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Bivariate Koornwinder–Sobolev Orthogonal Polynomials

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Abstract. The so-called Koornwinder bivariate orthogonal polynomials are generated by means of a non-trivial procedure involving two families of univariate orthogonal polynomials and a function $\rho(t)$ such that $\rho(t)^2$ is a polynomial of degree less than or equal to 2. In this paper, we extend the Koornwinder method to the case when one of the univariate families is orthogonal with respect to a Sobolev inner product. Therefore, we study the new Sobolev bivariate families obtaining relations between the classical original Koornwinder polynomials and the Sobolev one, deducing recursive methods in order to compute the coefficients. The case when one of the univariate families is classical is analysed. Finally, some useful examples are given.

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1. Introduction

An interesting non trivial tool for generating orthogonal polynomials in two variables was introduced in 1965 by S. A. Agahanov ([2]). Ten years later, T. H. Koornwinder used that procedure for classical Jacobi polynomials and introduced *Two variable analogues of the classical orthogonal polynomials* (see [6]). In fact, given two univariate weight functions $\omega_i(t)$ defined on the intervals $(a_i, b_i) \subset \mathbb{R}$, for $i = 1, 2$, and a function $\rho(t)$ on (a_1, b_1), such that $\rho(t)^2$ is a polynomial of degree less than or equal to 2, orthogonal polynomials in two variables associated with the weight function defined by

$$W(x, y) = \omega_1(x)\,\omega_2(y/\rho(x)) \tag{1.1}$$

were studied. Observe that $\rho(t)$ defined as above can be a polynomial of degree ≤ 1, or the square root of a non-negative polynomial of degree ≤ 2.

Despite its apparent simplicity, this method provides a key to the study of algebraic, differential, and analytical properties for a large class of bivariate orthogonal families of polynomials. In fact, the most usual bivariate families correspond to this scheme. For instance, eight of the nine different cases of Krall and Sheffer classical bivariate polynomials (see [8]) can be constructed in this way. Two-dimensional Krall and Sheffer polynomials are analogues of the classical orthogonal polynomials since they are eigenfunctions of second order linear partial differential operators and, moreover, they satisfy orthogonality conditions. In [5], Harnad et al. proved that all Krall–Sheffer polynomials are connected with two-dimensional superintegrable systems on spaces with constant curvature.

In [13], the authors exploited the structure of Koornwinder's weight function to obtain the coefficients of the three term relation satisfied by the bivariate polynomials when both weight functions ω_1 and ω_2 are classical. Moreover, in [11] differential properties for the weight function $W(x, y)$ are deduced. Those properties were the extension to the Koornwinder case of the Pearson's differential equation of univariate classical weight functions.

Recently, using a similar construction, Olver and Xu ([14]) presented explicit constructions of orthogonal polynomials inside quadratic bodies of revolution, including cones, hyperboloids, and paraboloids. They also constructed orthogonal polynomials on the boundary of quadratic surfaces of revolution. In such a construction, they replaced the univariate weight function ω_2 with the classical weight function on the d-dimensional ball or with the Lebesgue measure on the sphere, respectively.

In Koornwinder's construction, a family of bivariate polynomials orthogonal with respect to (1.1) can be defined by means of

$$P_{n,m}(x, y) = p^{(m)}_{n-m}(x)\, \rho^m(x)\, q_m\left(\frac{y}{\rho(x)}\right), \quad n \geq 0, \quad m = 0, 1, \ldots, n, \tag{1.2}$$

where $\{p^{(m)}_n(t)\}_{n\geq 0}$ is an orthogonal polynomial sequence (OPS in short) associated with the weight function $\rho^{2m+1}(t)\omega_1(t)$, $m \geq 0$, and $\{q_n(t)\}_{n\geq 0}$ is an OPS associated with $\omega_2(t)$. In this work, we consider an extension of Koornwinder's construction to the Sobolev realm. To this end, we modify the family of bivariate polynomials (1.2) replacing one of the univariate OPS by a univariate Sobolev OPS. First, we consider the case where $\{p^{(m)}_n(t)\}_{n\geq 0}$ is a Sobolev OPS and, next, we study the case where $\{q_n(t)\}_{n\geq 0}$ is a Sobolev OPS. In both cases, we show that the bivariate polynomials in (1.2) are orthogonal with respect to a bivariate Sobolev inner product involving first order partial derivatives. The Sobolev inner product obtained in the second case is quite similar to the inner product studied by Kwon and Littlejohn in [9]. Whatever, our most interesting result is the existence of connection formulas relating the bivariate Sobolev and standard orthogonal polynomials. Algorithms to obtain the coefficients in those connection formulas are provided.

We refer to the survey paper [12] as the most recent presentation of the state of the art on Sobolev orthogonal polynomials both in the univariate as well as the bivariate case.

In the one-dimensional case, the natural framework of application of Sobolev orthogonal polynomials seems to be the implementation of spectral methods for boundary value problems for elliptic differential operators. For instance, in [15] the second- and fourth-order elliptic boundary value problems with Dirichlet or Robin boundary conditions are considered, and generalized Jacobi spectral schemes are proposed. In fact, the authors construct an orthogonal basis of Jacobi–Sobolev polynomials which allows the diagonalization of the involved discrete systems. The corresponding error estimates and numerical results illustrate the effectiveness and the spectral accuracy of the method. In several variables, Sobolev orthogonal polynomials on the unit ball have been considered by Xu in the numerical solution of boundary value problems for elliptic partial differential operators. For details, we refer again to the survey paper [12] and the references therein.

The structure of the paper is as follows: In Sect. 2, we introduce the notation and basic results on orthogonal polynomials used throughout our work. Section 3 contains the definition and properties of the first type of bivariate Sobolev orthogonal polynomials. The case where the first family in (1.2) is a classical one is studied in Sect. 4. Section 5 contains the definition and properties of the second type of bivariate Sobolev orthogonal polynomials. In Sect. 6 we have included several explicit examples. Finally, we added an appendix containing the technical proofs of our results.

2. Preliminaries

We need to fix the notation and recall the basic results used throughout this work, in order to be self-contained. We will need univariate and bivariate tools, and we start briefly with the univariate ones.

2.1. Univariate Basic Tools

Let $\mathbf{u}$ be a linear functional defined on Π, the linear space of real polynomials in one variable, by means of its *moments*

$$\begin{aligned} \mathbf{u} : \Pi &\longrightarrow \mathbb{R} \\ t^n &\longmapsto \langle \mathbf{u}, t^n \rangle = \mu_n, \quad n \geq 0, \end{aligned}$$

and extended by linearity to Π. Hence, we say that $\mathbf{u}$ is a *moment functional.*

We will work with polynomial sequences on Π, $\{p_n(t)\}_{n\geq 0}$, such that $\deg p_n = n$, for $n \geq 0$, and then $\{p_n(t)\}_{n\geq 0}$ is always a basis of Π. In addition, if $\langle \mathbf{u}, p_n\, p_m\rangle = 0$, $n \neq m$, and $\langle \mathbf{u}, p_n^2\rangle = h_n \neq 0$, $n \geq 0$, we say that $\{p_n(t)\}_{n\geq 0}$ is an orthogonal polynomial sequence (OPS) associated with $\mathbf{u}$.

Following [3,16], given a moment functional $\mathbf{u}$ defined as above, there is not always an OPS associated with it. If an OPS associated with $\mathbf{u}$ exists, then $\mathbf{u}$ is called *quasi-definite.* It is well known that if $\mathbf{u}$ is quasi-definite, then its OPS is unique except for a constant factor. A moment functional $\mathbf{u}$ is *positive definite* if $\langle \mathbf{u}, p^2\rangle > 0$ for all non zero polynomial $p \in \Pi$; positive definite moment functionals are quasi-definite, and OPS associated with $\mathbf{u}$ exists. In addition, $\mathbf{u}$ is *symmetric* if all odd moments are zero, that is, $\langle \mathbf{u}, x^{2n+1}\rangle = 0$, $n \geq 0$.

In this paper, *classical moment functionals* play a central role. To recall this concept, we need to revise two usual operations for a moment functional $\mathbf{u}$

- the *distributional derivative*: $\langle D\mathbf{u}, p\rangle = -\langle \mathbf{u}, p'\rangle$,
- the *left multiplication by a polynomial*: $\langle q\,\mathbf{u}, p\rangle = \langle \mathbf{u}, q\,p\rangle$,

for $p, q \in \Pi$. Moreover, the product rule holds, that is, $D(q\,\mathbf{u}) = q'\,\mathbf{u} + q\,D\mathbf{u}$.

A quasi-definite moment functional $\mathbf{u}$ is called *classical* (see, for instance, [10]) if there exist non-zero polynomials $\phi(t)$ and $\psi(t)$ with $\deg \phi \le 2$ and $\deg \psi = 1$, such that $\mathbf{u}$ satisfies the distributional Pearson equation

$$D(\phi(t)\,\mathbf{u}) = \psi(t)\,\mathbf{u}. \tag{2.1}$$

Orthogonality can be defined similarly by using a bilinear form $(\cdot,\cdot)$ defined on Π. A polynomial sequence $\{p_n(t)\}_{n\ge 0}$ is orthogonal with respect to $(\cdot,\cdot)$ if

$$(p_n, p_m) = h_n\,\delta_{n,m}, \quad n, m \ge 0,$$

where $h_n \neq 0$. Quasi-definiteness and positive-definiteness are defined as above. Moreover, given a moment functional $\mathbf{u}$, we can define the bilinear form $(p, q) = \langle \mathbf{u}, p\,q\rangle$ for all $p, q \in \Pi$.

2.2. Bivariate Basic Tools

We turn our attention to bivariate orthogonal polynomials. Our main reference for the basic theory is [4].

As an extension of the univariate case, we denote by Π^2 the linear space of bivariate polynomials with real coefficients, and we define a bivariate moment functional $\mathbf{w}$ by using its moments

$$\langle \mathbf{w}, x^n y^m\rangle = \omega_{n,m}, \quad n, m \ge 0,$$

and extended by linearity to Π^2, where $\{\omega_{n,m}\}_{n,m\ge 0}$ is a sequence of real numbers.

A sequence of polynomials of Π^2 will be represented by $\{P_{n,m}(x,y) : n \ge 0,\ 0 \le m \le n\}$ where the set $\{P_{n,m}(x,y) : 0 \le m \le n\}$ for a fixed n consists of $n+1$ linearly independent polynomials of total degree n, that is, $\deg P_{n,m} = n$, $0 \le m \le n$. In this way, a sequence of polynomials as above is a basis of Π^2.

We will say that they are orthogonal with respect to $\mathbf{w}$ if

$$\langle \mathbf{w}, P_{n,m}\,Q\rangle = 0, \quad 0 \le m \le n,$$

for all polynomial Q of total degree less than or equal to $n-1$. Moreover, if

$$\langle \mathbf{w}, P_{n,m}\,P_{i,j}\rangle = H_{n,m}\,\delta_{n,i}\,\delta_{m,j},$$

where $H_{n,m} \neq 0$, $n \ge 0$, then we say that $\{P_{n,m}(x,y) : n \ge 0,\ 0 \le m \le n\}$ is a *mutually orthogonal polynomial sequence.*

Orthogonality of polynomial sequences can be defined with respect to a bilinear form $(\cdot,\cdot)$ acting on Π^2 in a similar way as for orthogonality in the univariate case.

2.3. Bivariate Koornwinder Orthogonal Polynomials

Bivariate Koornwinder polynomials are constructed from two univariate families of orthogonal polynomials associated with inner products defined by means of weight functions [2,4,6], and an auxiliary function. This kind of polynomials are orthogonal with respect to a new bivariate inner product defined by a specific weight function.

We use the extension of this construction to moment functionals. This extension, which was studied by the authors in [13], uses two univariate quasi-definite moment functionals in order to build a quasi-definite bivariate moment functional. We recall this construction.

Consider two univariate quasi-definite moment functionals $\mathbf{u}^{(x)}$ and $\mathbf{v}^{(y)}$. The superscript corresponds to the variable that each moment functional acts upon. Additionally, we need a non-zero function $\rho(x)$ such that $\rho^2(x)$ is a real polynomial of degree less than or equal to 2. We have two cases as follows:

$$\textit{Case I}: \ \ \rho(x) = r_1\, x + r_0, \text{ where } |r_1| + |r_0| > 0,$$

$$\textit{Case II}: \rho(x) = \sqrt{\ell_2\, x^2 + 2\,\ell_1\, x + \ell_0}, \text{ with } |\ell_2| + |\ell_1| + |\ell_0| > 0, \text{ and } \mathbf{v}^{(y)} \textit{ is symmetric.}$$

In both cases, we also impose that the functional $\mathbf{u}_m^{(x)} = \rho(x)^{2m+1}\,\mathbf{u}^{(x)}$, for $m \geq 0$, is quasi-definite, and let $\{p_n^{(m)}(x)\}_{n\geq 0}$ be an OPS associated with $\mathbf{u}_m^{(x)}$. Moreover, let $\{q_n(y)\}_{n\geq 0}$ be an OPS associated with $\mathbf{v}^{(y)}$.

Define the bivariate polynomials

$$P_{n,m}(x,y) = p_{n-m}^{(m)}(x)\,\rho(x)^m q_m\left(\frac{y}{\rho(x)}\right), \quad 0 \leq m \leq n, \quad n \geq 0.$$

It was shown in [13] that the set $\{P_{n,m}(x,y) \,:\, 0 \leq m \leq n, n \geq 0\}$ is a mutually orthogonal basis with respect to the moment functional $\mathbf{w}$ defined by

$$\langle \mathbf{w}, Q(x,y)\rangle = \langle \mathbf{u}_0^{(x)}, \langle \mathbf{v}^{(y)},\ Q(x, y\,\rho(x))\,\rangle\,\rangle, \quad \forall Q \in \Pi^2. \tag{2.2}$$

3. First Type of Bivariate Sobolev Orthogonal Polynomials

Here, we extend the construction described in Sect. 2.3 to include a class of Sobolev bivariate orthogonal polynomials generated from univariate orthogonal polynomials.

Hence, let $\mathbf{u} \equiv \mathbf{u}^{(x)}$ and $\mathbf{v} \equiv \mathbf{v}^{(y)}$ be univariate quasi-definite moment functionals acting on the variables x and y, respectively. When there is no confusion, we remove the corresponding superscript in the notation.

For $m \geq 0$, let $\{p_n^{(m)}(x)\}_{n\geq 0}$ be an orthogonal polynomial sequence associated with the quasi-definite moment functional $\mathbf{u}_m = \rho(x)^{2m+1}\,\mathbf{u}$, and let $\{q_m(y)\}_{m\geq 0}$ be an orthogonal polynomial sequence associated with $\mathbf{v}$. In addition, we denote

$$h_n^{(m)} = \langle \mathbf{u}_m, (p_n^{(m)})^2 \rangle, \tag{3.1}$$

$$h_m^{(q)} = \langle \mathbf{v}, q_m^2 \rangle, \tag{3.2}$$

with $h_n^{(m)} \neq 0$ and $h_m^{(q)} \neq 0$, for $n, m \geq 0$.

Let $\rho \equiv \rho(x)$ be a function satisfying the conditions of either Case I or Case II. Furthermore, we introduce the univariate bilinear form

$$(p, q)_{\mathbf{u}} = \langle \mathbf{u},\ p\, q + \lambda\, \rho^{\varsigma+2}\, p'\, q' \rangle, \quad \lambda \in \mathbb{R},$$

where

$$\varsigma = \begin{cases} 0, & \text{Case I}, \\ 2, & \text{Case II}. \end{cases}$$

For $m \geq 0$, let us define the Sobolev bilinear form

$$(p, q)_m = (\, \rho^m\, p,\ \rho^m\, q\,)_{\rho\, \mathbf{u}},$$

in both Case I and Case II, or equivalently,

$$(p, q)_m = \left\langle \mathbf{u}_m,\ p\, q + \lambda\, \rho^{\varsigma}\, \begin{pmatrix} p\; p' \end{pmatrix} \begin{pmatrix} (m\, \rho')^2 & m\, \rho\, \rho' \\ m\, \rho\, \rho' & \rho^2 \end{pmatrix} \begin{pmatrix} q \\ q' \end{pmatrix} \right\rangle, \tag{3.3}$$

and suppose that (3.3) is quasi-definite for $m \geq 0$, that is, there exists a sequence of polynomials $\{s_n^{(m)}\}_{n\geq 0}$ such that $\deg s_n^{(m)} = n$, for all $n \geq 0$. If the moment functional $\mathbf{u}_m$ is positive definite for $m \geq 0$ and $\lambda \geq 0$, then (3.3) defines an inner product and there exists a sequence of orthogonal polynomials associated with it. When $\mathbf{u}_m$ is quasi-definite and $\lambda \in \mathbb{R}$, some additional reasonable conditions are needed in order to guarantee the existence of such sequence of orthogonal polynomials.

Define

$$(s_n^{(m)}, s_i^{(m)})_m = \widetilde{h}_n^{(m)}\, \delta_{n,i}, \tag{3.4}$$

with $\widetilde{h}_n^{(m)} \equiv \widetilde{h}_n^{(m)}(\lambda) \neq 0$, for $n, i, m \geq 0$.

Since $\{s_n^{(m)}\}_{n\geq 0}$ is unique up to a constant factor, we choose it such that, for $n \geq 0$, $s_n^{(m)}(x)$ has the same leading coefficient as $p_n^{(m)}(x)$, and therefore $s_0^{(m)}(x) = p_0^{(m)}(x)$. We say that $\{s_n^{(m)}\}_{n\geq 0}$ is a univariate Sobolev orthogonal polynomial sequence. We remark that when $\lambda = 0$, we recover the original univariate standard orthogonal polynomial sequence $\{p_n^{(m)}\}_{n\geq 0}$, and $\widetilde{h}_n^{(m)}(0) = h_n^{(m)}$, $n, m \geq 0$.

