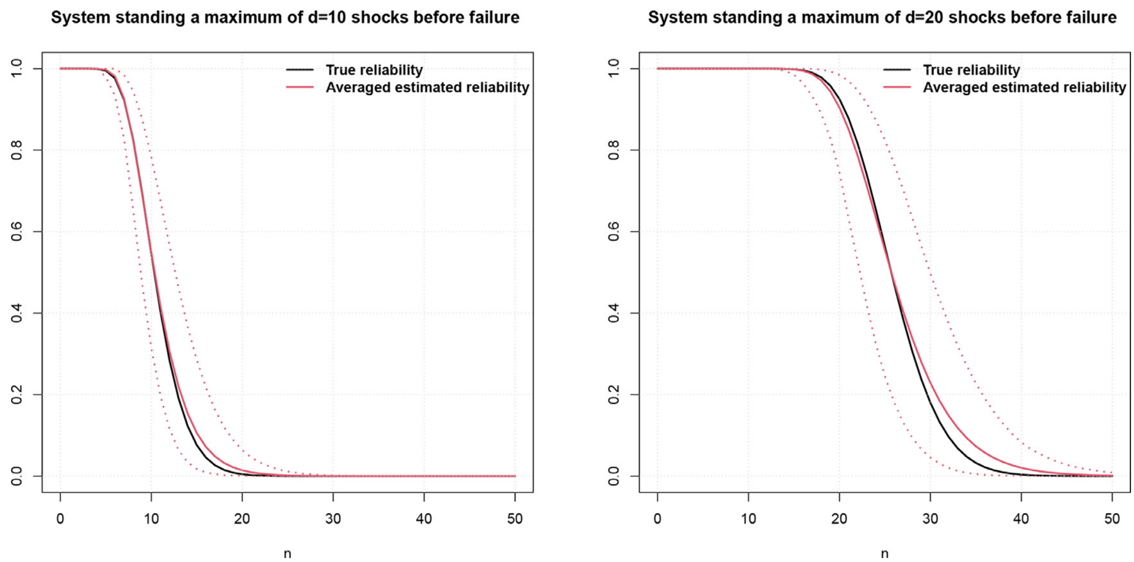

Fig. 3. A system with shocks. Left panel: $d = 10$ and $s = 20$. Right panel: $d = 20$ and $s = 30$. For all graphs, the black line is true reliability curve; the red line is the averaged reliability estimations from 500 samples. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

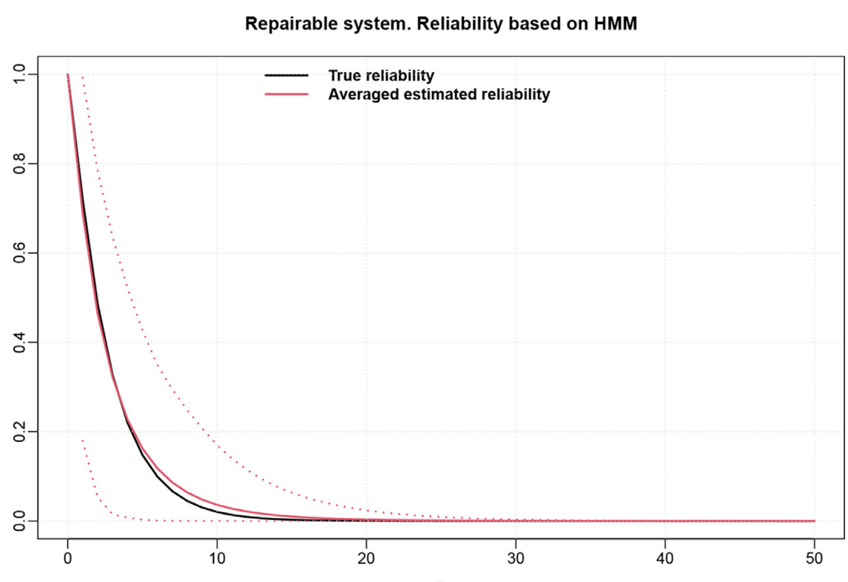

Fig. 4. Repairable system.

