mathematics

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Superconvergent Nyström and Degenerate Kernel Methods for Integro-Differential Equations

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Abstract: The aim of this paper is to carry out an improved analysis of the convergence of the Nyström and degenerate kernel methods and their superconvergent versions for the numerical solution of a class of linear Fredholm integro-differential equations of the second kind. By using an interpolatory projection at Gauss points onto the space of (discontinuous) piecewise polynomial functions of degree $\leqslant r-1$, we obtain convergence order $2r$ for degenerate kernel and Nyström methods, while, for the superconvergent and the iterated versions of theses methods, the obtained convergence orders are $3r+1$ and $4r$, respectively. Moreover, we show that the optimal convergence order $4r$ is restored at the partition knots for the approximate solutions. The obtained theoretical results are illustrated by some numerical examples.

Keywords: degenerate kernel method; Nyström method; Fredholm integro-differential equation

Citation: Saou, A.; Sbibih, D.; Tahrichi, M.; Barrera, D. Superconvergent Nyström and Degenerate Kernel Methods for Integro-Differential Equations. *Mathematics* **2022**, *10*, 893. https://doi.org/10.3390/math10060893

Academic Editor: Mirosław Lachowicz

Received: 30 January 2022
Accepted: 21 February 2022
Published: 11 March 2022

1. Introduction

Integro-differential equations emerged at the beginning of the twentieth century thanks to the work of Vito Volterra. The applications of these equations have proved worthy and effective in the fields of engineering, mechanics, physics, chemistry, astronomy, biology, economics, potential theory, electrostatics, etc. (see [1–4] and references therein).

Many numerical methods have been developed for solving integro-differential equations. Each of these methods has its inherent advantages and disadvantages, and the search for easier and more accurate methods is a continuous and ongoing process. Among the existing methods in the literature, we cite the Adomian decomposition [5], homotopy analysis [2], Chebyshev and Taylor collocation [6], Taylor series expansion [7,8], integral mean value [9], and decomposition method [10]. For other methods to solve integro-differential equations, see [11–14].

Recently, many authors have used spline functions for the numerical solution of integro-differential equations; in particular, a semi-orthogonal spline wavelets approximation method for Fredholm integro-differential equations was proposed in [15]. In [16], the authors used a fast multiscale Galerkin method for solving second order linear Fredholm integro-differential equation with Dirichlet boundary conditions. In [17], the authors applied B-spline collocation method for solving numerically linear and nonlinear Fredholm and Volterra integro-differential equations, and in [18] an exponential spline method for approximating the solution of Fredholm integro-differential equation was studied. More recently, in [19] Kulkarni introduced an efficient method called modified projection method or multi-projection method to solve Fredholm integral equations of the second kind. Inspired in Kulkarni's method, authors in [20] have introduced superconvergent Nyström and degenerate kernel methods to solve the same type of equations.

This work is concerned with numerical methods to solve a class of linear Fredholm integro-differential equations of the form

$$\begin{cases} y'(x) + a(x)y(x) = \int_0^1 k(x,t)y(t)\,dt + f(x), \quad x \in [0\,,1], \\ y(0) = y_0, \end{cases} \tag{1}$$

where $y_0 \in \mathbb{R}$, a, f, and k are continuous functions, and y is the function to be determined.

The paper is organised as follows. In Section 2, the proposed methods to solve (1) are defined along with relevant notations. In Section 3, error estimates are given and precise convergence orders are obtained. Implementation details on the linear systems are discussed in Section 4. Finally, in Section 5, we provide some numerical results that illustrate the convergence orders of the proposed methods and we give a comparison with other known approaches in the literature.

2. Methods and Notations

Consider the following partition of the interval $[0,1]$

$$0 = x_0 < x_1 < \cdots < x_n = 1. \tag{2}$$

Let $I_i = [x_{i-1}, x_i], h_i = x_i - x_{i-1}, i = 1,2,\ldots,n$, and let $h = \max_{1\le i\le n} h_i$ be the maximum step size of the partition. We assume that $h \to 0$ as $n \to \infty$. For $r \geq 1$, we denote by $\mathcal{P}_r$ the space of all polynomials of degree $\leq r-1$. Let

$$\mathcal{S}_{r,n} := \left\{ u : [0,1] \mapsto \mathbb{R} : u|_{I_i} \in \mathcal{P}_r, 1 \leq i \leq n \right\},$$

be the space of piecewise polynomials of degree $\leq r-1$, with breakpoints at $x_1, x_2, \ldots, x_{n-1}$. No continuity conditions are imposed at the breakpoints. Let $B_r := \{\tau_1, \ldots, \tau_r\}$ be the set of r Gauss points, i.e., the zeros of the Legendre polynomials $p_r(t) = (d^r/dt^r)(t^2-1)^r$ in $[-1,1]$. Define $f_i : [-1,1] \to [x_{i-1}, x_i]$ as follows:

$$f_i(t) = \frac{1-t}{2}x_{i-1} + \frac{1+t}{2}x_i, \quad t \in [-1,1].$$

Then

$$A = \bigcup_{i=1}^{n} f_i(B_r) = \left\{ \tau_{ij} = f_i(\tau_j) : 1 \leq i \leq n, 1 \leq j \leq r \right\} := \{t_i\,,\, i = 1\,,\ldots,\, nr\}\,,$$

is the set of $N_h := nr$ Gauss points in $[0,1]$. Let

$$\ell_i(x) := \prod_{\substack{k=1 \\ k\neq i}}^{r} \frac{x - \tau_k}{\tau_i - \tau_k}, \quad i = 1,2,\ldots,r, \quad x \in [-1,1]\,,$$

be the Lagrange polynomials of degree $r-1$ on $[-1,1]$, which satisfy $\ell_i(\tau_j) = \delta_{ij}$.

Define

$$\varphi_{kp}(x) := \begin{cases} \ell_k\left(f_p^{-1}(x)\right), & x \in [x_{p-1}, x_p], \\ 0, & \text{otherwise}. \end{cases}$$

It is easy to verify that $\varphi_{kp} \in \mathcal{S}_{r,n}$ and $\varphi_{kp}(\tau_{ij}) = \delta_{jk}\delta_{ip}$, $i,p = 1,2,\ldots,n, j,k = 1,2,\ldots,r$.

