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# Isoperimetric inequalities in cylinders with density[☆]

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ABSTRACT

Given a compact Riemannian manifold with density $M$ without boundary and the real line $\mathbb{R}$ with constant density, we prove that isoperimetric regions of large volume in $M \times \mathbb{R}$ with the product density are slabs of the form $M \times [a, b]$. We previously prove, as a necessary step, the existence of isoperimetric regions in any manifold of density where a subgroup of the group of transformations preserving weighted perimeter and volume acts cocompactly.



## 1. Introduction and preliminaries

In recent years, isoperimetric problems have been considered in manifolds with density. One of the most interesting spaces of this type is the Gauss space, the Euclidean space $\mathbb{R}^n$ with the Gaussian density $\Psi(x) := \exp(-\pi|x|^2)$. Borell [6] and Sudakov and Tirel'son [29] independently proved in 1974 and 1975 that half-spaces minimize perimeter under a volume constraint for this density. A new proof was given in 1983 by Ehrhard [10] using symmetrization. In 1997 Bobkov [4] proved a functional version of this isoperimetric inequality, later extended to the sphere and used to prove isoperimetric estimates for the unit cube by Barthe and Maurey [2]. Following [4], Bobkov and Houdré [5] considered "unimodal densities"with finite total measure on the real line. These authors explicitly computed the isoperimetric profile for such densities and found some of the isoperimetric solutions. Gromov [14,15] studied manifolds with density as "metric measure spaces" and mentioned the natural generalization of mean curvature obtained by the first variation of weighted area. Bakry and Ledoux [1] and Bayle [3] proved generalizations of the Lévy–Gromov isoperimetric inequality and other geometric comparisons depending on a lower bound on the generalized Ricci curvature of the manifold. Isoperimetric comparison results in manifolds with density were considered by Maurmann and Morgan [17]. Existence of isoperimetric sets in $\mathbb{R}^n$ with density under various hypotheses

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on the growth of the density were proven by Morgan and Pratelli [23] and Milman [18], see also De Philippis, Franzina and Pratelli [8]. For regularity of isoperimetric regions with density see Sect. 3.10 in paper of Morgan [19] and see also Pratelli and Saracco [24]. Boundedness of isoperimetric regions was studied by Cinti and Pratelli [7] and Pratelli and Saracco [25]. Symmetrization techniques in manifolds with density were developed by Ros [27] and Morgan et al. [22].

For nice surveys on manifolds with density the reader is referred to [3,20,21] and the references therein.

In this paper, $(M, g, \Psi)$ will denote a manifold with density without boundary, where $g$ is a Riemannian metric on $M$ and $\Psi : M \to \mathbb{R}$ is a smooth function. We define the weighted *volume* of a set by

$$\mathrm{vol}_\Psi(E) := \int_E e^\Psi dM, \tag{1.1}$$

where $dM$ is the Riemannian volume element on $(M, g)$. The weighted *area* of a smooth hypersurface $\Sigma$ is defined by

$$\mathrm{area}_\Psi(\Sigma) := \int_\Sigma e^\Psi d\Sigma, \tag{1.2}$$

where $d\Sigma$ is the Riemannian area element on $\Sigma$.

If $E \subset M$, we define the weighted *perimeter* of $E$ in the manifold with density $(M, g, \Psi)$ by

$$P_\Psi(E) := \sup\left\{\int_E \mathrm{div}_\Psi X\, dM : X \in \mathfrak{X}_0^\infty(M),\ |X| \leqslant 1\right\}, \tag{1.3}$$

where $\mathfrak{X}_0^\infty(M)$ is the set of smooth vector fields on $M$ with compact support and

$$\mathrm{div}_\Psi(X) = \mathrm{div}(e^\Psi X), \tag{1.4}$$

and div is the Riemannian divergence in $(M, g)$. If $E$ has smooth boundary $\Sigma$, then $P_\Psi(E) = \mathrm{area}_\Psi(\Sigma)$, see [23].

Given a manifold with density, we shall denote by $\mathrm{Isom}(M, g, \Psi)$ the group of isometries of $(M, g)$ preserving the function $\Psi$ (i.e., maps $f : M \to M$ such that $\Psi \circ f = \Psi$). Such isometries preserve the weighted area and volume.

The isoperimetric profile of $(M, g, \Psi)$ is the function $I : [0, +\infty) \to \mathbb{R}^+$ defined by

$$I(v) = \inf\{P_\Psi(E) : \mathrm{vol}_\Psi(E) = v\}. \tag{1.5}$$

A set $E \subset M$ of finite weighted perimeter is isoperimetric if $P_\Psi(E) = I(\mathrm{vol}_\Psi(E))$. This means that $E$ minimizes the weighted perimeter under a weighted volume constraint. Regularity of isoperimetric sets was considered by Morgan and Pratelli [23].

