

Article

An Iterative Algorithm for Approximating the Fixed Point of a Contractive Affine Operator

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Abstract: First of all, in this paper we obtain a perturbed version of the geometric series theorem, which allows us to present an iterative numerical method to approximate the fixed point of a contractive affine operator. This result requires some approximations that we obtain using the projections associated with certain Schauder bases. Next, an algorithm is designed to approximate the solution of Fredholm's linear integral equation, and we illustrate the behavior of the method with some numerical examples.

Keywords: iterative numerical methods; Schauder bases; Fredholm integral equation

MSC: 65R20; 46B15; 45B05



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1. Introduction

The idea of iterative numerical methods is, given a complete metric space X (typically a Banach space) and a contractive operator $T : X \rightarrow X$, or at least one which guarantees the convergence of the Picard iterates, to construct a sequence of approximations of the fixed point of that operator $x_0 = T(x_0)$. The calculation of the Picard iterates is not generally easy or even feasible, so several methods which allow us to approximate the elements of the Picard sequence have been proposed. Therefore, a part of the Picard-type iterative algorithms are focused on determining, for an element $x \in X$, a value close to $T(x)$ and in this way, successively approximating the iterates. The numerical techniques used are very diverse, and the resulting algorithms have numerous applications. Proof of all this are the recent references [1–16].

However, our approach here is completely different: given x , instead of approximating successively $T(x)$, $T^2(x)$, $T^3(x)$, \dots , which necessarily involves an accumulation of errors, in this paper, we approximate directly $T^n(x)$ by means of the use of suitable Schauder bases, transforming it into a simple calculation which, for example, does not involve the resolution of systems of algebraic equations or the use of any quadrature formulae because simply linear combinations of certain values associated with the operator are calculated. What is more, motivated by its application for the numerical resolution of the linear Fredholm integral equation, the operator T is considered to be affine and continuous. This affine and continuous nature means that, instead of using a fixed-point language, we opted for resorting to an equivalent version using the geometric series theorem, and more specifically, our first contribution is to obtain a perturbed version of the same which is susceptible to presenting approximations by means of certain Schauder bases related to the operator. Such an approximation will imply a low computational cost as mentioned above. Thus, we are going to design an iterative-type algorithm which allows the approximation of the fixed point of a suitable continuous affine operator.

As we have mentioned, the application that we are presenting consists of a numerical algorithm to solve the linear Fredholm integral equation, which is chosen for its great versatility.

The structure of this paper is as follows. In Section 2 we establish an analytical–numerical result, which provides us with an approximation of the fixed point of a suitable continuous affine operator in a Banach space. To continue, Section 3 interprets the previous result in terms of an algorithm when a Schauder basis is introduced into the considered space. Section 4 derives a specific algorithm in the case of the linear Fredholm integral equation in two distinct contexts. Next, Section 5 shows some illustrative examples of equations or a classic model of electrostatics (Love’s equation), and finally, Section 6 rounds up with some conclusions.

2. Approximating Fixed Points of Affine Operators

The following result provides us with an approximation of the fixed point of a suitable continuous affine operator, as well as an estimation of the error. It addresses a version of the geometric series theorem, which we can label as perturbed: it presents the possibility of converting the precise calculations into approximate ones, in exchange for making the calculations possible.

Before establishing this, we present some standard notation. Given a (real) Banach space X , $L(X)$ will denote the Banach space (usual operator norm) of those bounded and linear operators from X to X . For $T \in L(X)$ and $n \in \mathbb{N}$, T^n denotes the power operator $T \circ \overbrace{\dots}^{n \text{ times}} \circ T$, while $T^0 = I$, the identity map on X .

Theorem 1. *Let X be a Banach space, $y \in X$ and $L \in L(X)$ with $\|L\| < 1$, and consider the continuous affine operator $A : X \rightarrow X$ defined by*

$$Ax := y + Lx, \quad (x \in X).$$

Let $y_0 \in X$, $n \in \mathbb{N}$ and $L_0, L_1, \dots, L_n \in L(X)$. Then, the equation $Ax = x$ has a unique solution $x^\bullet \in X$ and

$$\left\| \sum_{j=0}^n L_j y_0 - x^\bullet \right\| \leq \sum_{j=0}^n \|L_j y_0 - L^j y_0\| + \left(\frac{1 - \|L\|^{n+1}}{1 - \|L\|} \right) \|y_0 - y\| + \frac{\|L\|^{n+1}}{1 - \|L\|} \|y\|.$$

