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Towards a mathematical theory of behavioral human crowds

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The first part of our paper presents a general survey on the modeling, analytic problems, and applications of the dynamics of human crowds, where the specific features of living systems are taken into account in the modeling approach. This critical analysis leads to the second part which is devoted to research perspectives on modeling, analytic problems, multiscale topics which are followed by hints towards possible achievements. Perspectives

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include the modeling of social dynamics, multiscale problems and a detailed study of the link between crowds and swarms modeling.

Keywords: Crowd dynamics; living systems; complexity; multiscale problems; social dynamics; swarm dynamics.

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1. Plan of the Paper

The complex dynamical behavior of human crowds has fascinated researchers from various scientific fields for decades. Academic studies started with empirical observations and continued with the development of models in the field of applied physics and mathematics. Modeling, qualitative analysis, and computations of human crowd dynamics have captured a wide interest in recent years, due to their challenging nature and the potential benefits that the study of these systems can bring to our society. For instance, mathematical models can serve as decision-making tools for crisis managers to support safe evacuation in the presence of fires or contrasts between antagonist groups. Interesting recent developments motivated by the COVID-19 pandemic are in the direction of understanding the complex interactions between crowd congestions and contagion by virus spreading.

Challenging analytic and computational problems arise from the application of models to study real flow conditions within the general framework of the study of systems of many living, self-propelled entities undergoing nonlinearly additive and nonlocal interactions. This explains the growing interest in this field observed over the last few years. Some applications have generated new analytical and computational tools.

Individual behaviors significantly affect interactions, and, hence, play an important role on the emerging collective dynamics. This means that the modeling approach cannot be based on a straightforward application of methods of classical mechanics because the dynamics of a crowd does not simply rely on deterministic causality principles. For a realistic representation of crowd dynamics, the heterogeneous behavior of individual entities and the related modeling of complex interactions should be taken into account. In addition, one specific feature of all living systems must be accounted for: the ability to develop a self-organizing intelligence, i.e. a collective learning ability[31–33] that progressively modifies the rules of the interactions. Hence, collective behaviors cannot be simply related to those of a few entities.

As it is known,[7] the modeling approach can be developed at the three usual scales, namely microscopic (individual based), macroscopic (hydrodynamic), and mesoscopic (kinetic). The latter is intermediate between the small and the large scale. The interested reader is addressed to Ref. 66 for a survey of the literature on the physics and modeling of self-propelled particles, while the mathematical literature on crowd modeling by the individual based and by the hydrodynamic approach has been reviewed in Ref. 15. The book[44] is mainly focused on the modeling at the macroscopic scale with some visions on multiscale problems. The mesoscopic representation, which is typical of the mathematical kinetic theory, is delivered by a probability distribution function over the state of the individual entities, namely their state at the microscopic scale. The kinetic theory approach has been reviewed in Ref. 2 jointly with the mathematical approach to vehicular traffic and swarms. Finite treatment of space for mesoscopic and macroscopic models can be developed at each scale by classical methods of numerical analysis, e.g. finite difference,[51,77–79] finite element,[104] or finite volume[37,62,74] methods, or by computational tools such as cellular automata,[49] and Monte Carlo particle methods in the case of kinetic equations.[5,22,87]

It is worth observing that mathematicians have effectively heard the message delivered in Ref. 15, which urges modelers to consider heterogeneous behavioral features in crowds and the specific influence on the interactions of pedestrians to be interpreted as active, rather than classical, particles. Indeed, the recent literature witnesses increasing attention on the behavioral features of human crowds; detailed references will be given in Sec. 3.

Let us focus on a number of topics that, according to the authors' research knowledge represent key features and offer an interesting framework for future research perspectives.

(1) **Modeling from mathematical structures representing complexity:** Models can be derived within a framework offered by a general structure suitable to describe the dynamics in time and space of the variables deemed to describe the state of the system. In the book,[11] this general structure comes from a differential system suitable to capture the most relevant complexity features of a crowd viewed as a living, hence complex, system. This study, however, is limited to the kinetic theory framework, while the approach should be adopted at each scale pursuing a unified modeling rationale at each scale.
(2) **Scaling and a critical analysis of models:** Chasing the unified approach, mentioned in Item 1, requires a critical analysis of the modeling at each scale aimed at highlighting the merits and drawbacks of the present modeling state of the art in representing the aforementioned complexity features. A common modeling strategy at each scale consists first in the design of a general framework able to capture the complexity features of human crowds and subsequently in the derivation of models by implementing a mathematical description of interactions into the said structure.[7]
(3) **Social behaviors:** It is well understood that behavioral features of humans, for instance stress and the related propagation dynamics, can play an important role on the overall dynamics of crowds. This development has been motivated by human safety problems[14,65,68,69,73,84,108] induced by, as an example, forced evacuation in the presence of fire incidents.[91–93,95] Indeed, it has been shown that the walking strategy developed by pedestrians in distressed conditions is subject to important modifications that might even lead to unsafe conditions.[17]

These three key issues guide the presentation of the contents of our paper which aims at developing a conceptual strategy towards the derivation of a mathematical theory of multiscale behavioral crowds. The theory is required to account for the complexity features of living systems consistently with the general framework proposed in Ref. 13. This objective may be chased within an interdisciplinary vision that considers the complex interaction between different scientific areas from mechanics to psychology,[53,81] learning,[31,32] social sciences,[1] and various others according to the hints in Ref. 64.

The first part of the paper focuses on a review and critical analysis of the existing literature, while the second part is devoted to topics which, according to the authors' research knowledge, will form the mainstream of future research activity in the field. In more details, with the derivation of a mathematical theory of crowd dynamics with an outlook to applications. In general, the presentation is not limited to a concise outline of the aforementioned topics as detailed hints to their development are given.

In detail, the contents of the following sections is as follows.

Section 2 defines the complexity features of human crowds which should be considered in the modeling approach also in view of the contribution that modeling and simulations can give to crisis managers in charge of safety problems in crowds. These complexity features should be captured by the mathematical structures mentioned in Item 1.

Section 3 is devoted to the derivation of the aforementioned mathematical structures at each of the three scales, namely microscopic (individual-based), meso-scale by kinetic theory methods, and macroscopic (hydrodynamic). The rationale towards their derivation is the same at each scale, while the structures are deemed to provide, at each specific scale, the conceptual framework for the derivation of models. This section also provides a review and critical analysis of the existing literature.

Section 4 shows how the structures introduced in Sec. 3 naturally lead to the derivation of models referred to applications. Specifically, models are derived by inserting into the said structures a mathematical description of interactions, which are nonlocal and nonlinearly additive. It is worth stressing that these structures depict the dynamics in time and space of the dependent variables, specialized at each scale, deemed to describe the state of the crowd. An important feature is that the dependent variables include an additional variable, called *activity*, which models the non-mechanical state (e.g. a social state). A simplified approach is obtained under the assumption that the activity is a parameter constant in time.

Section 5 focuses on a number of research perspectives alongside hints to tackle them within an appropriate research program. Indeed, the critical analysis of the existing literature presented throughout the paper offers the conceptual basis to define new challenging research objectives. The selection of these topics does not claim to be exhaustive, as it is the result of the authors research experience.

The final aim to our paper consists in proposing a mathematical theory of human crowds within a multiscale vision by a modeling approach suitable to consider the complexity features of living systems.[13]

2. Complexity Features Towards Multiscale Behavioral Modeling

This section focuses on the main features of human crowds viewed as a living system constituted of many interacting entities. The objective is the derivation of a general mathematical structure able to capture the specific complexity features of living systems in general and human crowds in particular. These features should be defined for all scales in a common and unified way, as the key objective of modeling consists in designing models that have the potential to be multiscale, i.e. to provide a description covering all scales. Unlike inert matter, pedestrians possess the behavioral ability to develop walking strategies and to adapt them to the context. This leads to observable emerging behaviors which are generated by causes that often do not appear evident.

Let us define precisely two terms: *behavioral ability* and *walking strategy*. We need to borrow concepts from the mathematical theory of active particles (a-particles),[11] according to which each individual in every living system is a carrier of different purposes and *abilities* to chase them. In general, it is impossible to account for all of these abilities within a single mathematical framework, but we can state that, in the case of human crowds, such *purposes and abilities* lead to the selection of waling direction and adjustment of the speed to the local flow conditions, briefly density and stream. This selection leads to the development of a *walking strategy*. This complex dynamics can be simplified, as we shall see in full details in Sec. 3, by introducing:

— The *activity*: A vector variable that collects a number of behavioral variables, such as emotional state (stress/awareness induced by perception of danger), leadership attitude, etc. The activity is modified by interactions with other individuals in the crowd.

— The *walking strategy*: The way by which individuals select the trajectory to follow in order to reach the desired target and the speed by which they move along the trajectory. The activity has an important influence on the walking strategy, which accounts also for the physical features of the venue where the crowd moves.

Bearing all the above in mind, let us list five key specific features that should be accounted for in the modeling approach, being — in our opinion — of paramount importance.

(1) *Behavioral–emotional state*: Each individual has a behavioral (emotional) state that can play an important role in the development of the walking strategy. Possible examples are the *mental concentration and rationality* to reach a specific target, the *stress* induced by the perception of a danger, but even an *aggressive attitude* to contrast an antagonist group.

(2) *Ability to express a strategy*: Living entities have the ability to develop specific *strategies* related to their *organization ability*. These strategies depend on the state of the entities in their surrounding environment and on the physical fea-

tures of the areas where the crowd moves, as well as on non-predictable external events.

(3) *Heterogeneity*: The ability to express a strategy is *heterogeneously distributed among individuals* since it can include different objectives and possibly the presence of leaders who aim at driving all other pedestrians towards their own strategy. All types of heterogeneity induce various stochastic features in the interactions. In particular, irrational behaviors of a few entities can generate large deviations from the usual dynamics observed in situations driven by rationality.
(4) *Nonlinear interactions*: Interactions are *nonlinearly additive* and *nonlocal* as they involve not only immediate neighbors, but in some cases, also distant entities. Specifically, the topological distribution of a fixed number of neighbors can play a prominent role in the development of a strategy, as living entities interact with a fixed number of entities rather than with all those in their visibility domain.
(5) *Role of environment and venues*: The dynamics is affected by the *quality of the environment*, i.e. weather conditions for outdoor venues, geometry of the venue, luminosity, and various others. Pedestrians receive inputs from their environments and have the ability to learn from past experience. Hence, their rules of interactions evolve in time and space.

It appears to us that the above items are the most relevant. However, our choice does not claim to be exhaustive and additional aspects can be considered as well. Accordingly, we have to find a modeling approach flexible enough to include possible additions if consistent with the specific physical situation that is being modeled. The key difficulty is the lack of a field theory that would offer a natural support, as it happens in the sciences of the inert matter. This fact is strongly related to the problem of linking a rigorous mathematical approach to the study of living systems in general.

A common target, shared by all models, consists in describing collective motions based on interactions to be modeled consistently at each specific scale. Hence, the search of a general structure requires the preliminary design of substructures suitable to model interactions consistently with the aforementioned complexity features. These structures differ at each scale, namely: ordinary differential equations at the microscopic scale to model the individual state of each pedestrian; kinetic theory equations to describe, at the mesoscopic scale, the dynamics of probability distribution functions over the individual state at the microscopic scale of pedestrians; and partial differential equations corresponding to macroscopic variables at the macroscopic scale.