Given a manifold with density $(M, g, \Psi)$, we shall consider the cylinders with density $(M \times \mathbb{R}^k, g \times g_0, \Psi \times 1)$, where $\mathbb{R}^k$ is $k$-dimensional Euclidean space with its standard Riemannian metric $g_0$, and $(\Psi \times 1)(p, x) = \Psi(p)$ for every $(p, x) \in M \times \mathbb{R}^k$. Given $v \in \mathbb{R}^k$, we define $t_v : M \times \mathbb{R}^k \to M \times \mathbb{R}^k$ by $t_v(p, x) := (p, x + v)$ for any $(p, x) \in M \times \mathbb{R}^k$. The set $G := \{t_v : v \in \mathbb{R}^k\}$ is contained in $\mathrm{Isom}(M \times \mathbb{R}^k, g \times g_0, \Psi \times 1)$. In case $M$ is a compact manifold, the quotient of $M \times \mathbb{R}^k$ by $\mathrm{Isom}(M \times \mathbb{R}^k, g \times g_0, \Psi \times 1)$ is the compact base $M$ of the product. We focus in this paper in the case $k = 1$.

The aim of this paper is to prove that isoperimetric sets in $(M \times \mathbb{R}, g \times g_0, \Psi \times 1)$ are slabs of the form $M \times [a, b]$, where $a, b \in \mathbb{R}$, $a < b$. This result is proven in Theorem 3.3 in Section 3. As a necessary previous step in our proof we must show existence of isoperimetric regions in manifolds with density such that the action of $\mathrm{Isom}(M, g, \Psi)$ is cocompact, that is, the quotient $(M, g, \Psi)/\mathrm{Isom}(M, g, \Psi)$ is compact, like in the case of the cylinders considered in Section 3. The proof of existence is based on Galli and Ritoré's in contact sub-Riemannian manifolds, see [12]. Since this proof has now become standard, we check in Section 2 that the main ingredients are available: a relative isoperimetric inequality for uniform radii, see Theorem 2.3; the doubling property, see Theorem 2.4; and a deformation result for sets of finite perimeter, see Theorem 2.5. In Theorem 3.3 we characterize the isoperimetric regions of *large volume* in a cylinder with density $M \times \mathbb{R}$, where $M$ is a compact Riemannian manifold with density, the real line $\mathbb{R}$ is endowed with a constant density, and the product with the product density. In the non-weighted Riemannian case Duzaar and Steffen [9] proved that in $M \times \mathbb{R}$ isoperimetric sets of large volume are of the form $M \times [a, b]$, where $a, b \in \mathbb{R}$. For higher dimensional Euclidean factors the problem was considered in [26], where the authors proved that the isoperimetric solutions of large volume in the Riemannian product $M \times \mathbb{R}^k$ are of the form $M \times B(x, r)$, where $B(x, r)$ is an Euclidean ball, see also [13].

For more results about variational problems in cylinders the reader is referred to [11] and the references in [28].

## 2. Existence of isoperimetric regions in $M$

In this section we prove the existence of isoperimetric sets, for any volume, in a manifold with density $(M, g, \Psi)$ such that $\operatorname{Isom}(M, g, \Psi)$ acts cocompactly. The scheme of proof devised by Galli and Ritoré in [12] applies to our situation, provided we are able to show that

- There exists $r_0 > 0$ such that a relative isoperimetric inequality holds in all balls $B(p, r)$, with $p \in M$, $0 < r \leqslant r_0$, with a uniform constant.
- The manifold is doubling. This means the existence of $r_0 > 0$ and a uniform constant $C_D > 0$ such that $\operatorname{vol}_\Psi(B(p, 2r)) \leqslant C_D \operatorname{vol}_\Psi(B(p, r))$ for all $p \in M$ and $0 < r \leqslant r_0$.
- A deformation result for finite perimeter sets, see Theorem 2.5, holds in $(M, g, \Psi)$.

Assuming these results hold in $(M, g, \Psi)$, and using the well-known techniques in [12] we have the following result.

**Theorem 2.1.** *In a manifold with density $(M, g, \Psi)$ such that $\operatorname{Isom}(M, g, \Psi)$ acts cocompactly, isoperimetric sets exist for any given volume.*

To prove the required ingredients needed for Theorem 2.1 we start with a preliminary result. We recall that the *convexity radius* $\operatorname{conv}(K)$ of a subset $K$ of a Riemannian manifold M is the infimum of positive numbers $r$ such that the geodesic open ball $B(p, r)$ is convex for every $p \in K$. We call $d$ the Riemannian distance in $(M, g)$.

