

RIEMANN ZERO MEAN CURVATURE EXAMPLES IN LORENTZ-MINKOWSKI SPACE

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ABSTRACT. Riemann zero mean curvature examples in the Lorentz-Minkowski space are surfaces with zero mean curvature foliated by circles contained in parallel planes. In contrast to the Euclidean case, this family of surfaces presents new and rich features because of the variety of types of circles. In this paper, we give a geometric description of these examples when the circles are contained in spacelike planes and timelike planes.

1. INTRODUCTION AND MOTIVATION

In 1867 Riemann constructed a family of non-rotational minimal surfaces in Euclidean space foliated by circles contained in parallel planes ([17]). In the literature, they are known as Riemann minimal examples and play a remarkable role in the theory of minimal surfaces ([4, 13]).

When we extend this type of surfaces in Lorentz-Minkowski space \mathbb{L}^3 , we find two main differences regarding to the Euclidean space. First, the mean curvature is defined only in those surfaces where the induced metric from \mathbb{L}^3 is non-degenerated, that is, for spacelike surfaces (Riemannian metric) and for timelike surfaces (Lorentzian metric). Both types of surfaces have different behaviors. For example, the Weingarten endomorphism is real diagonalizable for spacelike surfaces but it is not for timelike surfaces. On the other hand, spacelike surfaces with zero mean curvature share similar properties with the minimal surfaces of Euclidean space, for example, they have a variational characterization in terms of its area, being locally a maximum for the area. For this reason, spacelike zero mean curvature surfaces are called *maximal surfaces*. By contrast, it does not make sense to define the area of a timelike surface.

A second difference comparing with the Euclidean context is the notion of a circle. In Euclidean space, a circle is a planar curve whose points are equidistant from a given point called the center of the circle. Such a definition can not extend to \mathbb{L}^3 because of the existence of planes whose metric is degenerated. For this reason, it is more convenient to define a circle as a planar non-degenerate curve with nonzero constant curvature ([7, 12]). Since there are three types of planes according to its metric, there are three types of circles

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in \mathbb{L}^3 : see Section 2 for details. Definitively, the family of zero mean curvature surfaces in \mathbb{L}^3 foliated by circles contained in parallel planes is richer than the Euclidean case and this makes the interest to its study.

To precise our terminology, we give the next definition.

Definition 1.1. A Riemann zero mean curvature example (shortly a Riemann ZMC example) is a non-rotational surface in \mathbb{L}^3 with zero mean curvature and foliated by pieces of circles contained in parallel planes.

In this paper, we exclude the rotational surface which are well known: see [3, 6, 11]. The first example of a spacelike Riemann ZMC example was discovered by the second author in [8] where, following ideas of Jagy ([5]), were described all spacelike Riemann ZMC examples foliated by circles contained in spacelike planes. Later, and for any causal character of the circles, spacelike Riemann ZMC examples were studied in [7] from the point of view of complex analysis obtaining the Weierstrass representation. More recently, Akamine has studied the Riemann ZMC examples in terms of their causal characters ([1]) and he observed the existence of timelike Riemann ZMC examples foliated by circles with constant radii.

The Riemann ZMC examples also share a property with Riemann minimal examples ([2]). If a zero mean curvature surface in \mathbb{L}^3 is foliated by circles, then these circles are contained in parallel planes and, consequently, the surface is rotational or it is a Riemann ZMC example ([9, 11, 7]).

The aim of the present paper is to give a new approach of the Riemann ZMC examples when the circles of the foliation are included in spacelike planes or in timelike planes. In contrast to [7], where the investigation was made only for spacelike surfaces in terms of the Weierstrass representation, we see the Riemann ZMC examples as the zeroes of a regular function. Here we follow similar ideas of Nitsche ([15, pp.85-90]), and more recently, of Meeks and Pérez ([14]). In particular, we obtain parametrizations of the Riemann ZMC examples without the use of complex notation and, consequently, we will derive some of their geometric properties. Although the precise statements will appear in the subsequent sections, our main results are the following.

- (1) The parametrizations of the Riemann ZMC examples are given in terms of elliptic integrals: see Theorems 3.5, 4.2 and 5.2.
- (2) We find the existence of particular examples where these integrals can be solved by quadratures, finding explicit parametrizations of Riemann ZMC examples in terms of elementary functions: see Theorems 3.3 and 3.4 and Propositions 4.4, 4.6 and 5.5.
- (3) In contrast to the Euclidean case, we will find all Riemann ZMC examples where the radii of the circles of the foliation are constant: see Propositions 3.2, 4.1 and 5.1.

We organize this paper as follows. In Section 2, we will obtain the expression of the mean curvature of a surface in given as the zeroes of a smooth function. In Section 3, we study the Riemann ZMC examples foliated by circles contained in spacelike planes. We will

establish properties of the symmetries of these surfaces in Corollaries 3.7, 3.8, 3.9 and 3.10. If the circles are included in timelike planes, its study is separated in two sections, depending if the circles are spacelike (Section 4) or timelike (Section 5).

2. PRELIMINARIES

The Lorentz-Minkowski space \mathbb{L}^3 is the vector space \mathbb{R}^3 with canonical coordinates (x_1, x_2, x_3) and endowed with the metric $\langle, \rangle = dx_1^2 + dx_2^2 - dx_3^2$. A vector $\vec{v} \in \mathbb{R}^3$ is spacelike, timelike or lightlike if $\langle \vec{v}, \vec{v} \rangle$ is positive, negative or zero, respectively. The norm of \vec{v} is $|\vec{v}| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$ if \vec{v} is spacelike and $|\vec{v}| = \sqrt{-\langle \vec{v}, \vec{v} \rangle}$ if \vec{v} is timelike. A curve or a surface $A \subset \mathbb{L}^3$ is called spacelike, timelike or lightlike if the induced metric on A is Riemannian, Lorentzian or degenerated, respectively. This property of A is called the causal character of A . We refer the reader to [12] for some basics of \mathbb{L}^3 . In \mathbb{R}^3 , as affine space, we shall utilize the terminology horizontal and vertical as usual, where the x_3 -coordinate indicates the vertical direction and a horizontal direction is a direction parallel to the plane of equation $x_3 = 0$.

A *circle* in \mathbb{L}^3 is defined as a non-degenerate planar curve with nonzero constant curvature ([7, 12]). After a rigid motion of \mathbb{L}^3 , we assume that the plane P containing the circle is the plane of equation $x_3 = 0$, $x_1 = 0$ or $x_2 - x_3 = 0$, if P is spacelike, timelike or lightlike, respectively. Consequently, a circle $C \subset \mathbb{L}^3$ can be described as follows:

- (1) If $P \equiv \{x_3 = 0\}$, then C is an Euclidean circle $\alpha(s) = p + r(\cos(s), \sin(s), 0)$, with center $p \in P$ and radius $r > 0$.
- (2) If $P \equiv \{x_1 = 0\}$, then C is a hyperbola $\alpha(s) = p + r(0, \sinh(s), \cosh(s))$ if α is spacelike or $\alpha(s) = p + r(0, \cosh(s), \sinh(s))$ if α is timelike. Here $p \in P$ is the center and $r > 0$ is the radius.
- (3) If $P \equiv \{x_2 - x_3 = 0\}$, then C is a parabola $\alpha(s) = p + (s, rs^2, rs^2)$, $p \in P$ and $r > 0$.

We say that a surface is foliated by (pieces of) circles if it is constructed by a smooth one-parameter family of (pieces of) circles. In the case of the Riemann ZMC examples, the planes containing the circles are parallel.

Let M be an orientable surface in \mathbb{L}^3 whose induced metric \langle, \rangle is non-degenerated. If $X = X(u, v)$ is a local parametrization of M , let $g_{11} = \langle X_u, X_u \rangle$, $g_{12} = \langle X_u, X_v \rangle$ and $g_{22} = \langle X_v, X_v \rangle$ be the coefficients of the first fundamental form with respect to X and $W = g_{11}g_{22} - g_{12}^2$. Then M is spacelike (resp. timelike) if $W > 0$ (resp. $W < 0$). In both types of surfaces, the mean curvature H is defined as the trace of the second fundamental form. If $H = 0$ everywhere, we say that M has zero mean curvature (ZMC in short).

