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A geometric characterization of Powell-Sabin triangulations allowing the construction of C^2 quartic splines

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ABSTRACT

The paper deals with the characterization of Powell-Sabin triangulations allowing the construction of bivariate quartic splines of class C^2. The result is established by relating the triangle and edge split points provided by the refinement of each triangle. For a triangulation fulfilling the characterization obtained, a normalized representation of the splines in the C^2 space is given.

1. Introduction

The polynomial spline functions defined on triangulations are tools widely used in many different fields, both theoretical and applied. The book by Lai and Schumaker [1] presents an in-deph study of this type of functions, focusing mainly on the theoretical aspects.

As shown in [2], class C^m on an arbitrary triangulation of a polygonal domain is obtained if all derivatives up to order $2m$ at the vertices of the triangles are given. In particular, to get C^1 triangular splines on an arbitrary triangulation the values of the derivatives of order less than or equal to 2 at the vertices and the lowest degree is equal to 5 (see [2, Thm. 2] and the references therein).

In order to reduce the degree of the spline, it was proposed in [3] to refine each triangle by joining its vertices to an interior point. The Clough-Tocher refinement thus obtained allows to determine a C^1 spline of degree 3 and also a macro-triangle whose nodal parameters yield a C^1 piecewise polynomial of degree 4 (see [4] and the references therein). Introduced more than 50 years ago, C^1 cubic splines on Clough-Tocher partitions are still a subject of interest. For example, in [5] Gaussian quadrature for C^1 cubic Clough-Tocher macro-triangles is studied.

In [6], Powell and Sabin introduced a new refinement with the specific objective of contour plotting, managing to define a C^1 piecewise quadratic function from the values at the nodes of the function to be approximated and its gradient. Since then, interest in C^1 quadratic Powell-Sabin (PS-) splines has been maintained: for instance, in [7], the construction of normalized B-spline bases was addressed; in [8], differential and discrete quasi-interpolants were defined; and in [9] Gaussian quadrature was studied. Blossoming was also used to build C^1 quadratic quasi-interpolants on PS-partitions [10].

In [11] and [12] the study of cubic splines was carried out, constructing a normalized basis and quasi-interpolants that yield the optimal approximation order, respectively. Quasi-interpolation in C^1 cubic PS-splines was also addressed in [13]. New interesting results for this type of splines were published in [14–16].

The construction of C^2 PS-splines needs to consider a degree equal to five. In [17], normalized bases are constructed for these spaces, and polar forms are used in [18] to construct discrete and differential quasi-interpolants reproducing quintic polynomials. Interpolation with quintic PS-splines is addressed in [19].

The construction of C^2 quartic PS-splines has only been studied very recently, using the idea proposed in [20,14] to deal with the cubic case, namely to impose additional smoothness conditions on the nodes or inside each triangle.

In [21], this strategy is adopted to construct PS-splines that are almost C^2 continuous. Actually, the resulting functions are only C^1 continuous, although they are of class C^2 except across some edges of the refinement.

In some sense, the characterization obtained here can be seen as a continuation of the work [22]. Indeed, in [22] C^2 quartic splines on a modified Morgan-Scott refinement is discussed. The linear functionals involved in the Hermite interpolation problems in [22] and in this paper are the same, only the refinements are different. Unfortunately, the space developed in [22] is only defined under specific geometrical

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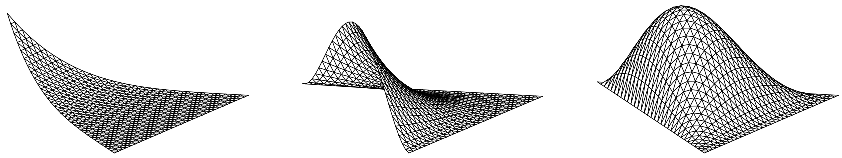

Fig. 1. From left to right, plots of the quartic Bernstein polynomials $\mathfrak{B}^4_{400,T}$, $\mathfrak{B}^4_{220,T}$ and $\mathfrak{B}^4_{211,T}$.

conditions. When the three inner points used to define the refinement collapse, this space is not defined, and this is the starting point for the work done in this paper. The construction of C^2 quartic splines over refined triangulations with modified Morgan-Scott split is also studied in [23] (see [24] in the case of C^1 quadratic splines). The authors in [23], first, they analysed the construction of C^2 quartic splines on a single macro-triangle endowed with a modified Morgan-Scott split. Then, they examined the problem of how to join the local C^2 interpolating splines on macro-triangles to a quartic spline that is C^2 continuous everywhere. Unfortunately, this results in a global system of linear equations, whose solvability, in general, is very difficult to analyse theoretically. This is because that, the linear system depends on the positions of the triangle split points and the edge split points that determine the modified Morgan-Scott split. The relations between the triangle split points and the edge split points involved in [22] can be viewed as a special case where this linear system has a unique solution.

Several families of PS-super splines of arbitrary degree (and corresponding regularity) have been introduced in the literature [25,26], and also quasi-interpolation operators based on PS-splines of arbitrary class r and degree $3r-1$ have been defined [27].

In this article, the main objective is to characterize the geometry of Powell-Sabin triangulations that allows C^2 class bivariate quartic splines to be defined.

The remainder of this paper is organized as follows. Section 2 recalls some concepts about the Bernstein-Bézier representation of bivariate polynomials defined on triangles. In Section 3, the quartic spline space is defined and the interpolation problem that uniquely determines each element of the spline space is stated and analyzed. Section 4 is devoted to define a basis of the Powell-Sabin space relative to a triangle, characterizing Powell-Sabin refinements allowing C^2 quartic splines on that triangle and to define an appropriate basis. In section 5, the dimension of the reduced space is given and C^2-continuity on the whole triangulation is addressed. Thus, the construction of a B-spline-like basis is outlined. Finally, in Section 6, we summarize the results obtained.

2. Preliminaries and notations

Given a non-negative integer d, let $\mathbb{P}_d$ be the linear space of all bivariate polynomials. They can be expressed in terms of the standard monomial basis $\left\{x^i y^j, i \geq 0, j \geq 0, i+j \leq d\right\}$. The construction of spline functions on triangulations makes it advisable to represent the polynomials on any triangle on the basis of the Bernstein polynomials. Let $T\langle V_1, V_2, V_3\rangle$ be a non-degenerated triangle in $\mathbb{R}^2$ with vertices $V_\ell := \left(x_\ell, y_\ell\right)$, $\ell = 1, 2, 3$. Any point $V = (x, y)$ in the plane has a unique representation of the form $V = \sum_{\ell=1}^{3} \tau_\ell V_\ell$, where the barycentric coordinates $\tau := \left(\tau_1, \tau_2, \tau_3\right)$ satisfy that $\tau_1 + \tau_2 + \tau_3 = 1$. The Bernstein polynomials of degree d, relative to T are defined as

$$\mathfrak{B}^d_{\beta,T}(\tau) := \frac{d!}{\beta_1!\,\beta_2!\,\beta_3!}\,\tau_1^{\beta_1}\,\tau_2^{\beta_2}\,\tau_3^{\beta_3},$$

where $\beta := \left(\beta_1, \beta_2, \beta_3\right) \in \mathbb{N}_0^3$ with length $|\beta| := \beta_1 + \beta_2 + \beta_3 = d$. They form a partition of unity, i.e. it holds $\sum_{|\beta|=d} \mathfrak{B}^d_{\beta,T}(\tau) = 1$ for all $\tau \in \mathbb{R}^3$. If V belongs to the triangle T, then $\tau_\ell \geq 0$ and any polynomial p can be represented on T as

$$p(x, y) = \sum_{|\beta|=d} b^d_{\beta,T} \mathfrak{B}^d_{\beta,T}(\tau), \tag{1}$$

$b^d_{\beta,T}$ being its Bézier (B-) ordinates or Bernstein-Bézier (BB-) coefficients of p. We refer to (1) as BB-representation of p. The BB-coefficients $b^d_{\beta,T}$ of the polynomial p with respect to the triangle T are associated with the domain points $\xi_{\beta,T} := \left(\frac{\beta_1}{d}, \frac{\beta_2}{d}, \frac{\beta_3}{d}\right)$, $|\beta| = d$, which define the lattice $\mathcal{D}_{d,T} := \left\{\xi_{\beta,T}, |\beta| = d\right\}$. If there is no possibility of confusion, any reference to the triangle T is omitted in writting the Bernstein polynomials, the BB-coefficients, the domain points and the lattice that they define. It is well-known that any point $(x, y, p(x, y))$, lying in the graph of the surface $z = p(x, y)$ on the triangle T can be written as

$$(x, y, p(x, y)) = \sum_{|\beta|=d} \left(\xi_{\beta,T}, b^d_{\beta,T}\right) \mathfrak{B}^d_{\beta,T}(\tau),$$

so that the graph of the surface is obained as convex linear combinations of the so-called control points. At the vertices of T, the surface interpolates the control points. The Fig. 1 shows the typical view of some quartic Bernstein polynomials.

Let Δ be a triangulation of a simply connected polygonal domain $\Omega \subset \mathbb{R}^2$. Given $0 \leq r < d$, we consider the spline space of degree d on Δ with global C^r continuity, defined as

$$S^r_d(\Delta) := \left\{s \in C^r(\Omega) \,:\, s_{|T} \in \mathbb{P}_d,\ T \in \Delta\right\}.$$

The smoothness conditions between adjacent polynomial patches are easily expressed in terms of the BB-coefficients relative to the triangles. Let $\hat{T} := \langle V_4, V_2, V_3\rangle$ be an adjacent triangle to T and $\hat{p}$ a polynomial of total degree d defined on $\hat{T}$. Assume that V_4 has $\hat{\tau} := \left(\hat{\tau}_1, \hat{\tau}_2, \hat{\tau}_3\right)$ as vector of barycentric coordinates with respect to T. Then the function defined by assembling p and $\hat{p}$ is of class C^r across the edge $\langle V_2, V_3\rangle$ if the B-ordinates $\hat{b}_{\beta,\hat{T}}$ of $\hat{p}$ satisfy for $\beta_1 = 0, \ldots, r$ and $\beta_2 + \beta_3 = d - r$ the conditions

$$\hat{b}_{\beta,\hat{T}} = \sum_{|\alpha|=\beta_1} b_{\alpha+\beta_2 e_2+\beta_3 e_3,T} \mathfrak{B}^r_{\alpha,T}(\hat{\tau}), \tag{2}$$

where $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$.

