

Required mathematical properties and behaviors of uncertainty measures on belief intervals

Serafin Moral-García  | Joaquín Abellán 

Department of Computer Science and Artificial Intelligence, University of Granada, Granada, Spain

Correspondence

Serafin Moral-García, Department of Computer Science and Artificial Intelligence, University of Granada, Granada 18071, Spain.

Email: seramoral@decsai.ugr.es

Abstract

The Dempster–Shafer theory of evidence (DST) has been widely used to handle uncertainty-based information. It is based on the concept of basic probability assignment (BPA). Belief intervals are easier to manage than a BPA to represent uncertainty-based information. For this reason, several uncertainty measures for DST recently proposed are based on belief intervals. In this study, we carry out a study about the crucial mathematical properties and behavioral requirements that must be verified by every uncertainty measure on belief intervals. We base on the study previously carried out for uncertainty measures on BPAs. Furthermore, we analyze which of these properties are satisfied by each one of the uncertainty measures on belief intervals proposed so far. Such a comparative analysis shows that, among these measures, the maximum of entropy on the belief intervals is the most suitable one to be employed in practical applications since it is the only one that satisfies all the required mathematical properties and behaviors.

KEYWORDS

behavioral requirements, belief intervals, conflict, mathematical properties, non-specificity, uncertainty measures

Serafin Moral-García and Joaquín Abellán contributed equally to this study.

This is an open access article under the terms of the Creative Commons Attribution License, which permits use, distribution and reproduction in any medium, provided the original work is properly cited.

© 2021 The Authors. *International Journal of Intelligent Systems* Published by Wiley Periodicals LLC

1 | INTRODUCTION

The Dempster–Shafer theory of evidence (DST)^{1,2} has been widely used to represent and handle uncertainty-based information in many areas such as *medical diagnosis*,³ *decision making support*,^{4,5} *face recognition*,⁶ or *pattern classification*.^{7,8} DST is an extension of Probability Theory (PT); it is based on the concept of *basic probability assignment* (BPA), a generalization of the probability distribution concept in PT.

In DST, it is essential to quantify the uncertainty-based information associated with a BPA, and it is still an open problem. Many measures to quantify the uncertainty involved in a BPA have been developed so far. A summary of most of them can be found in Reference [9]. Klir and Wierman¹⁰ provided a set of mathematical properties that have to be satisfied by every measure that quantifies the uncertainty involved in a BPA. That study was extended by Abellán and Masegosa in Reference [11], where a set of behavioral requirements for uncertainty measures in DST was also proposed.

So far, the only uncertainty measure on BPAs that verifies all the crucial mathematical properties and behaviors is the upper entropy.¹² However, the procedures developed so far to compute this measure¹²⁻¹⁵ are pretty complex. For this reason, many alternative measures have been proposed in the last years, such as the Deng entropy,^{16,17} or the one of Jiroušek and Shenoy.¹⁸ But none of these measures satisfies all the necessary mathematical properties and behaviors.

Belief intervals for singletons, whose lower and upper bounds are, respectively, the minimum and maximum support of information represented by the BPA in the corresponding singleton, have recently received considerable attention for calculating the uncertainty-based information involved in a BPA.¹⁹⁻²² In fact, they are easier to manage than the BPA to represent the uncertainty-based information, as explained in Reference [19]. Belief intervals have also been used in Reference [23] for quantifying the uncertainty-based information involved in a D-number,²⁴ a generalization of the concept of BPA in DST.

Hence, in recent works, several uncertainty measures on belief intervals have been developed. More specifically, Yang and Han²⁵ proposed an uncertainty measure based on a distance function of intervals; such a measure was improved by Xinyang, Fuyuan, and Yong²⁶; Wang and Song²⁷ proposed an uncertainty measure based on the relations among the central values of the belief intervals; in Reference [21], we developed a new uncertainty measure that consists of the maximum of entropy on the set of probability distributions compatible with the belief intervals.

In this study, we carry out a study about the mathematical properties that have to be satisfied by every uncertainty measure on belief intervals. We also analyze the crucial behavioral requirements for measures of this category. This study is based on the one carried out by Abellán and Masegosa,¹¹ but such a study is valid for uncertainty measures on BPAs, and the one carried out in this study is for uncertainty measures on belief intervals.

Moreover, we analyze which of the mathematical properties and behaviors established as essential in our study are satisfied by each one of the uncertainty measures on belief intervals proposed so far. Such a comparative analysis is useful to select the most suitable uncertainty measure on belief intervals in practical applications. It is shown that the maximum of entropy on the set of probability distributions consistent with the belief intervals is the only one that verifies all the crucial mathematical properties and behavioral requirements for this type of measure.

This paper is organized as follows: Section 2 describes DST, the main uncertainty measures developed so far on BPAs and belief intervals, and the set of essential mathematical properties and behaviors for uncertainty measures on BPAs. The study about the required mathematical properties and behaviors for uncertainty measures on belief intervals is detailed in Section 3. Section 4 analyzes the mathematical properties and behavioral requirements of the uncertainty measures on belief intervals proposed so far. Conclusions are given in Section 5.

2 | BACKGROUND

2.1 | Dempster–shafer theory of evidence

Let us assume that we have a finite set of possible alternatives $X = \{x_1, \dots, x_n\}$. Let $\wp(X)$ denote the set of all the subsets of X .

The DST^{1,2} is based on the BPA concept. A BPA is a mapping $m: \wp(X) \rightarrow [0, 1]$ such that $m(\emptyset) = 0$ and $\sum_{A \in \wp(X)} m(A) = 1$.

If $A \subseteq X$ verifies that $m(A) > 0$, A is said to be a *focal element* of m .

A given BPA m on X has a *belief function*, Bel_m , and a *plausibility function*, Pl_m , associated with it. They are defined as follows:

$$Bel_m(A) = \sum_{B|B \subseteq A} m(B), \quad Pl_m(A) = \sum_{B|B \cap A \neq \emptyset} m(B), \quad \forall A \in \wp(X). \quad (1)$$

Clearly, $\forall A \subseteq X$, $Bel_m(A) \leq Pl_m(A)$. The interval $[Bel_m(A), Pl_m(A)]$ is known as the *belief interval* of A .

In addition,

$$Pl_m(A) = 1 - Bel_m(\bar{A}), \quad \forall A \subseteq X, \quad (2)$$

where \bar{A} denotes the complement of A .

For a given BPA m on X , there exists a credal set[†] associated with it. It is determined by:

$$\mathcal{P}_m = \{p \in \mathcal{P}(X) | Bel_m(A) \leq p(A) \quad \forall A \subseteq X\}, \quad (3)$$

where $\mathcal{P}(X)$ is the set of all probability distributions on X . We may note that the condition $Bel_m(A) \leq p(A) \quad \forall A \in \wp(X)$ is equivalent to $Bel_m(A) \leq p(A) \leq Pl_m(A) \quad \forall A \in \wp(X)$ due to the equality expressed in Equation (2).

Let us suppose now that X and Y are finite sets. Let m be a BPA on the product space $X \times Y$. The marginal BPA of m on X , $m^{\downarrow X}$, is defined in the following way:

$$m^{\downarrow X}(A) = \sum_{R|A=R_X} m(R), \quad \forall A \subseteq X, \quad (4)$$

being R_X the projection of R on X . The definition of the marginal BPA of m on Y , $m^{\downarrow Y}$, is analogous.

2.2 | Uncertainty measures in DST

Let $X = \{x_1, \dots, x_n\}$ be a finite set of possible alternatives.

The Shannon entropy²⁸ is a well-established measure of uncertainty in classical PT. It is defined, for a probability distribution p on X , in the following way:

$$S(p) = \sum_{i=1}^n p(\{x_i\}) \log_2(p(\{x_i\})). \quad (5)$$

In classical possibility theory, the Hartley measure²⁹ is known to be suitable to quantify uncertainty. It is defined as follows:

$$H(A) = \log_2(|A|), \quad \forall A \subseteq X. \quad (6)$$

The type of uncertainty captured by S is usually called *conflict*, while the one quantified by H is known as *non-specificity*.

Yager³⁰ established that, in DST, both conflict and non-specificity coexist; conflict appears when the information is focused on disjunct sets, whereas non-specificity appears when the information is focused on sets with cardinality greater than 1. In consequence, both Shannon entropy and Hartley measure must be properly extended to DST.

The Hartley measure was generalized to DST by Dubois and Prade.³¹ Such a generalization is defined in the following way:

$$GH(m) = \sum_{A \in \wp(X)} m(A) \log_2(|A|). \quad (7)$$

Several uncertainty measures were proposed for the generalization of the Shannon entropy to DST. One of the most remarkable of them was the Dissonance measure,³⁰ defined as follows:

$$Diss(m) = - \sum_{A \subseteq X} m(A) \log_2(Pl_m(A)). \quad (8)$$

But none of the candidates proposed for the generalization of S satisfies all the crucial properties for this type of measure in DST.

Afterward, a measure of total uncertainty in DST was proposed by Harmanec and Klir.¹² It consists of the maximum of entropy on the credal set associated with m , $\mathcal{P}(m)$, defined in Equation (3):

$$S^*(m) = \max_{p \in \mathcal{P}(m)} \{S(p)\}. \quad (9)$$

This measure was established by Klir and Wierman¹⁰ as suitable to quantify uncertainty-based information in DST because it verifies a set of desirable properties.

Nonetheless, the algorithms proposed so far in References [12-15] for the computation of S^* are very complex. Thus, alternative measures to S^* have been introduced in the last years. For instance, the Deng entropy was proposed in References [16,17]. It is determined by:

$$E_d(m) = - \sum_{A \subseteq X} m(A) \log_2 \left(\frac{m(A)}{2^{|A|} - 1} \right). \quad (10)$$

However, as shown by Abellán,³² this function does not satisfy most of the essential mathematical properties and presents some problematic behaviors. Modifications of the Deng entropy were developed in References [33,34], but they also violate most of the crucial mathematical properties for uncertainty measures in DST.³⁵

Some recent total uncertainty measures use the plausibility transformation,^{36,37} defined in the following way:

$$Pt(x_i) = \frac{Pl_m(\{x_i\})}{\sum_{j=1}^n Pl_m(\{x_j\})}, \quad \forall i = 1, 2, \dots, n. \quad (11)$$

A total uncertainty measure consisting of the sum of GH and the Shannon entropy of the plausibility transformation was introduced by Jiroušek and Shenoy¹⁸:

$$H_{JS}(m) = -\sum_{i=1}^n Pt(x_i) \log_2(Pt(x_i)) + GH(m). \quad (12)$$

Pan et al.,³⁸ proposed a total uncertainty measure also based on the plausibility transformation. It is defined as follows:

$$H_{PQ}(m) = -\sum_{A \subseteq X} m(A) \log_2(Pm(A)) + GH(m), \quad (13)$$

where $Pm(A) = \sum_{x \in A} Pt(x)$, $\forall A \subseteq X$.