The Sobolev bilinear form (3.3) is a particular case of what is usually known in the literature as a non diagonal Sobolev bilinear form (see, for instance, [12] and the references therein), and the analytic and algebraic properties of the associated orthogonal polynomials constitute an interesting topic in itself.

Now, we present the announced extension to the Sobolev case as follows:

Theorem 3.1. *Let* $\mathbf{w}$ *be a bivariate moment functional defined as in* (2.2). *The two-variable polynomials defined as*

$$S_{n,m}(x, y) = s_{n-m}^{(m)}(x)\, \rho(x)^m\, q_m\left(\frac{y}{\rho(x)}\right), \quad 0 \leq m \leq n, \quad n \geq 0, \tag{3.5}$$

form a mutually orthogonal sequence with respect to the Sobolev bivariate bilinear form

$$(P,Q) = \left\langle \mathbf{w},\ P\,Q + \lambda\,\rho(x)^{\varsigma}\,(\nabla P)^t \begin{pmatrix} \rho(x)^2 & y\,\rho'(x)\,\rho(x) \\ y\,\rho'(x)\,\rho(x) & y^2\,\rho'(x)^2 \end{pmatrix} \nabla Q \right\rangle, \tag{3.6}$$

where $\lambda \in \mathbb{R}$ *and* ∇ *is the usual gradient operator* $\nabla = (\partial_x,\, \partial_y)^t$.

Moreover,

$$H_{n,m} = (S_{n,m},\, S_{n,m}) = \widetilde{h}_{n-m}^{(m)}\, h_m^{(q)},$$

where $\widetilde{h}_{n-m}^{(m)}$ *and* $h_m^{(q)}$ *were defined in* (3.4) *and* (3.2), *respectively.*

Proof. Let us define the change of variables $x = s$ and $y = t\,\rho(s)$. Thus, for any polynomial $P(x,y) \in \Pi^2$, we get

$$\begin{aligned} \partial_s P(s, t\,\rho(s)) &= \partial_x P(s, t\,\rho(s)) + t\,\rho'(s)\,\partial_y P(s, t\,\rho(s)), \\ \partial_t P(s, t\,\rho(s)) &= \rho(s)\,\partial_y P(s, t\,\rho(s)). \end{aligned}$$

Then,

$$\begin{aligned} \partial_x P(x,y) &= \partial_s P(s, t\,\rho(s)) - t\,\frac{\rho'(s)}{\rho(s)}\,\partial_t P(s, t\,\rho(s)), \\ \partial_y P(x,y) &= \frac{1}{\rho(s)}\,\partial_t P(s, t\,\rho(s)). \end{aligned}$$

Using this jointly with (2.2) and (3.5), we compute

$$\begin{aligned} (S_{n,m},\, S_{i,j}) &= \langle \mathbf{w}, S_{n,m}\,S_{i,j} + \lambda\rho^{\varsigma}(\rho\,\partial_x S_{n,m} + y\rho'\partial_y S_{n,m})(\rho\,\partial_x S_{i,j} + y\rho'\partial_y S_{i,j})\rangle \\ &= \langle \rho(s)\,\mathbf{u}^{(s)} \langle \mathbf{v}^{(t)}, (\rho^m(s)\,s_{n-m}^{(m)}(s))(\rho^j(s)\,s_{i-j}^{(j)}(s)) q_m(t)\,q_j(t) \\ &\quad + \lambda\rho^{\varsigma}(s)(\partial_s(\rho^m(s)\,s_{n-m}^{(m)}(s)) q_m(t)(\partial_s(\rho^j(s)\,s_{i-j}^{(j)}(s)) q_j(t)\rangle\rangle \\ &= \langle \rho\,\mathbf{u}^{(s)}, (\rho^m\,s_{n-m}^{(m)})(\rho^j\,s_{i-j}^{(j)}) + \lambda\rho^{\varsigma}(\partial_s(\rho^m\,s_{n-m}^{(m)})(\partial_s(\rho^j\,s_{i-j}^{(j)})\rangle \\ &\quad \times \langle \mathbf{v}^{(t)}, q_m(t) q_j(t)\rangle \\ &= (s_{n-m}^{(m)},\, s_{i-j}^{(j)})_m\,\langle \mathbf{v}, q_m\,q_j\rangle = (s_{n-m}^{(m)},\, s_{i-m}^{(m)})_m\,h_m^{(q)}\,\delta_{m,j}. \end{aligned}$$

Since $\{s_n^{(m)}\}_{n\geq 0}$ is orthogonal with respect to $(\cdot\,,\cdot)_m$, the theorem is proved. □

Notice that the 2×2 matrix in (3.6) is positive semidefinite. In Case I, the entries of this matrix are polynomials, but in Case II, this matrix has rational functions as entries since $\rho'(x)$ is of the following form:

$$\rho'(x) = \frac{2\,\ell_2\,x + 2\,\ell_1}{2\sqrt{\ell_2\,x^2 + 2\,\ell_1\,x + \ell_0}} = \frac{\ell_2\,x + \ell_1}{\rho(x)}.$$

Then, multiplication by $\rho(x)^{\varsigma} = \rho(x)^2$ cancels out the denominators in the matrix of the bilinear form (3.6).

4. The Classical First-Type Univariate Sobolev Orthogonal Polynomials

In this section, we show that if the univariate moment functional $\mathbf{u} \equiv \mathbf{u}^{(x)}$ involved in the construction presented in Sect. 2.3 is classical and ρ is related to the polynomial coefficients of the Pearson equation (2.1) satisfied by $\mathbf{u}$, then the orthogonal sequence $\{s_n^{(m)}(x)\}_{n\geq 0}$ associated with the bilinear form $(\cdot,\cdot)_m$ defined in (3.3) can be computed recursively.

First, we introduce some notation to be used in the sequel. We write the three-term recurrence relation satisfied by following: $\{p_n^{(m)}(x)\}_{n\geq 0}$ as ([3,16])

$$\begin{aligned} &x\, p_n^{(m)}(x) = a_n^{(m)}\, p_{n+1}^{(m)}(x) + b_n^{(m)} p_n^{(m)}(x) + c_n^{(m)}\, p_{n-1}^{(m)}(x), \quad n \geq 0, \\ &p_{-1}^{(m)}(x) = 0, \;\; p_0^{(m)}(x) = 1, \qquad m \geq 0, \end{aligned} \tag{4.1}$$

and if

$$p_n^{(m)}(x) = k_n^{(m)}\, x^n + \text{lower degree terms},$$

then

$$a_n^{(m)} = \frac{k_n^{(m)}}{k_{n+1}^{(m)}}, \quad b_n^{(m)} = \frac{\langle \mathbf{u}_m, x\,(p_n^{(m)})^2\rangle}{h_n^{(m)}}, \quad n \geq 0,$$

and

$$c_n^{(m)} = \frac{k_{n-1}^{(m)}}{k_n^{(m)}}\frac{h_n^{(m)}}{h_{n-1}^{(m)}} = a_{n-1}^{(m)}\frac{h_n^{(m)}}{h_{n-1}^{(m)}}, \quad n \geq 1,$$

where $h_n^{(m)}$ was defined in (3.1).

We work with classical moment functionals $\mathbf{u}$, that is, moment functionals satisfying Pearson equation (2.1). If $\rho(x)$ and $\phi(x)$ are related, the classical character is inherited by the moment functional $\mathbf{u}_m$.

We have the following preliminary result:

Lemma 4.1. *Let $\mathbf{u}$ be a classical univariate moment functional satisfying the Pearson equation (2.1).*

(i) *In Case I, if $\rho(x) = r_1\, x + r_0$ divides $\phi(x)$,*
(ii) *In Case II, if $\rho(x)^2 = \ell_2\, x^2 + 2\,\ell_1\, x + \ell_0$ divides $\phi(x)$,*

then, for $k \geq 0$, the moment functional

$$\widehat{\mathbf{u}}_k = \rho(x)^k\, \mathbf{u}$$

is classical.

Proof. If a moment functional $\mathbf{u}$ is classical, then it satisfies the Pearson equation $D(\phi(x)\,\mathbf{u}) = \psi(x)\,\mathbf{u}$, with $\deg\phi \leq 2$ and $\deg\psi = 1$. For $k \geq 1$, we compute

$$\begin{aligned} D(\phi(x)\,\widehat{\mathbf{u}}_k) &= D(\phi(x)\,\rho(x)^k\,\mathbf{u}) = k\,\rho(x)^{k-1}\,\rho'(x)\,\phi(x)\,\mathbf{u} + \rho(x)^k\,D(\phi(x)\,\mathbf{u}) \\ &= \left(k\,\frac{\rho'(x)\,\phi(x)}{\rho(x)} + \psi(x)\right)\rho(x)^k\mathbf{u} = \widehat{\psi}(x)\,\widehat{\mathbf{u}}_k, \end{aligned}$$

where $\widehat{\psi}(x) = k\,\frac{\rho'(x)\,\phi(x)}{\rho(x)} + \psi(x)$.

In Case I, since $\rho(x)$ divides $\phi(x)$, then

$$\widehat{\psi}(x) = k\, r_1\, \frac{\phi(x)}{\rho(x)} + \psi(x),$$

is a polynomial of degree equal to 1. In *Case II*, we have

$$\widehat{\psi}(x) = k\,(\ell_2\, x + \ell_1)\, \frac{\phi(x)}{\rho(x)^2} + \psi(x).$$

Again, since $\rho(x)^2$ divides $\phi(x)$ and $\deg \rho(x)^2 \leq 2$, $\deg \phi \leq 2$, and $\deg \psi = 1$, $\widehat{\psi}(x)$ is a polynomial of degree equal to 1.

As a consequence, in both cases $\widehat{\mathbf{u}}_k$ is classical. □

Notice that the Hermite, Laguerre, Jacobi, and Bessel are the only families of classical orthogonal polynomials as it was shown in [10], among others. In these cases, the polynomial $\phi(x)$ is usually normalized as 1, x, $1 - x^2$, and x^2, respectively. Recall that the Hermite, Laguerre, and Jacobi polynomials are associated with a positive definite moment functional, and the Bessel polynomials are associated with a quasi-definite moment functional. By Lemma 4.1, in Case I, $\rho(x)$ must divide $\phi(x)$ and, in Case II, $\rho(x)^2$ must divide $\phi(x)$. Then, non trivial cases are obtained by taking $\rho(x) = a\, x$ in the Laguerre and Bessel cases, and $\rho(x) = a\,(1 \pm x)$ in the Jacobi case, for $a \in \mathbb{R}$. In Case II, non trivial cases are obtained by taking $\rho(x)^2 = a\, x$ in the Laguerre and Bessel cases, and $\rho(x)^2 = a(1 - x^2)$ in the Jacobi case. We remark that in Case II the functional $\mathbf{v}$ must be symmetric.

We return to the construction presented in Sect. 2.3. Under the hypotheses of Lemma 4.1, $\mathbf{u}_m = \rho(x)^{2m+1}\mathbf{u}$ is classical for $m \geq 0$. As proved in [10], this means that $\{p_n^{(m)}(x)\}_{n\geq 0}$ satisfies the so-called *Second Structure Relation*

$$p_n^{(m)}(x) = \xi_n^{(m)} \frac{\mathrm{d}}{\mathrm{d}x} p_{n+1}^{(m)}(x) + \sigma_n^{(m)} \frac{\mathrm{d}}{\mathrm{d}x} p_n^{(m)}(x) + \tau_n^{(m)} \frac{\mathrm{d}}{\mathrm{d}x} p_{n-1}^{(m)}(x), \quad n \geq 0, \tag{4.2}$$

for some constants $\xi_n^{(m)}$, $\sigma_n^{(m)}$, and $\tau_n^{(m)}$, with

$$\xi_n^{(m)} = \frac{k_n^{(m)}}{(n+1)k_{n+1}^{(m)}} = \frac{a_n^{(m)}}{n+1} \neq 0, \quad n, m \geq 0.$$

For $m \geq 0$, denote by $\pi_n^{(m)}(x)$ the polynomial of degree n defined by

$$\begin{aligned} \pi_n^{(m)}(x) &= \xi_{n-1}^{(m)}\, p_n^{(m)}(x) + \sigma_{n-1}^{(m)}\, p_{n-1}^{(m)}(x) + \tau_{n-1}^{(m)}\, p_{n-2}^{(m)}(x), \quad n \geq 1, \\ \pi_0^{(m)}(x) &= 1. \end{aligned} \tag{4.3}$$

It follows by definition that

$$\frac{\mathrm{d}}{\mathrm{d}x} \pi_{n+1}^{(m)}(x) = p_n^{(m)}(x), \quad n \geq 0. \tag{4.4}$$

Now, we study Case I and Case II separately.

4.1. Case I

In this case, $\rho(x) = r_1 x + r_0$, $|r_1| + |r_0| > 0$, and the bilinear form $(\cdot,\cdot)_m$ reads

$$(p,q)_m = (1+\lambda m^2 r_1^2)\langle \mathbf{u}_m, p\,q\rangle + \lambda\, m\, r_1 \langle \mathbf{u}_m, (r_1 x + r_0)\,(p'\,q + p\,q')\rangle + \lambda\,\langle \mathbf{u}_m,\, (r_1\,x + r_0)^2\, p'\,q'\rangle. \quad (4.5)$$

Next, we can express the polynomial $\pi_n^{(m)}(x)$ in terms of the Sobolev polynomials. The proof is given at the Appendix A.

Proposition 4.2. *For $m \geq 0$, let $s_{-1}^{(m)}(x) = 0$. Then,*

$$\begin{aligned} &\pi_n^{(m)}(x) = \xi_{n-1}^{(m)}\, s_n^{(m)}(x) + d_{n,1}^{(m)}\, s_{n-1}^{(m)}(x) + d_{n,2}^{(m)}\, s_{n-2}^{(m)}(x), \quad n \geq 1,\\ &\pi_0^{(m)}(x) = 1, \end{aligned} \quad (4.6)$$

where

$$d_{n,2}^{(m)} = \frac{1}{\widetilde{h}_{n-2}^{(m)}} \mathcal{A}_{n,2}^{(I)} h_{n-2}^{(m)}, \quad n \geq 2, \quad (4.7)$$

$$d_{n,1}^{(m)} = \frac{1}{\widetilde{h}_{n-1}^{(m)}} \left[\mathcal{B}_{n,1}^{(I)}\, h_{n-1}^{(m)} + \left[\mathcal{B}_{n,2}^{(I)} - \mathcal{A}_{n,2}^{(I)}\, d_{n-1,1}^{(m)} \right] \frac{h_{n-2}^{(m)}}{\xi_{n-2}^{(m)}} \right], \quad n \geq 1, \quad (4.8)$$

where $d_{n,i}^{(m)} = \mathcal{A}_{n,i}^{(I)} = \mathcal{B}_{n,i}^{(I)} = 0$, for $n < i$,

$$\mathcal{A}_{n,2}^{(I)} = \tau_{n-1}^{(m)} + \lambda\, r_1^2\, (m+n-2) \left(m\, \tau_{n-1}^{(m)} + c_{n-1}^{(m)} \right),$$

$$\mathcal{B}_{n,1}^{(I)} = \sigma_{n-1}^{(m)} + \lambda\, r_1\, (m+n-1)\, \rho \left(m\, \sigma_{n-1}^{(m)} + b_{n-1}^{(m)} \right),$$

$$\mathcal{B}_{n,2}^{(I)} = \tau_{n-1}^{(m)}\, \sigma_{n-2}^{(m)} + \lambda\, r_1 \left(m\, \tau_{n-1}^{(m)} + c_{n-1}^{(m)} \right) \rho \left(m\, \sigma_{n-2}^{(m)} + b_{n-2}^{(m)} \right),$$

and $h_n^{(m)}$, $\widetilde{h}_n^{(m)}$ were defined in (3.1) and (3.4), respectively.

Therefore, from (4.2) and (4.6), we deduce a short relation between univariate Sobolev orthogonal polynomials and the first family of orthogonal polynomials.

Corollary 4.3. *For $n \geq 1$, the following relation holds:*

$$s_n^{(m)}(x) + \widetilde{d}_{n,1}^{(m)}\, s_{n-1}^{(m)}(x) + \widetilde{d}_{n,2}^{(m)}\, s_{n-2}^{(m)}(x) = p_n^{(m)}(x) + \widetilde{\sigma}_{n-1}^{(m)}\, p_{n-1}^{(m)}(x) + \widetilde{\tau}_{n-1}^{(m)}\, p_{n-2}^{(m)}(x), \quad (4.9)$$

where

$$\widetilde{d}_{n,1}^{(m)} = \frac{d_{n,1}^{(m)}}{\xi_{n-1}^{(m)}}, \quad \widetilde{d}_{n,2}^{(m)} = \frac{d_{n,2}^{(m)}}{\xi_{n-1}^{(m)}}, \quad \widetilde{\sigma}_{n-1}^{(m)} = \frac{\sigma_{n-1}^{(m)}}{\xi_{n-1}^{(m)}}, \quad \widetilde{\tau}_{n-1}^{(m)} = \frac{\tau_{n-1}^{(m)}}{\xi_{n-1}^{(m)}}, \quad (4.10)$$

and $s_{-1}^{(m)}(x) = p_{-1}^{(m)}(x) = 0$, $s_0^{(m)}(x) = p_0^{(m)}(x)$.