The mathematical approach always needs a multiscale vision,[7] as only one observation and representation scale is not sufficient to describe the overall collective dynamics of living systems. For instance, the dynamics at the microscopic scale defines the conceptual basis towards the derivation of models at the higher scales, where observable macroscopic quantities correspond to the collective dynamics.

3. From Complexity to Mathematical Modeling

This section shows how the modeling rationale proposed in Sec. 2 can be transferred into a differential framework able to capture the complexity features therein defined. Subsequently, we develop a critical analysis to understand how far the present state of the art on crowds modeling has effectively contributed to derivation and application of behavioral models. In more detail, on the contents of the following sections, we first design a strategy towards the modeling approach. Next, we define, at each scale, the mathematical structures which can provide conceptual basis for the derivation of models. The latter section includes a critical analysis of the consistency of such structures with the complexity features of human crowds. Finally, we propose additional reasonings on modeling and validation.

The main novelty of the aforementioned mathematical structures is the modeling of social dynamics and their influence over the collective motion. This topic, which has not been exhaustively treated in the literature, is developed in this section and is further discussed in the next one.

3.1. *Strategy towards modeling living crowds*

The rationale followed in the modeling approach is shown in Fig. 1, where each block represents a milestone towards the derivation and application of models. The

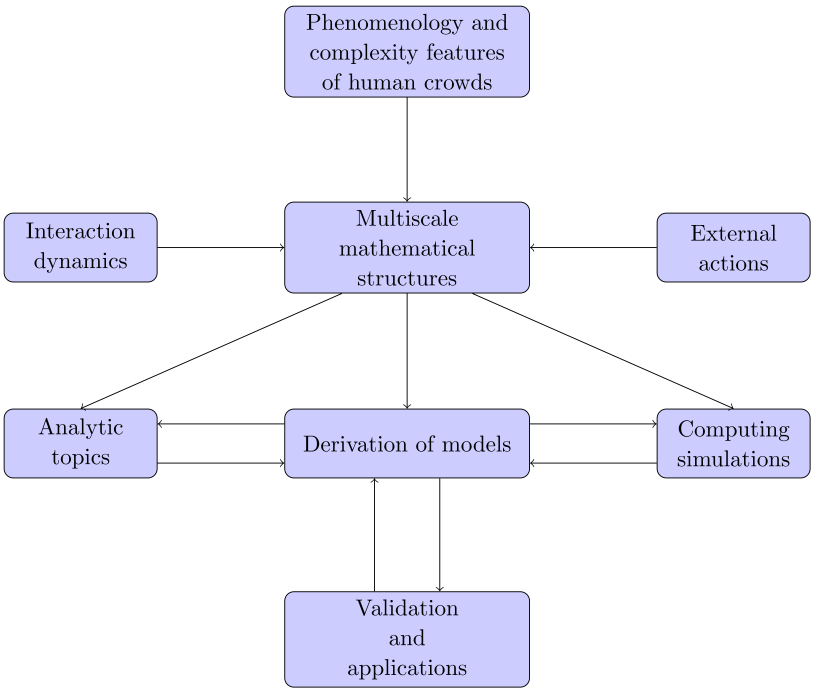

Fig. 1. (Color online) Rationale of the modeling approach.

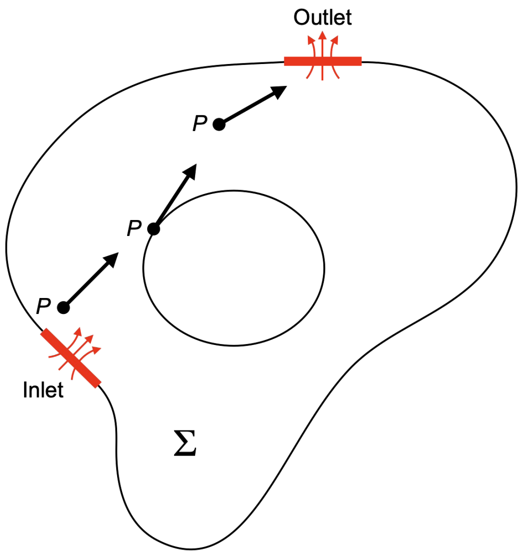

Fig. 2. (Color online) Venue with inlet–outlet doors and obstacles.

path moves from the phenomenological interpretation of the class of systems under consideration to the validation and application of models.

Figure 2 shows a schematic picture of the geometry of a sample venue, where the crowd moves, which includes inlet and outlet doors and internal obstacles. Point P represents a pedestrian along a trajectory to the exit by avoiding obstacles, while the associated arrow represents her/his walking direction. The motion can also take place across several interconnected areas, each of them characterized by a different geometry and a specific quality of the venue.

Let us introduce the following quantities that will be useful in the derivation of the mathematical structures:

— Σ_v denotes the venue where the crowd moves. If Σ includes walls, internal obstacles, and inlet–outlet doors, $\Sigma \subset \mathbb{R}^2$ is the walkable area, while $\Sigma_0 \subseteq \Sigma$ denotes the domain containing the whole crowd at the initial time. The boundary of Σ is denoted by $\partial\Sigma$. If the crowd is in an unbounded domain, then Σ_v includes either preferred directions or meeting points.

— ℓ is a characteristic length to be taken as the diameter of the circle containing Σ for problems in domains with boundaries or Σ_0 for problems in unbounded domains.

— $\alpha \in [0, 1]$ is a parameter modeling the quality of the venue, where $\alpha = 0$ corresponds to very low quality (i.e. motion is prevented) and $\alpha = 1$ to very high quality (i.e. fast motion is allowed).

— v_M is the highest speed an individual can reach by walking fast in free flow in high quality venues.

— v_ℓ is the highest, venue-dependent, speed an individual can reach. It is related to v_M by the following simple model $v_\ell \cong \alpha\, v_M$.
— ξ_M is the highest *mean* speed that can be reached by pedestrians in free flow in a high quality venue.
— $T = \ell/v_M$ is the characteristic time corresponding to the time a fast pedestrian can cover the distance ℓ.
— ρ_M is the maximal crowd density, i.e. the number of pedestrians packed in a square meter.
— $u \in [0, 1]$ is a scalar variable that models the level of the social states specifically considered in the model, for instance the level of stress, with $u = 0$ and $u = 1$ corresponding to lowest and highest levels of u, respectively.
— $\beta \in [0, 1]$ is a constant parameter that replaces u when this variable is equally shared by all pedestrians and is not modified by interactions. As above, $\beta = 0$ and $\beta = 1$ correspond to lowest and highest levels of, e.g. level of stress, respectively.
— Ω is the visibility domain, where a pedestrian can see the other pedestrians. Ω is a local quantity, generally an arc of circle referred to the local velocity direction. A more precise definition is provided later referring to each representation scale.

Figure 3 shows the visibility domain Ω_i in a two-dimensional space for the ith pedestrian. The representation is analogous at the other two scales. Ω_i is, in the figure, symmetric, but it might be shaded by obstacles and/or walls.

It is worth stressing that the vast majority of the models make use of β, i.e. a fixed parameter, instead of the variable u. On the other hand, physical reality suggests that a micro-scale variable, which is modified by interactions, should be used. Models accounting for this feature have been recently proposed.[17,20,77,106]

Remark 3.1. More in general, the social variable can be a vector $\boldsymbol{u} = \{u_k\}$, with $k = 1, 2, \ldots$ and $u_k \in [0, 1]$. In the kinetic theory of a-particles, $\boldsymbol{u}$ has been called

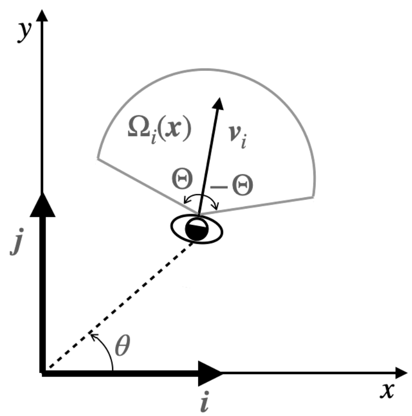

Fig. 3. Visibility domain Ω_i.

"activity". Correspondingly also $\boldsymbol{\beta}$ is a vector. These terms can be used at all scales. The activity has a direct influence on the motion of the crowd and motivates the use of "behavioral dynamics" to define the collective motion of a crowd.

3.2. *On the derivation of mathematical structures*

In this section, we derive, at each scale, the mathematical structures for the dynamics of a crowd in unbounded domains. The statement of boundary conditions, which is an essential step towards simulations, is treated in the last section. Dimensionless quantities are used to define variables and parameters according to criteria unified for all scales. In more details, all linear space quantities are divided by ℓ, all speeds by v_ℓ, and local densities by ρ_M. Hence, the related physical quantities have physical meaning only within the domain $[0, 1]$, by bounds that can be precisely defined according to the real geometry of the venue where the crowd moves.

We consider a heterogeneous human crowd in a two-dimensional domain Σ. Borrowing some definitions from the kinetic theory for a-particles,[7] pedestrians are considered *a-particles*, whose state is identified by mechanical variables, typically position and velocity, and vector variable (called *activity*) modeling their emotional or social state. These particles can be subdivided into *functional subsystems* (FSs) grouping a-particles that share the same *activity* and mechanical purposes. However, individual behaviors are heterogeneously distributed within each FS.

Remark 3.2. In the following, the derivation of the mathematical structures is obtained in the case of one subsystem only. The more general case of several interacting FSs is developed in Sec. 4 referring to well-defined classes of models.

We proceed according to the following common rules: (i) definition of the variables deemed to describe the state of the system; (ii) derivation of mathematical structures capturing the complexity features of human crowds; and (iii) a concise review of the achievements available in the literature, which is limited to the last decade in order to avoid repetitions with respect to the survey[15] and the book.[44]

3.3. *Microscopic (individual-based) scale*

We consider a crowd of N individuals and the representation of the overall state of the system. The dependent variables, deemed to define the overall state of each i-pedestrian, are defined by position $\boldsymbol{x}_i = \boldsymbol{x}_i(t) = (x_i(t), y_i(t))$, velocity $\boldsymbol{v}_i = \boldsymbol{v}_i(t) = (v_{ix}(t), v_{iy}(t))$, and activity $\boldsymbol{u}_i$, with $i \in \{1, \ldots, N\}$. The independent variable is the dimensionless time t, obtained by scaling the dimensional time by the characteristic time T. Polar coordinates $\boldsymbol{v}_i = \{v_i, \theta_i\}$ can be used to define the velocity of the individual motion, where v_i is the dimensionless speed and θ_i is the direction of the i-pedestrian.

The mathematical framework is derived within a pseudo-Newtonian mechanics somehow inspired by the mathematical theory of behavioral swarms.[19] This conceptual approach can be derived according to the following rationale:

(1) Each particle is able to develop a specific strategy which is heterogeneously distributed.
(2) A decisional hierarchy is applied supposing that interactions first modify the activity and subsequently the motion which in turn depends also on the activity.
(3) Each a-particle has a *visibility domain* and interacts with all particles within such a domain by nonlocal and nonlinearly additive interactions.
(4) The visibility domain is supposed to be an arc of circle symmetric with respect to the pedestrian's velocity direction. At the *microscopic scale*, it is denoted by $\Omega_i = \Omega_i(\boldsymbol{x}_i, \theta_i)$ for each i-pedestrian located at $\boldsymbol{x}_i$ and walking with direction θ_i.
(5) The action that produces a pseudo-acceleration of the activity variable of the i-particle by all particles in Ω_i is denoted by $\boldsymbol{\psi}_i$, while the psycho-mechanical acceleration over the velocity variable is denoted by φ_i.