3. Quartic Powell-Sabin splines

A Powell-Sabin (PS-) refinement Δ_{PS} of Δ is obtained by decomposing each macro triangle T into six micro-triangles as follows [6]:

1. Choose an interior point Z_j in each triangle T_j and connect it to the three vertices of T_j by straight lines.
2. For each pair of triangles T_i and T_j with a common edge, connect the two points Z_i and Z_j, and let $R_{i,j}$ be the intersection point with the common edge.
3. If T_j is a boundary triangle, then also connect Z_j to an arbitrary point on each of the boundary edges.

The choice of the triangle split points determines the position of the edge split points. Reciprocally, the location of the edge-split points of a given triangle impose restrictions on the positions of the triangle-split points of the surrounding triangles. Fig. 2 shows on the right a triangulation along with the Powell-Sabin refinement of its triangles, and on the left a single triangle to introduce the notation to be used. In general, the triangle split point Z and the edge split points $R_{2,3}$, $R_{3,1}$ and $R_{1,2}$ are not collinear with V_1, V_2 and V_3, respectively. However, for

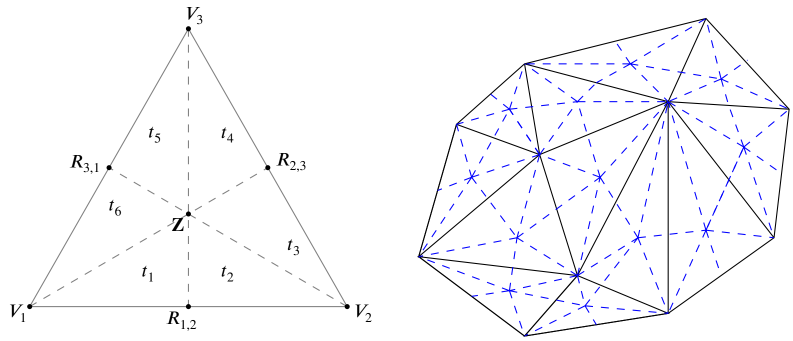

Fig. 2. (Left) Powell-Sabin split of a single triangle $T\langle V_1, V_2, V_3\rangle$. (Right) Powell-Sabin refinement of Δ.

each triangle of the partition the schematic representation on the left in Fig. 2 is adopted.

Let $\mathcal{V} := \{V_i\}_{i=1}^{nv}$, $\mathcal{E} := \{\mathfrak{e}_i\}_{i=1}^{ne}$ and $\mathcal{Z} := \{Z_i\}_{i=1}^{nt}$ be the subsets of vertices and edges in Δ, and split points in Δ_{PS}, respectively. Moreover, let $\mathcal{E}^*$ be the subset of edges in Δ_{PS} that connect the split points in $\mathcal{Z}$ to the associated edge split points after applying the steps 1 and 2 in the previous algorithm. Denote by $\mathcal{R}$ the subset composed by the edge split points $R_{i,j}$.

In [21], a normalized basis of the subspace

$$S_4^{1,2}\left(\Delta_{\mathrm{PS}}\right) := \left\{ s \in S_4^1\left(\Delta_{\mathrm{PS}}\right) : \ s \in C^2\left(\mathcal{V} \cup \mathcal{Z} \cup \mathcal{E} \cup \mathcal{E}^*\right)\right\}.$$

of $S_4^1\left(\Delta_{\mathrm{PS}}\right)$ is constructed. Its dimension is equal to $6nv + 3ne$. The splines in this subspace are C^2 continuous everywhere except across the edges that connect the split points and the vertices.

In this work, we consider the following subspace of $S_4^{1,2}\left(\Delta_{\mathrm{PS}}\right)$ [21]:

$$S_4^{1,2,3}\left(\Delta_{\mathrm{PS}}\right) := \left\{ s \in S_4^{1,2}\left(\Delta_{\mathrm{PS}}\right) : s \in C^3\left(\mathcal{E}^*\right)\right\}. \tag{3}$$

Here, $C^3(\mathcal{E}^*)$ means that for any edge $\mathfrak{e} \in \mathcal{E}^*$ the polynomials over the two micro-triangles sharing $\mathfrak{e}$ have common derivatives up to order three along $\mathfrak{e}$. Splines in $S_4^{1,2,3}\left(\Delta_{\mathrm{PS}}\right)$ are C^3 continuous at the set of edge split points and C^2 at the set of triangle split points.

This is not a classical super spline space because additional continuity has been imposed across certain, but not all, interior edges of Δ_{PS}.

A spline $s \in S_4^{1,2,3}\left(\Delta_{\mathrm{PS}}\right)$ can be defined by means of the following Hermite interpolation problem.

Theorem 1. *There exists a unique spline $s \in S_4^{1,2,3}\left(\Delta_{PS}\right)$ solving the interpolation problem*

$$\begin{aligned} &D_x^a D_y^b s\left(V_i\right) = f_i^{a,b}, \ i = 1, \dots, nv, \ a \geq 0, \ b \geq 0, \ a + b \leq 2, \\ &D^2_{\omega_{m,n,q}} s\left(R_{m,n}\right) = g_{m,n} \quad \text{for all } R_{m,n} \in \mathcal{R}, \ \ R_{m,n} \in \langle V_m, V_n\rangle, \end{aligned} \tag{4}$$

for given values $f_i^{a,b}$ and $g_{m,n}$, $\omega_{m,n,q}$ being a unit direction parallel to $\left\langle R_{m,n}, Z_q\right\rangle$, where Z_q is the triangle split point of a triangle T_q having $\langle V_m, V_n\rangle$ as an edge.

Proof. The proof will be done on a single macro-triangle. Its extension to the whole triangulation is deduced from Theorem 1 in [21]. To prove the unisolvency of the interpolation problem on a macro-triangle T, we only need to determine the BB-coefficients on T of a spline s satisfying (4). For the sake of simplicity, and without loss of generality, consider a single macro-triangle $T\langle V_1, V_2, V_3\rangle$. On each micro-triangle in T, the spline s is a quartic polynomial (see Fig. 3). We will show how the BB-coefficients of s are uniquely determined by conditions (4) and the smoothness requirements.

Since the spline s is C^2 continuous at vertices V_i, $i = 1, 2, 3$, then the values and derivatives up to order 2 at each vertex in (4) are uniquely determined by the BB-coefficients relative to the domain points lying in the disks of radius 2 associated with the vertices of T, i.e. the subsets

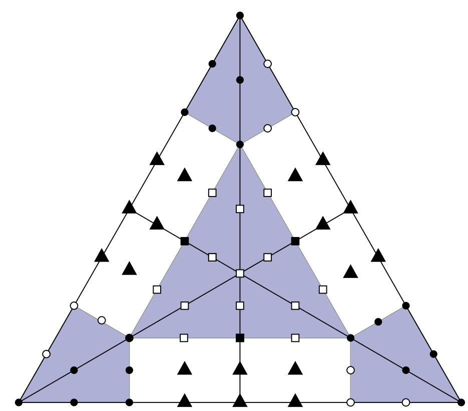

Fig. 3. The subset $\mathcal{D}_{4,T}$ relative to a macro-triangle T of Δ_{PS}. The B-ordinates of the restriction to T of a spline $s \in S_4^{1,2,3}\left(\Delta_{\mathrm{PS}}\right)$ are determined for the specified subsets of domain points from the interpolation conditions at the vertices and the regularity of s.

each consisting of the nine domain points lying in each of the coloured neighbouring regions of the vertices shown in Fig. 3, and which are represented by the symbols • and ∘.

To deal with C^2 smoothness at triangle split point Z, we define the triangle with vertices

$$W_i := \frac{V_i + Z}{2}, \ i = 1, 2, 3. \tag{5}$$

The BB-coefficients relative to the domain points in this triangle are computed by our construction. Also the BB-coefficients marked with ■ are determined from the second derivative of s in the specified direction given in (4), to give six independent constraints that yield a quadratic polynomial p_2 in $\tilde{T}\langle W_1, W_2, W_3\rangle$ from which the BB-coefficients related to the domain points ordinates indicated by □ in Fig. 3 are determined.

The remaining BB-coefficients, indicated by ▲, and placed in the 0th and 1st rows parallel to edge $\left\langle V_i, V_j\right\rangle$ are computed from C^3 smoothness conditions along $\left\langle R_{i,j}, Z\right\rangle$. Fot $\ell = 0$, let b_k^0, $k = 1, \dots, 7$, be the seven central BB-coefficients placed on 0th row parallel to edge $\left\langle V_i, V_j\right\rangle$. They can be considered as the BB-coefficients of the univariate cubic polynomial p_3^0 defined on the segment $\left[\hat{W}_{i,j}^0, \tilde{W}_{i,j}^0\right]$ with

$$\hat{W}_{i,j}^0 := \frac{3}{4}V_i + \frac{1}{4}R_{i,j} \quad \text{and} \quad \tilde{W}_{i,j}^0 := \frac{3}{4}V_j + \frac{1}{4}R_{i,j}$$

having BB-coefficients b_1^0, b_2^0, b_6^0 and b_7^0 (see Fig. 4). After subdivision, b_3^0, b_4^0 and b_5^0 result. This construction ensures that the spline is C^3 at $R_{i,j}$. To determine the BB-coefficients b_k^1, $k = 1, \dots, 7$, associated with the domain points lying on the 1st row paralell to edge $\left\langle V_i, V_j\right\rangle$, a similar approach is applied, by considering the points

$$\hat{W}_{i,j}^1 := \frac{3}{4}V_i + \frac{1}{4}Z \quad \text{and} \quad \tilde{W}_{i,j}^1 := \frac{3}{4}V_j + \frac{1}{4}Z,$$

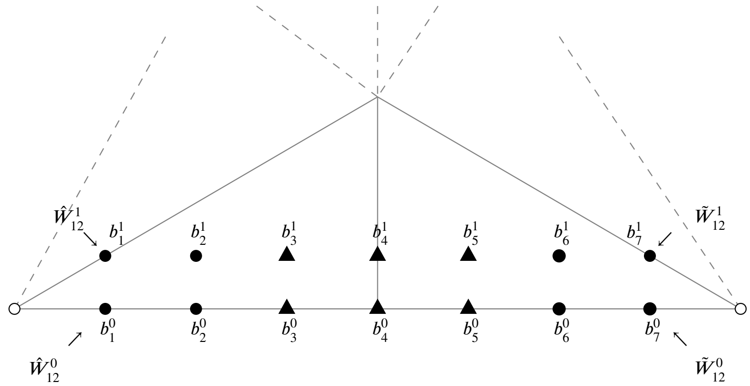

Fig. 4. The seven central BB-coefficients placed on ℓth ($\ell = 0, 1$) row parallel to edge $\langle V_1, V_2\rangle$.

and the polynomial p_3^1 defined on $\left[\hat{W}^1_{i,j}, \tilde{W}^1_{i,j}\right]$ with BB-coefficients b_1^1, b_2^1, b_6^1 and b_7^1. The BB-coefficients b_1^ℓ, b_2^ℓ, b_6^ℓ and b_7^ℓ, $\ell = 0, 1$, have been already determined by the interpolation conditions (4) at V_i and V_j. This construction ensures that the spline is C^3 across the edge $\left\langle R_{i,j}, Z\right\rangle$.