Let

$$\phi_{(p-1)r+k} := \varphi_{kp}\,, \quad k = 1,\ldots,r \quad \text{and} \quad p = 1,\ldots,n.$$

For a fixed p, the family of functions $\left\{ \varphi_{kp} : k = 1,2,\ldots,r \right\}$ form a basis (Lagrange basis) for the space of polynomials functions of degree $r-1$ in $[x_{p-1}, x_p]$. As, in the space

$\mathcal{S}_{r,n}$, no continuity conditions are imposed at the breakpoints, we deduce that the set $\left\{\varphi_{kp} : k = 1, \ldots, r\,,\; p = 1, \ldots, n\right\} = \{\phi_j : j = 1, \ldots, nr\}$ form a basis of this space.

Let $\pi_n : \mathcal{C}[0,1] \to \mathcal{S}_{r,n}$ be the interpolatory operator defined by

$$\pi_n u(x) := \sum_{i=1}^{N_h} u(t_i)\phi_i(x)\,. \tag{3}$$

It follows that $\pi_n u \in \mathcal{S}_{r,n}\,,\quad \pi_n u(t_i) = u(t_i),\quad i = 1, 2, \ldots, N_h\,.$ Then $\pi_n u \to u$ as $n \to \infty$ for each $u \in \mathcal{C}[0,1]$. By using a result in [21], π_n can be extended to a projection from $\mathcal{L}^\infty[0,1]$ to $\mathcal{S}_{r,n}$.

Equation (1) can be written as

$$\begin{cases} y'(x) + a(x)y(x) = \mathcal{K}y(x) + f(x), & x \in [0\,,1], \\ y(0) = y_0, \end{cases} \tag{4}$$

where $\mathcal{K}$ is the integral operator defined by

$$\mathcal{K}(u)(s) := \int_0^1 k(s,t)u(t)dt\,. \tag{5}$$

Under the regularity assumptions on $a\,, f$, and k, it is well known that (see e.g., [22]) the initial value problem (4) has a unique solution y that satisfies the integral equation

$$y(x) = y_0 e^{A(0)-A(x)} + \int_0^x \Big(\mathcal{K}y(t) + f(t)\Big)e^{A(t)-A(x)}dt\,, \tag{6}$$

where A is a primitive function of a.

We consider the following Volterra operator

$$\mathcal{V}u(x) := \int_0^x u(t)e^{A(t)-A(x)}dt\,, \tag{7}$$

and we define

$$g(x) := y_0 e^{A(0)-A(x)} + \mathcal{V}f(x).$$

Then, Equation (6) becomes

$$y - \mathcal{V}\mathcal{K}y = g\,. \tag{8}$$

In this paper, we propose to solve the above equation by using the four following methods based on the projection π_n given in (3).

1. Degenerate kernel method, where the operator $\mathcal{K}$ is approximated by the following degenerate kernel operator

$$\mathcal{K}_{n,1}(u)(s) := \int_0^1 k_n(s,t)u(t)dt,$$

with

$$k_n(s,t) := \pi_n k(s,\cdot) = \sum_{i=1}^{N_h} k(s,t_i)\phi_i(t).$$

The approximate equation of (8) is then given by

$$y_{n,1} - \mathcal{V}\mathcal{K}_{n,1}y_{n,1} = g\,. \tag{9}$$

2. Nyström method, where the operator $\mathcal{K}$ is approximated by the Nyström operator based on π_n and defined by

$$\mathcal{K}_{n,2}(u)(s) := \sum_{i=1}^{N_h} w_i k(s, t_i) u(t_i),$$

with $w_i := \int_0^1 \phi_i(t)dt, i = 1, 2, \ldots, N_h$. The corresponding approximate equation of (8) is then given by

$$y_{n,2} - \mathcal{V}\mathcal{K}_{n,2} y_{n,2} = g. \quad (10)$$

3. Superconvergent degenerate kernel method, where the operator $\mathcal{K}$ is approximated by the following finite rank operator

$$\mathcal{K}_{n,1}^{\mathcal{S}} := \pi_n \mathcal{K} + \mathcal{K}_{n,1} - \pi_n \mathcal{K}_{n,1}.$$

The corresponding approximation of (8) becomes

$$y_{n,1}^{\mathcal{S}} - \mathcal{V}\mathcal{K}_{n,1}^{\mathcal{S}} y_{n,1}^{\mathcal{S}} = g. \quad (11)$$

Furthermore, we define the iterated solution by

$$\tilde{y}_{n,1}^{\mathcal{S}} := \mathcal{V}\mathcal{K} y_{n,1}^{\mathcal{S}} + g. \quad (12)$$

4. Superconvergent Nyström method, where the operator $\mathcal{K}$ is approximated by the following finite rank operator

$$\mathcal{K}_{n,2}^{\mathcal{S}} := \pi_n \mathcal{K} + \mathcal{K}_{n,2} - \pi_n \mathcal{K}_{n,2}.$$

The corresponding approximation of (8) becomes

$$y_{n,2}^{\mathcal{S}} - \mathcal{V}\mathcal{K}_{n,2}^{\mathcal{S}} y_{n,2}^{\mathcal{S}} = g. \quad (13)$$

Additionally, we define the iterated solution by

$$\tilde{y}_{n,2}^{\mathcal{S}} := \mathcal{V}\mathcal{K} y_{n,2}^{\mathcal{S}} + g. \quad (14)$$

We show later that, for $i = 1, 2$, the iterated solutions $\tilde{y}_{n,i}^{\mathcal{S}}$ converge to y faster than $y_{n,i}^{\mathcal{S}}$. The reduction of (9)–(11) and (13) to systems of linear equations is presented in Section 4.

3. Convergence Analysis

In addition to the assumptions about a, f, and k required previously to insure the existence and the uniqueness of the exact solution of (1), we assume in the subsequent considerations that the operator $\mathcal{I} - \mathcal{V}\mathcal{K}$ is invertible with a bounded inverse. Therefore, it is easy to verify that, for the above four methods, the operators $\mathcal{I} - \mathcal{V}\mathcal{K}_{n,i}$ and $\mathcal{I} - \mathcal{V}\mathcal{K}_{n,i}^{\mathcal{S}}$ are invertible for enough large n and we have

$$\left\| (\mathcal{I} - \mathcal{V}\mathcal{K}_{n,i})^{-1} \right\|_\infty \leqslant L_i < \infty \text{ and } \left\| \left(\mathcal{I} - \mathcal{V}\mathcal{K}_{n,i}^{\mathcal{S}}\right)^{-1} \right\|_\infty \leqslant L_i' < \infty,$$

where L_i and L_i' are constants independent of n [20,21].