The polynomials $\{s_n^{(m)}(x)\}_{n\geq 0}$ can be deduced recursively from (4.9) if we know how to compute $\widetilde{h}_n^{(m)}$. The following result provides an effective way to implement it:

Proposition 4.4. *Let* $\widetilde{h}_{-1}^{(m)} = 0$, *then* $\widetilde{h}_0^{(m)} = (1+\lambda\, m^2\, r_1^2)\, h_0^{(m)}$ *and, for* $n \geq 1$,

$$\widetilde{h}_n^{(m)} = \mathcal{C}_n^{(I)} h_n^{(m)} + \mathcal{D}_n^{(I)} h_{n-1}^{(m)} + \mathcal{E}_n^{(I)} h_{n-2}^{(m)} - (\widetilde{d}_{n,1}^{(m)})^2\, \widetilde{h}_{n-1}^{(m)} - (\widetilde{d}_{n,2}^{(m)})^2\, \widetilde{h}_{n-2}^{(m)}, \tag{4.11}$$

where

$$\mathcal{C}_n^{(I)} = 1 + \lambda\, r_1^2 (m+n)^2,$$
$$\mathcal{D}_n^{(I)} = \frac{(\sigma_{n-1}^{(m)})^2 + \lambda\, \rho (m\, \sigma_{n-1}^{(m)} + b_{n-1}^{(m)})^2}{(\xi_{n-1}^{(m)})^2},$$
$$\mathcal{E}_n^{(I)} = \frac{(\tau_{n-1}^{(m)})^2 + \lambda\, r_1^2 (m\, \tau_{n-1}^{(m)} + c_{n-1}^{(m)})^2}{(\xi_{n-1}^{(m)})^2}.$$

Proof. On the one hand, using (4.6), we get

$$(\pi_n^{(m)}, \pi_n^{(m)})_m = (\xi_{n-1}^{(m)})^2\, \widetilde{h}_n^{(m)} + (d_{n,1}^{(m)})^2\, \widetilde{h}_{n-1}^{(m)} + (d_{n,2}^{(m)})^2\, \widetilde{h}_{n-2}^{(m)}.$$

On the other hand, using (4.1), (4.3), (4.4), and the explicit expression of the Sobolev bilinear form (4.5), we obtain

$$(\pi_n^{(m)}, \pi_n^{(m)})_m = \mathcal{C}_n^{(I)} h_n^{(m)} + \mathcal{D}_n^{(I)} h_{n-1}^{(m)} + \mathcal{E}_n^{(I)} h_{n-2}^{(m)}.$$

Moreover, using directly (4.5), we get $\widetilde{h}_0^{(m)} = (s_0^{(m)}, s_0^{(m)})_m = (p_0^{(m)}, p_0^{(m)})_m = (1+\lambda\, m^2\, r_1^2)\, h_0^{(m)}$. □

Now, we want to analyse how to compute explicitly the Sobolev orthogonal polynomials $s_n^{(m)}(x)$, its norms

$$\widetilde{h}_n^{(m)} = (s_n^{(m)}, s_n^{(m)})_m,$$

as well as the real numbers $\widetilde{d}_{n,1}^{(m)}$, for $n \geq 1$, and $\widetilde{d}_{n,2}^{(m)}$, for $n \geq 2$, assuming that we know all the coefficients for the standard polynomials $\{p_n^{(m)}(x)\}_{n\geq 0}$ involved in relations (4.1) and (4.2).

First of all, we know that

$$\widetilde{h}_0^{(m)} = (1+\lambda\, m^2\, r_1^2)\, h_0^{(m)},$$

and we can compute $\widetilde{d}_{2,2}^{(m)}$ and $\widetilde{d}_{1,1}^{(m)}$ by using (4.7) and (4.8), respectively, and (4.10). Then, we use (4.11) for $n=1$, and we deduce $\widetilde{h}_1^{(m)}$. Next, we can compute $\widetilde{d}_{3,2}^{(m)}$ and $\widetilde{d}_{2,1}^{(m)}$, and from (4.11) for $n=2$ using $\widetilde{d}_{2,2}^{(m)}$ and $\widetilde{d}_{2,1}^{(m)}$ we obtain $\widetilde{h}_2^{(m)}$, and so on. In Fig. 1 we can see how the algorithm generates all the *tilde* constants.

Finally, using expression (3.5), relation (4.9) can be extended to the bivariate case multiplying by $\rho(x)^m q_m\left(\frac{y}{\rho(x)}\right)$. Therefore, bivariate Sobolev orthogonal polynomials are related to the standard bivariate orthogonal polynomials, and we can compute them recursively.

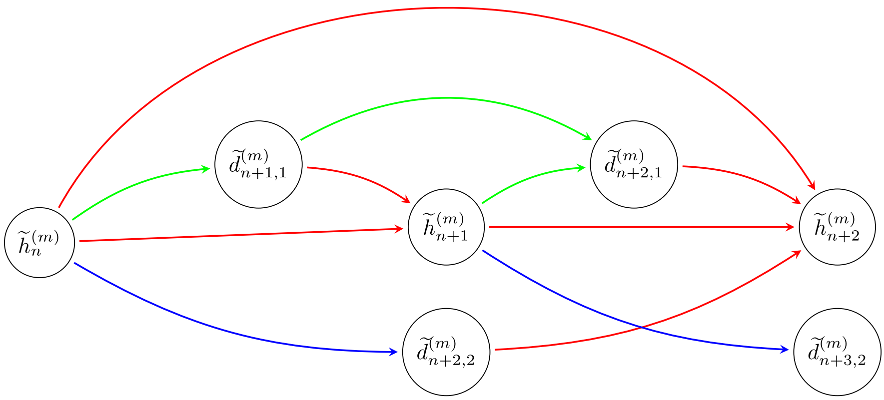


Figure 1. Relation between norms and coefficients for $n \geq 0$ in Case I. Color code for the arrows: blue arrows: we use (4.7); green arrows: we use (4.8); red arrows: we use (4.11)

Theorem 4.5. *For $n \geq 1$ and $0 \leq m \leq n$, the following relation holds:*

$$\begin{aligned} S_{n,m}(x,y) + \widetilde{d}_{n-m,1}^{(m)} S_{n-1,m}(x,y) + \widetilde{d}_{n-m,2}^{(m)} S_{n-2,m}(x,y) \\ = P_{n,m}(x,y) + \widetilde{\sigma}_{n-m-1}^{(m)} P_{n-1,m}(x,y) + \widetilde{\tau}_{n-m-1}^{(m)} P_{n-2,m}(x,y), \end{aligned}$$

and $S_{0,0}(x,y) = P_{0,0}(x,y)$.

4.2. Case II

In this case, $\rho(x) = \sqrt{\ell_2 x^2 + 2\ell_1 x + \ell_0}$, with $|\ell_2| + |\ell_1| + |\ell_0| > 0$, and the bilinear form (3.3) reads

$$\begin{aligned} (p,q)_m =& \langle \mathbf{u}_m, p\,q\rangle + \lambda\, m^2 \langle \mathbf{u}_m, (\ell_2 x + \ell_1)^2 p\,q\rangle \\ & + \lambda\, m \langle \mathbf{u}_m, (\ell_2 x + \ell_1)\, \rho^2(x)\, (p'\,q + p\,q')\rangle + \lambda \langle \mathbf{u}_m,\, \rho^4(x)\, p'\,q'\rangle \\ =& \langle \mathbf{u}_m, p\,q\rangle + \lambda\, m^2 \langle \mathbf{u}_m, (\ell_2 x + \ell_1)^2 p\,q\rangle \\ & + \lambda\, m \langle \mathbf{u}_m, (\ell_2 x + \ell_1)\,(\ell_2 x^2 + 2\ell_1 x + \ell_0)\,(p'\,q + p\,q')\rangle \\ & + \lambda \langle \mathbf{u}_m,\, (\ell_2 x^2 + 2\ell_1 x + \ell_0)^2\, p'\,q'\rangle. \end{aligned} \tag{4.12}$$

Recall that $\mathbf{v}$ must be a symmetric moment functional.

A similar reasoning as in Proposition 4.2 allows us to prove the next result, taking into account the explicit expression of the bilinear form in this case (4.12). The explicit expressions of the coefficients are showed in the Appendix.

Proposition 4.6. *For $m \geq 0$, let $s_{-1}^{(m)}(x) = s_{-2}^{(m)} = s_{-3}^{(m)} = 0$ and $s_0^{(m)}(x) = p_0^{(m)}(x)$. Then, for $n \geq 1$, the polynomials $\pi_n^{(m)}(x)$ defined in (4.3) satisfy*

$$\begin{aligned} \pi_n^{(m)}(x) = \xi_{n-1}^{(m)} s_n^{(m)}(x) + e_{n,1}^{(m)} s_{n-1}^{(m)}(x) + e_{n,2}^{(m)} s_{n-2}^{(m)}(x) \\ + e_{n,3}^{(m)} s_{n-3}^{(m)}(x) + e_{n,4}^{(m)} s_{n-4}^{(m)}(x), \end{aligned} \tag{4.13}$$

where $e_{n,i}^{(m)} = 0$ *for* $n < i$, *and*

$$e_{n,4}^{(m)}\widetilde{h}_{n-4}^{(m)} = \mathcal{A}_{n,1}^{(II)}\, h_{n-4}^{(m)},$$

$$e_{n,3}^{(m)}\widetilde{h}_{n-3}^{(m)} = \mathcal{B}_{n,1}^{(II)}\, h_{n-3}^{(m)} + \mathcal{B}_{n,2}^{(II)}\, \frac{h_{n-4}^{(m)}}{\xi_{n-4}^{(m)}} - \frac{[e_{n,4}^{(m)}\widetilde{h}_{n-4}^{(m)}]}{\xi_{n-4}^{(m)}}\, e_{n-3,1}^{(m)}, \tag{4.14}$$

$$\begin{aligned} e_{n,2}^{(m)}\widetilde{h}_{n-2}^{(m)} = {} & \mathcal{C}_{n,1}^{(II)}\, h_{n-2}^{(m)} + \mathcal{C}_{n,2}^{(II)}\, \frac{h_{n-3}^{(m)}}{\xi_{n-3}^{(m)}} + \mathcal{C}_{n,3}^{(m)}\, \frac{h_{n-4}^{(m)}}{\xi_{n-3}^{(m)}} \\ & - \frac{[e_{n,3}^{(m)}\widetilde{h}_{n-3}^{(m)}]}{\xi_{n-3}^{(m)}}\, e_{n-2,1}^{(m)} - \frac{[e_{n,4}^{(m)}\widetilde{h}_{n-4}^{(m)}]}{\xi_{n-3}^{(m)}}\, e_{n-2,2}^{(m)}, \end{aligned} \tag{4.15}$$

$$\begin{aligned} e_{n,1}^{(m)}\widetilde{h}_{n-1}^{(m)} = {} & \mathcal{D}_{n,1}^{(II)}\, h_{n-1}^{(m)} + \mathcal{D}_{n,2}^{(II)}\, \frac{h_{n-2}^{(m)}}{\xi_{n-2}^{(m)}} + \mathcal{D}_{n,3}^{(II)}\, \frac{h_{n-3}^{(m)}}{\xi_{n-2}^{(m)}} \\ & - \frac{[e_{n,2}^{(m)}\widetilde{h}_{n-2}^{(m)}]}{\xi_{n-2}^{(m)}} e_{n-1,1}^{(m)} - \frac{[e_{n,3}^{(m)}\widetilde{h}_{n-3}^{(m)}]}{\xi_{n-2}^{(m)}} e_{n-1,2}^{(m)} - \frac{[e_{n,4}^{(m)}\widetilde{h}_{n-4}^{(m)}]}{\xi_{n-2}^{(m)}} e_{n-1,3}^{(m)}. \end{aligned} \tag{4.16}$$

As in Case I, we can establish a finite relation between Sobolev orthogonal polynomials and the first family of orthogonal polynomials joining (4.2) and (4.13).

Corollary 4.7. *For* $n \geq 1$, *the following relation holds:*

$$\begin{aligned} & s_n^{(m)}(x) + \widetilde{e}_{n,1}^{(m)}\, s_{n-1}^{(m)}(x) + \widetilde{e}_{n,2}^{(m)}\, s_{n-2}^{(m)}(x) + \widetilde{e}_{n,3}^{(m)}\, s_{n-3}^{(m)}(x) + \widetilde{e}_{n,4}^{(m)}\, s_{n-4}^{(m)}(x) \\ & \quad = p_n^{(m)}(x) + \widetilde{\sigma}_{n-1}^{(m)}\, p_{n-1}^{(m)}(x) + \widetilde{\tau}_{n-1}^{(m)}\, p_{n-2}^{(m)}(x), \end{aligned} \tag{4.17}$$

where

$$\widetilde{e}_{n,i}^{(m)} = \frac{e_{n,i}^{(m)}}{\xi_{n-1}^{(m)}}, \quad i = 1, 2, 3, 4,$$

and $s_0^{(m)}(x) = p_0^{(m)}(x)$.

Relation (4.17) and the explicit expressions of the coefficients $e_{n,i}^{(m)}$, for $i = 1, 2, 3, 4$, given in Proposition 4.6, can be used to compute $\{s_n^{(m)}(x)\}_{n\geq 0}$. The norms $\widetilde{h}_n^{(m)}$ are computed recursively as follows (Fig. 2):

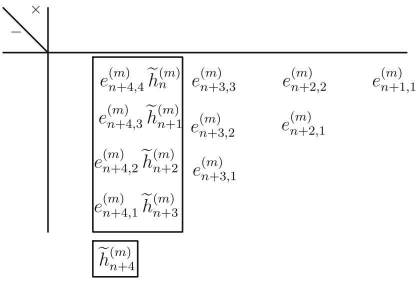
Proposition 4.8. *Let* $\widetilde{h}_{-1}^{(m)} = 0$. *Then,*

$$\widetilde{h}_0^{(m)} = \left[1 + \lambda\, m^2 \big((\ell_2\, b_0^{(m)} + \ell_1)^2 + \ell_2^2\, a_0^{(m)}\, c_1^{(m)}\big)\right] h_0^{(m)},$$

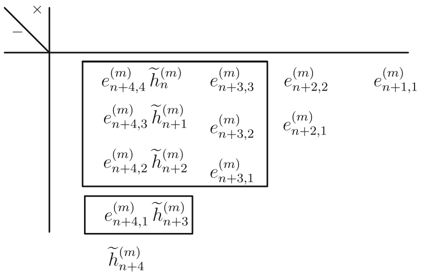
and, for $n \geq 1$,

$$\widetilde{h}_n^{(m)} = \mathcal{E}_n^{(II)} h_n^{(m)} + \mathcal{F}_n^{(II)} h_{n-1}^{(m)} + \mathcal{G}_n^{(II)} h_{n-2}^{(m)} - \sum_{i=1}^{4} (\widetilde{e}_{n,i}^{(m)})^2\, \widetilde{h}_{n-i}^{(m)}, \tag{4.18}$$

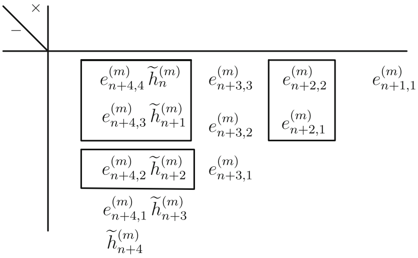
where the explicit expressions of the coefficients $\mathcal{E}_n^{(II)}$, $\mathcal{F}_n^{(II)}$ *and* $\mathcal{G}_n^{(II)}$ *are given in the Appendix.*



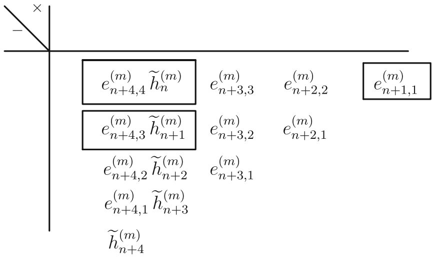
(A) Terms involved in (4.18)



(B) Terms involved in (4.16)



(C) Terms involved in (4.15)



(D) Terms involved in (4.14)

Figure 2. In each diagram, the coefficients and norms indicated across the same row are multiplied. The resulting products are terms appearing in the referenced equations

Finally, relation (4.17) can be translated to the bivariate case by multiplying by $\rho(x)^m q_m\left(\frac{y}{\rho(x)}\right)$.

Theorem 4.9. *We get* $S_{0,0}(x,y)=P_{0,0}(x,y)$*, and for* $n\geq 1$ *and* $0\leq m\leq n$*, the following relation holds:*

$$\begin{aligned} S_{n,m}(x,y) &+ \widetilde{e}^{(m)}_{n-m,1}S_{n-1,m}(x,y) + \widetilde{e}^{(m)}_{n-m,2}S_{n-2,m}(x,y) \\ &+ \widetilde{e}^{(m)}_{n-m,3}S_{n-3,m}(x,y) + \widetilde{e}^{(m)}_{n-m,4}S_{n-4,m}(x,y) \\ &= P_{n,m}(x,y) + \widetilde{\sigma}^{(m)}_{n-m-1}P_{n-1,m}(x,y) + \widetilde{\tau}^{(m)}_{n-m-1}P_{n-2,m}(x,y). \end{aligned}$$

An interesting case appears when $\rho(x)^2=\phi(x)$. In this case, not only $\mathbf{u}_m$ is classical for $m\geq 0$ (Lemma 4.1), and $\{p_n^{(m)}(x)\}_{n\geq 0}$ satisfies (4.2), but also $\{p_n^{(m)}(x)\}_{n\geq 0}$ satisfies the structure relation (see, for instance, [10]) as follows:

$$\rho(x)^2\,\frac{\mathrm{d}}{\mathrm{d}x}p_n^{(m)}(x) = \vartheta_n^{(m)}\,p_{n+1}^{(m)}(x) + \nu_n^{(m)}\,p_n^{(m)}(x) + \varpi_n^{(m)}\,p_{n-1}^{(m)}(x), \quad n\geq 1, \tag{4.19}$$

where $\rho(x)^2=\ell_2\,x^2+2\,\ell_1\,x+\ell_0$, and $\vartheta_n^{(m)}=\ell_2\,n\,a_n^{(m)}$.

