Linear Algebra and its Applications 643 (2022) 39–79

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Surjective isometries between unitary sets of unital JB*-algebras

María Cueto-Avellaneda [a], Yuta Enami [b], Daisuke Hirota [b], Takeshi Miura [c], Antonio M. Peralta [d,*]

[a] *School of Mathematics, Statistics and Actuarial Science, University of Kent, Canterbury, Kent CT2 7NX, UK*
[b] *Graduate School of Science and Technology, Niigata University, Niigata 950-2181, Japan*
[c] *Department of Mathematics, Faculty of Science, Niigata University, Niigata 950-2181, Japan*
[d] *Instituto de Matemáticas de la Universidad de Granada (IMAG), Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Granada, 18071 Granada, Spain*

ARTICLE INFO

Article history:
Received 31 May 2021
Accepted 3 February 2022
Available online 7 February 2022
Submitted by P. Semrl

MSC:
primary 47B49, 46B03, 46B20, 46A22, 46H70
secondary 46B04, 46L05, 17C65

Keywords:
Isometry
Jordan *-isomorphism
Unitary
JB*-algebra
JBW*-algebra
Extension of isometries

ABSTRACT

This paper is, in a first stage, devoted to establishing a topological–algebraic characterization of the principal component, $\mathcal{U}^0(M)$, of the set of unitary elements, $\mathcal{U}(M)$, in a unital JB*-algebra M. We arrive to the conclusion that, as in the case of unital C*-algebras,

$$\begin{aligned}\mathcal{U}^0(M) &= M_1^{-1} \cap \mathcal{U}(M) \\ &= \left\{ U_{e^{ih_n}} \cdots U_{e^{ih_1}}(\mathbf{1}) : \begin{array}{c} n \in \mathbb{N},\ h_j \in M_{sa} \\ \forall\ 1 \leq j \leq n \end{array} \right\} \\ &= \{ u \in \mathcal{U}(M) : \text{there exists } w \in \mathcal{U}^0(M) \text{ with} \\ &\qquad \|u - w\| < 2 \}\end{aligned}$$

is analytically arcwise connected. Actually, $\mathcal{U}^0(M)$ is the smallest quadratic subset of $\mathcal{U}(M)$ containing the set $e^{iM_{sa}}$. Our second goal is to provide a complete description of the

* Corresponding author.
E-mail addresses: mcueto@ugr.es, emecueto@gmail.com (M. Cueto-Avellaneda), enami@m.sc.niigata-u.ac.jp (Y. Enami), f20j008d@mail.cc.niigata-u.ac.jp (D. Hirota), miura@math.sc.niigata-u.ac.jp (T. Miura), aperalta@ugr.es (A.M. Peralta).

https://doi.org/10.1016/j.laa.2022.02.003

surjective isometries between the principal components of two unital JB*-algebras M and N. Contrary to the case of unital C*-algebras, we shall deduce the existence of connected components in $\mathcal{U}(M)$ which are not isometric as metric spaces. We shall also establish necessary and sufficient conditions to guarantee that a surjective isometry $\Delta : \mathcal{U}(M) \to \mathcal{U}(N)$ admits an extension to a surjective linear isometry between M and N, a conclusion which is not always true. Among the consequences it is proved that M and N are Jordan *-isomorphic if, and only if, their principal components are isometric as metric spaces if, and only if, there exists a surjective isometry $\Delta : \mathcal{U}(M) \to \mathcal{U}(N)$ mapping the unit of M to an element in $\mathcal{U}^0(N)$. These results provide an extension to the setting of unital JB*-algebras of the results obtained by O. Hatori for unital C*-algebras.

1. Introduction

Why are we so attracted by the unitary group of unital C*-algebras? The celebrated Russo–Dye theorem, asserting that the convex hull of the unitary elements in a unital C*-algebra is norm dense in its closed unit ball (cf. [47]), is probably one of the eldest tools leading our attention to the unitary group. It is perhaps unnecessary to recall that an element, u, in a unital C*-algebra, A, lies in the subgroup, $\mathcal{U}(A)$, of all unitaries in A if $uu^* = u^*u = 1$. As we have already advanced, $\mathcal{U}(A)$ is a subgroup of A containing the unit element. Applications of the Russo-Dye theorem and subsequent generalizations have been employed along the last fifty five years, recent usages on problems about extensions of isometries appear, for example, in [45] and [6,15].

The unitary group of a unital C*-algebra was employed as a complete invariant to classify von Neumann factors and certain unital C*-algebras. Highlighting the group structure of the set of unitaries, S. Sakai established that if A and B are AW*-factors and $\rho : \mathcal{U}(A) \to \mathcal{U}(B)$ is a uniformly continuous group isomorphism between their respective unitary groups, then there is a unique map T from A onto B which is either a linear or a conjugate linear *-isomorphism and agrees with ρ on $\mathcal{U}(A)$ (see [48]). A similar conclusion was proved by H.A. Dye for W*-factors in [19] with an independent argument which also gives results when no continuity assumption is made on the group isomorphism. More recently, Al-Rawashdeh, Booth and Giordano show that if the unitary groups of two simple unital AH-algebras of slow dimension growth and of real rank zero are isomorphic as abstract groups, then their K_0-ordered groups are isomorphic (cf. [3]).

In a surprising turn, O. Hatori and L. Molnár considered the set $\mathcal{U}(A)$, of unitary elements in a unital C*-algebra A, as a metric space equipped with the metric given by the C*-norm and forgot about its group structure (see [32]). It should be noted that the metric space $(\mathcal{U}(A), \|\cdot\|)$ is almost never compact (cf. [50, Exercise II.2.1]).

In the case of von Neumann algebras, the metric space given by the set of unitaries is a complete invariant since, as shown by Hatori and Molnár, every surjective isometry between the unitary groups of two von Neumann algebras admits an extension to a surjective real linear isometry between these algebras (see [32, Corollary 3]). Actually if $\Delta : \mathcal{U}(A) \to \mathcal{U}(B)$ is a surjective isometry between the sets of unitary elements of two unital C*-algebras, then there is a central projection $p \in B$ and a Jordan *-isomorphism $J : A \to B$ satisfying

$$\Delta(e^{ix}) = \Delta(1)(pJ(e^{ix}) + (1-p)J(e^{ix})^*),$$

for all $x \in A_{sa}$ (see [32, Theorem 1]). In particular A and B are Jordan *-isomorphic. Here, and henceforth, the self-adjoint part of a C*-algebra A will be denoted by A_{sa}.

The framework of von Neumann algebras is a very favorable scenario since, among other things, the set of unitaries in a von Neumann algebra W is precisely the set $\exp(iW_{sa})$, which is an analytically arcwise connected set (cf. [38, Theorem 5.2.5]). However, it is well known that in the case of a general unital C*-algebra A the set $\mathcal{U}(A)$ is not always connected (cf. [50, Exercise I.11.3], [38, Exercises 4.6.6 and 4.6.7], [32, pages 162-164], [28]). The connected component of $\mathcal{U}(A)$ containing the unit element is called the principal component, and it is denoted by $\mathcal{U}^0(A)$.

Hatori and Molnár pointed out in [32, Corollary 8] the existence of surjective isometries between the sets of unitaries in two unital commutative C*-algebras which do not admit an extension to a surjective real linear isometry between these algebras. The problem of determining the structure of all surjective isometries between the sets of unitaries in two arbitrary unital C*-algebras was carried out by O. Hatori in [28], where two main results are established: Suppose A and B are two unital C*-algebras.

(a) Suppose $\Delta : \mathcal{U}^0(A) \to \mathcal{U}^0(B)$ is a mapping. Then Δ is a surjective isometry if and only if there exist a central projection $p \in B$ and a Jordan *-isomorphism $J : A \to B$ satisfying

$$\Delta(a) = \Delta(1)\Big(pJ(a) + (1-p)J(a)^*\Big),$$

for all $a \in \mathcal{U}^0(A)$. In particular A and B are Jordan *-isomorphic. The principal components $\mathcal{U}^0(A)$ and $\mathcal{U}^0(B)$ can be somehow replaced by any two connected components (see [28, Theorem 3.1 and Corollary 3.5]).

(b) Each surjective isometry $\Delta : \mathcal{U}(A) \to \mathcal{U}(B)$ can be written as a direct sum of a family of surjective real linear isometries between A and B restricted to the corresponding connected components, and the extensibility of Δ to a surjective real linear isometry is only possible under additional conditions ([28, Theorem 4.1 and Corollary 5.1]).

From a strict mathematical point of view, C*-algebras are contained in the strictly wider class of JB*-algebras, a class of Jordan-Banach algebras determined by an appropriate version of the Gelfand-Naimark axiom. The class of JB*-algebras is nowadays a

consolidated object of study, whose origins go back to the works of P. Jordan, J. von Neumann, E. Wigner, I. Kaplansky, J.D.M. Wright, M. Youngson, H. Hanche-Olsen, E. Størmer, E.M. Alfsen and F.W. Shultz, among others. Detailed definitions and complete sources of references can be found in subsection 1.1 and in the monographs [1,2,26,11,12]. The notions of invertibility, spectrum and unitaries admit a perfect translation to the setting of unital JB*-algebras, and there is a version of the Russo–Dye theorem established by J.D.M. Wright and M. Youngson (see subsection 1.1 for more details). In the case of a unital JB*-algebra M, the subset, $\mathcal{U}(M)$, of all unitaries in M is not a Jordan-sub-semigroup of M –actually the Jordan product of two unitary elements need not be, in general, a unitary element even in the case of unital C*-algebras–. So, in the Jordan setting we can no longer speak about the unitary group.

For any couple of elements a, b in a Jordan algebra M, we shall write $U_{a,b}$ and M_a for the linear operators on M defined by

$$U_{a,b}(x) = (a \circ x) \circ b + (b \circ x) \circ a - (a \circ b) \circ x, \quad \text{and } M_a(x) = a \circ x$$

for all $x \in M$. We shall simply write U_a for $U_{a,a}$. If M is obtained from an associative algebra (with product denoted by juxtaposition) equipped with its natural Jordan product $a \circ b = \frac{1}{2}(ab + ba)$, it can be easily checked that $U_{a,b}(x) = \frac{1}{2}(axb + bxa)$ $(a, b, x \in M)$.

In the same way that a celebrated theorem, due to Sakai, identifies von Neumann algebras with those C*-algebras which are dual Banach spaces, in the setting of JB*-algebras, those which are dual Banach spaces are called JBW*-algebras and enjoy additional geometric properties. For example, each JBW*-algebra contains a unit element, and every unitary element in a JBW*-algebra M is of the form $\exp(ih)$, where h lies in the self-adjoint part, M_{sa}, of M. In [16], the first and fifth authors of this note extended the previously-commented result by Hatori and Molnár, and proved that the set of unitaries in a JBW*-algebra is a complete invariant too. More concretely, suppose $\Delta : \mathcal{U}(M) \to \mathcal{U}(N)$ is a surjective isometry, where M and N are two JBW*-algebras. Then there exist a unitary ω in N, a central projection $p \in N$, and a Jordan *-isomorphism $\Phi : M \to N$ such that

$$\begin{aligned}\Delta(u) &= U_{\omega^*}(p \circ \Phi(u)) + U_{\omega^*}((\mathbf{1}_N - p) \circ \Phi(u)^*) \\ &= P_2(U_{\omega^*}(p))U_{\omega^*}(\Phi(u)) + P_2(U_{\omega^*}(\mathbf{1}_N - p))U_{\omega^*}(\Phi(u^*)),\end{aligned}$$

for all $u \in \mathcal{U}(M)$. Consequently, Δ admits a (unique) extension to a surjective real linear isometry from M onto N (see [16, Theorem 3.9]).

Since unital C*-algebras are unital JB*-algebras when equipped with the natural Jordan product $a \circ b = \frac{1}{2}(ab + ba)$, and the same involution and norm; and when both structures are possible, the notions of unitary coincide, we know that the set of unitaries in a unital JB*-algebra M need not be, in general, connected. The connected component of $\mathcal{U}(M)$ containing the unit element will be also called the *principal component*, and will be denoted by $\mathcal{U}^0(M)$.

Despite nowadays we have excellent and thorough academic monographs covering the theory of JB*-algebras (cf., for example, [1,2,14,26] and the recent books [11,12]), some geometric aspects have not been explored yet. This is the case of the principal component of the set of unitaries. Section 2 is aimed to complete our knowledge on the principal component. The starting point is a result by O. Hatori which asserts that for each unital C*-algebra A we have

$$\mathcal{U}^0(A) = \{e^{ih_1} \cdot \ldots \cdot e^{ih_n} \mathbf{1} e^{ih_n} \cdot \ldots \cdot e^{ih_1} : n \in \mathbb{N},\ h_1, \ldots, h_n \in A_{sa}\}$$

is open, closed, and path connected, in the (relative) norm topology on $\mathcal{U}(A)$ (see [28, Lemma 3.2]). The advantage of the previous description is that it admits an expression in terms of expressions given by Jordan products. In Theorem 2.3 we prove that for each unital JB*-algebra M, the principal component of the set of unitaries in M admits the following description

$$\begin{aligned} \mathcal{U}^0(M) &= M_{\mathbf{1}}^{-1} \cap \mathcal{U}(M) \\ &= \{U_{e^{ih_n}} \cdots U_{e^{ih_1}}(\mathbf{1}) \colon n \in \mathbb{N},\ h_j \in M_{sa}\ \forall\ 1 \leq j \leq n\} \\ &= \left\{u \in \mathcal{U}(M) :\ \text{there exists } w \in \mathcal{U}^0(M) \text{ with } \|u - w\| < 2\right\} \end{aligned},$$

where $M_{\mathbf{1}}^{-1}$ stands for the principal component of the set M^{-1} of all invertible elements in M. Consequently, $\mathcal{U}^0(M)$ is analytically arcwise connected. A central notion in the theory of Jordan algebras has already appeared in the previous description where the U_a operator associated with an element a in a Jordan algebra M is defined by

$$U_a(x) = 2(a \circ x) \circ a - (a \circ a) \circ x, \quad (x \in M).$$

When an associative algebra A is regarded as a Jordan algebra with respect to the natural Jordan product we have $U_a(x) = axa$ for all $a, x \in A$.

We should admit that the instability of $\mathcal{U}(M)$ under Jordan products is a handicap to find a description of this set in terms of algebraic properties. A similar instability affects the set of invertible elements in a unital Jordan-Banach algebra. In any case, given two invertible (respectively, unitary) elements a, b in a unital Jordan-Banach algebra (respectively, in a unital JB*-algebra) M the element $U_a(b)$ is invertible (respectively, unitary). Let M be a unital Jordan-Banach algebra. Following [46], we shall say that a subset $\mathfrak{M} \subseteq M^{-1}$ is a *quadratic subset* if $U_{\mathfrak{M}}(\mathfrak{M}) \subseteq \mathfrak{M}$. Let us observe that $\mathcal{U}(M)$ is a self-adjoint subset of M whenever M is a unital JB*-algebra, in such a case we also prove that $\mathcal{U}^0(M)$ is the smallest quadratic subset of $\mathcal{U}(M)$ containing the set $e^{iM_{sa}}$ (see Proposition 2.6).

Surjective isometries between different connected components of the unitary sets of two unital JB*-algebras are studied in section 3. In a first result we establish that for each surjective isometry $\Delta : \mathcal{U}^0(M) \to \mathcal{U}^0(N)$ between the principal components of two unital JB*-algebras, there exist $k_1, \ldots, k_n \in N_{sa}$, a central projection $p \in N$ and a Jordan *-isomorphism $\Phi : M \to N$ such that

$$\Delta(u) = p \circ U_{e^{ik_n}} \dots U_{e^{ik_1}} \Phi(u) + (\mathbf{1}_N - p) \circ (U_{e^{-ik_n}} \dots U_{e^{-ik_1}} \Phi(u))^*,$$

for all $u \in \mathcal{U}^0(M)$. Consequently, M and N are Jordan *-isomorphic, and there exists a surjective real linear isometry (i.e., a real linear triple isomorphism) from M onto N whose restriction to $\mathcal{U}^0(M)$ is Δ (see Theorem 3.4). Surjective isometries between connected components of the unitary sets of two unital JB*-algebras are described in Theorem 3.6.

Corollary 3.8 in [16] proves that two unital JB*-algebras M and N are isometrically isomorphic as (complex) Banach spaces if, and only if, they are isometrically isomorphic as real Banach spaces if, and only if, there exists a surjective isometry between their unitary sets. Corollary 3.5 in this note improves this conclusion by showing that the following statements are equivalent:

(a) M and N are Jordan *-isomorphic;
(b) There exists a surjective isometry $\Delta : \mathcal{U}(M) \to \mathcal{U}(N)$ satisfying $\Delta(\mathbf{1}_M) \in \mathcal{U}^0(N)$;
(c) There exists a surjective isometry $\Delta : \mathcal{U}^0(M) \to \mathcal{U}^0(N)$.

The extendibility of a surjective isometry between the unitary sets of two unital JB*-algebras is discussed in section 3.1 (see Corollary 3.8 and Proposition 3.10).

We cannot conclude this section without pointing out the exceptional particularities naturally linked to the category of JB*-algebras. For example, in a unital C*-algebra A, any connected component $\mathcal{U}^c(A)$ of $\mathcal{U}(A)$ is isometrically isomorphic to the principal component $\mathcal{U}^0(A)$ –it suffices to consider the left or right multiplication operator on A by an element in $\mathcal{U}^c(A)$. In the setting of unital JB*-algebras this geometric property is not always true, combining our results with a construction due to R. Braun, W. Kaup and H. Upmeier in [8], we shall exhibit an example of a unital JB*-algebra M admitting a unitary w such that the connected component of $\mathcal{U}(M)$ containing w is not isometrically isomorphic to the principal component (see Remark 3.7). This counterexample reinforces the fact that a unital JB*-algebra admits many different Jordan products and involutions, at least as many as unitaries, and some of them produce structures which are not Jordan *-isomorphic. However, by a fascinating result due to W. Kaup, each JB*-algebra admits an essentially unique structure of JB*-triple (see [39, Proposition 5.5] and subsection 1.1). This is essentially the reason for which we employ JB*-triple theory in some of our arguments.

1.1. Definitions and background

A complex (respectively, real) *Jordan algebra* M is a (non-necessarily associative) algebra over the complex (respectively, real) field whose product "$\circ$" is commutative and satisfies the so-called *Jordan identity*:

$$(a \circ b) \circ a^2 = a \circ (b \circ a^2) \ (a, b \in M). \tag{1}$$

A *normed Jordan algebra* is a Jordan algebra M equipped with a norm, $\|.\|$, satisfying $\|a \circ b\| \leq \|a\| \, \|b\|$ $(a, b \in M)$. A *Jordan-Banach algebra* is a normed Jordan algebra whose norm is complete. Every real or complex associative Banach algebra is a real Jordan-Banach algebra with respect to its natural Jordan product. All Jordan algebras in this paper are assumed to be unital.

