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Let u denote a solution to a rotationally invariant Hessian equation $F(D^2u) = 0$ on a bounded simply connected domain $\Omega \subset \mathbb{R}^2$, with constant Dirichlet and Neumann data on $\partial\Omega$. We prove that if u is real analytic and not identically zero, then u is radial and Ω is a disk. The fully nonlinear operator $F \not\equiv 0$ is of general type and, in particular, not assumed to be elliptic. We also show that the result is sharp, in the sense that it is not true if Ω is not simply connected, or if u is C^∞ but not real-analytic.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a C^2 smooth bounded domain, and let $u \in C^2(\overline{\Omega})$ be a solution to $\Delta u + 1 = 0$ that satisfies overdetermined boundary conditions

$$u = 0, \qquad |Du| = \text{constant} \quad \text{on } \partial\Omega. \tag{1-1}$$

In his famous paper, Serrin [1971] proved that under these conditions Ω is a ball and u is a radial function. Starting with Serrin's paper (see also the influential work [Weinberger 1971]), there has been great interest in extending Serrin's result to more general PDEs that satisfy the overdetermined conditions (1-1). Ellipticity has typically been an essential component in all these extensions of Serrin's theorem.

In this paper we consider Serrin's overdetermined problem for general (not necessarily elliptic) fully nonlinear Hessian equations, i.e.,

$$\begin{cases} F(D^2u) = 0 & \text{in } \Omega, \\ u = 0, \qquad |Du| = c & \text{on } \partial\Omega, \end{cases} \tag{1-2}$$

where F is a function on the space $\mathcal{M}_n$ of all symmetric $n \times n$ matrices. To avoid meaningless situations, *we will assume from now on that F is never locally zero*, i.e., $F \not\equiv 0$ on any open set. A natural and necessary hypothesis on F dictated by the nature of the boundary conditions (1-1) (see, e.g., [Silvestre and Sirakov 2015]) is that F is *rotationally invariant*, i.e., $F(Q^t M Q) = F(M)$ for any $M \in \mathcal{M}_n$ and any orthogonal matrix Q. Equivalently, F is a symmetric function of the eigenvalues of the Hessian D^2u.

When F is elliptic in a suitable sense, the existence of a solution u to (1-2) forces Ω to be a ball and u to be a radial function (see [Silvestre and Sirakov 2015]). It is not surprising that, if F is not elliptic, this

is not true anymore; some simple counterexamples will be given in Example 2.1. So, in some sense, the rigidity given by ellipticity seems fundamental for the desired radial symmetry result to hold.

This situation makes our main result here somehow unexpected. We prove that if $\Omega \subset \mathbb{R}^2$ is simply connected, and u is a real-analytic solution to (1-2), then Ω is a disk and u is radial. No ellipticity assumption is made on F, and no sign assumption is made on u:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded, simply connected domain, let $F : \mathcal{M}_2 \to \mathbb{R}$ be rotationally invariant, and let $u \in C^\omega(\overline{\Omega})$ be a nonzero solution to (1-2).*

Then Ω is a disk and u is a radial function with respect to the center of Ω.

Remarkably, the topological hypothesis that Ω is simply connected cannot be weakened. Indeed, in Section 2 we will show that there exist positive real-analytic solutions u to (1-2) for which u is nonradial and $\Omega \subset \mathbb{R}^2$ is diffeomorphic to an annulus. This example also shows the global nature of Theorem 1.1, and in particular indicates that it cannot follow from a local application of the Cauchy–Kowalevsky theorem along the boundary. In addition, there exist nonradial, C^∞ solutions u to (1-2) for Ω simply connected (see Example 2.1). Thus, Theorem 1.1 is sharp in these directions.

Theorem 1.1 is inspired by classical surface theory, and in particular by a theorem of K. Voss [1959], according to which any compact analytic Weingarten surface of genus zero immersed in $\mathbb{R}^3$ is a rotational sphere. The proof of Theorem 1.1 is an application of the Poincaré–Hopf theorem to an adequate line field with singularities in $\overline{\Omega}$. We emphasize that this line field is not given by the gradient of a solution to $F(D^2u) = 0$, so in this sense the application of the Poincaré–Hopf theorem here is not very usual in PDE theory. This strategy was used by the second author in [Mira 2018] in order to solve overdetermined problems with nonconstant boundary data for fully nonlinear elliptic equations, and also by Espinar and Mazet [2019] for solving the classification problem of f-extremal disks in the two-sphere $\mathbb{S}^2$. Both of these works are inspired by our previous paper [Gálvez and Mira 2020] about uniqueness of immersed spheres modeled by elliptic PDEs in three-manifolds. The key tool in all these works is to use ellipticity in order to construct a line field on the surface with isolated singularities of negative index, and derive from there a contradiction with the Poincaré–Hopf theorem. However, in our present situation, the lack of ellipticity makes this approach unsuitable; the natural line field that we construct may have nonisolated singularities, and even at the isolated ones its index can be positive.

The results in the present paper strengthen the connection between overdetermined problems and hypersurface theory, a connection already present in Serrin's theorem, and which has been exploited in many works; see, e.g., [Del Pino et al. 2015; Domínguez-Vázquez et al. 2019; Espinar et al. 2021; Espinar and Mazet 2019; Farina and Valdinoci 2013; Farina et al. 2013; Hauswirth et al. 2011; Mira 2018; Ros et al. 2017; 2020; Ros and Sicbaldi 2013; Schlenk and Sicbaldi 2012; Sicbaldi 2010; Traizet 2014]. Nonetheless, to the authors' best knowledge, Theorem 1.1 might be the first example of such connection for nonelliptic equations.

We next provide an outline of the proof of Theorem 1.1. Let u be a nonradial, real-analytic solution to (1-2) on $\overline{\Omega}$. Then, the eigenlines of D^2u define two analytic line fields L_1, L_2 on $\overline{\Omega} - \mathcal{U}$, where $\mathcal{U}$ is the set of points in $\overline{\Omega}$ where D^2u is proportional to the identity, i.e., the set of points where D^2u has a double eigenvalue. We wish to analyze how L_1, L_2 extend across $\mathcal{U}$.