We showed in Reference [21] that neither H_{JS} nor H_{PQ} satisfy all the required mathematical properties and behaviors for uncertainty measures in DST.

2.2.1 | Uncertainty measures on belief intervals

Belief intervals have attracted considerable interest to quantify uncertainty-based information in DST.

Let m be a BPA on X , and Bel_m and Pl_m its associated belief and plausibility functions, respectively.

Let us consider the corresponding set of belief intervals for singletons:

$$\mathcal{I}_m = \{[Bel_m(\{x_i\}), Pl_m(\{x_i\})], \forall i = 1, \dots, n\}. \quad (14)$$

The set of belief intervals \mathcal{I}_m has associated the following credal set²¹:

$$\mathcal{P}(\mathcal{I}_m) = \{p \in \mathcal{P}(X) | p(\{x_i\}) \in [Bel_m(\{x_i\}), Pl_m(\{x_i\})], \forall i = 1, \dots, n\}, \quad (15)$$

where $\mathcal{P}(X)$ is the set of all probability distributions on X .

As shown in References [21,27], this set of probability intervals is reachable, which means that, $\forall i = 1, \dots, n$, and $\forall v_i \in [Bel_m(\{x_i\}), Pl_m(\{x_i\})]$, there exists $p_{v_i} \in \mathcal{P}(\mathcal{I}_m)$ such that $p_{v_i}(\{x_i\}) = v_i$.

We show below the total uncertainty measures proposed so far on \mathcal{I}_m .

- The total uncertainty measure defined by Yang and Han in Reference [25] $TUM^I(\mathcal{I}_m)$, utilizes the following distance measure for intervals:

$$d^I([a_1, b_1], [a_2, b_2]) = \sqrt{\left[\frac{a_1 + b_1}{2} - \frac{a_2 + b_2}{2}\right]^2 + \frac{1}{3}\left[\frac{b_1 - a_1}{2} - \frac{b_2 - a_2}{2}\right]^2}. \quad (16)$$

Then, TUM^I is defined as follows:

$$TUM^I(\mathcal{I}_m) = 1 - \frac{\sqrt{3}}{n} \sum_{i=1}^n d^I([Bel_m(\{x_i\}), Pl_m(\{x_i\})], [0, 1]). \quad (17)$$

- In Reference [26], Xinyang, Fuyuan, and Yong developed a total uncertainty measure on belief intervals to solve some drawbacks of the previous one. Such a measure employs the following distance function for intervals:

$$d_E^I([a_1, b_1], [a_2, b_2]) = \sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2}. \quad (18)$$

The uncertainty measure introduced by Xinyang, Fuyuan, and Yong is defined in the following way:

$$TUM_E^I(\mathcal{I}_m) = \sum_{i=1}^n \left[1 - d_E^I([Bel_m(\{x_i\}), Pl_m(\{x_i\})], [0, 1]) \right]. \quad (19)$$

- The total uncertainty measure proposed by Wang and Song in Reference [27] is defined as follows:

$$SU(\mathcal{I}_m) = \sum_{i=1}^n \left[-\frac{Bel_m(\{x_i\}) + Pl_m(\{x_i\})}{2} \log_2 \frac{Bel_m(\{x_i\}) + Pl_m(\{x_i\})}{2} + \frac{Pl_m(\{x_i\}) - Bel_m(\{x_i\})}{2} \right]. \quad (20)$$

- We proposed in Reference [21] a total uncertainty measure on belief intervals that consists of the maximum of entropy on the credal set associated with the belief intervals:

$$S^*(\mathcal{P}(\mathcal{I}_m)) = \max_{p \in \mathcal{P}(\mathcal{I}_m)} S(p). \quad (21)$$

2.2.2 | Properties and behaviors for total uncertainty measures on BPAs

According to the research carried out by Abellán and Masegosa in Reference [11], every total uncertainty measure on BPAs TUM must verify the following mathematical properties:

1. **Probabilistic consistency:** If m is a BPA on X such that all its focal elements are singletons, then TUM must coincide with the Shannon entropy:

$$TUM(m) = \sum_{i=1}^n m(\{x_i\}) \log_2 m(\{x_i\}). \quad (22)$$

2. **Set consistency:** If m is a BPA on X such that $m(A) = 1$ for some $A \subseteq X$, then TUM must collapse to the Hartley measure:

$$TUM(m) = \log_2 |A|. \quad (23)$$

3. **Range:** TUM must take values in $[0, \log_2(n)]$.
4. **Subadditivity:** Let X and Y be finite sets and m a BPA on the product space $X \times Y$. Let $m^{\downarrow X}$ and $m^{\downarrow Y}$ be the marginal BPAs of m on X and Y , respectively. Then, it must hold that:

$$TUM(m) \leq TUM(m^{\downarrow X}) + TUM(m^{\downarrow Y}). \quad (24)$$

5. Additivity: Let us suppose that m is a BPA on a product space $X \times Y$, where X and Y are finite sets. Let $m^{\downarrow X}$ and $m^{\downarrow Y}$ be the marginal BPAs of m on X and Y , respectively. Let us assume that the marginal BPAs are not-interactive, that is, $m(A \times B) = m^{\downarrow X}(A)m^{\downarrow Y}(B) \forall A \subseteq X, B \subseteq Y$, and $m(C) = 0$ if $C \neq A \times B$. Then, TUM must verify that:

$$TUM(m) = TUM(m^{\downarrow X}) + TUM(m^{\downarrow Y}). \quad (25)$$

6. Monotonicity: TUM has to highlight consistently a decrease or increase of uncertainty-based information. Formally, if m_1 and m_2 are two BPAs on X such that $\mathcal{P}_{m_1} \subseteq \mathcal{P}_{m_2}$, then it must hold that:

$$TUM(m_1) \leq TUM(m_2). \quad (26)$$

Abellán and Masegosa, in Reference [11], concluded that a total uncertainty measure on BPAs TUM , in addition to these mathematical properties, must satisfy the following behavioral requirements:

1. The complexity of the calculation of TUM must not be too high.
2. It has to be possible to decompose TUM into two measures that respectively quantify conflict and non-specificity.
3. TUM has to be sensitive to changes in the BPA, directly or via its parts of conflict and non-specificity.
4. The extension of TUM to more general theories than DST must be possible.

3 | MATHEMATICAL PROPERTIES AND BEHAVIORAL REQUIREMENTS OF TOTAL UNCERTAINTY MEASURES ON BELIEF INTERVALS

We must take into account the following issues for a total uncertainty measure on the set of belief intervals \mathcal{I}_m , determined via Equation (14):

- When there is a single probability distribution compatible with this set of intervals, which occurs if, and only if, $Bel_m(\{x_i\}) = Pl_m(\{x_i\}) \forall i = 1, \dots, n$, then a total uncertainty measure on \mathcal{I}_m has to coincide with the well-established uncertainty measure in PT, the Shannon entropy.
- If it is only known that the information is focused on a single subset $A \subseteq X$ with $|A| \geq 2$, that is, $Bel_m(\{x_i\}) = 0 \forall i = 1, \dots, n$, $Pl_m(\{x_i\}) = 0 \forall x_i \notin A$ and $Pl_m(\{x_i\}) = 1 \forall x_i \in A$, then a total uncertainty measure on \mathcal{I}_m may have to coincide with the one established as appropriate in classical possibility theory. Nevertheless, as pointed by Wang and Song in Reference [5], it should be considered that the uncertainty in a classical set depends on its cardinality. Consequently, in these cases, it is only crucial that a total uncertainty measure on \mathcal{I}_m is an increasing function of $|A|$.
- In the study carried out by Abellán and Masegosa in Reference [11], it was established that the range of a total uncertainty measure on BPAs has to be equal to $[0, \log_2 n]$, as in PT. However, this point is debatable since in DST there are more kinds of uncertainty than in PT and, thus, arguments for a larger range might be reasonable. Nonetheless, a total uncertainty

measure on \mathcal{I}_m must be nonnegative. The value 0 must be reached if, and only if, the information is focused on a singleton, that is, $Bel_m(\{x_i\}) = Pl_m(\{x_i\}) = 1$ for some $i \in \{1, \dots, n\}$ and $Bel_m(\{x_j\}) = Pl_m(\{x_j\}) = 0 \forall j \in \{1, \dots, n\}$ with $j \neq i$. It can be stated that it is the only case in which there is no uncertainty. Furthermore, where there is an absolute lack of information, that is, when $Bel_m(\{x_i\}) = 0$ and $Pl_m(\{x_i\}) = 1 \forall i = 1, \dots, n$, a total uncertainty measure on \mathcal{I}_m must attain its maximum value.

- As happens with BPAs, a total uncertainty measure on \mathcal{I}_m has to be consistent when an increase or decrease of information is produced. In terms of belief intervals, the set of belief intervals associated with a BPA m_1 , $\mathcal{I}_{m_1} = \{[Bel_{m_1}(\{x_i\}), Pl_{m_1}(\{x_i\})], \forall i = 1, \dots, n\}$, involves more uncertainty-based information than the one corresponding to another BPA m_2 , $\mathcal{I}_{m_2} = \{[Bel_{m_2}(\{x_i\}), Pl_{m_2}(\{x_i\})], \forall i = 1, \dots, n\}$, if

$$[Bel_{m_1}(\{x_i\}), Pl_{m_1}(\{x_i\})] \subseteq [Bel_{m_2}(\{x_i\}), Pl_{m_2}(\{x_i\})], \quad \forall i = 1, \dots, n. \quad (27)$$

As we show in the following proposition, the condition given in Equation (27) is equivalent to the fact that the set of probability distributions consistent with \mathcal{I}_{m_1} is contained in the one compatible with \mathcal{I}_{m_2} .