ing state whereas $d.$ refers to the down states.The true transition and emission matrices are given by

$$\mathbb{P} = \begin{pmatrix} 0.4 & 0.4 & 0.1 & 0.1 \\ 0.3 & 0.4 & 0.2 & 0.1 \\ 0.2 & 0.3 & 0.3 & 0.2 \\ 0 & 0.3 & 0.4 & 0.3 \end{pmatrix}, \text{ and,}$$

$$\mathbb{M} = \begin{pmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.3 & 0.5 & 0.2 & 0 \\ 0.1 & 0.2 & 0.5 & 0.2 \\ 0 & 0.2 & 0.3 & 0.5 \end{pmatrix}.$$

In this case we have simulated a total of 500 samples of size $n = 150$ each. The results are illustrated in Fig. 4 where also bootstrap confidence intervals have been added based on the empirical quantiles.

We use this example to check the asymptotic properties of the estimator of the reliability function. We have simulated samples of size $N = 100, 500, 1000, 5000$ respectively. The plot in Fig. 5 shows the estimated curves for each sample and it is noticeable how the estimation errors decrease as the sample size increases.

Table 1
Example 3: Relative errors of the estimated values of the transition matrix **P** and the emission matrix **M**.

N	$Err_{\mathbf{P}}$	$Err_{\mathbf{M}}$
100	0.1187	0.2126
500	0.0300	0.1008
1000	0.0070	0.0857
5000	0.0014	0.0080

To summarize the results obtained in this example we present in Table 1 an overall measure of the relative errors reported by the estimations of matrices **P** and **M**. Specifically we have considered the following measurements of the error

$$Err_{\mathbf{P}} = \sum_{i,j \in E} \frac{\left(\widehat{P}(i,j) - P(i,j)\right)^2}{P(i,j)}; \text{ and,}$$

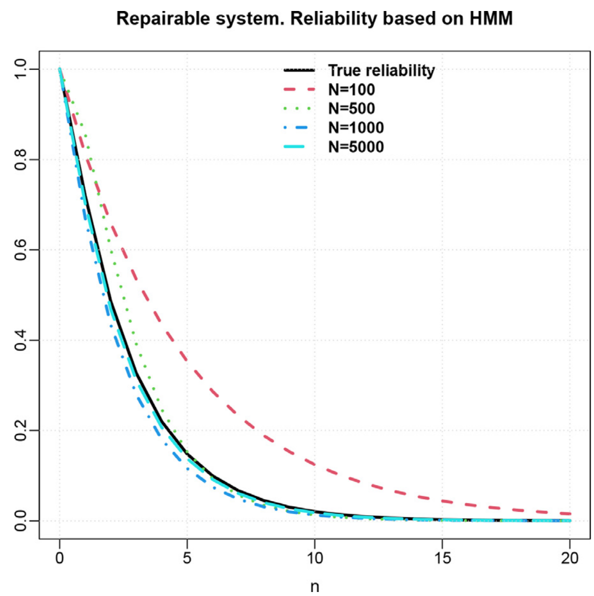

Fig. 5. Example 3. Consistency of the reliability estimator.

$$Err_{\mathbf{M}} = \sum_{i \in E, a \in A} \frac{\left(\widehat{M}(i,a) - M(i,a)\right)^2}{M(i,a)}.$$

The consistency properties of the estimators can be numerically evaluated by noticing the significant reduction of the errors (increasing accuracy) as N increases.

A graphical inspection element-by-element of the two matrices is given in Figs. 6 and 7 that display, respectively, summary statistics for the estimations of the matrices **P** and **M**. We have considered in each case samples of sizes $N = 100, 500, 1000, 5000$. Each boxplot on the graphs gives the results for each element of matrix **P** in Fig. 6 and matrix **M** in Fig. 7. The red symbols inside the boxplots represent the true values. The blue symbols represent, for each probability value, the average of the estimated values. These averages have been obtained by considering 500 samples for each sample size. As expected the accuracy of the estimators increases with the sample size. The distance between the red and blue points inside the plots provides a graphical impression of the bias of the estimator for each element of the corresponding matrix, **P** and **M**. As can be seen even for small samples the bias is reasonable and it almost vanishes for the biggest samples. On the other hand, the number of parameters to be estimated is large in this case, 24 unknown parameters in total, and then the estimators show high variability for the smallest samples ($N = 100$) especially in the case of the emission matrix. However, this variability shows a remarkable descending trend as the sample size increases, as it can be appreciated on the plots.

