Asymptotic behaviour of some nonlocal equations in mathematical biology and kinetic theory

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Bu çalışmayı birini henüz üç yaşımdayken,
diğer üçünü de geçtiğimiz üç yıl içinde kaybettiğim
büyükanne ve büyükbabalarıma ithaf ediyorum.

I hereby declare that except where specific reference is made to the work of others, the contents of this dissertation are original and have not been submitted in whole or in part for consideration for any other degree or qualification at the University of Granada, or any other university or similar institution. This dissertation is the result of my own work and contains nothing which is the outcome of work done in collaboration with others, except as declared below.

Chapter 1 is an introduction and contains a brief literature review, main results of this work and structure of the dissertation. Chapter 2 is another introduction about the main probabilistic techniques which are extensively used in the subsequent chapters to achieve the original conclusions. Chapter 3 is based on a joint work with José A. Cañizo at the University of Granada and published in the journal *Nonlinearity* [28]. Chapter 4 is based on two joint works in progress: The first one is a result of a collaboration with José A. Cañizo at the University of Granada and Pierre Gabriel at Versailles Saint-Quentin-en-Yvelines University and the second one is with José A. Carrillo de la Plata at Imperial College London. Chapter 5 is a joint work with José A. Cañizo at the University of Granada, Chuqi Cao and Josephine Evans at University Paris Dauphine. This work is accepted for publication to the journal *Kinetic and Related Models* [39]. Chapter 6 contains conclusion, perspectives, future possible applications of the aforementioned methods and works in progress. This dissertation is complemented with two appendices consisting of reviews of some mathematical concepts which are used throughout the thesis.

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Abstract

We study the long-time behaviour of solutions to some partial differential equations arising in modeling of several biological and physical phenomena. In this work, the type of the equations we consider is mainly nonlocal, in the sense that they involve integral operators. Moreover, the equations we consider describe the time evolution of either some populations structured by several traits like age, elapsed-time and size or the distribution of the dynamical states of a single particle, depending on time, space and velocity. In the latter case, they are called kinetic equations.

We are interested in showing quantitatively the asynchronous behaviour of interacting neuron populations which are composed of large and fully connected networks. Neurons undergo a charging period followed by a sudden discharge in the form of firing a spike. We consider two nonlinear models structured by the time elapsed since the last discharge and nonlinearity comes from the dependence of firing rate on the total neural activity at a time. In the second model, there is an addition of a fragmentation term to include the effect of the past activity of neurons by displaying adaptation and fatigue. With this addition, the equation shares many common properties with another class of integro-partial differential equations called the growth-fragmentation equation. This is the second type of equation we look at the convergence rate to a universal profile in a quantitative way. The growth-fragmentation equation describes a system of growing and dividing particles which may be used as a model for many processes in ecology, neuroscience, telecommunications and cell biology. We consider two types of fragmentation processes, namely mitosis and constant fragmentation and include nonconservative cases where eigenelements cannot be computed explicitly. We present quantitative exponential convergence speeds in the weighted total variation norm. Furthermore, we also study hypocoercivity of some space inhomogeneous linear kinetic equations including linear relaxation Boltzmann(linear BGK) and linear Boltzmann equations which are posed either on the torus or on the whole space with a confining potential. We prove exponential convergence in the torus or on the whole space with a potential growing quadratically at infinity. Moreover, for the weaker confining potentials (subquadratic) we present subgeometric convergence rates quantitatively.

The physiologically structured population models and the space inhomogeneous linear kinetic equations we deal with in this work are well-studied from various aspects in the already-existing literature. We provide the references later. What differs from the past plentiful studies on the asymptotic behaviour of these equations is the techniques we use here. We consider a probabilistic approach which is first developed for studying ergodic properties of discrete-time Markov processes. The method is due to Doeblin and Harris; based on establishing a combination of a minorisation (irreducibility) and a geometric drift (Lyapunov) conditions for a Markovian process. This method gives a quantitative convergence speed and existence of a unique steady state even without having to calculate it explicitly. Application of Harris's Theorem into the aforementioned partial differential equations to study the long-time behaviour of solutions is the core of this thesis.

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Chapter 1

Introduction

"Adına dünya dediğimiz bu kitabı oku."
— İhsan Oktay Anar, Puslu Kıtalar Atlası

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1.1 Overview

In this thesis, we are concerned with the long-time behaviour of solutions for some nonlocal partial differential equations which describe the dynamics of various biological and physical phenomena. Local models involve differential operators and they are designed to capture some properties of a function at a given point. In this case, one only needs to know the values of the function in an arbitrarily small neighborhood to ensure that all partial derivatives are defined. However, nonlocal models involve integral operators which describe some modeling structure of a function so that; in order to determine the value at a desired point, information about the values of the function in a larger interval is needed. These types of models are interesting to study for several reasons. One can use nonlocal equations to deal with the discontinuities or singularities of a function. Moreover, under suitable conditions if the integral operator is a limiting case of the local operator, then the nonlocal equation can be considered as an approximation of the local equation.

In this work, we give quantitative rates of convergence to a stationary state for some linear nonlocal models coming from structured population dynamics. In some cases, we are able to present some results in the nonlinear setting as well. Moreover, we also study hypocoercivity of some linear, space inhomogeneous kinetic equations. Hypocoercivity is a term used in the study of convergence to stationary state for certain classes of kinetic equations, which are the linear relaxation Boltzmann and the linear Boltzmann equations in our case. A common trait of such nonlocal population models and kinetic equations is that the underlying dynamics is a special type of stochastic process, called Markov process. A Markov process is a random process whose future state depends only on its current state, not the past. The main innovation in this work is to use a method coming from probability, from the study of convergence of Markov processes. This technique is based on Doeblin's/Harris's Theorems and it can be easily adapted to linear, nonlocal equations. In the next two sections, we describe the mathematical models we consider for structured population dynamics and kinetic theory. We conclude the introduction after giving an outline for the thesis.

1.2 Structured population dynamics

Population dynamics aims to have predictions about certain properties like the size of a biological population in time. This problem has captured the attention of scientists since long time. If the population consists of large number of members, tracking the time evolution of some averaged quantity makes it possible to construct a mathematical model by using partial differential equations. Studying many properties of this PDEs mathematically, provides a lot of information about the underlying population. One main question in this kind of study concerns how the growth of a population will be regulated in the long-time. Particularly being able to show the exponential growth as a natural tendency of a biological population is the core of this study. It is often referred as the *asymptotic behaviour* or the *asynchronous behaviour* or the *Malthusian behaviour* of the population (after Thomas Robert Malthus).

Mathematical models for a structured population dynamics could be inspired from various disciplines in biological, environmental and medical sciences like ecology, epidemiology, cell dynamics, genetics and evolution. Deriving useful analytical properties for a mathematical model, most of the time, involves simplifying assumptions which might be severe from a biologists' point of view. Even so, sometimes it is still hard to capture very fundamental biological properties of a population. That is what makes the mathematical study of real-life-inspired models challenging yet beautiful and interesting.