Hence for large enough n, the approximate equations have unique solutions. Moreover, in the following lemma, we give some error estimates essential in the proof of the convergence orders.

Lemma 1. *For a sufficiently large integer n and for $i = 1, 2$, the following estimates hold:*

$$\|y - y_{n,i}\|_\infty \leqslant L_i \|(\mathcal{V}\mathcal{K} - \mathcal{V}\mathcal{K}_{n,i})y\|_\infty, \tag{15}$$

$$\left\|y - y^{\mathcal{S}}_{n,i}\right\|_\infty \leqslant L'_i \|\mathcal{V}(\mathcal{I} - \pi_n)(\mathcal{K} - \mathcal{K}_{n,i})y\|_\infty, \tag{16}$$

$$\begin{aligned}\left\|y - \tilde{y}^{\mathcal{S}}_{n,i}\right\|_\infty \leqslant C_i (&\|\mathcal{K}\mathcal{V}(\mathcal{I} - \pi_n)(\mathcal{K} - \mathcal{K}_{n,i})y\|_\infty \\ &+ \|\mathcal{K}\mathcal{V}(\mathcal{I} - \pi_n)(\mathcal{K} - \mathcal{K}_{n,i})\|_\infty \left\|y - y^{\mathcal{S}}_{n,i}\right\|_\infty \Big),\end{aligned} \tag{17}$$

where L_i, L'_i, and C_i are constants independent of n.

Proof. The proof can be investigated in a similar way with the proof of Theorem 4 of [20]. □

In the rest of this section the following estimates are crucial. For $y \in C^r[0,1]$, (see [23], Corollary 7.6, p. 328), it holds

$$\|(I - \pi_n)y\|_\infty \leqslant C_1 h^r \left\|y^{(r)}\right\|_\infty. \tag{18}$$

For $y \in C^{2r}[0,1]$ and $g \in C^r[0,1]$, we find

$$\left|\int_{x_{i-1}}^{x_i} g(t)(I - \pi_n)y(t)dt\right| \leqslant C_2 h^{2r+1} \left\|g^{(r)}\right\|_\infty \left\|y^{(2r)}\right\|_\infty, \; i = 1, \dots, n, \tag{19}$$

where C_1 and C_2 are constants independent of n.

The following results provide the convergence orders associated with each approximate solution defined above.

Theorem 1. *Let $y_{n,1}$ and $y_{n,2}$ be the approximate solutions defined, respectively, by (9) and (10). In the case of the degenerate kernel method, we assume that $k(\cdot,\cdot) \in \mathcal{C}^{r-1,2r}([0,1] \times [0,1])$, $a \in \mathcal{C}^{r-1}[0,1]$, and $f \in \mathcal{C}^{r-1}[0,1]$, while in the case of the Nyström method, we assume that $k(\cdot,\cdot) \in \mathcal{C}^{2r,2r}([0,1] \times [0,1])$, $a \in \mathcal{C}^{2r}[0,1]$, and $f \in \mathcal{C}^{2r}[0,1]$. Then*

$$\|y - y_{n,i}\|_\infty = \mathcal{O}\left(h^{2r}\right), \quad i = 1, 2. \tag{20}$$

Proof. Let $i = 1$. From (15), we find

$$\|y - y_{n,1}\|_\infty \leqslant L_1 \|\mathcal{V}(\mathcal{K} - \mathcal{K}_{n,1})y\|_\infty \leqslant L_1 \|\mathcal{V}\| \|(\mathcal{K} - \mathcal{K}_{n,1})y\|_\infty. \tag{21}$$

Moreover, by using (19) we have

$$|(\mathcal{K} - \mathcal{K}_{n,1})y(x)| = \left|\int_0^1 y(t)(\mathcal{I} - \pi_n)k(x,.)(t)dt\right| \leq C_2 h^{2r} \|y^{(r)}\|_\infty \left\|\frac{\partial^{2r}}{\partial t^{2r}} k(x,t)\right\|_\infty.$$

By taking a supremum over x in the last inequality and by using (21), estimate (20) follows.

For $i = 2$, the proof is similar. □

Theorem 2. *Let $y^{S}_{n,1}$ and $y^{S}_{n,2}$ be the approximate solutions defined, respectively, by (11) and (13). Let $\tilde{y}^{S}_{n,1}$ and $\tilde{y}^{S}_{n,2}$ be the iterated versions defined respectively by (12) and (14). For both methods, we assume that $k(\cdot,\cdot) \in \mathcal{C}^{2r,2r}([0,1] \times [0,1])$, $a \in \mathcal{C}^{r-1}[0,1]$, and $f \in \mathcal{C}^{r-1}[0,1]$. Then for $i = 1, 2$, we have*

$$\left\|y - y_{n,i}^{\mathcal{S}}\right\|_{\infty} = \mathcal{O}\left(h^{3r+1}\right), \tag{22}$$

$$\left\|y - \tilde{y}_{n,i}^{\mathcal{S}}\right\|_{\infty} = \mathcal{O}\left(h^{4r}\right). \tag{23}$$

Proof. We only consider the case of superconvergent degenerate kernel method $(i = 1)$. For the case of superconvergent Nyström method $(i = 2)$, the proof can be investigated in a similar way. Let $x \in [0, 1]$ and let m $(0 \leq m \leq n-1)$ be an integer such that $x \in [x_m, x_{m+1}]$. We have

$$\begin{aligned}
\mathcal{V}\mathcal{K}y(x) - \mathcal{V}\mathcal{K}_{n,1}^{\mathcal{S}}y_n(x) &= \mathcal{V}(\mathcal{I} - \pi_n)(\mathcal{K} - \mathcal{K}_{n,1})y(x) \\
&= \int_0^x e^{A(t)-A(x)}(\mathcal{I} - \pi_n)(\mathcal{K} - \mathcal{K}_{n,1})y(t)dt \\
&= \sum_{j=1}^{m} \int_{x_{j-1}}^{x_j} e^{A(t)-A(x)}(\mathcal{I} - \pi_n)G(t)dt \\
&+ \int_{x_m}^{x} e^{A(t)-A(x)}(\mathcal{I} - \pi_n)G(t)dt,
\end{aligned} \tag{24}$$

where $G(t) := (\mathcal{K} - \mathcal{K}_{n,1})y(t)$.