5.4. Maintenance analysis

In this section we discuss the maintenance problem for the system in Example 1, that is, the deteriorating system. We consider in particular that $d_0 = 5$, so the hidden chain has $d = 6$ states of which $r = 5$ are operative states. The subset of critical states is $U_2 = \{4, 5\}$, so $r_2 = 2$. With respect to the observations (signals), we consider that $s = 7$ is the total number of possible signals emitted by the system, with $A_2 = \{4, 5, 6\}$ the set of warning signals among the set of good signals, that is, $A = \{1, 2, 3, 4, 5, 6\}$. We assume that the system is allowed to be in operation for a maximum of $N_0 = 50$ transitions. Our aim is to calculate the expected cost considering two different PM strategies as explained in Section 4.

Table 2
The Virkler's dataset. The observations are the increments of the crack size in successive intervals of length equal to 2000 cycles of functioning.

Clusters	Estimated signal	Frequency
1	0.1023	56
1	0.2609	27
2	0.606	38
3	1.6907	5

The vector of cost is defined as follows. $C_{CM} = 1$ is the cost associated to a CM action. The vector of cost associated to PM maintenance is $C_{PM} = (0, 0, 0, 0.5, 0.75)'$. In Fig. 8 the solid line reports the results of the total cost that entails the operation of the system until time $N_0 = 50$ under CM and PM based on critical states for a range of values of the threshold probability q. The dashed line give the results of the cost generated by the system operating until $N_0 = 50$ under CM and PM based on critical signals. We can see that when the critical probability increases, the PM based on states get smaller values, while the opposite is true for the PM strategy based on signals.

6. A real case study: The Virkler's dataset

As illustrative example we consider the fatigue crack growth problem in a degradation mechanism analyzed in Chiquet, Limnios, & Eid (2009), where a piecewise deterministic Markov process is proposed for the degradation modelling. The data consist of an aluminum alloy specimen that was tested to investigate fatigue crack propagation.[1]

Starting from an initial crack of length 9*mm* for a particular item in test, the number of cycles for the size of the crack to reach a predetermined value was recorded successively. That is, it is registered the number of cycles every time an increment of size 0.2 *mm* in length occurs. The experiment finishes once a critical size of the crack is reached, meaning the failure of the item. The data were first published in Virkler, Hillberry, & Goel (1979). The random factor here is the inhomogeneity in the material.

Fatigue crack growth in materials may exhibit high variability due to among other causes material inhomogeneity or environmental conditions and thus an HMM is a good model to explain such variability. Certain information on the state of the piece is recorded regularly in terms of the size of the crack, however full understanding of the real degradation also needs to account for random factors that are involved in the underlying process. So, the fatigue crack growth is assumed to fit into different regimes with different crack propagation rates. One can consider that these regimes are in a one-to-one correspondence with the actual deterioration states of the piece, so that a state transition of the hidden model means a regime-switching. The transition between regimes (states) may happen at an arbitrary random time.

One item is followed until the size of the crack exceeds 49.8 *mm*, which occurs at time $\tau = 247251$ (cycles of functioning) for the selected item. Assume that the piece is observed every 2000 *cycles* and that the increment of the crack size between two consecutive inspections is recorded while the true state of the piece remains unobserved. Let us denote $Y_1, Y_2, \dots, Y_N$ the sequence of the crack increments observed in the item and $X_1, X_2, \dots, X_N$, the corresponding sequence of hidden states. For simplicity the observation space is divided into four categories using the k-means clustering method. In Table 2 it is detailed the

[1] Although the original dataset contains a total of 68 trajectories, we have considered only one of them to develop this application example. There is no particular preference for the one utilized here and similar results would have been obtained had it been selected a different case of the dataset.