In order to do this, we consider a point $p \in \mathcal{U}$, and we let $w(x, y)$ be the first nonzero homogeneous polynomial of degree $n \geq 3$ in its series expansion around that point (if w does not exist, the result is trivial). There will be three cases to consider.

If $w(x, y)$ is radially symmetric, we will prove in Proposition 3.1 that $u(x, y)$ is also radially symmetric with respect to p, up to a translation. In that case, the result follows easily, and so we discard this situation. If $w(x, y)$ is some power of a linear function, we will prove in Section 4 that $\mathcal{U}$ is a real-analytic regular curve around p, and that the eigenfields L_1, L_2 extend analytically across p. Finally, if $w(x, y)$ is not of either of these two types, then we will show in Section 2 that it is a harmonic polynomial; in that case p is isolated in $\mathcal{U}$, and the Poincaré–Hopf index of the line fields L_1, L_2 around p is negative.

Once there, the proof ends as follows. By the previous discussion, both L_1, L_2 can be extended to line fields on $\overline{\Omega}$ with only isolated singularities, all of them of negative index. Also, the overdetermined conditions (1-1) imply that one of L_1 or L_2 is tangent to $\partial\Omega$ at each boundary point. Since Ω is simply connected, this provides a contradiction with the Poincaré–Hopf index theorem. The contradiction proves that u is radial, and from there, that Ω is a disk.

2. Necessity of the hypotheses

We will first show that the hypothesis that u is real-analytic in Theorem 1.1 is necessary by constructing a C^∞ solution to (1-2) that is not a radial function. In this construction, the domain $\Omega \subset \mathbb{R}^2$ is an arbitrary simply connected smooth (or even real-analytic) bounded domain.

Example 2.1. Let $\Omega \subset \mathbb{R}^2$ be the simply connected domain bounded by a real-analytic regular Jordan curve γ in $\mathbb{R}^2$. Given any $\rho > 0$, let f denote a smooth function on the closed disk $D_\rho = \overline{D(0, \rho)}$, with the following properties:

(1) f is a radial function with respect to the origin.

(2) The values of f and all its derivatives vanish at every point of ∂D_ρ.

For instance, we can choose

$$f(x, y) = \exp\left(\frac{-1}{\rho^2 - (x^2 + y^2)}\right).$$

Let now $D_1, \ldots, D_k$ denote a collection of mutually disjoint closed disks in Ω of radius $\rho > 0$. For each $i \in \{1, \ldots, k\}$, let $u_i \in C^\infty(D_i)$ be given by

$$u_i(x, y) := f(x - a_i, y - b_i),$$

where (a_i, b_i) is the center of D_i.

Define now the function $u \in C^\infty(\overline{\Omega})$ by

$$u(x, y) = u_i(x, y) \quad \text{if } (x, y) \in D_i, \qquad u(x, y) = 0 \quad \text{otherwise.}$$

Note that $u = |Du| = 0$ along $\partial\Omega$; i.e., u satisfies the overdetermined boundary conditions in (1-2) for the choice $c = 0$.

Since each function u_i is radial with respect to some point in $\mathbb{R}^2$, it follows that the Jacobian satisfies

$$J[\Delta u, \mathcal{H}(u)] := (\Delta u)_x (\mathcal{H}(u))_y - (\Delta u)_y (\mathcal{H}(u))_x = 0 \tag{2-1}$$

on Ω, where we let $\mathcal{H}(u) := \det(D^2 u)$. This implies by a classical theorem of Brown and Sard, see, e.g., [Newns 1967], that there exists a smooth function $\Phi(s, t)$ with $\Phi(0, 0) = 0$ such that

$$\Phi(\Delta u, \mathcal{H}(u)) = 0. \tag{2-2}$$

By considering the function $F \in C^\infty(\mathcal{M}_2)$ associated to Φ in the obvious way, this implies that u is a nonradial C^∞ solution to (1-2) for $c = 0$ and the above choices of Ω and F.

We remark that even though the function u constructed in Example 2.1 is not real-analytic, the function Φ (and so, the function F in (1-2)) can be chosen to be real-analytic in many cases.

Example 2.1 satisfies the boundary conditions in (1-2) for the choice $c = 0$. It is also possible to construct smooth nonradial solutions to (1-2) with $c \neq 0$. For example, let v be a smooth radial function in the unit disk $\mathbb{D}$, with $v = 0$ and $|Dv| = c \neq 0$ along $\partial\mathbb{D}$, that is constant in some smaller disk $D(0, \rho) \subset \mathbb{D}$. Then, it is straightforward to modify v inside $D(0, \rho)$ as in Example 2.1 to obtain a nonradial smooth function u in $\mathbb{D}$ that solves (1-2) for $c \neq 0$ and for an adequate choice of F.

The next example shows that the hypothesis that Ω is simply connected in Theorem 1.1 cannot be removed. For that, we will construct real-analytic, nonradial solutions to overdetermined problems of the form (1-2), with $F \not\equiv 0$ real-analytic and rotationally symmetric, on planar domains $\Omega \subset \mathbb{R}^2$ diffeomorphic to an annulus but that are not radially symmetric in general.

Example 2.2. Let $\gamma(s) := (\alpha(s), \beta(s))$, $s \in [0, L]$, be a real-analytic, regular Jordan curve in $\mathbb{R}^2$ parametrized by arc-length, and let $\nu(s)$ be its unit normal. Assume that the normal map

$$\Psi(s, t) := \gamma(s) + t\nu(s) : \mathbb{R}/(L\mathbb{Z}) \times [-1, 1] \to \mathbb{R}^2$$

is a real-analytic diffeomorphism onto the compact planar region

$$\overline{\Omega} := \{\gamma(s) + t\nu(s) : |t| \leq 1\} \subset \mathbb{R}^2.$$

Note that $\overline{\Omega}$ is real-analytic and diffeomorphic to an annulus. We remark that the condition that Ψ is a local diffeomorphism is equivalent to the curvature $\kappa(s)$ of γ satisfying $|\kappa| < 1$ at every point. If the curve $\gamma(s)$ is chosen to be convex, the condition $|\kappa| < 1$ is also sufficient for Ψ being a real-analytic diffeomorphism.