In general, all the above quantities depend on all variables $\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{u}, \boldsymbol{z}$, where $\boldsymbol{z}$ is the set of all speeds $d\boldsymbol{u}_i/dt$ by which $\boldsymbol{u}_i$ increase/decreases. Each i-pedestrian is sensitive all pedestrians in the visibility domain. The formal structure of the framework is as follows:

$$\begin{cases} \dfrac{d\boldsymbol{u}_i}{dt} = \boldsymbol{z}_i, \\[2ex] \dfrac{d\boldsymbol{z}_i}{dt} = \displaystyle\sum_{j\in\Omega_i} \boldsymbol{\psi}_i(\boldsymbol{x}_i, \boldsymbol{v}_i, \boldsymbol{u}_i, \boldsymbol{x}_j, \boldsymbol{v}_j, \boldsymbol{u}_j; \alpha, \Sigma), \\[2ex] \dfrac{d\boldsymbol{x}_i}{dt} = \boldsymbol{v}_i, \\[2ex] \dfrac{d\boldsymbol{v}_i}{dt} = \displaystyle\sum_{j\in\Omega_i} \varphi_i(\boldsymbol{x}_i, \boldsymbol{v}_i, \boldsymbol{u}_i, \boldsymbol{x}_j, \boldsymbol{v}_j, \boldsymbol{u}_j; \alpha, \Sigma), \end{cases} \tag{3.1}$$

where the notation $j \in \Omega_i$ indicates that summation refers to all j-particles in the domain Ω_i.

The solution of mathematical problems provides the evolution of the dependent variables corresponding to position, velocity, and activity. Macroscopic quantities can be obtained by a local averaging at each point in the domain where the crowd moves. In practice, in a domain σ surrounding the point $\boldsymbol{x}$, the local density $\rho(t, \boldsymbol{x})$ and the mean velocity $\boldsymbol{\xi}(t, \boldsymbol{x})$ are given by

$$\rho(t, \boldsymbol{x}) \cong \frac{\sum_{i\in\sigma} i}{\rho_M|\sigma|}, \quad \boldsymbol{\xi}(t, \boldsymbol{x}) \cong \frac{\sum_{i\in\sigma} \boldsymbol{v}_i}{\rho(t, \boldsymbol{x})\,|\sigma|}, \tag{3.2}$$

where $|\sigma|$ denotes the measure of σ. Notice that the calculation is approximate as the limit $\sigma \to 0$ is not allowed because the system is not continuous.

If $\boldsymbol{u}_i \cong \boldsymbol{\beta}$ is a constant parameter shared by all pedestrians, then the dimension of the framework simplifies to

$$\begin{cases} \dfrac{d\boldsymbol{x}_i}{dt} = \boldsymbol{v}_i, \\ \dfrac{d\boldsymbol{v}_i}{dt} = \displaystyle\sum_{j \in \Omega_i} \varphi_i(\boldsymbol{x}_i, \boldsymbol{v}_i, \boldsymbol{\beta}, \boldsymbol{x}_j, \boldsymbol{v}_j; \beta, \alpha, \Sigma). \end{cases} \tag{3.3}$$

Remark 3.3. The key problem consists in modeling the two terms $\boldsymbol{\psi}_i$ and $\boldsymbol{\varphi}_i$ that refer to interactions. Notice that both $\boldsymbol{\psi}_i$ and $\boldsymbol{\varphi}_i$ depend on the quality of the venue, modeled by α, and on the overall geometry Σ as pedestrians modify their trajectories to avoid walls and obstacles. This matter will be treated in Sec. 3.5.

An overview of the literature on individual-based modeling indicates a variety of empirical studies on human behaviors under stress conditions due to perception of danger. The main objective of the research activity consists in developing computational models capable of capturing the dynamics of a crowd in emergencies conditions (see Ref. 102 for a recent review). Various authors use the definition *models of panicking crowds* to define models focused on a dynamics under stress conditions. Actually, we prefer to refer to these latter models as *models crowds under perception of danger conditions*.

Microscopic models can be roughly grouped into two categories depending on whether the focus is on how stress impacts human decision making and how stress spreads through the crowd. In the former case, special emphasis is put on how the irrational behavior of pedestrians in emergency affects crowd safety. It is worth mentioning that typically in all these contributions it is assumed that pedestrians are already under stress, i.e. the activity variable corresponds to a constant parameter $u \cong \beta$. Therefore, the onset of stress conditions and their spreading over the crowd is not investigated. The seminal paper by Helbing *et al.*[67] still constitutes the main contribution to this line of research, and provides invaluable insights on pedestrians' behaviors not present in a calm crowd (e.g. crushing behavior).

Studies that focus on how stress sets in and spreads through a crowd can mainly be found in the psychological literature. An interesting example is provided by Ref. 60 where three key conditions are identified, namely perception of an immediate threat, sense of powerlessness, and the belief that escape routes exist but are rapidly closing.

Many microscopic computational models have been proposed which try to embed these psychological aspects as well as social dynamics and collective learning[1,32,33,90] to capture the spreading of stress conditions. These studies can be roughly divided into two groups. To the first group belong epidemiological-like models where it is supposed that interactions with emotionally "infected" pedestrians increase the chance that others are getting infected (see for instance Ref. 53). It is worth mentioning that the COVID-19 pandemic has spurred a great interest in this modeling approach (see for instance some recent investigations of the interplay between pedestrian movement and virus spreading[77–79]). To the second group belong models that draw their inspiration from the phenomenon of heat dissipation in thermodynamics, where energy (in this case panic) is transferred between neighboring entities (see for instance Ref. 26).

It is worth mentioning that a growing body of studies is devoted to the ambitious aim to account for both the spread of panic and the subsequent change in pedestrians' behavior (see for instance Refs. 107, 109 and 111) and even generation and contagion of panic under multi-hazard circumstances (see for instance Ref. 54).

Various papers carry out empirical studies to understand how different models of individual behaviors lead to different pattern formations, e.g. considering the perception of the environment[39] or asymmetric interactions.[59] However, the modeling approach of a different variety of emotional states by a variable modified by interactions still appears to be an open problem which is tackled in Sec. 4 of our paper.

3.3.1. *On the approach of the kinetic theory for active particles*

Let us consider a system of interacting pedestrians who, according to the kinetic theory approach, are viewed — as mentioned — as *a-particles*. The dynamics is in the area Σ and, similarly to the micro-scale approach, we consider a system constituted by one FS only.

The *mesoscopic* (*kinetic*) *representation* is delivered by the one-particle distribution function at time t, over the microscopic state:

$$f = f(t, \boldsymbol{x}, v, \theta, \boldsymbol{u}), \quad \boldsymbol{x} \in \Sigma, \quad v \in [0,1], \quad \theta \in [0, 2\pi), \quad \boldsymbol{u} \in D_{\boldsymbol{u}}, \tag{3.4}$$

where polar coordinates have been used, namely v is the *speed*, $\theta \in [0, 2\pi)$ is the velocity direction related to an orthogonal plane frame and the activity $\boldsymbol{u}$ is a vector. Let $\boldsymbol{i}$ and $\boldsymbol{j}$ denote the unit vectors of the coordinate axes. The following notation can also be used:

$$\boldsymbol{v} = v(\cos\theta \boldsymbol{i} + \sin\theta \boldsymbol{j}) = v\boldsymbol{\omega}, \tag{3.5}$$

where $\boldsymbol{\omega}$ is the unit vector denoting the velocity direction.

The distribution function f is linked to the so-called test particle (pedestrian) assumed to be representative of the whole system. Therefore, f is the dependent variable defined over the *micro-state* $\{\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{u}\}$, while time and space are the independent variables. If f is locally integrable then $f(t, \boldsymbol{x}, \boldsymbol{v}, \boldsymbol{u}) d\boldsymbol{x}\, d\boldsymbol{v}\, d\boldsymbol{u}$ is the (expected) infinitesimal number of pedestrians whose micro-state, at time t, is comprised in the elementary volume

$$[\boldsymbol{x}, \boldsymbol{x} + d\boldsymbol{x}] \times [\boldsymbol{v}, \boldsymbol{v} + d\boldsymbol{v}] \times [\boldsymbol{u}, \boldsymbol{u} + d\boldsymbol{u}]$$

of the space of the micro-states.

Consistently with the physics of the system, the function f may be divided by ρ_M, which is the maximal full packing density of pedestrians as defined in Sec. 3.1.

Macroscopic observable quantities can be obtained, under suitable integrability assumptions, by velocity weighted moments of the distribution functions. As an example, the local *density* and *mean velocity* read

$$\rho(t, \boldsymbol{x}) = \int_{D_{\boldsymbol{v}}} \int_{D_{\boldsymbol{u}}} f(t,\, \boldsymbol{x},\, v, \theta, \boldsymbol{u}) v \, dv \, d\theta \, d\boldsymbol{u}, \tag{3.6}$$

where $D_{\boldsymbol{v}} = [0, 2\pi) \times [0, 1]$, and

$$\boldsymbol{\xi}(t, \boldsymbol{x}) = \frac{1}{\rho(t, \boldsymbol{x})} \int_{D_{\boldsymbol{v}}} \int_{D_{\boldsymbol{u}}} \boldsymbol{v} f(t,\, \boldsymbol{x},\, v, \theta, \boldsymbol{u}) v \, dv \, d\theta \, d\boldsymbol{u}, \tag{3.7}$$

respectively.

Interactions involve three types of a-particles: *test*, *field*, and *candidate* particles whose distribution functions are $f(t, \boldsymbol{x}, \boldsymbol{v}, u)$, $f(t, \boldsymbol{x}^*, \boldsymbol{v}^*, \boldsymbol{u}^*)$, and $f(t, \boldsymbol{x}^*, \boldsymbol{v}_*, \boldsymbol{u}_*)$, respectively. As mentioned before, test particles are representative of the whole system, while candidate particles can acquire, in probability, the micro-state of the test particle after interaction with the field particles. At the same time, test particles lose their state from this interaction with field particles.

In general, walking strategy of each pedestrian is determined by interactions with pedestrians in the interaction domain. These lead to a modification of activity, velocity direction, and speed depending on the micro-state and distribution function of the pedestrians in the interaction domain. Interactions can be modeled using the following quantities:

— *Short-range interaction domain*: Pedestrians interact with the other pedestrians in the visibility domain Ω, which is a circular sector with radius R, symmetric with respect to the velocity direction being defined by the visibility angles Θ and $-\Theta$. The a-particle perceives in Ω local density and density gradients.
— *Perceived density*: Particles moving along the direction θ perceive a density ρ_θ^p different from the local density ρ. Models should account that $\rho_\theta^p > \rho$ when the density increases along θ, while $\rho_\theta^p < \rho$ when the density decreases.
— *Interaction rate*: $\eta[f](\boldsymbol{x}, \boldsymbol{x}^*, \boldsymbol{v}_*, \boldsymbol{v}^*, \boldsymbol{u}_*, \boldsymbol{u}^*; \alpha, \Sigma)$ models the frequency at which a candidate (or test) particle at $\boldsymbol{x}$ enters in contact with the field particles in Ω.
— *Transition probability density*: $\mathcal{A}[f](\boldsymbol{v}_* \to \boldsymbol{v}, \boldsymbol{u}_* \to \boldsymbol{u} \,|\, \boldsymbol{x}, \boldsymbol{x}^*, \boldsymbol{v}_*, \boldsymbol{v}^*, \boldsymbol{u}_*, \boldsymbol{u}^*; \alpha, \Sigma)$ models the probability density that a candidate particle in $\boldsymbol{x}$ with state $\{\boldsymbol{v}_*, \boldsymbol{u}_*\}$ shifts to the state of the test particle due to the interaction with field particles in Ω with state $\{\boldsymbol{v}^*, \boldsymbol{u}^*\}$.