The Jordan identity implies that Jordan algebras are power associative, that is, each subalgebra generated by a single element a in a Jordan algebra M is associative. More concretely, let us set $a^0 = \mathbf{1}$, $a^1 = a$, and $a^{n+1} = a \circ a^n$ $(n \geq 1)$. In this case the identity $a^{m+n} = a^m \circ a^n$, holds for all natural numbers n, m (cf. [26, Lemma 2.4.5]). In the case that M is a Jordan-Banach algebra, the completeness of the norm assures that the series $\exp(a) = e^a = \sum_{n=0}^{\infty} \frac{a^n}{n!}$ is uniformly convergent on bounded subsets of M. Furthermore, the mapping $a \mapsto e^a$ defines an analytic mapping on M. The Jordan-Banach subalgebra C of M generated by an element a is a commutative associative subalgebra with respect to the inherited Jordan product [36, 1.1]. Suppose A is an associative Banach algebra. For each element $a \in A$, the powers of A with respect to the associative and to the Jordan product coincide, and thus e^a is just the usual exponential in the usual sense for associative Banach algebras. It follows that $\exp(a)$ has its usual meaning in the Jordan-Banach subalgebra C, and, in particular we have

$$e^{sa} \circ e^{ta} = e^{(s+t)a}, \text{ for all } s, t \in \mathbb{C}. \tag{2}$$

Two elements a, b in a Jordan algebra M are said to *operator commute* if the identity $(a \circ c) \circ b = a \circ (c \circ b)$ holds for all $c \in M$ (cf. [26, 4.2.4]). The *centre* of M is formed by all elements in M which operator commute with any other element in M.

In a Jordan algebra the left or right multiplication operator by a fixed element does not play the role played by the left or right multiplication in an associative Banach algebra. The protagonist in the Jordan setting is associated with the U operator. Let M be a Jordan-Banach algebra. One of the fundamental identities in Jordan theory assures that

$$U_{U_a(b)} = U_a U_b U_a, \text{ for all } a, b \text{ in a Jordan algebra } M \tag{3}$$

(see [26, 2.4.18]). It can be easily proved, via an induction argument, that the following generalisation of (3) holds in every Jordan algebra M:

$$U_{U_{a_n} \cdots U_{a_1}(b)} = U_{a_n} \cdots U_{a_1} U_b U_{a_1} \cdots U_{a_n}, \tag{4}$$

for all $n > 0$, and all $a_n, \ldots, a_1, b \in M$.

An element a in a unital Jordan-Banach algebra M is called *invertible* whenever there exists $b \in M$ satisfying $a \circ b = \mathbf{1}$ and $a^2 \circ b = a$. The element b is unique and it will be denoted by a^{-1} (cf. [26, 3.2.9] and [11, Definition 4.1.2]). We know from [11, Theorem 4.1.3] that an element $a \in M$ is invertible if and only if U_a is a bijective

mapping, and in such a case $U_a^{-1} = U_{a^{-1}}$ (cf. (3)). If an associative Banach algebra A is regarded with its natural Jordan product, the notion of invertibility in the Jordan setting is precisely the usual notion in the associative context. As in the associative setting, the set M^{-1} of all invertible elements in M is open (cf. [11, Theorem 4.1.7]), and its principal component (i.e. the connected component containing the unit element) will be denoted by $M_{\mathbf{1}}^{-1}$.

Additional geometric axioms are required to define JB- and JB*-algebras. The definition of JB*-algebras is essentially due to I. Kaplansky, who introduced them as a Jordan generalization of C*-algebras. A *JB*-algebra* is a complex Jordan-Banach algebra M equipped with an algebra involution * satisfying the following geometric axiom:

$$\|U_a(a^*)\| = \|a\|^3, \quad (a \in M). \tag{5}$$

Every C*-algebra A is a JB*-algebra when equipped with its natural Jordan product and the original norm and involution. We observe that in this case the geometric axiom (5) writes in the form $\|aa^*a\| = \|a\|^3$ $(a \in A)$, which is known to be an equivalent reformulation of the Gelfand-Naimark axiom. M.A. Youngson proved in [54, Lemma 4] that the involution of every JB*-algebra is in fact an isometry.

A *JB-algebra* is a real Jordan-Banach algebra J in which the norm satisfies the following two axioms for all $a, b \in J$:

(i) $\|a^2\| = \|a\|^2$;
(ii) $\|a^2\| \leq \|a^2 + b^2\|$.

As pointed out by Kaplansky, the hermitian part, M_{sa}, of a JB*-algebra, M, is always a JB-algebra. The converse implication was open for some time, and it was shown to be true in a celebrated result due to J.D.M. Wright, asserting that the complexification of every JB-algebra is a JB*-algebra (see [51]). The reader who may feel the necessity of additional details and explanations for any of the basic results on JB-algebras, JB*-algebras and JB*-triples (whose definition appears below) can consult the monographs [26,14,1,2,11] and [12].

Let M and N be JB*-algebras. A linear mapping $\Phi : M \to N$ is called a *Jordan homomorphism* if $\Phi(a \circ b) = \Phi(a) \circ \Phi(b)$ $(a, b \in M)$, and a *Jordan *-homomorphism* if it is a Jordan homomorphism and $\Phi(a)^* = \Phi(a^*)$ for all $a \in M$. A Jordan *-isomorphism is a Jordan *-homomorphism which is also a bijection. A real linear mapping from M to N preserving Jordan products will be called a real linear Jordan homomorphism. Real linear Jordan *-homomorphisms are similarly defined. A mapping between unital JB*-algebras is called *unital* if it sends the unit in the domain to the unit in the codomain.

A JBW*-algebra is a JB*-algebra which is also a dual Banach space. Norm-closed Jordan *-subalgebras of C*-algebras are called *JC*-algebras*. JC*-algebras which are also dual Banach spaces are called *JW*-algebras*. Any JW*-algebra is a weak*-closed Jordan *-subalgebra of a von Neumann algebra. The reader should be warned about the

existence of exceptional JB*-algebras which cannot be embedded as Jordan *-subalgebras of some $B(H)$ (see [26, Corollary 2.8.5], [11, Example 3.1.56]).

We recall that an element u in a unital JB*-algebra M is a *unitary* if it is invertible and its inverse coincides with u^*. As in the associative setting, we shall denote by $\mathcal{U}(M)$ the set of all unitary elements in M. It is known that the identities

$$U_u(u^*) = u, \text{ and } U_u((u^*)^2) = \mathbf{1},$$

hold for every unitary element u in M (cf. [11, Theorem 4.1.3]). Furthermore, M becomes a unital JB*-algebra, with unit u, for the Jordan product and involution defined by

$$x \circ_u y := U_{x,y}(u^*) \text{ and } x^{*_u} := U_u(x^*), \text{ respectively.} \tag{6}$$

This new unital JB*-algebra $(M, \circ_u, *_u)$ is called the *u-isotope* of M, and will be denoted by $M(u)$ (see [11, Lemma 4.2.41(i)]). For each $a \in M$, we shall write $a^{(n,u)}$ for the n-th power of a in the JB*-algebra $M(u)$. Let us observe that if a unital C*-algebra is regarded as a JB*-algebra both notions of unitaries coincide. An element s in a unital JB-algebra J is called a *symmetry* if $s^2 = \mathbf{1}_J$. If M is a unital JB*-algebra, the symmetries in M are defined as the symmetries in its self-adjoint part M_{sa}.

In a JBW*-algebra M the set $\mathcal{U}(M)$ is path connected and coincides with the set $\{e^{ih} : h \in M_{sa}\}$ (cf. [16, Remark 3.2(9)]). However, as in the case of unital C*-algebras, in a general unital JB*-algebra the set of unitaries might contain many different connected components. The principal component of $\mathcal{U}(M)$ is the connected component containing the unit, and it will be denoted by $\mathcal{U}^0(M)$.

Beside the deep connections with holomorphic theory on arbitrary complex Banach spaces and the classification of bounded symmetric domains in this complex setting which led W. Kaup to introduce those complex Banach spaces called JB*-triples in [39], for the purposes of this note, we shall employ the JB*-triple structure associated with each JB*-algebra to simplify all possible Jordan products and involutions under a triple product which encompasses all of them. A *JB*-triple* is a complex Banach space E equipped with a continuous triple product $\{.,.,.\} : E \times E \times E \to E$, $(a, b, c) \mapsto \{a, b, c\}$, which is bilinear and symmetric in (a, c) and conjugate linear in b, and satisfies the following axioms for all $a, b, x, y \in E$:

(a) $L(a, b)L(x, y) = L(x, y)L(a, b)+L(L(a, b)x, y)-L(x, L(b, a)y)$, where $L(a, b) : E \to E$ is the operator defined by $L(a, b)x = \{a, b, x\}$;
(b) $L(a, a)$ is a hermitian operator with non-negative spectrum;
(c) $\|\{a, a, a\}\| = \|a\|^3$.

Examples of JB*-triples include all C*-algebras and JB*-algebras with the triple products of the form

$$\{x, y, z\} = \frac{1}{2}(xy^*z + zy^*x), \tag{7}$$

and

$$\{x, y, z\} = (x \circ y^*) \circ z + (z \circ y^*) \circ x - (x \circ z) \circ y^*, \tag{8}$$

respectively ([11, Fact 4.1.41 and Theorem 4.1.45]). Actually, when a C*-algebra is regarded as a JC*-algebra with its natural Jordan product, the expression in (8) coincides with the triple product in (7). The product in (7) is actually valid to define a structure of JB*-triple on spaces $B(H, K)$ of bounded linear operators between complex Hilbert spaces (infinite dimensional examples of rectangular matrices), complex Hilbert spaces, and J*-algebras in the sense introduced by L. Harris in [27]–i.e. closed complex-linear subspaces of $B(H, K)$ which are closed for the triple product in (7)–.

The triple product of every JB*-triple is a non-expansive mapping, that is,

$$\|\{a, b, c\}\| \leq \|a\|\|b\|\|c\| \quad \text{for all } a, b, c \tag{9}$$

(see [25, Corollary 3]).

A (real linear) triple homomorphism between JB*-triples E and F is a (real) linear mapping $\Phi : E \to F$ preserving triple products, i.e.,

$$T\{a, b, c\} = \{T(a), T(b), T(c)\}, \text{ for all } a, b, c \in E.$$

It is well known that every unital triple homomorphism between unital JB*-algebras is a Jordan *-homomorphism.

Among the amazing geometric properties of those complex Banach spaces in the class of JB*-triples we find a Banach-Stone type theorem, proved by W. Kaup, asserting that a linear bijection T between two JB*-triples is an isometry if and only if it is a triple isomorphism (cf. [39, Proposition 5.5], see also [11, Theorem 2.2.28] or [18] and [23] for alternative proofs). Thanks to this result we can understand now that, despite of the existence of many different Jordan products and involutions on a unital JB*-algebra, they all produce the same triple structure. Namely, if u_1 is any unitary in a unital JB*-algebra M, we know that the triple product on M given in (8) coincides with the one given by

$$\{x, y, z\}_1 = (x \circ_{u_1} y^{*_{u_1}}) \circ_{u_1} z + (z \circ_{u_1} y^{*_{u_1}}) \circ_{u_1} x - (x \circ_{u_1} z) \circ_{u_1} y^{*_{u_1}},$$

because the identity mapping is a surjective linear isometry.

It should be remarked that Kaup's Banach–Stone theorem is not, in general, true for surjective real linear isometries between JB*-triples (cf. [17, Remark 2.7], [33, §4] and [22]). However, surjective real linear isometries between C*-algebras and JB*-algebras are all real linear triple isomorphisms (see [17, Corollaries 3.2 and 3.3] and [22, Corollary 3.4]). We further know that every unital surjective real linear isometry between unital JB*-algebras is a real linear Jordan *-isomorphism [17, Corollary 3.2].

For each couple of elements a, b in a JB*-triple E, we denote by $Q(a, b)$ the conjugate linear operator on E defined by $Q(a, b)(x) = \{a, x, b\}$. We shall write $Q(a)$ for $Q(a, a)$.

In the case of a JB*-algebra M, the Q operator is intrinsically related to the U operator by the identity

$$Q(a,b)(x) = \{a,x,b\} = U_{a,b}(x^*), \quad \text{for all } x \in M.$$

We have already mentioned that the involution of every unital JB*-algebra M is an isometry. We can actually apply the previous Kaup's Banach-Stone theorem to the involution as a surjective linear isometry from M onto the Banach space M equipped with the same norm but replacing the product by scalars by the one given by $\lambda \odot x = \overline{\lambda} x$, or it can be directly checked that the involution is a (conjugate linear) triple isomorphism. Consequently, for any a and b in M, we have that

$$(U_a(b))^* = (\{a,b^*,a\})^* = \{a^*,b,a^*\} = U_{a^*}(b^*). \tag{10}$$

By an induction argument we have

$$(U_{a_n} \cdots U_{a_1}(a_0))^* = U_{a_n^*} \cdots U_{a_1^*}(a_0^*), \tag{11}$$

for all natural n and $a_n, \ldots, a_1, a_0 \in M$.

Each element e in a JB*-triple E satisfying $\{e,e,e\} = e$ is called a *tripotent*. If a C*-algebra A is regarded as a JB*-triple with the triple product in (7), the partial isometries in A are precisely the tripotents of A. Each tripotent $e \in E$ determines a *Peirce decomposition* of E in the form

$$E = E_2(e) \oplus E_1(e) \oplus E_0(e),$$

where $E_j(e) = \{x \in E : \{e,e,x\} = \frac{j}{2} x\}$ $(j = 0,1,2)$ is called the *Peirce j-subspace*.

Triple products among elements in Peirce subspaces satisfy the following *Peirce arithmetic*:

$$\{E_i(e), E_j(e), E_k(e)\} \subseteq E_{i-j+k}(e) \ \text{ if } i-j+k \in \{0,1,2\},$$
$$\{E_i(e), E_j(e), E_k(e)\} = \{0\} \text{ if } i-j+k \notin \{0,1,2\},$$

and $\{E_2(e), E_0(e), E\} = \{E_0(e), E_2(e), E\} = \{0\}$. Consequently, each Peirce subspace $E_j(e)$ is a JB*-subtriple of E.

The natural projection, $P_k(e)$, of E onto $E_k(e)$ is called the *Peirce k-projection*. It is known that $P_2(e) = Q(e)^2$, $P_1(e) = 2(L(e,e) - Q(e)^2)$, and $P_0(e) = \mathrm{Id}_E - 2L(e,e) + Q(e)^2$. Furthermore, $\|P_k(e)\| \leq 1$ for all $k = 0,1,2$ (cf. [24, Corollary 1.2]). It is worth remarking that if e is a tripotent in a JB*-triple E, the Peirce 2-subspace $E_2(e)$ is a unital JB*-algebra with unit e, product $x \circ_e y := \{x,e,y\}$ and involution $x^{*_e} := \{e,x,e\}$, respectively (cf. [11, Theorem 4.1.55]).

The adjective "unitary" is also employed to denote some special tripotents. More precisely, a tripotent e in a JB*-triple E is called *unitary* if $E_2(e) = E$. However, there

is no risk of controversy, since, as shown by R. Braun, W. Kaup and H. Upmeier in [8, Proposition 4.3], the unitaries in a unital JB*-algebra M are precisely the unitary tripotents in M when the latter is regarded as a JB*-triple. We can therefore conclude that every surjective real linear isometry T between two unital JB*-algebras M and N maps $\mathcal{U}(M)$ onto $\mathcal{U}(N)$. In particular, for each unitary $u \in M$, the mapping $U_u : M \to M$ is a triple isomorphism satisfying $U_u(\mathcal{U}(M)) = \mathcal{U}(M)$.

2. Range tripotents and the principal component of the unitary set in a JB*-algebra

The principal components of the sets of invertible elements, A^{-1}, and of the unitary group, $\mathcal{U}(A)$, of a unital C*-algebra A are the connected components of these subsets containing the unit $\mathbf{1} \in A$. These principal components will be denoted by $A_{\mathbf{1}}^{-1}$ and $\mathcal{U}^0(A)$, respectively. These two sets are well known, studied and described in the literature. More concretely, for any (associative) unital real or complex Banach algebra A, A^{-1} is an open subset of A which is stable under associative products. The principal component of A^{-1} is precisely the least subgroup of A^{-1} containing $\exp(A) = \{e^a : a \in A\}$, it is path connected, and it can be algebraically described in the following terms:

$$A_{\mathbf{1}}^{-1} = \{e^{a_1} \cdot \ldots \cdot e^{a_n} : n \in \mathbb{N},\ a_1, \ldots, a_n \in A\}$$

(see [7, Propositions 8.6 and 8.7], [50, page 12] and [42, Theorem 2.3.1]).

The principal component of the unitary group of a unital C*-algebra A has been also studied and described by different authors. It is known that

$$\begin{aligned} \mathcal{U}^0(A) &= \{e^{ih_1} \cdot \ldots \cdot e^{ih_n} : n \in \mathbb{N},\ h_1, \ldots, h_n \in A_{sa}\} \\ &= \{e^{ih_1} \cdot \ldots \cdot e^{ih_n} \mathbf{1} e^{ih_n} \cdot \ldots \cdot e^{ih_1} : n \in \mathbb{N},\ h_1, \ldots, h_n \in A_{sa}\} \end{aligned} \tag{12}$$

is open, closed, and path connected, in the (relative) norm topology on $\mathcal{U}(A)$ –the first equality in (12) can be found in Kadison–Ringrose book [38, Exercises 4.6.6 and 4.6.7], while the second equality, which is closer to the Jordan structure, is due to O. Hatori [28, Lemma 3.2]. Another interesting result asserts that, by the continuity of the module mapping $x \mapsto |x| := (x^*x)^{\frac{1}{2}}$, we have

$$A_{\mathbf{1}}^{-1} \cap \mathcal{U}(A) = \mathcal{U}^0(A) \quad \text{(see [50, Exercise 2, page 56])}.$$

The reader should be warned that "polar decompositions" are not available for elements in a JB*-algebra, so these arguments are not valid in the Jordan setting. To the best of our knowledge, in the setting of real C*-algebras a full description of the principal component of the unitary group remains unknown.

In the case of a unital C*-algebra A, the sets A^{-1}, $A_{\mathbf{1}}^{-1}$, $\mathcal{U}(A)$, $\mathcal{U}^0(A)$ are closed for the associative product but they are not, in general, stable under Jordan products.

In the case of Jordan-Banach algebras, B. Aupetit posed the question whether the principal component of the set of invertible elements in a Jordan-Banach algebra is analytically arcwise connected in [4, comments after the proof of Theorem 3.1]. A complete positive solution to this question was found by O. Loos in [43] (see also [11, Theorem 4.1.111]). However, as long as we know the description of the principal component of the set of unitaries in a unital JB*-algebras has not been explicitly treated in the literature. By employing some tools in the recent note [16], we shall complete the state-of-the-art in the Jordan setting. One of the main geometric properties derived from the local Gelfand theory in C*-algebras assures that for each unitary u in a unital C*-algebra A with $\|\mathbf{1} - u\| < 2$ we can always find a hermitian element $h \in A_{sa}$ satisfying $u = e^{ih}$ (see [38, Exercise 4.6.6]). Building upon the celebrated *Shirshov-Cohn theorem*, which proves that the JB*-subalgebra of a JB*-algebra generated by two self-adjoint elements (and the unit element) is a JC*-algebra, that is, a JB*-subalgebra of some $B(H)$ (cf. [26, Theorem 7.2.5] and [51, Corollary 2.2]), the just commented result has been extended to the setting of unital JB*-algebras in [16, Lemma 2.2].