Define $u \in C^\omega(\overline{\Omega})$ by

$$u(\Psi(s, t)) = 1 - t^2;$$

see Figure 1. A computation using $x = \alpha(s) - t\beta'(s)$, $y = \beta(s) + t\alpha'(s)$ shows that

$$u_{xx} + u_{yy} = -2 + \frac{2t\kappa(s)}{1 - t\kappa(s)}, \quad u_{xx}u_{yy} - u_{xy}^2 = \frac{-4t\kappa(s)}{1 - t\kappa(s)}. \tag{2-3}$$

Let $\phi_1(s, t)$ and $\phi_2(s, t)$ denote the right-hand sides in (2-3). A computation shows that the Jacobian determinant of the map $(s, t) \mapsto (\phi_1, \phi_2)$ vanishes identically. Consequently, by (2-3), we have that
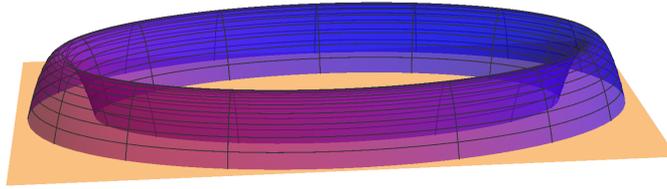
Figure 1. A nonradial, analytic solution to (1-2) in an elliptical region, with base on the ellipse $4x^2 + y^2 = 64$.

$u(x, y)$ satisfies (2-1). Arguing as in Example 2.1, we conclude that u is a solution to (2-2) on $\bar{\Omega}$ for some real-analytic function Φ.

Finally, we note that u satisfies the overdetermined boundary conditions (1-1). That $u = 0$ along $\partial\Omega$ is clear by construction. A computation shows that $Du = (u_x, u_y) = (2t\beta'(s), -2t\alpha'(s))$, which has constant length for each fixed value of t. In particular, $|Du| = 2$ along $\partial\Omega$, which corresponds to $t = \pm 1$.

To sum up, $u \in C^\omega(\bar{\Omega})$ is a real-analytic, nonradial solution to (1-2) for the real-analytic choice of Φ above and for the real-analytic annulus Ω in $\mathbb{R}^2$. Note that, in general, Ω is not radially symmetric, and the two closed curves in $\partial\Omega$ are not necessarily convex.

We conjecture that Theorem 1.1 does not hold in higher dimensions. Indeed, note that the techniques used in the proof are essentially two-dimensional. This is coherent with the somewhat related geometric situation of constant mean curvature spheres in Euclidean space, which are necessarily round in dimension two by Hopf's theorem, but can be nonembedded in higher dimensions (see [Hsiang 1982]).

3. Behavior around points with a double eigenvalue of $D^2 u$

Let $u = u(x, y)$ denote a real-analytic solution to $F(D^2 u) = 0$ on a regular planar domain $\bar{\Omega} \subset \mathbb{R}^2$, where $F : \mathcal{M}_2 \to \mathbb{R}$ is rotationally invariant. Setting $\mathcal{H}(u) := \det(D^2 u) = u_{xx} u_{yy} - u_{xy}^2$, this equation can be rewritten as

$$\Phi(\Delta u, \mathcal{H}(u)) = 0, \tag{3-1}$$

where Φ is not identically zero on any open set of $\mathbb{R}^2$, by the related hypothesis on F. Define

$$\mathcal{U} = \{p \in \bar{\Omega} : D^2 u(p) = \lambda \operatorname{Id} \text{ for some } \lambda \in \mathbb{R}\} = \{p \in \Omega : (\Delta u(p))^2 = 4\mathcal{H}(p)\}. \tag{3-2}$$

Note that in general $(\Delta u(p))^2 \geq 4\mathcal{H}(p)$ for every $p \in \bar{\Omega}$.

Choose $p_0 \in \mathcal{U}$. Since problem (1-2) is invariant by translations in the (x, y)-variables, we may assume without loss of generality that $p_0 = (0, 0)$. We will also assume: *u is not a polynomial of degree at most* 2 (note that the statement of Theorem 1.1 is trivial if u is such a polynomial). Observe that since u is real-analytic in $\bar{\Omega}$, it can be extended to a real-analytic function on an open set Ω_0 containing $\bar{\Omega}$; in particular, u can be assumed to be well-defined and real-analytic around $(0, 0)$, even if this point lies in the boundary $\partial\Omega$.

It follows from (3-1) and the fact that $\Phi \not\equiv 0$ on open sets that u satisfies in Ω the equation

$$J[\Delta u, \mathcal{H}(u)] = 0, \tag{3-3}$$

where we are defining for f, g of class C^1

$$J[f, g] := f_x g_y - f_y g_x.$$

Indeed, if $J[\Delta u, \mathcal{H}(u)] \neq 0$ around some point in Ω, then $(s, t) := (\Delta u, \mathcal{H}(u))$ are local parameters around that point, and from (3-1) we would obtain that Φ vanishes in an open set of $\mathbb{R}^2$, a contradiction.

Therefore, by real-analyticity, u satisfies (3-3) on $\Omega_0 \supset \bar{\Omega}$. Thus, there exists a nonconstant real-analytic function $\sigma(x, y)$ defined in a neighborhood of $(0, 0)$, with $\sigma(0, 0) = 0$, and nonconstant real-analytic functions of one variable $\alpha(t), \beta(t)$ such that

$$\Delta u = \alpha \circ \sigma, \quad \mathcal{H}(u) = \beta \circ \sigma. \tag{3-4}$$

Therefore, there exist nonnegative real-analytic functions $\phi(t), \varphi(t)$ with $\phi(0) = \varphi(0) = 0$ (since $(0, 0) \in \mathcal{U}$) such that

$$(\Delta u)^2 - 4\mathcal{H}(u) = \phi \circ \sigma, \quad (\Delta u - \Delta u(p_0))^2 = \varphi \circ \sigma. \tag{3-5}$$