**Lemma 2.2.** *Let $(M, g)$ be a Riemannian manifold, and $K \subset M$ a compact subset. Let $r_0 = \operatorname{conv}(K)$. Then there exist functions $\lambda$, $\Lambda : [0, r_0] \to \mathbb{R}$ such that $1 + \lambda$, $1 + \Lambda$ are positive, $\lim_{r\to 0} \lambda(r) = \lim_{r\to 0} \Lambda(r) = 0$, and*

$$(1 + \lambda(r))\,|x - y| \leqslant d(\exp_p(x), \exp_p(y)) \leqslant (1 + \Lambda(r))\,|x - y|, \tag{2.1}$$

*for any $p \in K$ and $x, y \in B(0, r_0) \subset T_pM$.*

**Proof.** Given $p \in K$, consider a compact coordinate neighborhood $U$ around $p$ with a global orthonormal basis defined on $U$. Given $q \in U$, let $g^q_{ij}$ be the components of the Riemann tensor in the coordinate neighborhood defined by the exponential map $\exp_q : B(0, r_0) \subset T_qM \to B(q, r_0)$ and the global orthonormal basis. The functions $g^q_{ij}$ depend smoothly on $q$. We define

$$\begin{aligned}
\alpha_U(r) &:= \min_{q\in U}\Big\{ \Big( \sum_{i,j=1}^n g^q_{ij}(z)\, v_i v_j \Big)^{\frac12} : |z| \leqslant r < r_0, \sum_{i=1}^n v_i^2 = 1 \Big\},\\
\beta_U(r) &:= \max_{q\in U}\Big\{ \Big( \sum_{i,j=1}^n g^q_{ij}(z)\, v_i v_j \Big)^{\frac12} : |z| \leqslant r < r_0, \sum_{i=1}^n v_i^2 = 1 \Big\}.
\end{aligned} \tag{2.2}$$

It is easy to check that $\alpha_U(r)$ is decreasing, $\beta_U(r)$ is increasing, and that

$$\lim_{r\to 0}\alpha_U(r)=\lim_{r\to 0}\beta_U(r)=1.$$

Given $q\in U$, we take $x,y\in B(0,r_0)$. To compute the distance $d$ between the points $\exp_q(x)$, $\exp_q(y)$, it is enough to consider curves inside the convex ball $B(0,r)$. Let $\gamma: I\to B(0,r_0)$ be a curve joining $x$ and $y$. Then

$$d(\exp_q(x),\exp_q(y))=L(\exp_q\circ\gamma)=\int_I\left(\sum_{i,j=1}^n g_{ij}^q(\gamma(t))\,\gamma_i'(t)\gamma_j'(t)\right)^{\frac{1}{2}}dt. \tag{2.3}$$

Observe that

$$\alpha_U(r)\left(\sum_{i=1}^n\gamma_i'(t)^2\right)^{1/2}\leqslant\sum_{i,j=1}^n g_{ij}^q(\gamma(t))\,\gamma_i'(t)\gamma_j'(t)\leqslant\beta_U(r)\left(\sum_{i=1}^n\gamma_i'(t)^2\right)^{1/2}. \tag{2.4}$$

The left quantity is larger than or equal to $\alpha_U(r)\,|x-y|$. Since $\gamma$ is an arbitrary curve joining $x$ and $y$, this implies $\alpha_U(r)\,|x-y|\leqslant d(\exp_q(x),\exp_q(y))$. On the other hand,

$$d(\exp_q(x),\exp_q(y))\leqslant L(\exp_q\circ\gamma)\leqslant\beta_U(r)\,|x-y|,$$

so we have

$$\alpha_U(r)\leqslant\frac{d(\exp_q(x),\exp_q(y))}{|x-y|}\leqslant\beta_U(r). \tag{2.5}$$

The result follows by covering the compact set $K$ by a finite number of coordinate neighborhoods $U$, and taking $\alpha(r)$ as the minimum of the $\alpha_U(r)$ and $\beta(r)$ as the maximum of the $\beta_U(r)$. Setting $\lambda(r)=\alpha(r)-1$, $\Lambda(r)=\beta(r)-1$ the result follows. $\square$

Using Lemma 2.2 we obtain as corollaries the existence of a uniform relative isoperimetric inequality and the existence of a doubling constant.