Let M be a unital JB*-algebra. Let us consider a unitary element u in M, and the u-isotope $M(u) := (M, \circ_u, *_u)$ as defined in subsection 1.1. It is known that $\mathcal{U}(M) = \mathcal{U}(M(u))$ (cf. [16, Lemma 2.1], [49, Lemma 4.2(ii) and Theorem 4.6] or [11, Lemma 4.2.41(ii)]). Let us observe that the unital JB*-algebras M and $M(u)$ share the same underlying Banach spaces, the quoted unitary sets are equipped with the same norm and distance. Let us suppose that u lies in $\mathcal{U}^0(M)$. Since the connected components of a topological space are disjoint, we can affirm that

$$\mathcal{U}^0(M(u)) = \mathcal{U}^0(M), \text{ for every } u \in \mathcal{U}^0(M). \tag{13}$$

Section 3 below is devoted to studying and describing the surjective isometries between the principal components of the unitary sets of two unital JB*-algebras. For this purpose, we shall first establish an algebraic characterization of the set of unitary elements in terms of U operators of exponentials of skew symmetric elements, like the one commented in the case of unital C*-algebras (cf. (12)).

Let us recall the local Gelfand theory to the readers. In a C*-algebra, the C*-subalgebra A generated by a non-normal element cannot be identified, via Gelfand theory, with a commutative C*-algebra of the form $C_0(L)$, for a locally compact Hausdorff space L. However, if we regard A as a JB*-triple, the JB*-subtriple generated by a single element is always representable as a $C_0(L)$ space. Let $0 \neq a$ be an element in a JB*-triple E. The JB*-subtriple of E generated by a (denoted by E_a) coincides with the norm closure of the linear span of the odd powers of a defined as $a^{[1]} = a$, $a^{[3]} = \{a, a, a\}$, and $a^{[2n+1]} := \left\{a, a, a^{[2n-1]}\right\}$, $(n \in \mathbb{N})$. The local Gelfand theory for JB*-triples assures that E_a is isometrically JB*-triple isomorphic to some $C_0(\Omega_a)$ for a unique compact Hausdorff space Ω_a contained in the set $[0, \|a\|]$, such that 0 cannot be an isolated point in Ω_a, where, along this note, the symbol $C_0(\Omega_a)$ will stand for the Banach space of all complex-valued continuous functions on Ω_a vanishing at 0 in case that $0 \in \Omega_a$, and

all continuous functions on Ω_a otherwise. It is further known that there exists a triple isomorphism $\Psi : E_a \to C_0(\Omega_a)$ satisfying $\Psi(a)(t) = t$ $(t \in \Omega_a)$ (cf. [40, Lemma 3.2, Corollary 3.4 and Proposition 3.5], [39, Corollary 1.15], see also [11, Theorem 4.2.9] or [14, Theorem 3.1.12]). The set Ω_a is called *the triple spectrum* of a (in E), and it does not change when computed with respect to any JB*-subtriple $F \subseteq E$ containing a [40, Proposition 3.5]. In some reference, like [14, Theorem 3.1.12] and [11, Theorem 4.2.9], the local Gelfand theory is built upon the locally compact set $\Omega\backslash\{0\}$. Here we follow the notation in [40], where, in a more natural terminology, the triple spectrum is a compact set.

As in the associative setting, the local Gelfand theory opens the door to apply a *continuous triple functional calculus*. Given an element a in a JB*-triple E, let Ω_a and $\Psi : E_a \to C_0(\Omega_a)$ denote the triple spectrum of a and the triple isomorphism given in the previous paragraph. For each continuous function $f \in C_0(\Omega_a)$, we set $f_t(a) := \Psi^{-1}(f)$, and we call it the *continuous triple functional calculus* of f at a. If $p(\lambda) = \alpha_1\lambda + \alpha_3\lambda^3 + \ldots + \alpha_{2n-1}\lambda^{2n-1}$ is an odd polynomial with complex coefficients, it is easy to check that $p_t(a) = \alpha_1 a + \alpha_3 a^{[3]} + \ldots + \alpha_{2n-1}a^{[2n-1]}$. However, for more general functions in $C_0(\Omega_a)$ the continuous triple functional calculus is not so obvious. For example if a is a non-self-adjoint element in a C*-algebra A, the element $a^{[2]} = h_t(a)$ with $h(t) = t^2$ does not coincide with a^2 in A. Another example: the function $g(t) = \sqrt[3]{t}$ produces $g_t(a) = a^{[\frac{1}{3}]}$. Let us note that $g_t(a) = a^{[\frac{1}{3}]}$ is the unique *cubic root* of a in E_a, i.e. the unique element $a^{[\frac{1}{3}]} \in E_a$ satisfying

$$\left\{a^{[\frac{1}{3}]}, a^{[\frac{1}{3}]}, a^{[\frac{1}{3}]}\right\} = a. \tag{14}$$

The sequence $(a^{[\frac{1}{3^n}]})_n$ can be recursively defined by $a^{[\frac{1}{3^{n+1}}]} = \left(a^{[\frac{1}{3^n}]}\right)^{[\frac{1}{3}]}$, $n \in \mathbb{N}$. It is easy to see that the sequence $(a^{[\frac{1}{3^n}]})_n$ need not be, in general, norm convergent in E (consider, for example, the element $a(t) = t$ in $E = C([0,1])$).

The functional analysis provides another topology to assure the convergence of the above sequence. Let us recall that a *JBW*-triple* is a JB*-triple which is also a dual Banach space (with a unique isometric predual [5]). The triple product of every JBW*-triple is separately weak*-continuous (cf. [5]), and the second dual, E^{**}, of a JB*-triple E is always a JBW*-triple under a triple product which extends the original triple product in E (cf. [12, Proposition 5.7.10]).

If W is a JBW*-triple (in particular when we regard E as a JB*-subtriple of E^{**}), the sequence $(a^{[\frac{1}{3^n}]})_n$ converges in the weak*-topology of W to a (unique) tripotent denoted by $r(a)$. The tripotent $r(a)$ is called the *range tripotent* of a in W. This range tripotent $r(a)$ can be characterized as the smallest tripotent $e \in W$ satisfying that a is positive in the JBW*-algebra $W_2(e)$ (compare [20, Lemma 3.3]).

Given an element a in a JB*-triple E, we shall denote by $r(a)$ the range tripotent of a in E^{**}. There exist examples in which we can guarantee that the element $r(a)$ lies in E. According to the usual notation followed in [10,21,40,37], an element a in a JB*-triple E

is called *von Neumann regular* if $a \in Q(a)(E)$; if $a \in Q(a)^2(E)$ we say that a is *strongly von Neumann regular*. For each von Neumann regular element a in a JB*-triple E there might exist many different elements c in E satisfying $Q(a)(c) = a$. However, it is known (cf. [21, Theorem 1] and [40, Lemma 4.1, Lemma 3.2 and comments after its proof], [10, Theorem 2.3 and Corollary 2.4]) that the following statements are equivalent:

(a) a is von Neumann regular;
(b) There exists $b \in E$ such that $Q(a)(b) = a$, $Q(b)(a) = b$ and $[Q(a), Q(b)] := Q(a)\,Q(b) - Q(b)\,Q(a) = 0$;
(c) 0 does not belong to the triple spectrum, Ω_a, of a;
(d) $Q(a)$ has norm-closed range;
(e) The range tripotent $r(a)$ of a lies in E and a is positive and invertible in the JB*-algebra $E_2(r(a))$.

The element b appearing in statement (b) above is unique, and it will be called the *generalized inverse* of a in E (denoted by $a^\dagger$). Henceforth we shall write $\mathrm{Reg}_{vN}(E)$ for the set of all von Neumann regular elements in E. It is further known that $L(a, a^\dagger) = L(a^\dagger, a) = L(r(a), r(a))$ and $Q(a)Q(a^\dagger) = Q(a^\dagger)Q(a) = P_2(r(a))$ (see, for example, [37, page 589], [10, comments in page 192] or [41, Lemma 3.2 and subsequent comments]).

If a is an invertible element in a unital JB*-algebra M and the latter is regarded as a JB*-triple, then a is von Neumann regular and its range tripotent, $r(a)$, is a unitary element in M, that is,

$$a \in M^{-1} \Rightarrow r(a) \in \mathcal{U}(M) \quad \text{(cf. [37, Lemma 2.2 and Remark 2.3])}. \tag{15}$$

Namely, in this case, by denoting z^{-1} the inverse of z in M we have

$$\begin{aligned}
Q(z)((z^{-1})^*) &= \{z, (z^{-1})^*, z\} = U_z(z^{-1}) = z;\\
Q((z^{-1})^*)(z) &= \{(z^{-1})^*, z, (z^{-1})^*\} = U_{(z^{-1})^*}(z^*) = (z^{-1})^*;\\
Q(z)Q((z^{-1})^*)(x) &= U_z\left(U_{(z^{-1})^*}(x^*)\right)^* = U_z U_{z^{-1}}(x) = U_z U_z^{-1}(x) = x;\\
Q((z^{-1})^*)Q(z)(x) &= U_{(z^{-1})^*}\left(U_z(x^*)\right)^* = U_{(z^*)^{-1}} U_{z^*}(x) = U_{z^*}^{-1} U_{z^*}(x) = x,
\end{aligned}$$

for all $x \in M$, which proves that z is von Neumann regular with $z^\dagger = (z^{-1})^* = (z^*)^{-1}$, and since $P_2(r(z)) = Q(z)Q(z^\dagger) = Id$, the tripotent $r(z)$ is a unitary in M.

We explore next certain properties of the continuous triple functional calculus which might result surprising. Let us fix an element a in a JB*-triple E. Let us consider the isometric triple isomorphism $\Psi : E_a \to C_0(\Omega_a)$ satisfying $\Psi(a)(t) = t$ ($t \in \Omega_a$). The function $g(t) = t^2$ lies in $C_0(\Omega_a)$, and hence $g_t(a) = a^{[2]}$ is an element in E_a. The reader should be warned that $a^{[2]}$ is only a square for a local binary product depending on the element a. Suppose, additionally, that a is von Neumann regular, that is, $\Omega_a \subset \mathbb{R}^+$. In this case the function $h(t) = \frac{1}{t}$ lies in $C_0(\Omega_a)$ and $a^{[-1]} = h_t(a) \in E_a$. It is easy to see that

$$\{a, a, a^{[-1]}\} = a, \{a, r(a), a^{[-1]}\} = r(a), \text{ and } \{r(a), a^{[-1]}, r(a)\} = a^{[-1]}.$$

By definition $a^{[2]}$ belongs to the subtriple of E generated by a. Since the range tripotent of a satisfies that $a \in E_2(r(a))$, it follows that $E_a \subseteq E_2(r(a))$ because the latter is a JB*-subtriple of E. The elements $a, a^{[-1]}$, $a^{[2]}$ and $a^\dagger$ belong to the unital JB*-algebra $E_2(r(a))$. Having in mind that, by Kaup's theorem, when the triple product of E is restricted to $E_2(r(a))$ it coincides with

$$\{x, y, z\} = (x \circ_{r(a)} y^{*_{r(a)}}) \circ_{r(a)} z + (z \circ_{r(a)} y^{*_{r(a)}}) \circ_{r(a)} x - (x \circ_{r(a)} z) \circ_{r(a)} y^{*_{r(a)}},$$

for all $x, y, z \in E_2(r(a))$. Then

$$r(a) = \{a^\dagger, a, r(a)\} = L(a^\dagger, a)(r(a)) = a^\dagger \circ_{r(a)} a,$$

$$\begin{aligned} \text{and } a &= L(a^\dagger, a)(a) = \{a^\dagger, a, a\} \\ &= (a^\dagger \circ_{r(a)} a^{*_{r(a)}}) \circ_{r(a)} a + (a \circ_{r(a)} a^{*_{r(a)}}) \circ_{r(a)} a^\dagger - (a^\dagger \circ_{r(a)} a) \circ_{r(a)} a^{*_{r(a)}} \\ &= (a \circ_{r(a)} a) \circ_{r(a)} a^\dagger. \end{aligned}$$

Therefore $a^\dagger = a^{[-1]}$.

We make a pause at the next Jordan version of [50, Proposition I.4.10]. The proof provides the readers with a better understanding of the continuous triple functional calculus.

Proposition 2.1. *Let $\Omega \subset \mathbb{R}_0^+$ be a compact set, and let E be a JB*-triple. Suppose $\mathcal{E}_\Omega$ denotes the set of all elements $a \in E$ with $\Omega_a \subseteq \Omega$. If $f : \Omega \to \mathbb{C}$ is a continuous function, then the continuous triple functional calculus: $a \in \mathcal{E}_\Omega \mapsto f_t(a) \in E$ is (norm) continuous. Consequently, the mapping $a \in E \mapsto a^{[2]} \in E$ is continuous.*

Proof. A variant of the Stone-Weierstrass theorem established in [39] and [11, Lemma 4.2.8] assures that if $\iota : \Omega \to \mathbb{C}$ denotes the inclusion mapping, then the set

$$\{p(\iota^2)\iota : p \text{ a polynomial with complex coefficients}\}$$

is dense in $C_0(\Omega)$. Therefore, given $\varepsilon > 0$ there exists a polynomial p with complex coefficients such that for $q(\lambda) = p(\lambda^2)\lambda$ we have $\sup_{t\in\Omega} |q(t) - f(t)| < \varepsilon$. By the continuity of q –via the continuity of the triple product– we can find $\delta > 0$ such that $\|q_t(a) - q_t(b)\| < \varepsilon$ for all $a, b \in E$ with $\|a - b\| < \delta$ and $\|a\|, \|b\| \leq \max \Omega$. Taking $a, b \in \mathcal{E}_\Omega$ with $\|a - b\| < \delta$ we have

$$\|f_t(a) - f_t(b)\| \leq \|f_t(a) - q_t(a)\| + \|q_t(a) - q_t(b)\| + \|q_t(b) - f_t(b)\| < 3\varepsilon.$$

The last statement is clear. □

The next corollary is our tool to avoid polar decompositions in the Jordan setting.

Corollary 2.2. *Let M be a unital JB*-algebra. Then the mapping*

$$a \in M^{-1} \mapsto r(a) \in \mathcal{U}(M) \text{ is continuous.}$$

Proof. It is not hard to check, from the properties of the continuous triple functional calculus, that for each von Neumann regular element $a \in E$ the identity $r(a) = Q(a^\dagger)(a^{[2]})$ holds.

We recall that the set of invertible elements in a unital Jordan-Banach algebra M is an open set, and the mapping $a \in M^{-1} \mapsto a^{-1} \in M$ is continuous (cf. [11, Theorem 4.1.7 and Proposition 4.1.6]). Therefore, if M is a unital JB*-algebra, the mapping $a \in M^{-1} \mapsto a^\dagger = (a^{-1})^* \in M^{-1}$ is continuous. Consequently, by the continuity of the triple product, the mapping

$$a \in M^{-1} \mapsto r(a) = Q(a^\dagger)(a^{[2]})$$

is continuous. Let us finally note that $r(a) \in \mathcal{U}(M)$ for all $a \in M^{-1}$ (cf. (15)). □

We have already employed the notion of u-isotope associated with each unitary element u in a unital JB*-algebra M. This device also makes sense for invertible elements in M. More concretely, for each invertible element c in a unital Jordan–Banach algebra M then the vector space M becomes a Jordan–Banach algebra $M_{(c)}$ with unit element c^{-1} and quadratic operators

$$U_a^{(c)} = U_a U_c. \tag{16}$$

The linear Jordan product in $M_{(c)}$ is actually given by $x \circ_c y = x \circ (c \circ y) + y \circ (c \circ x) - c \circ (x \circ y)$ (cf. [36, §2, 1.7]). We should note that, according to the notation introduced in page 47, $M(w) = M_{(w^*)}$.

The algebraic characterization of the principal component of the set of unitaries in a unital JB*-algebra reads as follows.

Theorem 2.3. *Let M be a unital JB*-algebra, let $\mathcal{U}^0(M)$ denote the principal component of the set of unitaries in M and let u be an element in $\mathcal{U}(M)$. Then the following statements are equivalent:*

(a) $u \in M_{\mathbf{1}}^{-1} \cap \mathcal{U}(M)$;
(b) *There exists a continuous path* $\Gamma : [0,1] \to \mathcal{U}(M)$ *with* $\Gamma(0) = \mathbf{1}$ *and* $\Gamma(1) = u$;
(c) $u \in \mathcal{U}^0(M)$;
(d) $u = U_{e^{ih_n}} \cdots U_{e^{ih_1}}(\mathbf{1})$, *for some* $n \in \mathbb{N}$, $h_1, \ldots, h_n \in M_{sa}$;
(e) *There exists* $w \in \mathcal{U}^0(M)$ *such that* $\|u - w\| < 2$.

Consequently,

$$\begin{aligned}\mathcal{U}^0(M) &= M_{\mathbf{1}}^{-1} \cap \mathcal{U}(M) \\ &= \{U_{e^{ih_n}} \cdots U_{e^{ih_1}}(\mathbf{1}) \colon n \in \mathbb{N},\ h_j \in M_{sa}\ \forall\ 1 \leq j \leq n\} \\ &= \left\{u \in \mathcal{U}(M) :\ \textit{there exists } w \in \mathcal{U}^0(M) \textit{ with } \|u - w\| < 2\right\}\end{aligned} \tag{17}$$

is analytically arcwise connected.

Proof. $(a) \Rightarrow (b)$ Suppose $u \in M_{\mathbf{1}}^{-1} \cap \mathcal{U}(M)$. Since by [43, Corollary], the set $M_{\mathbf{1}}^{-1}$ is analytically arcwise connected, there exists a continuous path $\gamma : [0,1] \to M_{\mathbf{1}}^{-1}$ satisfying $\gamma(0) = \mathbf{1}$ and $\gamma(1) = u$. Corollary 2.2 assures that the mapping $\Gamma : [0,1] \to \mathcal{U}(M)$, $\Gamma(t) := r(\gamma(t))$ is continuous with $\Gamma(0) = \mathbf{1}$ and $\Gamma(1) = u$.

The implication $(b) \Rightarrow (c)$ is clear.

$(c) \Rightarrow (a)$ It follows from $(a) \Rightarrow (b) \Rightarrow (c)$ that $M_{\mathbf{1}}^{-1} \cap \mathcal{U}(M) \subseteq \mathcal{U}^0(M)$. We know from [43, Corollary] that $M_{\mathbf{1}}^{-1}$ is a clopen subset of M^{-1}, and thus $M_{\mathbf{1}}^{-1} \cap \mathcal{U}(M)$ is a clopen subset of $\mathcal{U}(M)$ which contains the unit and is contained in $\mathcal{U}^0(M)$. Having in mind that $\mathcal{U}^0(M)$ is a connected set, we deduce that $M_{\mathbf{1}}^{-1} \cap \mathcal{U}(M) = \mathcal{U}^0(M)$.