Proof. Let us first observe that, according to the geometric series theorem, there exists a unique solution $x^\bullet \in X$ for the equation $Ax = x$,

$$x^\bullet = (I - L)^{-1}y,$$

which satisfies for any $k \in \mathbb{N}$,

$$\left\| \sum_{j=0}^k L^j y - x^\bullet \right\| \leq \frac{\|L\|^{k+1}}{1 - \|L\|} \|y\|.$$

Therefore,

$$\begin{aligned} \left\| \sum_{j=0}^n L_j y_0 - x^\bullet \right\| &\leq \left\| \sum_{j=0}^n L_j y_0 - \sum_{j=0}^n L^j y_0 \right\| + \left\| \sum_{j=0}^n L^j y_0 - \sum_{j=0}^n L^j y \right\| + \left\| \sum_{j=0}^n L^j y - x^\bullet \right\| \\ &\leq \sum_{j=0}^n \|L_j y_0 - L^j y_0\| + \sum_{j=0}^n \|L\|^j \|y_0 - y\| + \frac{\|L\|^{n+1}}{1 - \|L\|} \|y\| \\ &= \sum_{j=0}^n \|L_j y_0 - L^j y_0\| + \left(\frac{1 - \|L\|^{n+1}}{1 - \|L\|} \right) \|y_0 - y\| + \frac{\|L\|^{n+1}}{1 - \|L\|} \|y\|, \end{aligned}$$

as announced. \square

It is worth mentioning that when $y_0 = y$ and for all $j = 0, 1, \dots, m$, we have that $L_j = L^j$, and we recover a well-known algorithm associated with the geometric series theorem. However, iterative procedures such as this, used to involve difficult and even impossible calculations from a practical perspective, so the idea behind this theorem is to choose the operators L_0, L_1, \dots, L_n in such a way that L_0y_0, L_1y_0, L_ny_0 are not only calculable, but also have a low computational cost. In addition, if y_0 represents an approximation of y —normally due to a certain type of error—the previous result shows how y_0 influences the final approximation. Finally, we can obtain an approximation for x^\bullet for some adequate $n \in \mathbb{N}$, and for each $j = 0, 1, \dots, n$, L_jy_0 is close to L^jy_0 . More specifically:

Corollary 1. *Suppose that X is a Banach space, $L \in L(X)$ and $\|L\| < 1$, $y \in X$, and that $A : X \rightarrow X$ is the continuous and affine operator $A(\cdot) := y + L(\cdot)$, whose unique fixed point is denoted by $x^\bullet \in X$. Additionally, assume that for some $y_0 \in X$, $n \in \mathbb{N}$, $L_0, L_1, \dots, L_n \in L(X)$ and $\varepsilon, \varepsilon_0, \varepsilon_n > 0$, we have that*

$$\sum_{j=0}^n \varepsilon_j < \frac{\varepsilon}{2},$$

$$j = 0, \dots, n \Rightarrow \|L_jy_0 - L^jy_0\| < \varepsilon_j, \tag{1}$$

and that

$$\left(\frac{1 - \|L\|^{n+1}}{1 - \|L\|}\right)\|y_0 - y\| + \frac{\|L\|^{n+1}}{1 - \|L\|}\|y\| < \frac{\varepsilon}{2}. \tag{2}$$

Then

$$\left\| \sum_{j=0}^n L_jy_0 - x^\bullet \right\| < \varepsilon.$$

Obviously, (2) is valid as soon as n is large enough and $\|y_0 - y\|$ is small. For condition (1), we present some analytical tools in the next section.