**Theorem 2.3** (*Relative Isoperimetric Inequality in* $(M,g,\Psi)$). *Let $(M,g,\Psi)$ be an $n$-dimensional manifold with density and $K\subset M$ a compact subset. Let $r_0>0$ be the radius obtained in Lemma 2.2. Then for all $p\in K$ and $E\subset B(p,r)$ with $0<r\leqslant r_0$, there exists a positive constant $C>0$ not depending on $p$, such that*

$$P_\Psi(E,B(p,r))\geqslant C\cdot\min\left\{\mathrm{vol}_\Psi(E),\,\mathrm{vol}_\Psi(B(p,r)\setminus E)\right\}^{(n-1)/n}. \tag{2.6}$$

*In particular, (2.6) holds in the whole manifold if* $\mathrm{Isom}(M,g,\Psi)$ *acts cocompactly on $M$, see Lemma 3.5 in [12].*

**Proof.** By Lemma 2.2, for $0<r\leqslant r_0$, the exponential map $f=\exp_p: B(0,r)\to B(p,r)$ is a diffeomorphism. Let $F=f^{-1}(E)$. Then

$$\mathrm{vol}_\Psi(f(F))=\mathrm{vol}_\Psi(E)=\int_E e^\Psi dH^n, \tag{2.7}$$

where $H^n$ is the $n$-dimensional Hausdorff measure in $(M,g)$. Consider positive function constants $a,b>0$ so that $a\leqslant e^\Psi\leqslant b$ in $\overline{B}(p,r_0)$ for all $p\in K$. So we have

$$a\cdot H^n(E)=a\int_E dH^n\leqslant\int_E e^\Psi dH^n\leqslant b\int_E dH^n=b\cdot H^n(E). \tag{2.8}$$

As $f$ is Lipschitz, Lemma 2.2 in dimension $n$ implies

$$a(1+\lambda(r))^n H_0^n(F)\leqslant a\cdot H^n(E)\leqslant\mathrm{vol}_\Psi(E)\leqslant b\cdot H^n(E)\leqslant b(1+\Lambda(r))^n H_0^n(F), \tag{2.9}$$

where $H_0^n$ is the $n$-dimensional Hausdorff measure with respect to the Euclidean metric.

On the other hand, by § 2 in [23],

$$P_{\Psi}(E, B(r,p)) = \int_{\partial_\star E \cap B(p,r)} e^{\Psi} dH^{n-1},$$

where $\partial_\star E$ is the reduced boundary of $E$. And therefore,

$$\begin{aligned} P_{\Psi}(E, B(p,r)) &\leqslant bH^{n-1}(\partial_\star E \cap B(p,r)) \\ &\leqslant b(1+\Lambda(r))^{n-1} H_0^{n-1}(\partial_\star F \cap B(0,r)) \\ &= b(1+\Lambda(r))^{n-1} P_0(F, B(0,r)), \end{aligned}$$

where $P_0$ is the Euclidean perimeter. By a similar computation we obtain

$$P_{\Psi}(E, B(p,r)) \geqslant a(1+\lambda(r))^{n-1} P_0(F, B(0,r)).$$

Then

$$a(1+\lambda(r))^{n-1} P_0(F, B(0,r)) \leqslant P_{\Psi}(E, B(p,r)) \leqslant b(1+\Lambda(r))^{n-1} P_0(F, B(0,r)).$$

Observe that

$$\begin{aligned} P_{\Psi}(E,B(p,r)) &\geqslant a(1+\lambda(r))^{n-1} P_0(F, B(0,r)) \\ &\geqslant a(1+\lambda(r))^{n-1} \cdot C_0 \cdot \min\left\{H^n(F),\, H^n(B(0,r) \setminus F)\right\}^{(n-1)/n} \\ &\geqslant a(1+\lambda(r))^{n-1} \cdot C_0 \cdot \min\left\{\frac{\mathrm{vol}_{\Psi}(E)}{(b(1+\Lambda(r)))^n}, \frac{\mathrm{vol}_{\Psi}(B(p,r) \setminus E)}{(b(1+\Lambda(r)))^n}\right\}^{\frac{n-1}{n}}, \end{aligned}$$

where $C_0$ is the constant in the relative isoperimetric (Poincaré) inequality in Euclidean balls. Thus,

$$P_{\Psi}(E, B(p,r)) \geqslant \frac{a(1+\lambda(r))^{n-1}}{b(1+\Lambda(r))^{n-1}} \cdot C_0 \cdot \min\left\{\mathrm{vol}_{\Psi}(E), \mathrm{vol}_{\Psi}(B(p,r) \setminus E)\right\}^{\frac{n-1}{n}}. \quad \square$$

In the following result we prove that $(M, g, \Psi)$ is a doubling metric space.

**Theorem 2.4** (*Doubling Property*). *Let $(M, g, \Psi)$ be an $n$-dimensional manifold with density and $K \subset M$ a compact subset. Let $r_0 > 0$ be the radius obtained in Lemma 2.2. Then there exists a constant $C_D > 0$, only depending on $K$, such that for all $x_0 \in K$ and $0 < r \leqslant r_0/2$ we have*

$$\mathrm{vol}_{\Psi}(B(x_0, 2r)) \leqslant C_D \mathrm{vol}_{\Psi}(B(x_0, r)). \tag{2.10}$$

*In particular, (2.10) holds in the whole manifold if* $\mathrm{Isom}(M, g, \Psi)$ *acts cocompactly on $M$.*

**Proof.** From Eq. (2.9) we know that, for $r_0$, $0 < r < r_0$, we must have

$$AH_0^n(F) \leqslant \mathrm{vol}_{\Psi}(E) \leqslant BH_0^n(F),$$

where $A = a(1+\lambda(r))^n$, $B = b(1+\Lambda(r))^n$ and $\exp_{x_0}(F) = E$ for all $x_0 \in M$.