The construction above is carried out on the macro-triangle T. The rest of the proof runs as in [21, Thm. 1]. $\square$

In what follows, we divide the work into two parts. In the first one, we discuss the space of quartic Powell-Sabin splines on a single macro-triangle T, wherein we investigate the necessary and sufficient conditions to achieve global C^2 smoothness on T. The second part is devoted to extend the results obtained for a macro-triangle to the whole triangulation.

4. The Powell-Sabin space on a single triangle

As mentioned earlier, we are looking for geometrical conditions ensuring that $S_4^{1,2,3}\left(\Delta_{\rm PS}\right)$ becomes of C^2 continuity. To do that, we start by analysing the Powell-Sabin space relative to a single triangle by defining an appropriate basis for it, then, we will generalize the obtained results on the whole triangulation.

Consider the macro-triangle $T\langle V_1, V_2, V_3\rangle$, with $V_1 = \left(x_1, y_1\right)$, $V_2 = \left(x_2, y_2\right)$ and $V_3 = \left(x_3, y_3\right)$ (see Fig. 2 (left)). The barycentric coordinates of the vertices V_1, V_2 and V_3 w.r.t. T are $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$, respectively. Suppose that the barycentric coordinates of $Z = \left(x_z, y_z\right)$ are $\left(z_1, z_2, z_3\right)$, and let $\left(\lambda_{1,2}, \lambda_{2,1}, 0\right)$, $\left(0, \lambda_{2,3}, \lambda_{3,2}\right)$ and $\left(\lambda_{1,3}, 0, \lambda_{3,1}\right)$ be coordinates of $R_{1,2} = \left(x_{1,2}, y_{1,2}\right)$, $R_{2,3} = \left(x_{2,3}, y_{2,3}\right)$ and $R_{3,1} = \left(x_{3,1}, y_{3,1}\right)$, respectively. Moreover, we can write

$$R_{1,2} = \tau_{1,1}\, V_2 + \tau_{2,1}\, R_{2,3} + \tau_{3,1}\, Z,\ \ R_{2,3} = \tau_{1,2}\, V_3 + \tau_{2,2}\, R_{3,1} + \tau_{3,2}\, Z,$$
$$R_{3,1} = \tau_{1,3}\, V_1 + \tau_{2,3}\, R_{1,2} + \tau_{3,3}\, Z,$$

where

$$\left(\tau_{1,1},\, \tau_{2,1},\, \tau_{3,1}\right) := \left(\frac{\lambda_{1,2} z_3 + \lambda_{3,2}\left(\lambda_{2,1} + z_1 - 1\right)}{\lambda_{3,2} z_1}, -\frac{\lambda_{1,2} z_3}{\lambda_{3,2} z_1}, \frac{\lambda_{1,2}}{z_1}\right),$$
$$\left(\tau_{1,2},\, \tau_{2,2},\, \tau_{3,2}\right) := \left(\frac{-z_3 \lambda_{2,3} + \lambda_{3,2} z_2 - \lambda_{31}\left(z_2 - \lambda_{2,3}\right)}{\lambda_{1,3} z_2}, -\frac{\lambda_{2,3} z_1}{\lambda_{1,3} z_2}, \frac{\lambda_{2,3}}{z_2}\right), \tag{6}$$
$$\left(\tau_{1,3},\, \tau_{2,3},\, \tau_{3,3}\right) := \left(\frac{\lambda_{3,1}\left(z_2 - \lambda_{2,1}\right)}{\lambda_{2,1} z_3} + 1, -\frac{\lambda_{3,1} z_2}{\lambda_{2,1} z_3}, \frac{\lambda_{3,1}}{z_3}\right).$$

Let us suppose that T is decomposed into the following micro-triangles t^ℓ, $\ell = 1, \ldots, 6$:

$$t^1\left\langle V_1, R_{1,2}, Z\right\rangle, t^2\left\langle R_{1,2}, V_2, Z\right\rangle, t^3\left\langle V_2, R_{2,3}, Z\right\rangle, t^4\left\langle R_{2,3}, V_3, Z\right\rangle,$$
$$t^5\left\langle V_3, R_{3,1}, Z\right\rangle, t^6\left\langle R_{3,1}, V_1, Z\right\rangle.$$

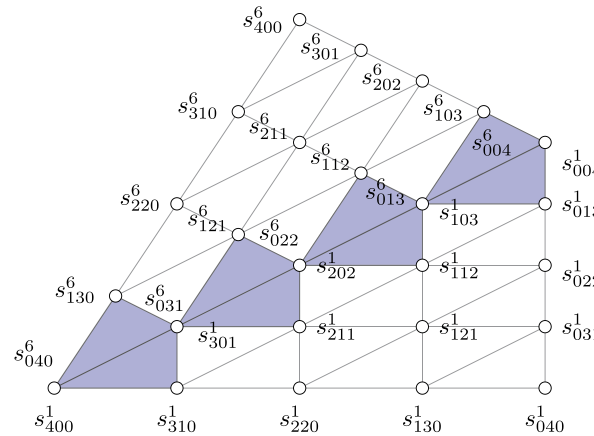

Fig. 5. The B-ordinates relative to micro-triangles t^1 and t^6 sharing vertex V_1 are shown. The other follow cyclically. The control net triangles involved in the C^1 continuity conditions between s^1 and s^6 are shown in blue.

Let s^ℓ be the restriction of s to t^ℓ, and $s^\ell_{i,j,k}$, $i + j + k = 4$, be its BB-coefficients.

The continuity of s on T is easily expressed in terms of the BB-coefficients. For instance, the continuity across the micro-edge $\langle Z, V_1\rangle$ is equivalent to the fulfilment of conditions

$$s^1_{4-j,0,j} = s^6_{0,4-j,j},\ j = 0, \ldots, 4.$$

The conditions yielding the continuity across $\langle Z, R_{1,2}\rangle$, $\langle Z, V_2\rangle$, $\langle Z, R_{2,3}\rangle$, $\langle Z, V_3\rangle$ and $\langle Z, R_{3,1}\rangle$ are similar and involve the BB-coefficients of $\left\{s^1, s^2\right\}$, $\left\{s^2, s^3\right\}$, $\left\{s^3, s^4\right\}$, $\left\{s^4, s^5\right\}$ and $\left\{s^5, s^6\right\}$, respectively (see Fig. 5).

We also recall that the C^1 continuity of s across $\langle Z, V_1\rangle$ is expressed as

$$s^6_{1,3-j,j} = \tau_{1,3} s^1_{4-j,0,j} + \tau_{2,3} s^1_{3-j,1,j} + \tau_{3,3} s^1_{3-j,0,j+1}, \quad j = 0, 1, 2, 3,$$

where the barycentric coordinates $\left(\tau_{1,3},\, \tau_{2,3},\, \tau_{3,3}\right)$ of $R_{3,1}$ with respect to $t^1\left\langle V_1,\, R_{1,2},\, Z\right\rangle$ are given in (6). Similar expressions are obtained for the C^1 continuity across the micro-edges $\left\langle Z, R_{1,2}\right\rangle$, $\langle Z, V_2\rangle$, $\left\langle Z, R_{2,3}\right\rangle$, $\langle Z, V_3\rangle$ and $\left\langle Z, R_{3,1}\right\rangle$ that use the barycentric coordinates of V_2, $R_{2,3}$, V_3, $R_{3,1}$ and V_1 w.r.t. t^1, t^2, t^3, t^4 and t^5, respectively.

Definition 2. Let C_1, C_2 and C_3 be the unique solutions given by Theorem 1 associated with the interpolation data $f_i^{a,b} = 0$, $i = 1, 2, 3$, $a, b \geq 0$, $a + b \leq 2$, and

$$g_{1,2} = \frac{24 \lambda_{1,2} \lambda_{2,1}}{\|Z - R_{1,2}\|^2}, \quad g_{2,3} = g_{3,1} = 0,$$
$$g_{2,3} = \frac{24 \lambda_{2,3} \lambda_{3,2}}{\|Z - R_{2,3}\|^2}, \quad g_{1,2} = g_{3,1} = 0,$$

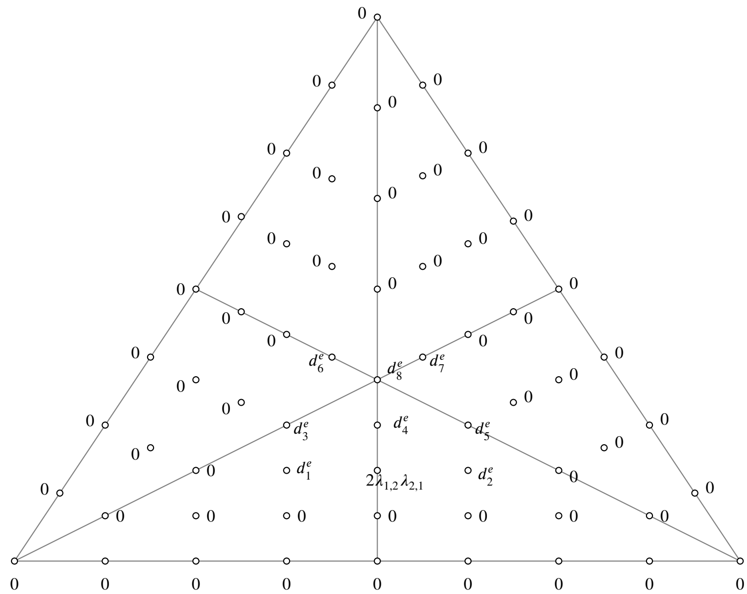

Fig. 6. Bernstein-Bézier coefficients of blending function $\mathcal{C}_1$.

$$g_{3,1} = \frac{24\lambda_{3,1}\lambda_{1,3}}{\|Z - R_{3,1}\|^2}, \quad g_{1,2} = g_{2,3} = 0,$$

respectively. We call $\mathcal{C}_\ell$, $\ell = 1,2,3$, the blending functions of the first kind relative to V_ℓ.