Remark 3.4. In the above notations, round and square parenthesis distinguish the argument of linear and nonlinear interactions, respectively. In more details, linear interactions involve only microscopic and independent variables, while non-linearity involves also the dependent variables. These nonlinear terms are nonlocal and depend on the quality of the environment/venue. This concept will be further clarified in the following.

The mathematical structure can be obtained by a balance of particles in the elementary volume of the space of the micro-states: $[\boldsymbol{x}, \boldsymbol{x}+d\boldsymbol{x}] \times [\boldsymbol{v}, \boldsymbol{v}+d\boldsymbol{v}] \times [\boldsymbol{u}, \boldsymbol{u}+d\boldsymbol{u}]$.

This equation is derived by equating the variation rate of the number of a-particles plus the transport due to the velocity variable to net flux rates within the elementary volume. It consists in an integro-differential equation that describes the time dynamics of the distribution functions f as follows:

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} f = \mathcal{G}[f,\, f] - f\mathcal{L}[f], \tag{3.8}$$

where the dot product denotes the standard inner product in $\mathbb{R}^2$ and $\nabla_{\boldsymbol{x}}$ denotes the gradient operator with respect to the space variables only. Moreover, $\mathcal{G}$ and $\mathcal{L}$ represent *gain* and *loss*, both nonlinearly acting on f, of pedestrians in the elementary volume of the phase space about the test microscopic state $(\boldsymbol{x},\, \boldsymbol{v})$, respectively. The detailed expression of these terms correspond to different ways of modeling pedestrian interactions at the microscopic scale.

In the case of *one FS only*, the aforementioned balance of particles yields:

$$\begin{aligned}
&(\partial_t + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}}) f(t, \boldsymbol{x}, \boldsymbol{v}, \boldsymbol{u}) = J[f](t, \boldsymbol{x}, \boldsymbol{v}, \boldsymbol{u}) \\
&\quad = \int_{\Gamma \times D_{\boldsymbol{v}} \times D_{\boldsymbol{u}}} \eta[f](\boldsymbol{x}, \boldsymbol{x}^*, \boldsymbol{v}_*, \boldsymbol{v}^*, \boldsymbol{u}_*, \boldsymbol{u}^*; \alpha, \Sigma) \\
&\qquad \times \mathcal{A}[f](\boldsymbol{v}_* \to \boldsymbol{v}, \boldsymbol{u}_* \to \boldsymbol{u} \,|\, \boldsymbol{x}, \boldsymbol{x}^*, \boldsymbol{v}_*, \boldsymbol{v}^*, \boldsymbol{u}_*, \boldsymbol{u}^*; \alpha, \Sigma) \\
&\qquad \times f(t, \boldsymbol{x}, \boldsymbol{v}_*, \boldsymbol{u}_*) f(t, \boldsymbol{x}^*, \boldsymbol{v}^*, \boldsymbol{u}^*) d\boldsymbol{x}^*\, d\boldsymbol{v}_*\, d\boldsymbol{v}^*\, d\boldsymbol{u}_*\, d\boldsymbol{u}^* \\
&\qquad - f(t, \boldsymbol{x}, \boldsymbol{v}, \boldsymbol{u}) \int_{\Gamma} \eta[f](\boldsymbol{x}, \boldsymbol{x}^*, \boldsymbol{v}, \boldsymbol{v}^*, \boldsymbol{u}, \boldsymbol{u}^*; \alpha, \Sigma) \\
&\qquad \times f(t, \boldsymbol{x}, \boldsymbol{x}^*, \boldsymbol{v}^*, \boldsymbol{u}^*) d\boldsymbol{x}^*\, d\boldsymbol{v}^*\, d\boldsymbol{u}^*,
\end{aligned} \tag{3.9}$$

where $\Gamma = \Omega \times D_{\boldsymbol{v}} \times D_{\boldsymbol{u}}$.

If the activity variable can be viewed as a constant parameter $\boldsymbol{u} \cong \boldsymbol{\beta}$, then the corresponding mathematical structure takes the simplified form:

$$\begin{aligned}
&(\partial_t + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}}) f(t, \boldsymbol{x}, \boldsymbol{v}) = J[f](t, \boldsymbol{x}, \boldsymbol{v}) \\
&\quad = \int_{\Gamma} \eta[f](\boldsymbol{x}, \boldsymbol{x}^*, \boldsymbol{v}_*, \boldsymbol{v}^*; \alpha, \boldsymbol{\beta}, \Sigma) \mathcal{A}[f](\boldsymbol{v}_* \to \boldsymbol{v} \,|\, \boldsymbol{x}, \boldsymbol{x}^*, \boldsymbol{v}_*, \boldsymbol{v}^*; \alpha, \boldsymbol{\beta}, \Sigma) \\
&\qquad \times f(t, \boldsymbol{x}, \boldsymbol{v}_*) f(t, \boldsymbol{x}^*, \boldsymbol{v}^*) d\boldsymbol{x}^*\, d\boldsymbol{v}_*\, d\boldsymbol{v}^* \\
&\qquad - f(t, \boldsymbol{x}, \boldsymbol{v}) \int_{\Omega \times D_{\boldsymbol{v}}} \eta[f](\boldsymbol{x}, \boldsymbol{x}^*, \boldsymbol{v}, \boldsymbol{v}^*; \alpha, \boldsymbol{\beta}, \Sigma) f(t, \boldsymbol{x}, \boldsymbol{x}^*, \boldsymbol{v}^*) d\boldsymbol{x}^*\, d\boldsymbol{v}^*.
\end{aligned} \tag{3.10}$$

Hints towards the modeling of crowd dynamics by the kinetic theory approach were given in Ref. 15, where it is shown how the modeling tools of vehicular traffic[88,89] can be further developed to describe human crowds. A systematic study of human crowds accounting for nonlinear interactions between pedestrians has been started in Ref. 12 with a model that assumes discrete velocity directions and speed depending on the local density. An application was technically developed to describe the contrast of two groups moving in opposite directions. A qualitative analysis of the initial value problem and the derivation of macro-scale equations from the underlying description at the micro-scale where presented in Ref. 10.

This pioneering approach generated further developments. In particular, modeling by continuous velocity distributions and the study of fingering problem has been developed in Ref. 16. In Ref. 78, the kinetic model presented in Ref. 16 is further developed to study the dynamics in bounded domains with obstacles. A survey and critical analysis on the modeling literature can be found in Ref. 2.

Articles devoted to safety problems clearly indicate that crisis management can take advantage of models that account for human behaviors.[92,93,95,108] These articles have motivated recent works focused on human psychology in decision making[35,50] and related empirical studies on the calibration of models.[40,41] In the modeling of the social dynamic in crowds,[17,20] interacting pedestrians modify their psychological status and, in turn, the walking strategy. The emotional state significantly affects the overall crowd dynamics in extreme real-life situations such as a peaceful demonstration that turns violent[55] and the spreading of panic in emergency evacuations.[65]

One of the first works on a multiscale (from microscopic to macroscopic) approach to crowd dynamics with emotional contagion is in Ref. 106. Therein, fear is propagated by a Bhatnagar-Gross-Krook (BGK)-like model and results are limited to one space dimension, (see for BGK models of classical particles, e.g. Ref. 38). Studies on the impact of social dynamics on individual interactions and their influence at a higher scale are carried out in Refs. 47 and 48.

More recently, a kinetic approach to modeling pedestrian dynamics in the presence of social phenomena (e.g. propagation of stress) is presented in Ref. 17. The numerical results in Ref. 17 show that stress propagation significantly affects crowd density patterns and overall crowd dynamics in Ref. 77, the kinetic model presented in Ref. 16 is further extended to account for the propagation of stress conditions in time and space. As for the control of crowds, this topic is studied in Ref. 3 by means of the social influence of leaders, namely trained personnel that may guide pedestrians to egress from a complex environment whose connectivity is not known or modified by incidents. Besides its theoretical interest, this topic is of practical importance as it may significantly contribute to crowd management in emergency situations where overcrowding may cause fatal accidents, see Ref. 9.

A closely related problem is that of epidemics spread. A hybrid approach, that couples the kinetic model of crowd dynamics from Ref. 16 with one of contagion spreading inspired from the work on emotional contagion in Refs. 20 and 106, has been proposed in Ref. 79. The kinetic model proposed in Ref. 79 introduces an activity that denotes the level of exposure to people spreading the disease, with the underlying idea that the more a person is exposed the more likely her/his is to get infected. Such a model includes a parameter that describes the contagion interaction strength and a kernel function that is a decreasing function of the distance between a person and a spreading individual.

In spite of the similarity between models that deal with evacuation and virus transmission, a remarkable difference must be pointed out. In the former the key social state is the level of stress, whereas in the latter is the level of awareness. The resulting pedestrians' behavior is completely different in the two cases. Indeed, the level of stress promotes aggregation of pedestrians and leads to the herd behavior under panic conditions,[81] while the level of awareness pushes pedestrians to follow social distancing guidelines.

The research program on these challenging topics has just started, and many contributions are still to come due to the many complex aspects of human psychology as well as the inherent system heterogeneity. A multiscale framework, like the one proposed in Ref. 7 for the modeling of human crowds, needs to be formulated because, for practical applications, the crowd must be described at all the three possible modeling scales (i.e. microscopic, mesoscopic, macroscopic) by a consistent approach, namely, models must be derived at each scale using the same principles and similar parameters.

3.4. *Macroscopic hydrodynamic modeling*

The Eulerian description at the macroscopic scale can be adopted for large-scale systems, in which the local behavior of groups is sufficient to capture the global dynamics. The following macro-scale variables define the state of the system:

— $\rho = \rho(t, \boldsymbol{x})$ is the dimensionless crowd density at the point $\boldsymbol{x}$ and time t, normalized with respect to the maximum packing density of pedestrians ρ_M.
— $\boldsymbol{\xi} = \boldsymbol{\xi}(t, \boldsymbol{x})$ is the dimensionless mean velocity at the point $\boldsymbol{x}$ and time t, normalized with respect to the maximum average speed ξ_M. The mean velocity can also be expressed in polar coordinates as follows: $\boldsymbol{\xi} = \xi(t,\, \boldsymbol{x})\boldsymbol{\omega}(t,\, \boldsymbol{x})$, where ξ is the dimensionless mean speed and $\boldsymbol{\omega}$ is the unit vector giving the direction of the local mean velocity.
— $\boldsymbol{u} = \boldsymbol{u}(t, \boldsymbol{x})$, with $\boldsymbol{u} \in D_{\boldsymbol{u}}$, is a dimensionless variable deemed to model the specific social-emotional state considered in each case study. It can be viewed as a local mean activity.
— $\Omega = \Omega(t, \boldsymbol{x}; \boldsymbol{\omega}(t, \boldsymbol{x}))$ is the local visibility domain. The pedestrians at $\boldsymbol{x}$ perceive the action of all pedestrians in Ω, which makes interactions nonlocal.