In particular,

$$AH_0^n(B(0,2r)) \leqslant \mathrm{vol}_{\Psi}(B(x_0,2r)) \leqslant BH_0^n(B(0,2r))$$

and

$$AH_0^n(B(0,r)) \leqslant \mathrm{vol}_{\Psi}(B(x_0,r)) \leqslant BH_0^n(B(0,r)).$$

Thus

$$\frac{\mathrm{vol}_{\Psi}(B(x_0,2r))}{\mathrm{vol}_{\Psi}(B(x_0,r))} \leqslant \frac{BH_0^n(B(0,2r))}{AH_0^n(B(0,r))} \leqslant \frac{B}{A}\frac{H_0^n(B(0,1)) \cdot (2r)^n}{H_0^n(B(0,1)) \cdot (r)^n} = 2^n \frac{B}{A}.$$

Therefore, $\mathrm{vol}_{\Psi}(B(x_0,2r)) \leqslant C_D \mathrm{vol}_{\Psi}(B(x_0,r))$ with $C_D = 2^n \sup\limits_K \frac{B}{A}$. Note that $C_D$ is finite by Lemma 2.2 and strictly positive by Eq. (2.1). $\square$

**Theorem 2.5** (*Deformation of Finite Perimeter Sets*). *Let $E \subset M$ be a set of locally finite weighted perimeter. Assume that $P_\Psi(E, B(p,r)) > 0$ for some $p \in M$ and $r > 0$. Then there exists a deformation $\{E_t\}_{t\in(-\delta,\delta)}$ of $E$, with $E_0 = E$, by sets of locally finite perimeter, and a constant $C = C(p,r,\delta)$ such that*

(1) $\mathrm{vol}_\Psi(E_t) = \mathrm{vol}_\Psi(E) + t$,
(2) $E \triangle E_t \subset B(p,r)$,
(3) $|P_\Psi(E) - P_\Psi(E_t)| \leqslant C\,|\mathrm{vol}_\Psi(E) - \mathrm{vol}_\Psi(E_t)| \leqslant C\,|\mathrm{vol}_\Psi(E \triangle E_t)|$.

**Proof.** Since $E$ is a set of locally finite perimeter and $P_\Psi(E, B(p,r)) > 0$ there exists a vector field $X$, with $\|X\| \leqslant 1$, such that $X \in \mathfrak{X}_0^\infty(B(p,r))$ and $\int_E \mathrm{div}_\Psi(X)\,dM > 0$. Let $\{\varphi_s\}_{s\in\mathbb{R}}$ be the flow associated to $X$. Since the map

$$s \mapsto \mathrm{vol}_\Psi(\varphi_s(E))$$

is differentiable and its derivative at $s = 0$ is $\int_E \mathrm{div}_\Psi(X) dM > 0$, we can apply the inverse function theorem to find $\delta > 0$ and a function $g : (-\delta,\delta) \to \mathbb{R}$ such that $g(0) = 0$ and $\mathrm{vol}_\Psi(\varphi_{g(t)}(E)) = \mathrm{vol}_\Psi(E) + t$. Let $E_t = \varphi_{g(t)}(E)$. This proves (1). If necessary we can reduce $\delta$ so that, for $|t| \leqslant \delta$ we have

$$\left|\mathrm{vol}_\Psi(E_t \cap B(p,r)) - \mathrm{vol}_\Psi(E \cap B(p,r))\right| \geqslant \left|\frac{t}{2}\left(\int_{E\cap B(p,r)} \mathrm{div}_\Psi X\,dM\right)^{-1}\right|. \tag{2.11}$$

As $\varphi_t(q) = q$ for all $q \notin B(p,r)$ and $t \in \mathbb{R}$ we trivially have $E \triangle E_t \subset B(p,r)$ for all $t \in \mathbb{R}$. This proves (2).