We have proved that statements (a), (b) and (c) are equivalent and $\mathcal{U}^0(M) = M_{\mathbf{1}}^{-1} \cap \mathcal{U}(M)$ is analytically arcwise connected.

$(b) \Rightarrow (d)$ Suppose there exists a continuous path $\Gamma : [0,1] \to \mathcal{U}(M)$ with $\Gamma(0) = \mathbf{1}$ and $\Gamma(1) = u$. Let us find, by continuity and compactness, $0 = t_0 < t_1 < \ldots < t_n < 1 = t_{n+1}$ such that $\|\Gamma(t_i) - \Gamma(t_{i+1})\| < 2$ for all $0 \leq i \leq n$. Set $u_i = \Gamma(t_i)$ for all $0 \leq i \leq n+1$. By applying [16, Lemma 2.2] to the element u_1 –which satisfies $\|u_1 - \mathbf{1}\| < 2$– we deduce the existence of $h_1 \in M_{sa}$ such that $U_{e^{ih_1}}(\mathbf{1}) = u_1$. Since $2 > \|u_1 - u_2\| = \|\mathbf{1} - U_{e^{-ih_1}}(u_2)\|$, a new application of [16, Lemma 2.2] shows the existence of $h_2 \in M_{sa}$ satisfying $U_{e^{-ih_1}}(u_2) = U_{e^{ih_2}}(\mathbf{1})$, or equivalently, $u_2 = U_{e^{ih_1}} U_{e^{ih_2}}(\mathbf{1})$. Suppose, by an induction argument on n, that we have found $h_1, \ldots, h_n \in M_{sa}$ with $u_n = U_{e^{ih_1}} \cdots U_{e^{ih_n}}(\mathbf{1})$. As before, the condition

$$2 > \|u_{n+1} - u_n\| = \|U_{e^{-ih_n}} \cdots U_{e^{-ih_1}}(u_{n+1}) - \mathbf{1}\|,$$

implies, via [16, Lemma 2.2], the existence of $h_{n+1} \in M_{sa}$ such that $u = u_{n+1} = U_{e^{ih_1}} U_{e^{ih_2}} \cdots U_{e^{ih_{n+1}}}(\mathbf{1})$.

The implication $(d) \Rightarrow (c)$ is also easy because given $u = U_{e^{ih_n}} \cdots U_{e^{ih_1}}(\mathbf{1})$, with $n \in \mathbb{N}$, $h_1, \ldots, h_n \in M_{sa}$, the mapping $\Gamma : [0,1] \to \mathcal{U}(M) \subseteq M^{-1}$, $\Gamma(t) = U_{e^{ith_n}} \cdots U_{e^{ith_1}}(\mathbf{1})$ is an analytic curve with $\Gamma(0) = \mathbf{1}$ and $\Gamma(1) = u$. This also proves the second equality in (17).

The implication $(c) \Rightarrow (e)$ is clear.

Suppose finally that statement (e) holds, that is, there exists $w \in \mathcal{U}^0(M)$ satisfying $\|w - u\| < 2$. By the first two equalities in (17), $w = U_{e^{ih_n}} \cdots U_{e^{ih_1}}(\mathbf{1})$ for some $h_1, \ldots, h_n$ in M_{sa}. A similar argument to that in the proof of $(b) \Rightarrow (d)$ implies that $u = U_{e^{ih_{n+1}}} U_{e^{ih_n}} \cdots U_{e^{ih_1}}(\mathbf{1}) \in \mathcal{U}^0(M)$. □

In case that u is a unitary in an associative unital C*-algebra A, the left multiplication operator $L_u : A \to A$ is a surjective linear isometry mapping the unit to u. Consequently, $L_u(\mathcal{U}^0(A)) = u\,\mathcal{U}^0(A)$ is precisely the connected component of $\mathcal{U}(A)$ containing u. However, the left multiplication operator admits no direct Jordan analogue. We shall see next that assuming that a unitary element u in a unital JB*-algebra M writes as the square of another unitary, the connected component containing it can be described.

Remark 2.4. Let u be a unitary element in a unital JB*-algebra M. We cannot always guarantee that $u = e^{ih}$ for some $h \in M_{sa}$ (cf. [38, Exercise 4.6.9]). Suppose that u satisfies the weaker condition that $u = v^2$ for some unitary $v \in M$ –this automatically holds if $u = e^{ih}$. Let $\mathcal{U}^c(M)$ denote the connected component of $\mathcal{U}(M)$ containing u. Since the mapping $U_v : M \to M(u)$ is a unital surjective linear isometry –and hence a Jordan *-isomorphism between these two algebras– $U_v(\mathcal{U}^0(M))$ is a connected component of $\mathcal{U}(M) = \mathcal{U}(M(u))$ and contains u. It then follows from Theorem 2.3 that

$$\mathcal{U}^c(M) = U_v(\mathcal{U}^0(M)) = \{U_v U_{e^{ih_n}} \cdots U_{e^{ih_1}}(\mathbf{1}) \colon n \in \mathbb{N},\ h_j \in M_{sa}\ \forall\ 1 \leq j \leq n\}\,.$$

The existence of a unitary square root does not necessarily hold for any unitary element, even in the setting of commutative C*-algebras. Let $\mathbb{T}$ denote the unit sphere of the complex plane. The principal component of $\mathcal{U}(C(\mathbb{T}))$ is precisely the subgroup $\exp(iC_{\mathbb{R}}(\mathbb{T}))$, which is the set of all functions $u : \mathbb{T} \to \mathbb{T}$ which are deformable or homotopic to the unit element (cf. [38, Exercise 4.6.7]). The quotient group $\mathcal{U}(C(\mathbb{T}))/\mathcal{U}^0(C(\mathbb{T})) = \mathcal{U}(C(\mathbb{T}))/\exp(iC_{\mathbb{R}}(\mathbb{T}))$ –known as the Bruschlinsky group– identifies with the ring $\mathbb{Z}$ of integers (see [34, §II.3] or [50, Exercise I.11.3]). It is known that the maps $u_1, u_2 : \mathbb{T} \to \mathbb{T}$, $u_1(\lambda) = \lambda$ and $u_2(\lambda) = \lambda^2$ are not in the principal component of $\mathcal{U}(C(\mathbb{T}))$, they are actually in two different connected components (cf. [34, §II, Lemma 3.2]). Therefore, $u_2 \notin \exp(iC_{\mathbb{R}}(\mathbb{T}))$ but $u_2 = u_1^2$ admits a unitary square root.

The hypothesis assuring that a unitary element u in a JB*-algebra M admits a square root has been already considered in the literature, for example, in [8, Problem 5.1 and Lemma 5.2], [30], [13] and [44]. The good behavior exhibited by unital C*-algebras, where the connected components of the unitary group can be derived from the principal component by just multiplying by a determined unitary on the left or on the right, fails in the setting of unital JB*-algebras, not only due to the lacking of a left (or right) multiplication operator, but because of the possibility that two different connected components of the unitary set of a unital JB*-algebra are not isometric as metric spaces (see Remark 3.7).

Remark 2.5. Let w be a unitary element in a unital JB*-algebra M. Suppose $\mathcal{U}^c(M)$ is a connected component of $\mathcal{U}(M)$, $w \in \mathcal{U}^c(M)$ and M_w^{-1} is the connected component of M^{-1} containing w. Let $M(w)$ denote the w-isotope admitting w as unit element. Since $\mathcal{U}(M) = \mathcal{U}(M(w))$ and $M^{-1} = (M(w))^{-1}$ as sets (cf. [16, Lemma 2.1]), $\mathcal{U}^c(M)$ and

M_w^{-1} are the principal components of $\mathcal{U}(M(w))$ and $(M(w))^{-1}$, respectively. Therefore the previous Theorem 2.3 applied to the w-isotope $M(w)$ gives

$$\begin{aligned}\mathcal{U}^c(M) &= M_w^{-1} \cap \mathcal{U}(M) \\ &= \left\{ U^{(w)}_{e^{ih_n}_w} \cdots U^{(w)}_{e^{ih_1}_w}(w) \colon n \in \mathbb{N},\ h_j \in (M(w))_{sa},\ \forall\ 1 \leq j \leq n \right\} \qquad (18) \\ &= \{u \in \mathcal{U}(M) :\ \text{there exists } v \in \mathcal{U}^c(M) \text{ with } \|u - v\| < 2\}\end{aligned}$$

is analytically arcwise connected. We can actually take any $\widetilde{w} \in \mathcal{U}^c(M)$ and the same conclusion holds for the connected component $\mathcal{U}^c(M)$ and the $\widetilde{w}$-isotope $M(\widetilde{w})$.

It will be useful to establish several further consequences of the previous Theorem 2.3.

According to [46], given a unital JB*-algebra M, a subset $\mathfrak{M} \subseteq M^{-1}$ will be called a *quadratic subset* if $U_{\mathfrak{M}}(\mathfrak{M}) \subseteq \mathfrak{M}$. Let us observe that $\mathcal{U}(M)$ is a self-adjoint subset of M.

Proposition 2.6. *Let M be a unital JB*-algebra. Then $\mathcal{U}^0(M)$ is a self-adjoint quadratic subset of $\mathcal{U}(M)$, that is, for all $u, w \in \mathcal{U}^0(M)$ the elements w^* and $U_w(u)$ belong to $\mathcal{U}^0(M)$. Furthermore, $\mathcal{U}^0(M)$ is the smallest quadratic subset of $\mathcal{U}(M)$ containing the set $e^{iM_{sa}}$. In particular,*

$$U_{\mathcal{U}^0(M)}\left(\mathcal{U}^0(M)\right) = \mathcal{U}^0(M) = \left\{ U_{e^{ih_n}} \cdots U_{e^{ih_1}}(e^{ih_0}) \colon n \in \mathbb{N},\ h_j \in M_{sa}\ \forall\ 0 \leq j \leq n \right\}.$$

Proof. Since $u, w \in \mathcal{U}^0(M)$, Theorem 2.3 guarantees the existence of hermitian elements $h_m, \ldots, h_1, k_n, \ldots, k_1 \in M_{sa}$ such that $w = U_{e^{ih_m}} \cdots U_{e^{ih_1}}(\mathbf{1})$ and $u = U_{e^{ik_n}} \cdots U_{e^{ik_1}}(\mathbf{1})$ with $n, m \in \mathbb{N}$. In what concerns the involution we have

$$w^* = (U_{e^{ih_m}} \cdots U_{e^{ih_1}}(\mathbf{1}))^* = (\text{by (11)}) = U_{e^{-ih_m}} \cdots U_{e^{-ih_1}}(\mathbf{1}),$$

which proves that $w^* \in \mathcal{U}^0(M)$ (cf. Theorem 2.3).

Concerning the remaining property, we apply the identity in (4) to deduce that

$$\begin{aligned}U_w(u) &= U_w\left(U_{e^{ik_n}} \cdots U_{e^{ik_1}}(\mathbf{1})\right) = U_{U_{e^{ih_m}} \cdots U_{e^{ih_1}}(\mathbf{1})}\left(U_{e^{ik_n}} \cdots U_{e^{ik_1}}(\mathbf{1})\right) \\ &= U_{e^{ih_m}} \cdots U_{e^{ih_1}} U_{\mathbf{1}} U_{e^{ih_1}} \cdots U_{e^{ih_m}}\left(U_{e^{ik_n}} \cdots U_{e^{ik_1}}(\mathbf{1})\right) \\ &= U_{e^{ih_m}} \cdots U_{e^{ih_1}} U_{e^{ih_1}} \cdots U_{e^{ih_m}} U_{e^{ik_n}} \cdots U_{e^{ik_1}}(\mathbf{1}),\end{aligned}$$

which in view of Theorem 2.3 assures that $U_w(u) \in \mathcal{U}^0(M)$.

Fix an arbitrary w in $\mathcal{U}^0(M)$. Since U_w is an invertible mapping with inverse U_{w^*}, the above properties show that $w^* \in \mathcal{U}^0(M)$ and

$$\mathcal{U}^0(M) = U_w U_{w^*}(\mathcal{U}^0(M)) \subseteq U_w(\mathcal{U}^0(M)) \subseteq \mathcal{U}^0(M).$$

Finally, suppose that $\mathcal{V}$ is a quadratic subset of $\mathcal{U}(M)$ containing the set $e^{iM_{sa}}$. Clearly for each $h_1, \ldots, h_n$ in M_{sa} an induction argument shows that

$$U_{e^{ih_n}} \ldots U_{e^{ih_1}}(\mathbf{1}) \in \mathcal{V},$$

and thus, by Theorem 2.3, $\mathcal{U}^0(M) \subseteq \mathcal{V}$. The rest is clear. □

We conclude this section with a remark and an open problem.

Remark 2.7. Let M be a unital JB*-algebra and let u be an element in $\mathcal{U}(M)\backslash\mathcal{U}^0(M)$. Suppose $\mathcal{U}^c(M)$ is the connected component of $\mathcal{U}(M)$ containing u. The set

$$\mathcal{E}_u := \left\{ U_{e^{ih_1}} \ldots U_{e^{ih_n}}(u) : \begin{array}{c} n \in \mathbb{N},\ h_j \in M_{sa} \\ \text{for all } 1 \leq j \leq n \end{array} \right\}$$

is arcwise connected and contains u. Thus, $\mathcal{E}_u \subseteq \mathcal{U}^c(M)$. We do not know if the reverse inclusion holds, in general. If $M = A$ is a unital C*-algebra both inclusions hold, a fact which is essentially due to the existence of left and right multiplication operators. Namely, since $\mathcal{U}^0(A) = \{e^{ih_1} \cdot \ldots \cdot e^{ih_n} : n \in \mathbb{N},\ h_1, \ldots, h_n \in A_{sa}\}$ (cf. (12)), and the right multiplication operator $R_u : A \to A$, $x \mapsto xu$ is a surjective linear isometry mapping the unit to u, we deduce that

$$\left\{ e^{ih_1} \cdot \ldots \cdot e^{ih_n} u : n \in \mathbb{N},\ h_1, \ldots, h_n \in A_{sa} \right\} = R_u(\mathcal{U}^0(A)) = \mathcal{U}^c(A).$$

Similarly, the products of the form $we^{ih_n} \cdot \ldots \cdot e^{ih_1}$ lie in $\mathcal{U}^c(A)$ for all $w \in \mathcal{U}^c(A)$, $h_1, \ldots, h_n \in A_{sa}$. So, given $h_1, \ldots, h_n \in A_{sa}$, the mapping $R_{e^{ih_1} \cdot \ldots \cdot e^{ih_n}}$ is a surjective linear isometry on A mapping $\mathcal{U}^c(A)$ onto itself, and hence

$$\begin{aligned} \mathcal{U}^c(A) &= \left\{ e^{ih_1} \cdot \ldots \cdot e^{ih_n} u e^{ih_n} \cdot \ldots \cdot e^{ih_1} : n \in \mathbb{N},\ h_1, \ldots, h_n \in A_{sa} \right\} \\ &= \left\{ U_{e^{ih_1}} \ldots U_{e^{ih_n}}(u) : \begin{array}{c} n \in \mathbb{N},\ h_j \in A_{sa} \\ \text{for all } 1 \leq j \leq n \end{array} \right\}. \end{aligned}$$

Despite we do not know if the equality $\mathcal{E}_u = \mathcal{U}^c(M)$ holds in general, the inclusion $\mathcal{E}_u \subseteq \mathcal{U}^c(M)$ is enough to prove the following rule concerning the U-operators

$$U_{\mathcal{U}^0(M)}(\mathcal{U}^c(M)) = \mathcal{U}^c(M), \text{ for all connected component } \mathcal{U}^c(M) \text{ of } \mathcal{U}(M). \tag{19}$$

Indeed, by Theorem 2.3, every $w \in \mathcal{U}^0(M)$ is of the form $w = U_{e^{ih_n}} \cdots U_{e^{ih_1}}(\mathbf{1})$ with $n \in \mathbb{N}$, $h_j \in M_{sa}$. Therefore, given $u \in \mathcal{U}^c(M)$, it follows from (4) that

$$U_w(u) = U_{U_{e^{ih_n}} \cdots U_{e^{ih_1}}(\mathbf{1})}(u) = U_{e^{ih_n}} \ldots U_{e^{ih_1}} U_{e^{ih_1}} \ldots U_{e^{ih_n}}(u) \in \mathcal{E}_u \subseteq \mathcal{U}^c(M).$$

We finally observe that the equality $\bigcup_{w\in\mathcal{U}^c(M)} \mathcal{E}_w = \mathcal{U}^c(M)$ follows trivially from the above arguments.

3. Surjective isometries between principal components

We begin by observing that Theorem 2.9 in [16] decodes the main properties of a surjective isometry between the sets of unitary elements in two unital JB*-algebras. In the next theorem we establish an analogous version of the just quoted result for surjective isometries between the principal components of the sets of unitary elements. The original proof requires a modified argument.

Theorem 3.1. *Let $\Delta : \mathcal{U}^0(M) \to \mathcal{U}^0(N)$ be a surjective isometry, where M and N are unital JB*-algebras. Suppose $u, v \in \mathcal{U}^0(M)$ with $\|u-v\| < \frac{1}{2}$. Then the following statements hold:*

(1) *For all $x, y \in \mathcal{U}^0(M)$ we have*

$$\left\|U_v(x^{-1}) - U_v(y^{-1})\right\| = \|U_v(x^*) - U_v(y^*)\| = \|x^* - y^*\| = \|x - y\|.$$

(2) *The constant $K = 2 - 2\|u - v\| > 1$ satisfies that*

$$\|U_v(w^*) - w\| = \left\|U_v(w^{-1}) - w\right\| \geq K\|w - v\|,$$

for all w in the set

$$L^0_{u,v} = \{w \in \mathcal{U}^0(M) : \|u - w\| = \|U_v(u^{-1}) - w\| = \|U_v(u^*) - w\| = \|u - v\|\}.$$

(3) *There exists $w \in \mathcal{U}^0(N)$ satisfying*

$$U_w(\Delta(u)^*) = \Delta\left(U_v(u^*)\right),$$

and

$$\|U_w(x^*) - U_w(y^*)\| = \|x - y\|, \quad \forall x, y \in \mathcal{U}^0(N).$$

(4) *The equalities $\Delta(U_v(u^*)) = \Delta(U_v(u^{-1})) = U_{\Delta(v)}(\Delta(u)^{-1}) = U_{\Delta(v)}(\Delta(u)^*)$ hold.*

Proof. Let us note that Theorem 2.9 in [16] proves that the same conclusion hold for every surjective isometry $\Delta : \mathcal{U}(M) \to \mathcal{U}(N)$.

Since $\mathcal{U}^0(M) \subseteq \mathcal{U}(M)$, statements (1) and (2) follow directly from the same argument in [16, Theorem 2.9(a)] (see also [16, Lemma 2.8]).