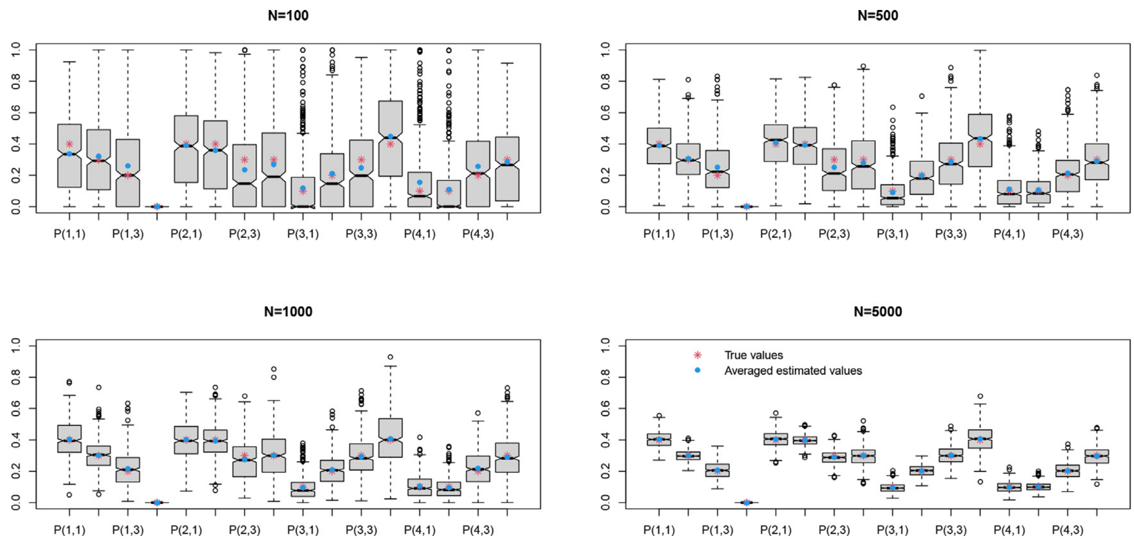

Fig. 6. Example 3. Estimation of the transition matrix **P**.

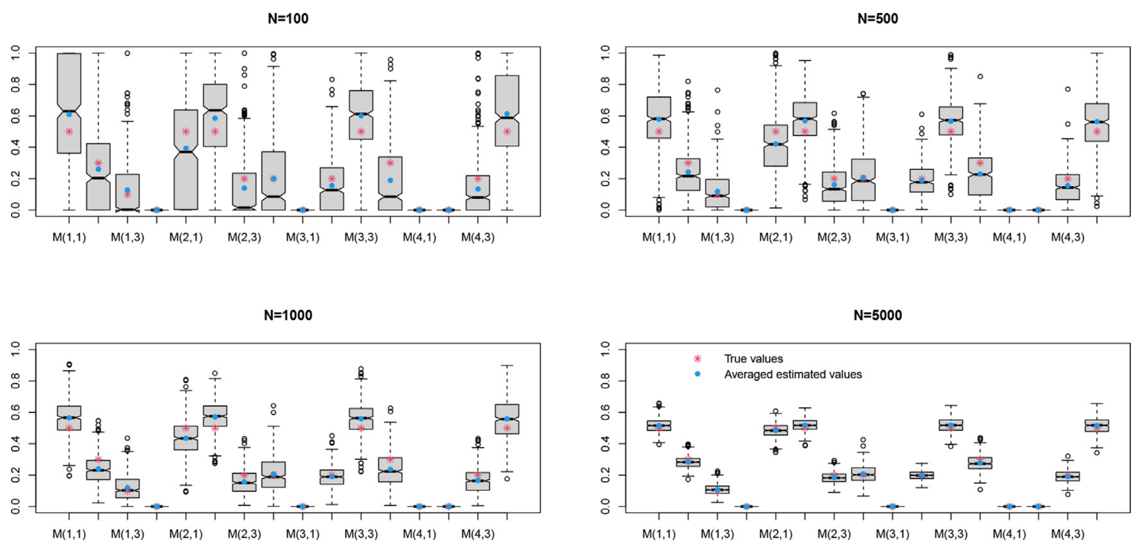

Fig. 7. Example 3. Estimation of the emission matrix **M**.