On one hand, from (19), it follows that

$$\left|\sum_{j=1}^{m} \int_{x_{j-1}}^{x_j} e^{A(t)-A(x)}(I - \pi_n)G(t)dt\right| \leqslant C_2 C_{r,x} h^{2r} \left\|G^{(2r)}\right\|_{\infty}, \tag{25}$$

and using (18) yields

$$\begin{aligned}
\left|\int_{x_m}^{x} e^{A(t)-A(x)}(\mathcal{I} - \pi_n)G(t)dt\right| &\leqslant C_{0,x} h \|(I - \pi_n)G\|_{\infty} \\
&\leqslant C_1 C_{0,x} h^{r+1} \left\|G^{(r)}\right\|_{\infty},
\end{aligned} \tag{26}$$

where $C_{j,x} := \sup_{t \in [0,1]} \left|\frac{\partial^j}{\partial t^j} e^{A(t)-A(x)}\right|$.

On the other hand, for $j = 0, \ldots, 2r$ and again using (19), we find

$$\begin{aligned}
\left|G^{(j)}(t)\right| &= \left|\int_0^1 y(s)(\mathcal{I} - \pi_n)\frac{\partial^j}{\partial t^j}k(t, \cdot)(s)ds\right| \\
&\leqslant C_2 C_{j,t} h^{2r} \left\|y^{(r)}\right\|_{\infty}.
\end{aligned} \tag{27}$$

where $C_{j,t} := \sup_{s \in [0,1]} \left|\frac{\partial^{2r}}{\partial s^{2r}} \frac{\partial^j}{\partial t^j} k(t,s)\right|$.

Taking supremum over $x, t \in [0, 1]$ in (25)–(27) and using (24), we deduce the error estimate (22).

Now, we prove (23). From (19), we can show that

$$\begin{aligned}
|\mathcal{K}\mathcal{V}(\mathcal{I} - \pi_n)(\mathcal{K} - \mathcal{K}_{n,1})y(x)| &= \left|\int_0^1 k(x,s)\mathcal{V}(\mathcal{I} - \pi_n)(\mathcal{K} - \mathcal{K}_{n,1})y(s)ds\right| \\
&= \left|\int_0^1 k(x,s)\left(\int_0^s (\mathcal{I} - \pi_n)(\mathcal{K} - \mathcal{K}_{n,1})y(t)e^{A(t)-A(s)}dt\right)ds\right| \\
&= \left|\int_0^1 v_x(t)(\mathcal{I} - \pi_n)G(t)dt\right| \\
&\leqslant Ch^{2r}\left\|v_x^{(r)}\right\|_{\infty}\left\|G^{(2r)}\right\|_{\infty},
\end{aligned}$$

where $v_x(t) := \int_t^1 k(x,s)e^{A(t)-A(s)}ds$. Using (27) for $j = 2r$, we deduce that

$$\|\mathcal{K}\mathcal{V}(\mathcal{I}-\pi_n)(\mathcal{K}-\mathcal{K}_{n,i})y\|_\infty = \mathcal{O}\left(h^{4r}\right). \tag{28}$$

Moreover, it is easy to prove that

$$\|\mathcal{V}\mathcal{K}(\mathcal{I}-\pi_n)(\mathcal{K}-\mathcal{K}_{n,1})\|_\infty = \mathcal{O}(h^r),$$

Then, from (22), it follows that

$$\|\mathcal{V}\mathcal{K}(\mathcal{I}-\pi_n)(\mathcal{K}-\mathcal{K}_{n,1})\|_\infty \left\|y - y_{n,1}^{\mathcal{S}}\right\|_\infty = \mathcal{O}\left(h^{4r+1}\right). \tag{29}$$

Now, by combining (17), (28), and (29) we find (23). □

In the following theorem, we give superconvergence results for the approximate solutions $y_{n,1}^{\mathcal{S}}$ and $y_{n,2}^{\mathcal{S}}$ at the partition knots.

Theorem 3. *Let $y_{n,1}^{\mathcal{S}}$ and $y_{n,2}^{\mathcal{S}}$ be the approximate solutions defined, respectively, by (11) and (13). According to the same assumptions of Theorem 2, the following superconvergence orders at the partition knots hold*

$$\left|y(x_j) - y_{n,i}^{\mathcal{S}}(x_j)\right| = \mathcal{O}\left(h^{4r}\right), \quad j = 1,\dots,n, \quad i = 1,2. \tag{30}$$

Proof. Let $i = 1$. The error function $e_{n,1} := y - y_{n,1}^{\mathcal{S}}$ satisfies the following equation

$$e'_{n,1}(x) + a(x)e_{n,1}(x) = \mathcal{K}e_{n,1}(x) + \delta_{n,1}(x), \tag{31}$$

where

$$\delta_{n,1}(x) = \left(\mathcal{K} - \mathcal{K}_{n,1}^{\mathcal{S}}\right)y_{n,1}^{\mathcal{S}}(x).$$

Under the regularity assumptions on a, f, and k, Equation (31) has a unique solution satisfying the initial condition $e_{n,1}(0) = 0$, which is given by

$$e_{n,1}(x) = \int_0^x \boldsymbol{r}(x,s)\delta_{n,1}(s)ds,$$

where $\boldsymbol{r}$ is the differential kernel (see [22]).