We now consider a surface given as an implicit equation and we calculate the expression of its mean curvature H . Let $F : O \subset \mathbb{R}^3 \rightarrow \mathbb{R}$ be a smooth function defined in an open set O of \mathbb{R}^3 and let $M = F^{-1}(\{0\})$ be a surface defined as the preimage of a regular value of F . Suppose that M endowed with the induced metric of \mathbb{L}^3 is a non-degenerate surface. Then the Lorentzian gradient $\nabla^L F$ of F

$$\nabla^L F = (F_{x_1}, F_{x_2}, -F_{x_3})$$

defines a normal vector field on M where the subscript x_i indicates the partial derivative with respect to the x_i -variable. Then $\nabla^L F/|\nabla^L F|$ is a unit normal vector field on M and

$$\operatorname{div}^L \left(\frac{\nabla^L F}{|\nabla^L F|} \right) = \left(\frac{F_{x_1}}{|\nabla^L F|} \right)_{x_1} + \left(\frac{F_{x_2}}{|\nabla^L F|} \right)_{x_2} - \left(\frac{F_{x_3}}{|\nabla^L F|} \right)_{x_3} = H,$$

where div^L is the Lorentzian divergence operator. As a consequence, the equation $H = 0$ is equivalent to

$$\frac{\Delta^L F}{|\nabla^L F|} + \frac{\epsilon}{|\nabla^L F|^3} (\nabla^L F)^t \cdot \operatorname{Hess} F \cdot \nabla^L F = 0,$$

where $\epsilon = 1$ if M is spacelike and $\epsilon = -1$ if M is timelike, $\Delta^L F = F_{x_1 x_1} + F_{x_2 x_2} - F_{x_3 x_3}$ and

$$\operatorname{Hess} F = \begin{pmatrix} F_{x_1 x_1} & F_{x_1 x_2} & F_{x_1 x_3} \\ F_{x_2 x_1} & F_{x_2 x_2} & F_{x_2 x_3} \\ F_{x_3 x_1} & F_{x_3 x_2} & F_{x_3 x_3} \end{pmatrix}.$$

Proposition 2.1. *If $M = F^{-1}(\{0\})$ is a non-degenerate surface in \mathbb{L}^3 , then M is a ZMC surface if and only if*

$$(1) \quad -\langle \nabla^L F, \nabla^L F \rangle \Delta^L F + (\nabla^L F)^t \cdot \operatorname{Hess} F \cdot \nabla^L F = 0.$$

3. RIEMANN ZMC EXAMPLES FOLIATED BY CIRCLES CONTAINED IN SPACELIKE PLANES

In this section, we study ZMC surfaces in \mathbb{L}^3 foliated by circles contained in parallel spacelike planes. After a rigid motion of \mathbb{L}^3 , we suppose that the foliating circles are contained in parallel planes to the plane of equation $x_3 = 0$, hence the circles are Euclidean circles. Let M be a such surface and consider the x_3 -coordinate as a parameter of the foliation. Let $z = x_3$ and write $(\alpha(z), z) = (\alpha_1(z), \alpha_2(z), z)$ the center of the circle $M \cap \{x_3 = z\}$ and by $r(z) > 0$ its radius. Here α and r are smooth functions defined in an interval $(a, b) \subset \mathbb{R}$. If $F : \mathbb{R}^2 \times (a, b) \rightarrow \mathbb{R}$ is the function

$$F(x, z) = (x_1 - \alpha_1(z))^2 + (x_2 - \alpha_2(z))^2 - r(z)^2,$$

where $x = (x_1, x_2)$, then $M \subset F^{-1}(\{0\})$. We observe that if $\vec{a} = (0, 0)$, then α is the x_3 -axis, which corresponds with the case that the surface is rotational.

Let $\alpha' = d\alpha/dz$. Here we identify the factor \mathbb{R}^2 of the domain of F with $\mathbb{R}^2 \times \{0\}$ endowed with the induced metric $\langle \cdot, \cdot \rangle = dx_1^2 + dx_2^2$. The computation of each one of the terms of (1) yields

$$\begin{aligned} \nabla^L F &= (2(x - \alpha(z)), -F_z) \\ \Delta^L F &= 4 - F_{zz} \\ (\nabla^L F)^t \cdot \operatorname{Hess} F \cdot \nabla^L F &= 8r^2 + 8F_z \langle x - \alpha, \alpha' \rangle + F_z^2 F_{zz}. \end{aligned}$$

Then equation (1) writes as

$$(2) \quad F_z^2 + r^2(F_{zz} - 2) + 2F_z \langle x - \alpha, \alpha' \rangle = 0.$$

By using the definition of F ,

$$(3) \quad F_z = -2\langle x - \alpha, \alpha' \rangle - (r^2)',$$

and (2) simplifies into $-2r^2 + r^2 F_{zz} - (r^2)' F_z = 0$. We divide by r^4 , obtaining easily

$$-\frac{2}{r^2} + \left(\frac{F_z}{r^2} \right)_z = 0.$$

Hence we integrate with respect to z ,

$$(4) \quad -2 \int^z \frac{1}{r(u)^2} du + \frac{F_z}{r^2} = c(x)$$

for some function $c = c(x)$ depending on x . The definition of F leads to

$$\left(\frac{F_z}{r^2} \right)_{x_i} = -\frac{2\alpha'_i}{r^2}, \quad i = 1, 2,$$

and thus

$$\frac{F_z}{r^2} = -\frac{2}{r^2} \langle x, \alpha' \rangle + h(z)$$

for some function $h = h(z)$. If we insert this in (4), it follows that

$$-2 \int^z \frac{1}{r(u)^2} du - \frac{2}{r^2} \langle x, \alpha' \rangle + h(z) = c(x).$$

Differentiating with respect to the variables x_1 and x_2 ,

$$c_{x_i} = -\frac{2\alpha'_i(z)}{r(z)^2}, \quad i = 1, 2.$$

Therefore there is a vector $\vec{a} = (a_1, a_2)$ such that

$$(c_{x_1}, c_{x_2}) = -\frac{2}{r(z)^2} \alpha'(z) = \vec{a},$$

that is,

$$\alpha'(z) = -r(z)^2 \vec{a}.$$

Hence integrating with respect to z ,

$$\alpha(z) = -m(z) \vec{a}, \quad m(z) = \int^z r(u)^2 du.$$

As immediate consequence of the expression of α , the curve α is a horizontal straight-line and thus the curve of the centers of the circles is contained in the plane containing the x_3 -axis.

Proposition 3.1. *The curve formed by the centers of the foliation circles of a Riemann ZMC example foliated by circles contained in spacelike planes is planar.*

We express the radius of the circle $M \cap \{x_3 = z\}$ as a function of z . We will identify \vec{a} with the vector $(\vec{a}, 0) \in \mathbb{L}^3$ in the $x_1 x_2$ -plane. Moreover, $\langle \vec{a}, \vec{a} \rangle = |\vec{a}|^2$. It follows from (3) that

$$\begin{aligned} F_z &= 2\langle x + m\vec{a}, r^2 \vec{a} \rangle - (r^2)' \\ F_{zz} &= 2r^4 \langle \vec{a}, \vec{a} \rangle + 2(r^2)' \langle x + m\vec{a}, \vec{a} \rangle - (r^2)'' \end{aligned}$$

Inserting F_z and F_{zz} in (2),

$$2|\vec{a}|^2 r^6 + (r^2)'^2 - r^2(2 + (r^2)'') = 0.$$

By taking $q = r^2$, we deduce

$$(5) \quad 2\langle \vec{a}, \vec{a} \rangle q^3 + q'^2 - q(2 + q'') = 0.$$

Definitively, equation (5) describe all ZMC surfaces foliated by circles contained in parallel spacelike planes.

As immediate consequence of (5) is the existence of solutions where the radius function r is constant, which is a novelty comparing with the Euclidean case.

Proposition 3.2 (Case of constant radii). *The only Riemann ZMC examples foliated by circles with constant radii contained in spacelike planes are parametrized by*

$$(6) \quad X(z, v) = z(-r^2\vec{a}, 1) + r(\cos(v), \sin(v), 0),$$

where $z, v \in \mathbb{R}$ and $\vec{a} \in \mathbb{R}^2 \setminus \{0\}$. The surfaces are timelike and can be extended to two lightlike straight-lines. See figure 1, left.

Proof. If r is constant, then $|\vec{a}|^2 q^2 - 1 = 0$. Thus $\vec{a} \neq 0$ and $r^2 = 1/|\vec{a}|$. Then $m(z) = r^2 z$ and $\alpha(z) = -r^2 z \vec{a}$ obtaining (6). In order to know the causal character of the surface, we have $X_z = (-r^2\vec{a}, 1)$ and $X_v = r(-\sin(v), \cos(v), 0)$. Hence, $g_{11} = 0$, $g_{22} = r^2$ and $g_{12} = -r^3 \langle \vec{a}, (-\sin(v), \cos(v)) \rangle$. We infer that M is timelike except at the points where $g_{12} = 0$, which are lightlike points.