Since u is not a polynomial of degree ≤ 2, we can define $w(x, y)$ as the first homogeneous polynomial of degree ≥ 3 in the Taylor series expansion of $u(x, y)$ around $(0, 0)$. Since $(0, 0) \in \mathcal{U}$, we have then around $(0, 0)$

$$u(x, y) = c_0 + ax + by + \frac{\lambda}{2}(x^2 + y^2) + w(x, y) + \cdots. \tag{3-6}$$

From here, a simple power series expansion around the origin shows that

$$(\Delta u)^2 - 4\mathcal{H}(u) = (\Delta w)^2 - 4\mathcal{H}(w) + \cdots, \quad (\Delta u - \Delta u(p_0))^2 = (\Delta w)^2 + \cdots. \tag{3-7}$$

By (3-5), this implies that both $(\Delta w)^2 - 4\mathcal{H}(w)$ and $(\Delta w)^2$ are proportional to $\hat{\sigma}(x, y)^l$ for some $l \geq 1$, where $\hat{\sigma}$ is the first nonzero term in the Taylor series expansion of σ at $(0, 0)$. In particular, since $(\Delta w)^2 - 4\mathcal{H}(w)$ is not zero, we have

$$(\Delta w)^2 = \mu^2((\Delta w)^2 - 4\mathcal{H}(w)) \tag{3-8}$$

for some $\mu \in \mathbb{R}$. Once here, a classical algebraic lemma by Hopf [1951] shows that if a homogenous polynomial $w(x, y)$ of degree $n + 2 \geq 3$ satisfies (3-8), then after a rotation in the (x, y)-coordinates, one of the following three situations happens, where we set $\zeta := x + iy$:

(C.1) $\mu = 0$, and $w(x, y) = a \operatorname{Re}(\zeta^{n+2})$ for $a \neq 0$.

(C.2) $\mu = 1$, and $w(x, y) = a x^{n+2}$ for $a \neq 0$.

(C.3) $\mu = 1 + 1/k$ for some positive integer k, and $w(x, y) = a|\zeta|^{2k+2}$, with $n = 2k$, for $a \neq 0$.

Let $\lambda \in \mathbb{R}$ be the value for which $D^2 u(0, 0) = \lambda \operatorname{Id}$, and let us write

$$u(x, y) = c_0 + ax + by + \frac{\lambda}{2}(x^2 + y^2) + u_1(x, y), \tag{3-9}$$

where $a, b, c_0 \in \mathbb{R}$ and $u_1(x, y)$ satisfies $u_1(0, 0) = Du_1(0, 0) = 0$. Note that also $D^2u_1(0, 0)$ vanishes, by the definition of λ. Also, by (3-3), we have

$$J[\Delta u_1, \mathcal{H}(u_1)] = 0. \tag{3-10}$$

The next proposition shows that case (C.3) above can only happen, even locally, under very restrictive conditions.

Proposition 3.1. *Assume that case* (C.3) *above happens; i.e.,* $\mu = 1 + 1/k$ *holds in* (3-8). *Then* $u_1(x, y)$ *is radially symmetric with respect to* $(0, 0)$; *i.e.,* $u_1(x, y)$ *only depends on* $\sqrt{x^2 + y^2}$.

Proof. Note that the leading homogeneous polynomial of u_1 is $w(x, y)$, of degree $n + 2 \geq 3$. As we are in case (C.3), we have

$$w(x, y) = a\varrho^{n+2}, \tag{3-11}$$

where $\varrho := \sqrt{x^2 + y^2}$ and $a \neq 0$.

Assume that u_1 is not radial. Then we can write $u_1(x, y) = h(\varrho) + \eta(x, y) + \cdots$, where $\eta(x, y)$ is a nonradial homogeneous polynomial of degree $m + 2 > n + 2$, and $h(\varrho)$ denotes a radial polynomial of degree less than $m + 2$. Define for f, g of class C^2 the operator

$$\{f, g\} := f_{xx}g_{yy} + f_{yy}g_{xx} - 2f_{xy}g_{xy}.$$

Then, if we write $\psi(x, y) := u_1(x, y) - h(\varrho)$, we have

$J[\Delta u_1, \mathcal{H}(u_1)]$

$\quad = J[\Delta h + \Delta\psi, \mathcal{H}(h) + \mathcal{H}(\psi) + \{h, \psi\}]$

$\quad = J[\Delta h, \mathcal{H}(h)] + J[\Delta h, \{h, \psi\}] + J[\Delta\psi, \mathcal{H}(h)] + J[\Delta h, \mathcal{H}(\psi)] + J[\Delta\psi, \mathcal{H}(\psi) + \{h, \psi\}].$ (3-12)

Note that $J[\Delta h, \mathcal{H}(h)] = 0$, since both $\Delta h, \mathcal{H}(h)$ are radial functions. Also, the lowest-order term in the series expansions of the right-hand side of (3-12) is given by $J[\Delta w, \{w, \eta\}] + J[\Delta\eta, \mathcal{H}(w)]$, which has degree $2n + m - 2$ (if it is not identically zero). But now, since u_1 satisfies (3-10), we obtain

$$J[\Delta w, \{w, \eta\}] + J[\Delta\eta, \mathcal{H}(w)] = 0. \tag{3-13}$$

We next compute the left-hand side of (3-13). By (3-11), we have

$$\Delta w = a(n+2)^2\varrho^n, \quad \mathcal{H}(w) = a^2(n+1)(n+2)^2\varrho^{2n}. \tag{3-14}$$

If we write $\eta(x, y)$ in polar coordinates as $\eta = c(\theta)\varrho^{m+2}$, then

$$\Delta\eta = \varrho^m(c''(\theta) + (m+2)^2c(\theta)). \tag{3-15}$$

A longer but also straightforward computation, again changing to polar coordinates, shows that

$$\{w, \eta\} = \frac{a}{2}(n+2)\varrho^{n+m}\big((n+4)(c''(\theta) + (m+2)^2c(\theta)) - 2n(m+1)(m+2)c(\theta)\big). \tag{3-16}$$

Moreover, if we express the Jacobians in (3-13) also in polar coordinates, we obtain

$$(\Delta w)_\rho(\{w, \eta\})_\theta - (\Delta w)_\theta(\{w, \eta\})_\rho = -(\Delta\eta)_\rho(\mathcal{H}(w))_\theta + (\Delta\eta)_\theta(\mathcal{H}(w))_\rho. \tag{3-17}$$