Then

$$e_{n,1}(x_j) = \int_0^{x_j} \boldsymbol{r}(x_j,s)\delta_{n,1}(s)ds = \sum_{\ell=1}^{j}\int_{x_{\ell-1}}^{x_\ell} \boldsymbol{r}(x_j,s)\delta_{n,1}(s)ds.$$

Next, for $1 \le \ell \le j$, we have

$$\begin{aligned}\left|\int_{x_{\ell-1}}^{x_\ell} \boldsymbol{r}(x_j,s)\delta_{n,1}(s)ds\right| \le & \left|\int_{x_{\ell-1}}^{x_\ell} \boldsymbol{r}(x_j,s)\left(\mathcal{K}-\mathcal{K}_{n,1}^{\mathcal{S}}\right)y(s)ds\right| \\ & + \left|\int_{x_{\ell-1}}^{x_\ell} \boldsymbol{r}(x_j,s)\left(\mathcal{K}-\mathcal{K}_{n,1}^{\mathcal{S}}\right)(y - y_{n,1}^{\mathcal{S}})(s)ds\right|.\end{aligned} \tag{32}$$

Using (19) and the regularity of the resolvent kernel $\boldsymbol{r}(x,s)$, it is easy to show that the first term on the right hand side of (32) is on $\mathcal{O}(h^{4r+1})$. For the second, using (18) and (22), we find

$$\begin{aligned}\left|\int_{x_{\ell-1}}^{x_\ell} \boldsymbol{r}(x_j,s)\left(\mathcal{K}-\mathcal{K}_{n,1}^{\mathcal{S}}\right)(y - y_{n,1}^{\mathcal{S}})(s)ds\right| &\le h\|\boldsymbol{r}(x_j,.)\|_\infty\|\mathcal{K}-\mathcal{K}_{n,1}^{\mathcal{S}}\|_\infty\|y - y_{n,1}^{\mathcal{S}}\|_\infty \\ &= \mathcal{O}(h^{4r+2}).\end{aligned}$$

We deduce that

$$\left| \int_{x_{\ell-1}}^{x_\ell} \boldsymbol{r}(x_j, s)\delta_{n,1}(s)ds \right| = \mathcal{O}(h^{4r+1}).$$

Hence

$$e_{n,1}(x_j) = \mathcal{O}(h^{4r}),$$

which proves (30). For $i = 2$, the proof is similar. □

4. Implementation Details

In this section, we consider the reduction of (9)–(11) and (13) to systems of linear equations. Let $\mathcal{X} := L^2[0,1]$, $k_i := k(\cdot, t_i)$, $\tilde{k}_i := k(t_i, \cdot)$ and let $\langle , \rangle$ denote the usual inner product on $\mathcal{X}$, we put

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Theorem 4. *Let B and $\widetilde{B}$ be the vectors with components*

$$B_i := \langle g, \phi_i \rangle \quad \text{and} \quad \widetilde{B}_i := g(t_i). \tag{33}$$

Let M and $\widetilde{M}$ be the matrices with entries

$$M_{i,j} := \langle \mathcal{V}k_j, \phi_i \rangle \quad \text{and} \quad \widetilde{M}_{i,j} := w_j \mathcal{V}k_j(t_i)\,. \tag{34}$$

The approximate solutions $y_{n,1}$ and $y_{n,2}$ of (9) and (10) are given by

$$y_{n,1} = g + \sum_{j=1}^{N_h} X_j \mathcal{V}k_j \quad \text{and} \quad y_{n,2} = g + \sum_{j=1}^{N_h} w_j Y_j \mathcal{V}k_j\,,$$

where $X := (X_1, \ldots, X_{N_h})^T$ and $Y := (Y_1, \ldots, Y_{N_h})^T$ are, respectively, the solutions of the linear systems of size N_h given by

$$(I - M)X = B \quad \text{and} \quad (I - \widetilde{M})Y = \widetilde{B}\,.$$

Proof. From Equation (9), the approximate solution $y_{n,1}$ can be written as

$$\begin{aligned} y_{n,1}(x) &= g(x) + \int_0^x \left(\int_0^1 y_{n,1}(s)\pi_n k(t,.)(s)ds \right) e^{A(t)-A(x)}dt \\ &= g(x) + \sum_{j=1}^{N_h} \left(\int_0^1 y_{n,1}(s)\phi_j(s)ds \right) \int_0^x k(t,t_j)e^{A(t)-A(x)}dt \\ &= g(x) + \sum_{j=1}^{N_h} X_j \mathcal{V}k_j(x). \end{aligned} \tag{35}$$

The coefficients $X_j, j = 1, \ldots, N_h$ are obtained by replacing $y_{n,1}$ into Equation (9) and by identifying the coefficients of the functions $k_j\,, j = 1, \ldots, N_h$, which we suppose to be linearly independent.

More precisely, we find the equations

$$\begin{aligned} &X_i - \sum_{j=1}^{N_h} \left(\int_0^1 \int_0^t k_j(s)e^{A(s)-A(t)}\phi_i(t)dsdt \right) X_j \\ &= \int_0^1 \left(y_0 e^{A(0)-A(t)} + \int_0^t f(s)e^{A(s)-A(t)}ds \right) \phi_i(t)dt\,, \quad i = 1, \ldots, N_h, \end{aligned}$$

which are expressed in matrix form as

$$(I - M)X = B\,,$$

where B and M are given by (33) and (34). This completes the proof for $y_{n,1}$.

By the same techniques, the form of $y_{n,2}$ and the corresponding linear system are derived. □

- Superconvergent degenerate kernel and Nyström approximate solutions

Theorem 5. *Let B and $\widetilde{B}$ be vectors with components*

$$B_i := \langle \tilde{k}_i, g\rangle - \sum_{\ell=1}^{N_h} \langle \phi_\ell, g\rangle k_\ell(t_i) \quad \text{and} \quad \widetilde{B}_i := \langle g\,, \phi_i\rangle, \tag{36}$$

and let F, $\widetilde{F}$, G, and $\widetilde{G}$ be matrices with entries

$$F_{i,j} := \langle \tilde{k}_i\,, \mathcal{V}\phi_j\rangle - \sum_{\ell=1}^{N_h} \langle \phi_\ell\,, \mathcal{V}\phi_j\rangle k_\ell(t_i) \quad \text{and} \quad \widetilde{F}_{i,j} := \langle \mathcal{V}k_j\,, \phi_i\rangle, \tag{37}$$

$$G_{i,j} := -\langle \tilde{k}_i\,, \mathcal{V}k_j\rangle + \sum_{\ell=1}^{N_h} \langle \phi_\ell\,, \mathcal{V}k_j\rangle k_\ell(t_i) \quad \text{and} \quad \widetilde{G}_{i,j} := -\langle \mathcal{V}\phi_j\,, \phi_i\rangle\,. \tag{38}$$