In such a situation, the relation between standard and Sobolev polynomials is shorter than (4.17).

Proposition 4.10. *For $m \ge 0$, let $s_{-1}^{(m)}(x) = 0$. Then,*

$$\begin{aligned} p_n^{(m)}(x) &= s_n^{(m)}(x) + f_{n,1}^{(m)}\, s_{n-1}^{(m)}(x) + f_{n,2}^{(m)}\, s_{n-2}^{(m)}(x), \quad n \ge 1, \\ s_0^{(m)}(x) &= p_0^{(m)}(x), \end{aligned} \tag{4.20}$$

where $f_{n,i}^{(m)} = \widehat{\mathcal{A}}_{n,i}^{(II)} = \widehat{\mathcal{B}}_{n,i}^{(II)} = 0$, for $n < i$,

$$f_{n,2}^{(m)} = \frac{1}{\widetilde{h}_{n-2}^{(m)}} \widehat{\mathcal{A}}_{n,2}^{(II)} h_{n-1}^{(m)}, \quad n \ge 2, \tag{4.21}$$

$$f_{n,1}^{(m)} = \frac{1}{\widetilde{h}_{n-1}^{(m)}} \left[\widehat{\mathcal{B}}_{n,1}^{(II)}\, h_n^{(m)} + [\widehat{\mathcal{B}}_{n,2}^{(II)} - \widehat{\mathcal{A}}_{n,2}^{(II)}\, f_{n-1,1}^{(m)}]\, h_{n-1}^{(m)} \right], \quad n \ge 1, \tag{4.22}$$

and, if we denote $\widehat{\rho}(x) = \ell_2\, x + \ell_1$, then

$$\begin{aligned} \widehat{\mathcal{A}}_{n,2}^{(II)} &= \lambda\, \ell_2\, a_{n-2}^{(m)}\, (m+n-2)(m\, \ell_2\, c_n^{(m)} + \varpi_n^{(m)}), \\ \widehat{\mathcal{B}}_{n,1}^{(II)} &= \lambda\, \ell_2\, a_{n-1}^{(m)}\, (m+n-1)\, (m\, \widehat{\rho}(b_n^{(m)}) + \nu_n^{(m)}), \\ \widehat{\mathcal{B}}_{n,2}^{(II)} &= \lambda (m\, \widehat{\rho}(b_{n-1}^{(m)}) + \nu_{n-1}^{(m)})(m\, \ell_2\, c_n^{(m)} + \varpi_n^{(m)}). \end{aligned}$$

Proof. Observe that

$$p_n^{(m)}(x) = s_n^{(m)}(x) + \sum_{i=0}^{n-1} f_{n,n-i}^{(m)}\, s_i^{(m)}(x),$$

where

$$f_{n,n-i}^{(m)} = \frac{(p_n^{(m)}, s_i^{(m)})_m}{\widetilde{h}_i^{(m)}}, \quad 0 \le i \le n-1.$$

Using the explicit expression of the Sobolev bilinear form (4.12) and the structure relation (4.19), we deduce that $f_{n,n-i}^{(m)} = 0$, for $0 \le i \le n-3$. The explicit expressions for the coefficients $f_{n,1}^{(m)}$ and $f_{n,2}^{(m)}$ are deduced after a straightforward computation similar to the proof of Proposition 4.2. □

Finally, the norms can be computed in a simpler way.

Proposition 4.11. *Let $\widetilde{h}_{-1}^{(m)} = 0$. If we denote $\widehat{\rho}(x) = \ell_2\, x + \ell_1$, then*

$$\widetilde{h}_0^{(m)} = [1 + \lambda\, m^2 (\widehat{\rho}(b_0^{(m)}))^2] h_0^{(m)} + \lambda m^2 \ell_2^2 (a_0^{(m)})^2 h_1^{(m)},$$

and, for $n \ge 1$ (Fig. 3),

$$\widetilde{h}_n^{(m)} = \widehat{\mathcal{C}}_n^{(II)}\, h_{n+1}^{(m)} + \widehat{\mathcal{D}}_n^{(II)}\, h_n^{(m)} + \widehat{\mathcal{E}}_n^{(II)}\, h_{n-1}^{(m)} - (f_{n,1}^{(m)})^2\, \widetilde{h}_{n-1}^{(m)} - (f_{n,2}^{(m)})^2\, \widetilde{h}_{n-2}^{(m)}, \tag{4.23}$$

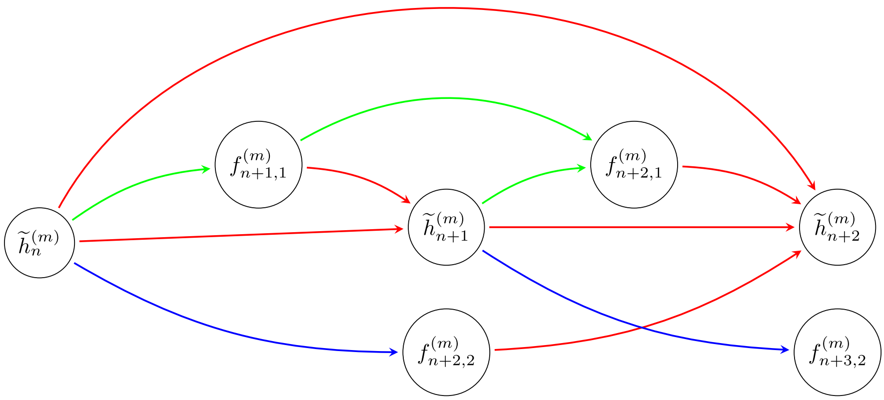


Figure 3. Relation between norms and coefficients for $n \geq 0$. Color code: blue–(4.21), green–(4.22), red–(4.23)

where

$$\widehat{\mathcal{C}}_n^{(II)} = \lambda \ell_2^2 (a_n^{(m)})^2 (m+n)^2,$$
$$\widehat{\mathcal{D}}_n^{(II)} = 1 + \lambda (m\,\widehat{\rho}(b_n^{(m)}) + \nu_n^{(m)})^2,$$
$$\widehat{\mathcal{E}}_n^{(II)} = \lambda (m\,\ell_2\, c_n^{(m)} + \varpi_n^{(m)})^2.$$

As in Case I, relation (4.20) can be translated to the bivariate case by multiplying by $\rho(x)^m q_m\left(\frac{y}{\rho(x)}\right)$. In this case, we deduce an expression of a standard bivariate polynomial as a linear combination of three consecutive first-type bivariate Sobolev orthogonal polynomials.

Corollary 4.12. *For $n \geq 1$ and $0 \leq m \leq n$, the following relation holds:*

$$P_{n,m}(x,y) = S_{n,m}(x,y) + f_{n-m,1}^{(m)}\, S_{n-1,m}(x,y) + f_{n-m,2}^{(m)}\, S_{n-2,m}(x,y),$$

with $S_{0,0}(x,y) = P_{0,0}(x,y)$, and $S_{i,j}(x,y) = 0$ for $i < j$.

5. Second Type of Bivariate Sobolev Orthogonal Polynomials

Now, we replace the univariate linear functional acting on the second variable y in the construction described in Sect. 2.3 by a univarite quasi-definite Sobolev bilinear form and study its associated sequence of orthogonal polynomials.

Again, let $\mathbf{u} \equiv \mathbf{u}^{(x)}$ and $\mathbf{v} \equiv \mathbf{v}^{(y)}$ be univariate quasi-definite moment functionals acting on the variables x and y, respectively, and let $\rho(x)$ be a function satisfying the conditions of either Case I or Case II above.

Define the univariate Sobolev bilinear form

$$(p,q)_{\mathbf{v}} = \langle \mathbf{v},\ p\,q + \lambda\, p'\, q' \rangle, \tag{5.1}$$

where λ is a real number such that $(\cdot,\cdot)_{\mathbf{v}}$ is quasi-definite. Observe that when $\mathbf{v}$ is positive definite and $\lambda \geq 0$, then (5.1) defines an inner product.

Let $\{s_m(y)\}_{m\geq 0}$ be the corresponding univariate orthogonal polynomial sequence, standardized in such a way the leading coefficient of $s_m(y)$ is the same as the leading coefficient of $q_m(y)$, for $m \geq 0$. Therefore, $s_0(y) = q_0(y)$. In addition, let

$$\widehat{h}_m^{(s)} = \widehat{h}_m^{(s)}(\lambda) = (s_m, s_m)_{\mathbf{v}} \neq 0, \quad m \geq 0. \tag{5.2}$$

Note that $\widehat{h}_m^{(s)}(0) \equiv h_m^{(q)}$.

Observe that in Case II the moment functional $\mathbf{v}$ is symmetric, and then, the bilinear form $(\cdot,\cdot)_{\mathbf{v}}$ is also symmetric. Moreover, if $\mathbf{v}$ is classical, then the only possibilities are Hermite and Gegenbauer moment functionals.

Therefore, we can construct bivariate polynomials as follows:

Theorem 5.1. *Let* $\mathbf{w}$ *be a bivariate moment functional defined as in* (2.2). *The polynomials defined as*

$$\widehat{S}_{n,m}(x,y) = p_{n-m}^{(m)}(x)\,\rho(x)^m\, s_m\left(\frac{y}{\rho(x)}\right), \quad n \geq 0, \quad 0 \leq m \leq n \tag{5.3}$$

constitute a mutually orthogonal basis with respect to the bivariate Sobolev bilinear form

$$(P,Q)_S = \left\langle \mathbf{w},\ P\,Q + \lambda\,\rho^2\,\partial_y P\,\partial_y Q\right\rangle, \tag{5.4}$$

where $\lambda \in \mathbb{R}$. *Moreover,*

$$\widehat{H}_{n,m} = (\widehat{S}_{n,m},\, \widehat{S}_{n,m})_S = h_{n-m}^{(m)}\,\widehat{h}_m^{(s)},$$

where $h_{n-m}^{(m)}$ *and* $\widehat{h}_m^{(s)}$ *were defined in* (3.1) *and* (5.2).

Proof. We compute

$$(\widehat{S}_{n,m},\, \widehat{S}_{i,j})_S = \langle \mathbf{w}, \widehat{S}_{n,m}\,\widehat{S}_{i,j}\rangle + \lambda\langle \mathbf{w}, \rho^2\,\partial_y\widehat{S}_{n,m}\,\partial_y\widehat{S}_{i,j}\rangle.$$

Using (2.2) and (5.3) in the first term of the right side, we get

$$(\widehat{S}_{n,m},\, \widehat{S}_{i,j})_S = \langle \rho^{m+j+1}\,\mathbf{u}, p_{n-m}^{(m)}\,p_{i-j}^{(j)}\rangle\langle \mathbf{v}, s_m\,s_j + \lambda\,s_m'\,s_j'\rangle.$$

Therefore, $(\widehat{S}_{n,m},\, \widehat{S}_{i,j})_S = h_{n-m}^{(m)}\widehat{h}_m^{(s)}\,\delta_{n,i}\,\delta_{m,j}$. □

Since $\{q_m(y)\}_{m\geq 0}$ is a standard OPS associated with the functional $\mathbf{v}$, it satisfies a three-term recurrence relation

$$y\,q_m(y) = a_m\,q_{m+1}(y) + b_m\,q_m(y) + c_m\,q_{m-1}(y), \quad m \geq 0,$$
$$q_{-1}(y) = 0, \quad q_0(y) = 1,$$

and if $q_m(y) = k_m\,y^m +$ lower degree terms, then

$$a_m = \frac{k_m}{k_{m+1}}, \quad b_m = \frac{\langle \mathbf{v}, y\,q_m^2\rangle}{h_m^{(q)}}, \quad c_{m+1} = \frac{k_m}{k_{m+1}}\frac{h_{m+1}^{(q)}}{h_m^{(q)}} = a_m\frac{h_{m+1}^{(q)}}{h_m^{(q)}},$$

for $m \geq 0$, and $h_m^{(q)}$ defined in (3.2). In Case II, since $\mathbf{v}$ is symmetric, then $b_m = 0$, $m \geq 0$, and therefore, polynomials $q_m(y)$ are symmetric, that is,

$$q_m(-y) = (-1)^m\,q_m(y), \quad m \geq 0.$$

As in Sect. 4, if $\mathbf{v}$ is a classical linear functional, then orthogonal polynomials $\{q_m(y)\}_{n\geq 0}$ satisfy a *Second Structure Relation* ([10]) as (4.2), that is, a polynomial can be written in terms of three consecutive derivatives in the form

$$q_m(y) = \xi_m \frac{\mathrm{d}}{\mathrm{d}y} q_{m+1}(y) + \sigma_m \frac{\mathrm{d}}{\mathrm{d}y} q_m(y) + \tau_m \frac{\mathrm{d}}{\mathrm{d}y} q_{m-1}(y), \quad m \geq 0, \quad (5.5)$$

for some constants ξ_n, σ_m, and τ_m, with

$$\xi_m = \frac{k_m}{(m+1)k_{m+1}} = \frac{a_m}{m+1} \neq 0, \quad m \geq 0.$$

In Case II, from the symmetry, we get $\sigma_m = 0$, for $m \geq 0$.

As in the above section, we define the polynomial

$$\begin{aligned} &\pi_m(y) = \xi_{m-1}\, q_m(y) + \sigma_{m-1}\, q_{m-1}(y) + \tau_{m-1}\, q_{m-2}(y), \qquad m \geq 0, \\ &\pi_0(y) = 1. \end{aligned} \quad (5.6)$$

Then,

$$\frac{\mathrm{d}}{\mathrm{d}y} \pi_{m+1}(y) = q_m(y), \quad m \geq 0.$$

Proposition 5.2. *Suppose that* $\mathbf{v}$ *is a classical linear functional. Then,*

$$\begin{aligned} &\pi_m(y) = \xi_{m-1}\, s_m(y) + d_{m,1}\, s_{m-1}(y) + d_{m,2}\, s_{m-2}(y), \quad m \geq 0, \\ &\pi_0(y) = 1, \end{aligned} \quad (5.7)$$

where

$$\begin{aligned} d_{m,2} &= \frac{\tau_{m-1}\, h_{m-2}^{(q)}}{\widehat{h}_{m-2}^{(s)}}, \quad m \geq 2, \\ d_{m,1} &= \frac{1}{\widehat{h}_{m-1}^{(s)}} \left[\sigma_{m-1}\, h_{m-1}^{(q)} + \frac{\tau_{m-1}\, h_{m-2}^{(q)}}{\xi_{m-2}} (\sigma_{m-2} - d_{m-1,1}) \right], \quad m \geq 1. \end{aligned}$$

In Case II, $d_{m,1} = 0$*,* $m \geq 0$*, and Sobolev orthogonal polynomials* $\{s_m(y)\}_{m\geq 0}$ *are also symmetric polynomials.*

Using (5.5) and (5.7) we can deduce a finite relation between the Sobolev orthogonal polynomials and the second family of orthogonal polynomials.

Corollary 5.3. *For* $m \geq 1$*, the following relation holds:*

$$\begin{aligned} s_m(y) + \widehat{d}_{m,1}\, s_{m-1}(y) + \widehat{d}_{m,2}\, s_{m-2}(y) = \; & q_m(y) + \widehat{\sigma}_{m-1}\, q_{m-1}(y) \\ & + \widehat{\tau}_{m-1}\, q_{m-2}(y), \end{aligned} \quad (5.8)$$

where

$$\widehat{d}_{m,i} = \frac{d_{m,i}}{\xi_{m-1}}, \quad i = 1, 2, \quad \widehat{\sigma}_{m-1} = \frac{\sigma_{m-1}}{\xi_{m-1}}, \quad \widehat{\tau}_{m-1} = \frac{\tau_{m-1}}{\xi_{m-1}},$$

and $s_0(y) = q_0(y)$*.*

In Case II we get $\widehat{d}_{m,1} = \widehat{\sigma}_{m-1} = 0$*, for* $m \geq 1$*.*

The polynomials $\{s_m(y)\}_{n\geq 0}$ can be deduced recursively from (5.7) if we know how to compute $\widehat{h}_n^{(s)}$. We can do so effectively as follows:

Proposition 5.4. *Let* $\widehat{h}_{-1}^{(s)} = 0$ *and* $\widehat{h}_{0}^{(s)} = h_0$. *Then, for* $m \geq 1$,

$$\widehat{h}_{m}^{(s)} = h_{m}^{(q)} + (\widehat{\sigma}_{m-1}^{2} + \lambda\, \xi_{m-1}^{-2}) h_{m-1}^{(q)} + \widehat{\tau}_{m-1}^{2}\, h_{m-2}^{(q)} - \widehat{d}_{m,1}^{2}\, \widehat{h}_{m-1}^{(s)} - \widehat{d}_{m,2}^{2}\, \widehat{h}_{m-2}^{(s)}.$$

Finally, we translate relation (5.8) to the bivariate case, relating Sobolev orthogonal polynomials to standard bivariate polynomials.

Theorem 5.5. *For* $n \geq 1$ *and* $0 \leq m \leq n$, *there exist real numbers such that the following relation holds:*

$$\sum_{i=0}^{4} \left[\widehat{\eta}_{n-i}^{(m)} \widehat{S}_{n+2-i,m} + \widehat{d}_{m,1} \widehat{\theta}_{n-i}^{(m)} \widehat{S}_{n+2-i,m-1} + \widehat{d}_{m,2} \widehat{\vartheta}_{n-i}^{(m)} \widehat{S}_{n+2-i,m-2}\right]$$
$$= \sum_{i=0}^{4} \left[\widehat{\eta}_{n-i}^{(m)} P_{n+2-i,m} + \widehat{\sigma}_{m-1} \widehat{\theta}_{n-i}^{(m)} P_{n+2-i,m-1} + \widehat{\tau}_{m-1} \widehat{\vartheta}_{n-i}^{(m)} P_{n+2-i,m-2}\right],$$

with $S_{0,0}(x,y) = P_{0,0}(x,y)$, *and* $S_{i,j}(x,y) = 0$ *for* $i < j$.