3. Numerical Ideas behind the Algorithm for the Equation $y + Lx = x$

In view of Corollary 2.2 and under its hypotheses, we can approximate the fixed point x^\bullet of A by a series close to the geometric one:

$$y_0 \in X \rightsquigarrow \begin{array}{l} \left(\begin{array}{ccccccc} y_0 & Ly_0 & L^2y_0 & \dots & L^ny_0 & \rightsquigarrow & \sum_{j=0}^n L^jy_0 & \xrightarrow{(n \rightarrow \infty)} & x^\bullet \\ L_0y_0 & L_1y_0 & L_2y_0 & \dots & L_ny_0 & \rightsquigarrow & \sum_{j=0}^n L_jy_0 & \approx & \sum_{j=0}^n L^jy_0 \end{array} \right) \end{array}.$$

In order to derive

$$\sum_{j=0}^n L_jy_0 \approx \sum_{j=0}^n L^jy_0$$

an approximation as that given in (1) is required. To this end, a possible tool appears provided by the Schauder bases, since they give an explicit linear approximation of any element of a Banach space by means of the associated projections, which is compatible with the continuity and affinity of the operator. What is more, in the case of classic bases, we easily obtain approximations of (the linear part of) A and its powers.

Thus, before continuing, we revise some of the basic notions of Schauder bases that we are going to need in the design of our algorithm. A sequence $\{e_j\}_{j \in \mathbb{N}}$ in a Banach space X is a *Schauder basis* if all the element $x \in X$ can be uniquely represented as

$$x = \sum_{j=1}^{\infty} \alpha_j e_j,$$

for a sequence of real $\{\alpha_j\}_{j \in \mathbb{N}}$. If we define for each $j \in \mathbb{N}$ the linear operator $P_j : X \rightarrow X$, known as the j -th projection associated with the basis, as

$$P_j x := \sum_{k=0}^j \alpha_k e_k,$$

for such an x , it is easy to prove, as a consequence of the Baire lemma, that it is a continuous operator and, in view of the representation of x in terms of the elements of the basis,

$$\lim_{j \rightarrow \infty} \|P_j x - x\| = 0.$$

With the aid of a Schauder basis, we can approximate Lx with $L(P_j x)$ which, on occasion, is easy to calculate. To summarize all of this, we focus on a type of affine equation, linear Fredholm integral equations, although this is the objective of the following section.

4. Algorithm to Approximate the Solution of a Linear Fredholm Integral Equation

In the rest of this paper, we focus our efforts on realizing everything that we explained thus far in order to address the study of a specific problem, the numerical resolution of a linear Fredholm integral equation, in two distinct settings.

Let $X = C[a, b]$ or $X = L^p[a, b]$, ($1 < p < \infty$), $k \in C[a, b]^2$ or $k \in L^\infty[a, b]^2$, respectively, and $y \in X$. Then we consider the corresponding linear Fredholm integral equation

$$x(t) = y(t) + \int_a^b k(t, s)x(s)ds, \tag{3}$$

where $x \in X$ is the unknown function. In view of the previous results, we consider the continuous and linear operator $L : X \rightarrow X$ defined at each $y_0 \in X$ as

$$Ly_0 := \int_a^b k(\cdot, s)y_0(s)ds.$$

Then, given $j \in \mathbb{N}$,

$$L^j y_0 = \int_a^b \left(\cdots \int_a^b k(\cdot, t_1)k(t_1, t_2) \cdots k(t_{j-1}, t_j)y_0(t_j)dt_j \right) \cdots dt_1.$$

From now on, in both cases ($X = C[a, b]$ or $X = L^p[a, b]$), we assume that

$$\|k\|(b - a) < 1,$$

since such a condition is sufficient for the validity of $\|L\| < 1$ and it is very easy to check. Furthermore, for each $d \in \mathbb{N}$, we fix a Schauder basis $\{e_j^{(d)}\}_{j \in \mathbb{N}}$ in $C[a, b]^d$ (if $X = C[a, b]$) or in $L^p[a, b]^d$ (if $X = L^p[a, b]$) and we denote the projections in this basis as $\{P_j^{(d)}\}_{j \in \mathbb{N}}$.

With all of this, we are now ready to define the approximate operators L_j : for each $x \in X$ and $j \in \mathbb{N}$, we take

$$\Phi_j(x)(t, t_1, \dots, t_j) := k(\cdot, t_1)k(t_1, t_2) \cdots k(t_{j-1}, t_j)y_0(t_j)$$

and fixed on $r_j \in \mathbb{N}$, thus $L_j : X \rightarrow X$ is given as

$$L_j y_0 := \int_a^b \left(\cdots \int_a^b P_{r_j}^{(j+1)}(\Phi_j(y_0)(\cdot, t_1, \dots, t_j))dt_j \right) \cdots dt_1. \tag{4}$$

Now we can apply the corollary 1 since without going any further, each r_j is big enough, $\|L_j x - L^j x\| < \varepsilon_j$.