The approximate solution $y^{\mathcal{S}}_{n,1}$ is given by

$$y^{\mathcal{S}}_{n,1} = g + \sum_{i=1}^{N_h} Z_i \mathcal{V}\phi_i + \sum_{j=1}^{N_h} \widetilde{Z}_j \mathcal{V}k_j\,,$$

where $\left[Z\ \widetilde{Z}\right]^T$ is the solution of the following linear system of size $2N_h$:

$$\begin{pmatrix} I - F & G \\ \widetilde{G} & I - \widetilde{F} \end{pmatrix} \begin{pmatrix} Z \\ \widetilde{Z} \end{pmatrix} = \begin{pmatrix} B \\ \widetilde{B} \end{pmatrix}.$$

Proof. From (11) and the explicit expression of $\mathcal{K}^{\mathcal{S}}_{n,1}$, it is easy to prove that $y_{n,1}$ takes the form

$$\begin{aligned} y_{n,1}(x) &= g(x) + \mathcal{V}\mathcal{K}^{\mathcal{S}}_{n,1} y_{n,1}(x) \\ &= g(x) + \mathcal{V}(\pi_n \mathcal{K} y_{n,1} + \mathcal{K}_{n,1} y_{n,1} - \pi_n \mathcal{K}_{n,1} y_{n,1})(x) \qquad (39) \\ &= g(x) + \sum_{i=1}^{N_h} Z_i \mathcal{V}\phi_i(x) + \sum_{j=1}^{N_h} \widetilde{Z}_j \mathcal{V}k_j(x), \qquad (40) \end{aligned}$$

where the coefficients Z_j and $\widetilde{Z}_j, j = 1, \ldots, N_h$, are obtained by replacing $y_{n,1}$ given by (40) into the approximate Equation (11) and by identifying coefficients of the family of functions $\{\phi_j, k_j\},\ j = 1, \ldots, N_h$, supposed to be linearly independent. More precisely, we find the following equations

$$Z_i = \sum_{j=1}^{N_h} \left(\int_0^1\int_0^t \tilde{k}_i(t)\phi_j(s)e^{A(s)-A(t)}dsdt - \sum_{\ell=1}^{N_h} \left(\int_0^1\int_0^t \phi_j(s)\phi_\ell(t)e^{A(s)-A(t)}dsdt \right) k_\ell(t_i) \right) Z_j$$
$$+ \sum_{j=1}^{N_h} \left(\int_0^1\int_0^t \tilde{k}_i(t)k_j(s)e^{A(s)-A(t)}dsdt - \sum_{\ell=1}^{N_h} \left(\int_0^1\int_0^t k_j(s)\phi_\ell(t)e^{A(s)-A(t)}dsdt \right) k_\ell(t_i) \right) \widetilde{Z}_j$$
$$+ \int_0^1 \tilde{k}_i(t)g(t)dt - \sum_{\ell=1}^{N_h} \int_0^1 g(t)\phi_\ell(t)dtk_\ell(t_i)\,, \quad i = 1,\dots,N_h\,,$$

and

$$\widetilde{Z}_i = \sum_{j=1}^{N_h} \left(\int_0^1\int_0^t k_j(s)e^{A(s)-A(t)}\phi_i(t)dsdt \right) \widetilde{Z}_j + \sum_{j=1}^{N_h} \left(\int_0^1\int_0^t \phi_j(s)e^{A(s)-A(t)}\phi_i(t)dsdt \right) Z_j$$
$$+ \int_0^1 g(t)\phi_i(t)dt\,, \quad i = 1,\dots,N_h\,.$$

In matrix form

$$\begin{pmatrix} I-F & G \\ \widetilde{G} & I-\widetilde{F} \end{pmatrix} \begin{pmatrix} Z \\ \widetilde{Z} \end{pmatrix} = \begin{pmatrix} B \\ \widetilde{B} \end{pmatrix}.$$

where B , $\widetilde{B}$, F , $\widetilde{F}$, G , and $\widetilde{G}$ are given by (36)–(38), respectively.

The proof is complete. □

Theorem 6. *Let F and $\widetilde{F}$ be the vectors with the components*

$$F_i := \langle \tilde{k}_i, g\rangle - \sum_{\ell=1}^{N_h} w_\ell \tilde{k}_i(t_\ell)g(t_\ell) \quad \textit{and} \quad \widetilde{F}_i := g(t_i),$$

and let M , $\widetilde{M}$, H , and $\widetilde{H}$ be the matrices with the entries

$$M_{i,j} = \langle \tilde{k}_i\,, \mathcal{V}\psi_j\rangle - \sum_{\ell=1}^{N_h} w_\ell \big(\tilde{k}_i\, \mathcal{V}\psi_j\big)(t_\ell) \quad \textit{and} \quad \widetilde{M}_{i,j} = w_j\mathcal{V}k_j(t_i)\,,$$

$$H_{i,j} = -w_j\left(\langle \tilde{k}_i\,, \mathcal{V}k_j\rangle - \sum_{\ell=1}^{N_h} w_\ell \big(\tilde{k}_i\, \mathcal{V}k_j\big)(t_\ell) \right) \quad \textit{and} \quad \widetilde{H}_{i,j} = -w_j\mathcal{V}\psi_j(t_i)\,.$$

The approximate solution is given by

$$y_n = g + \sum_{i=1}^{N_h} X_i\mathcal{V}\psi_i + \sum_{j=1}^{N_h} w_j\widetilde{X}_j\mathcal{V}k_j\,,$$

where $\left[X\,\widetilde{X}\right]^T$ is the solution of the following linear system of size $2N_h$:

$$\begin{pmatrix} I-M & H \\ \widetilde{H} & I-\widetilde{M} \end{pmatrix} \begin{pmatrix} X \\ \widetilde{X} \end{pmatrix} = \begin{pmatrix} F \\ \widetilde{F} \end{pmatrix}.$$

Proof. The proof can be presented in a similar way as that of Theorem 5 . □

Remark 1. *It should be noted that there are integrals in setting up the above systems and in evaluating the approximate solutions and their iterated versions. These integrals are evaluated numerically by suitable quadrature rules with high accuracy to imitate the exact integration.*

5. Numerical Results

In this section, we illustrate the accuracy and effectiveness of theoretical results established in the previous sections for numerically solving Fredholm integro-differential equations. More precisely, we consider four numerical examples of such equations defined on $[0,1]$ and given in the following table.