1.2.1 Age-structured population models

The study of age-structured populations dates back to the beginning of 20th century. McKendrick's paper in 1926 [79] on the renewal equation (also known as McKendrick-von Foerster equation) and nonlinear extensions of this work aroused much interest both in linear and nonlinear age-structured population models. *Asynchronous exponential growth* of an age structured population has been heuristically derived first by Alfred J. Lotka and F. R. Sharpe in 1911 (Part II of [97]). They showed that the population grows exponentially with a Malthusian parameter and it converges to a "fixed" and stable age-distribution.

Later another rigorous proof of exponential growth was given by William Feller in 1941 (Part II of [97]). A milestone in this area is the book of Webb [104] on age-structured population dynamics where he proved the stability of a stationary state by using semigroup approach. Later the study of linear age-structured population dynamics served as a step for developping general theory of positive semigroups [100, 50, 68].

A commonly used version of the conservative renewal equation is given by

$$\begin{aligned} \frac{\partial}{\partial t}n(t,a) + \frac{\partial}{\partial a}n(t,a) &= -\gamma(a)n(t,a), && t,a>0,\\ n(t,0) &= \int_0^\infty \beta(a)n(t,a)\ \mathrm{d}a, && t>0,\\ n(0,a) &= n_0(a), && a\geq 0. \end{aligned} \tag{1.1}$$

Here the structuring variable a, stands for *age*, grows as fast as the time variable and reset to zero with the rate γ. For (1.1) the stationary age distribution, $N_*(a)$, is explicit and given by

$$N_*(a) = N_*(0)e^{-\int_0^a \gamma(s)\,\mathrm{d}s}.$$

Now we look at different variations, including a nonlinear version, of this equation first; then some other models which are structured by different variables. Since some part of this work focuses on modelling of neuron population models, we give a brief biological introduction about the structure and the physiology of a single neuron.

1.2.2 Elapsed-time structured neuron population models

An integral part of an animal's body is the nervous system that helps for behavioral coordination by transmitting signals between different parts of the body. The nervous system is composed of neuron cells and the human brain alone contains roughly 100 millions of neurons. How do neurons function? They process the input which is received from the outside world; then according to the input they send commands to muscles by exchanging information among themselves and other cell types. The brain is such an important and unique organ yet it is very challenging to study due to the complexity of large neural networks it is composed of. Understanding the behaviour of interacting neuron populations gives rise to many mathematical problems as well. In this context, several mean-field models have been proposed to understand the electrical activity of a group of interacting neurons. They are all based on simplified models for the electrical activity of a single neuron, which can be described by an averaged partial differential equation or integro-differential equation if the number of neurons involved is large enough. Now we look at the physiology of a single neuron to understand better a population of neurons.

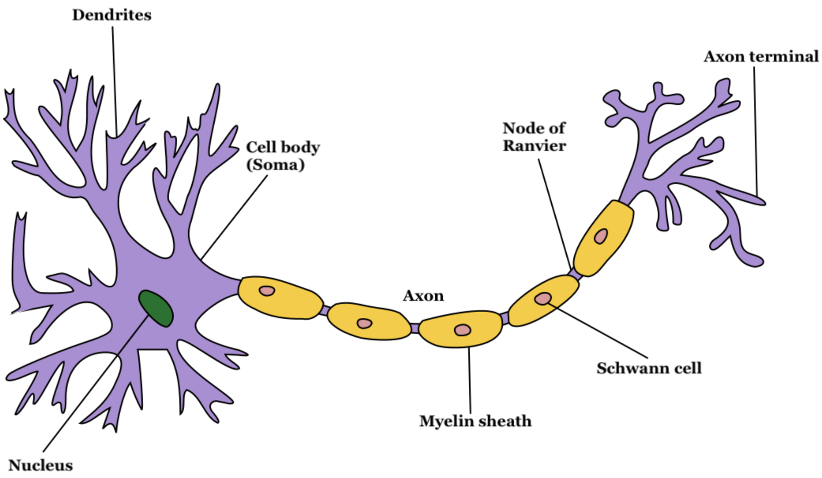


Fig. 1.1 Structure of a single neuron.
Image by Quasar Jarosz at English Wikipedia, CC BY-SA 3.0.

Physiology of a single neuron

A neuron consists mainly of dendrites, an axon, and a cell body or soma. The actual shape of a neuron can be thought as analogous to that of a tree: dendrites, the axon and the soma correspond to the branches, roots and trunk of a tree (See Figure 1.1). A neuron receives the input from other cells through dendrites and sends the output towards the axon via electrical signals called *action potentials*. Electrical signals are created in the axon and then the action potentials travel through the signals. This process initiates neurotransmitter release into the synapse enabling the neuron to communicate with other neurons (See Figure 1.2). Action potentials are sometimes also called spikes. After each spike, hundreds of synapses of a single neuron release neurotransmitters so that a single neuron can communicate with hundreds of other neuron cells.

Neurotransmitter is a chemical substance which can influence a neuron in an *excitatory* or an *inhibitory* way. If a neurotransmitter promotes generating an action potential then it is called an excitatory transmitter. On the contrary, if it prevents the generation of an action potential then it is called inhibitory. Communication between neurons depends on the balance between excitatory and inhibitory effects. When a neuron receives hundreds of inputs; they are either added or subtracted based on the type of the effect in the brain and this process is called *synaptic integration*. If the total input hits a threshold value where the excitatory effect surpass the inhibitory one then the neuron gets active and spikes.

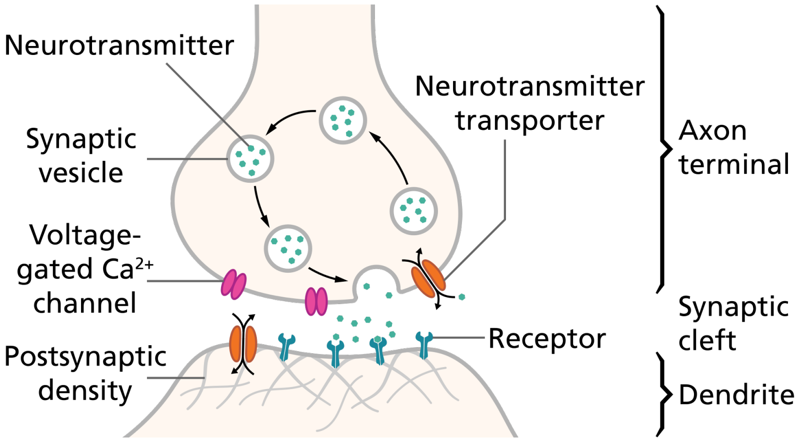


Fig. 1.2 Structure of a chemical synapse.
Image by Thomas Splettstoesser (www.scistyle.com) - Own work, CC BY-SA 4.0.

Roughly, the natural "charging" process of interacting neuron populations, followed by a sudden "discharge" takes place in a stochastic way depending on some variables like the current charge, the time since the last discharge, and the activity of other connected neurons.

One way of modelling the dynamics of neuron populations is to consider the membrane potential of neurons as the structuring variable. This means studying the evolution of the density $n(t,v)$ of neurons with potential v at time t. We refer to [101, 27, 26, 24, 37, 29, 32, 33] for some of the recent mathematical theory of these models.

We are interested in the family of models where the time evolution of the probability density of neurons is structured by the time passed since the last discharge. Therefore they are called *elapsed time structured* models and the models we look for here were proposed in [89–91].