Corollary 2. For any $\varepsilon > 0$ and $y_0 \in X$, there are natural numbers n and r_0, \dots, r_n in such a way that if x^\bullet is the unique solution to the linear Fredholm integral Equation (3), then

$$\left\| \sum_{j=0}^n L_j y_0 - x^\bullet \right\| < \varepsilon,$$

where $L_0 = I$ and for each $j \geq 1$, the operator L_j is defined by (4).

Thus, we have established the following (Algorithm 1)

Algorithm 1: Algorithm for approximating the solution of the linear Fredholm integral equation.

Choose $y_0, k, n, \varepsilon, \varepsilon_0, \dots, \varepsilon_n, r_0, \dots, r_n \in \mathbb{N}$, and $\{e_i^{(d)}\}_{i \in \mathbb{N}, d = 1, \dots, n+1}$;
 $L_0 \leftarrow I$;
 $j \leftarrow 1$;
while $\left\| \sum_{j=0}^n L_j y_0 - x^\bullet \right\| \geq \varepsilon$ and $j \leq n$
 $\Phi_j(y_0)(t, t_1, \dots, t_j) \leftarrow k(\cdot, t_1)k(t_1, t_2) \cdots k(t_{j-1}, t_j)y_0(t_j)$;
 $L_j y_0 \leftarrow \int_a^b \left(\cdots \int_a^b P_{r_j}^{(j+1)}(\Phi_j(y_0)(\cdot, t_1, \dots, t_j) dt_j \right) \cdots dt_1$;
 $j \leftarrow j + 1$; **end (while)**
sol_approx $\leftarrow \sum_{j=0}^n L_j y_0$.

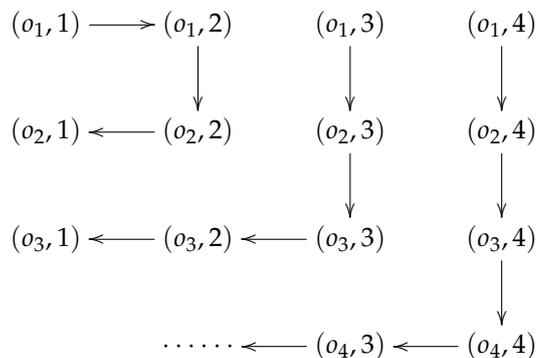
Observe that

$$\|\text{sol_approx} - x^\bullet\| < \varepsilon$$

and that for an appropriate choice of the bases $\{e_j^{(d)}\}_{j \in \mathbb{N}}$, the calculations are immediate, as justified below.

Returning to the considered spaces in order to study the linear Fredholm integral equation, $X = C[a, b]$ or $X = L^p[a, b]$, we remember how it is possible to tensorially construct bases $\{e_j^{(d)}\}_{j \in \mathbb{N}}$ in $X = C[a, b]^d$ or $X = L^p[a, b]^d$, respectively, from a basis $\{e_j^{(1)}\}_{j \in \mathbb{N}}$ in the aforementioned spaces.

Specifically, given $d \in \mathbb{N}, d \geq 2$, we consider in \mathbb{N}^d the square ordering introduced in [17] in an inductive form: for $d \geq 2, (1, 1), (1, 2), (2, 2), (2, 1), (1, 3), (2, 3), (3, 3), (3, 2), \dots$, and given the ordering o_1, o_2, \dots of \mathbb{N}^{d-1} , the order in \mathbb{N}^d is $(o_1, 1), (o_1, 2), (o_2, 2), (o_2, 1), (o_1, 3), (o_2, 3), (o_3, 3), \dots$. Graphically,



Thus, we establish a bijection $\tau : \mathbb{N} \rightarrow \mathbb{N}^d$, that for each $j \in \mathbb{N}$ a d -upla is assigned in the form

$$\tau(j) = (\alpha_1, \dots, \alpha_d)$$

and for such a j , we define

$$e_j^{(d)}(t_1, \dots, t_d) := e_{\alpha_1}(t_1) \cdots e_{\alpha_d}(t_d), \quad ((t_1, \dots, t_d) \in [a, b]^d). \tag{5}$$