Proof. First, for $m \geq 1$, let $\{p_n^{(m-1)}(x)\}_{n\geq 0}$ and $\{p_n^{(m)}(x)\}_{n\geq 0}$ be univariate sequences of orthogonal polynomials associated with the quasi-definite moment functionals $\mathbf{u}_{m-1}$ and $\mathbf{u}_m$. Then both sequences can be related by means of expression (4.4) in [13],

$$p_n^{(m-1)}(x) = \delta_n^{(m)}\, p_n^{(m)}(x) + \epsilon_n^{(m)}\, p_{n-1}^{(m)}(x) + \zeta_n^{(m)}\, p_{n-2}^{(m)}(x), \tag{5.9}$$

where $\delta_n^{(m)} \neq 0$.

Let us multiply (5.8) times $p_{n-m+2}^{(m-2)}(x)\, \rho(x)^m$, and we study each term of the sum. First, using (5.9) twice, we obtain

$$p_{n-m+2}^{(m-2)}(x) \rho(x)^m s_m\left(\frac{y}{\rho(x)}\right) = \left[\sum_{i=0}^{4} \widehat{\eta}_{n-i}^{(m)} p_{n-m+2-i}^{(m)}(x)\right] \rho(x)^m\, s_m\left(\frac{y}{\rho(x)}\right)$$
$$= \sum_{i=0}^{4} \widehat{\eta}_{n-i}^{(m)} \widehat{S}_{n+2-i,m}(x,y),$$

where $\widehat{\eta}_{n-i}^{(m)}$ are real numbers. A similar reasoning shows that

$$p_{n-m+2}^{(m-2)}(x) \rho(x)^m q_m\left(\frac{y}{\rho(x)}\right) = \sum_{i=0}^{4} \widehat{\eta}_{n-i}^{(m)} P_{n+2-i,m}(x,y).$$

Now, we consider the second term in both sides of (5.8). If $\rho(x)$ is the square root of a polynomial of degree no greater than 2, then $\mathbf{v}$ is a symmetric moment functional, and therefore, $\widehat{d}_{m,1} = \widehat{\sigma}_{m-1} = 0$ for every nonnegative integer m.

Suppose that $\rho(x)$ is a polynomial of degree ≤ 1, that is, $\rho(x) = r_1\, x + r_0$, with $|r_1| + |r_0| > 0$. In this case, using (5.9) and (4.1), we get

$$\begin{aligned}
&\rho(x)\, p_{n-m+2}^{(m-2)}(x)\, \rho(x)^{m-1} s_{m-1}\left(\frac{y}{\rho(x)}\right)\\
&\quad = \Big\{(r_1\, x + r_0)\left[\delta_{n-m+2}^{(m-1)}\, p_{n-m+2}^{(m-1)}(x) + \epsilon_{n-m+2}^{(m-1)}\, p_{n-m+1}^{(m-1)}(x) + \zeta_{n-m+2}^{(m-1)}\, p_{n-m}^{(m-1)}(x)\right]\\
&\qquad \times \rho(x)^{m-1} s_{m-1}\left(\frac{y}{\rho(x)}\right)\\
&\quad = \left[\sum_{i=0}^{4} \widehat{\theta}_{n-i}^{(m)}\, p_{n-m+3-i}^{(m-1)}(x)\right] \rho(x)^{m-1}\, s_{m-1}\left(\frac{y}{\rho(x)}\right)\\
&\quad = \sum_{i=0}^{4} \widehat{\theta}_{n-i}^{(m)} \widehat{S}_{n+2-i,m-1}(x,y),
\end{aligned}$$

with $\widehat{\theta}_{n-i}^{(m)}$ constants depending on r_1 and r_0, among other factors. In the same way,

$$\rho(x)\, p_{n-m+2}^{(m-2)}(x)\, \rho(x)^{m-1} q_{m-1}\left(\frac{y}{\rho(x)}\right) = \sum_{i=0}^{4} \widehat{\theta}_{n-i}^{(m)}\, P_{n+2-i,m-1}(x,y).$$

Next, for $m \ge 2$, we compute the third term of the sum in (5.8). Observe that, in both Cases I and II, $\rho(x)^2$ is a polynomial of degree less than or equal to 2, and we can denote $\rho(x)^2 = s_2 x^2 + s_1 x + s_0$ its explicit expression, with $s_2, s_1, s_0 \in \mathbb{R}$, and $|s_2| + |s_1| + |s_0| > 0$ (in Case I we have $s_2 = r_1^2$, $s_1 = 2 r_1 r_0$, and $s_0 = r_0^2$). Then, applying twice the three term relation (4.1) for polynomials $\{p_n^{(m-2)}(x)\}_{n\ge0}$, we deduce

$$\begin{aligned}
&\rho(x)^2\, p_{n-m+2}^{(m-2)}(x)\, \rho(x)^{m-2}\, s_{m-2}\left(\frac{y}{\rho(x)}\right)\\
&\quad = \left[\sum_{i=0}^{4} \widehat{\vartheta}_{n-i}^{(m)}\, p_{n-m+4-i}^{(m-2)}(x)\right] \rho(x)^{m-2}\, s_{m-2}\left(\frac{y}{\rho(x)}\right)\\
&\quad = \sum_{i=0}^{4} \widehat{\vartheta}_{n-i}^{(m)} \widehat{S}_{n+2-i,m-2}(x,y),
\end{aligned}$$

as well as

$$\rho(x)^2\, p_{n-m+2}^{(m-2)}(x)\, \rho(x)^{m-2}\, q_{m-2}\left(\frac{y}{\rho(x)}\right) = \sum_{i=0}^{4} \widehat{\vartheta}_{n-i}^{(m)} P_{n+2-i,m-2}(x,y).$$

Finally, from the above expressions we get the desired result. □

6. Examples

Here we present examples of bivariate Sobolev orthogonal polynomials and study the involved univariate Sobolev orthogonal polynomials.

The first two examples deal with two families of Sobolev orthogonal polynomials on the unit disk obtained by using our construction and Gegenbauer polynomials. Next two examples are devoted to construct Sobolev orthogonal polynomials on the biangle and the simplex, respectively. Then, we present an example defined on an unbounded domain which is based on Laguerre and Hermite classical orthogonal polynomials. In the last two examples we analyse quasi-definite families of first and second-type Sobolev orthogonal polynomials constructed with Bessel and Gegenbauer polynomials.

We will use the standard representation and properties for classical Jacobi and Gegenbauer polynomials considered in the literature (see for instance [1,3,16]).

6.1. First-Type Sobolev Orthogonal Polynomials on the Unit Disk

For $\mu > -1/2$, orthogonal polynomials on the unit disk can be defined as ([4, p. 31])

$$P_{n,m}^{(\mu)}(x,y) = C_{n-m}^{(\mu_m)}(x)\,(1-x^2)^{m/2}\,C_m^{(\mu)}\left(\frac{y}{\sqrt{1-x^2}}\right), \quad 0 \le m \le n, \quad n \ge 0, \tag{6.1}$$

where $\{C_n^{(\mu)}\}_{n\ge 0}$ denotes the sequence of classical Gegenbauer polynomials, and $\mu_m = \mu + m + 1/2$. These polynomials constitute a mutually orthogonal sequence with respect to the moment functional $\mathbf{w}_\mu$ defined by

$$\langle \mathbf{w}_\mu, P\rangle = \iint_{\mathbf{B}^2} P(x,y)\,(1-x^2-y^2)^{\mu-1/2}\,\mathrm{d}x\mathrm{d}y, \quad \forall P \in \Pi^2,$$

where $\mathbf{B}^2 = \{(x,y) \in \mathbb{R}^2 :\ x^2+y^2 \le 1\}$ (see [11,13] as well as [4,6]). This moment functional is constructed using the method described in Sect. 2.3 by taking $\rho(x) = \sqrt{1-x^2}$, and $\mathbf{u} = \mathbf{v} = \mathbf{u}_\mu$ the univariate Gegenbauer moment functional

$$\langle \mathbf{u}_\mu, p\rangle = \int_{-1}^{1} p(t)(1-t^2)^{\mu-1/2}dt, \quad \mu > -1/2. \tag{6.2}$$

We are in Case II with $\ell_2 = -1$, $\ell_1 = 0$, $\ell_0 = 1$, and $\widehat{\rho}(x) = -x$.

Here, the bilinear form (3.6) reads

$$(P,Q)_\mu = \left\langle \mathbf{w}_\mu, P\,Q + \lambda\,(\nabla P)^t \begin{pmatrix} (1-x^2)^2 & -x\,y\,(1-x^2) \\ -x\,y\,(1-x^2) & x^2\,y^2 \end{pmatrix} \nabla Q \right\rangle. \tag{6.3}$$

From Theorem 3.1, we have that the bivariate polynomials

$$S_{n,m}^{(\mu)}(x,y) = s_{n-m}^{(m)}(x)\,(1-x^2)^{m/2}\,C_m^{(\mu)}\left(\frac{y}{\sqrt{1-x^2}}\right), \quad n \ge 0, \quad 0 \le m \le n,$$

are a mutually orthogonal basis with respect to (6.3). Here, $\{s_n^{(m)}(x)\}_{n\geq 0}$ are univariate Sobolev orthogonal polynomials associated with the bilinear form

$$(p,q)_m = \left\langle \mathbf{u}_m^{(x)}, p\,q + \lambda\,\left(p\;p'\right)\begin{pmatrix} m^2\,x^2 & -m\,x\,(1-x^2) \\ -m\,x\,(1-x^2) & (1-x^2)^2 \end{pmatrix}\begin{pmatrix} q \\ q' \end{pmatrix}\right\rangle,$$

where $\mathbf{u}_m^{(x)} = (1-x^2)^{m+1/2}\mathbf{u}^{(x)}$ is the moment functional associated with the Gegenbauer weight function $\rho(x)^{2m+1}\,w(x) = (1-x^2)^{m+1/2}\,w(x) = (1-x^2)^{\mu+m}$.

Since $\mathbf{u}^{(x)}$ satisfies the Pearson equation for the Gegenbauer moment functional

$$D\,(\phi(x)\,\mathbf{u}_\mu) = -(2\,\mu+1)\,x\,\mathbf{u}_\mu,$$

with $\phi(x) = \rho(x)^2 = 1-x^2$, we can use Proposition 4.10. To this end, we need the three-term recurrence relation for Gegenbauer polynomials (4.7.17) in [16, p. 81], and the structure relation (4.19) given in (4.7.27) in [16, p. 83]). Therefore, by Proposition 4.10 and the symmetry of Gegenbauer polynomials, we deduce a relation between classical Gegenbauer polynomials and univariate Sobolev orthogonal polynomials

$$\begin{aligned} &C_n^{(\mu_m)}(x) = s_n^{(m)}(x) + f_{n,2}^{(m)}\,s_{n-2}^{(m)}(x), \quad n\geq 1, \\ &s_{-1}^{(m)} = 0, \quad s_0^{(m)}(x) = 1, \end{aligned}$$

where

$$f_{n,2}^{(m)} = -\lambda\frac{(n-1)\,(n+m-2)\,(n+2\mu_m-1)\,(n-m+2\mu_m)}{4\,(n+\mu_m-2)\,(n+\mu_m)}\frac{h_{n-1}^{(\mu_m)}}{\widetilde{h}_{n-2}^{(m)}}, \quad n\geq 2,$$

and $h_{n-1}^{(\mu_m)}$ denotes the square of the norms of Gegenbauer polynomials given in [16, p. 81].

Furthermore, by Proposition 4.11, the norms $\widetilde{h}_n^{(m)} = (s_n^{(m)}, s_n^{(m)})_m$ satisfy the recurrence relation (4.23) with

$$\begin{aligned} \widehat{\mathcal{C}}_n^{(II)} &= \frac{\lambda}{4}\,\frac{(n+1)^2\,(n+m)^2}{(n+\mu_m)^2}, \\ \widehat{\mathcal{D}}_n^{(II)} &= 1, \\ \widehat{\mathcal{E}}_n^{(II)} &= \frac{\lambda}{4}\,\frac{(n+2\mu_m-1)^2\,(n-m+2\mu_m)^2}{(n+\mu_m)^2}. \end{aligned}$$

and

$$\widetilde{h}_0^{(m)} = \left(1+\frac{\lambda\,m^2}{2\mu_m}\right)h_0^{(\mu_m)}.$$

6.2. Second-Type Sobolev Orthogonal Polynomials on the Unit Disk

Again, we consider the bivariate orthogonal polynomials on the disk $\{P_{n,m}^{(\mu)}(x,y) : 0\leq m\leq n,\ n\geq 0\}$, defined in (6.1), and the bivariate functional $\mathbf{w}_\mu$

defined in the previous example. Let $\{s_m(y)\}_{m\geq 0}$ be a family of univariate orthogonal polynomials with respect to the Sobolev bilinear form

$$(p,q)_{\mathbf{u}_\mu} = \langle \mathbf{u}_\mu, p\,q + \lambda\, p'\, q'\rangle\,, \quad \lambda\in\mathbb{R},$$

where $\mathbf{u}_\mu$ is the Gegenbauer moment functional (6.2). Then, by Theorem 5.1, the bivariate polynomials defined by

$$\widehat{S}_{n,m}(x,y) = C_{n-m}^{(\mu_m)}(x)\,(1-x^2)^{m/2}\, s_m\left(\frac{y}{\sqrt{1-x^2}}\right),$$

for $0\leq m\leq n$, where $\mu_m = \mu+m+1/2$, are mutually orthogonal with respect to the bivariate bilinear form (5.4)

$$(P,Q)_S = \left\langle \mathbf{w}_\mu, P\,Q + \lambda\,(1-x^2)\,\partial_y P\,\partial_y Q\right\rangle.$$

Since $\mathbf{u}_\mu$ is classical, the sequence of univariate Sobolev polynomials $\{s_m(y)\}_{m\geq 0}$ and the Gegenbauer polynomials satisfy relation (5.8). In order to find the coefficients, we need the second structure relation (5.5) for Gegenbauer polynomials that can be found in (4.7.29) in [16, p. 83], and, by the symmetry of the Gegenbauer and the Sobolev polynomials, relation (5.8) reads

$$\begin{aligned} &s_m(y) + \widehat{d}_{m,2}\, s_{m-2}(y) = C_m^{(\mu)}(y) - C_{m-2}^{(\mu)}(y), \quad m\geq 1,\\ &s_{-1}(y)=0, \quad s_0(y)=1, \end{aligned}$$

with $h_m^{(\mu)}$ given in [16, p. 81], and

$$\widehat{d}_{m,2} = -\frac{h_{m-2}^{(\mu)}}{\widehat{h}_{m-2}^{(s)}}, \quad m\geq 2.$$

Furthermore, by Proposition 5.4, the norms $\widehat{h}_m^{(s)} = (s_m, s_m)_{\mathbf{u}_\mu}$ satisfy the recurrence relation

$$\begin{aligned} &\widehat{h}_m^{(s)} = h_m^{(\mu)} + 4\,\lambda\,(m+\mu-1)^2\, h_{m-1}^{(\mu)} + h_{m-2}^{(\mu)} - \widehat{d}_{m,2}^{\,2}\,\widehat{h}_{m-2}^{(s)},\\ &\widehat{h}_{-1}^{(s)} = 0, \quad \widehat{h}_0^{(s)} = h_0^{(\mu)}. \end{aligned}$$

6.3. Sobolev Orthogonal Polynomials on the Biangle

In this case we consider the Jacobi and Gegenbauer univariate moment functionals

$$\mathbf{u}^{(x)} = \overline{\mathbf{u}}_{\alpha,\beta}, \quad \mathbf{v}^{(y)} = \mathbf{u}_{\beta+1/2}, \quad \alpha,\beta>-1,$$

where

$$\langle \overline{\mathbf{u}}_{\alpha,\beta}, p\rangle = \int_0^1 p(t)\,(1-t)^\alpha t^\beta dt,$$

and $\mathbf{u}_{\beta+1/2}$ is the Gegenbauer moment functional (6.2). If we take the function $\rho(x)=\sqrt{x}$ ($\ell_2=\ell_0=0$, $\ell_1=1/2$), we obtain the bivariate moment functional $\mathbf{w}_{\alpha,\beta}$ defined as

$$\langle \mathbf{w}_{\alpha,\beta}, P\rangle = \iint_\Omega P(x,y)\,(1-x)^\alpha\,(x-y^2)^\beta\,\mathrm{d}x\,\mathrm{d}y, \quad \forall P\in\Pi^2,$$

where $\Omega = \{(x,y) \in \mathbb{R}^2 \,:\, y^2 \le x \le 1\}$ is called the *biangle* on $\mathbb{R}^2$. Define the polynomials $\{P_{n,m}^{(\alpha,\beta)}(x,y) : 0 \le m \le n,\, n \ge 0\}$ by

$$P_{n,m}^{(\alpha,\beta)}(x,y) = \overline{P}_{n-m}^{(\alpha,\beta_m)}(x)\, \left(\sqrt{x}\right)^m\, C_m^{(\beta+1/2)}\left(\frac{y}{\sqrt{x}}\right),$$

where $\beta_m = \beta + m + 1/2$, and $\{\overline{P}_n^{(\alpha,\beta_m)}\}_{n\ge 0}$ are the Jacobi univariate orthogonal polynomials associated with $\overline{\mathbf{u}}_{\alpha,\beta_m}$. Then, they are mutually orthogonal with respect to $\mathbf{w}_{\alpha,\beta}$ (see [11,13] as well as [4,6]).