Therefore the surface can be extended to the points $\langle \vec{a}, (-\sin(v), \cos(v)) \rangle = 0$ as a region of lightlike points. These points form a set of straight-lines, such as was demonstrated in [1]. Indeed, there are exactly two values v_0 and v_1 , up to an integer 2π -multiple, such that $\langle \vec{a}, (-\sin(v_i), \cos(v_i)) \rangle = 0$, $i = 1, 2$. This set parametrizes as $z \mapsto X(z, v_i) = r(\cos(v_i), \sin(v_i), 0) + z(-r^2\vec{a}, 1)$, proving that they are two lightlike straight-lines. \square

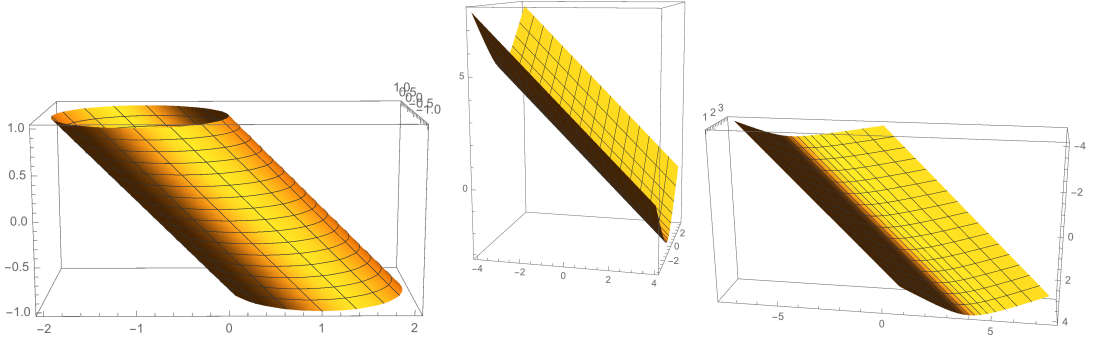


FIGURE 1. Riemann ZMC examples with constant radii. Left: Euclidean circles. Middle: spacelike hyperbola. Right: timelike hyperbola

From now, we suppose that the radii of the foliation are not constant. We write (5) as

$$\left(\frac{(q')^2}{q^2}\right)' = \frac{2q'}{q^3}(q''q - q'^2) = 4q' \left(|\vec{a}|^2 - \frac{1}{q^2}\right).$$

Integrating with respect to z ,

$$(7) \quad \frac{(q')^2}{q^2} = 4 \left(|\vec{a}|^2 q + \frac{1}{q}\right) + 4\lambda$$

for a constant $\lambda \in \mathbb{R}$. In particular, the right-hand side of (7) must be non-negative. Now we have

$$q' = \frac{dq}{dz} = 2\sqrt{|\vec{a}|^2 q^3 + \lambda q^2 + q},$$

or equivalently,

$$\frac{dz}{dq} = \frac{1}{2} \frac{1}{\sqrt{|\vec{a}|^2 q^3 + \lambda q^2 + q}}.$$

As the new parameter is q , the center of the circle $M \cap \{x_3 = z\}$ is $(-m(q)\vec{a}, z(q))$, where

$$(8) \quad z(q) = \frac{1}{2} \int^q \frac{du}{\sqrt{|\vec{a}|^2 u^3 + \lambda u^2 + u}}, \quad m(q) = \frac{1}{2} \int^q \frac{u du}{\sqrt{|\vec{a}|^2 u^3 + \lambda u^2 + u}}.$$

The parametrization of M is

$$X(q, v) = (-m(q)\vec{a}, z(q)) + \sqrt{q}(\cos(v), \sin(v), 0).$$

After a rotation about the x_3 -axis, suppose that $\vec{a} = a(1, 0)$, $a > 0$. With a change of variables, we see that $X_{a,\lambda}$ and $X_{1,\lambda/a}$ is $\sqrt{a}X_{a,\lambda}(q, v) = X_{1,\lambda/a}(aq, v)$. Hence the corresponding surfaces are equal up to a homothety of ratio \sqrt{a} . Fixing the value of a , say $a = 1$, we conclude from 8 that the family of Riemann ZMC examples only depends on the real parameter λ . Now

$$(9) \quad z(q) = \frac{1}{2} \int_{q_0}^q \frac{du}{\sqrt{u^3 + \lambda u^2 + u}}, \quad m(q) = \frac{1}{2} \int_{q_0}^q \frac{u du}{\sqrt{u^3 + \lambda u^2 + u}}.$$

Suppose now that the surface is spacelike. The first derivatives of X are

$$X_q = (-m', 0, z') + \frac{1}{2\sqrt{q}}(\cos(v), \sin(v), 0)$$

$$X_v = \sqrt{q}(-\sin(v), \cos(v), 0).$$

From (9), the spacelike condition $g_{11}g_{22} - g_{12}^2 > 0$ is

$$(10) \quad (1 + \cos(v)^2)q - 2\cos(v)\sqrt{q^2 + \lambda q + 1} + \lambda > 0.$$

The radicand of this two-degree inequation is $q^3 + \lambda q^2 + q$. We analyze if it is positive in order to determined the lower limit q_0 of the integrals (9). The equation $u^3 + \lambda u^2 + u = 0$ has three roots, namely, 0 and, if exist,

$$q_1 = \frac{-\lambda - \sqrt{\lambda^2 - 4}}{2}, \quad q_2 = \frac{-\lambda + \sqrt{\lambda^2 - 4}}{2},$$

with $q_1 \leq q_2$.

We now study the properties of the surface depending if $\lambda^2 - 4 < 0$, $\lambda^2 - 4 = 0$ and $\lambda^2 - 4 > 0$. Firstly, we distinguish the case $\lambda^2 - 4 = 0$ because we will obtain explicit

parametrizations of surfaces. This contrast to the Euclidean case, where the Riemann minimal examples are given in terms of elliptic integrals that can not be integrated by simple quadratures.

Theorem 3.3. *There are spacelike Riemann ZMC examples foliated by circles contained in spacelike planes with parametrizations given in terms of elementary functions for the cases $\lambda = 2$ and $\lambda = -2$.*

(1) *Case $\lambda = 2$. The surface is*

$$(11) \quad X(r, v) = (-r + \arctan(r), 0, \arctan(r)) + r(\cos(v), \sin(v), 0),$$

where $r > 0$, $v \in (0, 2\pi)$. See figure 2, left. The properties of this surface are the following.

- (a) *The circles of the foliation are punctured.*
- (b) *The surface is contained in the horizontal slab $0 < x_3 \leq \pi/2$.*
- (c) *The surface converges to a conelike point as $x_3 \rightarrow 0$ and converges to a straight-line L orthogonal to the plane Π of equation $x_2 = 0$ as $x_3 \rightarrow \pi/2$. The straight-line L is contained in the surface.*
- (d) *The surface is asymptotic to the horizontal plane of equation $x_3 = \pi/2$.*
- (e) *The surface can be extended to a lightlike straight-line.*

(2) *Case $\lambda = -2$. The surface is*

$$(12) \quad X(r, v) = \left(-r - \frac{1}{2} \log\left(\frac{r-1}{r+1}\right), 0, \frac{1}{2} \log\left(\frac{r-1}{r+1}\right)\right) + r(\cos(v), \sin(v), 0),$$

where $r \in [r_0, \infty)$ for any $r_0 > 1$ and $\cos(v) < 1 - 2/r^2$. See figure 2, right. The properties of this surface are the following.

- (a) *The surface is foliated by pieces of circles that can be extended to fully circles assuming that the surface is degenerated in the points $\cos(v) = 1 - 2/r^2$ and timelike when $\cos(v) > 1 - 2/r^2$.*
- (b) *The surface lies contained in the horizontal slab $z_0 < x_3 < 0$, where $z_0 = \log(\frac{r_0-1}{r_0+1})/2$.*
- (c) *If $r = r_0$, the surface has a boundary component that is (part of) a circle and it converges to a straight-line orthogonal to the plane Π as $x_3 \rightarrow 0$. This straight-line is contained in the surface.*

Proof. (1) Case $\lambda = 2$. Then $q_1 = q_2 = -1$ and we take $q_0 = 0$ as the lower limit of integration in (9). An integration by quadratures gives

$$z(q) = \frac{1}{2} \int_0^q \frac{du}{\sqrt{u}(u+1)} = \arctan(\sqrt{q})$$

$$m(q) = \frac{1}{2} \int_0^q \frac{\sqrt{u} du}{u+1} = \sqrt{q} - \arctan(\sqrt{q}).$$

Setting $r = \sqrt{q}$, the parametrization of the surface is (11). Inequality (10) writes as

$$q(1 - \cos(v))^2 + 2(1 - \cos(v)) > 0,$$

which holds if $\cos(v) \neq 1$. This proves that the circles of the foliation are punctured.

On the other hand, the excluded points $\cos(v) = 1$ form a curve where the metric is degenerated. Moreover, as the (extended) surface has only spacelike and lightlike points, the lightlike points correspond with a straight-line ([1]). Indeed, these points parametrize as $r \mapsto X(r, v) = \arctan(r)(1, 0, 1)$ and from (11), we have

$$\lim_{r \rightarrow 0} x_3(r) = 0, \quad \lim_{r \rightarrow \infty} x_3(r) = \frac{\pi}{2}.$$

This proves that the surface is included in the slab $0 < x_3 < \pi/2$. Finally, we consider the intersection of the plane Π with each circle of the foliation, that is, for the points where $\cos(v) = \pm 1$. If $r \rightarrow \infty$, the points where $\cos(v) = -1$ go to $-\infty$ and the points where $\cos(v) = 1$ (lightlike points) converge to the point $(\pi/2, 0, \pi/2)$. If $r \rightarrow \infty$, the surface converges to the straight-line L orthogonal to the plane Π through the point $(\pi/2, 0, \pi/2)$.