The f-values yielding the blending functions above are all equal to zero. New blending functions results when all g-values are zero.

Definition 3. Let $\mathcal{D}_1$ be the unique solution given by Theorem 1 associated with the values $g_{1,2} = g_{2,3} = g_{3,1} = 0$, $f_2^{a,b} = f_3^{a,b} = 0$ for $a, b \geq 0$ and $a + b \leq 2$, $f_1^{0,0} = 0$, and

$$f_1^{1,0} = \frac{4}{F_1}\left(y_1 - y_z\right),$$
$$f_1^{0,1} = -\frac{4}{F_1}\left(x_1 - x_z\right),$$
$$f_1^{2,0} = \frac{12}{F_1^2}\left(y_1 - y_z\right)\left(\lambda_{1,2}y_1 + \left(1 + \lambda_{2,1}\right)y_z - 2y_{1,2}\right),$$
$$f_1^{1,1} = \frac{12}{F_1^2}\left(-x_2\left(\lambda_{1,2}\left(y_z - y_1\right) - 2y_z + y_1 + y_{1.2}\right)\right.$$
$$\left.+x_1\left(-\lambda_{1,2}y_1 - \lambda_{2,1}y_z + y_r\right) + x_r\left(y_1 - y_z\right)\right),$$
$$f_1^{0,2} = \frac{12}{F_1^2}\left(x_1 - x_z\right)\left(\lambda_{1,2}x_1 + \left(1 + \lambda_{2,1}\right)x_z - 2x_{1,2}\right),$$

with

$$F_1 := x_z\left(y_{1,2} - y_1\right) + x_1\left(y_z - y_r\right) + x_{1,2}\left(y_1 - y_z\right).$$

We call $\mathcal{D}_1$ the blending function of the second kind relative to V_1.

For vertices V_2 and V_3, the blending functions of the second kind $\mathcal{D}_2$ and $\mathcal{D}_3$ are defined respectively as solutions of the Hermite interpolation problem in Theorem 1 with the following datasets:

1. $g_{1,2} = g_{2,3} = g_{3,1} = 0$, $f_1^{a,b} = f_3^{a,b} = 0$ for $a, b \geq 0$ and $a + b \leq 2$, $f_2^{0,0} = 0$, and

$$f_2^{1,0} = \frac{4}{F_2}\left(y_2 - y_z\right),$$
$$f_2^{0,1} = -\frac{4}{F_2}\left(x_2 - x_z\right),$$
$$f_2^{2,0} = \frac{12}{F_2^2}\left(y_2 - y_z\right)\left(\lambda_{1,2}y_2 + \left(1 + \lambda_{2,1}\right)y_z - 2y_{2,3}\right),$$
$$f_2^{1,1} = \frac{12}{F_2^2}\left(-x_z\left(\lambda_{1,2}\left(y_z - y_2\right) - 2y_z + y_2 + y_{2,3}\right)\right.$$
$$\left.+x_2\left(-\lambda_{1,2}y_2 - \lambda_{2,1}y_z + y_{2,3}\right) + x_{2,3}\left(y_2 - y_z\right)\right),$$
$$f_2^{0,2} = \frac{12}{F_2^2}\left(x_2 - x_z\right)\left(\lambda_{1,2}x_2 + \left(1 + \lambda_{2,1}\right)x_z - 2x_{2,3}\right),$$

with $F_2 := x_z\left(y_{2,3} - y_2\right) + x_2\left(y_z - y_{2,3}\right) + x_{2,3}\left(y_2 - y_z\right)$.

2. $g_{1,2} = g_{2,3} = g_{3,1} = 0$, $f_1^{a,b} = f_2^{a,b} = 0$ for $a, b \geq 0$ and $a + b \leq 2$, $f_3^{0,0} = 0$, and

$$f_3^{1,0} = \frac{4}{F_3}\left(y_3 - y_z\right),$$
$$f_3^{0,1} = -\frac{4}{F_3}\left(x_3 - x_z\right),$$
$$f_3^{2,0} = \frac{12}{F_3^2}\left(y_3 - y_z\right)\left(\lambda_{1,2}y_3 + \left(1 + \lambda_{2,1}\right)y_z - 2y_{3,1}\right),$$
$$f_3^{1,1} = \frac{12}{F_3^2}\left(-x_z\left(\lambda_{1,2}\left(y_z - y_3\right) - 2y_z + y_3 + y_{3,1}\right)\right.$$
$$\left.+x_3\left(-\lambda_{1,2}y_3 - \lambda_{2,1}y_z + y_{3,1}\right) + x_{3,1}\left(y_3 - y_z\right)\right),$$
$$f_3^{0,2} = \frac{12}{F_3^2}\left(x_3 - x_z\right)\left(\lambda_{1,2}x_3 + \left(1 + \lambda_{2,1}\right)x_z - 2x_{3,1}\right),$$

with $F_2 := x_z\left(y_{3,1} - y_3\right) + x_3\left(y_z - y_{3.1}\right) + x_{3,1}\left(y_3 - y_z\right)$.

On each micro-triangle t^ℓ, $\ell = 1, \ldots, 6$, the splines $\mathcal{C}_1$ and $\mathcal{D}_1$, are quartic polynomials that can be represented according to (1). The corresponding BB-coefficients are schematically represented in Figs. 6 and 7, respectively. They are given by

$$d_1^e = \lambda_{2,1},\ d_2^e = \lambda_{1,2},\ d_3^e = z_2,\ d_4^e = \lambda_{1,2}z_2 + \lambda_{2,1}z_1,\ d_5^e = z_1,\ d_6^e = \lambda_{1,3}z_2,$$
$$d_7^e = z_1\lambda_{2,3},\ d_8^e = 2z_1z_2,$$

and

$$d_1^v = 1,\ d_2^v = \lambda_{1,2},\ d_3^v = \lambda_{1,2}^2,\ d_4^v = \lambda_{1,2}^3,\ d_5^v = \tau_{2,3},\ d_6^v = \tau_{2,3}\lambda_{1,3},$$
$$d_7^v = \tau_{2,3}\lambda_{1,3}^2,\ d_8^v = \tau_{2,3}\lambda_{1,3}^3.$$

Fig. 8 shows the typical plots of blending functions.

$S_4^{1,2,3}(T)$ is a linear space with dimension equal to 21 and its subspace $\mathbb{P}_4$ has dimension 15, so we can think of extending a basis for $\mathbb{P}_4$ to one for $S_4^{1,2,3}(T)$.

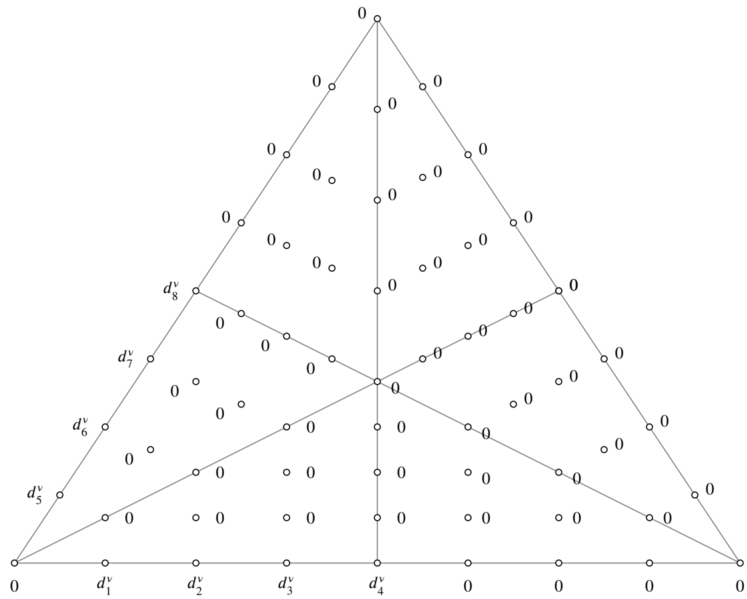

Fig. 7. Bernstein-Bézier coefficients of blending function $\mathcal{D}_1$.

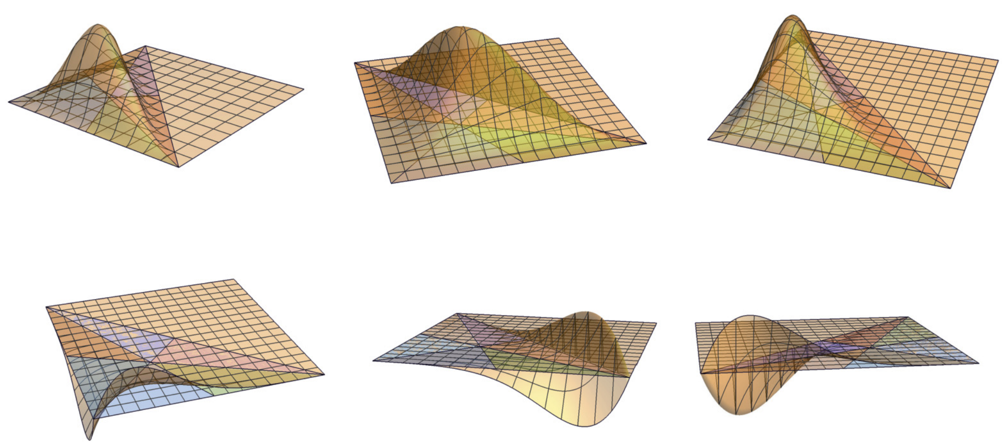

Fig. 8. (Top) Blending functions C_i and (bottom) $\mathcal{D}_i$.