	Kernel k	Function a	Function f	Exact Solution y
Example 1	$\frac{1}{t+\exp(s)}$	-1	$-\log\left(\frac{t+e}{t+1}\right)$	$\exp(t)$
Example 2	$\sin(t+s)$	$-\sin(t)$	$\frac{1}{4}(\cos(t+2)-\cos(t))-\frac{1}{2}(3\sin(t)+\sin(2t))$	$\cos(t)$
Example 3	ts	-1	$-2\sin(x)-x(-1+2\sin(1))$	$\cos(x)+\sin(x)$
Example 4	$\sin(4\pi t+2\pi s)$	-1	$-2\pi\sin(2\pi x)-\cos(2\pi x)(1+\sin(2\pi x))$	$\cos(2\pi t)$

Firstly, for Examples 1 and 2, we consider the space of piecewise constant functions $(r=1)$ and the space of piecewise linear functions $(r=2)$ defined on the interval $[0,1]$ endowed with the uniform partition

$$0<\frac{1}{n}<\frac{2}{n}<\cdots<\frac{n-1}{n}<1. \tag{41}$$

For different values of n and for $i=1,2$, we compute the maximum absolute errors

$$E_{i,\infty}:=\|y-y_{n,i}\|_\infty, \quad E^{\mathcal{S}}_{i,\infty}:=\|y-y^{\mathcal{S}}_{n,i}\|_\infty,$$

$$\widetilde{E}^{\mathcal{S}}_{i,\infty}:=\|y-\tilde{y}^{\mathcal{S}}_{n,i}\|_\infty, \quad E^{\mathcal{S}}_{i}:=\max_j|y(x_j)-y^{\mathcal{S}}_{n,i}(x_j)|.$$

Moreover, we present the corresponding numerical convergence orders denoted $\mathcal{NCO}$ and obtained by the logarithm to base 2 of the ratio between two consecutive errors. The obtained results are illustrated in the following tables.

Tables 1–4 show that the superconvergent Nyström and degenerate kernel methods are more accurate than the Nyström and degenerate kernel methods, and the computed NCOs match well with the expected values.

Next, in order to give a comparison, we illustrate in Tables 5 and 6 the punctual errors provided by the application of the superconvergent Nyström and degenerate kernel methods and other known errors obtained in [24,25]. In particular, for $i=1,2$ we denote by

$$E_{i,j}=|y(x_j)-y^{\mathcal{S}}_{n,i}(x_j)|, \quad x_j=j/10\,,\ j=0,\cdots,10,$$

the punctual errors obtained by our methods for $r=1$ and $n=4$, while $E_{Sp,j}$ denote the errors obtained in [24] by using a cubic spline interpolation, and $E_{Ad,j}$ are those obtained in [25] by using Adomian's decomposition with four iterations.

Table 1. Numerical methods based on piecewise constant functions ($r = 1$).

Example 1					$i = 1$			
n	$E_{i,\infty}$	$\mathcal{NCO}$	$E^{\mathcal{S}}_{i,\infty}$	$\mathcal{NCO}$	$\widetilde{E}^{\mathcal{S}}_{i,\infty}$	$\mathcal{NCO}$	$E^{\mathcal{S}}_{i}$	$\mathcal{NCO}$
2	2.37(−02)	−−	2.22(−04)	−−	1.02(−04)	−−	1.62(−04)	−−
4	5.82(−03)	2.02	1.23(−05)	4, 17	4.92(−06)	4.38	7.04(−06)	4.52
8	1.43(−03)	2.02	7.11(−07)	4.11	2.80(−07)	4.13	3.75(−07)	4.23
16	3.21(−04)	2.02	4.52(−08)	4.00	1.82(−08)	3.93	2.23(−08)	4.07
Theoretical order	−	2.0	−	4.0	−	4.0	−	4.0
					$i = 2$			
2	1.81(−03)	−−	2.81(−04)	−−	1.20(−04)	−−	2.81(−04)	−−
4	4.51(−04)	2.00	2.21(−05)	3.66	9.66(−06)	3.63	2.21(−05)	3.66
8	1.12(−04)	2.00	1.47(−06)	3.90	6.48(−07)	3.89	1.47(−06)	3.90
16	2.81(−05)	2.00	9.38(−08)	3.97	4.16(−08)	3.96	9.38(−08)	3.97
Theoretical order	−	2.0	−	4.0	−	4.0	−	4.0

Table 2. Numerical methods based on piecewise constant functions ($r = 1$).

Example 2					$i = 1$			
n	$E_{i,\infty}$	$\mathcal{NCO}$	$E^{\mathcal{S}}_{i,\infty}$	$\mathcal{NCO}$	$\widetilde{E}^{\mathcal{S}}_{i,\infty}$	$\mathcal{NCO}$	$E^{\mathcal{S}}_{i}$	$\mathcal{NCO}$
2	2.85(−02)	−−	1.43(−04)	−−	4.55(−05)	−−	5.07(−05)	−−
4	7.06(−03)	2.01	1.15(−05)	3.63	3.25(−06)	3.80	2.95(−06)	4.10
8	1.74(−03)	2.02	7.09(−07)	4.02	2.16(−07)	3.91	1.76(−07)	4.06
16	3.25(−04)	2.41	4.70(−08)	3.91	1.54(−08)	3.80	1.08(−08)	4.01
Theoretical order	−	2.0	−	4.0	−	4.0	−	4.0
					$i = 2$			
2	4.41(−02)	−−	4.98(−04)	−−	2.58(−04)	−−	4.98(−04)	−−
4	1.11(−02)	1.98	2.96(−05)	4.07	1.53(−05)	4.07	2.96(−05)	4.07
8	2.78(−03)	1.99	1.83(−06)	4.01	9.44(−07)	4.01	1.83(−06)	4.01
16	6.69(−04)	1.99	1.14(−07)	4.00	5.89(−08)	4.00	1.14(−07)	4.00
Theoretical order	−	2.0	−	4.0	−	4.0	−	4.0

Table 3. Numerical methods based on piecewise linear functions ($r = 2$).