The first model is based on stochastic simulations done in [94]. It is a nonlinear version of the conservative renewal (McKendrick-Von Foerster) equation which has been well-studied by many authors in the past as a model for a broad range of biological phenomena like epidemic spread and cell division [104, 92, 74, 68, 100, 50].

Thus, the dynamics of the interacting neuron populations governed by the following integro-differential PDE

$$\frac{\partial}{\partial t}n(t,s) + \frac{\partial}{\partial s}n(t,s) + p(N(t),s)n(t,s) = 0, \qquad t,s > 0, \tag{1.2}$$

coupled with a boundary and an initial condition given by

$$\begin{aligned} N(t) := n(t, s=0) = \int_0^{+\infty} p(N(t), s) n(t,s)\,\mathrm{d}s, &\quad t > 0, \\ n(t=0, s) = n_0(s), &\quad s \geq 0. \end{aligned} \tag{1.3}$$

The integro-PDE (1.2)-(1.3) models the evolution of a neuron population density $n(t, s)$ depending on time t and the time s elapsed since the last discharge. Neurons randomly fire at a rate p per unit of time, and they re-enter the cycle from $s = 0$ immediately after they fire, as imposed through the boundary condition at $s = 0$; the variable s can thus be regarded as the 'age' of neurons, making a parallel with models for birth and death processes. The global activity $N(t)$ denotes the density of neurons which are undergoing a discharge at time t. If the firing rate p increases with N, interactions are considered as *excitatory* and the firing of neurons makes it more likely that connected neurons will also fire. If p decreases with N then interactions are *inhibitory*.

The second model which we consider has many similarities with a larger class of integro differential equations called the *growth-fragmentation equations* which we will give more details in the next section. The nonlinear version we study here was introduced in [91], but on this general type of equations we also mention the works in [93, 82, 96, 57, 75, 59, 38, 55, 62].

This model is a modified version of the first one where s represents a generic "state" of the neuron, not necessarily the time elapsed since the last discharge now. It is assumed that neurons in a state u return to a certain state $s < u$ after firing, with a certain probability distribution $\kappa(s, u)$. The model reads as follows:

$$\frac{\partial}{\partial t} n(t,s) + \frac{\partial}{\partial s} n(t,s) + p(N(t), s) n(t,s) = \int_0^{+\infty} \kappa(s,u) p(N(t), u) n(t,u) du, \quad u, s, t > 0, \tag{1.4}$$

coupled with

$$\begin{aligned} n(t, s=0) = 0, &\quad t \geq 0. \\ n(t=0, s) = n_0(s), &\quad s \geq 0. \end{aligned} \tag{1.5}$$

The flux of neurons which are firing at time t is defined the same as before

$$N(t) := \int_0^{+\infty} p(N(t), s) n(t,s) ds, \quad t > 0.$$

The difference between (1.4)-(1.5) and the first one (1.2)-(1.3) is the addition of a kernel $\kappa = \kappa(s,u)$. For fixed u, the quantity $\kappa(\cdot, u)$ is a probability measure which gives the distribution of neurons which take the state s when they discharge at a state u. Hence, neurons do not necessarily start the cycle from $s = 0$ after firing, and so s cannot be considered as an 'age' variable in this model. We remark that that equation (1.2)-(1.5) is a limiting case of equation (1.4)–(1.3) when $\kappa(\cdot, u) = \delta_0(s)$, the Dirac delta at $x = 0$. We remark that the terms involving p and κ are mathematically close to the ones appearing in fragmentation processes.

These nonlinear models (1.2)-(1.5) and (1.4)–(1.3) preserve positivity and have a conservation property such that

$$\frac{d}{dt}\int_0^{+\infty} n(t,s)\,\mathrm{d}s = 0. \tag{1.6}$$

In particular, this ensures that if the density of neurons is a probability distribution initially, then it remains so. Whenever it is convenient we assume that n_0 is a probability distribution (which may be assumed after a suitable scaling).

These equations and similar models have been shown to exhibit many interesting phenomena which are consistent with the experimental behaviour of neurons: depending on the parameter p and the initial data one can find periodic solutions, apparently chaotic solutions, and solutions which approach an equilibrium state. The first two kinds of behaviour (periodic and chaotic solutions) are harder to study mathematically; numerical simulations have been performed in [89, 90] and some explicit solutions have been found. Regarding convergence to equilibrium, some regimes are studied in these works using perturbative techniques, such as the so-called *low-connectivity* and *high-connectivity* cases.

As a population balance equation of a form that appears often in mathematical biology, several techniques exist to study equations (1.2)-(1.3) and (1.4)-(1.5) rigorously. We refer to [92] for a good exposition of many of the relevant tools.

One of the main methods used so far in the study of convergence to equilibrium for equations (1.2) and (1.4) is the *entropy method*, which involves finding a suitable Lyapunov functional $H = H(n)$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}H(n(t,\cdot)) = -D(n(t,\cdot)) \leq 0 \tag{1.7}$$

along solutions $n = n(t,s)$ to (1.2)-(1.3) or (1.4)-(1.5), and then investigating whether one may prove inequalities of the type $\lambda H(n) \leq D(n)$ for some $\lambda > 0$ and a family of functions n sufficiently large to contain $n(t,\cdot)$ for all times t. If the answer is positive, one can apply the Gronwall inequality to (1.7) and deduce that $H(n(t,\cdot))$ decays exponentially with a rate proportional to $e^{-\lambda t}$. This in turn may give useful information on the approach to equilibrium, often implying that $n(t,\cdot)$ approaches equilibrium in the L^1 norm. This idea was followed in [89–91], using a specific Lyapunov functional obtained by integrating the primitive of $n - n_*$ (where n_* is an equilibrium state) against a suitable weight. A fundamental difficulty is that phenomenological equations motivated by biological considerations do not have any obvious Lyapunov functionals. This difficulty leads us to considering cases which are close to a linear regime, taking advantage of the fact that mass and positivity conserving linear equations (Markov evolutions).

Apart from the entropy method, for the time elapsed neuron network model (1.2), another approach has been developed in [105, 86]. This approach is based on spectral analysis theory for semigroups in Banach spaces. In [86], uniqueness of the steady state and its nonlinear exponential stability in the weak connectivity regime for the first model was proved. This approach is extended in [105] to the cases without delay and with delay both in the weak and strong connectivity regimes considering a particular step function as a firing rate. Furthermore, in [47] the link between several point process models (Poisson, Wold, Hawkes) that have been proved to statistically fit real spike trains data and age-structured partial differential equations which are introduced by [89] was investigated. This approach is extended to generalized Hawkes processes as microscopic models of individual neurons in [46].

In this work, we propose an alternative approach that is based on neither the entropy method nor the aforementioned approaches, but instead takes advantage of a set of results in the theory of Markov processes known as *Doeblin's theory*, with some extensions such as *Harris's theorem*; see [71], or [70, 63] for simplified recent proofs and [**?**][Chapter 2] for a basic exposition. The idea is still based on first studying the linear case and then carrying out a perturbation argument; the difference is that we study the spectral properties of the linear operator by Doeblin's theory, which is quite flexible and later simplifies the proofs. We obtain a spectral gap property of the linear equation in a set of measures, and this leads to a perturbation argument which naturally takes care of the boundary conditions in (1.2)- (1.3) and (1.4)-(1.5). Similar ideas are reviewed in [63] for the renewal equation, and have been recently used in [8] for neuron population models structured by voltage.