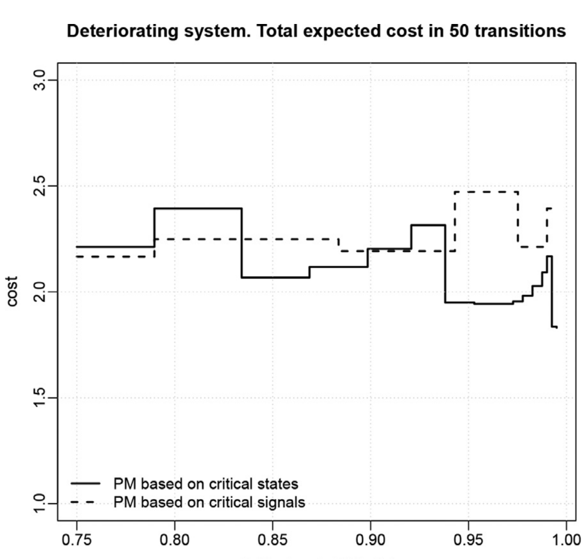

Fig. 8. Deteriorating system with maintenance.

estimated 4-dimensional signal-space as well as the frequency corresponding to each value (see Chiquet et al., 2009 for a similar discussion).

In an attempt to reveal the number of the actual degradations levels underlying the observations we study two possibilities. First, we distinguish only two internal (hidden) states, that is $E = \{1, 2\}$. It is reasonable that initially the item is occupying its less degraded state, that is $X_0 = 1$. The estimated transition matrix $\widehat{P}_2$ and emission matrix $\widehat{M}_2$ are given

$$\widehat{\mathbf{P}}_2 = \begin{pmatrix} 0.985 & 0.015 \\ 0.000 & 1.000 \end{pmatrix}; \text{ and,}$$

$$\widehat{\mathbf{M}}_2 = \begin{pmatrix} 0.842 & 0.158 & 0.000 & 0.000 \\ 0.000 & 0.445 & 0.471 & 0.084 \end{pmatrix}.$$

Now we fit a 3-state HMM model to the same dataset and obtain the following

$$\widehat{\mathbf{P}}_3 = \begin{pmatrix} 0.985 & 0.008 & 0.008 \\ 0.000 & 0.800 & 0.200 \\ 0.000 & 0.200 & 0.800 \end{pmatrix}; \text{ and,}$$

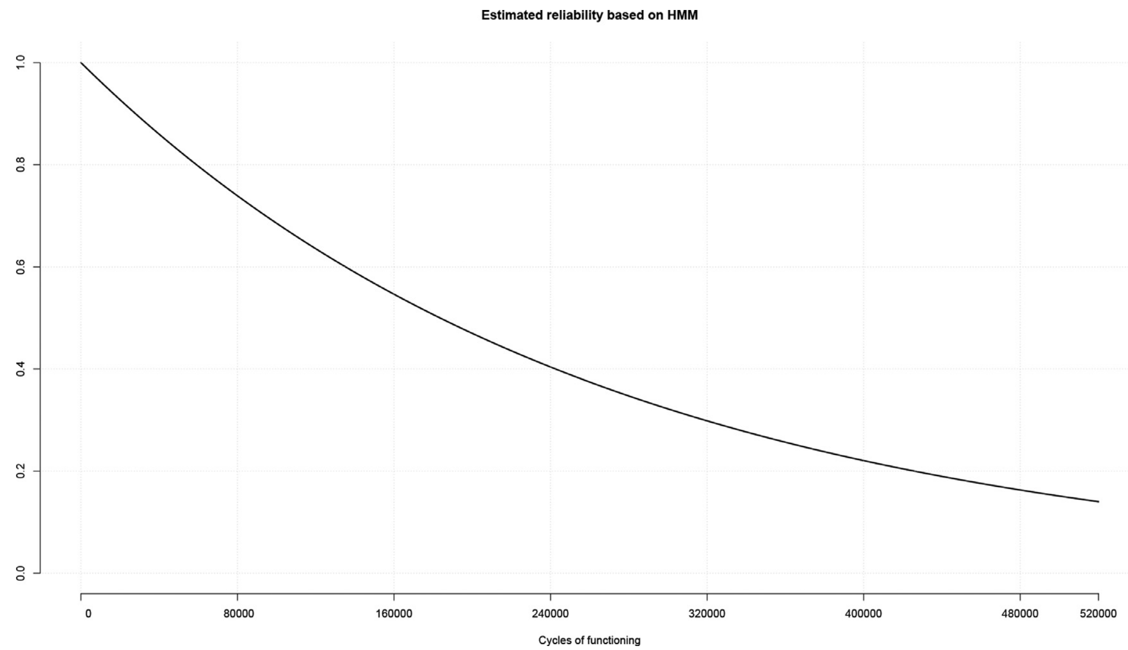

Fig. 9. Virkler dataset. A real case reliability study based on HMM.