The usual Schauder basis in $C[a, b]$ is the Faber–Schauder system, and in $L^p[a, b]$ is the Haar system [18]. More specifically, and assuming without loss of generality that $a = 0$ and $b = 1$, for the Faber–Schauder system, we start from the nodes $\{t_j\}_{j \in \mathbb{N}}$, which are the points of $[a, b]$ arranged dyadically, and the basis functions $\{e_j^{(1)}\}_{j \in \mathbb{N}}$ are continuous piecewise linear functions, the so-called *hat functions*, satisfying for each $j \in \mathbb{N}$

$$e_j^{(1)}(t_j) = 1$$

and

$$1 \leq k < n \Rightarrow e_k^{(1)}(t_j) = 0.$$

On the other hand, if A is a non-empty subset of $[0, 1]$ and $\delta_A : [0, 1] \rightarrow \mathbb{R}$ is the function defined in each $0 \leq t \leq 1$ as

$$\delta_A(t) := \begin{cases} 1, & \text{if } t \in A \\ 0, & \text{if } t \notin A \end{cases}$$

and $\varphi : [0, 1] \rightarrow \mathbb{R}$ is the function such that in each $0 \leq t \leq 1$

$$\varphi(t) := \delta_{[0,0.5]}(t) - \delta_{[0.5,1]}(t),$$

then the Haar system is given by

$$e_1^{(1)} := 1$$

and for $j \geq 2$, written uniquely as $j = 2^k + r + 1$, with $k = 0, 1, \dots$ and $r = 0, 1, \dots, 2^k - 1$,

$$e_j^{(1)}(\cdot) := \varphi(2^k(\cdot) - r).$$

In both cases, the tensorial sequences defined as (5) constitute Schauder bases in their respective spaces, $C[a, b]^d$ and $L^p[a, b]^d$ [17,19]. However, what really makes these bases useful when they are used in our Algorithm 1 is precisely that the calculation of the approximate operators L_j is very easy, since the basis functions $e_j^{(d)}$ are of separate variables and each factor is immediately integrable. Let us mention that these Schauder bases allow us to preserve the linearity of the convergence that it is guaranteed by the series geometric theorem.

5. Numerical Examples

We now show the numerical results obtained in several specific examples. Beforehand, let us mention that the reordering of a finite number of Schauder basis elements produces another new Schauder basis, which could be interesting from a computational point of view. Thus, for each $r \in \mathbb{N}$, we reordered the bases of $C[a, b]^d$ and $L^p[a, b]^d$ so that the r^d first elements correspond to $(\alpha_1, \alpha_2, \dots, \alpha_d)$ being $1 \leq \alpha_i \leq r$. For these reordered bases, we maintain the same previous notation, $\{e_j^{(d)}\}_{j \in \mathbb{N}^d}$ for the basis and $\{P_j^{(d)}\}_{j \in \mathbb{N}}$ for the sequence of projections. Furthermore, given $n, r \in \mathbb{N}$, we write

$$x^{(n,r)} := \sum_{j=0}^n L_j y_0,$$

where the indices r_j involved in the definition of L_j are given by $\tau(r_j) = (\overbrace{r, \dots, r}^{j+1 \text{ times}})$.

In each example, we consider $y_0 = y$ since another choice of y_0 is rather more theoretical, and since as we have indicated previously, it addresses the function y when some kind of error is produced in this function. Calculations were obtained by means of the Mathematica 12 software.

Example 1. We consider the equation of the Example 1 in [10]:

$$x(t) = \frac{30\pi t - \sin(\pi t)}{15} + \frac{1}{15} \int_0^1 t \cos(\pi t s^2) x(s) ds$$

whose solution is $x^\bullet(t) = 2\pi t$.

The errors obtained with our method are comparable to the order of those obtained in the reference taking $m = 4$ and $p = 2$, as shown in Table 1. The advantage in our case is that it is not necessary to start with an approximate solution “close enough” to the exact solution and it is not necessary either solve any system of linear equations.