- (2) Case $\lambda = -2$. Then $q_1 = q_2 = 1$. There are two cases, namely, $q_0 = 0$ and $q \in (0, 1)$, or $q_0 = 1$ and $q \in (1, \infty)$. We observe that if $q \in (0, 1)$, then (10) is equivalent to

$$(1 + \cos(v))(q(1 + \cos(v)) - 2) > 0.$$

It is clear that $\cos(v) \neq -1$. Then the above inequality writes as $q > 2/(1 + \cos(v))$, which is impossible since $q \in (0, 1)$. This proves definitively that if the surface is spacelike, then $q \in (1, \infty)$. The spacelike condition (10) is now

$$(1 - \cos(v))((1 - \cos(v))q - 2) > 0,$$

in particular, $\cos(v) \neq 1$, hence $\cos(v) < (q - 2)/q$. Consequently, the spacelike condition implies that in each leaf of the foliation the corresponding circle is not complete, and the spacelike character is only defined in arcs of this circle. Moreover, the lengths of these (spacelike) arcs go varying along the foliation. Assume that $\cos(v) < (q - 2)/q$, hence $v \in (\arccos((q - 2)/2), 2\pi \arccos((q - 2)/2))$. It follows

$$z(q) = \frac{1}{2} \log \left(\frac{\sqrt{q} - 1}{\sqrt{q} + 1} \right), \quad m(q) = \sqrt{q} + \frac{1}{2} \log \left(\frac{\sqrt{q} - 1}{\sqrt{q} + 1} \right).$$

It is obvious that $q_0 > 1$ since if $q_0 = 1$, the integrals are indefinite. If $r = \sqrt{q}$, the parametrization of the surface coincides with (12). Moreover, $z_0 \leq x_3 < 0$, where $z_0 = \log((r_0 - 1)/(r_0 + 1))/2$ and $r_0 = \sqrt{q_0}$.

If $r \rightarrow \infty$, the points satisfying $\cos(v) = -1$ go to $-\infty$ but the points with $\cos(v) = 1$ (lightlike points) converge to the point $(0, 0, 0)$. Since $r \rightarrow \infty$, the surface has a limit set the straight-line L orthogonal to the plane Π through the point $(0, 0, 0)$.

□

Until here, we have assumed that the surface is spacelike. However, the same arguments hold by changing the domain of the parametrization in order to ensure $W < 0$. Recall that if $\lambda = 2$, then the surface can not extend to timelike points. However, this differs if $\lambda = -2$. In such a case, if $q \in (1, \infty)$, there are regions of timelike points and if $q \in (0, 1)$,

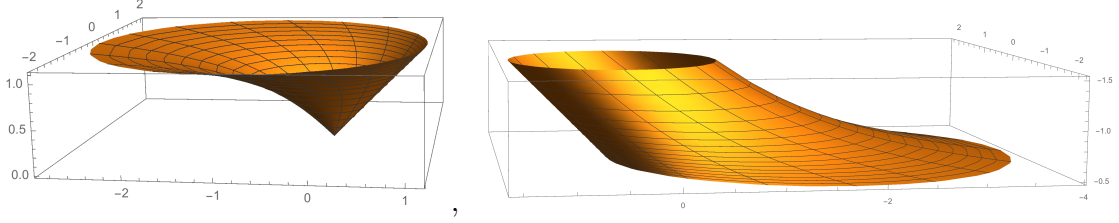


FIGURE 2. Riemann ZMC examples foliated by Euclidean circles of Theorem 3.3. Both surfaces extend to lightlike straight-lines

the surface is timelike except when $\cos(v) = -1$. Here we take $q_0 = 0$ to be the lower limit of integration in (9), obtaining

$$z(q) = \operatorname{artanh}(\sqrt{q}), \quad m(q) = -\sqrt{q} + \operatorname{artanh}(\sqrt{q}).$$

Theorem 3.4. *In case $\lambda = -2$, we have parametrizations of timelike Riemann ZMC examples foliated by circles contained in spacelike planes in terms of elementary functions:*

- (1) *The parametrization (12) for any $r_0 > 1$, when $r \in [r_0, \infty)$ and $1 - 2/r^2 < \cos(v) < 1$.*
- (2) *The parametrization*

$$X(r, v) = (r - \operatorname{artanh}(r), 0, \operatorname{artanh}(r)) + r(\cos(v), \sin(v), 0),$$

where $r \in (0, 1)$ and $v \in (-\pi, \pi)$. This surface extends to a lightlike straight-line by considering the points $\cos(v) = -1$. The surface is included in the halfspace $x_3 > 0$ with $\lim_{r \rightarrow \infty} x_3 = \infty$. See figure 3 and [10, Ex. 1].

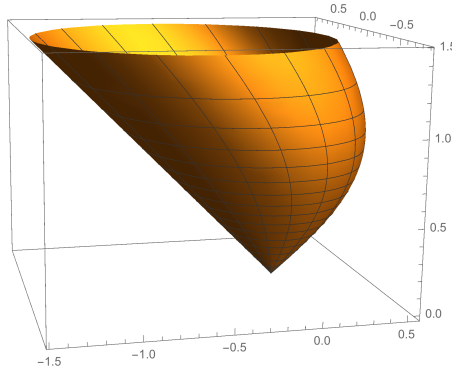


FIGURE 3. Theorem 3.4: a timelike Riemann ZMC example foliated by circles contained in spacelike planes and with explicit parametrization

We consider the general case for the parameter λ . In the following result, we show the geometric properties of these surfaces.

Theorem 3.5. *Riemann ZMC examples foliated by circles contained in spacelike planes form a one-parameter family of surfaces depending on a parameter λ and parametrize as*

$$X(q, v) = (m(q), 0, z(q)) + \sqrt{q}(\cos(v), \sin(v), 0),$$

where

$$z(q) = \frac{1}{2} \int^q \frac{du}{\sqrt{u^3 + \lambda u^2 + u}}, \quad m(q) = \frac{1}{2} \int^q \frac{u du}{\sqrt{u^3 + \lambda u^2 + u}}.$$

Depending on λ , we have the following cases.

- (1) Case $\lambda^2 = 4$. These surfaces have been described in Theorems 3.3 and 3.4.
- (2) Case $\lambda^2 < 4$. The surface contains regions of points with the three causal characters; it is included in a horizontal slab $z_0 \leq x_3 < 0$ with a conelike point at $x_3 = 0$; the surface contains a straight-line orthogonal to the plane Π . See figure 4, left.
- (3) Case $\lambda < -2$. The q -parameter belongs to $(0, q_1) \cup (q_2, \infty)$.
 - (a) In the interval $(0, q_1)$ the surface is timelike and included in a horizontal slab $0 < x_3 < z_0$. As $x_3 \rightarrow 0$, the surface converges to a point and if $x_3 \rightarrow c$, the surface converges to a circle.
 - (b) If $q \in (q_2, \infty)$, the surface has regions of points with the three causal characters. Moreover, it is included in a horizontal slab $0 \leq x_3 \leq z_0$. If $x_3 = 0$, the surface has a lower boundary component that is a circle and as $x_3 = z_0$, the surface is a straight-line orthogonal to the plane Π at the height z_0 . See figure 4, right.
- (4) Case $\lambda > 2$. The surface is spacelike and included in a horizontal slab $0 < x_3 \leq z_0$. As $x_3 \rightarrow 0$, the surface converges to a conelike point and if $x_3 = z_0$, the surface is a straight-line L orthogonal to the plane Π .

Proof. Let M be a Riemann ZMC example foliated by circles contained in parallel planes to the plane of equation $x_3 = 0$. The centers of the circles $M \cap \{x_3 = z(q)\}$ lie included in the plane Π of equation $x_2 = 0$. Assume $\lambda^2 \neq 4$.

- (1) Case $\lambda^2 - 4 < 0$. Since $u^2 + \lambda u + 1$ has not real roots, then $u^2 + \lambda u + 1 > 0$. This implies that $u^3 + \lambda u^2 + u$ only vanishes at $u = 0$, hence the radicand $u^3 + \lambda u^2 + u$ is positive when $u > 0$. Then we can choose $q_0 = 0$ to be the lower limit in the integrals in (9). In view of

$$\lim_{q \rightarrow \infty} \frac{1}{2} \int_0^q \frac{du}{\sqrt{u^3 + \lambda u^2 + u}} := z_0 < \infty,$$

the surface M lies contained in a slab of the form $0 < z < z_0$ and M is asymptotic to the horizontal plane of equation $x_3 = z_0$.

Each circle $M \cap \{x_3 = z(q)\}$ meets Π in two antipodal points, $A_+(q)$ and $A_-(q)$, by taking $\cos(v) = 1$ and $\cos(v) = -1$ respectively:

$$A_{\pm}(q) = \left(\pm \sqrt{q} + \frac{1}{2} \int_0^q \frac{u}{\sqrt{u^3 + \lambda u^2 + u}} du, 0, \frac{1}{2} \int_0^q \frac{1}{\sqrt{u^3 + \lambda u^2 + u}} du \right).$$

Then

$$\begin{aligned} \lim_{q \rightarrow \infty} \sqrt{q} + \frac{1}{2} \int_0^q \frac{u}{\sqrt{u^3 + \lambda u^2 + u}} du &= \infty \\ \lim_{q \rightarrow \infty} -\sqrt{q} + \frac{1}{2} \int_0^q \frac{u}{\sqrt{u^3 + \lambda u^2 + u}} du &:= c < \infty \end{aligned}$$

for some $c < 0$. Thus

$$\lim_{q \rightarrow \infty} A_-(q) := A = (c, 0, z_0), \quad \lim_{q \rightarrow \infty} A_+(q) = \infty.$$

This implies that $M \cap \{x_3 = z_0\} \neq \emptyset$. Since $A_+(q)$ diverges, then $M \cap \{x_3 = z_0\}$ is a straight-line L orthogonal to the plane Π through the point A .