(3) Let us consider the u-isotope $M(u)$, and observe that $v \in \mathcal{U}(M(u))$ (cf. [16, Lemma 2.1(b)]). Let $\mathcal{B}$ denote the JB*-subalgebra of $M(u)$ generated by v and the unit of $M(u)$,

that is, u. The Shirshov-Cohn theorem assures that $\mathcal{B}$ is a JB*-subalgebra of $B(H)$, for some complex Hilbert space H, and we can further assume that u is the unit in $B(H)$ (cf. [26, Theorem 7.2.5] and [51, Corollary 2.2]). Moreover, u can be identified with the unit of $B(H)$. We shall use juxtaposition for the product in $B(H)$, and the symbol $\sharp$ for its involution.

It is worth noting that when M and $\mathcal{B}$ are regarded as JB*-triples, the triple product in $\mathcal{B}$ is precisely $\{\cdot,\cdot,\cdot\}_M|_{\mathcal{B}\times\mathcal{B}\times\mathcal{B}}$. At the same time, the triple product in $\mathcal{B}$ can be expressed in terms of the associative product of $B(H)$. By Proposition 2.6 the element $U_v(u^*)$ lies in $\mathcal{U}^0(M)$, and hence we can apply Δ at this element and compute the next distances:

$$\begin{aligned}\|\Delta(u)-\Delta(U_v(u^*))\|_N &= \|u-U_v(u^*)\|_M = \|u-\{v,u,v\}_M\|_M = \|u-\{v,u,v\}_{\mathcal{B}}\|_{\mathcal{B}}\\ &= \|u-\{v,u,v\}_{B(H)}\|_{B(H)} = \|u-vu^\sharp v\|_{B(H)}\\ &= \|u-vv\|_{B(H)} = \|v^\sharp - v\|_{B(H)} \leq \|v^\sharp-u\|_{B(H)}+\|u-v\|_{B(H)}\\ &= 2\|u-v\|_{B(H)} = 2\|u-v\|_{\mathcal{B}} = 2\|u-v\|_M < 1.\end{aligned}.$$

Actually, there is another argument which avoids any use of the Shirshov-Cohn theorem. Namely, since $\mathcal{B}$ is the JB*-subalgebra of the u-isotope $M(u)$ generated by the unitary v –which is a unitary in $M(u)$ [16, Lemma 2.1(b)]– and the unit of $M(u)$ –i.e. u– it must be isometrically Jordan *-isomorphic to a unital commutative C*-algebra, that is, to some $C(\Omega)$ for an appropriate compact Hausdorff space Ω, and under this identification, u corresponds to the unit (cf. [26, 3.2.4. The spectral theorem] or [11, Proposition 3.4.2 and Theorem 4.1.3(v)]). So, working in this commutative unital C*-algebra $\mathcal{B}\cong C(\Omega)$, we can compute the above distances in an even easier setting.

We are in a position to apply Lemma 2.2(b) in [16] to assure the existence of a unitary $w\in\mathcal{U}(N)$ satisfying

$$U_w(\Delta(u)^*) = \Delta\left(U_v(u^*)\right).$$

However, by hypotheses $\|\Delta(u)-\Delta(v)\| = \|u-v\| < \frac{1}{2}$, and consequently, [16, Lemma 2.2(b)] also guarantees that

$$\|w-\Delta(v)\| \leq \|w-\Delta(u)\| + \|\Delta(u)-\Delta(v)\| \leq \|\Delta(u)-\Delta(U_v(u^*))\| + \frac{1}{2} < 2.$$

Since $\Delta(v)\in\mathcal{U}^0(N)$, Theorem 2.3 implies that w belongs to the principal component of $\mathcal{U}(N)$, that is, $w\in\mathcal{U}^0(N)$.

Now, it follows from the fact that the mapping U_w is an isometry that

$$\|U_w(x^*)-U_w(y^*)\| = \|U_w(x^{-1})-U_w(y^{-1})\| = \|x^{-1}-y^{-1}\| = \|x^*-y^*\| = \|x-y\|,$$

for all $x,y\in\mathcal{U}^0(N)\subseteq\mathcal{U}(N)$.

(4) Let $w\in\mathcal{U}^0(N)$ be the element given by (3).

Let us define the mappings $\varphi : \mathcal{U}^0(M) \to \mathcal{U}^0(M)$ and $\psi : \mathcal{U}^0(N) \to \mathcal{U}^0(N)$ given by

$$\varphi(x) := U_v(x^{-1}) = U_v(x^*), \quad x \in \mathcal{U}^0(M),$$

and

$$\psi(y) := U_w(y^{-1}) = U_w(y^*), \quad y \in \mathcal{U}^0(N),$$

respectively. They are well defined thanks to Proposition 2.6.

We aim to apply [29, Theorem 2.4] to φ, ψ and Δ at $a = u$ and $c = v$. For this purpose let us observe that, since v and w are unitaries lying in the respective principal components, φ and ψ are distance preserving bijections (cf. [16, Lemma 2.1(d)]).

On the other hand, the element v is invertible with v^* as its inverse in M. That implies that $\varphi(v) = U_v(v^{-1}) = U_v(v^*) = v$ and

$$\varphi \circ \varphi(x) = U_v((U_v(x^{-1}))^{-1}) = U_v((U_v(x^*))^*) = U_v(U_{v^*}(x)) = U_v(U_{v^{-1}}(x)) = x,$$

for every $x \in \mathcal{U}^0(M)$ (cf. (10)). That is, $\varphi \circ \varphi$ is the identity mapping on $\mathcal{U}^0(M)$. By applying that $U_w(\Delta(u)^*) = \Delta\left(U_v(u^*)\right)$ we have

$$\begin{aligned} \psi(\Delta(u)) &= U_w(\Delta(u)^*) = \Delta\left(U_v(u^*)\right) = \Delta\left(\varphi(u)\right), \text{ and} \\ \psi\left(\Delta\left(\varphi(u)\right)\right) &= U_w\left(\left(\Delta\left(U_v(u^*)\right)\right)^*\right) = \left(U_{w^*}\left(\Delta\left(U_v(u^*)\right)\right)\right)^* = \Delta(u)^{**} = \Delta(u). \end{aligned}$$

Therefore the mappings φ, ψ, Δ, $a = u$ and $c = v$ satisfy all hypotheses in [29, Theorem 2.4], and hence the mentioned result implies that $U_w(\Delta(v)^*) = \Delta(v)$. Since $\|w - \Delta(v)\| < 2$, [16, Lemma 2.3] guarantees that $w = \Delta(v)$. Having in mind that $U_w(\Delta(u)^*) = \Delta\left(U_v(u^*)\right)$ we get $\Delta(U_v(u^*)) = U_{\Delta(v)}(\Delta(u)^*)$, as desired. □

The next technical result is a refinement of [16, Lemma 3.3], which is in fact a Jordan version of [31, Lemma 7]. The main difference is that in the present result we have a surjective isometry between the principal components of the sets of unitaries in two unital JB*-algebras, while in [16, Lemma 3.3] we find a surjective isometry between the whole sets of unitaries of two unital JB*-algebras. Since the principal component are a self-adjoint quadratic subset of the set of unitaries (cf. Proposition 2.6), the arguments in the proof of [16, Lemma 3.3] are also valid in our setting. So, the details will be omitted.

Lemma 3.2. *Let M and N be two unital JB*-algebras. Let $\{u_k : 0 \leq k \leq 2^n\}$ be a subset of $\mathcal{U}^0(M)$ (with $n \in \mathbb{N}$) and let $\Phi : \mathcal{U}^0(M) \to \mathcal{U}^0(N)$ be a mapping such that*

$$U_{u_{k+1}}(u_k^*) = u_{k+2}, \text{ and } \Phi(U_{u_{k+1}}(u_k^*)) = U_{\Phi(u_{k+1})}(\Phi(u_k)^*),$$

for all $0 \leq k \leq 2^n - 2$. Then $U_{u_{2^n-1}}(u_0^*) = u_{2^n}$ and

$$\Phi\left(U_{u_{2^n-1}}(u_0^*)\right) = U_{\Phi(u_{2^n-1})}\Phi(u_0)^*.$$

Before addressing our main goal, we shall explore the behavior of those unital surjective isometries over U-products of exponentials of self-adjoint elements. The next result is the technical core of the arguments.

Proposition 3.3. *For each natural n the identity*

$$\Delta_0(U_{e^{ih_n}}\cdots U_{e^{ih_1}}(e^{ih_0})) = U_{\Delta_0(e^{ih_n})}\cdots U_{\Delta_0(e^{ih_1})}(\Delta(e^{ih_0})),$$

holds for every couple of unital JB-algebras M and N, every unital (i.e., $\Delta_0(\mathbf{1}_M)=\mathbf{1}_N$) surjective isometry $\Delta_0:\mathcal{U}^0(M)\to\mathcal{U}^0(N)$ and every $h_n,\ldots,h_1,h_0\in M_{sa}$. Furthermore, if we fix a couple of unital JB*-algebras M and N, and we denote by $Iso_1(\mathcal{U}^0(M),\mathcal{U}^0(N))$ the set of all surjective unital isometries between $\mathcal{U}^0(M)$ and $\mathcal{U}^0(N)$, then there exists a mapping $k: M_{sa}\times Iso_1(\mathcal{U}^0(M),\mathcal{U}^0(N))\to N_{sa}$ satisfying the following properties:*

(a) *If we fix $\Delta_0\in Iso_1(\mathcal{U}^0(M),\mathcal{U}^0(N))$, the mapping $k(\cdot,\Delta_0):M_{sa}\to N_{sa}$ is a surjective (real) linear isometry;*
(b) *$k(\cdot,\Delta_0)^{-1}=k(\cdot,\Delta_0^{-1})$, for every $\Delta_0\in Iso_1(\mathcal{U}^0(M),\mathcal{U}^0(N))$;*
(c) *$\Delta_0(e^{ith})=e^{itk(h,\Delta_0)}$ for all $\Delta_0\in Iso_1(\mathcal{U}^0(M),\mathcal{U}^0(N))$, $h\in M_{sa}$ and $t\in\mathbb{R}$.*

Proof. Concerning the first identity, we observe that it suffices to prove that for each natural n the identity

$$\Delta_0(U_{e^{ih_n}}\cdots U_{e^{ih_1}}(\mathbf{1}_M)) = U_{\Delta_0(e^{ih_n})}\cdots U_{\Delta_0(e^{ih_1})}(\mathbf{1}_N) \tag{20}$$

holds for every Δ_0 and $h_n,\ldots,h_1\in M_{sa}$ as in the hypotheses. Namely, if $h_0\in M_{sa}$ and the previous identity (20) is true we have

$$\begin{aligned}\Delta_0(U_{e^{ih_n}}\cdots U_{e^{ih_1}}(e^{ih_0})) &= \Delta_0(U_{e^{ih_n}}\cdots U_{e^{ih_1}}U_{e^{i\frac{h_0}{2}}}(\mathbf{1}_M))\\ = U_{\Delta_0(e^{ih_n})}\cdots U_{\Delta_0(e^{ih_1})}U_{\Delta_0(e^{i\frac{h_0}{2}})}(\mathbf{1}_N) &= U_{\Delta_0(e^{ih_n})}\cdots U_{\Delta_0(e^{ih_1})}\Delta_0(U_{e^{i\frac{h_0}{2}}}(\mathbf{1}_M))\\ &= U_{\Delta_0(e^{ih_n})}\cdots U_{\Delta_0(e^{ih_1})}(\Delta_0(e^{ih_0})),\end{aligned}$$

where at the penultimate equality we applied (20).

We shall prove (20) by induction on n. Let us first assume that $n=1$, and fix Δ_0 and $h_1\in M_{sa}$ as in the hypotheses. Fix an arbitrary $h_0\in M_{sa}$ and consider the continuous mapping $E_{h_0}:\mathbb{R}\to\mathcal{U}^0(M)$, $E_{h_0}(t)=e^{ith_0}$. We begin our argument by proving that the identity

$$\Delta_0(U_{v_t}(v_s)) = U_{\Delta_0(v_t)}(\Delta_0(v_s^*)^*) \tag{21}$$

holds for every $v_s=e^{ish_0}=E_{h_0}(s)$ and $v_t=e^{ith_0}=E_{h_0}(t)$ in $E_{h_0}(\mathbb{R})$.

Namely, choose a positive integer m such that

$$e^{\frac{\|i(s+t)h_0\|_M}{2^m}} - 1 < \frac{1}{2}, \text{ and hence } \left\| e^{\frac{i(s+t)h_0}{2^m}} - \mathbf{1}_M \right\|_M \leq e^{\frac{\|i(s+t)h_0\|_M}{2^m}} - 1 < \frac{1}{2}. \tag{22}$$

Consider now the family $\{u_l : 0 \leq l \leq 2^{m+1}\} \subseteq \mathcal{U}^0(M)$, where

$$u_l = v_s^* \circ e^{\frac{il(s+t)h_0}{2^m}} = e^{-ish_0} \circ e^{\frac{il(s+t)h_0}{2^m}} = (2) = e^{\left(-is + \frac{il(s+t)}{2^m}\right)h_0}, \quad 0 \leq l \leq 2^{m+1}.$$

Having in mind (2) it is not hard to see that

$$\begin{aligned} u_0 = v_s^*,\ u_{2^m} = v_t,\ u_{2^{m+1}} = U_{v_t}(v_s), \\ U_{u_{l+1}}(u_l{}^*) = U_{u_{l+1}}(u_l^{-1}) = u_{l+2}, \text{ for any } 0 \leq l \leq 2^{m+1} - 2, \\ \text{and } \|u_{l+1} - u_l\| = \left\| v_s{}^* \circ e^{\frac{i(l+1)(s+t)h_0}{2^m}} - v_s{}^* \circ e^{\frac{il(s+t)h_0}{2^m}} \right\| \\ \leq \|v_s{}^*\| \left\| e^{\frac{il(s+t)h_0}{2^m}} \circ e^{\frac{i(s+t)h_0}{2^m}} - e^{\frac{il(s+t)h_0}{2^m}} \right\| \\ \leq \left\| e^{\frac{il(s+t)h_0}{2^m}} \right\| \left\| e^{\frac{i(s+t)h_0}{2^m}} - \mathbf{1}_M \right\| \\ \leq e^{\frac{\|i(s+t)h_0\|}{2^m}} - 1 < \frac{1}{2}, \text{ for all } 0 \leq l \leq 2^{m+1} - 1. \end{aligned} \tag{23}$$

An application of Theorem 3.1(3) to Δ_0 and the unitaries u_{l+1} and u_l gives

$$\Delta_0(U_{u_{l+1}}(u_l{}^*)) = U_{\Delta_0(u_{l+1})}(\Delta_0(u_l)^*), \text{ for every } l \in \{0, \ldots, 2^{m+1} - 1\}. \tag{24}$$

It follows from (23) and (24) that we are in a position to apply Lemma 3.2 with $\Phi = \Delta_0$ and $n = m + 1$ to deduce that

$$\begin{aligned} \Delta_0(U_{v_t}(v_s)) = \Delta_0(U_{v_t}((v_s{}^*)^*)) = \Delta_0(U_{u_{2^m}}(u_0{}^*)) \\ = U_{\Delta_0(u_{2^m})}(\Delta_0(u_0)^*) = U_{\Delta_0(v_t)}(\Delta_0(v_s{}^*)^*), \end{aligned}$$

which concludes the proof of (21).

Let us see some consequences derived from (21). If in this identity we take $t = 1$, $s = 0$ and $h_0 = h_1 \in M_{sa}$ we get

$$\Delta_0(e^{i2h_1}) = \Delta_0(U_{e^{ih_1}}(\mathbf{1}_M)) = U_{\Delta_0(e^{ih_1})}(\mathbf{1}_N) = \Delta_0(e^{ih_1})^2, \tag{25}$$

which, in particular, proves (20) for $n = 1$.

Second, by replacing $t = 0$ and $h_0 = h_1 \in M_{sa}$ in (21) we get

$$\Delta_0(e^{ish_1}) = \Delta_0(e^{-ish_1})^*, \quad \forall h_1 \in M_{sa},\ \forall s \in \mathbb{R}. \tag{26}$$

The most important consequence derived from (21) (and (26)) assures that, for each $h_0 \in M_{sa}$, the mapping

$$\mathbb{R} \to \mathcal{U}^0(N),\ t \mapsto \Delta_0(e^{ith_0})$$

is a uniformly continuous one-parameter family of unitaries satisfying the hypotheses of the Jordan version of Stone's one-parameter theorem in [16, Theorem 3.1]. Namely,

$$\begin{aligned} U_{\Delta_0(e^{ith_0})}(\Delta_0(e^{ish_0})) = U_{\Delta_0(e^{ith_0})}(\Delta_0(e^{-ish_0})^*) &= \Delta_0(U_{e^{ith_0}}(e^{ish_0})) \\ &= \Delta_0(e^{i(2t+s)h_0}). \end{aligned}$$

Therefore there exists a unique $k(h_0, \Delta_0) \in N_{sa}$, depending on h_0 and Δ_0, satisfying

$$\Delta_0(e^{ith_0}) = e^{itk(h_0,\Delta_0)},\ \text{for all } t \in \mathbb{R}. \tag{27}$$

We observe that the mapping $k(\cdot, \Delta_0) : M_{sa} \to N_{sa}$ is well defined, and moreover, by (27), $\mathbf{1}_N = \Delta_0(e^{it0}) = e^{itk(0,\Delta_0)}$, for all $t \in \mathbb{R}$, which assures that $k(0, \Delta_0) = 0$. Let us prove that this mapping is a surjective isometry, and as a consequence of the Mazur–Ulam theorem, it must be a surjective (real) linear isometry.

Similarly, by considering the surjective isometry $\Delta_0^{-1} : \mathcal{U}^0(N) \to \mathcal{U}^0(M)$, the above arguments prove that for each $k_0 \in N_{sa}$ there exists a unique $h(k_0, \Delta_0^{-1}) \in M_{sa}$, depending on k_0 and Δ_0^{-1}, satisfying

$$\Delta_0^{-1}(e^{itk_0}) = e^{ith(k_0,\Delta_0^{-1})},\ \text{for all } t \in \mathbb{R}. \tag{28}$$

As before, the mapping $h(\cdot, \Delta_0^{-1}) : N_{sa} \to M_{sa}$ is well defined. By combining (27) and (28) we get

$$\begin{aligned} e^{ith_0} &= \Delta_0^{-1}\Delta_0(e^{ith_0}) = \Delta_0^{-1}\left(e^{itk(h_0,\Delta_0)}\right) = e^{ith(k(h_0,\Delta_0),\Delta_0^{-1})},\ \text{ and} \\ e^{itk_0} &= \Delta_0\Delta_0^{-1}(e^{itk_0}) = \Delta_0\left(e^{ith(k_0,\Delta_0^{-1})}\right) = e^{itk(h(k_0,\Delta_0^{-1}),\Delta_0)}, \end{aligned}$$

for all $t \in \mathbb{R}$, $h_0 \in M_{sa}$, $k_0 \in N_{sa}$. Taking a simple derivative at $t = 0$ we arrive to the identities

$$h(k(h_0, \Delta_0), \Delta_0^{-1}) = h_0,\ \text{and } k(h(k_0, \Delta_0^{-1}), \Delta_0) = k_0,$$

for all $h_0 \in M_{sa}$, $k_0 \in N_{sa}$, that is, $k(\cdot, \Delta_0)$ and $h(\cdot, \Delta_0^{-1})$ are bijections with $h(\cdot, \Delta_0^{-1}) = k(\cdot, \Delta_0)^{-1}$.