Table 3
Model selection.

Model	$d=2$	$d=3$
AIC	193.1830	234.7712

$$\widehat{\mathbf{M}}_3 = \begin{pmatrix} 0.842 & 0.158 & 0.000 & 0.000 \\ 0.000 & 0.445 & 0.471 & 0.084 \\ 0.000 & 0.445 & 0.471 & 0.084 \end{pmatrix}.$$

According to matrix $\widehat{\mathbf{M}}$, the probability distribution of signals emitted from state 2 is the same as from state 3. In other words, actually the model with three states does not allow to distinguish different behaviours between these two latent states. Finally, to compare the fitted HMMs on the basis of the number of states, we use the Akaike's information criterion as done in Votsi et al. (2013). The best model will be selected by means of the *AIC*, defined by $AIC = -2\log L + 2(d(d-1) + d(m-1))$, where $\log L$ is the estimated maximum log-likelihood function. The results are presented in Table 3, where it is shown that the 2-states HMM model fits better the data. This conclusion agrees with previous analyses of this dataset where it is argued that fatigue crack growth of this kind of material can be divided into two regimes with different crack propagation rates, see Abdessalema et al. (2012).

Using the fitted 2-dimensional HMM model we can define the reliability of the item in terms of *up* and *down* states as well as *good* and *bad* signals as explained in Section 3. In our case the hidden state space is $E = \{1,2\}$ and we can consider that 1 is denoting a good performance in the system while 2 refers to a bad functioning regime. Then, according to the previous notation we consider $U = \{1\}$ and $D = \{2\}$. In the same way, the observed signals can be split into two categories. If a crack size growth is detected near 0.606 *mm* or more during a single interval of 2000 cycles, then a danger situation is considered. So, we have $A_1 = \{1,2\}$ and $A_2 = \{3,4\}$, using the notation of Section 3. Then, we define the reliability of the piece in this terms as $R(t) = \mathbb{P}(X_{n_t} = 1; Y_{n_t} \in A_2)$, for $t > 0$ and $n_t = \sup\{n \in \mathbb{N} : n \le \frac{t}{2000}\}$. The results are displayed in Fig. 9.

7. Conclusions and future research

During the lifetime of most real complex systems, the real state of the system is unobservable most of the time, while indicators of this state, such as temperature, pressure, etc., are available via a control system. So, the real problem here is to be able to estimate the state of the system by considering those indicators.

This paper aims to validate the approach of the HMM models in reliability engineering. As we have seen in this paper, a hidden model can provide the key information about the system dependability such as the failed component of the system, the reliability of the system and related measures. Our approach focuses on the introduction of a new concept of the system reliability function when the true system degradation is not directly observed. The reliability function is expressed not only in terms of the internal (unobserved) states of the system but also in terms of the observed signal that is recorded and is treated as an indicator of the degradation level of the system. We have constructed a maximum-likelihood estimator of the reliability function based on a sample of observations of signals and have derived its theoretical (asymptotic) properties.

Maintenance is an important issue of system dependability. In this respect we have proposed for the first time in this context of missing information, two criteria for preventive maintenance. One is based on critical states probability criterion (CSPC) and the other in warning signals probability criterion (WSPC). We have studied the efficiency of these two criteria in terms of cost and we have illustrated our methodology through a simulation study where three systems of different nature have been analyzed.

The present work can be extended to:

- Hidden Semi-Markov reliability models;
- Define and develop other approaches of maintenance;
- Extend our study to the case of a continuous time follow-up of the system;
- Extend our model to the semi-Markov case via Phase-type distributions.

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References

Abdessalema, A. B., Azais, R., Touzet, M., Gegout-Petit, A., Puiggali, M., & Elegbede, C. (2012). Modelling of fatigue crack propagation using piecewise-deterministic Markov processes. In *International conference on engineering optimization, Rio de Janeiro, Brazil*.

Barbu, V. S., & Limnios, N. (2008). *Semi-Markov chains and hidden semi-Markov models toward applications*. Springer.

Baum, L. E., & Petrie, T. (1966). Statistical inference for probabilistic functions of finite state Markov chains. *The Annals of Mathematical Statistics, 37*(6), 1554–1563.