Proposition 4. *It holds that*

$$S_4^{1,2,3}(T) = \mathbb{P}_4 \oplus \operatorname{span}\left\{\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, C_1, C_2, C_3\right\}.$$

Proof. As all functions $\mathcal{D}_\ell$ and C_ℓ are in $S_4^{1,2,3}(T)$, it only remains to show that no non-trivial linear combination of those functions is in $\mathbb{P}_4$. Assume that there exist non-zero coefficients d_i and c_i such that

$$P := d_1\,\mathcal{D}_1 + d_2\,\mathcal{D}_2 + d_3\,\mathcal{D}_3 + c_1\,C_1 + c_2\,C_2 + c_3\,C_3 \in \mathbb{P}_4.$$

Then, in particular, P is of C^4 continuity across $\langle Z, R_{1,2}\rangle$, $\langle Z, R_{2,3}\rangle$ and $\langle Z, R_{3,1}\rangle$, so that

$$0 = \frac{d_1 \lambda_{1,2}^3}{\lambda_{2,1}^4} + \frac{d_2 z_3 \lambda_{1,2}}{z_1 \lambda_{2,1} \lambda_{3,2}},$$

$$0 = \frac{\lambda_{2,3}\left(d_3 z_1 \lambda_{3,2}^3 + d_2 z_2 \lambda_{1,3} \lambda_{2,3}^2\right)}{z_2 \lambda_{1,3} \lambda_{3,2}^4},$$

$$0 = -\frac{d_1 z_2 \lambda_{1,3}^3 + d_3 z_3 \lambda_{2,1} \lambda_{3,1}^2}{z_3 \lambda_{2,1} \lambda_{3,1}^3}.$$

The determinant of this system of linear equations is equal to

$$\frac{\left(1-\lambda_{2,1}\right)\left(1-\lambda_{3,2}\right)}{\lambda_{2,1}^4 \lambda_{3,1}^3 \lambda_{3,2}^4}\left(a\,\lambda_{3,2}^2 + b\,\lambda_{3,2} + c\right),$$

where

$$a := -2\lambda_{3,1}^2 \lambda_{2,1}^2 + 2\lambda_{3,1} \lambda_{2,1}^2 - \lambda_{2,1}^2 + 2\lambda_{3,1}^2 \lambda_{2,1} - \lambda_{3,1}^2,$$

$$b := 2\lambda_{2,1}^2 \lambda_{3,1}^2 - 4\lambda_{2,1} \lambda_{3,1}^2 + 2\lambda_{3,1}^2,$$

$$c : -\lambda_{2,1}^2 \lambda_{3,1}^2 + 2\lambda_{2,1} \lambda_{3,1}^2 - \lambda_{3,1}^2.$$

The discriminant of equation $a\,\lambda_{3,2}^2 + b\,\lambda_{3,2} + c = 0$ is given by

$$\Delta = -4\left(1-\lambda_{2,1}\right)^2 \lambda_{2,1}^2 \left(1-\lambda_{3,1}\right)^2 \lambda_{3,1}^2,$$

so that it is negative. Therefore, the unique solution is $d_1 = d_2 = d_3 = 0$.

Taking into account the latter, the polynomial function P can be rewritten as

$$P = c_1\,C_1 + c_2\,C_2 + c_3\,C_3.$$

The C^4 smoothness of C_ℓ across $\langle V_1, Z\rangle$, $\langle V_2, Z\rangle$ and $\langle V_3, Z\rangle$ yields

$$0 = \frac{2\lambda_{3,1}^3\left(z_3\left(\lambda_{3,1}\left(c_2 z_2 + c_3\left(-z_2+z_3+1\right)\right) - 2c_3 z_3\right) + c_1 z_2\left(4z_3 - \left(-3z_2+z_3+3\right)\lambda_{3,1}\right)\right)}{z_3^4},$$

$$0 = \frac{1}{z_1^4}\left(-2\lambda_{1,2}^3\left(-c_1 z_1\left(\left(z_2+2z_3-2\right)\lambda_{2,1} + z_2\right)\right.\right.$$

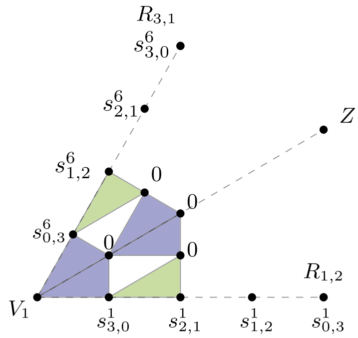

Fig. 9. BB-coefficients involved in the C^1 and C^2 continuity conditions between the restrictions of the spline to the micro-triangles t^1 and t^6.

$$
\begin{aligned}
&+z_3\left(c_2\left(\left(z_2+4z_3-4\right)\lambda_{2,1}+3z_2\right)-c_3z_1\lambda_{1,2}\right)\big)\big),\\
0=\,&\frac{1}{z_2^4}\left(-2\lambda_{2,3}^3\left(c_1z_1z_2\lambda_{2,3}+c_3z_1\left(-3z_3\lambda_{2,3}+4z_2\lambda_{3,2}\right)\right.\right.\\
&\left.\left.+c_2z_2\left(z_3-\left(2z_2+z_3\right)\lambda_{3,2}\right)\right)\right).
\end{aligned}
$$

This linear system has the following determinant

$$
\frac{32\left(\lambda_{2,1}-1\right)^3\lambda_{3,1}^3\left(\lambda_{3,2}-1\right)^3}{z_2^3z_3^3\left(z_2+z_3-1\right)^3}\left(\bar{a}+\bar{b}\lambda_{2,1}\right),
$$

where,

$$
\begin{aligned}
\bar{a}&:=z_2\left(-2\lambda_{3,1}+z_3\left(-\lambda_{3,1}\right)+2z_2\lambda_{3,1}+3z_3\right)\left(2z_3\lambda_{3,2}+3z_2\lambda_{3,2}-2z_3\right),\\
\bar{b}&:=-2z_3z_2^2\lambda_{3,1}-5z_3^2z_2\lambda_{3,1}+8z_3z_2\lambda_{3,1}\\
&\quad-3\left(z_3-1\right)z_3\left(\lambda_{3,2}-1\right)\left(\left(z_3+2\right)\lambda_{3,1}-3z_3\right)\\
&\quad-3z_3^2z_2+3z_2^3\lambda_{3,1}\lambda_{3,2}+5z_3z_2^2\lambda_{3,2}+3\left(3z_3-4\right)z_2^2\lambda_{3,1}\lambda_{3,2}\\
&\quad+z_3\left(17z_3-14\right)z_2\lambda_{3,2}\\
&\quad-12z_3z_2\lambda_{3,1}\lambda_{3,2}+9z_2\lambda_{3,1}\lambda_{3,2}.
\end{aligned}
$$

The determinant is equal to zero if and only if $\lambda_{2,1}=-\frac{\bar{a}}{\bar{b}}$. Since $\lambda_{2,1}$ is in $(0,1)$, the value of $\frac{\bar{a}}{\bar{b}}$ must be in $(-1,0)$ for all possible values of parameters z_2, z_3, $\lambda_{3,2}$ and $\lambda_{3,1}$, which is not true (for instance, for $z_2=0.802$, $z_3=0.493$, $\lambda_{3,2}=0.293$ and $\lambda_{3,1}=0.45$ it holds $\frac{\bar{a}}{\bar{b}}=1.07038\notin(-1,0)$). Then, it follows that $c_1=c_2=c_3=0$. The proof is complete. □

In general, the functions in $S_4^{1,2,3}(T)$ are not in $C^2(T)$ [21]. Therefore, it is reasonable to study under which conditions on the Powell-Sabin refinement of T the splines in $S_4^{1,2,3}(T)$ are C^2 continuous.

In order to achieve completely C^2 quartic Powell-Sabin splines, the blending functions need to be C^2 continuous across the micro-edges $\langle Z,V_1\rangle$, $\langle Z,V_2\rangle$ and $\langle Z,V_3\rangle$. We start by analyzing under what conditions the C^2 continuity of blending functions $\mathcal{D}_i$, $i=1,2,3$, is achieved. Then we will extract the relations between the first kind blending functions under the achieved configuration so that the spline becomes C^2 continuous.

In Fig. 9, a schematic representation of BB-coefficients involved in the C^2 smoothness across the edge $\langle Z,V_1\rangle$ is done.

Proposition 5. *Blending functions of the second kind are C^2 continuous on $T\langle V_1,V_2,V_3\rangle$ if and only if*

$$
\lambda_{2,1}=\frac{z_2}{1-z_3},\quad \lambda_{3,1}=\frac{z_3}{1-z_2},\quad \lambda_{3,2}=\frac{z_3}{1-z_1}.
$$

Proof. Consider $\mathcal{D}_1$ and the structure shown in Fig. 9. It is a C^2 continuous function across $\langle V_1, Z\rangle$ if and only if

$$
s^6_{1,2}=\tau_{2,3}^2\,s^1_{2,1}+2\,\tau_{2,3}\,\tau_{1,3}\,s^1_{3,0}.
$$

Note that $s^6_{1,2}=\tau_{2,3}\lambda_{1,3}\,s^1_{3,0}$ and $s^1_{2,1}=\lambda_{1,2}\,s^1_{3,0}$. That gives

$$
\tau_{2,3}=\frac{\lambda_{1,3}-2\tau_{1,3}}{\lambda_{1,2}}, \tag{7}
$$

Analogously, $\mathcal{D}_2$ and $\mathcal{D}_3$ are C^2 continuous functions across $\langle V_2,Z\rangle$ and $\langle V_3,Z\rangle$, respectively, if and only if

$$
\tau_{2,2}=\frac{\lambda_{3,2}-2\tau_{1,2}}{\lambda_{3,1}}\quad\text{and}\quad \tau_{2,3}=\frac{\lambda_{1,3}-2\tau_{1,3}}{\lambda_{1,2}}. \tag{8}
$$

Equations (7) and (8) can be reformulated as

$$
\begin{aligned}
\frac{\lambda_{3,1}z_2+\lambda_{2,1}\left(\lambda_{3,1}\left(z_2+z_3-2\right)+z_3\right)}{\lambda_{1,2}\lambda_{2,1}z_3}&=0,\\
\frac{\lambda_{3,2}\left(\lambda_{2,1}+\left(\lambda_{2,1}-2\right)z_2\right)-\left(\lambda_{2,1}+\lambda_{3,2}-1\right)z_3}{\lambda_{2,3}\lambda_{3,2}z_1}&=0,\\
\frac{\lambda_{3,1}z_2+\lambda_{2,1}\left(\lambda_{3,1}\left(z_2+z_3-2\right)+z_3\right)}{\lambda_{1,2}\lambda_{2,1}z_3}&=0.
\end{aligned} \tag{9}
$$

The unique solution of (9) provides the values in the claim. □

Conditions in Proposition 5 can be geometrically interpreted as follows.