The physical meaning of $\boldsymbol{u}$ and Ω is the same we used at the lower scales. However, the difference here consists in the fact that these quantities do not refer to individual entities but instead to the local density corresponding to the elementary physical element $d\boldsymbol{x}$. In addition, we refer to the parameters already defined at the lower scales in the search for a general mathematical framework to describe second-order dynamics, related to density ρ and linear momentum $\boldsymbol{v}$ and involving mechanical variables and activity $\boldsymbol{u}$. The formal structure of the framework is as follows:

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \nabla_{\boldsymbol{x}} \cdot (\rho \boldsymbol{\xi}) = 0, \\[2mm] \dfrac{\partial \boldsymbol{\xi}}{\partial t} + \boldsymbol{\xi} \cdot \nabla_{\boldsymbol{x}} \boldsymbol{\xi} = \boldsymbol{A}[\rho, \boldsymbol{\xi}, \boldsymbol{u}], \\[2mm] \dfrac{\partial \boldsymbol{u}}{\partial t} + \nabla_{\boldsymbol{x}} \cdot (\boldsymbol{u}\boldsymbol{\xi}) = \boldsymbol{S}[\rho, \boldsymbol{\xi}, \boldsymbol{u}], \end{cases} \tag{3.11}$$

where $\boldsymbol{A}$ is a pseudo-mechanical acceleration acting on pedestrians in the infinitesimal volume $d\boldsymbol{x}$ and $\boldsymbol{S}$ is a source term that implements locally the emotional state generated by the interaction with the surrounding pedestrians. Interactions are not only nonlocal, but also visual-based and nonlinearly additive. Like in the previous sections, the square brackets denotes that the dependence can be functional, for instance, dependent on the space derivatives of the variables in brackets.

If the activity is uniformly distributed in space, i.e. $\boldsymbol{u} \cong \boldsymbol{\beta}$ is constant in time, the structure becomes

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \nabla_{\boldsymbol{x}} \cdot (\rho \boldsymbol{\xi}) = 0, \\[2mm] \dfrac{\partial \boldsymbol{\xi}}{\partial t} + \boldsymbol{\xi} \cdot \nabla_{\boldsymbol{x}} \boldsymbol{\xi} = \boldsymbol{A}[\rho, \boldsymbol{\xi}, \boldsymbol{\beta}]. \end{cases} \tag{3.12}$$

Various authors (see, e.g. Ref. 44) have proposed a framework based on conservation of mass only and thus leading to first-order models. The equation of mass conservation is coupled to a phenomenological model linking the local mean velocity to the local density, as well as to the overall geometry of the venue, i.e. where the crowd moves. The corresponding structure is as follows:

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \nabla_{\boldsymbol{x}} \cdot (\rho \boldsymbol{\xi}) = 0, \\[2mm] \boldsymbol{\xi} = \xi[\rho]\boldsymbol{\omega}[\rho](\boldsymbol{x}; \alpha, \boldsymbol{\beta}), \end{cases} \tag{3.13}$$

where the scalar function $\xi[\rho]$ is often related to empirical data delivered by the so-called *fundamental diagram* and local gradients.[57,99–101] Recall that α is the parameter modeling the quality of the venue.

The main reference to the literature on crowd modeling by the macroscopic approach is the book.[44] This excellent book reports on the psychology of the crowd in Chap. 3 and on various first-order and second-order structures to be used in the modeling approach in Chap. 4. Such modeling part is followed by the study of a variety of challenging analytical and computational problems related to a multiscale framework.

Our paper is essentially focused on second-order models with the mathematical structure in (3.9), which includes the dynamics of the activity variable. Indeed, this is the main novelty with respect to the present state of the art. The main focus is on the modeling of pedestrians who, by rational behaviors, select optimal trajectories developing the pioneering ideas in Ref. 72. This class of equations has been the object of various studies devoted to qualitative analysis, e.g. Ref. 4. Instead, the dynamics of the activity variable has been developed essentially only by the kinetic theory approach.

In this respect, the study of crowd dynamics by hydrodynamic equations that include the dynamics of an activity variable, suitable to consider behavioral features, see Eq. (3.11), is still a research perspective. Then, looking ahead to forthcoming research activity we briefly report about a very few papers, where the approach proposed in our paper can bring added value.

Within a recent special issue devoted to the broad topic of mathematical models for collective dynamics,[36] a model of crowding and pushing in corridors with different motivation levels is proposed in Ref. 56. In the macroscopic approach developed therein, the motivation corresponds to a re-scaling in time accelerating or decelerating the dynamics.

Considering instead the latest engineering literature, it is worth mentioning the density-sensitive multiscale attempt by Ref. 21. The approach proposed in such a contribution is surely interesting, but the authors highlight some limitations in reproducing individual behavioral attributes, focusing on the walking strategy, at the macroscopic scale.

In Ref. 83, a novel model to quantitatively analyze pedestrian congestion in evacuation management is proposed based on the Hughes and social force models. In particular, the authors use the principle of virtual work to link the microscopic level of an individual pedestrian to the macroscopic level of a crowd in a multiscale attempt, but they do not hide, in their approach, in the modeling of the walking strategy.

Furthermore, Ref. 97 is devoted to the coupling of disease contagion models with crowd motion, a relevant and timely topic in the field; in this paper, the resulting coupled system is solved with hydrodynamic mesh-free methods.

We finally mention the macroscopic Lagrangian formulation of Ref. 110, consisting of mass conservation and an acceleration equation with a term of external forces which may include behavioral assumptions. All the aforementioned formulations might benefit from the ideas and the modeling framework discussed in this paper.

An additional aspect to be considered in the modeling is the study of the interaction between structures and crowds. This study is particularly important in the case of light mobile structures. A topic which has received attention in this context is the interactions between crowds and foot-bridges.[27,105] These applications have been treated by deterministic first-order macro-scale models. However, rapid changes in the motion suggest developing this approach to second-order models, where emotional behaviors are included specifically according to the framework proposed in Eq. (3.11).

3.5. *From mathematical structures to the derivation of models*

The mathematical structures derived in Secs. 3.1–3.4 are consistent with the complexity paradigms presented in Sec. 2 and provide the conceptual framework for the derivation of models. Particular models are obtained from the general framework by choosing how to represent interactions in $\boldsymbol{\varphi}_i$ and $\boldsymbol{\psi}_i$ for micro-scale model (3.1), η and $\mathcal{A}$ for kinetic theory approach (3.9), and $\boldsymbol{A}$ and $\boldsymbol{S}$ for macro-scale model (3.11). These terms refer, at each scale, to the ability of pedestrians to express walking strategies based on interactions with other individuals. Interactions are assumed to be nonlocal and nonlinearly additive as the strategy developed by a pedestrian is a nonlinear combination of different stimuli generated by the interactions with other pedestrians and with the environment.

The modeling of interactions is a key problem at all scales. Various authors have tackled this problem on the basis of heuristic assumptions somehow supported by empirical data, see for instance Refs. 76 and 85. Empirical and theoretical studies have been developed to contribute to modeling interactions,[40,41,45,86] with some studies focusing also on the role of emotional dynamics and self-organization.[52,71,82,86,96] Often, see for instance Refs. 28, 34, and 90, theoretical tools of game theory have been used to model interactions.

This section aims at defining the rationale to model interactions at the different scales in a consistent manner. The contents are limited to concepts, while the analytic formalization is postponed to Sec. 4, which shows how the general theory leads to the derivation of well-defined models corresponding to selected social dynamics.

3.5.1. *Heterogeneity and functional subsystems*

In general, the overall populations in the crowd is constituted by sub-populations, called FSs, which may differ due to their walking targets (either meeting points or exits) and/or ways of organizing their motion, e.g. antagonist groups contrasting each other within the crowd or leaders attracting pedestrians towards optimal trajectories. Then, the crowd can be divided into κ FSs labeled by superscript $\kappa = 1, 2, \dots, K$, which is applied, at each scale, to the dependent variables.

The generalization to multi-functional mathematical structures is obtained by adding all interactions across FSs, while treating each FS as in the case of one FS only. The same subdivision should be applied at each scale as a multiscale approach might require that the same models at different scales are used in the same venue, where the crowd presents coexistence of high density zones (where the hydrodynamical approach is appropriate) and rarefied areas (where a low-scale approach should be used).

3.5.2. *Decisional process towards a walking strategy*

A rationale, common to all scales, can be proposed towards a decisional process. This process is further specialized for the case studies treated in Sec. 4, where the specific emotional–social state, namely the activity, differs from case to case. The dynamics is induced by a collective learning[31] corresponding to *fast thinking*,[75] in some cases almost instantaneous.

The rationale is summarized in the following which includes also some technical indications in view of derivation of specific models:

(1) The decision process follows a *hierarchy* by which pedestrians first modify their *activity*, subsequently select the *velocity direction*, and finally modify their *speed* accounting for the local flow conditions. The dynamics is modeled by theoretical tools of game theory[1] describing how pedestrians *learn* the overall state of the crowd in the visibility domain Ω and modify their walking strategy accordingly.
(2) Pedestrians *select the velocity direction* accounting for a weighted contribution by the following trends: (i) reaching the nearest exit or meeting point; (ii) avoiding walls by nonlocal actions depending on the distance of pedestrians from walls; (iii) avoiding overcrowded areas; and (iv) attraction by the main stream which acts in contrast with the search of not congested areas. The weights by which the velocity direction is selected are supposed to depend, for each FS, on the local density, on the activity variable, and on the distance from the wall.
(3) Once the velocity direction has been selected, *the dynamics of the speed depends on the difference between the local density in the new direction and the local density in the direction before the change*. In detail, lower densities contribute to increasing the speed, while higher densities tend to decrease it. Local density has a role also in the selection of the velocity directions as it enhances trajectories through less congested areas.
(4) *The selection of the walking direction depends on a specific activity that plays a key role in the dynamics*. For instance, the *stress* contributes to increases the trend towards the mainstream over the trend towards the target. A different behavior is induced by the *awareness to contagion* as higher awareness increases the trend to avoid congested areas.
(5) *Walls and obstacles modify the selection of the velocity direction* with respect to the direction selected in unbounded domains. The action is nonlocal as the modification of the velocity increases by decreasing distance from the wall.

3.5.3. *On the statement of mathematical problems*

Mathematical problems are stated by assigning, at each scale, initial and boundary conditions. *Initial conditions* can be given at $t = 0$, for $\kappa = 1, \dots, K$, as follows:

Micro-scale: By the state of all pedestrians (viewed as a-particles)$\boldsymbol{u}_i^\kappa(t = 0)$, $\boldsymbol{z}_i^\kappa(t = 0)$, $\boldsymbol{x}_i^\kappa(t = 0)$, and $\boldsymbol{v}_i^\kappa(t = 0)$, for $i = 1, \dots, N$.

Meso-scale: By the distribution function over the micro-state $f^\kappa(t = 0, \boldsymbol{u}, \boldsymbol{x}, \boldsymbol{v})$, for $\boldsymbol{x} \in \Sigma$.

Macro-scale: By the local activity, density, and mean velocity $\boldsymbol{u}^k(t=0,\boldsymbol{x})$, $\rho^k(t=0,\boldsymbol{x})$, $\boldsymbol{\xi}^k(t=0,\boldsymbol{x})$, for $\boldsymbol{x} \in \Sigma$.

Let us now focus on the statement of *boundary conditions*. First, we observe that the presence of walls already modifies the trajectories of the motion, as pedestrians organize their motion to avoid getting close to the boundary $\partial\Sigma$. However, for those who do get in contact with the boundary a wall interaction law should be given linking the dynamics of pedestrians that leave the wall ($\boldsymbol{v}^+$ if $\boldsymbol{v} \times \boldsymbol{n} \geq 0$) to that of those who reach the wall ($\boldsymbol{v}^-$ if $\boldsymbol{v} \times \boldsymbol{n} < 0$), where $\boldsymbol{n}$ is the unit orthogonal to $\partial\Sigma$ directed towards the walking domain.