To prove (3) note that, for all $t \in (-\delta,\delta)$, we have

$$\begin{aligned} P_\Psi(E_t, B(p,r)) = P_\Psi(\varphi_{g(t)}(E), B(p,r)) &= \int_{\partial_\star \varphi_{g(t)}(E)\cap B(p,r)} e^\Psi dH^n \\ &= \int_{\partial_\star E\cap B(p,r)} \frac{(e^\Psi \circ \varphi_t)}{e^\Psi} e^\Psi |\mathrm{Jac}(\varphi_{g(t)})| dH^n. \end{aligned}$$

Hence we have

$$\begin{aligned} &|P_\Psi(E_t, B(p,r)) - P_\Psi(E, B(p,r))| = \\ &= \left|\int_{\partial_\star E\cap B(p,r)} \frac{(e^\Psi \circ \varphi_{g(t)})}{e^\Psi} e^\Psi |\mathrm{Jac}(\varphi_{g(t)})| dH^n - \int_{\partial_\star E\cap B(p,r)} e^\Psi dH^n\right| \\ &= \left|\int_{\partial_\star E\cap B(p,r)} \left(\frac{(e^\Psi \circ \varphi_{g(t)})}{e^\Psi} |\mathrm{Jac}(\varphi_{g(t)})| - 1\right) e^\Psi dH^n\right| \\ &\leqslant \sup_{\substack{t\in(-\delta,\delta)\\ q\in B(p,r)}} \left|\frac{e^\Psi \circ \varphi_{g(t)}(q)}{e^\Psi(q)} |\mathrm{Jac}(\varphi_{g(t)})(q)| - 1\right| P_\Psi(E, B(p,r)). \end{aligned}$$

Taking into account (2.11) we have, for $t \in (-\delta,\delta)\backslash\{0\}$,

$$\frac{|P_\Psi(E_t, B(p,r)) - P_\Psi(E, B(p,r))|}{|\mathrm{vol}_\Psi(E_t \cap B(p,r)) - \mathrm{vol}_\Psi(E \cap B(p,r))|} \leqslant \frac{2h(t,p,r,\delta) \cdot P_\Psi(E, B(p,r))}{\left|\left(\int_{E\cap B(p,r)} \mathrm{div}_\Psi X\,dM\right)^{-1}\right|} < C,$$

where

$$h(t,p,r,\delta) = \frac{1}{|t|} \sup_{\substack{q\in B(p,r)\\ t\in(-\delta,\delta)\backslash\{0\}}} \left|\frac{e^\Psi \circ \varphi_{g(t)}(q)}{e^\Psi(q)} |\,\mathrm{Jac}(\varphi_{g(t)})(q)| - 1\right|$$

and $C$ is a positive constant which depends on $p$, $r$ and $\delta$. Note that (3) is trivially true for $t = 0$ and any constant $C$. So we have

$$|P_\Psi(E_t, B(p,r)) - P_\Psi(E, B(p,r))| \leqslant C\left|\mathrm{vol}_\Psi(E_t \cap B(p,r)) - \mathrm{vol}_\Psi(E \cap B(p,r))\right|,$$

and this fact together with (2) implies the inequality of the left side of (3). Note that (3) is trivially true for $t = 0$ and any constant $C$.

On the other hand, for any positive measure $\mu$ we have

$$\left|\mu(E) - \mu(E')\right| \leqslant \mu(E \triangle E').$$

This completes the proof of (3). □

To conclude this section, we sketch the proof of Theorem 2.1. Since this proof has become standard after [12], we include some basic guidelines for reader's convenience.

**Proof of Theorem 2.1.** Using the relative isoperimetric inequality (2.6), the doubling property (2.10) and the hypothesis that $\mathrm{Isom}(M, g, \Psi)$ acts cocompactly on $M$, an isoperimetric inequality for small volumes can be obtained in a standard way, see Lemma 3.10 in [12]. Combining the latter with the deformation property of finite perimeter sets proven in Theorem 2.5, and using again that $\mathrm{Isom}(M, g, \Psi)$ acts cocompactly, we can prove that the isoperimetric solutions are bounded, see Lemma 4.6 in [12]. The Structure Theorem for minimizing sequences of sets of positive volume $v > 0$, see Proposition 5.1 in [12], works also in our case without modification. From them the Concentration Lemma 6.2 in [12], and the Existence Theorem 6.1 in [12] work without relevant modifications. □

## 3. Isoperimetric regions in $M \times \mathbb{R}$

In this section we prove existence of isoperimetric regions in a cylinder with density for large volumes. We shall need the following preliminary results in the proof of Theorem 3.3.