Bickel, B. J., Ritov, Y., & Rydén, T. (1998). Asymptotic normality of the maximum likelihood estimator for general hidden Markov models. *Annals of Statistics, 26*, 1614–1635.

Boussemart, M., Bickard, T., & Limnios, N. (2001). Markov decision processes with a constraint on the asymptotic failure rate. *Methodology and Computing in Applied Probability, 3*, 199–214.

Chiquet, J., Limnios, N., & Eid, M. (2009). Piecewise deterministic Markov processes applied to fatigue crack growth modelling. *Journal of Statistical Planning and Inference, 139*, 1657–1667.

Douc, R. (2005). Non singularity of the asymptotic Fisher information matrix in hidden Markov models. *École Polytechnique*. preprint

Doyen, L., & Gaudoin, O. (2004). Classes of imperfect repair models based on reduction of failure intensity or virtual age. *Reliability Engineering and System Safety, 84*, 45–56.

Durand, J. B., & Gaudoin, O. (2005). Software reliability modelling and prediction with hidden Markov chains. *Statistical Modelling, 5*, 75–93.

Fort, A., Mugnaini, M., & Vignoli, V. (2015). Hidden Markov models approach used for life parameters estimations. *Reliability Engineering & System Safety, 136*, 85–91.

Jiang, P., & Liu, X. (2015). Hidden Markov model for municipal waste generation forecasting under uncertainties. *European Journal of Operational Research, 250*(2), 639–651.

Jonge, B., & Scarf, P. (2020). A review on maintenance optimization. *European Journal of Operational Research, 285*(3), 805–824.

Kijima, M. (1989). Some results for repairable systems with general repair. *Journal of Applied Probability, 26*, 89–102.

Labeau, P. E., & Segovia, M. (2011). Effective age models for imperfect maintenance. *Proceedings of the Institution of Mechanical Engineers, Part O: Journal of Risk and Reliability, 225*(2), 117–130.

Landon, J., Ozekici, S., & Soyer, R. (2013). A Markov modulated poisson model for software reliability. *European Journal of Operational Research, 229*, 404–410.

Pham, H., & Wang, H. (1996). Imperfect maintenance. *European Journal of Operational Research, 94*(3), 425–438.

Rabiner, L. R. (1989). A tutorial on hidden Markov models and selected applications in speech recognition. *Proceedings of the IEEE, 77*(2), 257–286.

Sadek, A., & Limnios, N. (2002). Asymptotic properties for maximum likelihood estimators for reliability and failure rates of Markov chains. *Communications in Statistics. Theory Methods, 31*(10), 1837–1861.

Shey-Huei, S., Chin-Chih, C., Yen-Luan, C., & Zhe, G. Z. (2015). Optimal preventive maintenance and repair policies for multi-state systems. *Reliability Engineering and System Safety, 140*, 78–87.

Simoes, A., Viegas, J., Torres Farinha, J., & Fonseca, I. (2017). The state of the art of hidden Markov models for predictive maintenance of diesel engines: HMM for predictive maintenance of diesel engines. *Quality and Reliability Engineering, 33*, 2765–2779.

Trevezas, S., & Limnios, N. (2009). Maximum likelihood estimation for general hidden semi-Markov processes with backward recurrence time dependence. *Journal of Mathematical Sciences, 163*(3), 262–274.

Virkler, D. A., Hillberry, B. M., & Goel, P. K. (1979). The statistical nature of fatigue crack propagation. *Journal of Engineering Materials and Technology, 101*, 148–153.

Votsi, I., Limnios, N., Tsaklidis, G., & Papadimitriou, E. (2013). Hidden Markov models revealing the stress field underlying the earthquake generation. *Physica A, 392*, 2868–2885.

Vrignat, P., Avila, M., Duculty, F., & Kratz, F. (2015). Failure event prediction using hidden Markov model approaches. *IEEE Transactions on Reliability, 64*(3), 1038–1048.

Wang, H. (2002). A survey of maintenance policies of deteriorating systems. *European Journal of Operational Research, 139*(3), 469–489.

Zhou, Z. J., Hu, C. H., Xu, D. L., Chen, M. Y., & Zhou, D. H. (2010). A model for real-time failure prognosis based on hidden Markov model and belief rule base. *European Journal of Operational Research, 207*, 269–283.