(2) Case $\lambda < -2$. Then $0 < q_1 < q_2$ and the polynomial $u^3 + \lambda u^2 + u$ is positive in $(0, q_1) \cup (q_2, \infty)$.

(a) Case $q \in (0, q_1)$. The right-hand side of (10) is negative, hence the surface is timelike. As the integrals in (9) are finite, the surface lies contained in the slab $0 < x_3 < z_0$ where $z_0 = x_3(q_1)$.

(b) Case $q \in (q_2, \infty)$. We take q_2 to be the lower limit of the integrals in (9). This implies that the initial circle of the foliation has radius $\sqrt{q_2}$. Now

$$\lim_{q \rightarrow \infty} \frac{1}{2} \int_{q_2}^q \frac{1}{\sqrt{u^3 + \lambda u^2 + u}} du =: z_0 < \infty,$$

proving that M is included in the horizontal slab $0 < x_3 < z_0$. Moreover,

$$\begin{aligned} \lim_{q \rightarrow \infty} \sqrt{q} + \frac{1}{2} \int_{q_2}^q \frac{u}{\sqrt{u^3 + \lambda u^2 + u}} du &= \infty \\ \lim_{q \rightarrow \infty} -\sqrt{q} + \frac{1}{2} \int_{q_2}^q \frac{u}{\sqrt{u^3 + \lambda u^2 + u}} du &:= c \end{aligned}$$

for some $c \in \mathbb{R}$. A similar argument as in the case $\lambda^2 - 4 < 0$, proves that the surface is asymptotic to a straight-line orthogonal to the plane Π at the point $(c, 0, z_0)$.

(3) Case $\lambda > 2$. The roots of the radicand $u^3 + \lambda u^2 + u$ are $0, q_1$ and q_2 with $q_1 < q_2 < 0$. Since q is positive, we may choose $q_0 = 0$ to be the lower limit in the integrals (9). Now the spacelike condition (10) holds for any $q > 0$, indeed, (10) is equivalent to $((1 + \cos(v)^2)q + \lambda > 2 \cos(v) \sqrt{q^2 + \lambda q + 1})$. This inequality holds trivially if $\cos(v) \leq 0$. If $\cos(v) > 0$, squaring and simplifying, we obtain $\sin(v)^4 q^2 + 2\lambda q \sin(v)^2 + \lambda^2 - 4 \cos(v)^2$, which is positive if $\lambda > 2$. Again, we obtain

$$\begin{aligned} \lim_{q \rightarrow \infty} \frac{1}{2} \int_0^q \frac{1}{\sqrt{u^3 + \lambda u^2 + u}} du &:= z_0 < \infty \\ \lim_{q \rightarrow \infty} \sqrt{q} + \frac{1}{2} \int_0^q \frac{u}{\sqrt{u^3 + \lambda u^2 + u}} du &= \infty \\ \lim_{q \rightarrow \infty} -\sqrt{q} + \frac{1}{2} \int_0^q \frac{u}{\sqrt{u^3 + \lambda u^2 + u}} du &:= c \end{aligned}$$

for some $z_0, c \in \mathbb{R}$. This proves that M is included in the slab $0 < x_3 < z_0$ and M is asymptotic to a straight-line orthogonal to the plane Π at the height $x_3 = z_0$.

□

Focusing on the Riemann ZMC surfaces of spacelike type, we conclude:

Corollary 3.6. *Any spacelike Riemann ZMC surface foliated by circles contained in space-like planes is included in a horizontal slab $0 < x_3 \leq z_0$. If $x_3 \rightarrow 0$, the surface converges*

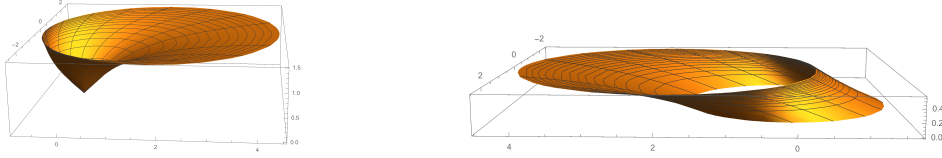


FIGURE 4. Theorem 3.5. Left: case $\lambda^2 < 4$. Right: case $\lambda < -2$ and $q \in (q_2, \infty)$

to a conelike point or a circle. At the height $x_3 = z_0$, the surface is a straight-line line L orthogonal to Π . Furthermore, if $x_3 \rightarrow z_0$, the surface is asymptotic to the horizontal plane of equation $x_3 = z_0$.

We finish this section obtaining properties of symmetries of the above Riemann ZMC examples. As a consequence of Proposition 3.1, we deduce the existence of symmetries about a vertical plane.

Corollary 3.7 (symmetry I). *Any Riemann ZMC example foliated by circles contained in spacelike planes is symmetric about the plane containing the centers of the circles of the foliation.*

Suppose that M is a spacelike Riemann ZMC example foliated by circles contained in spacelike planes. Up to rotations and dilations, we can assume that the plane containing the centers of the circles is the plane Π of equation $x_2 = 0$. By Corollary 3.6, M is asymptotic to the plane $x_3 = z_0$ and $M \cap \{x_3 = z_0\}$ is a straight-line L orthogonal to Π . We reflect M about L and we want to apply the Schwarz's reflection principle in order to extend analytically M along L . The Schwarz's reflection principle is due to the reflection principle of harmonic functions ([16]), which can be easily extended for maximal surfaces in \mathbb{L}^3 . We need to assure that M is spacelike around L . The straight-line L is obtained letting $q \rightarrow \infty$ with $W > 0$ except if $\cos(v) = 1$ and the parameter λ satisfies $0 \leq \lambda \leq 2$. In such a case, the surface is spacelike around L except at the point where $\cos(v) = 1$, which coincides with the intersection point $\Pi \cap L$. Definitively, we have established the following result.

Corollary 3.8 (symmetry II). *If M is a spacelike Riemann ZMC example foliated by circles contained in spacelike planes, then M contains a straight-line L orthogonal to the plane Π and M can be reflected analytically across L .*

In figure 5, the surface of Theorem 3.3, case (1), has been extended by a reflection about the line L .

Now, we focus in those spacelike Riemann ZMC examples converging to conelike points: Theorem 3.3, case (1) and Theorem 3.5, cases (2) and (4). We know that as $x_3 \rightarrow 0$, the surface converges to a conelike point P . Once that we have reflected the surface M about the straight-line L , if \mathcal{R} is the reflection across L , the surface $M^* = M \cup \mathcal{R}(M)$ is included in the slab $0 < x_3 < 2z_0$. If $x_3 \rightarrow 2z_0$, the surface M^* converges to the conelike point $\mathcal{R}(P)$. By means of the discrete group of translations generated by the vector $\overrightarrow{P\mathcal{R}(P)}$, we produce copies of M^* obtaining a periodic maximal surface: see figure 6.

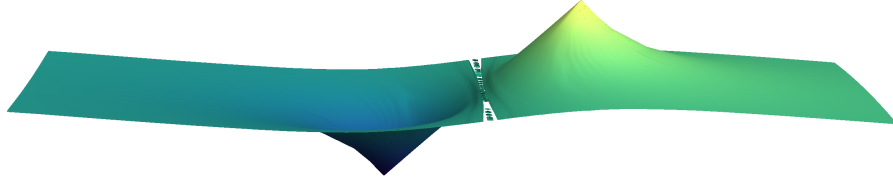


FIGURE 5. The surface parametrized by (11) has been extended by the Schwarz's reflection principle

Corollary 3.9 (symmetry III). *Let M be a spacelike Riemann ZMC example foliated by pieces of circles contained in spacelike planes. Suppose that M is bounded by a conelike point and a straight-line L orthogonal to the plane Π . Then M can be reflected across L and repeated by translations obtaining a periodic maximal surface foliated by pieces of circles, a discrete set of straight-lines in horizontal planes and a discrete set of conelike points. Furthermore, the surface is asymptotic to horizontal planes at the heights where are situated the straight-lines.*

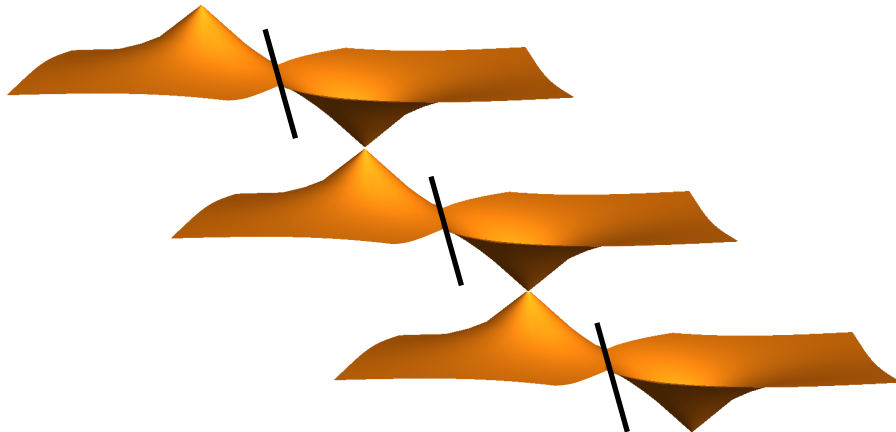


FIGURE 6. Case $\lambda > 2$ in Theorem 3.5: the surface has been extended by reflection across the line L and by translations

Finally, we study the spacelike Riemann ZMC examples that converge to a circle if $x_3 \rightarrow 0$: see Theorem 3.3, case (2) and Theorem 3.5, case (3). In such a case, the surface can be extended by reflection across L by the Schwarz's reflection principle, obtaining a spacelike Riemann ZMC example included in the slab $0 < x_3 < 2z_0$ and converging to two circles if $x_3 \rightarrow 0$ and if $x_3 \rightarrow 2z_0$. See figure 7.