Proposition 6. *Functions $\mathcal{D}_i$, $i=1,2,3$, are C^2 continuous if and only if the points in each of subsets $\{V_1,Z,R_{2,3}\}$, $\{V_2,Z,R_{3,1}\}$ and $\{V_3,Z,R_{1,2}\}$ are collinear.*

Proof. First, let us prove that the conditions are necessary. Without loss of generality, let us consider the third of the subsets. We have to prove that V_3, Z and $R_{1,2}$ are collinear. By Proposition 5, the barycentric coordinates of $R_{1,2}$ w.r.t. T are

$$
\begin{aligned}
\left(\lambda_{1,2},\lambda_{2,1},0\right)=\left(1-\lambda_{2,1},\lambda_{2,1}\right)&=\left(\frac{1-z_2-z_3}{1-z_3},\frac{z_2}{1-z_3},0\right)\\
&=\left(\frac{z_1}{1-z_3},\frac{z_2}{1-z_3},0\right).
\end{aligned}
$$

Then,

$$
R_{1,2}=\frac{z_1}{1-z_3}V_1+\frac{z_2}{1-z_3}V_2.
$$

Moreover, $Z=z_1V_1+z_2V_2+z_3V_3$. Taking into account the Cartesian coordinates of Z and the vertices, we get

$$
R_{1,2}-Z=\frac{z_3}{1-z_3}\left(z_1x_1+z_2x_2+\left(z_3-1\right)x_3,z_1y_1+z_2y_2+\left(z_3-1\right)y_3\right).
$$

Therefore, the slope of the straight line determined by Z and $R_{1,2}$ is equal to

$$
m_{1,2}:=\frac{z_1y_1+z_2y_2+\left(z_3-1\right)y_3}{z_1x_1+z_2x_2+\left(z_3-1\right)x_3}.
$$

On the other hand, the straight line determined by Z and V_3 has the direction of vector

$$
\begin{aligned}
Z-V_3&=z_1V_1+z_2V_2+\left(z_3-1\right)V_3\\
&=\left(z_1x_1+z_2x_2+\left(z_3-1\right)x_3,z_1y_1+z_2y_2+\left(z_3-1\right)y_3\right),
\end{aligned}
$$

so that its slope is also equal to $m_{1,2}$. Consequently, both the straight lines defined by $\{Z,R_{1,2}\}$ and $\{Z,V_3\}$ have the same slope and pass through the Z point, and V_3, Z and $R_{1,2}$ are collinear.

Conversely, suppose that V_3, Z and $R_{1,2}$ are collinear. As proved above, the slope of the straight line determined by V_3 and Z is equal to $m_{1,2}$, so that its equation is $y=m_{1,2}x+n_{1,2}$, where $n_{1,2}$ is computed by imposing that the line passes through V_3 to get

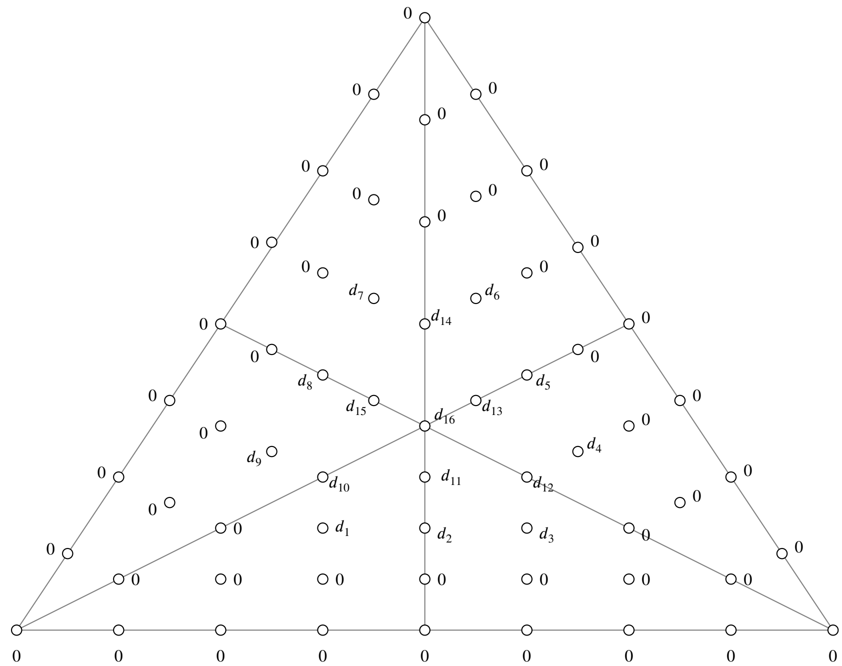

Fig. 10. B-ordinates of a reduced B-spline.

$$n_{1,2} = \frac{y_3\left(z_1x_1+z_2y_2\right)-x_3\left(z_1y_1+z_2y_2\right)}{z_1x_1+z_2x_2+\left(z_3-1\right)x_3}.$$

Since $R_{1,2} = \lambda_{1,2}V_1 + \lambda_{2,1}V_2$ can be written in Cartesian coordinates as

$$\begin{aligned}&\left(\lambda_{1,2}x_1+\lambda_{2,1}x_2, \lambda_{1,2}y_1+\lambda_{2,1}y_2\right)\\ &\quad = \left(\lambda_{1,2}x_1+\left(1-\lambda_{1,2}\right)x_2, \lambda_{1,2}y_1+\left(1-\lambda_{1,2}\right)y_2\right),\end{aligned}$$

it must be fulfilled that

$$m_{1,2}\left(\lambda_{1,2}x_1+\left(1-\lambda_{1,2}\right)x_2\right)+n_{1,2} = \lambda_{1,2}y_1+\left(1-\lambda_{1,2}\right)y_2.$$

A straightforward calculation gives

$$\lambda_{1,2} = \frac{z_1}{1-z_3}.$$

The proof is complete. □

Once C^2 continuity of blending functions of the second kind has been characterized, we need now to get C^2 continuity for the spline on T. To this end, we should derive the C^2 smoothness relations between the three blending functions of the first kind which are defined on a split triangle that meets the conditions in Proposition 5.

Theorem 7. *Assume that the PS-split T_{PS} of T meets the conditions in Proposition 5. Then, the spline $s = p_4 + \sum_{i=1}^{3}\left(d_i\mathcal{D}_i + c_i\mathcal{C}_i\right), p_4 \in \mathbb{P}_4(T)$, in $S_4^{1,2,3}(T)$ is fully C^2 continuous on T if and only if*

$$c_1z_2 = c_3z_3, \quad c_2z_3 = c_1z_1, \quad c_3z_1 = c_2z_2. \tag{10}$$

Proof. The C^2-smoothness conditions across the edge $\langle V_1, Z\rangle$ give the equality

$$0 = \tau_{3,3}^2\left(c_1z_2+c_3z_3\right)+2\tau_{3,3}\tau_{2,3}c_1\lambda_{2,1}.$$

Substituting $\tau_{3,3}$, $\tau_{2,3}$ and $\lambda_{2,1}$ respectively by their values $\frac{1}{1-z_2}$, $\frac{1-z_3}{1-z_2}$, and $\frac{z_2}{1-z_3}$, we get

$$c_1z_2 = c_3z_3.$$

The other two conditions are derived similarly. □

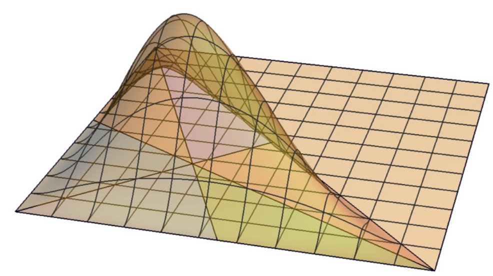

Fig. 11. Blending function $\mathcal{B}^t$.

Under the hypothesis in Proposition 5, the general solution of system (10) depends on one parameter $\alpha \in \mathbb{R}$ and can be written as $\left(c_1,c_2,c_3\right) = \alpha\left(z_3,z_1,z_2\right)$, so that any C^2 continuous spline $s \in S_4^{1,2,3}(T)$ can be expressed as

$$s = p_4 + \sum_{i=1}^{3} d_i\mathcal{D}_i + \alpha\mathcal{B}^t,$$

where $\mathcal{B}^t := z_3\mathcal{C}_1 + z_1\mathcal{C}_2 + z_2\mathcal{C}_3$ is a $C^2(T)$ continuous function associated to triangle T which will be called blending function of the third kind. The condition imposed on the Powell-Sabin refinement of T results in a lower dimension to $S_4^{1,2,3}(T)$, 19 instead of 21.

The B-ordinates of $\mathcal{B}^t$ are given by

$$\begin{aligned}
d_1 &= z_3\lambda_{2,1}, & d_9 &= z_2\lambda_{3,1},\\
d_2 &= 2z_3\lambda_{1,2}\lambda_{2,1}, & d_{10} &= 2z_3z_2,\\
d_3 &= z_3\lambda_{1,2}, & d_{11} &= 2z_3\left(\lambda_{1,2}z_2+\lambda_{2,1}z_1\right),\\
d_4 &= z_1\lambda_{3,2}, & d_{12} &= 2z_1z_3,\\
d_5 &= 2z_1\lambda_{2,3}\lambda_{3,2}, & d_{13} &= 2z_1\left(\lambda_{2,3}z_3+\lambda_{3,2}z_2\right),\\
d_6 &= z_1\lambda_{2,3}, & d_{14} &= 2z_1z_2,\\
d_7 &= z_2\lambda_{1,3}, & d_{15} &= 2z_2\left(\lambda_{1,3}z_3+\lambda_{3,1}z_1\right),\\
d_8 &= 2z_2\lambda_{1,3}\lambda_{3,1}, & d_{16} &= 6z_1z_2z_3.
\end{aligned}$$

They are shown in Fig. 10. The typical plot of a function $\mathcal{B}^t$ is shown in Fig. 11.

We have just proved that every spline $s \in S_4^{1,2,3}(T)$ is C^2 continuous on a triangle T for which its PS-split meets the condition in Proposition 5. When the refinement of T satisfies the conditions of Proposi-

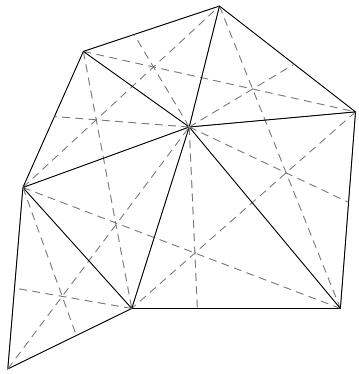

Fig. 12. Powell-Sabin triangulation satisfying conditions in Proposition 5.

tion 5, the dimension of $S_4^{1,2,3}(\mathcal{T})$ diminishes from 21 to 1 $\wp$, since three B-splines of the first kind give rise to a single B-spline of the third kind, $\mathcal{B}^t$.