Due to this strategy, studying solutions to (1.2)- (1.3) and (1.4)-(1.5) in the sense of measures comes as a natural setting for two important reasons: first, it fits well with the linear theory; and second, it allows us to treat the weakly nonlinear case as a perturbation of the linear one. Note that one difference between the weakly nonlinear case and the linear case for equation (1.2)- (1.3) is the boundary condition, and this is conveniently encoded as a difference in a measure source term. Measure solutions are also natural since a Delta function represents an initial population whose age (or structuring variable) is known precisely. There exist also recent works on numerical schemes for structured population models in the space of nonnegative measures [23, 44]. Entropy methods have also been extended to measure initial data by [66] for the renewal equation.

1.2.3 Size-structured population models

Sometimes, the age or the elapsed-time structure do not help in extracting meaningful information about the dynamics of a population. In this case, structuring variable can be *size, length, weight, DNA content, biochemical composition etc.* depending on the underlying dynamics which the equation describes. We call it as *size* for simplicity here. Cell division is one of the common examples of these type of models and they are governed via a large class of integro-partial differential equations called the *growth-fragmentation equations*.

The general form of the growth-fragmentation equation is given by

$$\frac{\partial}{\partial t}n(t,x) + \frac{\partial}{\partial x}(g(x)n(t,x)) = \int_x^{+\infty} \kappa(y,x)n(t,y)\,\mathrm{d}y - B(x)n(t,x), \quad t,x>0, \tag{1.8}$$

coupled with

$$\begin{aligned} n(t,0) &= 0, && t \geq 0, \\ n(0,x) &= n_0(x), && x > 0, \end{aligned} \tag{1.9}$$

where $n(t,x)$ is the population density of individuals structured by a variable $x > 0$ (size) at a time $t \geq 0$. The equation (1.8) is coupled with an initial condition $n_0(x)$ at time $t = 0$ and a 0 boundary condition which represents the fact that no individuals are newly created at size 0.

The function g is the *growth rate* and B the *total division/fragmentation rate* of individuals with size $x \geq 0$. The kernel $\kappa(y,x)$ represents the rate at which individuals of size x are obtained as the result of a fragmentation event of an individual of size y. The total fragmentation rate B is always obtained as

$$B(x) = \int_0^y \frac{y}{x}\kappa(x,y)\,\mathrm{d}y.$$

We consider two particular cases for the fragmentation kernel:

$$\kappa(x, y) = B(x)\frac{2}{x}\delta_{\{y=\frac{x}{2}\}},$$

which corresponds to the *mitosis* process, suitable for modelling of biological cells, where individuals can only break into two equal fragments; and

$$\kappa(x, y) = B(x)\frac{2}{x},$$

which is the case with *uniform fragment distribution*, where fragmentation gives fragments of any size less than the original one with equal probability. This case is used for example in modelling of fragmentation of polymer chains, like in [80].

Two opposite dynamics, groeth and fragmentation are balanced through equation (1.8)-(1.9). While growth term tends to increase the average size of the population, fragmentation term increases the total number of individuals but breaks the population into smaller sizes. If the growth rate $g(x) \equiv 0$, then only fragmentation takes place and the equation is known as the *pure fragmentation equation*. Similarly when B and κ are both 0, growth dominates the dynamics and the equation (1.8) is called the *pure growth equation*.

We are concerned here with the mathematical theory of this equation, and more precisely with its long-time behaviour as $t \to +\infty$. Under suitable conditions on the coefficients κ and g, the typical behaviour is that the total population tends to grow exponentially at a rate $e^{\lambda t}$, for some $\lambda > 0$, and the normalised population distribution tends to approach a *universal profile* for large times, independently of the initial condition. This has been investigated in a large amount of previous works, of which we give a short summary.

The first mathematical study of this type of equation was done in [51] for the mitosis case, in a work inspired by biophysical papers [9, 10]. In [51], authors considered the mitosis kernel with the size variable in a bounded space and proved exponential growth at a rate λ, and exponentially fast approach to the universal profile. In [83], the authors considered the size variable in $(0, \infty)$ and introduced *General Relative Entropy* method and proved exponential relaxation to equilibrium in L^p spaces without an explicit rate. Following the works [93] and [92], providing an explicit rate of convergence to a stationary state under reasonable assumptions became a topic of research for many other works. New functional inequalities were proved in [36, 35] in order to obtain explicit rates of convergence. Some authors provided explicit solutions like in [107, 106]; some authors used semigroup approach like in [2, 6, 7, 56, 64, 84]; and some authors used a probabilistic approach like in [16, 15, 17, 18, 22]. In [12], the authors proved that with a bounded fragmentation rate, exponential relaxation is not uniform with respect to the initial data. Later in [13], the same authors considered unbounded fragmentation rate and proved exponential growth in L^1 space. However, when the equal mitosis kernel is considered, there is a special case with a linear growth rate where solutions exhibit oscillatory behaviour in long time. This property was first proved mathematically by [64] when the equation is posed in a compact set. Recently in [14] this result is extended to $\mathbb{R}^+$ by the general relative entropy argument in a convenient weighted L^2 space, where well-posedness is obtained via semigroup analysis. They also propose a non-diffusive numerical scheme which can capture the oscillations.

Main tool when proving exponential relaxation to a stationary state is studying Perron eigenvalue problem. Specifically proving existence and uniqueness of the first positive eigenvalue associated to a positive eigenvector.

In [4], the authors gave some estimates on the principal eigenfunctions of the growth-fragmentation operator, giving their first order behavior close to 0 and $+\infty$. Then they proved a spectral gap result by means of entropy–entropy dissipation inequalities. They assumed that growth and fragmentation coefficients behave asymptotically like power laws. Similar method was previously used in [36].

Perron eigenvalue problem consists of finding suitable eigenelements $(\lambda, N(x), \phi(x))$ which satisfy the following:

$$\begin{aligned} \frac{\partial}{\partial x}(g(x)N(x)) + (B(x)+\lambda)N(x) &= \int_x^{+\infty} \kappa(y,x)N(y)\,\mathrm{d}y, \\ g(0)N(0) = 0, \quad N(x) \geq 0, \quad &\int_0^{+\infty} N(x)\,\mathrm{d}x = 1. \end{aligned} \tag{1.10}$$

$$\begin{aligned} -g(x)\frac{\partial}{\partial x}\phi(x) + (B(x)+\lambda)\phi(x) &= \int_0^{x} \kappa(x,y)\phi(y)\,\mathrm{d}y, \\ \phi(x) \geq 0, \quad &\int_0^{+\infty} \phi(x)N(x)\,\mathrm{d}x = 1. \end{aligned} \tag{1.11}$$