Proposition 1. *It holds that:*

$$\mathcal{P}(\mathcal{I}_{m_1}) \subseteq \mathcal{P}(\mathcal{I}_{m_2}) \Leftrightarrow [Bel_{m_1}(\{x_i\}), Pl_{m_1}(\{x_i\})] \subseteq [Bel_{m_2}(\{x_i\}), Pl_{m_2}(\{x_i\})], \quad \forall i = 1, \dots, n.$$

Proof. Let us suppose that $Bel_{m_1}(\{x_i\}) < Bel_{m_2}(\{x_i\})$, for some $i \in \{1, \dots, n\}$. As \mathcal{I}_{m_1} is reachable, $\exists p_i \in \mathcal{P}(\mathcal{I}_{m_1})$ such that $p_i(\{x_i\}) = Bel_{m_1}(\{x_i\}) < Bel_{m_2}(\{x_i\})$, which implies that $p_i \notin \mathcal{P}(\mathcal{I}_{m_2})$.

Likewise, if $Pl_{m_1}(\{x_i\}) > Pl_{m_2}(\{x_i\})$, then $\exists p'_i \in \mathcal{P}(\mathcal{I}_{m_1})$ such that $p'_i(\{x_i\}) = Pl_{m_1}(\{x_i\}) > Pl_{m_2}(\{x_i\})$, and, thus, $p'_i \notin \mathcal{P}(\mathcal{I}_{m_2})$.

In consequence, if $\mathcal{P}(\mathcal{I}_{m_1}) \subseteq \mathcal{P}(\mathcal{I}_{m_2})$, then the condition given in Equation (27) must be satisfied.

Let us assume now that

$$[Bel_{m_1}(\{x_i\}), Pl_{m_1}(\{x_i\})] \subseteq [Bel_{m_2}(\{x_i\}), Pl_{m_2}(\{x_i\})], \quad \forall i = 1, \dots, n.$$

If $p \in \mathcal{P}(\mathcal{I}_{m_1})$, then:

$$Bel_{m_2}(\{x_i\}) \leq Bel_{m_1}(\{x_i\}) \leq p(\{x_i\}) \leq Pl_{m_1}(\{x_i\}) \leq Pl_{m_2}(\{x_i\}), \quad \forall i = 1, \dots, n.$$

Hence,

$$Bel_{m_2}(\{x_i\}) \leq p(\{x_i\}) \leq Pl_{m_2}(\{x_i\}), \quad \forall i = 1, \dots, n,$$

and we conclude that $p \in \mathcal{P}(\mathcal{I}_{m_2})$. □

Deng and Jiang, in Reference [5], analyzed this requirement for total uncertainty measures on belief intervals by utilizing the criterion established to decide whether a certain BPA m_1

contains the uncertainty-based information involved by another one m_2 : $Bel_{m_1}(A) \geq Bel_{m_2}(A)$ and $Pl_{m_1}(A) \leq Pl_{m_2}(A)$, $\forall A \subseteq X$. We may note that this condition is stronger than the one imposed in Equation (27).

Let us suppose now that $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_{n'}\}$ are finite sets. Let m be a BPA on the product space $X \times Y$, and Bel_m and Pl_m the belief and plausibility functions associated with m , respectively. Let us consider the set of belief intervals for singletons corresponding to m :

$$\mathcal{I}_m = \{[Bel_m(x_i, y_j), Pl_m(x_i, y_j)], \quad i = 1, \dots, n, \quad j = 1, \dots, n'\}. \quad (28)$$

Let $\mathcal{P}(\mathcal{I}_m)$ denote the credal set consistent with \mathcal{I}_m . Let us consider the lower and upper probability functions corresponding to \mathcal{I}_m , determined as follows:

$$P_*(\mathcal{I}_m)(A) = \min_{p \in \mathcal{P}(\mathcal{I}_m)} p(A), \quad P^*(\mathcal{I}_m)(A) = \max_{p \in \mathcal{P}(\mathcal{I}_m)} p(A), \quad \forall A \subseteq X. \quad (29)$$

In Reference [21], we defined the projections of the set of belief intervals corresponding to a BPA defined on a product space on the marginal sets. Such a definition is based on the one given in Reference [39] for probability intervals.

The set of belief intervals resulting from projecting \mathcal{I}_m on X is defined in the following way:

$$\mathcal{I}_m^{\downarrow X} = \{[l_i, u_i] | l_i = P_*(\mathcal{I}_m)(x_i \times Y), \quad u_i = P^*(\mathcal{I}_m)(x_i \times Y), \quad \forall i = 1, 2, \dots, n\}. \quad (30)$$

The definition of the projection of \mathcal{I}_m on Y , $\mathcal{I}_m^{\downarrow Y}$, is analogous.

For a total uncertainty measure on the belief intervals associated with a BPA defined on a product space, it is important that, when it is projected in the marginal sets, the total uncertainty does not decrease. This is related to the subadditivity property for total uncertainty measures on BPAs. Nevertheless, if a total uncertainty measure is based on belief intervals, it is much more coherent that this requirement is imposed through the projections of the intervals rather than the ones of the BPA. We pointed this issue in Reference [21] for the total uncertainty measure on belief intervals that we introduced in that work, where we also showed via an example that the belief intervals of the marginal BPAs might not coincide with the marginalization of the belief intervals.

Let $\mathcal{P}(\mathcal{I}_m^{\downarrow X})$ and $\mathcal{P}(\mathcal{I}_m^{\downarrow Y})$ denote the credal sets corresponding to $\mathcal{I}_m^{\downarrow X}$ and $\mathcal{I}_m^{\downarrow Y}$, respectively. If these credal sets are independent, then the value of a total uncertainty measure on \mathcal{I}_m must coincide with the sum of the total uncertainty values on $\mathcal{I}_m^{\downarrow X}$ and $\mathcal{I}_m^{\downarrow Y}$. This is associated with the additivity property for total uncertainty measures on BPAs, but, again, it makes more sense to consider the marginal belief intervals than the marginal BPAs.

For independence of credal sets, the concept of *strong independence*⁴⁰ is commonly used in the literature. According to this concept, there is independence under $\mathcal{P}(\mathcal{I}_m)$ if, and only if,

$$\mathcal{P}(\mathcal{I}_m) = CH\left(\mathcal{P}(\mathcal{I}_m^{\downarrow X}) \times \mathcal{P}(\mathcal{I}_m^{\downarrow Y})\right), \quad (31)$$

where CH denotes the convex hull.

Hence, a total uncertainty measure on \mathcal{I}_m , $TUM(\mathcal{I}_m)$, must satisfy the following mathematical properties:

- 1. Probabilistic consistency:** When $Bel_m(\{x_i\}) = Pl_m(\{x_i\}) \forall i = 1, \dots, n$, $TUM(\mathcal{I}_m)$ has to collapse to the Shannon entropy:

$$TUM(\mathcal{I}_m) = -\sum_{i=1}^n Bel_m(\{x_i\}) \log_2(Bel_m(\{x_i\})). \quad (32)$$

- 2. Generalized set consistency:** If $\exists A \subseteq X$ with $|A| \geq 2$ such that $Bel_m(\{x_i\}) = 0 \forall i = 1, \dots, n$, $Pl_m(\{x_i\}) = 0 \forall x_i \notin A$, and $Pl_m(\{x_i\}) = 1 \forall x_i \in A$, then $TUM(\mathcal{I}_m)$ must take the form:

$$TUM(\mathcal{I}_m) = f(|A|), \quad (33)$$

being $f: \mathbb{N} \rightarrow \mathbb{R}$ an increasing function.

- 3. Coherent range:** $TUM(\mathcal{I}_m)$ has to be non-negative. It must hold that $TUM(\mathcal{I}_m) = 0 \Leftrightarrow Bel_m(\{x_i\}) = Pl_m(\{x_i\}) = 1$ for some $i \in \{1, \dots, n\}$ and $Bel_m(\{x_j\}) = Pl_m(\{x_j\}) = 0 \forall j = 1, \dots, n, j \neq i$.

The maximum value of $TUM(\mathcal{I}_m)$ must be attained when $Bel_m(\{x_i\}) = 0$ and $Pl_m(\{x_i\}) = 1, \forall i = 1, \dots, n$.

- 4. Monotonicity:** Let m_1 and m_2 be two BPAs on X whose respective sets of belief intervals are \mathcal{I}_{m_1} and \mathcal{I}_{m_2} . If it holds that:

$$[Bel_{m_1}(\{x_i\}), Pl_{m_1}(\{x_i\})] \subseteq [Bel_{m_2}(\{x_i\}), Pl_{m_2}(\{x_i\})], \quad \forall i = 1, \dots, n, \quad (34)$$

then TUM must verify that:

$$TUM(\mathcal{I}_{m_1}) \leq TUM(\mathcal{I}_{m_2}). \quad (35)$$

- 5. Subadditivity:** Let m be a BPA on a product space $X \times Y$ and \mathcal{I}_m its associated set of belief intervals. Let us assume that $\mathcal{I}_m^{\downarrow X}$ and $\mathcal{I}_m^{\downarrow Y}$ are the projections of \mathcal{I}_m on X and Y , respectively. Then, TUM must satisfy:

$$TUM(\mathcal{I}_m) \leq TUM(\mathcal{I}_m^{\downarrow X}) + TUM(\mathcal{I}_m^{\downarrow Y}). \quad (36)$$

- 6. Additivity:** Let m be a BPA on a product space $X \times Y$ and \mathcal{I}_m its corresponding set of belief intervals. Let $\mathcal{I}_m^{\downarrow X}$ and $\mathcal{I}_m^{\downarrow Y}$ be the projections of \mathcal{I}_m on X and Y , respectively. Let $\mathcal{P}(\mathcal{I}_m)$, $\mathcal{P}(\mathcal{I}_m^{\downarrow X})$, and $\mathcal{P}(\mathcal{I}_m^{\downarrow Y})$ denote the credal sets consistent with \mathcal{I}_m , $\mathcal{I}_m^{\downarrow X}$, and $\mathcal{I}_m^{\downarrow Y}$, respectively. If there is strong independence under $\mathcal{P}(\mathcal{I}_m)$, that is, $\mathcal{P}(\mathcal{I}_m) = CH(\mathcal{P}(\mathcal{I}_m^{\downarrow X}) \times \mathcal{P}(\mathcal{I}_m^{\downarrow Y}))$, then TUM must verify the following equality:

$$TUM(\mathcal{I}_m) = TUM(\mathcal{I}_m^{\downarrow X}) + TUM(\mathcal{I}_m^{\downarrow Y}). \quad (37)$$