Note that Δw and $\mathcal{H}(w)$ do not depend on θ, by (3-14). With this, a computation from (3-17) using (3-14), (3-15) and (3-16) shows that there exist positive constants $\alpha_1, \alpha_2 > 0$ such that

$$\alpha_1 c'''(\theta) = \alpha_2 c'(\theta).$$

Since $c'(\theta)$ is a periodic function, we necessarily have then $c'(\theta) = 0$; i.e., $c(\theta)$ is constant. This implies that the homogeneous polynomial $\eta = c(\theta)\varrho^{m+2}$ is radial, which contradicts our hypothesis. □

Note that Proposition 3.1 and the discussion preceding it imply the following consequence, of a local nature, which does not use the boundary conditions:

Corollary 3.2. *Let $u(x, y)$ be a real-analytic function satisfying $J[\Delta u, \mathcal{H}(u)] = 0$. Assume that near the origin u has the form*

$$u(x, y) = \frac{\lambda}{2}(x^2 + y^2) + w(x, y) + o(\varrho)^k, \quad \varrho := \sqrt{x^2 + y^2},$$

where $w(x, y)$ is a homogeneous polynomial of degree $k \geq 3$ that is neither harmonic nor a power of a linear function $(\alpha x + \beta y)^k$, $(\alpha, \beta) \neq (0, 0)$. Then u is radial; i.e., $u = u(\varrho)$.

4. Continuity of eigendirections when $\mu = 1$

In what follows we keep the notation of Section 3. In particular, $p_0 = (0, 0) \in \mathcal{U}$ is a point in $\overline{\Omega}$ where $D^2 u = \lambda \operatorname{Id}$ for some $\lambda \in \mathbb{R}$. Recall that we can extend u as a real-analytic function to an open set $\Omega_0 \supset \overline{\Omega}$; thus, u is real-analytic in a neighborhood of $(0, 0)$ even if this point lies in $\partial\Omega$.

Also, recall that we can define for each $p \in \overline{\Omega} \setminus \mathcal{U}$ the eigenlines $L_1(p), L_2(p)$ of $D^2 u(p)$. Thus, L_1, L_2 define two real-analytic line fields on $\overline{\Omega} \setminus \mathcal{U}$; they are given in coordinates with respect to the basis (dx, dy) as the solutions to

$$-u_{xy}(dx^2 - dy^2) + (u_{xx} - u_{yy}) \, dx \, dy = 0. \tag{4-1}$$

These eigenlines naturally extend to $\Omega_0 \setminus \mathcal{U}_0$, where

$$\mathcal{U}_0 := \{p \in \Omega_0 : D^2 u(p) = \lambda \operatorname{Id} \text{ for some } \lambda = \lambda(p) \in \mathbb{R}\} \supset \mathcal{U}.$$

The next result shows that L_1, L_2 can be analytically extended around the origin if $(0, 0) \in \mathcal{U}$ is in case (C.2) above:

Proposition 4.1. *Assume that $p_0 = (0, 0) \in \mathcal{U}$ and that case (C.2) above happens at p_0, i.e., $\mu = 1$ in (3-8). Then, there exists $\varepsilon > 0$ such that $\Gamma := \mathbb{D}(\varepsilon) \cap \mathcal{U}_0$ is a regular, real-analytic curve passing through $(0, 0)$, and L_1, L_2 extend analytically across Γ, i.e., they define real-analytic line fields on $\mathbb{D}(\varepsilon) := \{\xi \in \mathbb{R}^2 : |\xi| < \varepsilon\}$. Moreover, one of L_1, L_2 is tangent to Γ at $(0, 0)$.*

Proof. Let $u_1(x, y)$ be the real-analytic function defined by (3-9). By (3-9) and (3-6), the first term in the series expansion of $u_1(x, y)$ around the origin is equal to $w(x, y) = ax^{n+2}$. Let us define $\eta(x, y)$ as the lowest-order homogeneous polynomial in the Taylor series of $u_1(x, y)$ that is *not* divisible by x^{n+2}. Thus, in the case it exists, its degree is $m + 2 > n + 2$. We will consider three cases:

<u>Case 1</u>: $\eta(x, y)$ does not exist. Therefore, $u_1(x, y) = x^{n+2}\phi(x, y)$ for some real-analytic function ϕ around the origin, with $\phi(0, 0) = a \neq 0$. Thus, using (3-9), a simple computation shows that

$$(\Delta u)^2 - 4\mathcal{H}(u) = (\Delta u_1)^2 - 4\mathcal{H}(u_1) = x^{2n}G(x, y)$$

for some real-analytic function $G(x, y)$ with $G(0, 0) = (2 + 3n + n^2)a^2 > 0$ (since $\phi(0, 0) = a \neq 0$). This implies by (3-2) that there exists $\varepsilon > 0$ such that $\mathbb{D}(\varepsilon) \cap \mathcal{U}$ coincides with the $x = 0$ axis.

Using (3-9) in a similar way, (4-1) for the eigenlines L_1, L_2 is written as

$$x^n\big(\Phi_1(x, y)(dx^2 - dy^2) + \Phi_2(x, y)\,dx\,dy\big) = 0, \tag{4-2}$$

where

$$\Phi_1(x, y) := -(n+2)x\phi_y - x^2\phi_{xy}, \quad \Phi_2(x, y) := (n+1)(n+2)\phi + x^2(\phi_{xx} - \phi_{yy}).$$

Obviously, (4-2) defines for each $x \neq 0$ the same directions as

$$\Phi_1(x, y)(dx^2 - dy^2) + \Phi_2(x, y)\,dx\,dy = 0. \tag{4-3}$$

Moreover, for $x = 0$ and y small enough, (4-3) is just $dx\,dy = 0$, since $\phi(0, 0) = a \neq 0$. As a consequence, the eigenlines L_1, L_2 extend analytically across the $x = 0$ axis in $\mathbb{D}(\varepsilon)$ for $\varepsilon > 0$ small enough. More specifically, at each point of the form $(0, y) \in \mathbb{D}(\varepsilon)$, these eigenlines are precisely $x = 0$ and $y = 0$. This proves Proposition 4.1 in Case 1.