If such a triple exists then the equation (1.8)-(1.9) converges to a universal profile whose shape is given by the eigenfunction $N(x)$ and growth rate of the population is given by the dominated eigenvalue $\lambda > 0$. Moreover if we scale the equation by defining $m(t,x) := n(t,x)e^{-\lambda t}$ we obtain:

$$\begin{aligned} \frac{\partial}{\partial t}m(t,x) + \frac{\partial}{\partial x}(g(x)m(t,x)) + (B(x)+\lambda)m(t,x) &= \int_x^\infty \kappa(y,x)m(t,y)\,\mathrm{d}y, && t,x \geq 0, \\ m(t,0) &= 0, && t > 0, \\ m(0,x) &= n_0(x), && x > 0. \end{aligned} \tag{1.12}$$

We remark that $N(x)$, solution of (1.10)-(1.11), if exists, is the stationary distribution of (1.12). Existence and uniqueness of eigenelements prove many useful information about long-time behaviour of the growth-fragmentation equation (1.8)-(1.9). We refer to [55] for a detailed review. The reason we introduced the concept here is that we will mainly work on (1.12) instead of (1.10)-(1.11); since it is more straight-forward and we can easily recover the properties of the latter from the former. We also notice that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{+\infty} m(t,x)\phi(x)\,\mathrm{d}x = 0, \tag{1.13}$$

so that the quantity $f(t,x) := m(t,x)\phi(x)$ is conserved if there exist a solution to the Perron eigenvalue problem (1.10)-(1.11).

1.2.4 Main results

We introduced three different models (1.2)-(1.3); (1.4)-(1.5) and (1.8)-(1.9) from structured population dynamics and we are interested in long-time behaviour of solutions of these nonlocal PDEs. Since the time evolution for these type of models can also be described as a Markov process, we can naturally apply some results which already exist on the convergence of Markov processes. Here we use a couple of probabilistic results, namely Doeblin's-Harris's theorems, of which precise statements and proofs will be given in the Chapter 2.

We prove existence of solutions and steady states in the space of finite, nonnegative measures for (1.2)-(1.3) and (1.4)-(1.5). We also show that the solutions converge to the stationary state, which is unique up to scaling, exponentially in time, in the case of weak nonlinearity (i.e., weak connectivity). Concerning the asymptotic behaviour, we show the existence of a spectral gap property in the linear (no-connectivity) setting for both models by using Doeblin's theorem. Then by a constructive perturbation argument we give results on exponential relaxation to the steady state for the nonlinear models (1.2)-(1.3) and (1.4)-(1.5). The closest results in the literature are those of [89, 91].

Our equation (1.2)(1.3) is essentially the same as in [89], written in a slightly different formulation that does not include time delay and does not highlight the connectivity as a separate parameter (the connectivity of neurons in our case is measured in the size of $\partial_N p$). The results in [89] use entropy methods and show exponential convergence to equilibrium with a similar result to ours in a weighted L^1 space, for the case with delay and for a particular form of the firing rate p. As compared to this, our results work in a space of measures and can be easily written for general firing rates p; however, we have not considered the large-connectivity case (which would correspond to large $\partial_N p$ in our case) or the effects of time delay. Similar remarks apply to the results for equation (1.4)-(1.5) contained in [91]. In this case, our strategy works for general conditions which are simpler to state, and provide a general framework which may be applied to similar models. Similarly, we have not considered a time delay in the equation, which is a difference with the above work. There are numerical simulations and further results on regimes with a stronger nonlinearity in [89–91]. Details of our results can be found in Chapter 3.

For the growth-fragmentation equation (1.8)-(1.9), we prove the spectral gap property constructively under more general conditions on the total fragmentation and growth rates. We consider equal mitosis and uniform fragmentation kernels and provide quantitative rates of convergence by using Harris's Theorem. We also give an existence proof for eigenelements so that the spectral gap results do not require eigenelements to be known explicitly. Moreover, we provide some bounds on the dual eigenfunction ϕ in (1.11) so that we can use the scaled equation on $f(t,x) := \phi(x)n(t,x)e^{-\lambda t}$ which have a conservation property. Later we recover the properties of the original model. Some references for previous works on this equation was given earlier but we mention that a variation growth-fragmentation equation with bounded fragmentation rate is used in modeling of the dynamics of the carbon content of a forest whose deterministic growth is interrupted by natural disasters [25] recently. The authors used Harris's theorem to obtain quantitative convergence rates. This might be the closest work to ours in terms of the method and the model considered but we could obtain results in the case where the total fragmentation rate is unbounded. Details of our results can be found in Chapter 4.

1.3 Linear kinetic equations

Kinetic theory concerns the modelling of a particle system consisting of large number of particles by describing densities in the phase space via distribution functions. Some of the common examples are modelling of the dynamics of a gas or a plasma. Moreover, the phase space is not anymore composed only of the macroscopic variables like the position but also the microscopic variables describing the 'state' of a particle. Since we are interested neither in relativistic nor quantum systems here, we consider 'velocity' as a microscopic variable. Therefore the phase space will be (x, v), the position and the velocity of a particle at some time.

Main modelling assumption is that the gas or the particle system which we consider should be observed over some time $t \in [0, +\infty)$ and it is contained in a domain $\Omega \subset \mathbb{R}^d$ which can be bounded or unbounded. Then the nonnegative function $f(t, x, v)$ defined on $[0, +\infty) \times \Omega \times \mathbb{R}^d$ is the probability distribution function describing the particle system. Density of particles in the volume $\mathrm{d}x\,\mathrm{d}v$ for a fixed time t is defined by the quantity $f(t, x, v)\,\mathrm{d}x\,\mathrm{d}v$. Therefore f is at least assumed to be locally integrable or it is a bounded measure on some $\mathcal{X} \times \mathbb{R}^d$ where $\mathcal{X}$ is a compact subset of Ω.

The particle system can be treated as a continuum since we also assume that the system consists of large number of particles. We express macroscopic quantities in terms of integrals of the form $\int f(t, x, v)\varphi(v)\,\mathrm{d}v$, via observables that are macroscopic measures in terms of microscopic averages.

If we neglect the interaction between particles, then each particle will travel along a straight line with a constant velocity and the density will remain constant along the characteristics:

$$\dot{x} = v,$$
$$\dot{v} = 0,$$

so that the solution can be easily represented by the solution at time 0 by

$$f(t, x, v) = f(0, x - vt, v),$$

where f is a weak solution to the *free transport equation*:

$$\partial_t f + v \cdot \nabla_x f = 0, \tag{1.14}$$

and the operator $v \cdot \nabla_x$ is called the *transport operator*. In the presence of a macroscopic force acting on particles (1.14) becomes:

$$\partial_t f + v \cdot \nabla_x f + F(x) \cdot \nabla_v f = 0, \tag{1.15}$$

In this case, particles do not follow a straight line trajectories anymore under the influence of the force. The equation (1.15) is called the *linear Vlasov equation.* The force field can be interpreted as the action of the external potential via $F(x) = -\nabla_x \Phi(x)$. If we want to take into account any type of interaction, either between particles or with a background medium, we include an operator on the right-hand side of (1.14) or (1.15). In this work we consider an interaction term as well. We consider equations in $(x, v) \in \mathbb{T}^d \times \mathbb{R}^d$; thus of the type

$$\partial_t f + v \cdot \nabla_x f = \mathcal{L} f,$$

where $f = f(t, x, v)$ is the density of particles at position x (the), moving with velocity v at time $t \geq 0$. Here $\mathbb{T}^d$ denotes d-dimensional unit torus.