Now, it remains to prove that the spline s is also C^2 continuous over the whole triangulation Δ if the split point of each macro-element of Δ satisfies the conditions in Proposition 5 and the edge split points produced on common sides of two triangles coincide, i.e. if the opposite vertices of each pair of triangles sharing an edge are aligned with the corresponding triangle split points. Denote by $\widetilde{\Delta}_{\mathrm{PS}}$ this kind of triangulation. Fig. 12 shows a triangulation satisfying these requirements.

5. The Powell-Sabin space on the whole triangulation

This section aims to prove that each quartic spline space over $\widetilde{\Delta}_{\mathrm{PS}}$ is C^2 continuous everywhere and C^3 at the edge split points. To this end, we will provide a general representation of $S_4^{1,2,3}(\Delta_{\mathrm{PS}})$ over an arbitrary PS-split Δ_{PS} of Δ, and then we will prove that the provided representation is totally C^2 continuous over $\widetilde{\Delta}_{\mathrm{PS}}$. Moreover, the B-spline-like functions to be constructed in this section will enjoy the usual properties required when dealing with the construction of bases of spline function spaces. They will be non-negative, locally supported and form a unit partition. Furthermore, any spline represented in these bases have a meaningful geometric interpretation, can be locally controlled and evaluated in a stable way.

Since the dimension of $S_4^{1,2,3}(\Delta_{\mathrm{PS}})$ equals $6nv + ne$, then such a representation will be obtained by defining six B-spline-like functions $\mathcal{B}^v_{i,\alpha}$, $|\alpha| = 2$ associated with each vertex and another one, $\mathcal{B}^e_\ell$, for each edge. The B-spline-likes $\mathcal{B}^v_{i,\alpha}$ and $\mathcal{B}^e_\ell$ are called B-spline-likes with respect to vertices and edges, respectively. The procedure to construct them follows the technique in [11,14,17,21,26].

5.1. B-spline-like with respect to vertex

We outline the construction of $\mathcal{B}^v_{i,\alpha}$ in the spirit of [21]. For every vertex V_i, let $M_i := \cup_{T\in\Delta, V_i\in T} T$ be the molecule relative to V_i, i.e. the union of all triangles in Δ containing V_i. For all vertex V_ℓ lying on the boundary of M_i and for all $T_j \subset M_i$, let

$$S_{i,\ell} := \frac{1}{2}(V_i + R_{i,\ell}) \quad \text{and} \quad L_{i,j} := \frac{1}{2}(V_i + Z_j),$$

Points V_i, $S_{i,\ell}$ and $L_{i,j}$ are said to be PS4-points associated with V_i. Let $t_i := (Q_{i,1}, Q_{i,2}, Q_{i,3})$ be a triangle containing the PS4-points of V_i. It will be called PS4-triangle. Denote by $\mathfrak{B}^2_{t_i,\alpha}$, $|\alpha| = 2$, the Bernstein polynomials of degree 2 with respect to t_i, and define the values

$$\gamma^{a,b}_{i,\alpha} := \frac{12}{(4-a-b)(3-a-b)}\left(\frac{1}{2}\right)^{a+b} \partial^a_x \partial^b_y \mathfrak{B}^2_{t_i,\alpha}(V_i) \tag{11}$$

for all $a \geq 0, b \geq 0, 0 \leq a+b \leq 2$.

They are used to define the B-spline-like $\mathcal{B}^v_{i,\alpha}$ as follows.

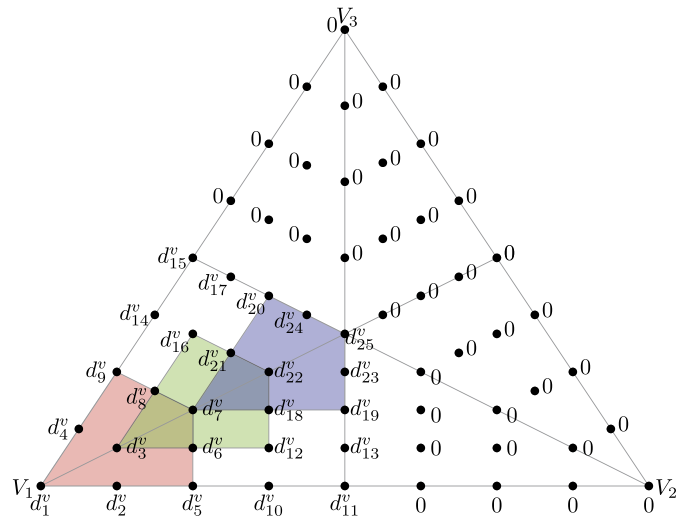

Fig. 13. B-ordinates of a B-spline-like with respect to vertex V_1.

Without loss of generality, consider the vertex V_1. $\mathcal{B}^v_{1,\alpha}$ is defined as the unique solution of the Hermite interpolation problem (4) with all f- and g-values equal to zero except $f_1^{a,b} = \gamma^{a,b}_{1,\alpha}$, $g_{1,2} = \beta^\alpha_{1,2}$ and $g_{3,1} = \beta^\alpha_{3,1}$, where the β-values are chosen as follows.

Let $T\langle V_1, V_2, V_3\rangle$ be a triangle included in the molecule M_1. In each of the six micro-triangles of T, $\mathcal{B}^v_{1,\alpha}$ is a quartic polynomial. The B-ordinates in its Bernstein-Bézier representation are shown in Fig. 13. Many of them are null. The non-zero B-ordinates are determined from the given data and the smoothness conditions. Note that

$$\beta^\alpha_{1,2} = \frac{12}{\|Z - R_{1,2}\|^2}\left(d^v_{11} - 2d^v_{13} + d^v_{19}\right) \quad \text{and}$$

$$\beta^\alpha_{3,1} = \frac{12}{\|Z - R_{3,1}\|^2}\left(d^v_{15} - 2d^v_{17} + d^v_{20}\right).$$

The B-ordinates $d^v_1, \ldots, d^v_9$ are computed from the chosen parameters $\gamma^{a,b}_{1,\alpha}$, $a \geq 0, b \geq 0, 0 \leq a+b \leq 2$. The ordinates $d^v_{18}, \ldots, d^v_{25}$ are computed from C^2 smoothness at the triangle split point Z. Let p_2 be the quadratic polynomial defined on the triangle $\langle W_1, W_2, W_3\rangle$ with vertices $W_i = \frac{1}{2}(V_i + Z)$ in such a way that all B-ordinates are equal to zero except $b_{2,0,0} = d^v_7$. Then, by subdivision, the following relationships result:

$$\begin{aligned}
d^v_{18} &= \lambda_{12}\, d^v_7, & d^v_{19} &= \lambda^2_{12}\, d^v_7, & d^v_{20} &= \lambda^2_{13}\, d^v_7, & d^v_{21} &= \lambda_{13}\, d^v_7, \\
d^v_{22} &= z_1\, d^v_7, & d^v_{23} &= \lambda_{12} z_1\, d^v_7, & d^v_{24} &= \lambda_{13} z_1\, d^v_7, & d^v_{25} &= z^2_1\, d^v_7.
\end{aligned}$$

The B-ordinates $d^v_{10}, \ldots, d^v_{17}$ are computed from C^3-smoothness across $\langle R_{1,2}, Z\rangle$ and $\langle R_{3,1}, Z\rangle$. Let us define the univariate cubic polynomials, p^0_3 and p^1_3, on the lines $\left\langle \frac{3V_1+R_{1,2}}{4}, \frac{3V_2+R_{1,2}}{4}\right\rangle$ and $\left\langle \frac{2V_1+R_{1,2}+Z}{4}, \frac{2V_2+R_{1,2}+Z}{4}\right\rangle$, respectively, having B-ordinates

$$b^0_{3,0} = d^v_2, \qquad b^0_{2,1} = \frac{d^v_5 - \lambda_{1,2} d^v_2}{\lambda_{2,1}} =: \tilde{d}^v_5, \qquad b^0_{1,2} = 0, \qquad b^0_{0,3} = 0,$$

and

$$b^1_{3,0} = d^v_3, \qquad b^1_{2,1} = \frac{d^v_6 - \lambda_{1,2} d^v_3}{\lambda_{2,1}} =: \tilde{d}^v_6, \qquad b^1_{1,2} = 0, \qquad b^1_{0,3} = 0.$$

Then, after subdivision,

$$d^v_{10} = \lambda^2_{1,2} d^v_2 + 2\lambda_{1,2}\lambda_{2,1}\tilde{d}^v_5, \qquad d^v_{11} = \lambda^3_{1,2} d^v_2 + 2\lambda^2_{1,2}\lambda_{2,1}\tilde{d}^v_5$$

and

$$d^v_{12} = \lambda^2_{1,2} d^v_3 + 2\lambda_{1,2}\lambda_{2,1}\tilde{d}^v_6, \qquad d^v_{13} = \lambda^3_{1,2} d^v_3 + 2\lambda^2_{1,2}\lambda_{2,1}\tilde{d}^v_6.$$

Similarly,

$$d^v_{14} = \lambda^2_{1,3} d^v_4 + 2\lambda_{1,3}\lambda_{3,1}\tilde{d}^v_9, \qquad d^v_{15} = \lambda^3_{1,3} d^v_4 + 2\lambda^2_{1,3}\lambda_{3,1}\tilde{d}^v_9,$$

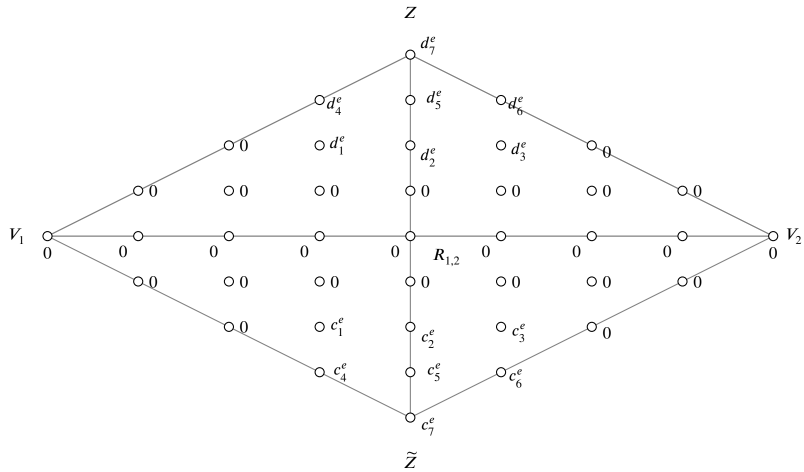

Fig. 14. B-ordinates of a B-spline-like $\mathcal{B}_1^e$ on the four micro triangles that have $\langle V_1, R_{1,2}\rangle$ or $\langle V_2, R_{1,2}\rangle$ as an edge.

and

$$d_{16}^v = \lambda_{1,3}^2 d_3^v + 2\lambda_{1,3}\lambda_{3,1}\tilde{d}_8^v, \qquad d_{17}^v = \lambda_{1,3}^3 d_3^v + 2\lambda_{1,3}^2\lambda_{3,1}\tilde{d}_8^v,$$

where $\tilde{d}_8^v := \frac{d_8^v - \lambda_{1,3} d_3^v}{\lambda_{3,1}}$ and $\tilde{d}_9^v := \frac{d_9^v - \lambda_{1,3} d_4^v}{\lambda_{3,1}}$.