We shall next prove that $k(\cdot, \Delta_0) : M_{sa} \to N_{sa}$ is an isometry. This is a standard procedure already in [32,28,16]. Given $h_0, h_0' \in M_{sa}$ and a real number t it is well known that

$$\|\cdot\|\text{-}\lim_{t\to 0^+} \frac{e^{ith_0} - e^{ith_0'}}{t} = \|\cdot\|\text{-}\lim_{t\to 0^+} \frac{e^{ith_0} - \mathbf{1}_M}{t} - \frac{e^{ith_0'} - \mathbf{1}_M}{t} = ih_0 - ih_0'.$$

It follows from (27) that

$$\|e^{itk(h_0,\Delta_0)} - e^{itk(h_0',\Delta_0)}\|_N = \|\Delta_0(e^{ith_0}) - \Delta_0(e^{ith_0'})\|_N = \|e^{ith_0} - e^{ith_0'}\|_M.$$

Therefore, we deduce from the uniqueness of the limits above that

$$\|k(h_0,\Delta_0) - k(h_0',\Delta_0)\|_N = \|h_0 - h_0'\|_M,$$

witnessing that $k(\cdot,\Delta_0) : M_{sa} \to N_{sa}$ is a surjective isometry, and hence a surjective linear isometry by the Mazur–Ulam theorem. We have shown the following: for each unital surjective isometry $\Delta_0 : \mathcal{U}^0(M) \to \mathcal{U}^0(N)$

$$\text{the mapping } k(\cdot,\Delta_0) : M_{sa} \to N_{sa} \text{ is a surjective linear isometry.} \tag{29}$$

We shall next prove the following identity

$$\Delta_0(U_{e^{ih_1}}(e^{ih_0})) = U_{\Delta_0(e^{ih_1})}(\Delta_0(e^{ih_0})), \text{ for all } h_1, h_0 \in M_{sa}. \tag{30}$$

Namely, fix $h_1 \in M_{sa}$ and consider the surjective isometry $\Delta_0^{h_1} := U_{\Delta_0(e^{ih_1})^*}\Delta_0 U_{e^{ih_1}} : \mathcal{U}^0(M) \to \mathcal{U}^0(N)$, which is well defined by Theorem 2.3. By (25) we obtain

$$\begin{aligned}\Delta_0^{h_1}(\mathbf{1}_M) &= U_{\Delta_0(e^{ih_1})^*}\Delta_0 U_{e^{ih_1}}(\mathbf{1}_M) = U_{\Delta_0(e^{ih_1})^*}\Delta_0(e^{2ih_1}) = U_{\Delta_0(e^{ih_1})^*}(\Delta_0(e^{ih_1})^2)\\ &= U_{\Delta_0(e^{ih_1})^*}U_{\Delta_0(e^{ih_1})}(\mathbf{1}_M) = \mathbf{1}_N,\end{aligned}$$

witnessing that the mapping $\Delta_0^{h_1} = U_{\Delta_0(e^{ih_1})^*}\Delta_0 U_{e^{ih_1}}$ is unital.

Let $k(\cdot,\Delta_0), k(\cdot,\Delta_0^{h_1}) : M_{sa} \to N_{sa}$ be the surjective linear isometries given by (27) and (29) for Δ_0 and $\Delta_0^{h_1}$, respectively.

Choose, by continuity, $\varepsilon > 0$ such that

$$\|e^{ih_1} - e^{-ith}\| < \frac{1}{2}, \ \forall t \in \mathbb{R}, \ h \in M_{sa} \text{ with } |t-1| < \varepsilon, \text{ and } \|h + h_1\| < \varepsilon.$$

By applying Theorem 3.1(3) to Δ_0 and the unitaries $e^{ih_1}, e^{-ith} \in \mathcal{U}^0(M)$ with t in $\mathbb{R}$ and $h \in M_{sa}$ satisfying $\|h + h_1\| < \varepsilon$ and $|t-1| < \varepsilon$ we conclude that

$$\Delta_0(U_{e^{ih_1}}(e^{ith})) = U_{\Delta_0(e^{ih_1})}((\Delta_0(e^{-ith}))^*) = (26) = U_{\Delta_0(e^{ih_1})}(\Delta_0(e^{ith})).$$

The previous identity implies, by the definition of $k(\cdot,\Delta_0)$ and $k(\cdot,\Delta_0^{h_1})$, that

$$e^{itk(h,\Delta_0)} = e^{itk(h,\Delta_0^{h_1})}, \ \forall t \in \mathbb{R}, \ h \in M_{sa} \text{ with } |t-1| < \varepsilon, \text{ and } \|h + h_1\| < \varepsilon.$$

If we fix an arbitrary $h \in M_{sa}$ with $\|h + h_1\| < \varepsilon$ and we take a simple derivative at $t = 1$ in the above equality we obtain

$$k(h,\Delta_0) = k(h,\Delta_0^{h_1}), \text{ for all } h \in M_{sa} \text{ with } \|h + h_1\| < \varepsilon,$$

and the linearity of $k(\cdot,\Delta_0)$, $k(\cdot,\Delta_0^{h_1})$ implies that $k(\cdot,\Delta_0)=k(\cdot,\Delta_0^{h_1})$.

Now, it follows from (27) that

$$\Delta_0(e^{ith_0})=e^{itk(h_0,\Delta_0)}=e^{itk(h_0,\Delta_0^{h_1})}=\Delta_0^{h_1}(e^{ith_0})=U_{\Delta_0(e^{ih_1})^*}\Delta_0 U_{e^{ih_1}}(e^{ith_0}),$$

for all real t and all $h_0\in M_{sa}$, which concludes the proof of (30).

Let us now proceed with the final part in the induction argument to get (20). Suppose by the induction hypothesis that (20) holds for every unital surjective isometry Δ_0 and $h_n,\ldots,h_1\in M_{sa}$ as in the hypotheses. Take an arbitrary unital surjective isometry $\Delta_1:\mathcal{U}^0(M)\to\mathcal{U}^0(N)$ and $h_{n+1},h_n,\ldots,h_1\in M_{sa}$. Set $w_0=e^{2ih_{n+1}}$ and $\widetilde{w}_0=\Delta_1(e^{2ih_{n+1}})\in\mathcal{U}^0(N)$. We consider the unital JB*-algebras given by the isotopes $M(w_0)$ and $N(\widetilde{w}_0)$ with the corresponding Jordan products $\circ_{w_0}$ and $\circ_{\widetilde{w}_0}$ and involutions $*_{w_0}$ and $*_{\widetilde{w}_0}$, respectively. Since $w_0\in\mathcal{U}^0(M)$ and $\widetilde{w}_0\in\mathcal{U}^0(N)$, it follows from (13) that

$$\mathcal{U}^0(M(w_0))=\mathcal{U}^0(M),\ \text{and}\ \mathcal{U}^0(N(\widetilde{w}_0))=\mathcal{U}^0(N),$$

and thus $\widetilde{\Delta}_1:\mathcal{U}^0(M(w_0))=\mathcal{U}^0(M)\to\mathcal{U}^0(N(\widetilde{w}_0))=\mathcal{U}^0(N)$, $\widetilde{\Delta}_1(a)=\Delta_1(a)$ is a unital (i.e., $\Delta_1(w_0)=\widetilde{w}_0$) surjective isometry.

The mapping $U_{e^{ih_{n+1}}}:M\to M(w_0)$ is a unital surjective linear isometry, and hence a Jordan *-isomorphism between these two unital JB*-algebras (cf. [16, Lemma 2.1(d)] or [53]). Therefore $\hat{h}_k:=U_{e^{ih_{n+1}}}(h_k)\in(M(w_0))_{sa}$ for all $1\le k\le n$.

By applying the induction hypothesis to $\widetilde{\Delta}_1:\mathcal{U}^0(M(w_0))\to\mathcal{U}^0(N(\widetilde{w}_0))$, and $\hat{h}_1,\ldots,\hat{h}_n$ we obtain

$$\begin{aligned}&\Delta_1 U^{(w_0)}_{\exp_{w_0}(i\hat{h}_n)}\ldots U^{(w_0)}_{\exp_{w_0}(i\hat{h}_1)}(w_0)=\widetilde{\Delta}_1 U^{(w_0)}_{\exp_{w_0}(i\hat{h}_n)}\ldots U^{(w_0)}_{\exp_{w_0}(i\hat{h}_1)}(w_0)\\&=U^{(\widetilde{w}_0)}_{\widetilde{\Delta}_1(\exp_{w_0}(i\hat{h}_n))}\ldots U^{(\widetilde{w}_0)}_{\widetilde{\Delta}_1(\exp_{w_0}(i\hat{h}_1))}(\widetilde{w}_0).\end{aligned}\tag{31}$$

Let us carefully decode the identity in (31). First by (16) and the fact that $M(w_0)=M_{(w_0^*)}$, for any $v_1,v_2\in\mathcal{U}^0(M)$ we have

$$U^{(w_0)}_{v_1}(v_2)=U_{v_1}U_{w_0^*}(v_2)=U_{v_1}U_{e^{-2ih_1}}(v_2).\tag{32}$$

Furthermore, since, as before, $\exp_{w_0}$ denotes the exponential in the unital JB*-algebra $M(w_0)$, it follows from the fact that $U_{e^{ih_{n+1}}}:M\to M(w_0)$ is a Jordan *-isomorphism that

$$\exp_{w_0}(i\hat{h}_k)=\exp_{w_0}(iU_{e^{ih_{n+1}}}(h_k))=U_{e^{ih_{n+1}}}(e^{ih_k}),\ \text{for all}\ 1\le k\le n,\tag{33}$$

and by the fundamental identity (3) we deduce

$$\begin{aligned}U^{(w_0)}_{\exp_{w_0}(i\hat{h}_k)}&=U_{\exp_{w_0}(i\hat{h}_k)}U_{w_0{}^*}=U_{U_{e^{ih_{n+1}}}(e^{ih_k})}U_{w_0{}^*}\\&=U_{e^{ih_{n+1}}}U_{e^{ih_k}}U_{e^{ih_{n+1}}}U_{e^{-2ih_{n+1}}}=U_{e^{ih_{n+1}}}U_{e^{ih_k}}U_{e^{-ih_{n+1}}},\end{aligned}\tag{34}$$

where in the last step we applied that $e^{ih_{n+1}}$ and $e^{-2ih_{n+1}}$ operator commute.

Having in mind (32) and (34), the left hand side of (31) writes in the form

$$\begin{aligned}
&\Delta_1 U^{(w_0)}_{\exp_{w_0}(i\hat{h}_n)} \dots U^{(w_0)}_{\exp_{w_0}(i\hat{h}_1)}(w_0) \\
&= \Delta_1 U_{e^{ih_{n+1}}} U_{e^{ih_n}} U_{e^{-ih_{n+1}}} \dots U_{e^{ih_{n+1}}} U_{e^{ih_1}} U_{e^{-ih_{n+1}}}(e^{2ih_{n+1}}) \\
&= \Delta_1 U_{e^{ih_{n+1}}} U_{e^{ih_n}} U_{e^{ih_{n-1}}} \dots U_{e^{ih_2}} U_{e^{ih_1}}(\mathbf{1}_M).
\end{aligned} \tag{35}$$

To deal with the right hand side of (31) we observe that, by (33), for $1 \leq k \leq n$ we have

$$\begin{aligned}
U^{(\widetilde{w}_0)}_{\widetilde{\Delta}_1(\exp_{w_0}(i\hat{h}_k))} &= U^{(\widetilde{w}_0)}_{\Delta_1(\exp_{w_0}(i\hat{h}_k))} = U_{\Delta_1(U_{e^{ih_{n+1}}}(e^{ih_k}))} U_{\widetilde{w}_0^*} \\
&= (\text{by (30)}) = U_{U_{\Delta_1(e^{ih_{n+1}})}\Delta_1(e^{ih_k})} U_{\widetilde{w}_0^*} = (\text{by (3)}) \\
&= U_{\Delta_1(e^{ih_{n+1}})} U_{\Delta_1(e^{ih_k})} U_{\Delta_1(e^{ih_{n+1}})} U_{\Delta_1(e^{2ih_{n+1}})^*} \\
&= (\text{by (25)}) = U_{\Delta_1(e^{ih_{n+1}})} U_{\Delta_1(e^{ih_k})} U_{\Delta_1(e^{ih_{n+1}})} U_{(\Delta_1(e^{ih_{n+1}})^2)^*} \\
&= U_{\Delta_1(e^{ih_{n+1}})} U_{\Delta_1(e^{ih_k})} U_{\Delta_1(e^{ih_{n+1}})^*} \\
&= U_{\Delta_1(e^{ih_{n+1}})} U_{\Delta_1(e^{ih_k})} U^{-1}_{\Delta_1(e^{ih_{n+1}})},
\end{aligned}$$

and hence the right hand side of (31) writes in the form

$$\begin{aligned}
&U^{(\widetilde{w}_0)}_{\widetilde{\Delta}_1(\exp_{w_0}(i\hat{h}_n))} \dots U^{(\widetilde{w}_0)}_{\widetilde{\Delta}_1(\exp_{w_0}(i\hat{h}_1))}(\widetilde{w}_0) \\
&= U_{\Delta_1(e^{ih_{n+1}})} U_{\Delta_1(e^{ih_n})} \dots U_{\Delta_1(e^{ih_1})} U^{-1}_{\Delta_1(e^{ih_{n+1}})}(\Delta_1(e^{2ih_{n+1}})) \\
&= (\text{by (25)}) = U_{\Delta_1(e^{ih_{n+1}})} U_{\Delta_1(e^{ih_n})} \dots U_{\Delta_1(e^{ih_1})} U^{-1}_{\Delta_1(e^{ih_{n+1}})}(\Delta_1(e^{ih_{n+1}})^2) \\
&= U_{\Delta_1(e^{ih_{n+1}})} U_{\Delta_1(e^{ih_n})} \dots U_{\Delta_1(e^{ih_1})}(\mathbf{1}_N).
\end{aligned} \tag{36}$$

Finally, (31), (35) and (36) give

$$\Delta_1 U_{e^{ih_{n+1}}} U_{e^{ih_n}} \dots U_{e^{ih_2}} U_{e^{ih_1}}(\mathbf{1}_M) = U_{\Delta_1(e^{ih_{n+1}})} U_{\Delta_1(e^{ih_n})} \dots U_{\Delta_1(e^{ih_1})}(\mathbf{1}_N).$$

By induction, we have (20) for every Δ_0, $n \in \mathbb{N}$, and $h_n, \dots, h_1 \in M_{sa}$. □

The notion of orthogonality has been studied and developed for general elements in a JB*-triple (cf. [9]). Elements a, b in a JB*-triple E are called *orthogonal* (written $a \perp b$) if $L(a, b) = 0$. It is known that $a \perp b \Leftrightarrow \{a, a, b\} = 0 \Leftrightarrow \{b, b, a\} = 0 \Leftrightarrow b \perp a$ (see [9, Lemma 1]).

Theorem 3.4. *Let $\Delta : \mathcal{U}^0(M) \to \mathcal{U}^0(N)$ be a surjective isometry between the principal components of the unitary sets of two unital JB*-algebras. Then there exist $k_1, \dots, k_n \in N_{sa}$, a central projection $p \in N$ and a Jordan *-isomorphism $\Phi : M \to N$ such that*

$$\Delta(u) = p \circ U_{e^{ik_n}} \dots U_{e^{ik_1}} \Phi(u) + (\mathbf{1}_N - p) \circ (U_{e^{-ik_n}} \dots U_{e^{-ik_1}} \Phi(u))^*,$$

for all $u \in \mathcal{U}^0(M)$. Consequently, M and N are Jordan *-isomorphic, and there exists a surjective real linear isometry (i.e., a real linear triple isomorphism) from M onto N whose restriction to $\mathcal{U}^0(M)$ is Δ.

Proof. Since $\Delta(\mathbf{1}_M) \in \mathcal{U}^0(N)$, Theorem 2.3 implies the existence of $k_1, \dots, k_n \in N_{sa}$ such that $\Delta(\mathbf{1}_M) = U_{e^{ik_n}} \dots U_{e^{ik_1}}(\mathbf{1}_N)$. It follows from Proposition 2.6 that the mapping $\Delta_0 = U_{e^{-ik_1}} \dots U_{e^{-ik_n}} \Delta : \mathcal{U}^0(M) \to \mathcal{U}^0(N)$ is a surjective isometry, and by construction $\Delta_0(\mathbf{1}_M) = \mathbf{1}_N$.

Let $k(\cdot, \Delta_0) : M_{sa} \to N_{sa}$ be the surjective linear isometry given by Proposition 3.3.

The self-adjoint part of any JB*-algebra is a JB-algebra. Thus, $k(\cdot, \Delta_0) : M_{sa} \to N_{sa}$ is a surjective linear isometry between JB-algebras. Theorem 1.4 and Corollary 1.11 in [35] guarantee the existence of a central symmetry $k(\mathbf{1}_M, \Delta_0)$ in N_{sa} and a Jordan *-isomorphism $\Phi : M \to N$ such that

$$k(h, \Delta_0) = k(\mathbf{1}_M, \Delta_0) \circ \Phi(h), \tag{37}$$

for every $h \in M_{sa}$.