<u>Case 2</u>: $\eta(x, y)$ has degree $n + 3$, i.e., $m = n + 1$. Since η is not divisible by x^{n+2}, we have that $\eta_{yy} \neq 0$. Using then that $u_1(x, y) = ax^{n+2} + \eta(x, y) + \cdots$ we see that the lowest-order nonzero Taylor polynomial of $\mathcal{H}(u_1)$ has degree $2n + 1$. Also, note that the lowest-order nonzero Taylor polynomial of Δu_1 has degree n.

Recall that $J[\Delta u_1, \mathcal{H}(u_1)] = 0$, by (3-10). Thus, there exists a nonconstant real-analytic function $\sigma_1(x, y)$ defined in a neighborhood of $(0, 0)$, with $\sigma_1(0, 0) = 0$, and nonconstant real-analytic functions of one variable $\alpha_1(t), \beta_1(t)$, with $\alpha_1(0) = \beta_1(0) = 0$, such that

$$\Delta u_1 = \alpha_1 \circ \sigma_1, \quad \mathcal{H}(u_1) = \beta_1 \circ \sigma_1. \tag{4-4}$$

Since, as explained above, Δu_1 (resp. $\mathcal{H}(u_1)$) has at the origin a zero of degree n (resp. $2n + 1$), and these two integers are coprime, we deduce from (4-4) that $\sigma_1(x, y)$ has a zero of order exactly 1 at the origin, i.e., $D\sigma_1(0, 0) \neq (0, 0)$; to see this, note that the vanishing order of σ_1 at the origin should necessarily be a divisor of both n and $2n + 1$, by (4-4). This also implies by (4-4) that α_1 (resp. β_1) has at the origin a zero of order n (resp. $2n + 1$). Thus, by the implicit function theorem and (4-4), we can choose local coordinates $(s(x, y), t(x, y))$, with $s(x, y) := \sigma_1(x, y)$ and $t(0, 0) = 0$, such that

$$\Delta u_1 = s^n g_1(s), \quad (\Delta u_1)^2 - 4\mathcal{H}(u_1) = s^{2n}g_2(s), \tag{4-5}$$

where $g_2(0) = g_1(0)^2 > 0$. The second equation in (4-5), together with the fact that $(\Delta u)^2 - 4\mathcal{H}(u) = (\Delta u_1)^2 - 4\mathcal{H}(u_1)$, shows that for $\varepsilon > 0$ small enough, $\mathbb{D}(\varepsilon) \cap \mathcal{U}_0$ agrees with the $s = 0$ curve. So, to prove

Proposition 4.1 in this situation we need to show that L_1, L_2 extend analytically across $s = 0$ and that one of them is tangent to $s = 0$ at the origin.

In order to prove this, let us observe that (4-5) implies that the eigenvalues μ_1, μ_2 of $D^2 u_1$ can be written in $\mathbb{D}(\varepsilon)$ as

$$\mu_i(s) = s^n \varphi_i(s), \quad i = 1, 2,$$

with $\varphi_1(0) = 0$ and $\varphi_2(0) = g_1(0) \neq 0$ (or vice versa). This implies that $D^2 u_1$ can be written as $s^n A(s, t)$, where $A(s, t)$ is a symmetric 2×2 matrix for each (s, t) close enough to the origin, and such that $A(0, 0)$ is not proportional to the identity. Consequently, from (3-9),

$$D^2 u = \lambda \operatorname{Id}_2 + s^n A(s, t).$$

Thus, if we denote by a_{ij} the coefficients of A, we see by (4-1) that the eigenlines L_1, L_2 are given as the solutions to the equation

$$s^n \left(-a_{12}(dx^2 - dy^2) + (a_{11} - a_{22}) \, dx \, dy \right) = 0. \tag{4-6}$$

Since A is not proportional to the identity at $(0, 0)$, the equation

$$-a_{12}(dx^2 - dy^2) + (a_{11} - a_{22}) \, dx \, dy = 0 \tag{4-7}$$

defines two real-analytic line fields in $\mathbb{D}(\varepsilon)$ for $\varepsilon > 0$ small enough, which by (4-6) agree with L_1, L_2 if $s \neq 0$. In other words, L_1, L_2 can be analytically extended across $s = 0$ around the origin, as wished.

Finally, we prove that L_1 or L_2 is tangent to $s = 0$ at the origin. From $D^2 u_1 = s^n A(s, t)$, we have $(u_1)_{xx} = s^n a_{11}$, $(u_1)_{xy} = s^n a_{12}$ and $(u_1)_{yy} = s^n a_{22}$. Therefore, $(s^n a_{11})_y = (s^n a_{12})_x$ and $(s^n a_{12})_y = (s^n a_{22})_x$. If we evaluate these equations at the origin, we obtain

$$s_y(0, 0) \, a_{11}(0, 0) = s_x(0, 0) \, a_{12}(0, 0), \quad s_y(0, 0) \, a_{12}(0, 0) = s_x(0, 0) \, a_{22}(0, 0).$$

Or equivalently, by the inverse function theorem,

$$-x_t(0, 0) \, a_{11}(0, 0) = y_t(0, 0) \, a_{12}(0, 0), \quad -x_t(0, 0) \, a_{12}(0, 0) = y_t(0, 0) \, a_{22}(0, 0). \tag{4-8}$$

Note that $(x_t(0, 0), y_t(0, 0))$ is tangent to $s = 0$ at the origin. From (4-8), we obtain at $(0, 0)$

$$-a_{12}(x_t^2 - y_t^2) + (a_{11} - a_{22}) x_t y_t = y_t(x_t a_{11} + y_t a_{12}) - x_t(x_t a_{12} + y_t a_{22}) = 0.$$

From (4-7), this implies that one of the analytic extensions of L_1 and L_2 is tangent to $s = 0$ at the origin, as wished. This proves Proposition 4.1 in Case 2.

Case 3: $\eta(x, y)$ has degree $> n + 3$, i.e., $m > n + 1$. We prove next that this case cannot happen, which together with the previous two cases will prove Proposition 4.1.