The statement of the boundary conditions for $\boldsymbol{x} \in \partial\Sigma$ and $\kappa = 1, \ldots, K$, requires some heuristic assumptions on the behavior of pedestrians consistent both with the fact that the flux of pedestrians at the wall is zero and the fact that the dynamics of pedestrians leaving a wall needs to be consistent with the paradigms stated in Sec. 3.5.2. The modeling has to account for a discontinuity of the dependent variables at the wall. Accordingly, these conditions can be stated, at each scale, as follows:

Micro-scale: Interactions with walls are local in space ($\boldsymbol{x}_i^{\kappa+} = \boldsymbol{x}_i^{\kappa-}$), pedestrians keep their activity ($\boldsymbol{u}_i^{\kappa+} = \boldsymbol{u}_i^{\kappa-}$) and speed, while moving along $\boldsymbol{n}$. Subsequently, the dynamics of each pedestrian can be modeled following the rules given in Sec. 3.5.2 depending on the local flow conditions.

Meso-scale: Interactions are local in space preserving the activity variable, while the distribution function of the pedestrians leaving the wall is related to that of the pedestrians moving towards the wall, as in the classical kinetic theory.[38] This is obtained by

$$f^{\kappa+}(\boldsymbol{u},\boldsymbol{x},\boldsymbol{v}^+) = \frac{|\boldsymbol{v}^- \times \boldsymbol{n}|}{|\boldsymbol{v}^+ \times \boldsymbol{n}|}\mathcal{R}(\boldsymbol{v}^- \to \boldsymbol{v}^+) f^{\kappa^-}(\boldsymbol{u},\boldsymbol{x},\boldsymbol{v}^-),$$

where the operator $\mathcal{R}$ denotes the probability density that a pedestrian moving towards the wall with velocity $\boldsymbol{v}^-$ then leaves the wall with velocity $\boldsymbol{v}^+$. The same heuristic assumption can be adopted at the micro-scale, i.e. reflection along $\boldsymbol{n}$ and subsequent dynamics as stated in Sec. 3.5.2.

Macro-scale: The statement of boundary conditions at the macro-scale can be stated using the same principles applied at the lower scales. In more details

$$\boldsymbol{u}^{\kappa+}(t,\boldsymbol{x}) = \boldsymbol{u}^{\kappa-}(t,\boldsymbol{x})$$

and

$$|\rho^{\kappa-}\boldsymbol{\xi}^{\kappa-}|(t,\boldsymbol{x}) = |\rho^{\kappa+}\boldsymbol{\xi}^{\kappa+}|(t,\boldsymbol{x}),$$

where $\boldsymbol{\xi}^{\kappa+} = |\boldsymbol{\xi}^{\kappa+}|\boldsymbol{n}$.

4. On the Derivation of Mathematical Models

The derivation of mathematical models can be developed within the framework of the structures proposed in Sec. 3. The presentation is mainly focused on concepts, methodology and, essential calculations. Further technical developments of the methods might be treated for specific applications, hopefully developed by interested readers.

The key step of the approach is the modeling of interactions to be properly referred to the specific social dynamics considered in each class of models. Interactions technically differ at each scale, but it is possible to model them according to common guidelines that are valid at all scales. Theoretical tools of game theory can be used to model them by further developing the methods already applied to the social dynamics.[1]

The hierarchy proposed in Sec. 3.5 suggests that pedestrians first modify their emotional state, then select a walking direction and, finally, adapt the speed to the flow conditions along the said direction. This section shows how mathematical models can be derived according to the guidelines given in Sec. 3. The modeling approach is developed for the aforementioned three representation scales. Full details are given for models at the micro-scale. Subsequently, the generalization to the other scales can be rapidly derived. We consider different types of emotional dynamics referred to a scalar activity variable. Then, we indicate how each specific emotional dynamics can be referred to real flow conditions. Overgeneralization of the role of the activity variable is avoided by limiting our study to a few case studies, i.e. stress in evacuation dynamics, perception of contagion risk, and leadership attraction towards optimal paths.

The literature on social dynamics is mainly based on consent dynamics. On the other hand, one of the open key problems posed in the last chapter of Ref. 23 indicates that different types of dynamics should be considered in the modeling of real systems. This idea was developed in Ref. 1 showing how the selection of consent or dissent may be related to the distance of the social state between interacting individuals. Focusing on the onset and propagation of stress in evacuation dynamics, it can happen that both *rational* and *irrational* behaviors are contextually observed. Therefore, both of them have to be taken into account by a dynamics which may depend also on the specific social variable that is object of modeling.

4.1. *Modeling at the microscopic (individual-based) scale*

Let us consider the modeling of the dynamics according to the aforementioned hierarchy, i.e. first the dynamics of the activity variable, then the selection of the velocity direction, and, finally, the dynamics of the speed. The movement of the crowd can, in various cases, be generated by a cluster of individuals localized in a small area within Σ. Then, it can pervade the whole area Σ due to interactions. Pedestrians move, as shown in Fig. 4, across zones with different local density which has an important influence over the interaction dynamics and, consequently, on the trajectories.

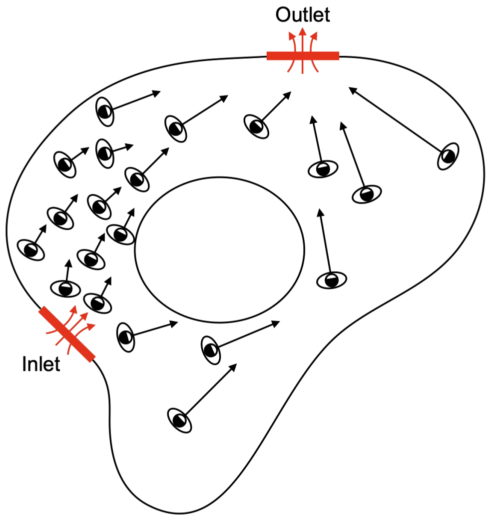

Fig. 4. (Color online) Flow with rarified and dense zones.

4.1.1. *Dynamics of the activity variable*

We consider the dynamics of the activity variable for the *micro-scale* referring to one FS only. Therefore, we consider interactions of the i-pedestrian, with state $\boldsymbol{x}_i, \boldsymbol{v}_i, u_i$ with all ij-pedestrians, i.e. the j-pedestrians which interact with the i-pedestrian as they are within the visibility domain Ω_i which, according to the definitions given in the preceding sections, depends on $\boldsymbol{v}_i$ and on Σ, while by ρ_i we denote the number of ij-pedestrians in Ω_i.

A very simple model relates the dynamics to the mean value $\mathbb{E}_i$ of the activity in Ω_i:

$$\mathbb{E}_i = \mathbb{E}_i(t; \boldsymbol{x}_i, \boldsymbol{v}_i, \Sigma) = \frac{1}{\rho_i[\Omega_i]} \sum_{j \in \Omega_i} u_j. \tag{4.1}$$

The consent/dissent dynamics can be modeled by the fractions $(1-\varepsilon)$ and ε denoting the trend towards and against $\mathbb{E}_i$, respectively. Therefore, the first two equations of (3.1) amount to the following model:

$$\begin{cases} \dfrac{d\boldsymbol{u}_i}{dt} = \boldsymbol{z}_i, \\[2ex] \dfrac{d\boldsymbol{z}_i}{dt} = \displaystyle\sum_{j \in \Omega_i} \boldsymbol{\psi}_i(\boldsymbol{x}_i, \boldsymbol{v}_i, \boldsymbol{u}_i, \boldsymbol{u}_j; \alpha, \Sigma) \\[2ex] \qquad = \eta(\rho_i)\big[(1-\varepsilon)(\mathbb{E}_i(t; \boldsymbol{x}_i, \boldsymbol{v}_i, \Sigma) - u_i) \\[1ex] \qquad\quad + \varepsilon(u_i - \mathbb{E}_i(t; \boldsymbol{x}_i, \boldsymbol{v}_i, \Sigma))\big], \end{cases} \tag{4.2}$$

where we recall that η denotes the interaction rate.

Let us now relate the above concepts to the following specific case studies, i.e.:

Stress in evacuation dynamics: Eq. (4.2) models the dynamics of a group of people divided into a fraction $(1-\varepsilon)$ experiencing high levels of stress and the remaining fraction ε that is unaffected by stress. Stress has a negative influence on the selection of safe trajectories. Therefore, one may classify as *rational* the trend of pedestrians in fraction ε, who will keep their previous walking strategy, while it is *irrational* the trend of the pedestrians in fraction $(1-\varepsilon)$, who will choose to "herd".

Awareness of contagion risk: Analogously to what we have seen above, Eq. (4.2) models the dynamics of a crowd divided into two groups. In this case, a fraction $(1-\varepsilon)$ of pedestrians show high levels of awareness, while a small fraction ε keeps the previous level of awareness. The classification is opposite though: *rational* is the trend of the fraction $(1-\varepsilon)$, while the fraction ε is *irrational*.

Leadership attraction: Leaders in a crowd are trained to express a rational behavior. They move along "optimal" trajectories, where optimality correspond to a compromise (to be mathematically formalized) between the search of less congested areas and the need to reach a well-defined target, i.e. an exit or a meeting point, in the shortest possible time. In practice, less congested areas correspond to low risk of incidents in the case of evacuation or contagion in the case of epidemics.

The modeling approach can be developed by adding n_ℓ leaders and a parameter $\sigma = n_\ell/N_0$ that quantifies the presence of leaders. In this case, one only FS is not sufficient as the crowd needs subdivided into two FSs, labeled by the subscripts $r=1$, corresponding to pedestrians, and $r=2$, corresponding to leaders. Interactions can be modeled by assuming that the leaders keep a constant, equally shared activity: $u_i^1 = u_i^1(t=0) =$ constant, while interactions among pedestrians and between pedestrians and leaders are described by Eq. (4.2), where the parameter ε has a different physical meaning in each type of interaction. The differential system can be rapidly obtained from Eq. (4.2).

4.1.2. *Selection of the velocity direction*

We consider the interactions of a i-pedestrian with the ij-pedestrians in Ω_i. These interactions depend on the activities u_i and u_{ij} of the pedestrians. The modeling can be viewed as a technical development of the approach proposed in Ref. 7. Let us start by summarizing the main ingredients.

(1) The i-pedestrian has a velocity direction and a visibility domain Ω_i which is the circular sector defined by the visibility angle and radius depending on the quality and shape of the venue, as the presence of obstacles or walls can reduce the area of Ω_i. Each i-pedestrian perceives in Ω_i the density ρ_i.

(2) All a-particles are subject to different stimuli towards well-defined directions:

$\boldsymbol{\nu}_i^{(E)} = \boldsymbol{\nu}_i^{(E)}[\boldsymbol{x}_i, \boldsymbol{x}_E]$: Walking direction from the location $\boldsymbol{x}_i$ of the i-pedestrian to a meeting point or exit $\boldsymbol{x}_E$. If the i-pedestrian can reach $\boldsymbol{x}_E$, then $\boldsymbol{\nu}_i^{(E)}$ is given by simple geometrical calculations, say the vector $\boldsymbol{x}_E - \boldsymbol{x}_i$ divided by its modulus $\|\boldsymbol{x}_E - \boldsymbol{x}_i\|$. Otherwise, if an obstacle does not allow the straight line, it is necessary accounting for the different tracts.