The rest of the proof follows similar arguments to those in the proof of [16, Theorem 3.4]. That is, by construction there exists a central projection p in N such that $k(\mathbf{1}_M, \Delta_0) = 2p - \mathbf{1}_N = p - (\mathbf{1}_N - p)$, where p and $(\mathbf{1}_N - p)$ clearly are orthogonal projections in N, and for any $n > 0$,

$$(2p - \mathbf{1}_N)^n = (p - (\mathbf{1}_N - p))^n = p + (-1)^n (\mathbf{1}_N - p).$$

The mapping Δ_0 can be expressed in terms of p and Φ by some simple computations. Namely, given an arbitrary $h \in M_{sa}$, we have

$$\Delta_0(e^{ih}) = e^{ik(h, \Delta_0)} = e^{ik(\mathbf{1}_M, \Delta_0) \circ \Phi(h)} = e^{i(2p - \mathbf{1}_N) \circ \Phi(h)} = \sum_{n=0}^{\infty} \frac{(i(2p - \mathbf{1}_N) \circ \Phi(h))^n}{n!}.$$

We now make use of the properties of Φ as Jordan *-isomorphism and the expression of $(2p - \mathbf{1}_N)^n$ given above. It is worth noting that $k(\mathbf{1}_M, \Delta_0)$ (and hence p) is central, and so it operator commutes with any element in N. Therefore,

$$\begin{aligned}\Delta_0(e^{ih}) &= \sum_{n=0}^{\infty} \frac{i^n (2p - \mathbf{1}_N)^n \circ \Phi(h)^n}{n!} = \sum_{n=0}^{\infty} \frac{i^n \left(p + (-1)^n (\mathbf{1}_N - p)\right) \circ \Phi(h^n)}{n!} \\ &= \sum_{n=0}^{\infty} \frac{i^n p \circ \Phi(h^n)}{n!} + \sum_{n=0}^{\infty} \frac{i^n (-1)^n (\mathbf{1}_N - p) \circ \Phi(h^n)}{n!} \\ &= p \circ \sum_{n=0}^{\infty} \frac{i^n \Phi(h^n)}{n!} + (\mathbf{1}_N - p) \circ \sum_{n=0}^{\infty} \frac{i^n (-1)^n \Phi(h^n)}{n!} \\ &= p \circ \Phi\left(\sum_{n=0}^{\infty} \frac{i^n h^n}{n!}\right) + (\mathbf{1}_N - p) \circ \Phi\left(\sum_{n=0}^{\infty} \frac{i^n (-1)^n h^n}{n!}\right) \\ &= p \circ \Phi(e^{ih}) + (\mathbf{1}_N - p) \circ \Phi(e^{-ih}) = p \circ \Phi(e^{ih}) + (\mathbf{1}_N - p) \circ \Phi(e^{ih})^*. \end{aligned} \tag{38}$$

Let us fix an arbitrary element $u \in \mathcal{U}^0(M)$. By Theorem 2.3, there exist a natural m and $h_1, \ldots, h_m \in M_{sa}$ such that $u = U_{e^{ih_m}} \cdots U_{e^{ih_1}}(\mathbf{1}_M)$. According to this expression, we can apply Proposition 3.3 and (38) to compute the evaluation of Δ_0 at the element u:

$$\begin{aligned}\Delta_0(u) &= \Delta_0(U_{e^{ih_m}} \cdots U_{e^{ih_1}}(\mathbf{1}_M)) = U_{\Delta_0(e^{ih_m})} \cdots U_{\Delta_0(e^{ih_1})}(\Delta_0(\mathbf{1}_M)) \\ &= U_{(p \circ \Phi(e^{ih_m}) + (\mathbf{1}_N - p) \circ \Phi(e^{ih_m})^*)} \cdots U_{(p \circ \Phi(e^{ih_1}) + (\mathbf{1}_N - p) \circ \Phi(e^{ih_1})^*)}(\mathbf{1}_N).\end{aligned}$$

Since p is a central projection in N, it is not hard to see that

$$U_{p \circ a + (\mathbf{1}_N - p) \circ b} = U_{p \circ a} + U_{(\mathbf{1}_N - p) \circ b} = p \circ U_a + (\mathbf{1}_N - p) \circ U_b,$$

for all $a, b \in N$. The orthogonality of p and $\mathbf{1}_N - p$ can be now combined with the fact that Φ is a Jordan *-isomorphism to conclude that

$$\begin{aligned}\Delta_0(u) &= p \circ \Phi(U_{e^{ih_m}} \cdots U_{e^{ih_1}}(\mathbf{1}_M)) + (\mathbf{1}_N - p) \circ \Phi(U_{e^{ih_m}} \cdots U_{e^{ih_1}}(\mathbf{1}_M))^* \\ &= p \circ \Phi(u) + (\mathbf{1}_N - p) \circ \Phi(u)^*.\end{aligned}$$

Finally, by the definition of Δ_0, we arrive at

$$\begin{aligned}\Delta(u) &= U_{e^{ik_n}} \ldots U_{e^{ik_1}} \Delta_0(u) = U_{e^{ik_n}} \ldots U_{e^{ik_1}} \left(p \circ \Phi(u) + (\mathbf{1}_N - p) \circ \Phi(u)^*\right), \\ &= p \circ U_{e^{ik_n}} \ldots U_{e^{ik_1}} \Phi(u) + (\mathbf{1}_N - p) \circ \left(U_{e^{-ik_n}} \ldots U_{e^{-ik_1}} \Phi(u)\right)^*,\end{aligned}$$

for all $u \in \mathcal{U}^0(M)$, which concludes the proof. □

We can now strengthen the conclusion in [16, Corollary 3.8] where it is deduced that two unital JB*-algebras M and N are isometrically isomorphic as (complex) Banach spaces if, and only if, they are isometrically isomorphic as real Banach spaces if, and only if, there exists a surjective isometry $\Delta : \mathcal{U}(M) \to \mathcal{U}(N)$.

Corollary 3.5. *Let M and N be two unital JB*-algebras. Then the following statements are equivalent:*

(a) M and N are Jordan *-isomorphic;
(b) There exists a surjective isometry $\Delta : \mathcal{U}(M) \to \mathcal{U}(N)$ satisfying $\Delta(\mathbf{1}_M) \in \mathcal{U}^0(N)$;
(c) There exists a surjective isometry $\Delta : \mathcal{U}^0(M) \to \mathcal{U}^0(N)$.

Proof. The implication $(a) \Leftrightarrow (b)$ is almost explicit in [16, Corollary 3.8], we include here an alternative argument.

The implication $(a) \Rightarrow (b)$ is clear because every Jordan *-isomorphism is an isometry and maps unitaries to unitaries.

$(b) \Rightarrow (c)$ Suppose there exists a surjective isometry $\Delta : \mathcal{U}(M) \to \mathcal{U}(N)$ satisfying $\Delta(\mathbf{1}_M) \in \mathcal{U}^0(N)$. Since every surjective isometry maps connected components to connected components, $\Delta(\mathcal{U}^0(M))$ is a connected component of $\mathcal{U}(N)$ and contains an element in $\mathcal{U}^0(N)$, the equality $\Delta(\mathcal{U}^0(M)) = \mathcal{U}^0(N)$ holds, and thus $\Delta|_{\mathcal{U}^0(M)} : \mathcal{U}^0(M) \to \mathcal{U}^0(N)$ is a surjective isometry.

Finally, the implication $(c) \Rightarrow (a)$ is a consequence of Theorem 3.4. □

If in Corollary 3.5 the set $\mathcal{U}(M)$ is connected all the statements are also equivalent to the existence of a surjective isometry $\Delta : \mathcal{U}(M) \to \mathcal{U}(N)$.

Let M be a unital JB*-algebra, let $\mathcal{U}^c(M)$ be a connected component of $\mathcal{U}(M)$, and let w be an element in $\mathcal{U}^c(M)$. We shall be mainly interested in the case in which $\mathcal{U}^c(M) \neq \mathcal{U}^0(M)$. We have already commented that $\mathcal{U}^c(M)$ is the principal component of the unitary set of the w-isotope $M(w)$ (cf. Remark 2.5). When particularized to the corresponding isotopes, the previous Theorem 3.4 can be re-stated as follows.

Theorem 3.6. *Let M and N be unital JB*-algebras. Let $\Delta : \mathcal{U}^c(M) \to \mathcal{U}^c(N)$ be a surjective isometry between two connected components of $\mathcal{U}(M)$ and $\mathcal{U}(N)$. Suppose $w_1 \in \mathcal{U}^c(M)$ and $w_2 \in \mathcal{U}^c(N)$. Then there exist $k_1, \ldots, k_n \in (N(w_2))_{sa}$, a central projection $p \in N(w_2)$ and a Jordan *-isomorphism $\Phi : M(w_1) \to N(w_2)$ such that*

$$\begin{aligned}\Delta(u) = {} & p \circ_{w_2} U^{(w_2)}_{\exp_{w_2}(ik_n)} \cdots U^{(w_2)}_{\exp_{w_2}(ik_1)} \Phi(u) \\ & + (w_2 - p) \circ_{w_2} \left(U^{(w_2)}_{\exp_{w_2}(-ik_n)} \cdots U^{(w_2)}_{\exp_{w_2}(-ik_1)} \Phi(u) \right)^{*_{w_2}},\end{aligned}$$

for all $u \in \mathcal{U}^c(M)$. In particular, the unital JB*-algebras $M(w_1)$ and $N(w_2)$ are Jordan *-isomorphic, and there exists a surjective real linear isometry (i.e., a real linear triple isomorphism) from M onto N whose restriction to $\mathcal{U}^c(M)$ is Δ.

We are now in position to exhibit additional, and deep, differences between the natural properties of the connected components of the unitary group in a unital C*-algebra with respect to those in the unitary set of a unital JB*-algebra.

Remark 3.7. R. Braun, W. Kaup and H. Upmeier show in [8, Example 5.7] that there exists a unital JB*-algebra M with unit $\mathbf{1}_M$ containing a unitary element w for which we cannot find a surjective linear isometry $T : M \to M$ mapping $\mathbf{1}_M$ to w, that is, the group of all surjective linear isometries on M –equivalently, triple automorphisms– on M is not transitive on $\mathcal{U}(M)$. It is further shown in [11, Antitheorem 3.4.34] that the unital JB*-algebras M and $M(w)$ are not Jordan *-isomorphic. For the concrete example it suffices to consider the transposition on $M_2(\mathbb{C})$, the unit circle $\mathbb{T}$ in $\mathbb{C}$, the JB*-algebra

$$M = \{a : \mathbb{T} \to M_2(\mathbb{C}) \text{ continuous} : a(\lambda)^t = a(\lambda) \text{ for all } \lambda \in \mathbb{T}\}$$

of all continuous functions from $\mathbb{T}$ to the JB*-algebra of all symmetric (complex) matrices in $M_2(\mathbb{C})$ equipped with the natural Jordan product $a \circ b := \frac{1}{2}(ab + ba)$, and the unitary $w : \mathbb{T} \to M_2(\mathbb{C})$, $w(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$.

Let $\mathcal{U}^c(M)$ denote the connected component of $\mathcal{U}(M)$ containing w. We can now conclude that there exists no surjective isometry from $\mathcal{U}^0(M)$ onto $\mathcal{U}^c(M)$. Otherwise, Theorem 3.6 would imply that M and $M(w)$ are Jordan *-isomorphic, which is impossible.

Consider now the unitary $v = w^2 \in M$. Let $\mathcal{U}^{\tilde{c}}(M)$ denote the connected component of $\mathcal{U}(M)$ containing v. Since v is the square of w and the mapping $U_w : M \to M(v)$ is a Jordan *-isomorphism mapping $\mathbf{1}_M$ to v (cf. Remark 2.4), it follows that $U_w\left(\mathcal{U}^0(M)\right) = \mathcal{U}^{\tilde{c}}(M)$.

3.1. Extensibility of surjective isometries between Jordan unitary sets

We have already noted the necessity of working with surjective real linear isometries when studying surjective isometries between connected components of the unitary sets of two unital JB*-algebras. More concretely, contrary to the conclusions in the setting of unital C*-algebras, Jordan *-isomorphisms (for the original Jordan product and involution) are not enough to describe these maps (cf. Theorem 3.6 and Remark 3.7).

As observed by O. Hatori and L. Molnár in the setting of commutative unital C*-algebras (see [32, Corollary 8]), and subsequently by Hatori in the more general case of unital C*-algebras A and B, a surjective isometry $\Delta : \mathcal{U}(A) \to \mathcal{U}(B)$ need not admit an extension to a surjective real linear isometry from A onto B (cf. [28, Corollary 5.1]). We are now in a position to consider the extendibility of a surjective isometry between the unitary sets of two unital JB*-algebras. We include here a new argument based on a variant of the Russo–Dye theorem for unital JB*-algebras due to J.D.M. Wright and M.A. Youngson [52].

Corollary 3.8. *Let $\Delta : \mathcal{U}(M) \to \mathcal{U}(N)$ be a surjective isometry between the unitary sets of two unital JB*-algebras. Let $\{\mathcal{U}^0(M)\} \cup \{\mathcal{U}^j(M) : j \in \Lambda\}$ be the collection of all connected components of $\mathcal{U}(M)$. For each $j \in \Lambda \cup \{0\}$, let $T_j : M \to N$ denote the surjective real*

linear isometry extending the mapping $\Delta|_{\mathcal{U}^j(M)} : \mathcal{U}^j(M) \to \Delta(\mathcal{U}^j(M))$ *whose existence is assured by Theorem 3.6. Then the following statements are equivalent:*

(*a*) Δ *admits an extension to a surjective (real linear) isometry from* M *onto* N*;*
(*b*) *The extensions* T_{j_1} *and* T_{j_2} *coincide for all* $j_1, j_2 \in \Lambda \cup \{0\}$.

Proof. (*a*) $\Rightarrow$ (*b*) Suppose there exists a surjective isometry $T : M \to N$ such that $T|_{\mathcal{U}(M)} = \Delta$. Fix $j \in \Lambda \cup \{0\}$, by hypothesis, $T(u) = \Delta(u) = T_j(u)$ for all $u \in \mathcal{U}^j(M)$.

Fix $u_0 \in \mathcal{U}^j(M)$. Let us consider the unital JB*-algebra given by the u_0-isotope $M(u_0)$. By the variant of the Russo–Dye theorem proved in [52, Corollary], the closed unit ball of the space $M = M(u_0)$ is the closed convex hull of the set

$$\{\exp_{u_0}(ih) : h \in (M(u_0))_{sa}\}.$$

By applying Remark 2.5 we deduce that the latter set is contained in the principal component of $\mathcal{U}(M(u_0))$ which is precisely $\mathcal{U}^j(M)$. It follows from the real linearity of T and T_j, combined with the fact that these two maps coincide on $\mathcal{U}^j(M)$, that $T = T_j$ on the closed unit ball of M, and hence on the whole M.

(*b*) $\Rightarrow$ (*a*) This implication is easy, it suffices to take $T = T_j$ for any $j \in \Lambda \cup \{0\}$. □

It is not hard to find surjective isometries between sets of unitary elements in two unital JB*-algebras which are not extendable to surjective real linear isometries. Namely, as in Remark 3.7 we can find a unital JB*-algebra M and a unitary element w_0 such that the connected components $\mathcal{U}^0(M)$ and $\mathcal{U}^c(M)$ are not isometric (where $\mathcal{U}^c(M)$ is the connected component of $\mathcal{U}(M)$ containing w_0). Let us pick two surjective linear isometries $T_1, T_2 : M \to M$ with $T_1 \neq T_2$ and $T_j(\mathcal{U}^0(M)) = \mathcal{U}^0(M)$ for all $j = 1, 2$ –consider, for example, $T_j = U_{u_j}$ with $u_1 \neq u_2$ in $\mathcal{U}^0(M)$. We define a mapping $\Delta : \mathcal{U}(M) \to \mathcal{U}(M)$ given by $\Delta(u) = T_1(u)$ if $u \in \mathcal{U}^0(M)$ and $\Delta(u) = T_2(u)$ otherwise. The surjectivity of Δ is clear because $T_j(\mathcal{U}^0(M)) = \mathcal{U}^0(M)$, and thus $T_2(\mathcal{U}(M) \backslash \mathcal{U}^0(M)) = \mathcal{U}(M) \backslash \mathcal{U}^0(M)$. Since, by Theorems 2.3 and 2.5, given u, v belonging to two different connected components of $\mathcal{U}(M)$ we have $\|u - v\| = 2$, it follows from the definition of Δ that it is an isometry. Obviously, Δ is non-extendable by Corollary 3.8.

Proposition 3.9. *Let* $T, T_0 : M \to N$ *be two real linear surjective isometries between two unital JB***-algebras. Let us assume that* T_0 *is a real linear Jordan* **-isomorphism, and suppose that* $U_{T_0(u_0)} = U_{T(u_0)}$ *for all* $u_0 \in \mathcal{U}^0(M)$*. Then there exists a central projection* $p \in M$ *such that* $T|_{M \circ p \equiv M_2(p)} = T_0|_{M \circ p \equiv M_2(p)}$ *and* $T|_{M \circ (\mathbf{1}-p) \equiv M_2(\mathbf{1}-p)} = -T_0|_{M \circ (\mathbf{1}-p) \equiv M_2(\mathbf{1}-p)}$.

Proof. Assume first that T is unital and hence a real linear Jordan *-isomorphism (cf. the comments in page 48). It follows from the assumptions that given $h \in M_{sa}$ we have

$$U_{T_0(e^{ith})}(\mathbf{1}) = U_{T(e^{ith})}(\mathbf{1}),$$

for all $t \in \mathbb{R}$. Taking derivatives at $t = 0$ in the above equality we arrive at

$$2T_0(ih) = 2U_{T_0(ih),\mathbf{1}}(\mathbf{1}) = 2U_{T(ih),\mathbf{1}}(\mathbf{1}) = 2T(ih),$$

which proves that $T_0(ih) = T(ih)$ for all $h \in M_{sa}$. We should recall that T and T_0 are merely real linear maps. Having in mind that T and T_0 are real linear Jordan *-isomorphisms we deduce that

$$-T(h^2) = T(ih)^2 = T_0(ih)^2 = -T_0(h^2), \text{ for all } h \in M_{sa},$$

witnessing that T and T_0 coincide on positive elements of M. Since every element in M_{sa} writes as the orthogonal sum of two positive elements in M, we conclude that T and T_0 coincide on M_{sa}, and for each $x \in M$ we set $x = h + ik$ with $h, k \in M_{sa}$ to prove that

$$T(x) = T(h + ik) = T(h) + T(ik) = T_0(h) + T_0(ik) = T_0(x).$$

We have therefore proved that $T = T_0$ in this case.

By hypothesis, $Id = U_{T_0(\mathbf{1})} = U_{T(\mathbf{1})}$, and hence $T(\mathbf{1})^* = U_{T(\mathbf{1})}(T(\mathbf{1})^*) = T(\mathbf{1})$, witnessing that $T(\mathbf{1})$ is a symmetry in N, that is, there exist two orthogonal projections $q_1, q_2 \in N$ such that $T(\mathbf{1}) = q_1 - q_2$ and $q_1 + q_2 = \mathbf{1}$. Since $U_{q_1} + U_{q_2} - 2U_{q_1,q_2} = U_{q_1-q_2} = Id = U_{q_1+q_2} = U_{q_1} + U_{q_2} + 2U_{q_1,q_2}$, it can be easily deduced that $U_{q_1,q_2} = 0$ and hence $N = N_2(q_1) \oplus^\infty N_2(q_2)$, which implies that q_1 and q_2 are central projections.

Since T_0 is a real linear Jordan *-isomorphism, the elements $p_1 = T_0^{-1}(q_1)$ and $p_2 = T_0^{-1}(q_2)$ are two orthogonal central projections in M with $p_1 + p_2 = \mathbf{1}$ and $M = M_2(p_1) \oplus^\infty M_2(p_2)$. Under these circumstances, every unitary u in M is of the form $u = u_1 + u_2$ with $u_j \in M_2(p_j)$, and $\mathcal{U}^0(M)$ identifies with $\mathcal{U}^0(M_2(p_1)) \times \mathcal{U}^0(M_2(p_2))$. It is clear from the hypotheses that T_0 is the orthogonal sum of the two real linear Jordan *-isomorphisms $T_0|_{M_2(p_1)} : M_2(p_1) \to N_2(q_1)$ and $T_0|_{M_2(p_2)} : M_2(p_2) \to N_2(q_2)$. Moreover, since T is a surjective real linear isometry (and hence a real linear triple isomorphism, cf. [17, Corollary 3.2] or [22, Corollary 3.4]) with $T(\mathbf{1}) = q_1 - q_2$. Therefore, the mapping $T : M \to N(q_1 - q_2)$ is a real linear Jordan *-isomorphism, and T writes as the orthogonal sum of the two real linear Jordan *-isomorphisms $T|_{M_2(p_1)} : M_2(p_1) \to N_2(q_1)$ and $T|_{M_2(p_2)} : M_2(p_2) \to N_2(-q_2)$.