With total uncertainty measures on BPAs, in some cases, depending on the form of the uncertainty measure, it makes more sense to consider the submultiplicativity and multiplicativity properties than subadditivity and additivity. Such properties, for total uncertainty measures on belief intervals, are defined in the following way taking into account the definition of additivity and subadditivity for this type of measures:

Submultiplicativity: Let m be a BPA on a product space $X \times Y$ and \mathcal{I}_m its associated set of belief intervals. Let $\mathcal{I}_m^{\downarrow X}$ and $\mathcal{I}_m^{\downarrow Y}$ be the projections of \mathcal{I}_m on X and Y , respectively. Then, TUM must verify that:

$$TUM(\mathcal{I}_m) \leq TUM(\mathcal{I}_m^{\downarrow X}) \times TUM(\mathcal{I}_m^{\downarrow Y}). \quad (38)$$

Multiplicativity: Let m be a BPA on a product space $X \times Y$ and \mathcal{I}_m its corresponding set of belief intervals. Let $\mathcal{I}_m^{\downarrow X}$ and $\mathcal{I}_m^{\downarrow Y}$ be the projections of \mathcal{I}_m on X and Y , respectively. Let $\mathcal{P}(\mathcal{I}_m)$, $\mathcal{P}(\mathcal{I}_m^{\downarrow X})$, and $\mathcal{P}(\mathcal{I}_m^{\downarrow Y})$ denote the credal sets associated with \mathcal{I}_m , $\mathcal{I}_m^{\downarrow X}$, and $\mathcal{I}_m^{\downarrow Y}$, respectively. If there is strong independence under $\mathcal{P}(\mathcal{I}_m)$, that is, $\mathcal{P}(\mathcal{I}_m) = CH(\mathcal{P}(\mathcal{I}_m^{\downarrow X}) \times \mathcal{P}(\mathcal{I}_m^{\downarrow Y}))$, then TUM must satisfy the following equality:

$$TUM(\mathcal{I}_m) = TUM(\mathcal{I}_m^{\downarrow X}) \times TUM(\mathcal{I}_m^{\downarrow Y}). \quad (39)$$

For total uncertainty measures on BPAs, submultiplicativity and multiplicativity are essentially equivalent to subadditivity and additivity, as pointed by Yang et al.⁴¹

Regarding the behavioral requirements for total uncertainty measures on belief intervals, the following points must be taken into consideration:

- As explained in References [19,21], belief intervals are easier to manage than BPAs to represent uncertainty-based information. So, uncertainty measures on belief intervals are, generally, faster to calculate than uncertainty measures on BPAs. Even so, a total uncertainty measure on belief intervals must not require a very complex calculation.
- When belief intervals are used to quantify uncertainty-based information, conflict and non-specificity also coexist, as pointed by Wang and Song.²⁷ Therefore, a total uncertainty measure on belief intervals must not conceal both kinds of uncertainty, as happens with total uncertainty measures on BPAs.
 - On the one hand, according to Wang and Song,²⁷ the non-specificity of a certain belief interval is measured via its span. But, indeed, it is wanted to measure the non-specificity of the whole set of belief intervals \mathcal{I}_m . The non-specificity value must be equal to 0 if, and only if, there is a unique probability distribution consistent with the belief intervals, that is, $Bel_m(\{x_i\}) = Pl_m(\{x_i\}) \forall i = 1, \dots, n$. The maximum value of non-specificity must be attained when all the probability distributions are compatible with the belief intervals, which happens if, and only if, $Bel_m(\{x_i\}) = 0$ and $Pl_m(\{x_i\}) = 1 \forall i = 1, \dots, n$. So, it makes sense that the non-specificity value of a total uncertainty measure on \mathcal{I}_m indicates how large is the set of probability distributions corresponding to \mathcal{I}_m .
 - On the other hand, the conflict of \mathcal{I}_m represents the distribution of the belief and plausibility values of the elements of X .²⁷ Thus, the conflict value of a total uncertainty measure on belief intervals should be related with the Shannon entropy. The maximum value of conflict must be obtained when only the uniform probability distribution belongs to $\mathcal{P}(\mathcal{I}_m)$ (in this case, there is no non-specificity). If the plausibility value for an element of X is equal to 1, then, due to the reachability of a set of belief intervals, the belief values for the rest of the elements of X are equal to 0. In these situations, a degenerate probability distribution is consistent with \mathcal{I}_m , and it can be considered that there is no conflict; the only type of uncertainty existing in these cases is non-specificity, which depends on how

large is $\mathcal{P}(\mathcal{I}_m)$. Hence, it is logical that the conflict value of \mathcal{I}_m coincides with the minimum conflict value of all the probability distributions compatible with \mathcal{I}_m .

- As happens with total uncertainty measures on BPAs, a total uncertainty measure on belief intervals must be sensitive to changes in the belief intervals. It must be remarked that, if a certain belief interval is widened (narrowed), then the non-specificity value may increase (decrease). In contrast, in these cases, the conflict value might decrease (increase). Consequently, it makes sense that, when there are changes in the belief intervals, the total uncertainty value keeps equal and the conflict and non-specificity values vary. Hence, a total uncertainty measure on belief intervals has to be sensitive to changes in the belief intervals, directly or via its parts of conflict and non-specificity.
- In the study carried out by Abellán and Masegosa,¹¹ it was pointed that every total uncertainty measure in DST must be extensible to more general theories. Nonetheless, in most of the more general theories than DST, the evidence can be expressed via a lower probability function, which always has associated an upper probability function (see Reference [42], for more details). In consequence, in more general theories than DST, the lower and upper probability values for singletons can be considered and, thus, the extension of a total uncertainty measure on belief intervals to more general theories than DST is always possible.

Therefore, every total uncertainty measure on \mathcal{I}_m , $TUM(\mathcal{I}_m)$, must satisfy the following behavioral requirements:

1. The calculation of $TUM(\mathcal{I}_m)$ must not be too complex.
2. It has to be possible to decompose $TUM(\mathcal{I}_m)$ into two measures that coherently indicate, respectively, the conflict and non-specificity values corresponding to \mathcal{I}_m .
3. $TUM(\mathcal{I}_m)$ must be sensitive to changes in the belief intervals, directly or through its components of conflict and non-specificity.

4 | ANALYSIS OF PROPERTIES AND BEHAVIORS FOR TOTAL UNCERTAINTY MEASURES ON BELIEF INTERVALS

Let $X = \{x_1, \dots, x_n\}$ be a finite set of possible alternatives. Let m be a BPA on X , Bel_m the belief function associated with m , and Pl_m its corresponding plausibility function. Let us consider the set of belief intervals for singletons \mathcal{I}_m , determined via Equation (14), and the credal set consistent with it $\mathcal{P}(\mathcal{I}_m)$, given by Equation (15).

In this section, we analyze which of the total uncertainty measures on belief intervals proposed so far, described in Section 2.2.1, satisfy each one of the essential mathematical properties for total uncertainty measures on belief intervals, exposed in Section 3.

- **Probabilistic consistency:** If $Bel_m(\{x_i\}) = Pl_m(\{x_i\}) \forall i = 1, \dots, n$, it is easy to deduce that both $TUM^I(\mathcal{I}_m)$ and $TUM_E^I(\mathcal{I}_m)$ may differ from the Shannon entropy. In contrast, in these situations, $SU(\mathcal{I}_m) = -\sum_{i=1}^n Bel_m(\{x_i\}) \log_2(Bel_m(\{x_i\}))$, i.e. $SU(\mathcal{I}_m)$ collapses to the Shannon entropy. Clearly, in these cases, $\mathcal{P}(\mathcal{I}_m)$ contains a single probability distribution and $S^*(\mathcal{P}(\mathcal{I}_m))$ coincides with the Shannon entropy.

- Generalized set consistency: Let us suppose that $\exists A \subseteq X$ such that $Bel_m(\{x_i\}) = 0 \forall i = 1, \dots, n, Pl_m(\{x_i\}) = 1 \forall x_i \in A$, and $Pl_m(\{x_j\}) = 0 \forall x_j \notin A$. For $x_i \in A$, it holds that $d^I([Bel_m(\{x_i\}), Pl_m(\{x_i\})], [0, 1]) = d^I([0, 1], [0, 1]) = 0$. For $x_i \notin A$, it is satisfied that $d^I([Bel_m(\{x_i\}), Pl_m(\{x_i\})], [0, 1]) = d^I([0, 0], [0, 1]) = \frac{1}{\sqrt{3}}$. Hence, in these scenarios, $TUM^I(\mathcal{I}_m) = 1 - \frac{1}{n}\sqrt{3} \sum_{x_i \notin A} \frac{1}{\sqrt{3}} = 1 - \frac{|A^c|}{n} = \frac{|A|}{n} \Rightarrow TUM^I$ satisfies generalized set consistency.

In these cases,

$$\begin{aligned} SU(\mathcal{I}_m) &= \sum_{i=1}^n \left[-\frac{Bel_m(\{x_i\}) + Pl_m(\{x_i\})}{2} \log_2 \left(\frac{Bel_m(\{x_i\}) + Pl_m(\{x_i\})}{2} \right) \right. \\ &\quad \left. + \frac{Pl_m(\{x_i\}) - Bel_m(\{x_i\})}{2} \right] \\ &= \sum_{x_i \in A} \left[-\frac{1}{2} \log_2 \frac{1}{2} + \frac{1}{2} \right] + \sum_{x_i \notin A} 0 = \sum_{x_i \in A} 1 = |A| \Rightarrow \end{aligned}$$

SU verifies the generalized set consistency property.

Concerning TUM^I_E , we have that, for $x_i \in A, d^I_E([Bel_m(\{x_i\}), Pl_m(\{x_i\})], [0, 1]) = d^I_E([0, 1], [0, 1]) = 0$. For $x_i \notin A, d^I_E([Bel_m(\{x_i\}), Pl_m(\{x_i\})], [0, 1]) = d^I([0, 0], [0, 1]) = 1$.