**Lemma 3.1.** *Let $(M, g, \Psi)$ be a compact manifold with density. Then there exist constants $c_1, c_2 > 0$, only depending on $M$, such that, for any set $E \subset M$ of finite perimeter with $\mathrm{vol}_\Psi(E) \leqslant \mathrm{vol}_\Psi(M)/2$ we have*

(1) *$P_\Psi(E) \geqslant c_1 \mathrm{vol}_\Psi(E)$, and*
(2) *$P_\Psi(E) \geqslant c_2 \mathrm{vol}_\Psi(E)^{(n-1)/n}$.*

**Proof.** It follows easily since the isoperimetric profile of $(M, g, \Psi)$ is strictly positive and asymptotic to the function $t \mapsto t^{(n-1)/n}$ for $t > 0$ small. □

We say that $E \subset N = M \times \mathbb{R}$ is a *normalized set* if the intersection $E_p = E \cap (\{p\} \times \mathbb{R})$ is either empty or a vertical segment centered at $(p, 0)$ for all $p \in M$. Notice that a normalized set is invariant by the reflection $\sigma : M \times \mathbb{R} \to M \times \mathbb{R}$ defined by $\sigma(p,t) = (p, -t)$, an isometry of $(M \times \mathbb{R}, g \times g_0)$ preserving the weighted volume. Given any set $E \subset N$, we denote by $E^*$ the projection of $E$ over $M$. From now on we denote $(E_t)^*$ by $E_t^*$ to simplify the notation. Notice that for normalized sets one has $E_t^* \subset E_s^*$ whenever $|s| \leqslant |t|$. We denote by $P$ and vol the perimeter and volume in the manifold with density $(M \times \mathbb{R}, g \times g_0, \Psi \times 1)$.

**Lemma 3.2.** *If $E \subset N$ is a normalized isoperimetric region and $\mathrm{vol}_\Psi(M \setminus E^*) > 0$ then there exists a constant $c > 0$ independent of $\mathrm{vol}(E)$ such that*

$$P(E) \geqslant c\,\mathrm{vol}(E). \tag{3.1}$$

**Proof.** For every $t \in \mathbb{R}$ we define $M_t = M \times \{t\}$ and $E_t = E \cap M_t$. As $E$ is normalized we can choose $\tau \geqslant 0$ so that $\mathrm{vol}_\Psi(E_t^*) \leqslant \mathrm{vol}_\Psi(M)/2$ for all $t \geqslant \tau$ and $\mathrm{vol}_\Psi(E_t^*) > \mathrm{vol}_\Psi(M)/2$ for all $t \in [0, \tau)$ if $\tau > 0$.

Let us consider first the case $\tau > 0$.

We apply the coarea formula and Lemma 3.1 (1) to obtain

$$\begin{aligned} P(E) \geqslant P(E, M \times [\tau, \infty)) &\geqslant \int_\tau^\infty P_\Psi(E_s^*) ds \\ &\geqslant c_1 \int_\tau^\infty \mathrm{vol}_\Psi(E_s^*) ds \\ &= c_1 \mathrm{vol}(E \cap (M \times [\tau, \infty))). \end{aligned} \tag{3.2}$$

On the other hand, for $t \in [0, \tau)$ we have

$$\mathrm{vol}_\Psi(M \setminus E_t^*) \geqslant P(E, M \times (0, t)), \tag{3.3}$$

since otherwise

$$\begin{aligned} \mathrm{vol}_\Psi(M) &= \mathrm{vol}_\Psi(M \setminus E_t^*) + \mathrm{vol}_\Psi(E_t^*) \\ &< P(E, M \times (0, t)) + P(E, M \times (t, \infty)) \\ &\leqslant P(E)/2. \end{aligned}$$

This is a contradiction since comparison of $E$ with a slab $M \times [a, b]$ of the same volume implies that $P(E) \leqslant P(M \times [a, b]) = 2\mathrm{vol}_\Psi(M)$. This proves (3.3). Calling $y(t) = \mathrm{vol}_\Psi(M \setminus E_t^*)$, using the coarea formula and Lemma 3.1(2), we may rewrite the inequality (3.3) as

$$y(t) \geqslant c_2 \int_0^t y(s)^{(n-1)/n} ds.$$

As $y(t) > 0$ for all $t \in [0, \tau)$ we have

$$y(t) \geqslant \left(\tfrac{c_2}{n}\right)^n t^n.$$

In particular, taking limits when $t \to \tau^-$ and using that $y(t)$ is non-decreasing

$$\mathrm{vol}_\Psi(M) \geqslant \mathrm{vol}_\Psi(M \setminus E_\tau^*) = y(\tau) \geqslant \left(\tfrac{c_2}{n}\right)^n \tau^n.$$

Hence

$$\tau \leqslant \frac{n \mathrm{vol}_\Psi(M)^{1/n}}{c_2}$$

and so

$$\begin{aligned} \mathrm{vol}_\Psi(E \cap (M \times (0, \tau))) &= \int_0^\tau \mathrm{vol}_\Psi(E_s^*) ds \leqslant \mathrm{vol}_\Psi(E_0^*)\tau \\ &\leqslant \mathrm{vol}_\Psi(E_0^*) \frac{n \mathrm{vol}_\Psi(M)^{1/n}}{c_2} \\ &\leqslant \frac{n \mathrm{vol}_\Psi(M)^{1/n}}{c_2} \frac{P(E)}{2}. \end{aligned} \tag{3.4}$$

The last inequality follows $2\mathrm{vol}_\Psi(E_0^*) \leqslant P(E)$, which holds since $E$ is normalized and so $P(E)$ is the sum of a lateral area that projects to some set of weighted measure zero on $M$ and the area of the graphs of two $C^1$ functions $u$ and $-u$ over some set $\Omega \subset M$ of full measure in $E^*$. So we have

$$\mathrm{vol}_\Psi(E^*) = \mathrm{vol}_\Psi(\Omega) = \int_\Omega e^\Psi dM \leqslant \int_\Omega e^\Psi \sqrt{1 + |\nabla u|^2} dM \leqslant \frac{P(E)}{2}.$$

Hence (3.1) follows from (3.2) and (3.4).