Table 1. $\|x^\bullet - x^{(n,r)}\|$ for Example 1 using the usual basis in $C[0, 1]$.

n	$r = 9$	$r = 17$	$r = 33$
1	0.00435474	0.00216401	0.00162257
2	0.00300495	0.000773636	0.000210816
3	0.00298439	0.000752736	0.00018924
4	0.00298394	0.000752272	0.000188771

Example 2. The following equation is also extracted from the same reference (Example 2, [10]):

$$x(t) = t^2 - t + 1 + \frac{1}{4} \int_0^1 e^{ts} x(s) ds.$$

As in the referenced paper, since the solution of this equation is not known, we consider the operator $F : C[0, 1] \rightarrow C[0, 1]$ given by

$$F(x)(t) = x(t) - t^2 + t - 1 - \frac{1}{4} \int_0^1 e^{ts} x(s) ds$$

and we show $\|F(x^{(n,r)})\|$ for different values of n and r in Table 2.

The errors obtained are similar to those reported in Table 2 of [10] but with the same advantage mentioned above.

Table 2. $\|F(x^{(n,r)})\|$ for Example 2 using the usual basis in $C[0, 1]$.

n	$r = 9$	$r = 17$	$r = 33$
1	0.122201	0.123111	0.123339
2	0.0399626	0.0413395	0.0416839
3	0.0120328	0.0136133	0.0140084
4	0.00258589	0.00422851	0.0046442

Example 3. The following equation,

$$x(t) = \frac{2t^2 - 1}{3} + \frac{2}{3} e^t (t - 1) + \frac{1}{3} \int_0^1 t^3 e^{ts} x(s) ds,$$

is taken from [20], Example 3. Its solution is $x^\bullet(t) = t^2 - 1$. See Table 3 for the error generated by Algorithm 1.

Table 3. $\|x^\bullet - x^{(n,r)}\|$ for Example 3 using the usual basis in $C[0, 1]$.

n	$r = 9$	$r = 17$	$r = 33$
1	0.0627248	0.0593289	0.0584785
2	0.0133747	0.0104771	0.00975651
3	0.00503508	0.00242016	0.00156699
4	0.00362812	0.00109348	0.000463489

Example 4. This is a standard test problem, and it arises in electrostatics (see [21]) where it is called Love’s equation.

$$x(t) = y(t) + \frac{\delta}{\pi} \int_0^1 \frac{x(s)}{\delta^2 + (t - s)^2} ds.$$

We consider $\delta = -1$ and $y(t) = 1 + \frac{1}{\pi}(\arctg(1 - t) + \arctg(t))$ as in Example 3.2 of [22]. In this case, the exact solution is $x^\bullet(t) = 1$.

The errors—see Tables 4 and 5—are similar to those obtained by the Haar wavelet method and rationalized Haar functions method (see Table 1 in [22]), although their computation requires to solve some high-order systems of linear equations.

Table 4. $\|x^\bullet - x^{(n,r)}\|$ for Example 4 using the usual basis in $C[0, 1]$.

n	$r = 9$	$r = 17$	$r = 33$
1	0.0819571	0.0825347	0.0826789
2	0.0241024	0.0234044	0.023227
3	0.0059927	0.00634808	0.00645712
4	0.00299427	0.00211501	0.0018917

Table 5. $\|x^\bullet - x^{(n,r)}\|$ for Example 4 using the usual basis in $L_2[0, 1]$.

n	$r = 8$	$r = 16$	$r = 32$
1	0.0785242	0.0785241	0.0785217
2	0.0235324	0.0235323	0.0235264
3	0.0101296	0.0101295	0.0101140
4	0.0083851	0.0083850	0.0083669

Example 5. Now considering Example 2 of [23] which has solution $x^\bullet(t) = \sin(2\pi t)$

$$x(t) = \sin(2\pi t) + \int_0^1 (t^2 - t - s^2 - s)x(s)ds.$$

We observe that the numerical results obtained with our method (Table 6) significantly improve those obtained in the reference.

Table 6. $\|x^\bullet - x^{(n,r)}\|$ for Example 5 using the usual basis in $L_2[0, 1]$.

n	$r = 8$	$r = 16$	$r = 32$
4	3.17949×10^{-11}	4.33093×10^{-11}	4.28502×10^{-9}

6. Conclusions

In this paper, we present an algorithm for iteratively approximating the fixed point of a continuous coercive affine operator. Its design is based on a perturbed version of the classic geometric series theorem, the error control that this provides, and the use of certain Schauder bases. All of this is illustrated for a wide group of affine problems, the linear Fredholm integral equations. The low computational cost that our algorithm entails makes it particularly efficient. All of this is illustrated by several examples. We consider that

future research could be focused on extending the algorithm to solve different types of integral and even integro-differential equations.

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