Similarly to our arguments in the proof of Proposition 3.1, let us start by noting that we can write $u_1(x, y) = h(x, y) + \psi(x, y)$, where $h(x, y)$ is a polynomial of degree at most $m + 1$ that is divisible by x^{n+2}, and $\psi := u_1 - h$ has $\eta(x, y)$ as the homogeneous polynomial of lowest degree in its series expansion. So, in our conditions, (3-12) holds. Next, consider the following facts:

(1) $J[\Delta u_1, \mathcal{H}(u_1)] = 0$, by (3-10). Thus, the left-hand side of (3-12) vanishes.

(2) $J[\Delta h, \mathcal{H}(h)]$ is divisible by x^{3n+1}, since h is divisible by x^{n+2}.

(3) The lowest-order term in the right-hand side of (3-12) not coming from $J[\Delta h, \mathcal{H}(h)]$ is given by

$$J[\Delta w, \{w, \eta\}] + J[\Delta \eta, \mathcal{H}(w)] \tag{4-9}$$

and has degree $2n + m - 2$ (if it is not identically zero).

Moreover, since in our situation $w(x, y) = ax^{n+2}$, we have that $\mathcal{H}(w) = 0$ and that (4-9) is a constant multiple of $x^{2n-1}\eta_{yyy}$. Since η is not divisible by x^{n+2} by hypothesis, we conclude from this discussion that the homogenous polynomial (4-9) is not divisible by x^{3n+1} (unless it is identically zero). Thus, adding this information to the three facts above, we conclude by (3-12) that $\eta_{yyy} = 0$. Consequently,

$$\eta(x, y) = x^m(a_1 x^2 + a_2 xy + a_3 y^2), \quad a_1, a_2, a_3 \in \mathbb{R}.$$

Since $m > n + 1$ by hypothesis, we see that x^{n+2} divides $\eta(x, y)$, a contradiction. So, Case 3 cannot happen, as claimed. □

5. Proof of Theorem 1.1

Following previous notation, let $p_0 = (0, 0)$ be a point in $\bar{\Omega}$ where $D^2 u = \lambda \, \mathrm{Id}$ for some $\lambda \in \mathbb{R}$ and for which situation (C.3) holds. Then, by (3-9) and Proposition 3.1, we have

$$u(x, y) = ax + by + c_0 + v(\sqrt{x^2 + y^2}), \tag{5-1}$$

globally on $\bar{\Omega}$ (by analyticity), where $v = v(r)$ is a real-analytic function, and a, b, c_0 are real constants.

If $a = b = 0$, then from (5-1) we see that $u = u(\varrho)$; i.e., u is a radial function with respect to the origin. From here, it is easy to check from the overdetermined conditions in (1-2) that Ω is a disk, and the result follows.

Assume next that $(a, b) \neq (0, 0)$. Up to a rotation in the (x, y)-coordinates, we can assume that $b = 0$, i.e., that

$$u(x, y) = ax + c_0 + v(\sqrt{x^2 + y^2}). \tag{5-2}$$

Moreover, let us observe that if u is a solution to (1-2) on Ω, and $t \neq 0$, then the function $u_t(x, y) := u(tx, ty)/t^2$ is a solution to (1-2) on $\Omega_t := t\Omega$ for the boundary constant $c_t := c/|t|$. Using this transformation, it is clear that we can assume without loss of generality that $a = 1$ holds in (5-2), i.e., that

$$u(x, y) = x + c_0 + v(\sqrt{x^2 + y^2}). \tag{5-3}$$

We will keep denoting by Ω the corresponding rotated and dilated domain in the plane; that is, Ω will be the simply connected planar domain for which (1-2) holds for u as in (5-3). By (5-3), we have

$$|Du|^2 = \frac{1}{\varrho^2}\left((\varrho + xv'(\varrho))^2 + y^2 v'(\varrho)^2\right) = 1 + v'(\varrho)^2 + \frac{2xv'(\varrho)}{\varrho}.$$

Since $u = 0$ along $\partial\Omega$, we conclude from (5-3) and the above equation that

$$|Du|^2 = 1 + v'(\varrho)^2 - \frac{2(c_0 + v(\varrho))}{\varrho} v'(\varrho) \quad \text{along } \partial\Omega.$$

So, since $|Du|^2 = c^2$ along $\partial\Omega$ by the Neumann condition in (1-2), we deduce that v is a solution to the ODE

$$1 + v'(\varrho)^2 - \frac{2(c_0 + v(\varrho))}{\varrho} v'(\varrho) = c^2. \tag{5-4}$$

Differentiating (5-4) we obtain

$$\frac{2(c_0 + v(\varrho) - \varrho v'(\varrho))(-v'(\varrho) + \varrho v''(\varrho))}{\varrho^2} = 0. \tag{5-5}$$

So, there are two options. If $c_0 + v(\varrho) - \varrho v'(\varrho) = 0$, then $v(\varrho) = -c_0 + t\varrho$, with $t \in \mathbb{R}$. Otherwise, we have $-v'(\varrho) + \varrho v''(\varrho) = 0$, from which we get $v(\varrho) = t_1\varrho^2 + t_2$, with $t_1, t_2 \in \mathbb{R}$.

In the first case, we have by (5-3) that $u(x, y) = x + t\sqrt{x^2 + y^2}$, and so its nodal set is contained in the union of two straight lines. Thus, this case is impossible, since $u = 0$ on $\partial\Omega$.

In the second case, we have by (5-3) that $u(x, y) = c_1 + x + c_2(x^2 + y^2)$ for some $c_1, c_2 \in \mathbb{R}$. Thus, $u(x, y)$ is radial with respect to the point $q_0 := (-1/(2c_2), 0) \in \mathbb{R}^2$, and Ω is a disk centered at q_0.