It remains to consider the case $\tau = 0$. In this case, Eq. (3.2) alone implies the linear isoperimetric inequality (3.1) since $\mathrm{vol}_\Psi(E \cap (M \times [0, +\infty))) = \frac{1}{2}\mathrm{vol}_\Psi(E)$ as $E$ is normalized. $\square$

**Theorem 3.3.** *Let $(M, g, \Psi)$ be a compact manifold with density. For large volumes, isoperimetric regions in the cylinder $(M \times \mathbb{R}, g \times g_0, \Psi \times 1)$ are slabs of the form $M \times [a, b]$, where $[a, b] \subset \mathbb{R}$ is a bounded interval.*

**Proof.** Existence of isoperimetric regions in $N = M \times \mathbb{R}$ is guaranteed by Theorem 2.1. If $E$ is an isoperimetric region in $N$, comparison with slabs implies

$$P(E) \leqslant 2\mathrm{vol}_{\Psi}(M), \tag{3.5}$$

for all volumes $v > 0$.

We take an isoperimetric set $E \subset M$. Let $\mathrm{sym}(E)$ be its Steiner symmetrization with respect to $M \times \{0\}$, see [16, § 14.1]. As

$$\mathrm{vol}(E) = \int_M \left\{ \int_{E_p} e^{\Psi \times 1} dt \right\} dM = \int_M e^{\Psi(p)} |E_p| dM(p) = \mathrm{vol}(\mathrm{sym}(E)),$$

where $|E_p|$ is the 1-dimensional Lebesgue measure of $E_p$, the volume is preserved when we pass to the Steiner symmetrization of $E$. To see that

$$P(\mathrm{sym}(E)) \leqslant P(E) \tag{3.6}$$

we consider a function $u : \Omega \subset M \to \mathbb{R}$ and the graph $G(u)$ of $u$ and we observe that the weighted area of $G(u)$ is given by

$$\mathrm{area}_{\Psi}(G(u)) = \int_{\Omega} e^{\Psi} \sqrt{1 + |\nabla u|^2} dM.$$

So we can reason as in the proof of the Euclidean case to verify (3.6), see again [16, § 14.1]. Equality holds if and only if $E = \mathrm{sym}(E)$. If $E$ is an isoperimetric region then also $\mathrm{sym}(E)$ is isoperimetric and, moreover, $\mathrm{sym}(E)^* = E^*$. So from now on we assume that $E$ is normalized replacing $E$ by $\mathrm{sym}(E)$ if necessary.

If $\mathrm{vol}_{\Psi}(M \setminus E^*) > 0$ then Lemma 3.2 provides a constant $c > 0$ independent of $\mathrm{vol}(E)$ so that $P(E) \geqslant c\mathrm{vol}(E)$. But this in contradiction to (3.5), since by hypothesis we are working with large volumes. Hence $\mathrm{vol}_{\Psi}(M \setminus E^*) = 0$ and $E^* = M$ and $E$ is the region between the graphs of two functions $u, v : M \to \mathbb{R}$. By regularity of isoperimetric regions, $\nabla u$, $\nabla v$ are defined a.e. on $M$ and

$$\begin{aligned} P(E) &= \int_M e^{\Psi} \sqrt{1 + |\nabla u|^2} dM + \int_M e^{\Psi} \sqrt{1 + |\nabla v|^2} dM \\ &\geqslant 2 \int_M e^{\Psi} dM = 2\mathrm{vol}_{\Psi}(M). \end{aligned}$$

Since $P(E) \leqslant 2\mathrm{vol}_{\Psi}(M)$ we should have equality in the above inequality, that implies $\nabla u = \nabla v = 0$ and so $E$ is a slab. This completes the proof of the theorem. □

**Remark 3.4.** Note that Theorem 3.3 does not hold when there is a non-trivial density in the vertical factor. For example, in $(S^n \times \mathbb{R}, g \times g_0, 1 \times e^{-t^2/2})$ the only isoperimetric regions are of the type $S^n \times (-\infty, a)$ or $S^n \times (a, \infty)$. See Example 4.6 in [28].

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