Example 1					$i = 1$			
n	$E_{i,\infty}$	$\mathcal{NCO}$	$E^{\mathcal{S}}_{i,\infty}$	$\mathcal{NCO}$	$\widetilde{E}^{\mathcal{S}}_{i,\infty}$	$\mathcal{NCO}$	$E^{\mathcal{S}}_{i}$	$\mathcal{NCO}$
2	7.25(−05)	−−	1.60(−07)	−−	7.83(−08)	−−	1.60(−07)	−−
4	4.51(−06)	4.00	1.40(−09)	6.83	4.68(−10)	7.38	9.51(−10)	7.40
8	2.82(−07)	4.00	1.28(−11)	6.77	2.07(−12)	7.82	4.27(−12)	7.79
16	1.76(−08)	4.00	1.01(−13)	6.97	8.02(−15)	8.01	1.70(−14)	7.97
Theoretical order	−	4.0	−	7.0	−	8.0	−	8.0
					$i = 2$			
2	1.94(−06)	−−	9.40(−08)	−−	4.19(−08)	−−	9.40(−08)	−−
4	1.20(−07)	4.01	6.56(−10)	7.16	3.07(−10)	7.09	6.56(−10)	7.16
8	7.50(−09)	4.00	6.52(−12)	6.65	1.43(−12)	7.73	3.07(−12)	7.73
16	4.68(−10)	4.00	3.68(−14)	7.46	4.88(−15)	8.20	1.24(−14)	7.95
Theoretical order	−	4.0	−	7.0	−	8.0	−	8.0

Table 4. Numerical methods based on piecewise linear functions ($r = 2$).

Example 2				$i = 1$				
n	$E_{i,\infty}$	$\mathcal{NCO}$	$E^{\mathcal{S}}_{i,\infty}$	$\mathcal{NCO}$	$\widetilde{E}^{\mathcal{S}}_{i,\infty}$	$\mathcal{NCO}$	$E^{\mathcal{S}}_{i}$	$\mathcal{NCO}$
2	1.97(−04)	−−	8.96(−08)	−−	1.14(−08)	−−	1.13(−08)	−−
4	1.21(−05)	4.02	6.36(−10)	7.13	4.29(−11)	8.05	4.90(−11)	7.85
8	7.52(−07)	4.00	4.97(−12)	6.99	1.57(−13)	8.09	2.00(−13)	7.93
16	4.69(−08)	4.00	3.78(−14)	7.03	6.02(−16)	8.03	7.91(−16)	7.98
Theoretical order	−	4.0	−	7.0	−	8.0	−	8.0
				$i = 2$				
2	2.55(−04)	−−	1.05(−07)	−−	1.29(−08)	−−	1.42(−08)	−−
4	1.56(−06)	4.03	7.69(−10)	7.10	4.87(−11)	8.05	5.85(−11)	7.93
8	9.70(−07)	4.00	6.08(−12)	6.98	1.81(−13)	8.06	2.40(−13)	7.92
16	6.05(−08)	4.00	4.56(−14)	7.05	3.33(−16)	9.09	1.11(−15)	7.76
Theoretical order	−	4.0	−	7.0	−	8.0	−	8.0

The results in Tables 5 and 6 show that the error obtained by our methods are comparable with those given in [24,25]. However, we notice that in [24] cubic spline functions (piecewise polynomials of degree three) are used, and in [25], four iterations were needed to obtain these errors, while in our case only piecewise constant polynomials defined on the partition (41) with $n = 4$ were enough to obtain the same accuracy.

Table 5. Comparison with results given in [24].

Example 3			
	Present Methods		**Method in [24]**
x_j	$E_{1,j}$	$E_{2,j}$	$E_{Sp,j}$
0	0	0	0
0.1	1.59×10^{-5}	1.71×10^{-6}	1.71×10^{-5}
0.2	1.27×10^{-5}	1.37×10^{-6}	3.27×10^{-5}
0.3	1.39×10^{-5}	1.50×10^{-6}	3.59×10^{-5}
0.4	2.12×10^{-5}	2.29×10^{-6}	4.17×10^{-5}
0.5	8.54×10^{-6}	8.87×10^{-7}	4.94×10^{-5}
0.6	2.59×10^{-5}	2.64×10^{-6}	5.88×10^{-5}
0.7	2.41×10^{-5}	2.18×10^{-6}	6.88×10^{-5}
0.8	2.65×10^{-5}	1.85×10^{-6}	8.49×10^{-5}
0.9	3.47×10^{-5}	1.59×10^{-6}	8.79×10^{-5}
1	2.20×10^{-5}	1.84×10^{-6}	1.48×10^{-4}

Table 6. Comparison with results given in [25].

Example 4			
	Present Methods		**Method in [25]**
x_j	$E_{1,j}$	$E_{2,j}$	$E_{Ad,j}$
0.1	2.37502×10^{-3}	7.18395×10^{-6}	6.77227×10^{-4}
0.2	3.24853×10^{-3}	1.32702×10^{-5}	3.57926×10^{-4}
0.3	3.78369×10^{-3}	5.49501×10^{-5}	7.20389×10^{-4}
0.4	3.64555×10^{-3}	1.14361×10^{-4}	1.65557×10^{-3}
0.5	1.69840×10^{-3}	2.07833×10^{-4}	2.33402×10^{-3}
0.6	4.39557×10^{-3}	3.47537×10^{-4}	3.76522×10^{-3}
0.7	5.61879×10^{-3}	4.93954×10^{-4}	6.78844×10^{-2}
0.8	6.51049×10^{-3}	6.49503×10^{-4}	1.09211×10^{-2}
0.9	6.91467×10^{-3}	1.02800×10^{-3}	1.49581×10^{-2}

6. Conclusions

In this paper, we have developed Nyström, degenerate kernel methods and their superconvergent/iterated superconvergent versions for the numerical solution of Fredholm linear integro-differential equations. We have proved that these methods exhibit high convergent orders. Finally, such methods turn out to be very effective, with low computational cost and comparable with other methods known in the literature.

Author Contributions: Conceptualization, D.S., M.T. and D.B.; methodology, M.T.; software, A.S.; validation, D.S. and D.B.; formal analysis, M.T. and D.B.; investigation, A.S. and M.T.; resources, D.S. and D.B.; data curation, A.S.; writing—original draft preparation, A.S.; writing—review and editing, A.S. and M.T.; visualization, D.S.; supervision, D.B. and D.S.; funding acquisition, D.B. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding and APC was funded by University of Granada.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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