Therefore,

$$\begin{aligned} TUM^I_E(\mathcal{I}_m) &= \sum_{i=1}^n \left[1 - d^I_E([Bel_m(\{x_i\}), Pl_m(\{x_i\})], [0, 1]) \right] \\ &= \sum_{x_i \in A} \left[1 - d^I_E([0, 1], [0, 1]) \right] + \sum_{x_i \notin A} \left[1 - d^I_E([0, 0], [0, 1]) \right] \\ &= \sum_{x_i \in A} 1 - \sum_{x_i \notin A} 0 = |A|, \end{aligned}$$

which implies that TUM^I_E satisfies the generalized set consistency property.

We may observe that, in these cases, the probability distribution \hat{p} of maximum entropy, among the ones belonging to $\mathcal{P}(\mathcal{I}_m)$, is given by:

$$\hat{p}(\{x_i\}) = \begin{cases} \frac{1}{|A|} & \text{if } x_i \in A \\ 0 & \text{if } x_i \notin A \end{cases}$$

It holds that

$$S^*(\mathcal{P}(\mathcal{I}_m)) = S(\hat{p}) = - \sum_{x_i \in A} \frac{1}{|A|} \log_2 \left(\frac{1}{|A|} \right) = |A| \frac{1}{|A|} \log_2(|A|) = \log_2(|A|).$$

Since \log_2 is an increasing function, it is deduced that S^* verifies generalized set consistency.

- **Coherent range:** The range of TUM^I is equal to $[0, 1]$.²⁵ The minimum value of $d^I([Bel_m(\{x_i\}), Pl_m(\{x_i\})], [0, 1])$ is reached when $Bel_m(\{x_i\}) = 0$ and $Pl_m(\{x_i\}) = 1$. Such a minimum value is equal to 0. In consequence, when $Bel_m(\{x_i\}) = 0$ and $Pl_m(\{x_i\}) = 1 \forall i = 1, \dots, n, TUM^I$ attains its maximum value, which is equal to 1. The maximum value of $d^I([Bel_m(\{x_i\}), Pl_m(\{x_i\})], [0, 1])$ is obtained when $Bel_m(\{x_i\}) = Pl_m(\{x_i\}) = 0$ or $Bel_m(\{x_i\}) = Pl_m(\{x_i\}) = 1$. In both cases, such a value is equal to $\sqrt{3}$.

In this way, TUM^I is equal to 0 if, and only if, $Bel_m(\{x_i\}) = Pl_m(\{x_i\}) = 1$ for some $i \in \{1, \dots, n\}$ and $Bel_m(\{x_j\}) = Pl_m(\{x_j\}) = 0 \forall j \in \{1, \dots, n\} \setminus \{i\}$.

It can be checked that $d_E^I([Bel_m(\{x_i\}), Pl_m(\{x_i\})], [0, 1]) = 0 \Leftrightarrow Bel_m(\{x_i\}) = 0$ and $Pl_m(\{x_i\}) = 1$. Thus, TUM_E^I obtains its maximum value, which is equal to n , when $Bel_m(\{x_i\}) = 0$ and $Pl_m(\{x_i\}) = 1 \forall i = 1, \dots, n$. Now, $d_E^I([Bel_m(\{x_i\}), Pl_m(\{x_i\})], [0, 1]) = 1$ if, and only if $\sqrt{(Bel_m(\{x_i\}))^2 + (1 - Pl_m(\{x_i\}))^2} = 1 \Leftrightarrow Bel_m(\{x_i\}) = Pl_m(\{x_i\}) = 0$ or $Bel_m(\{x_i\}) = Pl_m(\{x_i\}) = 1$. So, TUM_E^I is equal to 0 if, and only if, $\exists i \in \{1, \dots, n\}$ such that $Bel_m(\{x_i\}) = Pl_m(\{x_i\}) = 1$ and $Bel_m(\{x_j\}) = Pl_m(\{x_j\}) = 0 \forall j \in \{1, \dots, n\} \setminus \{i\}$. We may note that

$$\frac{Bel_m(\{x_i\}) + Pl_m(\{x_i\})}{2} \log_2 \left(\frac{Bel_m(\{x_i\}) + Pl_m(\{x_i\})}{2} \right) = 0 \Leftrightarrow \\ Bel_m(\{x_i\}) = Pl_m(\{x_i\}) = 0 \vee Bel_m(\{x_i\}) = Pl_m(\{x_i\}) = 1, \quad \forall i = 1, \dots, n.$$

Therefore, SU is equal 0 if, and only if, $\exists i \in \{1, \dots, n\}$ such that $Bel_m(\{x_i\}) = Pl_m(\{x_i\}) = 1$ and $Bel_m(\{x_j\}) = Pl_m(\{x_j\}) = 0 \forall j \in \{1, \dots, n\} \setminus \{i\}$. The maximum value of $\frac{Pl_m(\{x_i\}) - Bel_m(\{x_i\})}{2}$ is attained when $Bel_m(\{x_i\}) = 0$ and $Pl_m(\{x_i\}) = 1 \forall i = 1, \dots, n$. In these situations, $-\frac{Bel_m(\{x_i\}) + Pl_m(\{x_i\})}{2} \log_2 \left(\frac{Bel_m(\{x_i\}) + Pl_m(\{x_i\})}{2} \right)$ also reaches its maximum value, $\forall i = 1, \dots, n$, and, thus, the maximum value of SU is attained.

As we argued in Reference [21], the minimum value of $S^*(\mathcal{P}(\mathcal{I}_m))$, which is equal to 0, is obtained if, and only if, $\mathcal{P}(\mathcal{I}_m)$ just contains a degenerate probability distribution, which happens if, and only if, $Bel_m(\{x_i\}) = Pl_m(\{x_i\}) = 1$ for some $i \in \{1, \dots, n\}$ and $Bel_m(\{x_j\}) = Pl_m(\{x_j\}) = 0 \forall j \in \{1, \dots, n\}, j \neq i$. Furthermore, when all the probability distributions on X belong to $\mathcal{P}(\mathcal{I}_m)$, that is, when $Bel_m(\{x_i\}) = 0$ and $Pl_m(\{x_i\}) = 1 \forall i = 1, \dots, n$, $S^*(\mathcal{P}(\mathcal{I}_m))$ attains its maximum value. Thus, the four total uncertainty measures on belief intervals proposed so far have a coherent range.

- **Monotonicity:** Let m_1 and m_2 be two BPAs on X and \mathcal{I}_{m_1} and \mathcal{I}_{m_2} their respective sets of belief intervals. Let us assume that

$$[Bel_{m_1}(\{x_i\}), Pl_{m_1}(\{x_i\})] \subseteq [Bel_{m_2}(\{x_i\}), Pl_{m_2}(\{x_i\})], \quad \forall i = 1, \dots, n. \quad (40)$$

Deng and Jiang, in Reference [5], showed via counterexamples that, in these situations, it does not always hold that $SU(\mathcal{I}_{m_1}) \leq SU(\mathcal{I}_{m_2})$ nor that $TUM^I(\mathcal{I}_{m_1}) \leq TUM^I(\mathcal{I}_{m_2})$. In contrast, they demonstrated that $TUM_E^I(\mathcal{I}_{m_1}) \leq TUM_E^I(\mathcal{I}_{m_2})$ is always satisfied in these scenarios. In these cases, $\mathcal{P}(\mathcal{I}_{m_1}) \subseteq \mathcal{P}(\mathcal{I}_{m_2})$ and, obviously, $S^*(\mathcal{P}(\mathcal{I}_{m_1})) \leq S^*(\mathcal{P}(\mathcal{I}_{m_2}))$. Hence, TUM_E^I and S^* verify the monotonicity property, unlike TUM^I and SU .

- **Subadditivity/submultiplicativity and additivity/multiplicativity:** Since $\log(a \times b) = \log(a) + \log(b) \forall a, b \in \mathbb{R}$, for SU and S^* , the subadditivity and additivity properties make more sense than submultiplicativity and multiplicativity. In contrast, for the interval distance-based total uncertainty measures, the submultiplicativity and multiplicativity requirements are more appropriate than subadditivity and additivity.⁴¹ The following example shows that both TUM^I and TUM_E^I violate the submultiplicativity property.

Example 1. Let $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2\}$ be finite sets. We denote $z_{ij} = (x_i, y_j), \forall i = 1, 2, 3, j = 1, 2$. Let us consider the following BPA m on the product space $X \times Y$:

$$m(z_{11}) = 0.8, \quad m(X \times Y) = 0.2.$$

We have the following set of belief intervals for singletons \mathcal{I}_m :

$$\begin{aligned} z_{11} &\rightarrow [0.8, 1], \\ z_{12} &\rightarrow [0, 0.2], \\ z_{21} &\rightarrow [0, 0.2], \\ z_{22} &\rightarrow [0, 0.2], \\ z_{31} &\rightarrow [0, 0.2], \\ z_{32} &\rightarrow [0, 0.2]. \end{aligned}$$

The marginal set of belief intervals on X , $\mathcal{I}_m^{\downarrow X}$, is given by:

$$\begin{aligned} x_1 &\rightarrow [0.8, 1], \\ x_2 &\rightarrow [0, 0.2], \\ x_3 &\rightarrow [0, 0.2]. \end{aligned}$$

The set of the projections of the belief intervals on Y , $\mathcal{I}_m^{\downarrow Y}$, is the following one:

$$\begin{aligned} y_1 &\rightarrow [0.8, 1], \\ y_2 &\rightarrow [0, 0.2]. \end{aligned}$$

