



# Correction to: Equilibrium of Surfaces in a Vertical Force Field

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In the previous paper [2, Section 3.2], we describe the family of *titled*  $[\varphi, \vec{e}_3]$ -catenary cylinders as surfaces obtained from a  $[\varphi, \vec{e}_3]$ -catenary cylinder  $\Sigma$ , by rotation of angle  $\theta \in ]0, \pi/2[$  about the  $x$ -axis and dilation by scale factor  $1/\cos\theta$ . The authors state that the resulting surface,  $\tilde{\Sigma}$ , is always a  $[\varphi, \vec{e}_3]$ -minimal surface, but this is only true when the starting  $[\varphi, \vec{e}_3]$ -catenary cylinder is a grim reaper translating soliton, which follows directly from the relationship between their mean curvatures. In fact, following the same notation as in [2, Section 3.2],  $\tilde{H} = \cos\theta H$ , and  $\tilde{\Sigma}$  will be a  $[\varphi, \vec{e}_3]$ -minimal surface if and only if

$$\dot{\varphi}(u) = \dot{\varphi}(u + y \sin\theta) \quad \text{for any } y \in \mathbb{R},$$

that is, if and only if  $\Sigma$  is a grim reaper translating soliton. In this case,  $\tilde{\psi}$  will be a *titled grim reaper cylinder*.

The following result updates the classification of complete flat  $[\varphi, \vec{e}_3]$ -minimal surfaces in  $\mathbb{R}^3$  [2, Theorem 3.7]:

**Theorem 1.** *Let  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  be a diffeomorphism and  $\Sigma$  be a complete flat  $[\varphi, \vec{e}_3]$ -minimal surface in  $\mathbb{R}^3$ . Then, one of the following statements holds*

- $\Sigma$  is a vertical plane.
- $\Sigma$  is a grim reaper cylinder (maybe tilted).
- $\Sigma$  is a  $[\varphi, \vec{e}_3]$ -catenary cylinder.

*Proof.* From basic differential geometry,  $\Sigma = \gamma \times \Pi^\perp$  is a ruled surface and its Gauss map is constant along the rules, where  $\gamma$  is a complete regular curve in a plane  $\Pi \subset \mathbb{R}^3$ . Thus,  $\Sigma$  can be parametrized by  $\psi(s, t) = \gamma(s) + t\vec{v}$ , with  $\gamma(s)$  a complete regular curve contained in a plane  $\Pi$  and  $\vec{v}$  a unit vector

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orthogonal to  $\Pi$ . We may assume that  $(s, t) \in \mathbb{R}^2$  and  $|\gamma'| = 1$ , then, the Gauss map  $N$  of  $\psi$  and its mean curvature  $H$  are given by

$$N(s, t) = \gamma'(s) \wedge \vec{v}, \quad H(s, t) = \kappa_\gamma(s),$$

where  $\kappa_\gamma$  is the curvature of  $\gamma$ . Hence,  $\Sigma$  is  $[\varphi, \vec{e}_3]$ -minimal if and only if the following relation holds

$$\kappa_\gamma(s) = -\dot{\varphi}(\langle \gamma(s) + t\vec{v}, \vec{e}_3 \rangle) \langle \gamma'(s) \wedge \vec{v}, \vec{e}_3 \rangle.$$

Differentiating with respect to  $t$  in the above expression, we obtain that

$$0 = \ddot{\varphi}(\langle \gamma(s) + t\vec{v}, \vec{e}_3 \rangle) \langle \vec{v}, \vec{e}_3 \rangle \langle \gamma'(s) \wedge \vec{v}, \vec{e}_3 \rangle.$$

If  $\langle \vec{v}, \vec{e}_3 \rangle = 0$ , arguing as in [2, Theorem 3.7], for any horizontal rule  $\mathcal{L}$  of  $\Sigma$  there exists a  $[\varphi, \vec{e}_3]$ -catenary cylinder  $\mathcal{C}$  containing  $\mathcal{L}$  and tangent to  $\Sigma$  along  $\mathcal{L}$ . Thus, from standard theory of uniqueness of solution for an ODE, up to horizontal translation,  $\Sigma$  must coincide with  $\mathcal{C}$ .

If  $\langle \vec{v}, \vec{e}_3 \rangle \neq 0$ ,

$$0 = \ddot{\varphi}(\langle \gamma(s) + t\vec{v}, \vec{e}_3 \rangle) \langle \gamma'(s) \wedge \vec{v}, \vec{e}_3 \rangle$$

and we can assume that  $\ddot{\varphi}$  does not vanish everywhere otherwise, from [1],  $\varphi$  is a linear function and  $\Sigma$  is either a vertical plane or a grim reaper cylinder (maybe tilted). Thus, we have that  $\gamma' \wedge \vec{v}$  is orthogonal to  $\vec{e}_3$  and  $\Sigma$  must be a vertical plane. □

Now, the Corollary 3.8 in [2] updates to

**Corollary 2.** *Let  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  be a increasing diffeomorphism with  $\ddot{\varphi} \leq 0$  and  $\Sigma$  be a complete locally convex  $[\varphi, \vec{e}_3]$ -minimal surface in  $\mathbb{R}^3$ . If the Gauss curvature vanishes anywhere, then one of the following statements holds*

- $\Sigma$  is a vertical plane.
- $\Sigma$  is a grim reaper cylinder (maybe tilted).
- $\Sigma$  is a  $[\varphi, \vec{e}_3]$ -catenary cylinder.

*Remark 3.* Although it does not affect any of the results shown throughout the paper, the Eqs. (6) and (7) in [2, Lemma 2.1] must be change to

$$\nabla^2 H = -\eta \nabla^2 \dot{\varphi} - (\nabla \mathcal{A})(\nabla \varphi, \cdot, \cdot) - H \mathcal{A}^{[2]} - \mathcal{B} \tag{6}$$

$$\Delta \mathcal{A} + (\nabla \mathcal{A})(\nabla \varphi, \cdot, \cdot) + \eta \nabla^2 \dot{\varphi} + |\mathcal{A}|^2 \mathcal{A} + \mathcal{B} = 0, \tag{7}$$

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## References

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