For $m \ge 0$, let $\{s_n^{(m)}(x)\}_{n\ge 0}$ be a sequence of univariate orthogonal polynomials with respect to the bilinear form

$$(p,q)_m = \left\langle \overline{\mathbf{u}}_{\alpha,\beta_m}, p\,q + \lambda\left(p\ p'\right)\begin{pmatrix} m^2 & \frac{1}{2}m\,x \\ \frac{1}{2}m\,x & x^2 \end{pmatrix}\begin{pmatrix} q \\ q' \end{pmatrix}\right\rangle, \quad \lambda > 0,$$

where $\overline{\mathbf{u}}_{\alpha,\beta_m} = x^{m+1/2}\overline{\mathbf{u}}_{\alpha,\beta}$. Then, by Theorem 3.1, the polynomials defined by

$$S_{n,m}^{(\alpha,\beta)}(x,y) = s_{n-m}^{(m)}(x)\, \left(\sqrt{x}\right)^m\, C_m^{(\beta+1/2)}\left(\frac{y}{\sqrt{x}}\right), \quad n \ge 0, \quad 0 \le m \le n,$$

are mutually orthogonal with respect to the bivariate bilinear form

$$(P,Q) = \left\langle \mathbf{w}_{\alpha,\beta}, P\,Q + \lambda(\nabla P)^t \begin{pmatrix} x^2 & \frac{1}{2}x\,y \\ \frac{1}{2}x\,y & \frac{1}{4}y^2 \end{pmatrix} \nabla Q \right\rangle, \quad \forall p,q \in \Pi^2.$$

Observe that $\overline{\mathbf{u}}_{\alpha,\beta}$ satisfies the Pearson equation

$$D(\phi(x)\,\overline{\mathbf{u}}_{\alpha,\beta}) = \psi(x)\,\overline{\mathbf{u}}_{\alpha,\beta}, \tag{6.4}$$

with $\phi(x) = (1-x)\,x$, $\psi(x) = \beta + 1 - (\alpha+\beta+2)\,x$, and $\rho(x)^2 = x$ divides the polynomial $\phi(x)$. Using the three-term recurrence relation as well as the second structure relation for Jacobi polynomials on $[0,1]$, relation (4.17) allows to connect classical Jacobi polynomials to Sobolev polynomials $\{s_n^{(m)}(x)\}_{n\ge 0}$ with $e_{n,4}^{(m)} = 0$, $n \ge 4$, $e_{n,3}^{(m)} = 0$, $n \ge 3$, and

$$e_{n,2}^{(m)} = \frac{1}{\widetilde{h}_{n-2}^{(m)}}\left[\mathcal{C}_{n,1}^{(II)}\,\overline{h}_{n-2}^{(\alpha,\beta_m)} + \mathcal{C}_{n,2}^{(II)}\,\frac{\overline{h}_{n-3}^{(\alpha,\beta_m)}}{\overline{\xi}_{n-3}^{(\alpha,\beta_m)}}\right],$$

$$e_{n,1}^{(m)} = \frac{1}{\widetilde{h}_{n-1}^{(m)}}\left[\mathcal{D}_{n,1}^{(II)}\,\overline{h}_{n-1}^{(\alpha,\beta_m)} + \mathcal{D}_{n,2}^{(II)}\,\frac{\overline{h}_{n-2}^{(\alpha,\beta_m)}}{\overline{\xi}_{n-2}^{(\alpha,\beta_m)}} - \frac{[e_{n,2}^{(m)}\widetilde{h}_{n-2}^{(m)}]}{\xi_{n-2}^{(\alpha,\beta_m)}}e_{n-1,1}^{(m)}\right],$$

for $n \geq 1$, and $e_{n,i} = 0$, for $n < i$, where

$$
\begin{aligned}
\mathcal{C}_{n,1}^{(II)} &= \overline{\tau}_{n-1}^{(\alpha,\beta_m)} + \frac{\lambda}{4}(m+n-2)\left(m\,\overline{\tau}_{n-1}^{(\alpha,\beta_m)} + 2\,\overline{c}_{n-1}^{(\alpha,\beta_m)}\right),\\
\mathcal{C}_{n,2}^{(II)} &= \frac{\lambda}{4}\,\overline{c}_{n-2}^{(\alpha,\beta_m)}\left(m\,\overline{\tau}_{n-1}^{(\alpha,\beta_m)} + 2\,\overline{c}_{n-1}^{(\alpha,\beta_m)}\right),\\
\mathcal{D}_{n,1}^{(II)} &= \overline{\sigma}_{n-1}^{(\alpha,\beta_m)} + \frac{\lambda}{4}(m+n-1)\left(m\,\overline{\sigma}_{n-1}^{(\alpha,\beta_m)} + 2\,\overline{b}_{n-1}^{(\alpha,\beta_m)}\right),\\
\mathcal{D}_{n,2}^{(II)} &= \overline{\sigma}_{n-2}^{(\alpha,\beta_m)}\,\overline{\tau}_{n-1}^{(\alpha,\beta_m)} + \frac{\lambda}{2}\left(m\,\overline{\sigma}_{n-2}^{(\alpha,\beta_m)} + \overline{b}_{n-2}^{(\alpha,\beta_m)}\right)\left(m\,\overline{\tau}_{n-1}^{(\alpha,\beta_m)} + \overline{c}_{n-1}^{(\alpha,\beta_m)}\right)\\
&\quad + \frac{\lambda}{4}\,\overline{b}_{n-2}^{(\alpha,\beta_m)}\left(m\,\overline{\tau}_{n-1}^{(\alpha,\beta_m)} + 2\,\overline{c}_{n-1}^{(\alpha,\beta_m)}\right)\\
&\quad + \frac{\lambda}{4}\left(m\,\overline{\sigma}_{n-1}^{(\alpha,\beta_m)} + \overline{b}_{n-1}^{(\alpha,\beta_m)}\right)\overline{c}_{n-1}^{(\alpha,\beta_m)}.
\end{aligned}
$$

Furthermore, by Proposition 4.8, the norms $\widetilde{h}_n^{(m)} = (s_n^{(m)}, s_n^{(m)})_m$ satisfy the recurrence relation (4.18) with $\widetilde{h}_0^{(m)} = (1 + \frac{\lambda}{4}\,m^2)h_0^{(m)}$ and

$$
\begin{aligned}
\mathcal{E}_n^{(II)} &= 1 + \frac{\lambda}{2}\,(m+n)\,(m+2\,n),\\
\mathcal{F}_n^{(II)} &= \frac{\left[\overline{\sigma}_{n-1}^{(\alpha,\beta_m)}\right]^2 + \frac{\lambda}{4}\left[m\,\overline{\sigma}_{n-1}^{(\alpha,\beta_m)} + 2\,\overline{b}_{n-1}^{(\alpha,\beta_m)}\right]^2}{\left[\overline{\xi}_{n-1}^{(\alpha,\beta_m)}\right]^2},\\
\mathcal{G}_n^{(II)} &= \frac{\left[\overline{\tau}_{n-1}^{(\alpha,\beta_m)}\right]^2 + \frac{\lambda}{2}\left(m\,\overline{\tau}_{n-1}^{(\alpha,\beta_m)} + \overline{c}_{n-1}^{(\alpha,\beta_m)}\right)\left(m\,\overline{\tau}_{n-1}^{(\alpha,\beta_m)} + 2\,\overline{c}_{n-1}^{(\alpha,\beta_m)}\right)}{\left[\overline{\xi}_{n-1}^{(\alpha,\beta_m)}\right]^2}.
\end{aligned}
$$

6.4. Orthogonal Polynomials on the Simplex

For $\alpha, \beta, \gamma > -1$, the polynomials defined as

$$
P_{n,m}^{(\alpha,\beta,\gamma)}(x,y) = \overline{P}_{n-m}^{(\alpha,\beta_m)}(x)\,(1-x)^m\,\overline{P}_m^{(\beta,\gamma)}\left(\frac{y}{1-x}\right), \quad 0 \leq m \leq n, \quad n \geq 0,
$$

where $\beta_m = \beta + \gamma + 2m + 1$, and $\overline{P}_n^{(a,b)}$ denotes the n-th Jacobi polynomial orthogonal on the interval $[0,1]$, are mutually orthogonal with respect to the moment functional $\mathbf{w}$ defined as

$$
\langle \mathbf{w}, P \rangle = \int\int_{\mathbf{T}} P(x,y)\,x^\alpha y^\beta (1-x-y)^\gamma\,\mathrm{d}y\mathrm{d}x,
$$

where $\mathbf{T} = \{(x,y) \in \mathbb{R}^2 :\ x \geq 0,\ y \geq 0,\ 1-x-y \geq 0\}$. This functional is constructed by taking the univariate Jacobi functionals $\mathbf{u}^{(x)} = \overline{\mathbf{u}}_{\alpha,\beta+\gamma}$ and $\mathbf{v}^{(y)} = \overline{\mathbf{u}}_{\beta,\gamma}$, and the function $\rho(x) = 1 - x$.

For $\lambda > 0$, consider the bivariate Sobolev inner product

$$
(P,Q) = \left\langle \mathbf{w}, P\,Q + \lambda(\nabla P)^t \begin{pmatrix} (1-x)^2 & -y(1-x) \\ -y(1-x) & y^2 \end{pmatrix} \nabla Q \right\rangle, \qquad (6.5)
$$

and, for $m \geq 0$, let $\{\overline{s}_n^{(m)}(x)\}_{n\geq 0}$ be the sequence of orthogonal polynomials associated with the univariate Sobolev inner product

$$(p,q)_m = \left\langle \mathbf{u}_m^{(x)}, p\,q + \lambda\,(p\ p') \begin{pmatrix} m^2 & -m(1-x) \\ -m(1-x) & (1-x)^2 \end{pmatrix} \begin{pmatrix} q \\ q' \end{pmatrix} \right\rangle,$$

where $\mathbf{u}_m^{(x)} = \overline{\mathbf{u}}_{\alpha,\beta_m} = (1-x)^{2m+1}\overline{\mathbf{u}}_{\alpha,\beta+\gamma}$. By Theorem 3.1, the polynomials defined as

$$\overline{P}_{n,m}(x,y) = \overline{s}_{n-m}^{(m)}(x)\,(1-x)^m\,\overline{P}_m^{(\beta,\gamma)}\left(\frac{y}{1-x}\right), \qquad 0 \leq m \leq n, \qquad n \geq 0,$$

are mutually orthogonal with respect to (6.5).

Observe that $\mathbf{u}^{(x)}$ satisfies (6.4) and that $\rho(x) = 1-x$ divides $\phi(x) = (1-x)\,x$. Using the three-term relation and the structure relation for Jacobi polynomials on $[0,1]$, from Proposition 4.2, we have that the polynomials $\{\overline{s}_n^{(m)}(x)\}_{n\geq 0}$ satisfy (4.6) with

$$\mathcal{A}_{n,2}^{(I)} = \overline{\tau}_{n-1}^{(\alpha,\beta_m)} - \lambda\,(m+n-2)\left(m\,\overline{\tau}_{n-1}^{(\alpha,\beta_m)} + \overline{c}_{n-1}^{(\alpha,\beta_m)}\right),$$
$$\mathcal{B}_{n,1}^{(I)} = \overline{\sigma}_{n-1}^{(\alpha,\beta_m)} - \lambda\,(m+n-1)\left(1 - m\,\overline{\tau}_{n-1}^{(\alpha,\beta_m)} - \overline{c}_{n-1}^{(\alpha,\beta_m)}\right),$$
$$\mathcal{B}_{n,2}^{(I)} = \overline{\tau}_{n-1}^{(\alpha,\beta_m)}\,\sigma_{n-2}^{(\alpha,\beta_m)} - \lambda\left(m\,\overline{\tau}_{n-1}^{(\alpha,\beta_m)} + \overline{c}_{n-1}^{(\alpha,\beta_m)}\right)\left(1 - m\,\overline{\sigma}_{n-2}^{(\alpha,\beta_m)} - \overline{b}_{n-2}^{(\alpha,\beta_m)}\right).$$

Furthermore, by Proposition 4.4, the norms $\widetilde{h}_n^{(m)} = (s_n^{(m)}, s_n^{(m)})_m$ satisfy the recurrence relation (4.11) with $\widetilde{h}_0^{(m)} = (1+\lambda\,m^2)h_0^{(m)}$ and

$$\mathcal{C}_n^{(I)} = 1 + \lambda\,(m+n)^2,$$
$$\mathcal{D}_n^{(I)} = \frac{\left(\overline{\sigma}_{n-1}^{(\alpha,\beta_m)}\right)^2 + \lambda\left(1 - m\,\overline{\sigma}_{n-1}^{(\alpha,\beta_m)} - \overline{b}_{n-1}^{(\alpha,\beta_m)}\right)^2}{\left(\overline{\xi}_{n-1}^{(\alpha,\beta_m)}\right)^2},$$
$$\mathcal{E}_n^{(I)} = \frac{\left(\overline{\tau}_{n-1}^{(\alpha,\beta_m)}\right)^2 + \lambda\left(m\,\overline{\tau}_{n-1}^{(\alpha,\beta_m)} + \overline{c}_{n-1}^{(\alpha,\beta_m)}\right)^2}{\left(\overline{\xi}_{n-1}^{(\alpha,\beta_m)}\right)^2}.$$

6.5. Sobolev Orthogonal Polynomials on an Unbounded Domain

Let $\alpha > -1$. We define the bivariate Laguerre–Hermite Sobolev polynomials as

$$P_{n,m}^{(\alpha)}(x,y) \;=\; L_{n-m}^{(\alpha+2\,m+1)}(x)\,x^m\,H_m\left(\frac{y}{x}\right), \quad n \geq 0, \quad 0 \leq m \leq n, \quad (6.6)$$

where $\{L_n^{(\alpha)}(x)\}_{n\geq 0}$ and $\{H_n(x)\}_{n\geq 0}$ denote the sequence of univariate Laguerre and Hermite classical polynomials, respectively. Here, $\rho(x) = x$ and, therefore, we are in Case I of the Koornwinder construction.

The Laguerre–Hermite polynomials (6.6) are mutually orthogonal with respect to the linear functional

$$\langle \mathbf{w}, P\rangle \;=\; \int_0^{+\infty}\int_{-\infty}^{+\infty} P(x,y)\,x^\alpha\,e^{-x}\,e^{-y^2/x^2}\,\mathrm{d}y\,\mathrm{d}x, \quad P \in \Pi^2.$$

For $\lambda > 0$, consider the bivariate Sobolev inner product defined as

$$(P, Q) \;=\; \left\langle \mathbf{w}, P\,Q + \lambda\,(\nabla P)^t \begin{pmatrix} x^2 & x\,y \\ x\,y & y^2 \end{pmatrix} \nabla Q \right\rangle, \quad P, Q \in \Pi^2, \qquad (6.7)$$

and, for $m \geq 0$, let $\{s_n^{(m)}(x)\}_{n\geq 0}$ denote the sequence of univariate orthogonal polynomials associated with the univariate Sobolev inner product

$$(p, q)_m \;=\; \int_0^{+\infty} \left(p\,q + \lambda\,(p\;p') \begin{pmatrix} m^2 & m\,x \\ m\,x & x^2 \end{pmatrix} \begin{pmatrix} q \\ q' \end{pmatrix} \right) x^{\alpha+2\,m+1} e^{-x}\,\mathrm{d}x.$$

By Theorem 3.1, the bivariate polynomials defined as

$$S_{n,m}^{(\alpha)}(x, y) \;=\; s_{n-m}^{(m)}(x)\,x^m\,H_m\left(\frac{y}{x}\right), \quad n \geq 0, \quad 0 \leq m \leq n,$$

are mutually orthogonal with respect to (6.7).