Corollary 3.10. *Let M be a spacelike Riemann ZMC example foliated by pieces of circles contained in spacelike planes. Suppose that M converges to a circle if $x_3 \rightarrow 0$ and contains a straight-line L orthogonal to the plane Π at $x_3 = z_0$. Then M can be reflected across L obtaining a spacelike surface contained in the slab $0 < x_3 < 2z_0$ and foliated by pieces of circles. Furthermore, the surface contains a straight-line and converges to two circles as $x_3 \rightarrow 0$ and as $x_3 \rightarrow 2z_0$. The surface is asymptotic to the horizontal plane that contains L .*



FIGURE 7. A spacelike Riemann ZMC example of Theorem 3.3, case (2). Left: the surface is bounded by a straight-line L and a circle. Right: the same surface after a reflection across L

4. RIEMANN ZMC EXAMPLES FOLIATED BY SPACELIKE CIRCLES CONTAINED IN TIMELIKE PLANES

In this section, we study ZMC surfaces in \mathbb{L}^3 foliated by spacelike circles contained in parallel timelike planes. The arguments and computations follow the same steps than in the previous section. In order to be not repeated, we will omit the details.

Without loss of generality, we can assume that the planes of the foliation are parallel to the plane of equation $x_1 = 0$. Let M be a such surface and consider the height x_1 of the plane as a parameter of the foliation. Let $x = x_1$ and let $(x, \alpha(x)) = (x, \alpha_2(x), \alpha_3(x))$ be the center of the circle $M \cap \{x_1 = x\}$. A Lorentzian circle in the x_2x_3 -plane is a hyperbola that can be parametrized as $\alpha(s) = p+r(0, \sinh(s), \cosh(s))$, $s \in \mathbb{R}$, or $\beta(s) = p+r(0, \cosh(s), \sinh(s))$, $s \in \mathbb{R}$, where $p \in \mathbb{L}^3$ and $r > 0$. Since in this section we are considering spacelike circles, then right choice is the hyperbola α . We notice now that each circle of the foliation is not compact.

The surface M can be expressed as $M \subset F^{-1}(\{0\})$, where $F : (a, b) \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is

$$F(x, y) = (x_2 - \alpha_2(x))^2 - (x_3 - \alpha_3(x))^2 + r(x)^2.$$

Let $y = (x_2, x_3)$. We identify the factor \mathbb{R}^2 of the domain of F as $\{0\} \times \mathbb{R}^2$ endowed with the induced metric $\langle, \rangle = dx_2^2 - dx_3^2$. The equation (1) is

$$2r^2 + r^2 F_{xx} - (r^2)' F_x = 0.$$

Again, we divide this equation by r^4 , obtaining

$$2 \int^x \frac{1}{r(u)^2} du + \frac{F_x}{r(x)^2} = c(y)$$

for a function $c = c(y)$ depending only the variable y . We deduce that there is a vector $\vec{a} = (a_2, a_3)$ such that $\alpha'(x) = r(x)^2 \vec{a}$. Integrating with respect to x ,

$$\alpha(x) = m(x) \vec{a}, \quad m(x) = \int^x r(u)^2 du.$$

If $\vec{a} = (0, 0)$, the centers of the circles are included in the x_1 -axis, M is a surface of revolution and the x_1 -line is the rotational axis. This case was discarded from the beginning. By taking $q = r^2$, we obtain an ordinary differential equation on q , namely,

$$(13) \quad 2\langle \vec{a}, \vec{a} \rangle q^3 - q^2 + q(2 + q'') = 0.$$

As in the previous section, we study the case that the radii coincide in all circles of the foliation.

Proposition 4.1 (Case of constant radii). *The only Riemann ZMC examples foliated by spacelike circles with constant radii contained in timelike planes are parametrized by*

$$(14) \quad X(x, v) = r^2 x(0, \vec{a}) + (x, r \sinh(v), r \cosh(v)),$$

where $x, v \in \mathbb{R}$ and $\vec{a} \in \mathbb{R}^2 \setminus \{0\}$. The surfaces are timelike except at the points satisfying $\langle \vec{a}, (\cosh(v), \sinh(v)) \rangle = 0$, which form a lightlike straight-line.

Proof. Let us observe that the vector \vec{a} may have any causal character. Now (13) is $2\langle \vec{a}, \vec{a} \rangle q^3 + 2q = 0$, so there are not solutions if $\langle \vec{a}, \vec{a} \rangle \geq 0$. If $\langle \vec{a}, \vec{a} \rangle < 0$, then $q^2 = -1/\langle \vec{a}, \vec{a} \rangle$ and the parametrization of the surface is (14): see figure 1, right. The first derivatives of X are $X_x = r^2(0, \vec{a}) + (1, 0, 0)$ and $X_v = r(0, \cosh(v), \sinh(v))$, obtaining $g_{11} = 0$ and $g_{12} = r^3 \langle \vec{a}, (\cosh(v), \sinh(v)) \rangle$. Thus the surface is timelike, except at the points satisfying $\langle \vec{a}, (\cosh(v), \sinh(v)) \rangle = 0$. These points form a lightlike straight-line. Indeed, this curve parametrizes as $x \mapsto X(x, v)$ so $X'(x) = r^2(0, \vec{a}) + (1, 0, 0)$ and $X'(x)$ is lightlike because $1 + r^4 \langle \vec{a}, \vec{a} \rangle = 0$. \square

From now, we suppose that the radii of the foliation circles are not constant.

Theorem 4.2. *Riemann ZMC examples foliated by spacelike circles contained in timelike planes form a one-parameter family of surfaces depending on a parameter $\lambda \in \mathbb{R}$ and parametrize as*

$$X(q, v) = (x(q), m(q) \vec{a}) + \sqrt{q}(0, \sinh(v), \cosh(v)),$$

where

$$(15) \quad x(q) = \frac{1}{2} \int^q \frac{du}{\sqrt{-\langle \vec{a}, \vec{a} \rangle u^3 + \lambda u^2 + u}}, \quad m(q) = \frac{1}{2} \int^q \frac{u du}{\sqrt{-\langle \vec{a}, \vec{a} \rangle u^3 + \lambda u^2 + u}}.$$

The curve formed by the centers of the circles is contained in a plane and the surface is symmetric about this plane.

Proof. Using (13),

$$\left(\frac{(q')^2}{q^2}\right)' = -4q' \left(\langle \vec{a}, \vec{a} \rangle + \frac{1}{q^2}\right)$$

and integrating with respect to x ,

$$(16) \quad \frac{(q')^2}{q^2} = 4 \left(-\langle \vec{a}, \vec{a} \rangle q + \frac{1}{q}\right) + 4\lambda,$$

for a constant $\lambda \in \mathbb{R}$. In particular, the right-hand side of (16) must be non-negative. Now we have

$$q' = \frac{dq}{dx} = 2\sqrt{-\langle \vec{a}, \vec{a} \rangle q^3 + \lambda q^2 + q}$$

and

$$\frac{dx}{dq} = \frac{1}{2} \frac{1}{\sqrt{-\langle \vec{a}, \vec{a} \rangle q^3 + \lambda q^2 + q}}.$$

As our new parameter is q , the center of the circle $M \cap \{x_1 = x\}$ is $(x, \alpha(x)) = (x, m(x)\vec{a})$. This proves that this curve is contained in the plane determined by the x_1 -axis and the vector \vec{a} (recall $\vec{a} \neq (0, 0)$). \square

We identify the vector $\vec{a} = (a_2, a_3)$ with $(0, \vec{a}) \in \mathbb{L}^3$ contained in the x_2x_3 -plane. After a rotation about the x_1 -axis and a dilation, we may suppose that the vector \vec{a} is $(1, 0)$, $(0, 1)$ or $(1, 1)$. We discuss the three cases.

4.1. Case $\vec{a} = (1, 0)$. The parametrization of the surface is

$$X(q, v) = (x(q), m(q), 0) + \sqrt{q}(0, \sinh(v), \cosh(v)),$$

where

$$(17) \quad x(q) = \frac{1}{2} \int_{q_0}^q \frac{du}{\sqrt{-u^3 + \lambda u^2 + u}}, \quad m(q) = \frac{1}{2} \int_{q_0}^q \frac{u du}{\sqrt{-u^3 + \lambda u^2 + u}}.$$

The sign of W is determined by the expression

$$(18) \quad q - q \sinh^2(v) + 2\sqrt{-q^2 + \lambda q + 1} \sinh(v) - \lambda.$$

We analyze when $-u^2 + \lambda u + 1$ is positive in order to determine the lower limit q_0 in the integrals (17). The roots of the function $-u^3 + \lambda u^2 + u = 0$ are 0 and

$$q_1 = \frac{\lambda - \sqrt{\lambda^2 + 4}}{2}, \quad q_2 = \frac{\lambda + \sqrt{\lambda^2 + 4}}{2},$$

with $q_1 < 0 < q_2$. Then $q \in (0, q_2)$ and the surface contains regions with the three causal character according to (18). It is immediate that the integral $x(q)$ in (17) for $q_0 = 0$ is finite. Let $c = x_1(q_2)$.