We also consider the same equations posed on the whole space $(x, v) \in \mathbb{R}^d \times \mathbb{R}^d$ with an external potential Φ which has the confining effect;

$$\partial_t f + v \cdot \nabla_x f - (\nabla_x \Phi \cdot \nabla_v f) = \mathcal{L} f.$$

We note that the operator $\mathcal{L}$ acts only on the v variable. Particularly, we work with the cases $\mathcal{L}$ equal to the linear relaxation Boltzmann operator (sometimes known as linear BGK operator), and $\mathcal{L}$ equal to the linear Boltzmann operator. Therefore, we give a brief introduction for these equations next.

1.3.1 The linear Boltzmann equation

The Boltzmann equation is derived by Ludwig Boltzmann in 1872 to describe the statistical behaviour of a thermodynamical systems which is not in equilibrium. The equation can also be used to describe the evolution of some physical quantities of a fluid such as heat and momentum. In our context, the Boltzmann equation will describe the behaviour of a dilute gas. It is a nonlocal integro-differential equation gives the time evolution of a probability distribution in the phase space. This evolution depends on the external forces exerted on the particles, the diffusion of particles and the internal forces between particles during interactions.

First we specify the assumptions we need to make when considering the interactions between particles:

- Particles interact via *binary collisions* which implies that the particle system is *dilute* so that we can neglect the interaction of particles involving more than two particles since the probability of occurence of this type of encounters is much smaller than the occurence of binary collisions.

- Collisions are *localized* in space and time. They occur in a very short duration compared to time scale considered.
- Collisions are *elastic* in the sense that momentum and the kinetic energy is conserved during the collision. If we denote the velocities before the collision as v', v'_* and after the collision as v, v_* we obtain

$$\begin{aligned} v' + v'_* &= v + v_*, \\ |v'|^2 + |v'_*|^2 &= |v|^2 + |v_*|^2. \end{aligned} \tag{1.16}$$

  If d denotes the dimension, we end up with $d+1$ scalar equations for $2d$ unknown parameters.

  Sometimes it is more convenient to use *σ-representation* for (1.16) in the following way:

$$\begin{aligned} v' &= \frac{v+v_*}{2} + \frac{v-v_*}{2}\sigma, \\ v'_* &= \frac{v+v_*}{2} - \frac{v-v_*}{2}\sigma, \end{aligned} \tag{1.17}$$

  where $\sigma \in \mathbb{S}^{d-1}$ varying in the $d-1$ unit sphere (see Figure 1.3). Moreover, sometimes particles are assumed to interact via an *interaction potential* depending on the distance between each other.

- Collisions are *microreversible*. This deterministically means that the dynamics is time reversible. On the other hand, probabilistically this would mean that the probability of velocities (v', v'_*) changing into (v, v_*) is the same as the probability of velocities (v, v_*) changing into (v', v'_*).
- The last assumption is known also as the *molecular chaos hypothesis*. It means that the velocities of two particles at the same position x are uncorrelated before the collision.

Under these assumptions Ludwig Boltzmann (1872) derived the *quadratic collision operator* $Q(f, f)$ which describes how the collisions affect the distribution function f:

$$\frac{\partial}{\partial t} f(t,x,v) = Q(f,f)(t,x,v) = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} B(v-v_*,\sigma)\left(f(v')f(v'_*) - f(v)f(v_*)\right) \mathrm{d}\sigma\, \mathrm{d}v_*, \tag{1.18}$$

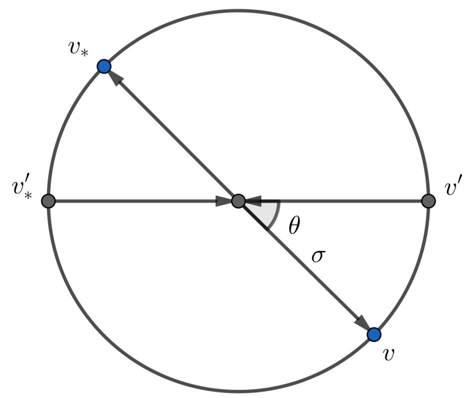


Fig. 1.3 A binary collision in the velocity phase space.
The angle between pre- and post- collisional velocities is θ.

where $f(v') = f(t, x, v')$, $f(v'_*) = f(t, x, v'_*)$ and $f(v_*) = f(t, x, v_*)$. The nonnegative function $B(z, \sigma)$ is called the *collision kernel* and depends only on $|z|$ and on the scalar product $\left\langle \frac{z}{|z|}, \sigma \right\rangle$ which is the cosine of the angle between pre- and post-collisional velocities. It can be thought as having the effect of replacing the two particles with given pre-collisional velocities with a cloud of particles whose post-collisional velocities are distributed over the d-dimensional sphere of radius σ in terms of a function of the relative velocity $v - v_*$. We write this dependence as

$$B(v - v_*, \sigma) = B(|v - v_*|, \cos\theta), \qquad \cos\theta = \left\langle \frac{v - v_*}{|v - v_*|}, \sigma \right\rangle$$

In the literature, there are two cases where B is explicit in terms of the interaction between particles:

- *Coulomb interactions*; where the interaction between particles depends on the distance between them by $\Psi(x) = 1/x$. In this case, the effect of *grazing* collisions are more stronger than the effect of other type of collisions and the solutions will converge towards the solution of the Fokker-Planck-Landau Equation which is used for describing the binary collisions between charged particles in a plasma and given by

$$\partial_t f + v \cdot \nabla_x f = Q(f, f),$$

where

$$Q(f,f) = \nabla_v \cdot \left(\int_{\mathbb{R}^d} A(v-v_*) \left(f_* \nabla_v f - f (\nabla_v f)_*\right) \mathrm{d}v_* \right),$$
$$A_{ij}(z) = (1/|z|) \left(\delta_{ij} - z_i z_j/|z|^2\right),$$

where $f = f(t,x,v)$, $t \geq 0$; $x, v \in \mathbb{R}^d$ and A_{ij} is a matrix valued function, for more details see [102, 42] and the references therein.

The grazing collisions are the type of collisions which result in an infinitesimal angle deflection of the trajectories.

- *Hard spheres*; which are impenetrable spheres which cannot overlap in space. They bounce back after a collision like billiard balls. In this case $B(|z|,\sigma)$ is proportional to $|z|$.

Considering inverse power law for the interaction potential is very common since it is used in modeling of some relevant physical phenomena. For example; if we consider a potential of the form

$$\Psi(x) = \frac{1}{x^{s-1}},$$

- $s = 7$ corresponds to Van der Waals interactions,
- $s = 5$ corresponds to ion-neutral interactions,
- $s = 3$ corresponds to Manev interactions,
- $s = 2$ corresponds to Coulomb interactions.