The classical Laguerre polynomials are orthogonal with respect to the moment functional

$$\langle \mathbf{u}_\alpha, p\rangle \;=\; \int_0^{+\infty} p(x)\,x^\alpha\,e^{-x}\,\mathrm{d}x,$$

which satisfies the Pearson equation

$$D(\phi(x)\,\mathbf{u}_\alpha) \;=\; \psi(x)\,\mathbf{u}_\alpha,$$

with $\phi(x) = x$ and $\psi(x) = \alpha + 1 - x$. Since $\rho(x) = x$ divides the coefficient $\phi(x)$ in the Pearson equation, we can use Proposition 4.2 to deduce the relation between the Laguerre polynomials and the univariate Sobolev orthogonal polynomials. To this end, using the explicit expression of the Laguerre polynomials ([16])

$$L_n^{(\alpha)}(x) \;=\; \sum_{k=0}^{n} (-1)^k \binom{n+\alpha}{n-k} \frac{x^k}{k!},$$

we get the three-term recurrence relation

$$x\,L_n^{(\alpha)}(x) \;=\; a_n^{(\alpha)}\,L_{n+1}^{(\alpha)}(x) + b_n^{(\alpha)}\,L_n^{(\alpha)}(x) + c_n^{(\alpha)}\,L_{n-1}^{(\alpha)}(x),$$

where

$$a_n^{(\alpha)} = -(n+1), \quad b_n^{(\alpha)} = 2\,n + \alpha + 1, \quad c_n^{(\alpha)} = -(n+\alpha),$$

and the structure relation

$$L_n^{(\alpha)}(x) \;=\; \xi_n^{(\alpha)}\,(L_{n+1}^{(\alpha)}(x))' + \sigma_n^{(\alpha)}\,(L_n^{(\alpha)}(x))' + \tau_n^{(\alpha)}\,(L_{n-1}^{(\alpha)}(x))',$$

where

$$\xi_n^{(\alpha)} = -1, \quad \sigma_n^{(\alpha)} = 1, \quad \tau_n^{(\alpha)} = 0.$$

From Proposition 4.2, we have that the Laguerre polynomials and the univariate Sobolev polynomials satisfy (4.6) with

$$\begin{aligned} \mathcal{A}_{n,2}^{(I)} &= -\lambda\,(m+n-2)\,(n+\alpha+2\,m), \\ \mathcal{B}_{n,1}^{(I)} &= 1 + \lambda\,(m+n-1)\,(2\,n+\alpha+3\,m)\,, \\ \mathcal{B}_{n,2}^{(I)} &= -\lambda\,(n+\alpha+2\,m)\,(2\,n+\alpha+3\,m-2)\,, \end{aligned}$$

and, from Proposition 4.4, we have that the norms $\widetilde{h}_n^{(m)} = (s_n^{(m)}, s_n^{(m)})_m$ satisfy (4.11) with

$$\mathcal{C}_n^{(I)} = 1 + \lambda\,(m+n)^2,$$
$$\mathcal{D}_n^{(I)} = 1 + \lambda\,(2\,n + \alpha + 3\,m)^2,$$
$$\mathcal{E}_n^{(I)} = \lambda\,(n + \alpha + 2\,m)^2,$$

and $\widetilde{h}_0^{(m)} = (1+\lambda\,m^2)\,h_0^{(\alpha+2\,m+1)}$, where ([16])

$$h_n^{(\alpha)} \;=\; \int_0^{+\infty} (L_n^{(\alpha)}(x))^2\,x^\alpha\,e^{-x}\,\mathrm{d}x = \Gamma(\alpha+1)\binom{n+\alpha}{n}.$$

Notice that the second type Sobolev orthogonal polynomials as discussed in Sect. 5 based on the Laguerre–Hermite polynomials (6.6) is a straightforward situation taking into account that the Hermite polynomials are orthogonal with respect to both the moment functional

$$\langle \mathbf{v}, p\rangle = \int_{-\infty}^{+\infty} p(t)\,e^{-t^2}\,dt,$$

and the Sobolev bilinear form in (5.1) associated with $\mathbf{v}$. That is, for $\lambda > 0$, the Laguerre–Hermite polynomials are bivariate Sobolev polynomials orthogonal with respect to the Sobolev inner product

$$(P,Q) \;=\; \int_0^{+\infty}\int_{-\infty}^{+\infty} (P\,Q + \lambda\,x^2\,\partial_y P\,\partial_y Q)\,x^\alpha\,e^{-x}\,e^{-y^2/x^2}\,\mathrm{d}y\,\mathrm{d}x, \quad P,Q\in\Pi^2.$$

6.6. A Quasi-Definite Family of First-Type Sobolev Orthogonal Polynomials

Let $\mu > -1/2$, $a \in \mathbb{R}$, such that $a \neq 0, -1, -2, \ldots$, and $b \neq 0$. We define the bivariate Bessel–Gegenbauer polynomials as

$$P_{n,m}^{(\mu,a,b)}(x,y) = B_{n-m}^{(a+2m+1,b)}(x)\,x^m\,C_m^{(\mu)}\left(\frac{y}{x}\right), \quad 0\le m\le n, \quad n\ge 0, \tag{6.8}$$

considering $\rho(x) = x$, and, therefore, we are in Case I of the Koornwinder construction.

Here $\{C_m^{(\mu)}\}_{m\ge 0}$ denotes the univariate classical Gegenbauer polynomials orthogonal with respect to the positive definite Gegenbauer moment functional defined in (6.2), and $\{B_n^{(a,b)}\}_{n\ge0}$ denote the univariate classical Bessel polynomials orthogonal with respect to the bilinear form

$$\langle f, g\rangle_{a,b} = \int_T f(z)\,g(z)\,w^{(a,b)}(z)\,\mathrm{d}z$$

with $w^{(a,b)}(z) = (2\,\pi\,i)^{-1} z^{a-2}\,e^{-b/z}$, $a \neq 0, -1, -2, \ldots$, $b \neq 0$, and T is the unit circle oriented in the counter-clockwise direction, standardized by the condition $B_n^{(a,b)}(0) = 1$ (see [7]). Note that the Bessel polynomials are associated with the quasi-definite linear functional defined as $\langle \mathbf{u}_{a,b}, p\rangle = \langle 1, p\rangle_{a,b}$, which satisfies the Pearson equation

$$D\left(x^2\,\mathbf{u}_{a,b}\right) \;=\; (a\,x+b)\,\mathbf{u}_{a,b}. \tag{6.9}$$

Bivariate Bessel–Gegenbauer polynomials (6.8) are orthogonal with respect to the quasi-definite bilinear form

$$(P,Q) = \int_T \int_{-1}^{1} P(z,y)\, Q(z,y)\, W_{\mu,a,b}(z,y)\, \mathrm{d}y\, \mathrm{d}z,$$

where $W_{\mu,a,b}(x,y) = (2\,\pi\, i)^{-1}\, x^{a-2\mu-1}\, e^{-b/x}\, (x^2-y^2)^{\mu-1/2}$.

For $m \geq 0$, let $\{s_n^{(m)}(x)\}_{n\geq 0}$ denote the sequence of univariate Bessel–Sobolev orthogonal polynomials with respect to the quasi-definite bilinear form

$$(p,q)_m \;=\; \int_T \left(p\, q + \lambda\, \begin{pmatrix} p & p' \end{pmatrix} \begin{pmatrix} m^2 & m\, z \\ m\, z & z^2 \end{pmatrix} \begin{pmatrix} q \\ q' \end{pmatrix} \right) w^{(a_m,b)}(z)\, \mathrm{d}z, \quad \lambda > 0,$$

where $a_m = a + 2m + 1$. By Theorem 3.1, the bivariate polynomials defined as

$$S_{n,m}^{(\mu,a,b)}(x,y) \;=\; s_{n-m}^{(m)}(x)\, x^m\, C_m^{(\mu)}\left(\frac{y}{x}\right), \quad 0 \leq m \leq n, \quad n \geq 0,$$

are orthogonal with respect to the quasi-definite Sobolev bilinear form

$$(P,Q) \;=\; \int_T \int_{-1}^{1} \left(P\, Q + \lambda\, (\nabla P)^t \begin{pmatrix} x^2 & x\, y \\ x\, y & y^2 \end{pmatrix} \nabla Q \right) W_{\mu,a,b}(z,y)\, \mathrm{d}y\, \mathrm{d}z.$$

The explicit expression for the Bessel polynomials ([7, (34), p. 108]) is

$$B_n^{(a,b)}(x) \;=\; \sum_{k=0}^{n} \binom{n}{k} (n+a-1)_k \left(\frac{x}{b}\right)^k. \tag{6.10}$$

Moreover, we have ([7, (58), p. 113])

$$h_n^{(a,b)} \;=\; \int_T \left(B_n^{(a,b)}(z) \right)^2 w^{(a,b)}(z)\, \mathrm{d}z \;=\; \frac{(-1)^{n+1}\, n!\, b}{(2\,n+a-1)\,(a)_{n-1}}. \tag{6.11}$$

The classical Bessel polynomials satisfy the three-term recurrence relation ([7, (51), p. 111])

$$x\, B_n^{(a,b)}(x) = a_n^{(a,b)}\, B_{n+1}^{(a,b)}(x) + b_n^{(a,b)}\, B_n^{(a,b)}(x) + c_n^{(a,b)}\, B_{n-1}^{(a,b)}(x), \tag{6.12}$$

where

$$a_n^{(a,b)} = \frac{(n+a-1)\, b}{(2n+a-1)\,(2n+a)},$$

$$b_n^{(a,b)} = -\frac{(a-2)\, b}{(2n+a-2)\,(2n+a)},$$

$$c_n^{(a,b)} = -\frac{n\, b}{(2n+a-2)\,(2n+a-1)}.$$

Using (6.10) and (6.11), we can deduce the structure relation

$$B_n^{(a,b)}(x) = \xi_n^{(a,b)} \frac{\mathrm{d}}{\mathrm{d}x} B_{n+1}^{(a,b)}(x) + \sigma_n^{(a,b)} \frac{\mathrm{d}}{\mathrm{d}x} B_n^{(a,b)}(x) + \tau_n^{(a,b)} \frac{\mathrm{d}}{\mathrm{d}x} B_{n-1}^{(a,b)}(x), \quad n \geq 0,$$

where

$$\xi_n^{(a,b)} = \frac{b\,(n+a-a)}{(n+1)\,(2n+a-1)_2},$$

$$\sigma_n^{(a,b)} = \frac{2\,b}{(2\,n+a-2)\,(2\,n+a)},$$

$$\tau_n^{(a,b)} = \frac{n\,b^3}{(a)_2\,(n+a-2)\,(2\,n+a-2)_2}.$$

Since $\mathbf{u}_{a,b}$ satisfies the Pearson equation (6.9) with $\phi(x) = x^2$ and $\rho(x) = x$ divides $\phi(x)$, we can use Proposition 4.2 to deduce the relation between the classical Bessel polynomials and the univariate Sobolev orthogonal polynomials. Indeed, these two sequences of polynomials satisfy (4.6) with

$$\mathcal{A}_{n,2}^{(I)} = \tau_{n-1}^{(a_m,b)} + \lambda\,(m+n-2)\left(m\,\tau_{n-1}^{(a_m,b)} + c_{n-1}^{(a_m,b)}\right),$$

$$\mathcal{B}_{n,1}^{(I)} = \sigma_{n-1}^{(a_m,b)} + \lambda\,(m+n-1)\left(m\,\sigma_{n-1}^{(a_m,b)} + b_{n-1}^{(a_m,b)}\right),$$

$$\mathcal{B}_{n,2}^{(I)} = \tau_{n-1}^{(a_m,b)}\,\sigma_{n-2}^{(a_m,b)} + \lambda\left(m\,\tau_{n-1}^{(a_m,b)} + c_{n-1}^{(a_m,b)}\right)\left(m\,\sigma_{n-2}^{(a_m,b)} + b_{n-2}^{(a_m,b)}\right).$$

Furthermore, by Proposition 4.4, the numbers $\widetilde{h}_n^{(m)} = (s_n^{(m)}, s_n^{(m)})_m$ satisfy the recurrence relation (4.11) with

$$\mathcal{C}_n^{(I)} = 1 + \lambda\,(m+n)^2,$$

$$\mathcal{D}_n^{(I)} = \frac{(\sigma_{n-1}^{(a_m,b)})^2 + \lambda\,(m\,\sigma_{n-1}^{(a_m,b)} + b_{n-1}^{(a_m,b)})^2}{(\xi_{n-1}^{(a_m,b)})^2},$$

$$\mathcal{E}_n^{(I)} = \frac{(\tau_{n-1}^{(a_m,b)})^2 + \lambda\,(m\,\tau_{n-1}^{(a_m,b)} + c_{n-1}^{(a_m,b)})^2}{(\xi_{n-1}^{(a_m,b)})^2},$$

and $\widetilde{h}_0^{(m)} = (1+\lambda\,m^2)\,h_0^{(a_m,b)}$.

6.7. A Quasi-Definite Family of Second-Type Sobolev Orthogonal Polynomials

Consider again the bivariate Bessel–Gegenbauer polynomials $\{P_{n,m}^{(\mu,a,b)}(x,y) : n \geq 0,\ 0 \leq m \leq n\}$ defined in (6.8).

As in Example 6.2, let $\{s_m(y)\}_{m\geq 0}$ be the sequence of Gegenbauer–Sobolev orthogonal with respect to

$$(p,q)_{\mathbf{u}_\mu} = \langle \mathbf{u}_\mu, p\,q + \lambda\,p'\,q' \rangle, \quad \lambda \in \mathbb{R},$$

where $\mathbf{u}_\mu$ is the Gegenbauer moment functional (6.2). Using Theorem 5.1, the bivariate polynomials defined by

$$\widehat{S}_{n,m}(x,y) = B_{n-m}^{(a+2m+1,b)}(x)\,x^m\,s_m\left(\frac{y}{x}\right), \quad 0 \leq m \leq n, \tag{6.13}$$

are mutually orthogonal with respect to the bivariate bilinear form (5.4)

$$(P,Q)_S = \left\langle \mathbf{w}_\mu, P\,Q + \lambda\,x^2\,\partial_y P\,\partial_y Q \right\rangle.$$

Since $\mathbf{u}_\mu$ is a symmetric classical moment functional, we have that the univariate Sobolev polynomials $\{s_m(y)\}_{m\geq 0}$ and the Gegenbauer polynomials satisfy relation (5.8), that reads

$$\begin{aligned}
&s_m(y) + \widehat{d}_{m,2}\, s_{m-2}(y) = C_m^{(\mu)}(y) - C_{m-2}^{(\mu)}(y), \quad m \geq 1,\\
&s_{-1}(y) = 0, \quad s_0(y) = 1,
\end{aligned}$$

as in Example 6.2. Finally, Theorem 5.5 provides the relation between bivariate Sobolev orthogonal polynomials (6.13) and Bessel–Gegenbauer polynomials (6.8)

$$\begin{aligned}
&\sum_{i=0}^{4} \left[\widehat{\eta}_{n-i}^{(m)}\, \widehat{S}_{n+2-i,m}(x,y) + \widehat{d}_{m,2}\, \widehat{\vartheta}_{n-i}^{(m)}\, \widehat{S}_{n+2-i,m-2}(x,y)\right]\\
&\quad = \sum_{i=0}^{4} \left[\widehat{\eta}_{n-i}^{(m)}\, P_{n+2-i,m}^{(\mu,a,b)}(x,y) + \widehat{\tau}_{m-1}\, \widehat{\vartheta}_{n-i}^{(m)}\, P_{n+2-i,m-2}^{(\mu,a,b)}(x,y)\right].
\end{aligned}$$

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Appendix A

A.1. Proof of Proposition 4.2

Proof. By orthogonality of $\{s_n^{(m)}\}_{n\geq 0}$, we can express

$$\pi_n^{(m)}(x) = \sum_{j=0}^{n} d_{n,n-j}^{(m)} \, s_j^{(m)}(x),$$

where

$$d_{n,n-j}^{(m)} = \frac{(\pi_n^{(m)}, s_j^{(m)})_m}{\widetilde{h}_j^{(m)}}, \quad 0 \leq j \leq n.$$

Since $s_n^{(m)}(x)$ has the same coefficient as $p_n^{(m)}(x)$, we have that $d_{n,0}^{(m)} = \xi_{n-1}^{(m)}$.

Using (4.3) and (4.4) in (4.5), we obtain

$$\begin{aligned}(\pi_n^{(m)}, s_j^{(m)})_m = \; & (1+\lambda\, m^2\, r_1^2)\langle \mathbf{u}_m, (\xi_{n-1}^{(m)}\, p_n^{(m)} + \sigma_{n-1}^{(m)}\, p_{n-1}^{(m)} + \tau_{n-1}^{(m)}\, p_{n-2}^{(m)}) s_j^{(m)}\rangle \\ & +\lambda\, m\, r_1 \langle \mathbf{u}_m, (r_1 x + r_0)\, p_{n-1}^{(m)}\, s_j^{(m)}\rangle \\ & +\lambda\, m\, r_1 \langle \mathbf{u}_m, (r_1 x + r_0)(\xi_{n-1}^{(m)}\, p_n^{(m)} \\ & +\sigma_{n-1}^{(m)}\, p_{n-1}^{(m)} + \tau_{n-1}^{(m)}\, p_{n-2}^{(m)})(s_j^{(m)})'\rangle \\ & +\lambda \langle \mathbf{u}_m, (r_1\, x + r_0)^2\, p_{n-1}^{(m)}\, (s_j^{(m)})'\rangle. \end{aligned} \tag{A.1}$$

Therefore, $d_{n,n-j}^{(m)} = 0$, for $3 \leq j \leq n$, and (4.6) holds.

We compute $d_{n,2}^{(m)}$ by taking $j = n-2$ in (A.1), and we study term by term. For the first term, by using the orthogonality, immediately we obtain

$$T_{n-2}^1 = (1+\lambda\, m^2\, r_1^2)\tau_{n-1}^{(m)}\, h_{n-2}^{(m)}.$$

For the second term, from (4.1), we get

$$\begin{aligned} T_{n-2}^2 &= \lambda\, m\, r_1^2 \langle \mathbf{u}_m, (a_{n-1}^{(m)}\, p_n^{(m)} + b_{n-1}^{(m)} p_{n-1}^{(m)} + c_{n-1}^{(m)}\, p_{n-2}^{(m)}) s_{n-2}^{(m)}\rangle \\ &= \lambda\, m\, r_1^2\, c_{n-1}^{(m)}\, h_{n-2}^{(m)}. \end{aligned}$$

Third term is computed taking into account that $(s_{n-2}^{(m)})'$ is a polynomial of degree $n-3$, obtaining

$$T_{n-2}^3 = \lambda\, m\, r_1^2 \tau_{n-1}^{(m)} \langle \mathbf{u}_m, x\, p_{n-2}^{(m)} (s_{n-2}^{(m)})'\rangle = \lambda\, m\, r_1^2 \tau_{n-1}^{(m)} (n-2) h_{n-2}^{(m)},$$

and finally, the fourth term is computed applying twice the three term relation and the fact that $(s_{n-2}^{(m)})'$ is a polynomial of degree $n-3$

$$\begin{aligned} T_{n-2}^4 =& \lambda \left\langle \mathbf{u}_m, (r_1\, x + r_0)^2\, p_{n-1}^{(m)}\, (s_{n-2}^{(m)})' \right\rangle = \lambda r_1^2 \left\langle \mathbf{u}_m, x^2 p_{n-1}^{(m)}\, (s_{n-2}^{(m)})' \right\rangle \\ =& \lambda\, r_1^2 (n-2)\, c_{n-1}^{(m)}\, h_{n-2}^{(m)}. \end{aligned}$$

Therefore, summing all terms, we get (4.7).