$\boldsymbol{\nu}_i^{(s)} = \boldsymbol{\nu}_i^{(s)}[\boldsymbol{\xi}_i]$: Attraction to the mainstream $\boldsymbol{\xi}_i$ computed in Ω_i, i.e. the motion of the other a-particles in Ω_i. Hence, $\boldsymbol{\nu}_i^{(s)}$ is given by $\boldsymbol{\xi}_i$ divided by $\|\boldsymbol{\xi}_i\|$.

$\boldsymbol{\nu}_i^{(v)} = \boldsymbol{\nu}_i^{(v)}[\rho_i]$: Attraction to less congested areas in order to avoid overcrowding corresponding to the local distribution of density ρ_i computed in Ω_i. In this case, one has to compute also local gradients:

$$\boldsymbol{\nu}_i^{(v)} = \boldsymbol{\nu}_i^{(v)}[\rho_i] = -\frac{\nabla_{\boldsymbol{x}}\rho_i}{\|\nabla_{\boldsymbol{x}}\rho_i\|}.$$

(3) The choice of the velocity direction, as mentioned in Item 2 of Sec. 3.5.2, corresponds to a weighted selection of the stimuli mentioned in Item 2 depending on the quality of the venue, the emotional state, and the local density. We propose an heuristic modeling approach based on the following hierarchy: Each i-pedestrian selects first a velocity direction $\boldsymbol{\nu}_i^{(sE)}$, which results from weighting $\boldsymbol{\nu}_i^{(s)}$ by u_i and $\boldsymbol{\nu}_i^{(E)}$ by $(1-u_i)$; then, the i-walker selects $\boldsymbol{\omega}_i$ by weighting $\boldsymbol{\nu}_i^{(v)}$ by ρ_i and $\boldsymbol{\nu}_i^{(sE)}$ by $(1-u_i)$. That is,

$$\boldsymbol{\nu}_i^{(sE)} = \boldsymbol{\nu}_i^{(sE)}[\rho_i, \boldsymbol{\xi}_i, \boldsymbol{x}_i, \boldsymbol{x}_E, u_i] = \frac{u_i\,\boldsymbol{\nu}_i^{(s)}[\boldsymbol{\xi}_i] + (1-u_i)\boldsymbol{\nu}_i^{(E)}[\boldsymbol{x}_i, \boldsymbol{x}_E]}{\|u_i\,\boldsymbol{\nu}_i^{(s)}[\boldsymbol{\xi}_i] + (1-u_i)\boldsymbol{\nu}_i^{(E)}[\boldsymbol{x}_i, \boldsymbol{x}_E]\|}, \tag{4.3}$$

and

$$\boldsymbol{\omega}_i = \boldsymbol{\omega}_i[\rho_i, \boldsymbol{\xi}_i, \boldsymbol{x}_i, \boldsymbol{x}_E, u_i] = \frac{\rho_i\,\boldsymbol{\nu}_i^{(v)}[\rho_i] + (1-\rho_i)\boldsymbol{\nu}_i^{(sE)}[\rho, \boldsymbol{\xi}_i, \boldsymbol{x}_i, \boldsymbol{x}_E]}{\|\rho_i\,\boldsymbol{\nu}_i^{(v)}[\rho_i] + (1-\rho_i)\boldsymbol{\nu}_i^{(sE)}[\rho, \boldsymbol{\xi}_i, \boldsymbol{x}_i, \boldsymbol{x}_E]\|}, \tag{4.4}$$

where the functional dependence is denoted in square brackets (as already mentioned) while all quantities refer to each individual $\boldsymbol{x}_i, \boldsymbol{\nu}_i, u_i$.

The following remark defines the phenomenological assumption on the role of the activity variable in the selection of $\boldsymbol{\omega}_i$.

Remark 4.1. A higher level of stress contributes to an increased attraction towards the mainstream perceived in the visibility domain, which is preferred over the trend towards the target. Awareness to contagion has a role opposite to that of the stress, i.e. it decreases the trend towards the mainstream. Increasing local density contributes to an increased trend towards less congested areas with respect to the stress-weighted trends towards the stream and the target.

The i-pedestrian, once moved to the new velocity direction $\boldsymbol{\omega}_i$, will perceive the local density $\rho_{\boldsymbol{\omega}_i}$ in the new visibility domain $\Omega_{\boldsymbol{\omega}_i} = \Omega(\boldsymbol{x}_i, \boldsymbol{\omega}_i)$.

4.1.3. *Modeling the acceleration term*

The local density $\rho_{\boldsymbol{\omega}_i}$ differs from the density previously perceived in the visibility domain $\Omega_i = \Omega(\boldsymbol{x}_i, \boldsymbol{\nu}_i)$. If $\rho_{\boldsymbol{\omega}_i} < \rho_i$, the i-pedestrian will tend to increase the speed, while if $\rho_{\boldsymbol{\omega}_i} > \rho_i$, the i-pedestrian will tend to reduce the speed.

The modeling of *the acceleration term* can be related to the above phenomenological description as follows:

$$\frac{d\boldsymbol{v}_i}{dt} = \mathbf{F}_i[\rho_i, \boldsymbol{\xi}_i, \boldsymbol{x}_i, \boldsymbol{x}_E, u_i] = \boldsymbol{\varphi}_i[\rho_i, \boldsymbol{\xi}_i, \boldsymbol{x}_i, \boldsymbol{x}_E, u_i] \cdot \boldsymbol{\omega}_i[\rho_i, \boldsymbol{\xi}_i, \boldsymbol{x}_i, \boldsymbol{x}_E, u_i], \quad (4.5)$$

where

$$\boldsymbol{\varphi}_i = \boldsymbol{\varphi}_i[\rho_i, \boldsymbol{\xi}_i, \boldsymbol{x}_i, \boldsymbol{x}_E, u_i] = \alpha u_i(\rho_i - \rho_{\boldsymbol{\omega}_i}), \quad (4.6)$$

where the parameter α has been introduced to account for the fact that high values of the quality of the venue promote increases of the speed, while $\alpha = 0$ totally prevents this specific dynamics.

Remark 4.2. This model corresponds to the following: If $u_i \to 1$, the pedestrian has a trend to accelerate and decelerate with probability 1, while if $u_i \to 0$, the pedestrian accelerates and decelerates with probability 0, i.e. an active pedestrian is supposed to act fast or slow according to u_i in both actions. More in general, asymmetries on the role of the activity can be introduced consistently with the specific type of behavioral variable included in the model.

Remark 4.3. The acceleration $\mathbf{F}_i$ is a nonlocal quantity that depends on the averaged state of all a-particles in the domains $\Omega_i(\boldsymbol{x}_i, \boldsymbol{\nu}_i)$ and $\Omega(\boldsymbol{x}_i, \boldsymbol{\omega}_i)$. Therefore, this modeling approach averages micro-scale quantities in the domains $\Omega_i(\boldsymbol{x}_i, \boldsymbol{\nu}_i)$ and $\Omega(\boldsymbol{x}_i, \boldsymbol{\omega}_i)$. Then, introducing (4.5) in the structure defined in Eq. (3.1) yields a new framework suitable to derive specific models.

4.2. *Modeling at the macroscopic scale*

The derivation of mathematical models at the macro-scale follows the same rationale presented in Sec. 4.1. Therefore, we do not repeat calculations, but simply highlight the key points. Let us first consider the derivation of hydrodynamical models which corresponds to the structure defined by Eq. (3.11).

Models can be obtained by specifying the source terms $\boldsymbol{A}$ and S, which correspond to the acceleration term and the dynamics of the activity variable, respectively. $\boldsymbol{A}$ can be modeled according to the same rationale proposed at the microscopic scale, which now corresponds to:

The a-particles in the elementary volume $d\boldsymbol{x}$ first select a direction $\boldsymbol{\omega}$ and subsequently accelerate or decelerate according to the local density conditions:

$$\boldsymbol{A} = \boldsymbol{A}[\rho, \boldsymbol{\xi}, \boldsymbol{x}_E, u; \alpha, \Sigma] = \boldsymbol{\varphi}[\rho, \boldsymbol{\xi}, \boldsymbol{x}_E, u; \alpha, \Sigma] \cdot \boldsymbol{\omega}[\rho, \boldsymbol{\xi}, \boldsymbol{x}_E, u; \alpha, \Sigma], \quad (4.7)$$

where the calculation of the local densities is restricted to $\Omega^M_{\boldsymbol{\xi}} = \Omega^M_{\boldsymbol{\xi}}(\boldsymbol{x}, \boldsymbol{\xi})$ and $\Omega^M_{\boldsymbol{\omega}} = \Omega^M_{\boldsymbol{\omega}}(\boldsymbol{x}, \boldsymbol{\omega})$, which define the visibility domains in the directions $\boldsymbol{\xi}$ and $\boldsymbol{\omega}$, respectively.

Let $\rho_{\boldsymbol{\omega}}$ be the density corresponding to the domain $\Omega^M_{\boldsymbol{\omega}}$ and $\rho_{\boldsymbol{\xi}}$ the density in $\Omega^M_{\boldsymbol{\xi}}$. Replacing ρ_i and $\rho_{\boldsymbol{\omega}_i}$ by $\rho_{\boldsymbol{\xi}}$ and $\rho_{\boldsymbol{\omega}}$, $\boldsymbol{\omega}$ is computed as in Eq. (4.4) and $\boldsymbol{\varphi}$ can be computed as in (4.6).

As for the modeling of the activity variable, it becomes

$$\frac{dz}{dt} = \eta(\rho_{\boldsymbol{\xi}})\big[(1-\varepsilon)(\mathbb{E}_{\boldsymbol{\xi}}(t;\boldsymbol{x},\boldsymbol{v},\Sigma) - u) + \varepsilon(u - \mathbb{E}_{\boldsymbol{\xi}}(t;\boldsymbol{x},\boldsymbol{v},\Sigma))\big]. \tag{4.8}$$

4.3. *Derivation of kinetic models*

The derivation of kinetic models moves from the structure defined by (3.9), which is related to the dynamics of the dependent variables defined in Eq. (3.4), where the micro-scale variable was represented in polar coordinates.

The key feature of this approach refers to the modeling of the interaction rate $\eta[f]$ and the transition probability density $\mathcal{A}[f]$. The concept of interaction rate is different from that of the classical kinetic theory, where only binary interactions are considered and η is delivered by the relative velocity of the interacting pairs multiplied by the cross-sectional area of the interacting classical particles. The mathematical theory of a-particles includes multiple interaction and a standard assumption consists in supposing that η depends on the local density which is computed in Ω. In the simplest case, $\eta = \eta_0$ is a constant, while in a more general case it grows with the density in the visibility domain.

The modeling of $\mathcal{A}$ in (3.9) can be developed following the same rationale proposed at the microscopic scale, namely *the a-particle first modifies the velocity direction and subsequently the speed*. Let us recall that the modeling approach considers that interactions involve three types of a-particles, i.e. *Test a-particles*, with distribution function $f(t,\boldsymbol{x},\theta,v,\boldsymbol{u})$; *candidate a-particles*, with distribution function $f(t,\boldsymbol{x}_*,\boldsymbol{v}_*,\boldsymbol{u}_*)$, which can acquire the state of test particles after interactions with field particles with distribution function $f(t,\boldsymbol{x}^*,\boldsymbol{v}^*,\boldsymbol{u}^*)$, while field a-particles lose their state as a consequence of interaction.