Moreover for $s > 2$, the collision kernel B cannot be computed explicitly but it can be written as a product

$$B(|v-v_*|,\sigma) = |v-v_*|^\gamma \, b(\cos\theta) = |v-v_*|^\gamma b\left(\frac{v-v_*}{|v-v_*|}\cdot\sigma\right), \tag{1.19}$$

where $\gamma = \frac{s-(2d-1)}{s-1}$, for dimension d. When $\gamma = 0$, the collision kernel is independent from the relative velocity and depends only on the angle θ. These type of molecules are *Maxwellian molecules*. If $0 < \gamma \leq 1$, this case corresponds to *hard potentials*. If $-d < \gamma < 0$, this case models the *soft potentials*.

We assume also that b is integrable and uniformly positive on $[-1,1]$; that is, there exists $C_b > 0$ such that

$$b(z) \geq C_b \text{ for all } z \in [-1,1]. \tag{1.20}$$

Under this setting the *Boltzmann equation* on the whole $\mathbb{R}^d$ reads as

$$\frac{\partial}{\partial t} f + v \cdot \nabla_x f = Q(f, f), \quad t \geq 0, x \in \mathbb{R}^d, v \in \mathbb{R}^d. \tag{1.21}$$

or if the macroscopic force is also considered then,

$$\frac{\partial}{\partial t} f + v \cdot \nabla_x f + F(x) \cdot \nabla_v f = Q(f, f), \quad t \geq 0,\ x \in \mathbb{R}^d,\ v \in \mathbb{R}^d. \tag{1.22}$$

The velocity distribution function of a gas in equilibrium is described by the *Maxwellian distribution function* which is derived by James Clerk Maxwell in 1860. It is given by the expression

$$\mathcal{M}(v) := (2\pi)^{-d/2} \exp(-|v|^2/2) \tag{1.23}$$

where d denotes the dimension.

Note that (1.21) and (1.22) are nonlinear but in this work we consider the linear Boltzmann equation modelling the dynamics of interacting gas particles with the background which is already in equilibrium. The reason is the probabilistic methods that we use here are valid only for linear equations. The linear Boltzmann equation is either posed in the d- dimensional torus so that it has periodic boundary conditions or posed on the whole space $\mathbb{R}^d$ with a confining external potential. If $x \in \mathbb{T}^d$, the equation takes the form;

$$\partial_t f + v \cdot \nabla_x f = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} B\left(|v - v_*|, \frac{v - v_*}{|v - v_*|} \cdot \sigma\right) (f(v')\mathcal{M}(v'_*) - f(v)\mathcal{M}(v_*))\, \mathrm{d}\sigma \mathrm{d}v_*. \tag{1.24}$$

We assume that B splits as

$$B\left(||v - v_*|, \sigma\right) = |v - v_*|^\gamma b\left(\frac{v - v_*}{|v - v_*|} \cdot \sigma\right).$$

We make a cutoff assumption that b is integrable in σ. Moreover, we assume also that b is bounded below by a constant. We also consider the hard spheres and Maxwell molecules regime in this work, that is to suppose $\gamma \geq 0$, since for the soft potentials it is expected that the convergence rates will be worse and difficult to obtain quantitatively. Moreover there might be a need of extra cutoff assumptions not only on the angle, see [42, 30] and references therein. We have

$$\partial_t f + v \cdot \nabla_x f = \mathcal{L}^+ f - \kappa(v) f,$$

where $\kappa(v) \geq 0$ and $\kappa(v)$ behaves like $|v|^\gamma$ for large v; that is,

$$0 \leq \kappa(v) \leq (1 + |v|^2)^{\gamma/2}, \qquad v \in \mathbb{R}^d. \tag{1.25}$$

See [30] Lemma 2.1 for example.

We also look at the situation where the spatial variable is in $\mathbb{R}^d$ and we have a confining potential. In this case the equation is

$$\partial_t f + v \cdot \nabla_x f - (\nabla_x \Phi(x) \cdot \nabla_v f) = Q(f, \mathcal{M}). \tag{1.26}$$

This equation models gas particles interacting with a background medium which is already in equilibrium. Moreover, it has been used in describing many other systems like radiative transfer, neutron transportation, cometary flow and dust particles. The spatially homogeneous case has been studied in [76, 21, 30]. The kinetic equations (5.7) or (5.6) fit into the general framework in [87, 53], so convergence to equilibrium in weighted L^2 norms may be proved by using the techniques described there.

1.3.2 The linear relaxation Boltzmann equation

Sometimes the collision term in the Boltzmann equation is too complex to deal with, some toy models are introduced to simplify it. The *linear relaxation Boltzmann equation* or the *linear BGK equation* due to P. L. **B**hatnagar, E. P. **G**ross and M. **K**rook [20] is the best known of such models. This modified version of the Boltzmann equation in the linear setting is then given by

$$\partial_t f + v \cdot \nabla_x f = \mathcal{L}^+ f - f. \tag{1.27}$$

when it is considered on the torus; that is, for $x \in \mathbb{T}^d$, $v \in \mathbb{R}^d$, assuming periodic boundary conditions.

If we consider the equation on the whole space $x \in \mathbb{R}^d$ then it reads

$$\partial_t f + v \cdot \nabla_x f - (\nabla_x \Phi \cdot \nabla_v f) = \mathcal{L}^+ f - f. \tag{1.28}$$

in both the cases (5.4) and (5.5), $\mathcal{L}^+$ is defined as

$$\mathcal{L}^+ f = \left(\int f(t, x, u) \mathrm{d}u \right) \mathcal{M}(v),$$

and $\mathcal{M}(v) := (2\pi)^{-d/2} \exp(-|v|^2/2)$, Maxwellian. We assume that the potential $\Phi \colon \mathbb{R}^d \to \mathbb{R}$ is a $\mathcal{C}^2$ function of x and it is bounded.

This simple equation is well studied in kinetic theory and can be thought of as a toy model with similar properties to either the non-linear BGK equation or linear Boltzmann equation. It is also one of the simplest examples of a hypocoercive equation. Convergence to equilibrium in H^1 for this equation has been shown in [34], at a rate faster than any function of t. It was then shown to converge exponentially fast in both H^1 and L^2 using hypocoercivity techniques in [72, 87, 53].

1.3.3 Main results

The study of the speed of relaxation to equilibrium for kinetic equations is a well known problem, both for linear and nonlinear models. The central obstacle is that dissipation happens only on the v variable via the effect of the operator $\mathcal{L}$, while only transport takes place in x. The transport then "mixes" the dissipation into the x variable, and one has to find a way to estimate this effect. The theory of hypocoercivity was developed in [103, 72, 73] precisely to overcome these problems for linear operators. In a landmark result, [49] proved that the full nonlinear Boltzmann equation converges to equilibrium at least at an algebraic rate. Exponential convergence results for the (linear) Fokker-Planck equation were given in [48], and a theory for a range of linear kinetic equations has been given in [53]. All of these results give convergence in exponentially weigthed L^2 norms or H^1 norms; convergence to equilibrium in weigthed L^1 norms can then be proved for several kinetic models by using the techniques in [65].