Now, we compute the numerator of $d_{n,1}^{(m)}$ term by term using (A.1) for $j = n-1$. Observe that, using (4.6) for $n-1$

$$s_{n-1}^{(m)}(x) = \frac{1}{\xi_{n-2}^{(m)}} \left[\pi_{n-1}^{(m)}(x) - d_{n-1,1}^{(m)}\, s_{n-2}^{(m)}(x) - d_{n-1,2}^{(m)}\, s_{n-3}^{(m)}(x)\right],$$

and (4.3), as a consequence,

$$\langle \mathbf{u}_m, p_{n-2}^{(m)} s_{n-1}^{(m)}\rangle = \frac{1}{\xi_{n-2}^{(m)}} \left[\sigma_{n-2}^{(m)} - d_{n-1,1}^{(m)}\right] h_{n-2}^{(m)},$$

$$\langle \mathbf{u}_m, x\, p_{n-2}^{(m)} (s_{n-1}^{(m)})'\rangle = \frac{1}{\xi_{n-2}^{(m)}} \left[b_{n-2}^{(m)} - (n-2) d_{n-1,1}^{(m)}\right] h_{n-2}^{(m)}.$$

Again, we compute $(\pi_n^{(m)}, s_{n-1}^{(m)})_m$ term by term, and we deduce

$$\begin{aligned}(\pi_n^{(m)}, s_{n-1}^{(m)})_m &= \mathcal{B}_{n,1}^{(m)}\, h_{n-1}^{(m)} + \mathcal{B}_{n,2}^{(m)}\, \frac{h_{n-2}^{(m)}}{\xi_{n-2}^{(m)}} - \mathcal{A}_{n,2}^{(I)}\, \frac{h_{n-2}^{(m)}}{\xi_{n-2}^{(m)}}\, d_{n-1,1}^{(m)} \\ &= \mathcal{B}_{n,1}^{(m)}\, h_{n-1}^{(m)} + \left[\mathcal{B}_{n,2}^{(m)} - \mathcal{A}_{n,2}^{(I)}\, d_{n-1,1}^{(m)}\right] \frac{h_{n-2}^{(m)}}{\xi_{n-2}^{(m)}},\end{aligned}$$

because

$$(n-1) h_{n-1}^{(m)} = \frac{a_{n-2}^{(m)}}{\xi_{n-2}^{(m)}} h_{n-1}^{(m)} = \frac{c_{n-1}^{(m)}}{\xi_{n-2}^{(m)}} h_{n-2}^{(m)}.$$

Observe that since $s_n^{(m)}(x)$ has the same leading coefficient as $p_n^{(m)}(x)$, we have that $s_0^{(m)}(x) = p_0^{(m)}(x)$, and $\widetilde{h}_0^{(m)} = (1+\lambda\, m^2\, r_1^2)\, h_0^{(m)}$. Moreover, by (A.1), we get

$$\begin{aligned}(\pi_1^{(m)}, s_0^{(m)})_m &= (1+\lambda\, m^2\, r_1^2) \langle \mathbf{u}_m, (\xi_0^{(m)}\, p_1^{(m)} + \sigma_0^{(m)}\, p_0^{(m)}) s_0^{(m)}\rangle \\ &\quad + \lambda\, m\, r_1 \langle \mathbf{u}_m, (r_1 x + r_0)\, p_0^{(m)}\, s_0^{(m)}\rangle \\ &= [(1+\lambda\, m^2\, r_1^2)\sigma_0^{(m)} + \lambda\, m\, r_1 \rho(b_0)] h_0^{(m)}.\end{aligned}$$

□

A.2. Explicit Expressions of the Coefficients of Proposition 4.6.

In order to simplify the expressions, let us denote

$$\widehat{a}_n^{(m)} = a_n^{(m)} + m\, \xi_n^{(m)}, \quad \widehat{b}_n^{(m)} = b_n^{(m)} + m\, \sigma_n^{(m)}, \quad \widehat{c}_n^{(m)} = c_n^{(m)} + m\, \tau_n^{(m)},$$

and $\widehat{\rho}(x) = \ell_2\, x + \ell_1$.

Then, the explicit expressions of the coefficients of Proposition 4.6 are

$$
\begin{aligned}
\mathcal{A}_{n,1}^{(II)} &= \lambda\,\ell_2^2\,(m+n-4)\,c_{n-3}^{(m)}\,c_{n-2}^{(m)}\,\widehat{c}_{n-1}^{(m)},\\
\mathcal{B}_{n,1}^{(II)} &= \lambda\,\ell_2\,(m+n-3)\,c_{n-2}^{(m)}\left[\left(\widehat{\rho}(b_{n-3}^{(m)})+\widehat{\rho}(b_{n-2}^{(m)})\right)\,\widehat{c}_{n-1}^{(m)}+\widehat{\rho}(\widehat{b}_{n-1}^{(m)})\,c_{n-1}^{(m)}\right],\\
\mathcal{B}_{n,2}^{(II)} &= \lambda\,\ell_2\,c_{n-3}^{(m)}\,c_{n-2}^{(m)}\,\widehat{c}_{n-1}^{(m)}\,\widehat{\rho}(\widehat{b}_{n-4}^{(m)}),\\
\mathcal{C}_{n,1}^{(II)} &= \tau_{n-1}^{(m)}+\lambda\,\ell_2^2\,(m+n-2)\,\widehat{c}_{n-1}^{(m)}\left[a_{n-3}^{(m)}\,c_{n-2}^{(m)}+a_{n-2}^{(m)}\,c_{n-1}^{(m)}\right]\\
&\quad+\lambda\,\ell_2^2\,(m+n-2)\,\widehat{a}_{n-1}^{(m)}\,c_{n-1}^{(m)}\,c_{n}^{(m)}+\lambda\,\ell_2\,(m+n-2)\,c_{n-1}^{(m)}\left(\ell_1\,b_{n-1}^{(m)}+\ell_0\right)\\
&\quad+\lambda\,\ell_2\,(m+n-2)\,\widehat{b}_{n-1}^{(m)}\,c_{n-1}^{(m)}\left(\widehat{\rho}(b_{n-2}^{(m)})+\widehat{\rho}(b_{n-1}^{(m)}\right)\\
&\quad+\lambda\,\ell_1\,(m+n-2)\,c_{n-1}^{(m)}\,\widehat{\rho}(b_{n-2}^{(m)})+\lambda\,(m+n-2)\,\widehat{c}_{n-1}^{(m)}\,\widehat{\rho}(b_{n-2})^2,\\
\mathcal{C}_{n,2}^{(II)} &= \lambda\,\ell_2\,\widehat{b}_{n-3}^{(m)}\,c_{n-1}^{(m)}\left[\widehat{\rho}(b_{n-3}^{(m)})+\widehat{\rho}(b_{n-2}^{(m)})\right]+\lambda\,\ell_2\,c_{n-2}^{(m)}\,\widehat{c}_{n-1}^{(m)}\left(\ell_1\,b_{n-2}^{(m)}+\ell_0\right)\\
&\quad+\lambda\,\ell_1\,c_{n-2}^{(m)}\,\widehat{c}_{n-1}^{(m)}\,\widehat{\rho}(b_{n-3}^{(m)})+\lambda\,c_{n-2}^{(m)}\,c_{n-1}^{(m)}\,\widehat{\rho}(\widehat{b}_{n-3}^{(m)})\,\widehat{\rho}(\widehat{b}_{n-1}^{(m)}),\\
\mathcal{C}_{n,3}^{(II)} &= \lambda\,\ell_2^2\,c_{n-3}^{(m)}\,\widehat{c}_{n-3}^{(m)}\,c_{n-2}^{(m)}\,\widehat{c}_{n-1}^{(m)},\\
\mathcal{D}_{n,1}^{(II)} &= \sigma_{n-1}^{(m)}+\lambda\,\ell_2\,(m+n-1)\,\widehat{a}_{n-1}^{(m)}\,c_{n}^{(m)}\left[\widehat{\rho}(b_{n-1}^{(m)})+\widehat{\rho}(b_{n}^{(m)})\right]\\
&\quad+\lambda\,\ell_2\,(m+n-1)\,\widehat{\rho}(\widehat{b}_{n-1}^{(m)})\left[a_{n-2}^{(m)}\,c_{n-1}^{(m)}+a_{n-1}^{(m)}\,c_{n}^{(m)}\right]\\
&\quad+\lambda\,\ell_2\,(m+n-1)\,a_{n-2}^{(m)}\,\widehat{c}_{n-1}^{(m)}\left[\widehat{\rho}(b_{n-2}^{(m)})+\widehat{\rho}(b_{n-1}^{(m)})\right]\\
&\quad+\lambda\,(m+n-1)\,\widehat{\rho}(b_{n-1}^{(m)})\left(\ell_1\,b_{n-1}^{(m)}+\ell_0\right)+\lambda\,(m+n-1)\,\widehat{b}_{n-1}^{(m)}\,\widehat{\rho}(b_{n-1}^{(m)})^2,\\
\mathcal{D}_{n,2}^{(II)} &= \sigma_{n-2}^{(m)}\,\tau_{n-1}^{(m)}+\lambda\,\ell_1\,\ell_0\,c_{n-1}^{(m)}+\lambda\,\ell_2\,\widehat{b}_{n-2}^{(m)}\,\widehat{b}_{n-1}^{(m)}\,c_{n-1}^{(m)}\left[\widehat{\rho}(b_{n-2}^{(m})+\widehat{\rho}(b_{n-1}^{(m)})\right]\\
&\quad+\lambda\,\ell_2\,\widehat{c}_{n-1}^{(m)}\,\widehat{\rho}(\widehat{b}_{n-2}^{(m)})\left[a_{n-3}^{(m)}\,c_{n-2}^{(m)}+a_{n-2}^{(m)}\,c_{n-1}^{(m)}\right]+\lambda\,\ell_2\,\widehat{a}_{n-1}^{(m)}\,c_{n-1}^{(m)}\,c_{n}^{(m)}\,\widehat{\rho}(\widehat{b}_{n-2}^{(m)})\\
&\quad+\lambda\,\ell_1\,\widehat{b}_{n-2}^{(m)}\,\widehat{c}_{n-1}^{(m)}\,\widehat{\rho}(b_{n-1}^{(m)})+\lambda\,\ell_1\,c_{n-1}^{(m)}\left[b_{n-2}^{(m)}\,\widehat{\rho}(\widehat{b}_{n-2}^{(m)})+\widehat{\rho}(b_{n-2}^{(m})\,\widehat{b}_{n-1}^{(m)}\right]\\
&\quad+\lambda\,\widehat{c}_{n-1}^{(m)}\,\widehat{\rho}(b_{n-2}^{(m)})\left(\ell_1\,b_{n-2}^{(m)}+\ell_0\right)+\lambda\,c_{n-1}^{(m)}\,\widehat{\rho}(\widehat{b}_{n-1}^{(m)})\left(\ell_1\,b_{n-1}^{(m)}+\ell_0\right)\\
&\quad+\lambda\,\widehat{b}_{n-2}^{(m)}\,\widehat{c}_{n-1}^{(m)}\,\widehat{\rho}(b_{n-2}^{(m)})^2,\\
\mathcal{D}_{n,3}^{(II)} &= \lambda\,\ell_2\,c_{n-2}^{(m)}\,\widehat{c}_{n-2}^{(m)}\,\widehat{c}_{n-1}^{(m)}\left[\widehat{\rho}(b_{n-3}^{(m)})+\widehat{\rho}(b_{n-2}^{(m)})\right]+\lambda\,\ell_2\,c_{n-2}^{(m)}\widehat{c}_{n-2}^{(m)}\,c_{n-1}^{(m)}\,\widehat{\rho}(\widehat{b}_{n-1}^{(m)}).
\end{aligned}
$$

A.3. Explicit Expressions of the Coefficients of Proposition 4.8.

$$
\begin{aligned}
\mathcal{E}_{n}^{(II)} &= 1+\lambda\,(m+n)^2\,\widehat{\rho}(b_{n}^{(m)})^2+\lambda\,\ell_1^2\,(m+n)^2\\
&\quad+\lambda\,\ell_2^2\,(m+n)^2\left[a_{n-1}^{(m)}\,c_{n}^{(m)}+a_{n}^{(m)}\,c_{n+1}^{(m)}\right]\\
&\quad+2\,\lambda\,\ell_2^2\,(m+n)\,n\,a_{n-2}^{(m)}\,\widehat{c}_{n-1}^{(m)}\\
&\quad+2\,\lambda\,\ell_2\,(m+n)\,n\,\widehat{b}_{n-1}^{(m)}\left[\widehat{\rho}(b_{n-1}^{(m)})+\widehat{\rho}(b_{n}^{(m)})\right]\\
&\quad+2\,\lambda\,\ell_2\,(m+n)\,n\left(\ell_1\,b_{n}^{(m)}+\ell_0\right)+2\,\lambda\,\ell_1\,(m+n)\,n\,\widehat{\rho}(b_{n-1}^{(m)}),\\
\mathcal{F}_{n}^{(II)} &= \frac{1}{[\xi_{n-1}^{(m)}]^2}\left[\left(\sigma_{n-1}^{(m)}\right)^2+2\lambda\,\ell_2\,a_{n-2}^{(m)}\,\widehat{b}_{n-1}^{(m)}\,\widehat{c}_{n-1}^{(m)}\left(\widehat{\rho}(b_{n-2}^{(m)})+\widehat{\rho}(b_{n-1}^{(m)})\right)\right.
\end{aligned}
$$

$$+\lambda\,\ell_2\,\widehat{b}_{n-1}^{(m)}\left(a_{n-2}^{(m)}\,c_{n-1}^{(m)}+a_{n-1}^{(m)}\,c_{n}^{(m)}\right)\left(\widehat{\rho}(\widehat{b}_{n-1}^{(m)})+\ell_1\right)$$
$$+\lambda\left(\widehat{b}_{n-1}^{(m)}\,\widehat{\rho}(b_{n-1}^{(m)})+\ell_1\,b_{n-1}^{(m)}+\ell_0\right)^2\Bigg],$$
$$\mathcal{G}_n^{(II)}=\frac{1}{[\xi_{n-1}^{(m)}]^2}\Bigg[\left(\tau_{n-1}^{(m)}\right)^2+\lambda\left(\widehat{c}_{n-1}^{(m)}\right)^2\widehat{\rho}(b_{n-2}^{(m)})^2+\lambda\,\ell_1^2\left(\widehat{c}_{n-1}^{(m)}\right)^2$$
$$+\lambda\,\ell_2^2\left(\widehat{c}_{n-1}^{(m)}\right)^2\left(a_{n-3}^{(m)}\,c_{n-2}^{(m)}+a_{n-2}^{(m)}\,c_{n-1}^{(m)}\right)$$
$$+2\,\lambda\,\ell_2\,\widehat{c}_{n-1}^{(m)}\,c_{n-1}^{(m)}\left(\ell_1\,b_{n-1}^{(m)}+\ell_0\right)+2\,\lambda\,\ell_1\,\widehat{c}_{n-1}^{(m)}\,c_{n-1}^{(m)}\,\widehat{\rho}(b_{n-2}^{(m)})\Bigg].$$

References

[1] Abramowitz, M., Stegun, I.A.: Handbook of Mathematical Functions, 9th printing. Dover, New York (1972)

[2] Agahanov, S.A.: A method of constructing orthogonal polynomials in two variables for a certain class of weight functions (Russian). Vestnik Leningrad Univ. **20**, 5–10 (1965)

[3] Chihara, T.S.: An Introduction to Orthogonal Polynomials, Mathematics and its Applications, vol. 13. Gordon and Breach, New York (1978)

[4] Dunkl, C.F., Xu, Y.: Orthogonal Polynomials of Several Variables, 2nd edition, Encyclopedia of Mathematics and its Applications, vol. 155. Cambridge University Press, Cambridge (2014)

[5] Harnad, J., Vinet, L., Yermolayeva, O., Zhedanov, A.: Two-dimensional Krall–Sheffer polynomials and integrable systems. J. Phys. A **34**, 10619–10625 (2001) [**In: Symmetries and Integrability of Difference Equations (Tokyo, 2000)**]

[6] Koornwinder, T.H.: Two variable analogues of the classical orthogonal polynomials. In: Askey, R. (ed.) Theory and Application of Special Functions, pp. 435–495. Academic Press, New York (1975)

[7] Krall, H.L., Frink, O.: A new class of orthogonal polynomials: the Bessel polynomials. Trans. Am. Math. Soc. **65**, 100–115 (1949)

[8] Krall, H.L., Sheffer, I.M.: Orthogonal polynomials in two variables. Ann. Math. Pura Appl. **76**, 325–376 (1967)

[9] Kwon, K.H., Littlejohn, L.L.: Sobolev orthogonal polynomials and second-order differential equations. Rocky Mt. J. Math. **28**, 547–594 (1998)

[10] Marcellán, F., Branquinho, A., Petronilho, J.: Classical orthogonal polynomials: a functional approach. Acta Appl. Math. **34**, 283–303 (1994)

[11] Marcellán, F., Marriaga, M.E., Pérez, T.E., Piñar, M.A.: Matrix Pearson equations satisfied by Koornwinder weights in two variables. Acta Appl. Math. **153**, 81–100 (2018)

[12] Marcellán, F., Xu, Y.: On Sobolev orthogonal polynomials. Expo. Math. **33**(3), 308–352 (2015)

[13] Marriaga, M.E., Pérez, T.E., Piñar, M.A.: Three term relations for a class of bivariate orthogonal polynomials. Mediterr. J. Math. **14**(2), 26 (2017) (**Art. 54**)

[14] Olver, S., Xu, Y.: Orthogonal polynomials in and on a quadratic surface of revolution. Math. Comput. **89**, 2847–2865 (2020)

[15] Yu, X., Wang, Z., Li, H.: Jacobi–Sobolev orthogonal polynomials and spectral methods for elliptic boundary value problems. Commun. Appl. Math. Comput. **1**, 283–308 (2019)

[16] Szegő, G.: Orthogonal Polynomials, 4th edn., vol. 23. American Colloquium Publications, American Mathematical Society, Providence RI (1975)

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