Proposition 4.3. *If $\vec{a} = (1, 0)$, then the surface lies contained in the vertical slab $0 < x_1 < c$ and the surface converges to one point if $x_1 \rightarrow 0$.*

4.2. **Case $\vec{a} = (0, 1)$.** The parametrization of the surface is

$$X(q, v) = (x(q), 0, m(q)) + \sqrt{q}(0, \sinh(v), \cosh(v)),$$

where

$$(19) \quad x(q) = \frac{1}{2} \int_{q_0}^q \frac{du}{\sqrt{u^3 + \lambda u^2 + u}}, \quad m(q) = \frac{1}{2} \int_{q_0}^q \frac{u du}{\sqrt{u^3 + \lambda u^2 + u}}.$$

The roots of $u^3 + \lambda u^2 + u$ are 0 and

$$q_1 = \frac{-\lambda - \sqrt{\lambda^2 - 4}}{2}, \quad q_2 = \frac{-\lambda + \sqrt{\lambda^2 - 4}}{2}.$$

The spacelike condition $W > 0$ is equivalent to

$$(20) \quad (1 + \cosh^2(v))q + 2 \cosh(v) \sqrt{q^2 + \lambda q + 1} + \lambda < 0.$$

Therefore λ is negative. As in the previous section, the integrals (19) can be explicitly integrated if $\lambda = \pm 2$.

Proposition 4.4. *If $\vec{a} = (0, 1)$, the special cases $\lambda = \pm 2$ provide parametrizations of Riemann ZMC examples in terms of elementary functions, namely,*

(1) *Case $\lambda = 2$. The surface is*

$$X(r, v) = (\arctan(r), 0, r - \arctan(r)) + r(0, \sinh(v), \cosh(v)),$$

where $r > 0$, $v \in \mathbb{R}$. The surface is timelike converging to a point if $r \rightarrow 0$. See figure 8, left.

(2) *Case $\lambda = -2$. The surface is*

$$X(r, v) = \left(\log \left| \frac{1-r}{r+1} \right|, 0, r + \frac{1}{2} \log \left| \frac{1-r}{r+1} \right| \right) + r(0, \sinh(v), \cosh(v)),$$

where $r > 0$, $r \neq 1$ and $v \in \mathbb{R}$. If $0 < r < 1$ the surface contains regions of spacelike and timelike points, but if $r > 1$, the surface is timelike. See figure 8, right.

If we now consider the integrals (19), the discussion is similar as in Section 3. We need to distinguish on the parameter λ according if $\lambda > 2$, $\lambda^2 - 4 < 0$ and $\lambda < -2$.

Proposition 4.5. *If $\vec{a} = (0, 1)$ and $\lambda^2 \neq 4$, then the surface is included in a vertical slab $0 < x_1 < c$. If $\lambda < -2$ and $q \in (q_2, \infty)$, then the surface converges to a hyperbola if $q \rightarrow q_2$. In the rest of cases, the surface converges to a point as $x_1 \rightarrow 0$.*

Proof. (1) Case $\lambda > 2$. Then $q_1 < q_2 < 0$ and we take $q_0 = 0$ to be the lower limit in (19). From (20) we deduce that the surface is timelike. It is also immediate that at $q = 0$, the surface converges to a point and if $q \rightarrow \infty$, the integral for $x(q)$ is finite, that is, $\lim_{q \rightarrow \infty} x(q) < \infty$.

(2) Case $\lambda^2 < 4$. Since the polynomial $u^2 + \lambda u + 1$ has not real roots, $u^3 + \lambda u^2 + u$ is positive for $u > 0$. We infer that we can take $q_0 = 0$ to be the lower limit in (19). Again, we have $\lim_{q \rightarrow \infty} x(q) < \infty$.

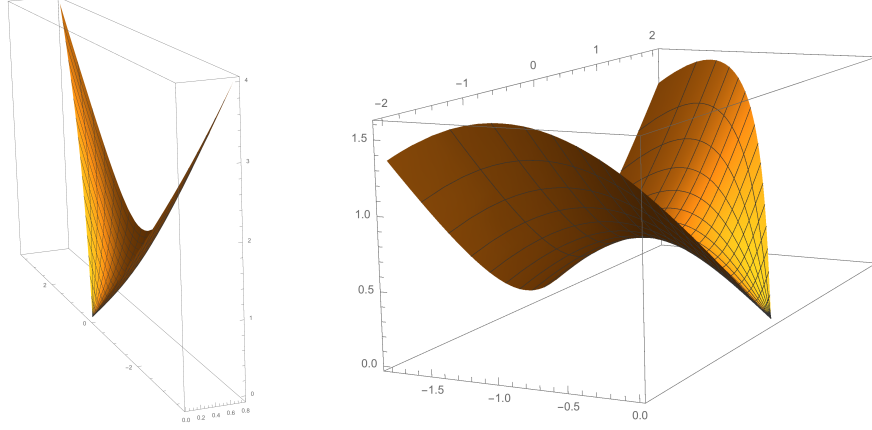


FIGURE 8. Surfaces of Proposition 4.4: Riemann ZMC surfaces in \mathbb{L}^3 foliated by spacelike hyperbolas in timelike planes

- (3) Case $\lambda < -2$. Then $0 < q_1 < q_2$, hence that $u^3 + \lambda u^2 + u > 0$ in $(0, q_1) \cup (q_2, \infty)$. If $q \in (0, q_1)$, let $q_0 = 0$ and if $q \in (q_2, \infty)$, the lower integration limit is $q_0 = q_2$. Moreover, the improper integral $x(\infty)$ is finite.

□

4.3. **Case $\vec{a} = (1, 1)$.** In this case $\langle \vec{a}, \vec{a} \rangle = 0$. The parametrization of the surface is

$$(21) \quad X(r, v) = (x(r), m(r), m(r)) + r(0, \sinh(v), \cosh(v)),$$

where if $q = r^2$, then

$$(22) \quad x(q) = \frac{1}{2} \int_{q_0}^q \frac{du}{\sqrt{\lambda u^2 + u}}, \quad m(q) = \frac{1}{2} \int_{q_0}^q \frac{u du}{\sqrt{\lambda u^2 + u}}$$

and $\lambda q + 1 > 0$. The computation of W yields

$$(23) \quad W = -\frac{qe^{-2v}}{4(1 + \lambda q)} (\lambda e^{2v} + 2e^v \sqrt{1 + \lambda q} + q).$$

In particular, if the surface is spacelike if $\lambda < 0$, and the surface if timelike if $\lambda > 0$. This case $\vec{a} = (1, 1)$ provides explicit parametrizations by integrating (22).

Proposition 4.6. *Riemann ZMC examples foliated by spacelike circles contained in timelike planes corresponding to the choice $\vec{a} = (1, 1)$ have the following explicit parametrizations (21), where the functions $x(r)$ and $m(r)$ are the following.*

- (1) Case $\lambda = 0$. Here $x(r) = r$ and $m(r) = r^3/3$. The surface is timelike and converges to a point as $r \rightarrow 0$.
(2) Case $\lambda > 0$. The surface converges to a point as $r \rightarrow 0$. Here

$$x(r) = \frac{1}{\sqrt{\lambda}} \operatorname{arsinh}(\sqrt{\lambda}r), \quad m(r) = \frac{1}{2\lambda^{3/2}} \left(r\sqrt{\lambda r^2 + 1} - \operatorname{arsinh}(\sqrt{\lambda}r) \right).$$

(3) *Case $\lambda < 0$. Here*

$$x(r) = \frac{1}{\sqrt{-\lambda}} \arcsin(\sqrt{-\lambda}r), \quad m(r) = \frac{1}{2\lambda} r \sqrt{\lambda r^2 + 1} + \frac{1}{4(-\lambda)^{3/2}} \arcsin(-2\lambda r^2 - 1).$$

Depending on the values v in (23), the surface contains spacelike, lightlike and timelike regions. The surface is included in the slab $0 < x_1 < \pi/(2\sqrt{-\lambda})$ and converges to a point as $r \rightarrow 0$.