The modeling approach can be developed first by assuming a hierarchy in the sequence of interactions, i.e. activity dynamics, selection of the velocity directions, and adaptation of the speed. Each interaction refers to the three types of a-particles mentioned above. The technical difference, with respect to the approach we have presented in Sec. 4.1, corresponds to three types of visibility domains, say Ω, Ω_*, and Ω^*, as well as to the respective densities in these domains. The interested reader can find the details of the technical calculations in the short book,[6] specifically focused on the kinetic theory approach, and in Ref. 7 within a multiscale framework.

4.4. *Further reasonings on modeling interactions*

As shown in the preceding sections, the derivation of a mathematical model can be developed by inserting heuristic models of interactions into the mathematical structures derived in Sec. 3. The rationale to develop these models is the same at each scale. This strategy has the advantage that models can make use of parameters with analogous meaning so that empirical data can be used for their assessment at all scales. Figure 5 provides a representation of these concepts, and of their sequence.

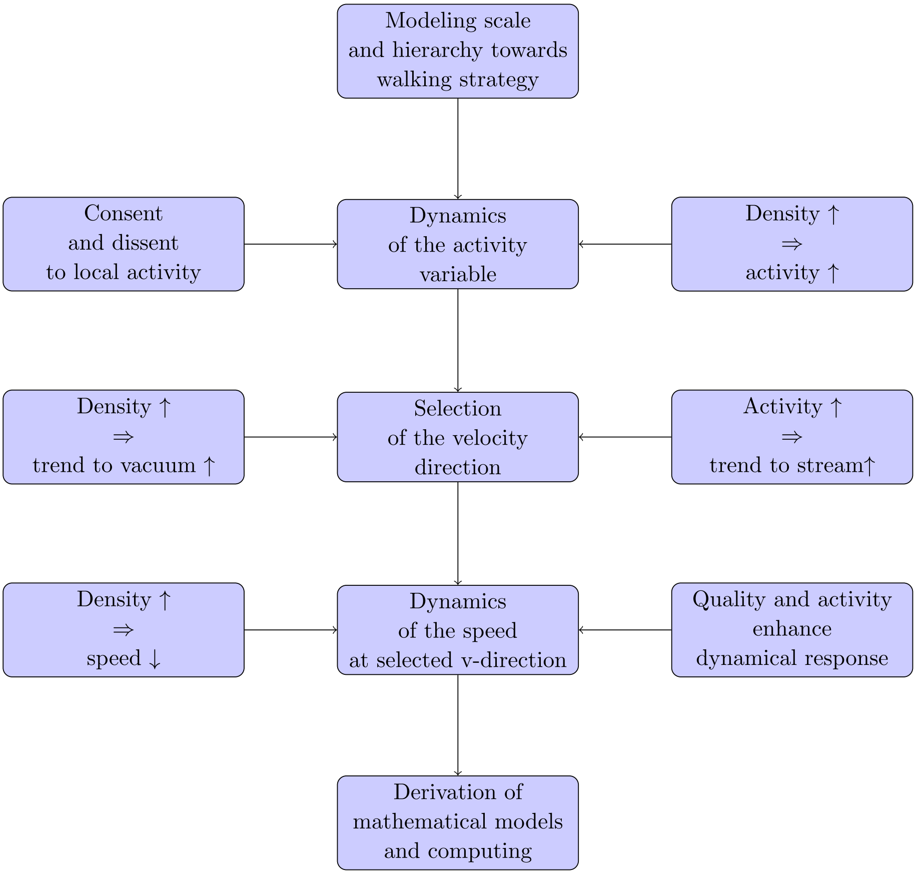

Fig. 5. (Color online) From modeling interactions to derivation of models.

Refinements of the models can be obtained by a progressive improvement of the models deemed to describe interactions. Therefore, we present, in the following, some perspective ideas, selected among various ones, that may contribute to an improved derivation of models. Actually, the preliminary reasonings in this section are a prelude to the overview on the research perspective presented in the last section.

Let us first consider the concept of *topological interactions* that differ from interactions restricted to entities in the visibility domain. We recall that the visibility domain corresponds to an arc of a circle with a radius depending on qualitative properties of the environment. This is a simplification of physical reality since in the study of animal swarms it has been conjectured that the radius of the circle depends on a critical density of individuals. Such a density corresponds to the number of individuals needed to elaborate a consistent movement strategy.[8] This conjecture has been formalized in a mathematical framework in a paper focused on swarm dynamics,[18] where the interaction domain Ω_s can differ from Ω.

Arguably, this conjecture can be transferred to crowds, where the modeling of Ω and Ω_s should also consider the influence of rational and irrational behaviors. Actually, if $\Omega_s \subset \Omega$, then each pedestrian receives sufficient information for a rational strategy. On the other hand, if $\Omega \subset \Omega_s$ then the information is not sufficient and may generate non-symmetrical interactions.

In general, topological interactions induce *asymmetric interactions*. Empirical data have been collected to investigate how the environment can modify visibility conditions[94] and how heterogeneity of the environment can contribute to the understanding of non-symmetric reactions to interactions.

Research activity to produce empirical data to inform the modeling of pedestrian behaviors and interactions is witnessed in a variety of interesting papers, for instance,[40–42] with the aim of contributing to model improvements.[98] On the other hand, this research field definitely needs further developments, for instance to understand how the psychology of the crowds[103] and collective learning[32] may contribute to the dynamics of the activity variable. Further improvements of the modeling approach should consider the case of vector activity variables and, consequently, how the components of the vector act on the hierarchy of the decisional process.

All reasonings briefly presented in this section apply to the mathematical structures proposed in our paper at all scales. Arguably, further development of said structures or alternative ones, such as the Fokker–Planck approach,[25] gives rise to research perspectives as presented in the following section.

5. A Forward Look at Research Perspectives

Various research perspectives have already been indicated in the preceding sections. Rather than adding new ones, we will focus on a selection of topics pertaining to the conceptual strategy that has guided our paper and has contributed to the framework supporting the derivation of models. Specifically, we refer to the multiscale vision that, as we have seen, leads to models derived by the same principles at each scale. The following topics are selected according to the authors' knowledge derived form their research activity, however, consistently with the aforementioned vision.

In more details, in the next paragraphs we focus on: A further discussion on the merits and pitfalls of the selection of a specific scale with respect to the others; discrete velocity models; possible contributions of the studies of crowd dynamics to modeling swarm dynamics.

• **Further reasoning on multiscale methods:** Human crowds should be viewed as discrete systems with finite degrees of freedom corresponding to the finite number of pedestrians. Therefore, the microscopic scale appears to offer the most appropriate choice. This selection leads to systems of ordinary differential equations. On the other hand, describing crowds at this scale requires to keep track of each individual i.e. of their individual behaviors accounting for multiple, nonlocal interactions.

The approach at the macroscopic scale is prone to criticisms, as crowds do not fit the paradigm of continuity of the matter. Focusing on the mesoscopic scale, the assumption of a continuous distribution function over the micro-states, borrowed from the classical kinetic theory of gases, is questionable as the number of pedestrians in a crowd is far less than the number of molecules in a gas.

In addition to the aforementioned conceptual difficulties, the representation scale has to fit the need to model pedestrians behavioral features, which modify the overall dynamics of this specific living system. We have devoted this paper to develop a unified modeling approach at all scale including the role of social dynamics. A key feature of this approach consists in including a behavioral variable (scalar or vector) modeling the social and emotional state of individuals in their micro-state. Thus, our approach belongs to the general framework of the so-called *behavioral dynamics* (see Ref. 80), where dynamics and behaviors are considered contextually in the modeling approach.

A somehow related challenging research perspective consists in deriving macro-scale models from the underlying description at the micro-scale. Some results have been proposed for vehicular traffic and crowd dynamics,[10] however limited to models where the behavioral variable is reduced to a constant parameter. Therefore, further studies are necessary for a more general class of models such as those proposed in this paper. The general methodological approach, developed in Refs. 29, 30 and inspired to the Hilbert problem,[70] can be further developed in the aforementioned research perspective.

• **Semi-discrete models:** *Discrete velocity models* refer to a class of models where the velocity can take only a finite number of directions and of speeds. Therefore, the velocity $\boldsymbol{v}$ is defined by a set of velocity directions $\{\theta_1, \dots, \theta_h, \dots, \theta_m\}$ and a set of speeds $\{v_1, \dots, v_k, \dots, v_n\}$. The velocity has a different meaning at each scale, i.e. it refers to the individual pedestrian at the micro-scale, to the test/candidate/field statistical particles at the meso-scale, and to the mean velocity at the macro-scale.

Discrete velocity models have been developed within the classical kinetic theory of gases[61] with the aim of simplifying the analytic and mechanical complexity of the classical Boltzmann equation. In crowd dynamics[12] and vehicular traffic,[43] discrete models can be developed to account for the fact that the number of individuals (or vehicles) is not large enough to justify the continuity assumption for both hydrodynamical and kinetic models.

At the meso-scale, discrete models have been introduced in Ref. 12 through a simplified model that heuristically relates the speed to the local density. This type of models has been further developed in Ref. 78, which focuses on the contagion risk during pandemics.[79] A sharp idea was proposed in Ref. 43 by referring the grid of the discrete values of the speed to the local density, namely the grid shrinks when the density increases. Surprisingly, this idea was not further developed for vehicular traffic modeling and has not been developed yet in the case of crowd dynamics. In the latter case, one would have to tackle the difficulty related to the two-dimensional representation of the velocity. Similar reasonings apply to models coupling vehicular traffic and crowds.[24,58]

Discrete velocity models have been developed within the framework of the kinetic theory approach. This approach might also be applied to the modeling at the micro-scale to cope with the need of keeping track of the number of pedestrians in a large crowd which otherwise would lead to simulations computational demanding (if not unfeasible). Therefore, an interesting challenge would be developing discrete velocity models also at the micro-scale. Then, modeling velocity dynamics should consider jumps over the nodes of the grid of velocity directions and speed.

• **From crowds to swarms:** Another interesting research perspective consists in extending the multiscale vision presented in this paper to the modeling of swarms related to the celebrated work by Cucker and Smale,[46] which has rapidly attracted the interest of applied mathematicians and generated a huge amount of scientific articles devoted to analytic, computational, and modeling topics. An account of this literature is given in the survey paper.[2]

Generally, swarm models refer to a pseudo-Newtonian framework in unbounded domains. The modeling of interactions (which include also the description delivered by interaction potentials) provides the models for the acceleration terms, so that the system giving the dynamics of all particles is well defined. The general approach has been further developed to include internal parameters with thermodynamical meaning,[63] while models accounting for the dynamics of internal variables have been proposed in Ref. 19. A kinetic theory formalization of the dynamics of swarms has been developed in Ref. 18.

Mathematical models have been developed for swarm dynamics in unbounded domains, but only some simple case studies have included dynamical behavioral variables in the micro-state. Further developments in the modeling of animal swarms might arise from the conceptual contribution of the mathematical theory of behavioral crowds on the treatment of boundary conditions and on the interactions between behavioral and mechanical dynamics.

• **Closure:** The plan of the paper presented in Sec. 1 was closed by the statement that the main purpose of our paper was a contribution to a mathematical theory of behavioral crowds. Indeed the contents have been developed towards this objective. Hopefully, it will contribute to the development of new models as well as to perspective ideas in parallel field, for instance animal swarms and, more in general, collective motion of living interacting entities.

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