In conclusion, we have proved: *if there exists some point $p_0 \in \overline{\Omega}$ such that $D^2u = \lambda\,\mathrm{Id}$ for some $\lambda \in \mathbb{R}$, and for which case* (C.3) *holds, then Theorem 1.1 is true.*

So, to finish the proof, *we assume next that there is no point $p_0 \in \overline{\Omega}$ with $p_0 \in \mathcal{U}$ and for which case* (C.3) *holds* and reach a contradiction. Let L_1, L_2 denote, as usual, the line fields given by the eigenlines of D^2u. As explained previously, they are well-defined and analytic in $\overline{\Omega} \setminus \mathcal{U}$. Moreover, since by hypothesis there are no points in $\mathcal{U}$ for which case (C.3) happens, and by Proposition 4.1 the line fields L_1, L_2 are well-defined and analytic around any point for which case (C.2) holds, we deduce that L_1, L_2 are well-defined and analytic in $\overline{\Omega} \setminus \mathcal{U}_1$, where $\mathcal{U}_1 := \{p \in \mathcal{U} : \text{case (C.1) holds at } p\}$.

We next prove that the points in $\mathcal{U}_1$ are isolated. Indeed, let $p_0 \in \mathcal{U}_1$, and assume for simplicity that $p_0 = (0, 0)$. Again, we recall that u can be extended analytically to a neighborhood of the origin, even if $(0, 0) \in \partial\Omega$. Following the notation of Section 3, let w denote the first nonzero homogeneous polynomial of degree ≥ 3 of the series expansion of u at $(0, 0)$. Since $(0, 0) \in \mathcal{U}_1$, we know w is a homogeneous harmonic polynomial. By the first equation in (3-7), we have

$$(\Delta u)^2 - 4\mathcal{H}(u) = -4(w_{xx}w_{yy} - w_{xy}^2) + \cdots. \tag{5-6}$$

By harmonicity of w, we have $w_{xx}w_{yy} - w_{xy}^2 < 0$ in $\mathbb{R}^2 \setminus \{(0, 0)\}$. Thus, from (5-6) we conclude that $(\Delta u)^2 - 4\mathcal{H}(u) > 0$ in a punctured neighborhood of $(0, 0)$. In particular, $(0, 0)$ is isolated in $\mathcal{U}_1$, and the line fields L_1, L_2 are well-defined around $(0, 0)$, with an isolated singularity at the origin.

This fact, together with the boundary conditions, implies the following:

Claim. *L_1, L_2 are two analytic line fields with isolated singularities in $\overline{\Omega}$, the singularities being the points in $\mathcal{U}_1$. Moreover, each L_i, $i = 1, 2$, is either tangent or normal to $\partial\Omega$ at each $p \in \partial\Omega \setminus \mathcal{U}_1$.*

Proof of the Claim. The first statement follows directly from the previous discussion. For the second one, let $\gamma(s) = (x(s), y(s))$ be a unit speed parametrization of $\partial\Omega$, and let $\nu(s) := (-y'(s), x(s))$ denote the inner unit normal of $\partial\Omega$. By the overdetermined boundary conditions in (1-2), we have $Du(\gamma(s)) = \hat{c}\nu(s)$ for some constant $\hat{c} \in \mathbb{R}$. Differentiating this expression, we obtain that

$$D^2u(\gamma'(s), \nu(s)) = 0.$$

This implies that the tangent and normal lines to $\partial\Omega$ are eigenlines of D^2u at every $p \in \partial\Omega$. Thus, each L_i is tangent or normal to $\partial\Omega$ at each $p \in \partial\Omega \setminus \mathcal{U}$.

Consider finally a point $p \in \partial\Omega$ that lies in $\Sigma := \{p \in \partial\Omega \cap \mathcal{U} : p \notin \mathcal{U}_1\}$. Then, case (C.2) happens at p. By Proposition 4.1, we have two possible situations: either Σ is a finite set or $\Sigma = \partial\Omega$. In the first case, again by Proposition 4.1, we deduce by continuity that L_1, L_2 are well-defined at any $p \in \Sigma$ and are tangent or normal to $\partial\Omega$ at that point. Thus, the statement of the Claim follows. In the second case, all points of $\Sigma = \partial\Omega$ are in $\mathcal{U}$, and by Proposition 4.1 we deduce that L_1 or L_2 is globally tangent to $\partial\Omega$. This finishes the proof of the Claim.   □

In these conditions, it follows from the Claim and a standard application of the Poincaré–Hopf theorem that the sum of all rotation indices of each L_i at the isolated singularities of $\mathcal{U}_1$ is equal to 1.

Next, we will compute the rotation index of L_1, L_2 at $(0, 0) \in \mathcal{U}_1$. In order to do this, let us first look at the eigenlines L_1^w, L_2^w of D^2w. These eigenlines are given as the solutions to the equation

$$-w_{xy}(dx^2 - dy^2) + (w_{xx} - w_{yy})\, dx\, dy = 2\,\mathrm{Im}(w_{\zeta\zeta}\, d\zeta^2) = 0, \tag{5-7}$$

where $\zeta := x + iy$ and $\partial_\zeta := (\partial_x - i\partial_y)/2$. Since $w = a\zeta^{n+2}$ for $n \geq 1$ by (C.1), we see that the eigenlines L_1^w, L_2^w given by (5-7) have an isolated singularity at $(0, 0)$, and their rotation index at the origin is negative, equal to $-n/2$.

Assume next that $p_0 = (0, 0)$ is an interior point, i.e., $p_0 \in \Omega$. From (3-6) and the previous arguments, we see then that the eigenlines L_1, L_2 of D^2u, given by (4-1), are arbitrarily well-approximated around the origin by the eigenlines L_1^w, L_2^w, given by (5-7). In particular, the rotation index of both line fields L_1, L_2 around $(0, 0)$ is also negative and equal to $-n/2$.

In the case that $(0, 0) \in \partial\Omega$, a similar argument shows that the (boundary) index of L_1 and L_2 at $(0, 0)$ coincides with the *half-rotation index* of L_1^w, L_2^w at the origin in a half-plane, which is given by the value $-n/4$.

This proves that both L_1, L_2 only have isolated singularities of negative index in $\overline{\Omega}$, which contradicts that the sum of all such indices must be equal to 1, as explained above. This contradiction proves that there exists a point $p_0 \in \mathcal{U}$ for which (C.3) holds. So, by previous arguments, u is radial and Ω is a disk. This concludes the proof of Theorem 1.1.

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