5. RIEMANN ZMC EXAMPLES FOLIATED BY TIMELIKE CIRCLES CONTAINED IN TIMELIKE PLANES

In this section, we study ZMC surfaces in \mathbb{L}^3 foliated by timelike circles contained in parallel timelike planes. Recall that any curve contained in a spacelike surface is a spacelike curve. Thus all surfaces of this section will be timelike and, perhaps, can be extended to lightlike regions. Again, we will assume that the planes of the foliation are parallel to the plane of equation $x_1 = 0$. We know that a timelike circle in the plane $x_1 = 0$ is parametrized by $\beta(s) = p + r(0, \cosh(s), \sinh(s))$. Let M be a such surface and take the x_1 -coordinate as the parameter of the circles of the foliation. Then M is included in $F^{-1}(\{0\})$, where $F : (a, b) \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is

$$F(x, y) = -(x_2 - \alpha_2(x))^2 + (x_3 - \alpha_3(x))^2 + r(x)^2.$$

Again, let $y = (x_2, x_3)$, $\alpha = (\alpha_2, \alpha_3)$ and $\langle, \rangle = dx_2^2 - dx_3^2$ the induced metric in the x_2x_3 -plane. The zero mean curvature equation (1) is

$$F_x^2 + r^2(2 - F_{xx}) - 2F_x \langle y - \alpha, \alpha' \rangle = 0.$$

If we repeat the same steps of the previous section, there is a vector $\vec{a} = (a_2, a_3)$, which we identify in the x_2x_3 -plane as $\vec{a} = (0, a_2, a_3) \in \mathbb{L}^3$, such that

$$\alpha(x) = m(x)\vec{a}, \quad m(x) = \int^x r(u)^2 du$$

and

$$(24) \quad 2\langle \vec{a}, \vec{a} \rangle r^6 + (r^2)'^2 + r^2(2 - (r^2)'') = 0.$$

When the radii of the circles are constant, the above equation is $\langle \vec{a}, \vec{a} \rangle r^4 + 1 = 0$. In particular, $\langle \vec{a}, \vec{a} \rangle < 0$, so the vector \vec{a} is timelike.

Proposition 5.1 (Case of constant radii). *The only Riemann ZMC examples foliated by timelike circles with constant radii contained in timelike planes are parametrized by*

$$X(x, v) = x(1, r^2\vec{a}) + r(0, \cosh(v), \sinh(v)),$$

where $x, v \in \mathbb{R}$ and $\vec{a} \in \mathbb{R}^2 \setminus \{0\}$. The surfaces are timelike except at the points $v = 0$, which form a straight-line of lightlike points. See figure 1, right.

We assume that the radii of the foliation circles are not constant.

Theorem 5.2. *Riemann ZCM examples foliated by timelike circles form a one-parameter family of surfaces depending on a parameter $\lambda \in \mathbb{R}$ and parametrize as*

$$(25) \quad X(q, v) = (x(q), m(q)\vec{a}) + \sqrt{q}(0, \cosh(v), \sinh(v)),$$

where \vec{a} is any timelike vector in the x_2x_3 -plane and

$$(26) \quad x(q) = \frac{1}{2} \int^q \frac{1}{\sqrt{\langle \vec{a}, \vec{a} \rangle u^3 + \lambda u^2 - u}} du, \quad m(q) = \frac{1}{2} \int^q \frac{u}{\sqrt{\langle \vec{a}, \vec{a} \rangle u^3 + \lambda u^2 - u}} du.$$

The curve formed by the centers of the foliation is contained in a plane and the surface is symmetric about this plane.

Proof. Let $q = r^2$. Then (24) is $2\langle \vec{a}, \vec{a} \rangle q^3 + (q')^2 + q(2 - q'') = 0$. Using this equation, we obtain

$$\left(\frac{(q')^2}{q^2} \right)' = 4q' \left(\langle \vec{a}, \vec{a} \rangle + \frac{1}{q^2} \right),$$

and integrating with respect to x ,

$$\frac{(q')^2}{q^2} = 4 \left(\langle \vec{a}, \vec{a} \rangle q - \frac{1}{q} \right) + 4\lambda, \quad \lambda \in \mathbb{R}.$$

In particular, the right-hand side of this equation must be non-negative. Now we have

$$q' = \frac{dq}{dx} = 2\sqrt{\langle \vec{a}, \vec{a} \rangle q^3 - q + \lambda q^2}$$

and

$$\frac{dx}{dq} = \frac{1}{2} \frac{1}{\sqrt{\langle \vec{a}, \vec{a} \rangle q^3 + \lambda q^2 - q}}.$$

Using q as a new parameter is q , it follows (26), hence (25). \square

Again the arguments are similar with Section 4. The family of timelike Riemann ZMC examples foliated by timelike circles contained in timelike planes depends on a real parameter λ . After a rigid motion and a dilation, we have three cases according the value of the vector \vec{a} .

5.1. **Case $\vec{a} = (1, 0)$.** The parametrization of the surface is

$$(27) \quad X(q, v) = (x(q), m(q), 0) + \sqrt{q}(0, \cosh(v), \sinh(v)),$$

where

$$(28) \quad x(q) = \frac{1}{2} \int_{q_0}^q \frac{du}{\sqrt{u^3 + \lambda u^2 - u}}, \quad m(q) = \frac{1}{2} \int_{q_0}^q \frac{u du}{\sqrt{u^3 + \lambda u^2 - u}}.$$

Then

$$(29) \quad W = -\frac{q \left((1 + \cosh^2(v))q - 2\sqrt{q^2 + \lambda q - 1} \cosh(v) + \lambda \right)}{4(q^2 + \lambda q - 1)}.$$

The roots of $u^3 + \lambda u^2 - u = 0$ are 0 and

$$q_1 = \frac{-\lambda - \sqrt{\lambda^2 + 4}}{2}, \quad q_2 = \frac{-\lambda + \sqrt{\lambda^2 + 4}}{2}.$$

Since $q_1 < 0 < q_2$, then $q \in (q_2, \infty)$. Hence we can take the lower limit in (28) as q_2 . From (29), we have $W < 0$ and the surface is timelike. On the other hand, the improper integral $x(q)$ in (28) with $q = \infty$ is convergent, that is, $\lim_{q \rightarrow \infty} x(q) = c < \infty$, and the same occurs if $q \rightarrow q_2$. Accordingly, in the limit $x_1(q_2)$, the surface is a hyperbola.

Proposition 5.3. *If $\vec{a} = (1, 0)$, the surface is timelike and parametrized as (27). Here the lower integration limit is q_2 and the functions $x(q)$ and $m(q)$ are determined in (28). The surface is contained in a vertical slab $0 \leq x_1 < c$ and the surface converges to a hyperbola as $x_1 \rightarrow 0$.*

5.2. **Case $\vec{a} = (0, 1)$.** The parametrization of the surface is

$$(30) \quad X(q, v) = (x(q), 0, m(q)) + \sqrt{q}(0, \cosh(v), \sinh(v)),$$

where

$$(31) \quad x(q) = \frac{1}{2} \int_{q_0}^q \frac{du}{\sqrt{-u^3 + \lambda u^2 - u}}, \quad m(q) = \frac{1}{2} \int_{q_0}^q \frac{u du}{\sqrt{-u^3 + \lambda u^2 - u}}.$$

The roots of the equation $-u^3 + \lambda u^2 - u$ are 0 and

$$q_1 = \frac{\lambda - \sqrt{\lambda^2 - 4}}{2}, \quad q_2 = \frac{\lambda + \sqrt{\lambda^2 - 4}}{2}.$$

Then

$$(32) \quad W = -\frac{q \left((-1 + \sinh^2(v))q - 2\sqrt{-q^2 + \lambda q - 1} \sinh(v) + \lambda \right)}{4(-q^2 + \lambda q - 1)}.$$

Here the case $\lambda^2 = 4$ must be discarded because $-u^3 + \lambda u^2 - u \leq 0$. If $\lambda^2 \neq 4$, we distinguish the cases $\lambda > 2$, $\lambda^2 - 4 < 0$ and $\lambda < -2$. However, for the cases $\lambda^2 - 4 < 0$ and $\lambda < -2$, the polynomial $-u^3 + \lambda u^2 - u$ is negative when $u > 0$. Consequently, the only possibility is that $\lambda > 2$. In such a case, $0 < q_1 < q_2$ and $-u^3 + \lambda u^2 - u > 0$ if $q \in (q_1, q_2)$. We take q_1 to be the lower limit in (31). From (32), we deduce that the surface is timelike.

Proposition 5.4. *If $\vec{a} = (0, 1)$, then $\lambda > 2$. The surface is parametrized as (30), where the lower integration limit is q_1 and the functions $x(q)$ and $m(q)$ are given by (31). The surface is contained in a vertical slab $0 < x_1 < c$, with $c = x(q_2)$, and in the limits $x_1 = 0$ and $x_1 = c$, the surface is formed by two hyperbolas.*

5.3. **Case $\vec{a} = (1, 1)$.** The integrals in (26) are now

$$x(q) = \frac{1}{2} \int_{q_0}^q \frac{du}{\sqrt{\lambda u^2 - u}}, \quad m(q) = \frac{1}{2} \int_{q_0}^q \frac{u du}{\sqrt{\lambda u^2 - u}}.$$

In particular, λ must be a positive number. From a direct integration, we prove the following proposition.

Proposition 5.5. *Riemann ZMC examples foliated by timelike circles contained in timelike planes corresponding to the case $\vec{a} = (1, 1)$ parametrize as*

$$X(r, v) = (x(r), m(r), m(r)) + r(0, \cosh(v), \sinh(v)),$$

where $r > 1/\sqrt{\lambda}$, $v \in \mathbb{R}$, $\lambda > 0$ and

$$x(r) = \frac{1}{\sqrt{\lambda}} \operatorname{arcosh}(\sqrt{\lambda}r), \quad m(r) = \frac{\sqrt{\lambda}r\sqrt{\lambda r^2 - 1} + \operatorname{arsinh}(\sqrt{\lambda r^2 - 1})}{2\lambda^{3/2}}.$$

The surface is contained in the halfspace $x_1 > 0$ and converges to a hyperbola as $x_1 \rightarrow 0$.

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