We have:

$$\begin{aligned} TUM^I(\mathcal{I}_m) &= 1 - \frac{\sqrt{3}}{6} (d^I([0.8, 1], [0, 1]) + 5 \times d^I([0, 0.2], [0, 1])) \\ &= 1 - \frac{\sqrt{3}}{6} \left(\frac{0.8}{\sqrt{3}} + 5 \times \frac{0.8}{\sqrt{3}} \right) = 0.2, \\ TUM^I(\mathcal{I}_m^{\downarrow X}) &= 1 - \frac{\sqrt{3}}{3} (d^I([0.8, 1], [0, 1]) + 2 \times d^I([0, 0.4], [0, 1])) \\ &= 1 - \frac{\sqrt{3}}{3} \times \left(\frac{0.8}{\sqrt{3}} + \frac{2 \times 0.6}{\sqrt{3}} \right) = \frac{1}{3}, \\ TUM^I(\mathcal{I}_m^{\downarrow Y}) &= 1 - \frac{\sqrt{3}}{2} (d^I([0.8, 1], [0, 1]) + d^I([0, 0.6], [0, 1])) \\ &= 1 - \frac{\sqrt{3}}{2} \times \left(\frac{0.8}{\sqrt{3}} + \frac{0.4}{\sqrt{3}} \right) = 0.4, \\ TUM_E^I(\mathcal{I}_m) &= \left(1 - d_E^I([0.8, 1], [0, 1]) \right) + 5 \times \left(1 - d_E^I([0, 0.2], [0, 1]) \right) \\ &= (1 - 0.8) + 5 \times (1 - 0.8) = 6 \times 0.2 = 1.2, \\ TUM_E^I(\mathcal{I}_m^{\downarrow X}) &= \left(1 - d_E^I([0.8, 1], [0, 1]) \right) + 2 \times \left(1 - d_E^I([0, 0.2], [0, 1]) \right) \\ &= (1 - 0.8) + 2 \times (1 - 0.8) = 3 \times 0.2 = 0.6, \\ TUM_E^I(\mathcal{I}_m^{\downarrow Y}) &= \left(1 - d_E^I([0.8, 1], [0, 1]) \right) + \left(1 - d_E^I([0, 0.2], [0, 1]) \right) \\ &= 0.2 + 0.2 = 0.4. \end{aligned}$$

Hence,

$$TUM^I(\mathcal{I}_m^{\downarrow X}) \times TUM^I(\mathcal{I}_m^{\downarrow Y}) = \frac{0.4}{3} < 0.2 = TUM^I(\mathcal{I}_m).$$

$$TUM_E^I(\mathcal{I}_m^{\downarrow X}) \times TUM_E^I(\mathcal{I}_m^{\downarrow Y}) = 0.24 < 1.2 = TUM_E^I(\mathcal{I}_m).$$

In the following example, it is shown that TUM^I and TUM_E^I do neither satisfy the multiplicativity property:

Example 2. Let us suppose that $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2\}$ are finite sets and that we have the following BPA m on $X \times Y$:

$$m(\{z_{11}\}) = \frac{1}{3}, \quad m(\{z_{21}\}) = \frac{1}{3}, \quad m(\{z_{31}\}) = \frac{1}{3},$$

where $z_{ij} = (x_i, y_j)$, $\forall i = 1, 2, 3, j = 1, 2$. The set of belief intervals for singletons, \mathcal{I}_m , is given by:

$$z_{11} \rightarrow \left[\frac{1}{3}, \frac{1}{3} \right],$$

$$z_{12} \rightarrow [0, 0],$$

$$z_{21} \rightarrow \left[\frac{1}{3}, \frac{1}{3} \right],$$

$$z_{22} \rightarrow [0, 0],$$

$$z_{31} \rightarrow \left[\frac{1}{3}, \frac{1}{3} \right],$$

$$z_{32} \rightarrow [0, 0].$$

Let $\mathcal{I}_m^{\downarrow X}$ and $\mathcal{I}_m^{\downarrow Y}$ denote the projections of \mathcal{I}_m on X and Y , respectively. They are determined by:

$$x_1 \rightarrow \left[\frac{1}{3}, \frac{1}{3} \right],$$

$$x_2 \rightarrow \left[\frac{1}{3}, \frac{1}{3} \right],$$

$$x_3 \rightarrow \left[\frac{1}{3}, \frac{1}{3} \right],$$

$$y_1 \rightarrow [1, 1],$$

$$y_2 \rightarrow [0, 0],$$

It is easy to observe that, in this case, $\mathcal{P}(\mathcal{I}_m) = CH\left(\mathcal{P}(\mathcal{I}_m^{\downarrow X}) \times \mathcal{P}(\mathcal{I}_m^{\downarrow Y})\right)$. It holds that:

$$\begin{aligned} TUM^I(\mathcal{I}_m) &= 1 - \frac{\sqrt{3}}{6} \left(3 \times d^I \left(\left[\frac{1}{3}, \frac{1}{3} \right], [0, 1] \right) + 3 \times d^I([0, 0], [0, 1]) \right) \\ &= 1 - \frac{\sqrt{3}}{6} \times \left(3 \times \sqrt{\left(\frac{1}{6} \right)^2 + \frac{(0.5)^2}{3}} + \frac{3}{\sqrt{3}} \right) = 0.3354, \end{aligned}$$

$$\begin{aligned} TUM^I(\mathcal{I}_m^{\downarrow X}) &= 1 - \frac{\sqrt{3}}{3} \left(3 \times d^I \left(\left[\frac{1}{3}, \frac{1}{3} \right], [0, 1] \right) \right) \\ &= 1 - \sqrt{3} \times \left(\sqrt{\left(\frac{1}{6} \right)^2 + \frac{(0.5)^2}{3}} \right) = 0.6709, \end{aligned}$$

$$\begin{aligned} TUM^I(\mathcal{I}_m^{\downarrow Y}) &= 1 - \frac{\sqrt{3}}{2} (d^I([1, 1], [0, 1]) + d^I([0, 0], [0, 1])) \\ &= 1 - \frac{\sqrt{3}}{2} \times \frac{2}{\sqrt{3}} = 0. \end{aligned}$$

$$TUM_E^I(\mathcal{I}_m) = 3 \times \left(1 - d_E^I \left(\left[\frac{1}{3}, \frac{1}{3} \right], [0, 1] \right) \right) + 3 \times \left(1 - d_E^I([0, 0], [0, 1]) \right) = 0.7639,$$

$$TUM_E^I(\mathcal{I}_m^{\downarrow X}) = 3 \times \left(1 - d_E^I \left(\left[\frac{1}{3}, \frac{1}{3} \right], [0, 1] \right) \right) = 0.7639,$$

$$TUM_E^I(\mathcal{I}_m^{\downarrow Y}) = 1 - d_E^I([1, 1], [0, 1]) + 1 - d_E^I([0, 0], [0, 1]) = 0.$$

Consequently,

$$TUM^I(\mathcal{I}_m^{\downarrow X}) \times TUM^I(\mathcal{I}_m^{\downarrow Y}) = 0.6709 \times 0 = 0 \neq 0.3354 = TUM^I(\mathcal{I}_m),$$

$$TUM_E^I(\mathcal{I}_m^{\downarrow X}) \times TUM_E^I(\mathcal{I}_m^{\downarrow Y}) = 0.7639 \times 0 = 0 \neq 0.7639 = TUM_E^I(\mathcal{I}_m).$$

The following example shows that SU does not satisfy the subadditivity requirement.

Example 3. Let $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2\}$ be finite sets and m the following BPA on the product space $X \times Y$:

$$m(z_{11}, z_{12}, z_{21}, z_{22}) = 0.7, \quad m(z_{31}, z_{32}) = 0.1, \quad m(X \times Y) = 0.2,$$

where $z_{ij} = (x_i, y_j)$, $\forall i = 1, 2, 3, j = 1, 2$.

We have the following set of belief intervals for singletons \mathcal{I}_m :

$$z_{11} \rightarrow [0, 0.9],$$

$$z_{12} \rightarrow [0, 0.9],$$

$$z_{21} \rightarrow [0, 0.9],$$

$$z_{22} \rightarrow [0, 0.9],$$

$$z_{31} \rightarrow [0, 0.3],$$

$$z_{32} \rightarrow [0, 0.3].$$

We consider the projections of the belief intervals on X and Y , which we denote $\mathcal{I}_m^{\downarrow X}$ and $\mathcal{I}_m^{\downarrow Y}$, respectively:

$$\begin{aligned}
x_1 &\rightarrow [0, 1], \\
x_2 &\rightarrow [0, 1], \\
x_3 &\rightarrow [0, 0.6], \\
y_1 &\rightarrow [0, 1], \\
y_2 &\rightarrow [0, 1],
\end{aligned}$$

It holds that:

$$\begin{aligned}
SU(\mathcal{I}_m) &= 4 \times \left(-\frac{0.9}{2} \log_2 \left(\frac{0.9}{2} \right) + \frac{0.9}{2} \right) + 2 \times \left(-\frac{0.3}{2} \log_2 \left(\frac{0.3}{2} \right) + \frac{0.3}{2} \right) = 4.9947, \\
SU(\mathcal{I}_m^{\downarrow X}) &= 2 \times \left(-\frac{1}{2} \log_2 \left(\frac{1}{2} \right) + \frac{1}{2} \right) - 0.3 \log_2 0.3 = 2.5211, \\
SU(\mathcal{I}_m^{\downarrow Y}) &= 2 \times \left(-\frac{1}{2} \log_2 \left(\frac{1}{2} \right) + \frac{1}{2} \right) = 2.
\end{aligned}$$

In this way:

$$SU(\mathcal{I}_m) = 4.9947 > 4.5211 = 2.5211 + 2 = SU(\mathcal{I}_m^{\downarrow X}) + SU(\mathcal{I}_m^{\downarrow Y}).$$

We show with an example below that SU does also not verify additivity.

Example 4. Let $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2\}$ be finite sets. Let us suppose that we have the following BPA m on the product space $X \times Y$:

$$m(X \times Y) = 1.$$

We denote $z_{ij} = (x_i, y_j) \forall i = 1, 2, 3, j = 1, 2$. The set of belief intervals for singletons, \mathcal{I}_m , is given by:

$$\begin{aligned}
z_{11} &\rightarrow [0, 1], \\
z_{21} &\rightarrow [0, 1], \\
z_{12} &\rightarrow [0, 1], \\
z_{22} &\rightarrow [0, 1], \\
z_{31} &\rightarrow [0, 1], \\
z_{32} &\rightarrow [0, 1].
\end{aligned}$$

The projections of \mathcal{I}_m on X and Y , which we denote $\mathcal{I}_m^{\downarrow X}$ and $\mathcal{I}_m^{\downarrow Y}$, respectively, are the following ones:

$$\begin{aligned}
x_1 &\rightarrow [0, 1], \\
x_2 &\rightarrow [0, 1], \\
x_3 &\rightarrow [0, 1], \\
y_1 &\rightarrow [0, 1], \\
y_2 &\rightarrow [0, 1],
\end{aligned}$$

It is obvious that, in this case, $\mathcal{P}(\mathcal{I}_m) = CH\left(\mathcal{P}(\mathcal{I}_m^{\downarrow X}) \times \mathcal{P}(\mathcal{I}_m^{\downarrow Y})\right)$. We have that:

$$\begin{aligned}
SU(\mathcal{I}_m) &= 6 \times \left(-\frac{1}{2} \log_2 \left(\frac{1}{2} \right) + \frac{1}{2} \right) = 6 \times 1 = 6, \\
SU(\mathcal{I}_m^{lX}) &= 3 \times \left(-\frac{1}{2} \log_2 \left(\frac{1}{2} \right) + \frac{1}{2} \right) = 3 \times 1 = 3, \\
SU(\mathcal{I}_m^{lY}) &= 2 \times \left(-\frac{1}{2} \log_2 \left(\frac{1}{2} \right) + \frac{1}{2} \right) = 2 \times 1 = 2.
\end{aligned}$$

Therefore,

$$SU(\mathcal{I}_m) = 6 \neq 5 = SU(\mathcal{I}_m^{lX}) + SU(\mathcal{I}_m^{lY}).$$

We demonstrated in Reference [21] that S^* verifies both subadditivity and additivity under the definitions given here for such properties.