The restriction of $\mathcal{B}_{1,\alpha}^v$ on T can be written in terms of $\mathcal{D}_i$, $i = 1, 2, 3$, and $\mathcal{B}^t$. Then, $\mathcal{B}_{1,\alpha}^v$ is C^2 continuous on T, if and only if T_{PS} meets the conditions in Proposition 5. In what follows, we will confirm this result.

The BB-coefficients involved in C^2 continuity conditions between the restrictions of $\mathcal{B}_{1,\alpha}^v$ to the micro-triangles t^1 and t^6 are divided into three categories. The BB-coefficients lying in the area in light red colour satisfy the C^2 smoothness because they are computed from the derivative values up to order two of $\mathcal{B}_{1,\alpha}^v$. The BB-coefficients lying in the area in blue colour also satisfy the C^2 smoothness. By construction, they are computed throughout the values of a quadratic polynomial defined on the triangle with vertices W_i in (5). It remains to check the C^2 smoothness conditions between the BB-coefficients lying in the area in green colour. Using equation (2), the remaining C^2 condition between the BB-coefficients lying in the area in green colour is given by

$$d_{16}^v = \tau_{2,3}^2 d_{12}^v + 2\tau_{2,3}\tau_{3,3} d_{18}^v + \tau_{3,3}^2 d_{22}^v + 2\tau_{3,3}\tau_{1,3} d_7^v + \tau_{1,3}^2 d_3^v + 2\tau_{1,3}\tau_{2,3} d_6^v.$$

By substituting the relevant BB-coefficients by their values, it is verified that the condition is fulfilled. By Theorem 1, it follows that $\mathcal{B}_{1,\alpha}^v$ is globally C^2 continuous over $\widetilde{\Delta}_{\mathrm{PS}}$.

5.2. B-spline-like with respect to edge

Let $T\langle V_1, V_2, V_3\rangle$ and $\widetilde{T}\langle V_1, V_2, V_4\rangle$ be two triangles sharing the common edge $\mathfrak{e}_1 = \langle V_1, V_2\rangle$. Let $\mathcal{B}_1^e$ be the B-spline-like with respect to the edge $\mathfrak{e}_1$. It is defined as the unique solution of the Hermite interpolation problem (4) with all f- and g-values equal to zero except $g_{1,2} = \beta_{1,2}$. For the sake of simplicity, we chose $\omega_{m,n,q} = \frac{Z - R_{1,2}}{\|Z - R_{1,2}\|}$ (see Theorem 1). The β-values can be chosen as in Definition 2. For instance we consider an arbitrary value for $\beta_{1,2}$.

Let $\widetilde{Z}$ be the inner spilt point of $\widetilde{T}$. The BB-coefficients of $\mathcal{B}_1^e$ on T are computed in a similar way to those of $\mathcal{C}_1$. Now we deal only with the BB-coefficients associated with the domain points located in the four micro-triangles that have $\langle V_1, R_{1,2}\rangle$ or $\langle V_2, R_{1,2}\rangle$ as an edge. They are schematically presented in Fig. 14. In order to prove that $\mathcal{B}_1^e$ is C^2 continuous across $\langle V_1, V_2\rangle$, we need to provide the value of d_1^e, d_2^e, d_3^e, c_1^e, c_2^e and c_3^e. The first ones are

$$d_2^e = \frac{\beta_{1,2}}{12}\|Z - R_{1,2}\|^2, \quad d_1^e = \frac{\beta_{1,2}}{24\lambda_{1,2}}\|Z - R_{1,2}\|^2, \quad d_3^e = \frac{\beta_{1,2}}{24\lambda_{2,1}}\|Z - R_{1,2}\|^2.$$

If $R_{1,2} = \lambda Z + (1-\lambda)\widetilde{Z}$, then, for the remaining ones we have

$$c_2^e = \left(\frac{\lambda}{1-\lambda}\right)^2 \frac{\beta_{1,2}}{12}\|Z - R_{1,2}\|^2, \quad c_1^e = \left(\frac{\lambda}{1-\lambda}\right)^2 \frac{\beta_{1,2}}{24\lambda_{1,2}}\|Z - R_{1,2}\|^2,$$

$$c_3^e = \left(\frac{\lambda}{1-\lambda}\right)^2 \frac{\beta_{1,2}}{24\lambda_{2,1}}\|Z - R_{1,2}\|^2.$$

The C^2 smoothness conditions across $\langle V_1, V_2\rangle$ are

$$c_1^e = \left(\frac{\lambda}{1-\lambda}\right)^2 d_1^e, \quad c_2^e = \left(\frac{\lambda}{1-\lambda}\right)^2 d_2^e \quad \text{and} \quad c_3^e = \left(\frac{\lambda}{1-\lambda}\right)^2 d_3^e.$$

The conditions are all fulfilled, which confirms that $\mathcal{B}_1^e$ is C^2 continuous across $\langle V_1, V_2\rangle$.

The value of $\beta_{1,2}$ must be fixed in order to ensure that the B-splines form a partition of unity. To this end, it suffices to chose $\beta_{1,2} = \frac{24\lambda_{1,2}\lambda_{2,1}}{\|Z - R_{1,2}\|^2}$.

The blending function of the third kind $\mathcal{B}^t$ associated with T is written as a convex combination of B-spline-like functions with respect to the edges of T with a suitable choice of coefficients which guarantees that it is C^2 continuous on T. Indeed, if we chose $g_{2,3} = \beta_{2,3} = \frac{24\lambda_{2,3}\lambda_{3,2}}{\|Z - R_{2,3}\|^2}$ and $g_{3,1} = \beta_{3,1} = \frac{24\lambda_{3,1}\lambda_{1,3}}{\|Z - R_{3,1}\|^2}$ for the other two edges, then $\mathcal{B}^t = z_3\mathcal{B}_1^e + z_1\mathcal{B}_2^e + z_2\mathcal{B}_3^e$, and the C^2 smoothness is ensured.

Hence, it is stated that the B-spline-like functions with respect to the vertices and the blending functions of the third kind are all C^2 everywhere. Furthermore, each quartic spline defined on $\tilde{\Delta}_{\mathrm{PS}}$ is C^2 continuous everywhere and C^3 at the edge split points, so that it would be appropriate to write $S_4^{2,3}(\tilde{\Delta}_{\mathrm{PS}})$ for the spline space. Its dimension is reduced to $6nv + nt$ because of the conditions imposed on $\tilde{\Delta}_{\mathrm{PS}}$, which on a single triangle give way to a blending function on the third kind $\mathcal{B}^t$ instead of three B-spline-likes with respect to edges.

6. Conclusions and discussions

We have proved that under certain geometrical conditions regarding the triangle and edge split points associated with an arbitrary triangulation of a polygonal domain Ω, the space of almost $C^2(\Omega)$ continuous Powell-Sabin splines introduced in [21] becomes a subspace of a $C^2(\Omega)$. This has been done by constructing for an arbitrary triangle T endowed with a Powell-Sabin refinement a specific basis and deriving the conditions that must be verified for the global regularity to be C^2 instead of C^1. For a triangulation whose triangles satisfy those conditions, the dimension of the corresponding space of C^2 quartic splines is reduced.

Except in exceptional cases (including type-1 and criss-cross triangulations), the sub-triangulation obtained by connecting the opposite vertices of each pair of triangles sharing an edge of the triangulation does not satisfy the conditions in Proposition 5, which characterizes

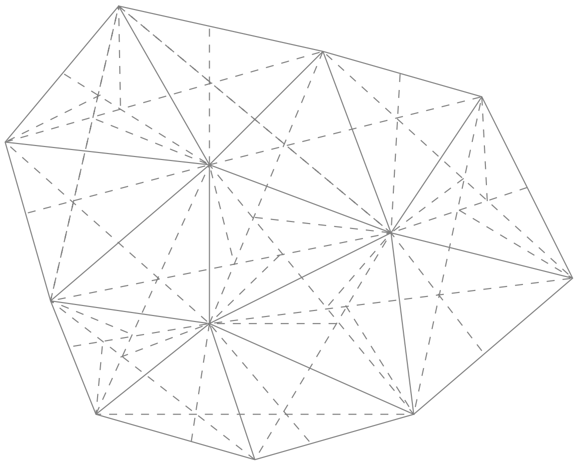

Fig. 15. Example of a mixed triangulation arising when the procedure to get a Powell-Sabin sub-triangulation allowing C^2-quartic splines is applied.

C^2 continuity. In some cases it will be possible, resulting in a Powell-Sabin sub-triangulation such that for each triangle the interior edges intersect at a point, as shown in Fig. 12. In other cases, Morgan-Scott sub-triangulations will be obtained, which easily give rise to modified Morgan-Scott sub-triangulations [22]. In other cases, mixed sub-triangulations will appear, as Fig. 15 shows.

It has been proved that, when the triangulation fulfils the conditions of Proposition 5, it is possible to construct C^2 quartic splines. If a Morgan-Scott sub-triangulation is obtained, then it is also possible to construct such splines on the corresponding modified Morgan-Scott sub-triangulation (see [22]). Otherwise, a mixed refinement will result. The work in progress deals with the geometrical construction of a B-spline-like basis for the space of quartic splines that can be defined over this sub-triangulation in order to get a normalized B-spline-like representation, whose coefficients will be expressed in terms of polar forms.

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