We claim that the pairs $T_0|_{M_2(p_1)}, T|_{M_2(p_1)} : M_2(p_1) \to N_2(q_1)$ and $-T_0|_{M_2(p_2)}$, $T|_{M_2(p_2)} : M_2(p_2) \to N_2(-q_2)$ (which are clearly unital) satisfy the hypothesis assumed for the pair T_0, T. Namely, given $u_j \in \mathcal{U}^0(M_2(p_j))$, the element $u = u_1 + u_2$ is a unitary in $\mathcal{U}^0(M)$, and by our hypotheses and the fact that $M = M_2(p_1) \oplus^\infty M_2(p_2)$ and $N = N_2(q_1) \oplus^\infty N_2(q_2)$ we obtain that

$$U_{T(u_1)} + U_{T(u_2)} = U_{T(u_1+u_2)} = U_{T_0(u_1+u_2)} = U_{T_0(u_1)} + U_{T_0(u_2)},$$

which implies that $U_{T_0(u_1)} = U_{T(u_1)}$ and $U_{T_0(u_2)} = U_{T(u_2)}$, because these maps have orthogonal ranges and supports. This finishes the proof of the claim.

We can therefore apply the conclusion in the first paragraph of this proof to the pairs $T_0|_{M_2(p_1)}$ and $T|_{M_2(p_1)}$ and $-T_0|_{M_2(p_2)}$ and $T|_{M_2(p_2)}$ to obtain that $T_0|_{M_2(p_1)} = T|_{M_2(p_1)}$ and $-T_0|_{M_2(p_2)} = T|_{M_2(p_2)}$. The proof concludes by taking $p = p_1$. □

We have seen in the previous results that the extendibility of a surjective isometry Δ between the sets of unitary elements in two unital JB*-algebras M and N is not always an easy task. Actually, the extendibility is, in general, hopeless. It is natural to ask whether an additional algebraic hypothesis can be added to guarantee a linear extension. The instability of the set of unitaries under Jordan products induces us to discard the preservation of the Jordan product as extra algebraic hypothesis. In view of the properties of surjective real linear isometries between JB*-algebras and the conclusion in [46, Proposition 3.9], it seems natural to assume the following extra premise:

$$\Delta\{u,v,u\} = \Delta U_u(v^*) = U_{\Delta(u)}\left(\Delta(v)^*\right) = \{\Delta(u),\Delta(v),\Delta(u)\}, \text{ for all } u,v \in \mathcal{U}(M).$$

We have been able to find a positive answer under some extra suppositions on the connected components of the unitary set. We shall only remark that the extra assumption is weaker than the property asserting that every unitary element admits a unitary square root.

Proposition 3.10. *Let $\Delta : \mathcal{U}(M) \to \mathcal{U}(N)$ be a surjective isometry between the unitary sets of two unital JB*-algebras. Let $\{\mathcal{U}^0(M)\} \cup \{\mathcal{U}^j(M) : j \in \Lambda\}$ be the collection of all connected components of $\mathcal{U}(M)$. Assume that for each $j \in \Lambda$ there exists $u_j \in \mathcal{U}(M)$ with $u_j^2 \in \mathcal{U}^j(M)$. Suppose additionally that*

$$\Delta\{u,v,u\} = \Delta U_u(v^*) = U_{\Delta(u)}\left(\Delta(v)^*\right) = \{\Delta(u),\Delta(v),\Delta(u)\}, \tag{39}$$

for all $u,v \in \mathcal{U}(M)$. Then Δ admits an extension to a surjective (real linear) isometry from M onto N.

Proof. Up to replacing the original Jordan product and involution on N by the one in $\Delta(\mathbf{1})$-isotope, we can assume that Δ is unital, and hence $\Delta(\mathcal{U}^0(M)) = \mathcal{U}^0(N)$. Let $T_j : M \to N$ denote the surjective real linear isometry extending the mapping $\Delta|_{\mathcal{U}^j(M)} : \mathcal{U}^j(M) \to \Delta(\mathcal{U}^j(M))$ whose existence is assured by Theorem 3.6.

Fix $j,k \in \Lambda$ satisfying $\left(\mathcal{U}^j(M)\right)^* = \mathcal{U}^k(M)$. For each $u_j \in \mathcal{U}^j(M)$, it follows from (39) that

$$T_k(u_j^*) = \Delta(u_j^*) = \Delta U_{\mathbf{1}}(u_j^*) = U_{\mathbf{1}}(\Delta(u_j)^*) = T_j(u_j)^*,$$

which proves that $T_k(u_j^*) = T_j(u_j)^*$ for all $u_j \in \mathcal{U}^j(M)$. We fix $w_j \in \mathcal{U}^j(M)$. Since, by the Wright-Youngson-Russo-Dye theorem [52, Corollary], the convex hull of the set $\{\exp_{w_j}(ih) : h \in (M(w_j))_{sa}\} (\subset \mathcal{U}^j(M) = \mathcal{U}^0(M(w_j)))$ is norm dense in the closed unit ball of $M(w_j)$, it follows from the linearity and continuity of T_k and T_j that

$$T_k(x^*) = T_j(x)^* \text{ for all } x \in M. \tag{40}$$

Take now an arbitrary $u_0 \in \mathcal{U}^0(A)$. By (19) we have $U_{u_0}(u_j^*) \in \mathcal{U}^k(M)$, and hence by (39) we get

$$\begin{aligned} U_{T_k(u_0)}(T_j(u_j)^*) &= U_{T_k(u_0)}(T_k(u_j^*)) = T_k U_{u_0}(u_j^*) = \Delta U_{u_0}(u_j^*) = U_{\Delta(u_0)}\left(\Delta(u_j^*)\right) \\ &= U_{T_0(u_0)}\left(T_k(u_j^*)\right) = (40) = U_{T_0(u_0)}\left(T_j(u_j)^*\right) \end{aligned},$$

for all $u_j \in \mathcal{U}^j(M)$. A new application of the Wright-Youngson-Russo-Dye theorem as in the previous paragraph –together with the surjectivity of T_j– implies that $U_{T_k(u_0)} = U_{T_0(u_0)}$ for all $u_0 \in \mathcal{U}^0(M)$. Proposition 3.9 assures the existence of a central projection $p_k \in M$ such that

$$T_k = T_0|_{M_2(p_k)} \oplus T_0|_{M_2(\mathbf{1}-p_k)}. \tag{41}$$

Fix an arbitrary $l \in \Lambda$. By assumptions, there exists $u_l \in \mathcal{U}(M)$ with $u_l^2 \in \mathcal{U}^l(M)$. We can assume that $u_l \in \mathcal{U}^j(M)$ and $u_l^* \in \mathcal{U}^k(M)$ for some $j, k \in \Lambda$. By hypotheses,

$$T_l(u_l^2) = \Delta(u_l^2) = \Delta U_{u_l}(\mathbf{1}) = U_{\Delta(u_l)}\Delta(\mathbf{1}) = \Delta(u_l)^2 = T_j(u_l)^2. \tag{42}$$

By considering the central projections p_l and p_j given by (41), by orthogonality, we have

$$T_j(u_l)^2 = \left(T_0(u_l \circ p_j) - T_0(u_l \circ (\mathbf{1} - p_j))\right)^2 = T_0(u_l^2 \circ p_j) + T_0(u_l^2 \circ (\mathbf{1} - p_j)) = T_0(u_l^2)$$

and

$$T_l(u_l^2) = T_0(u_l^2 \circ p_l) - T_0(u_l^2 \circ (\mathbf{1} - p_l)),$$

which combined with (42) give $T_0(u_l^2 \circ (\mathbf{1} - p_l)) = 0$, and hence $u_l^2 \circ (\mathbf{1} - p_l) = 0$ (by the injectivity of T_0). Having in mind that p_l is a central projection, we have

$$0 = u_l^2 \circ (\mathbf{1} - p_l) = U_{u_l}(\mathbf{1} - p_l),$$

witnessing that $\mathbf{1} - p_l = 0$, and consequently, $T_l = T_0$. The arbitrariness of $l \in \Lambda$ and Corollary 3.8 give the desired statement. □

Declaration of competing interest

There is no competing interest.

Acknowledgements

We would like to thank Prof. Lajos Molnár for encouraging us to explore this problem. We are also indebted to the anonymous reviewer for several useful comments.

First and fifth authors partially supported by the Spanish Ministry of Science, Innovation and Universities (MICINN) and European Regional Development Fund project no. PGC2018-093332-B-I00, Programa Operativo FEDER 2014-2020 and Consejería de Economía y Conocimiento de la Junta de Andalucía grant numbers A-FQM-242-UGR18 and FQM375. First author partially supported by EPSRC (UK) project "Jordan Algebras, Finsler Geometry and Dynamics" ref. no. EP/R044228/1. Second author partially supported by JSPS KAKENHI Grant Number JP 21J21512. Fourth author partially supported by JSPS KAKENHI (Japan) Grant Number JP 20K03650.

* Funding for open access charge: Universidad de Granada / CBUA

References

[1] E.M. Alfsen, F.W. Shultz, State Spaces of Operator Algebras. Basic Theory, Orientations, and C*-Products, Mathematics: Theory & Applications, Birkhäuser Boston, Inc., Boston, MA, 2001.
[2] E.M. Alfsen, F.W. Shultz, Geometry of State Spaces of Operator Algebras, Mathematics: Theory & Applications, Birkhäuser Boston, Inc., Boston, MA, 2003.
[3] A. Al-Rawashdeh, A. Booth, T. Giordano, Unitary groups as a complete invariant, J. Funct. Anal. 262 (11) (2012) 4711–4730.
[4] B. Aupetit, Spectral characterization of the socle in Jordan-Banach algebras, Math. Proc. Camb. Philos. Soc. 117 (1995) 479–489.
[5] T. Barton, R.M. Timoney, Weak*-continuity of Jordan triple products and applications, Math. Scand. 59 (1986) 177–191.
[6] J. Becerra Guerrero, M. Cueto-Avellaneda, F.J. Fernández-Polo, A.M. Peralta, On the extension of isometries between the unit spheres of a JBW*-triple and a Banach space, J. Inst. Math. Jussieu 20 (1) (2021) 277–303, https://doi.org/10.1017/S1474748019000173.
[7] F.F. Bonsall, J. Duncan, Complete Normed Algebras, Erg. der Math., vol. 80, Springer-Verlag, Berlin, 1973.
[8] R. Braun, W. Kaup, H. Upmeier, A holomorphic characterisation of Jordan-C*-algebras, Math. Z. 161 (1978) 277–290.
[9] M. Burgos, F.J. Fernández-Polo, J. Garcés, J. Martínez, A.M. Peralta, Orthogonality preservers in C*-algebras, JB*-algebras and JB*-triples, J. Math. Anal. Appl. 348 (2008) 220–233.
[10] M. Burgos, A. Kaidi, A. Morales Campoy, A.M. Peralta, M. Ramírez, Von Neumann regularity and quadratic conorms in JB*-triples and C*-algebras, Acta Math. Sin. 24 (2) (2008) 185–200.
[11] M. Cabrera García, A. Rodríguez Palacios, Non-associative Normed Algebras. Vol. 1: The Vidav-Palmer and Gelfand-Naimark Theorems, Encyclopedia of Mathematics and Its Applications, vol. 154, Cambridge University Press, Cambridge, 2014.
[12] M. Cabrera García, A. Rodríguez Palacios, Non-associative Normed Algebras. Vol. 2: Representation Theory and the Zel'manov Approach, Encyclopedia of Mathematics and Its Applications, vol. 167, Cambridge University Press, Cambridge, 2018.
[13] A. Chigogidze, A. Karasev, K. Kawamura, V. Valov, On C*-algebras with the approximate n-th root property, Bull. Aust. Math. Soc. 72 (2) (2005) 197–212.
[14] Ch.-H. Chu, Jordan Structures in Geometry and Analysis, Cambridge University Press, Cambridge, 2012.
[15] M. Cueto-Avellaneda, A.M. Peralta, On the Mazur–Ulam property for the space of Hilbert-space-valued continuous functions, J. Math. Anal. Appl. 479 (1) (2019) 875–902.
[16] M. Cueto-Avellaneda, A.M. Peralta, Can one identify two unital JB*-algebras by the metric spaces determined by their sets of unitaries?, accepted in Linear Multilinear Algebra (2021), https://doi.org/10.1080/03081087.2021.2003745, arXiv:2005.04794.
[17] T. Dang, Real isometries between JB*-triples, Proc. Am. Math. Soc. 114 (1992) 971–980.

[18] T. Dang, Y. Friedman, B. Russo, Affine geometric proofs of the Banach-Stone theorems of Kadison and Kaup, in: Proceedings of the Seventh Great Plains Operator Theory Seminar, Lawrence, KS, 1987, Rocky Mt. J. Math. 20 (2) (1990) 409–428.
[19] H.A. Dye, On the geometry of projections in certain operator algebras, Ann. Math. (2) 61 (1955) 73–89.
[20] C.M. Edwards, G.T. Rüttimann, Compact tripotents in bi-dual JB*-triples, Math. Proc. Camb. Philos. Soc. 120 (1996) 155–173.
[21] A. Fernández López, E. García Rus, E. Sánchez Campos, M. Siles Molina, Strong regularity and generalized inverses in Jordan systems, Commun. Algebra 20 (7) (1992) 1917–1936.
[22] F.J. Fernández-Polo, J. Martínez, A.M. Peralta, Surjective isometries between real JB*-triples, Math. Proc. Camb. Philos. Soc. 137 (2004) 709–723.
[23] F.J. Fernández-Polo, J. Martínez, A.M. Peralta, Geometric characterization of tripotents in real and complex JB*-triples, J. Math. Anal. Appl. 295 (2004) 435–443.
[24] Y. Friedman, B. Russo, Structure of the predual of a JBW*-triple, J. Reine Angew. Math. 356 (1985) 67–89.
[25] Y. Friedman, B. Russo, The Gelfand–Naimark theorem for JB*-triples, Duke Math. J. 53 (1986) 139–148.
[26] H. Hanche-Olsen, E. Størmer, Jordan Operator Algebras, Pitman, London, 1984.
[27] L.A. Harris, A generalization of C*-algebras, Proc. Lond. Math. Soc. (3) 42 (2) (1981) 331–361.
[28] O. Hatori, Isometries of the unitary groups in C*-algebras, Stud. Math. 221 (1) (2014) 61–86.
[29] O. Hatori, G. Hirasawa, T. Miura, L. Molnár, Isometries and maps compatible with inverted Jordan triple products on groups, Tokyo J. Math. 35 (2) (2012) 385–410.
[30] O. Hatori, T. Miura, On a characterization of the maximal ideal spaces of commutative C*-algebras in which every element is the square of another, Proc. Am. Math. Soc. 128 (4) (2000) 1185–1189.
[31] O. Hatori, L. Molnár, Isometries of the unitary group, Proc. Am. Math. Soc. 140 (6) (2012) 2127–2140.
[32] O. Hatori, L. Molnár, Isometries of the unitary groups and Thompson isometries of the spaces of invertible positive elements in C^*-algebras, J. Math. Anal. Appl. 409 (2014) 158–167.
[33] J.M. Isidro, W. Kaup, A. Rodríguez-Palacios, On real forms of JB*-triples, Manuscr. Math. 86 (1995) 311–335.
[34] S.T. Hu, Homotopy Theory, Academic Press, New York, 1959.
[35] J.M. Isidro, A. Rodríguez-Palacios, Isometries of JB-algebras, Manuscr. Math. 86 (1995) 337–348.
[36] N. Jacobson, Structure Theory of Jordan Algebras, Lecture Notes in Math., vol. 5, The University of Arkansas, 1981.
[37] F.B. Jamjoom, A.A. Siddiqui, H.M. Tahlawi, A.M. Peralta, Approximation and convex decomposition by extremals and the λ-function in JBW*-triples, Q. J. Math. (Oxford) 66 (2015) 583–603.
[38] R.V. Kadison, J.R. Ringrose, Fundamentals of the Theory of Operator Algebras. Vol. I: Elementary Theory, Pure and Applied Mathematics, vol. 100, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1983.
[39] W. Kaup, A Riemann mapping theorem for bounded symmetric domains in complex Banach spaces, Math. Z. 183 (1983) 503–529.
[40] W. Kaup, On spectral and singular values in JB*-triples, Proc. R. Ir. Acad. A 96 (1) (1996) 95–103.
[41] W. Kaup, On Grassmannians associated with JB*-triples, Math. Z. 236 (3) (2001) 567–584.
[42] B. Li, Real Operator Algebras, World Scientific Publishing Co., Inc., River Edge, NJ, 2003.
[43] O. Loos, On the set of invertible elements in Banach Jordan algebras, Results Math. 29 (1–2) (1996) 111–114.
[44] H. Matui, M. Nagisa, M. Yamamoto, Square root closed C*-algebras, Kyushu J. Math. 62 (1) (2008) 107–122.
[45] M. Mori, N. Ozawa, Mankiewicz's theorem and the Mazur–Ulam property for C*-algebras, Stud. Math. 250 (3) (2020) 265–281.
[46] A.M. Peralta, Surjective isometries between sets of invertible elements in unital Jordan-Banach algebras, J. Math. Anal. Appl. 502 (2021) 125284.
[47] B. Russo, H.A. Dye, A note on unitary operators in C*-algebras, Duke Math. J. 33 (1966) 413–416.
[48] S. Sakai, On the group isomorphism of unitary groups in AW*-algebras, Tohoku Math. J. 7 (1955) 87–95.
[49] A.A. Siddiqui, JB*-algebras of topological stable rank 1, Int. J. Math. Math. Sci. 2007 (2007) 37186.
[50] M. Takesaki, Theory of Operator Algebras I, Springer, New York, 2003.
[51] J.D.M. Wright, Jordan C*-algebras, Mich. Math. J. 24 (1977) 291–302.

[52] J.D.M. Wright, M.A. Youngson, A Russo–Dye theorem for Jordan C*-algebras, in: Functional Analysis: Surveys and Recent Results, Proc. Conf., Paderborn, 1976, in: North-Holland Math. Studies, vol. 27, North-Holland, Amsterdam, 1977, pp. 279–282, Notas de Mat., No. 63.
[53] J.D.M. Wright, M.A. Youngson, On isometries of Jordan algebras, J. Lond. Math. Soc. 17 (1978) 339–344.
[54] M.A. Youngson, A Vidav theorem for Banach Jordan algebras, Math. Proc. Camb. Philos. Soc. 84 (2) (1978) 263–272.