Concerning the behavioral requirements, we must remark the following issues:

- Once it is disposed of the belief intervals, the computation of TUM^l , TUM_E^l , and SU is direct. The procedure that we presented in Reference [21] to calculate S^* is not as direct as the calculation of the other total uncertainty measures on belief intervals, although the complexity is also not very high.
- So far, it has not been possible to decompose the total uncertainty measures that employ distance functions of belief intervals, TUM^l and TUM_E^l , into two measures that respectively quantify conflict and non-specificity.

In contrast, SU can be rewritten as follows:

$$\begin{aligned}
SU(\mathcal{I}_m) &= \sum_{i=1}^n - \frac{Bel_m(\{x_i\}) + Pl_m(\{x_i\})}{2} \log_2 \left(\frac{Bel_m(\{x_i\}) + Pl_m(\{x_i\})}{2} \right) \\
&\quad + \sum_{i=1}^n \frac{Pl_m(\{x_i\}) - Bel_m(\{x_i\})}{2}.
\end{aligned}$$

The first term of the previous expression indicates conflict, whereas the second one corresponds to non-specificity. In fact, the second term is equal to 0 if, and only if, $Bel_m(\{x_i\}) = Pl_m(\{x_i\}) \forall i = 1, \dots, n$ (the span of all the belief intervals is equal to 0). Also, the first term indicates how the belief and plausibility values for singletons are distributed. However, when $\exists i \in \{1, \dots, n\}$ such that $Pl_m(\{x_i\}) = 1$, the conflict value indicated by SU might not be equal to 0. It is undesirable because, in these cases, there is no conflict in the belief intervals, as argued in Section 3.

In Reference [21], we proposed the following decomposition for S^* :

$$S^*(\mathcal{P}(\mathcal{I}_m)) = (S^* - S_*)(\mathcal{P}(\mathcal{I}_m)) + S_*(\mathcal{P}(\mathcal{I}_m)), \quad (41)$$

where $S_*(\mathcal{P}(\mathcal{I}_m))$ is the minimum of entropy on $\mathcal{P}(\mathcal{I}_m)$. The non-specificity value of S^* is quantified by the first term of Equation (41). Indeed, it is equal to 0 if, and only if, $\mathcal{P}(\mathcal{I}_m)$ contains a unique probability distribution, and $(S^* - S_*)(\mathcal{P}(\mathcal{I}_m))$ attains its maximum value when all the probability distributions on X belong to $\mathcal{P}(\mathcal{I}_m)$. $S_*(\mathcal{P}(\mathcal{I}_m))$ captures the conflict part of $S^*(\mathcal{P}(\mathcal{I}_m))$. As we argued in Reference [21], S_* reaches its minimum value, 0, when a degenerate probability distribution belongs to $\mathcal{P}(\mathcal{I}_m)$. Thus, the decomposition of S^* into conflict and non-specificity measures is pretty logical.

- It is easy to observe that the distance functions utilized in TUM^I and TUM_E^I are sensitive to changes in the belief and plausibility values. So, both uncertainty measures are directly sensitive to changes in the belief intervals. Also, the values $-\frac{Bel_m(\{x_i\}) + Pl_m(\{x_i\})}{2} \log_2 \left(\frac{Bel_m(\{x_i\}) + Pl_m(\{x_i\})}{2} \right)$ and $\frac{Pl_m(\{x_i\}) - Bel_m(\{x_i\})}{2}$ may vary when the belief and plausibility values for singletons change, $\forall i = 1, \dots, n$. In consequence, SU is sensitive to changes in the belief intervals via its parts of conflict and non-specificity. We showed in Reference [21] that the value of $S^*(\mathcal{P}(\mathcal{I}_m))$ might not vary when changes in the belief intervals are produced. Nevertheless, we illustrated that, in those situations, the conflict and non-specificity values vary even though the total uncertainty value keeps constant. Therefore, $S^*(\mathcal{P}(\mathcal{I}_m))$ is sensitive to changes in the belief intervals via its parts of conflict and non-specificity, although not directly.

Table 1 summarizes the mathematical properties satisfied by the total uncertainty measures on belief intervals developed so far. Likewise, Table 2 shows a summary of the behavioral requirements of such measures.

We must remark the following issues about the mathematical properties:

- The four total uncertainty measures on belief intervals provide a logical result when it is only known that the information is focused on a subset of the set of possible alternatives since all of them verify generalized set consistency.
- The ranges of TUM^I , TUM_E^I , SU , and S^* are all coherent.
- When the belief intervals are reduced to a single probability distribution, both SU and S^* obtain a logical result, which coincides with the well-established uncertainty value for

TABLE 1 Summary of the mathematical properties satisfied by the total uncertainty measures on belief intervals proposed so far

Property	TUM^J	TUM_E^J	SU	S^*
Probabilistic consistency	No	No	Yes	Yes
Generalized set consistency	Yes	Yes	Yes	Yes
Coherent range	Yes	Yes	Yes	Yes
Monotonicity	No	Yes	No	Yes
Subadditivity/submultiplicativity	No	No	No	Yes
Additivity/multiplicativity	No	No	No	Yes

TABLE 2 Summary of the behavioral requirements of the total uncertainty measures on belief intervals proposed so far

Behavioral requirement	TUM^J	TUM_E^J	SU	S^*
Complexity	Low	Low	Low	Medium
Separation	No	No	Improbable	Coherent
Sensitivity	Yes	Yes	Yes	Yes

probability distributions (the Shannon entropy). It does not occur with the total uncertainty measures based on distance functions of intervals.

- Unlike TUM^I and SU , S^* and TUM_E^I are consistent when an increase or decrease of uncertainty-based information expressed via the belief intervals is produced (monotonicity property).
- S^* is the only uncertainty measure on belief intervals that verifies subadditivity and additivity, which implies that it is the only one that produces coherent results when it is defined over a set of belief intervals on a product space that can be decomposed into more simple sets.
- Consequently, only S^* satisfies all the required mathematical properties for total uncertainty measures on belief intervals.

Regarding the behavioral requirements, the following points are remarkable:

- The calculation of S^* is a little bit more complex than TUM^I , TUM_E^I , and SU , although the complexity is also not very high.
- Both SU and S^* can be decomposed into two measures that respectively capture conflict and non-specificity, unlike the intervals distance-based total uncertainty measures. The separation of SU is not as coherent as the one of S^* since the conflict value of SU might not be equal to 0 when the plausibility value for a singleton is equal to 1, which is not very logical.
- The four total uncertainty measures on belief intervals are sensitive to changes in the belief intervals, directly or through their parts of conflict and non-specificity.

Therefore, even though the computation of S^* is a little bit more complex than the other total uncertainty measures on belief intervals, S^* is the only total uncertainty measure on belief intervals so far that satisfies all the essential mathematical properties and behavioral requirements for this kind of measure.

5 | CONCLUSIONS

Belief intervals are easier to manage than BPAs to represent uncertainty-based information in DST. For this reason, they have recently received considerable attention for developing uncertainty measures in DST. In this study, total uncertainty measures on belief intervals have been considered.

On the one hand, we have carried out a study about the crucial mathematical properties and behavioral requirements for total uncertainty measures on belief intervals. Such a study has been based on the one previously carried out for total uncertainty measures on BPAs. It has been highlighted that, when the belief intervals are reduced to a single probability distribution, a total uncertainty measure on the belief intervals must coincide with the one well-established in classical PT, that is, the Shannon entropy; if it is only known that the information expressed via the belief intervals is focused on a subset of possible alternatives, then a total uncertainty measure must take the form of an increasing function with respect to the cardinality of that subset; the range of a total uncertainty measure on belief intervals must be coherent: the minimum value, which has to be equal to 0, must be attained if, and only if, the information is focused on a singleton, and the maximum value when all the probability distributions are consistent with the belief intervals; a total uncertainty measure

on belief intervals has to be consistent with an increase or decrease of information expressed by the belief intervals; the values of a total uncertainty measure on a set of belief intervals corresponding to a BPA defined over a joint space that can be decomposed into more simple sets must be coherent. Our proposed set of behavioral requirements for total uncertainty measures on belief intervals reveals that the computation of a measure of this type must not be too complex; it has to be possible to separate a total uncertainty measure on belief intervals into two ones that coherently indicate conflict and non-specificity, respectively; a total uncertainty measure on belief intervals has to be sensitive to changes in the belief intervals, directly or via conflict and non-specificity.

On the other hand, we have analyzed which of the essential mathematical properties and behavioral requirements proposed in our study are satisfied by each one of the total uncertainty measures on belief intervals proposed so far. We have shown that all of these measures have a coherent range, and produce logical results when it is just known that the information expressed through the belief intervals is focused on a subset of alternatives. Nonetheless, the interval distance-based total uncertainty measures may not coincide with the Shannon entropy when a unique probability distribution is compatible with the belief intervals, unlike the total uncertainty measure of Wang and Song (*SU*) and the maximum of entropy. One of the interval distance-based total uncertainty measures and *SU* might not be consistent with an increase or decrease of information expressed the belief intervals, while the other total uncertainty measure based on intervals distance and the maximum of entropy always coherently reflect that increase or decrease of information; the maximum of entropy is the only total uncertainty measure so far that provides coherent results with belief intervals associated with BPAs defined over joint spaces that can be decomposed into more simple sets. The maximum of entropy is more complex to calculate than the other total uncertainty measures on belief intervals developed so far, although the complexity is also not very high. Furthermore, we have shown that *SU* and the maximum of entropy can be separated into two measures that respectively quantify conflict and non-specificity, whereas, so far, it has not been possible to decompose the interval distance-based total uncertainty measures in this way. It must be remarked that the separation of the maximum of entropy is more coherent than the one of *SU* since the conflict value of the second measure is not always equal to 0 when the plausibility value of a certain singleton is equal to 1, which is not desirable. Also, it has been argued that all the total uncertainty measures on belief intervals proposed so far are sensitive to changes in the belief intervals, directly or through its parts of conflict and non-specificity.

Therefore, it can be concluded that, even though the maximum of entropy requires a higher computational complexity than the other total uncertainty measures on belief intervals, it is the most appropriate total uncertainty measure on belief intervals to be employed in practical applications because it is the only one so far that satisfies all the necessary mathematical properties and behavioral requirements for this kind of measure.

ENDNOTE

†A credal set is a closed and convex set of probability distributions.

ORCID

Serafín Moral-García  <https://orcid.org/0000-0002-8513-9081>

Joaquín Abellán  <https://orcid.org/0000-0001-9018-5165>

REFERENCES

1. Dempster AP. Upper and lower probabilities induced by a multivalued mapping. *Ann Math Stat.* 1967; 38(2):325-339. <https://doi.org/10.1214/aoms/1177698950>
2. Shafer G. *A Mathematical Theory of Evidence*. Princeton, NJ: Princeton University Press; 1976.
3. Beynon M, Curry B, Morgan P. The Dempster-Shafer theory of evidence: an alternative approach to multicriteria decision modelling. *Omega.* 2000;28(1):37-50. [https://doi.org/10.1016/S0305-0483\(99\)00033-X](https://doi.org/10.1016/S0305-0483(99)00033-X)
4. Yager RR, Alajlan N. Dempster-shafer belief structures for decision making under uncertainty. *Knowl-Based Syst.* 2015;8:58-66. <https://doi.org/10.1016/j.knosys.2014.12.031>
5. Deng X, Jiang W. An evidential axiomatic design approach for decision making using the evaluation of belief structure satisfaction to uncertain target values. *Int J Intell Syst.* 2018;3(1):15-32. <https://doi.org/10.1002/int.21929>
6. Ip H, Ng J. Human face recognition using Dempster-Shafer theory. *Proc Int Conf Image Proc.* 1994;2: 292-295. <https://doi.org/10.1109/ICIP.1994.413578>
7. Liu Z, Pan Q, Dezert J, Martin A. Adaptive imputation of missing values for incomplete pattern classification. *Pattern Recogn.* 2016;52:85-95. <https://doi.org/10.1016/j.patcog.2015.10.001>
8. Liu Z, Pan Q, Dezert J, Mercier G. Credal classification rule for uncertain data based on belief functions. *Pattern Recogn.* 2014;47(7):2532-2541. <https://doi.org/10.1016/j.patcog.2014.01.011>
9. Deng Y. Uncertainty measure in evidence theory. *Sci China Inform Sci.* 2020;63(11):1-19. <https://doi.org/10.1007/s11432-020-3006-9>
10. Klir G, Wierman M. *Uncertainty-Based Information: Elements of Generalized Information Theory, Studies in Fuzziness and Soft Computing*. Physica-Verlag HD; 1999.
11. Abellán J, Masegosa A. Requirements for total uncertainty measures in Dempster-Shafer theory of evidence. *Int J General Syst.* 2008;37(6):733-747. <https://doi.org/10.1080/03081070802082486>
12. Harmanec D, Klir GJ. Measuring total uncertainty in Dempster-Shafer theory: a novel approach. *Int J General Syst.* 1994; 22(4): 405-419. <https://doi.org/10.1080/03081079408935225>
13. Maeda Y, Nguyen HT, Ichihashi H. Maximum entropy algorithms for uncertainty measures. *Int J Uncertain Fuzziness Knowl-Based Syst.* 1993; 01(01): 69-93. <https://doi.org/10.1142/S021848859300005X>
14. Huynh V-N, Nakamori Y. Notes on reducing algorithm complexity for computing an aggregate uncertainty measure. *IEEE Trans Syst Man Cybern Part A: Syst Human.* 2010;40(1):205-209. <https://doi.org/10.1109/TSMCA.2009.2030962>
15. Meyerowitz A, Richman F, Walker E. Calculating maximum-entropy probability densities for belief functions. *Int J Uncertain Fuzziness Knowl-Based Syst.* 1994;2(4):377-389. <https://doi.org/10.1142/S0218488594000316>
16. Deng Y. Deng entropy. *Chaos Soliton Fractals.* 2016;91:549-553.
17. Kang B, Deng Y. The maximum Deng entropy. *IEEE Access.* 2019;7:120758-120765. <https://doi.org/10.1109/ACCESS.2019.2937679>
18. Jiroušek R, Shenoy PP. On properties of a new decomposable entropy of Dempster-Shafer belief functions. *Int J Approx Reason.* 2020;119:260-279. <https://doi.org/10.1016/j.ijar.2020.01.004>
19. Sun G, Guan X, Yi X, Zhao J. Belief intervals aggregation. *Int J Intell Syst.* 2018;33(12):2425-2447. <https://doi.org/10.1002/int.22046>
20. Deng Z, Wang J. Measuring total uncertainty in evidence theory. *Int J Intell Syst.* 2021;36(4):1721-1745. <https://doi.org/10.1002/int.22358>
21. Moral-García S, Abellán J. Maximum of entropy for belief intervals under evidence theory. *IEEE Access.* 2020; 8: 118017-118029. <https://doi.org/10.1109/ACCESS.2020.3003715>
22. Deng X. Analyzing the monotonicity of belief interval based uncertainty measures in belief function theory. *Int J Intell Syst.* 2018;33(9):1869-1879. <https://doi.org/10.1002/int.21999>
23. Deng X, Jiang W. A total uncertainty measure for d numbers based on belief intervals. *Int J Intell Syst.* 2019; 34(12): 3302-3316. <https://doi.org/10.1002/int.22195>
24. Deng Y. D numbers: theory and applications. *J Inform Comput Sci.* 2012; 9(9): 2421-2428.
25. Yang Y, Han D. A new distance-based total uncertainty measure in the theory of belief functions. *J Knowl-Based Syst.* 2016;94(C):114-123. <https://doi.org/10.1016/j.knosys.2015.11.014>

26. Xinyang D, Fuyuan X, Yong D. An improved distance-based total uncertainty measure in belief function theory. *Appl Intell*. 2017;46(4):898-915. <https://doi.org/10.1007/s10489-016-0870-3>
27. Wang X, Song Y. Uncertainty measure in evidence theory with its applications. *Appl Intell*. 2017;48:1672-1688. <https://doi.org/10.1007/s10489-017-1024-y>
28. Shannon CE. A mathematical theory of communication. *Bell Syst Tech J*. 1948;27(3):379-423. <https://doi.org/10.1002/j.1538-7305.1948.tb01338.x>
29. Hartley RVL. Transmission of information. *Bell Syst Techn J*. 1928;7(3):535-563. <https://doi.org/10.1002/j.1538-7305.1928.tb01236.x>
30. Yager RR. Entropy and specificity in a mathematical theory of evidence. *Int J General Syst*. 1983;9(4):249-260. <https://doi.org/10.1080/03081078308960825>
31. Dubois D, Prade H. A note on measures of specificity for fuzzy sets. *Int J General Syst*. 1985;10(4):279-283. <https://doi.org/10.1080/03081078508934893>
32. Abellán J. Analyzing properties of Deng entropy in the theory of evidence. *Chaos Solitons Fractals*. 2017;9:195-199. <https://doi.org/10.1016/j.chaos.2016.12.024>
33. Cui H, Liu Q, Zhang J, Kang B. An improved Deng entropy and its application in pattern recognition. *IEEE Access*. 2019;7:18284-18292. <https://doi.org/10.1109/ACCESS.2019.2896286>
34. Zhou D, Tang Y, Jiang W. A modified belief entropy in Dempster-Shafer framework. *PLOS One*. 2017;12(5):1-17. <https://doi.org/10.1371/journal.pone.0176832>
35. Moral-García S, Abellán J. Critique of modified deng entropies under the evidence theory. *Chaos Solitons Fractals*. 2020;140:110112. <https://doi.org/10.1016/j.chaos.2020.110112>
36. Voorbraak F. A computationally efficient approximation of Dempster-Shafer theory. *Int J Man-Mach Studies*. 1989;30(5):525-536. [https://doi.org/10.1016/S0020-7373\(89\)80032-X](https://doi.org/10.1016/S0020-7373(89)80032-X)
37. Cobb BR, Shenoy PP. On the plausibility transformation method for translating belief function models to probability models. *Int J Approx Reason*. 2006; 41(3): 314-330. <https://doi.org/10.1016/j.ijar.2005.06.008>
38. Pan Q, Zhou D, Tang Y, Li X, Huang J. A novel belief entropy for measuring uncertainty in Dempster-Shafer evidence theory framework based on plausibility transformation and weighted hartley entropy. *Entropy*. 2019;21(2):163. <https://doi.org/10.3390/e21020163>
39. De Campos LM, Huete JF, Moral S. Probability intervals: a tool for uncertain reasoning. *Int J Uncertain Fuzziness Knowl-Based Syst*1994;2(2):167-196.<https://doi.org/10.1142/S0218488594000146>
40. Couso I, Moral S, Walley P. A survey of concepts of independence for imprecise probabilities. *Risk Decis Policy*. 2000;5(2):165-181. <https://doi.org/10.1017/S1357530900000156>
41. Yang Y, Han D, Dezert J. A new non-specificity measure in evidence theory based on belief intervals. *Chinese J Aeronaut*. 2016; 29(3): 704-713. <https://doi.org/10.1016/j.cja.2016.03.004>
42. Suppes P. The measurement of belief. *J Royal Stat Society. Series B (Methodological)*. 1974;36(2):160-191.

How to cite this article: Moral-García S, Abellán J. Required mathematical properties and behaviors of uncertainty measures on belief intervals. *Int J Intell Syst*. 2021; 1-24. <https://doi.org/10.1002/int.22432>