We give exponential convergence results on the d-dimensional torus, or with confining potentials growing at least quadratically at ∞, always in total variation or weighted total variation norms (alternatively, L^1 or weighted L^1 norms). For subquadratic potentials we give algebraic convergence rates, again in the same kind of weighted L^1 norms. Some results were already available for these equations [34, 87, 53, 72, 58]. Previous proofs of convergence to equilibrium used strongly weighted L^2 norms (typically with a weight which is the inverse of a Gaussian), so one advantage of our method is that it directly yields convergence for a much wider range of initial conditions. The result works, in particular, for initial conditions with slow decaying tails, and for measure initial conditions with very bad local regularity. The method gives also existence of stationary solutions under quite general conditions; in some cases these are explicit and easy to find, but in other cases they can be nontrivial. We also note that our results for subquadratic potentials are to our knowledge new. Apart from these new results, our aim is to present a new application of a probabilistic method, using mostly PDE arguments, and which is probably useful for a wide range of models.

1.4 Structure of the thesis

In this thesis, we present quantitative rates of relaxation to equilibrium for some linear kinetic equations and nonlocal models for structured population dynamics. The common feature of these equations is that they are Markov evolutions and thus; we can use the results on the convergence of Markov processes.

In Chapter 2, we introduce the main techniques which are used in the subsequent chapters in order to achieve quantitative convergence results for some nonlocal models. This probabilistic method is first developed for the study of ergodic behaviour of discrete-time Markov processes and it dates back to [52]. Then in [71], the author extended this result to unbounded state space and gave the necessary and sufficient conditions of having a unique equilibrium, or an *invariant measure*, in the case of Markov processes. Later in [81] and in [99] this method is mentioned as Doeblin's Theorem and Harris's Theorem respectively. The method is based on verifying a minorisation condition and a geometric drift condition quantitatively in order to achieve quantitative rates of convergence. In [70], the authors gave simplified proofs of these theorems by using mass transport distances and it is an inspiration of this dissertation.

We consider two nonlinear models for elapsed-time structured describing neuron population models. The first one is a nonlinear version of the renewal equation and the structuring variable can also be considered as age. The second model is a modification of the first model with an addition of an integral term modeling the fact that neurons have a memory effect in the sense that they exhibit fatigue and adaptation. Both models were introduced in [89] and [91] respectively. Results concerning with the asymptotic behaviour of these models presented in Chapter 3. This part of the thesis is a joint work with José A. Cañizo at the University of Granada and it is published in the journal *Nonlinearity* [28].

We also consider a large class of integro-PDEs called the *growth-fragmentation equation* which is studied widely in the past literature due to the fact that it models various ecological, biological, telecommunicational and neuroscience related processes. In Chapter 4, we present the spectral gap results for this equation in some limiting cases of integral kernels. Besides we present some incomplete results in the discrete setting for the same equation. This part of the dissertation is based on two joint works.

First one is done jointly with José A. Cañizo at the University of Granada and Pierre Gabriel at Versailles Saint-Quentin-en-Yvelines University [40]; and the the second one is a joint work in progress with José A. Carrillo at Imperial College London.

In Chapter 5, we present results on hypocoercivity of the linear relaxation Boltzmann and the linear Boltzmann equations either on the torus $(x, v) \in \mathbb{T}^d \times \mathbb{R}^d$ or on the whole space $(x, v) \in \mathbb{R}^d \times \mathbb{R}^d$ with a confining potential. We obtain explicit convergence results in total variation or weighted total variation norms (alternatively L^1 or weighted L^1 norms) by using Harris's Theorem. This chapter is based on a joint work with José A. Cañizo at the University of Granada, Chuqi Cao and Josephine A. Evans both at University Paris Dauphine [39]. This work is recently accepted for publication to the journal *Kinetic and Related Models*.

Finally, Chapter 6 contains conclusions, perspectives for each chapter and brief introduction of some ongoing works where the aforementioned methods can be applicable. Some of these works started jointly with José A. Carrillo, Josephine Evans and Angeliki Menegaki.

Chapter 2

Convergence of Markov processes

"The ability to theorize is highly personal; it involves art, imagination, logic, and something more."

— Edwin Hubble

Contents

In this chapter, we revise some basic definitions and concepts about Markov processes. Eventually we are interested in convergence properties of Markov processes with a special focus on Doeblin's and Harris's Theorems since these theorems will be used for achieving the original results in Chapters 3-5. We give proofs of the theorems as well, mainly based on [70] but various versions can be found in [99, 81, 95, 71, 70, 69] and the references therein.

2.1 An introduction to Markov processes

A *discrete-time stochastic process* $\mathbf{X}$ on a state space $\mathcal{S}$ is a collection $\{X_n : n \in I\}$ of $\mathcal{S}$-valued random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathcal{S} \subset \Omega$ and I is a discrete index set $I = \{0, 1, 2, \cdots\}$. For a given n; X_n is the value of the process at time n. A *continuous-time stochastic process* $\{X(t) : t \geq 0\}$ on a state space $\mathcal{S}$ is defined in a similar way. The only difference here is that the increments now depend on the time variable which is continuous.

We remark that the state space $\mathcal{S}$ can be

- A finite set $\{1, \cdots, n\}$ (In this case, the state space is discrete).
- $\mathbb{R}^n$ or $\mathbb{Z}^n$.
- A manifold such as the d-dimensional sphere $\mathbb{S}^d$ or the d-dimensional torus $\mathbb{T}^d$.
- A Hilbert space such as $L^2([0, 1])$ or l^2.

Markov processes are stochastic processes whose distribution of increments does not depend on where they were in the past, but where they are at the present. This property is called *Markov property* and random processes satisfying this property are called *Markov processes*.

We give a formal definition of the Markov property in both discrete-time and continuous-time settings for a discrete state space $\mathcal{S}$:

2.1.1 Discrete-time Markov processes

The Markov property is given by the following:

For any $j, i_0, i_1, \cdots, i_{n-1} \in \mathcal{S}$ and any $n \geq 1$,

$$\mathbb{P}(X_n = j \mid X_0 = i_0, \ldots, X_{n-1} = i_{n-1}) = \mathbb{P}(X_n = j \mid X_{n-1} = i_{n-1}), \tag{2.1}$$

so that the distribution of X_n depends only on the immediate past, given the entire past of the process. We note that this does not mean that the distribution of the process does not depend on the time index n. In order to eliminate this possibility we make another assumption called *time-homogeneity* meaning that every time the process is at a state i then the distribution of where it is going to be at the next state is the same. Formally this means

$$\mathbb{P}(X_{n+m} = j \mid X_m = i) = \mathbb{P}(X_{n+m+k} = j \mid X_{m+k} = i), \tag{2.2}$$

for any $i, j \in \mathcal{S}$ and for any nonnegative n, m, k. The conditional probability (2.2) is called the *n-step transition probabilities* since they give where will the process end up n time after where it is right now. Similarly the 1-step transition probabilities defined as

$$\mathbb{P}(X_n = j \mid X_{n-1} = i) = \mathbb{P}(X_1 = j \mid X_0 = i) =: p_{i,j},$$

and $p_{i,j}$ does not depend on time because of the time-homogeneity assumption. We construct a matrix P, a *transition probability matrix*, by the 1-step transition probabilities where (i, j)th entry of P is $p_{i,j}$ such that

$$\sum_{j\in\mathcal{S}} p_{i,j} = \sum_{j\in\mathcal{S}} \mathbb{P}(X_1 = j \mid X_0 = i) = 1.$$

Since each row of P is a probability distribution over $\mathcal{S}$, P is a *stochastic matrix*. Moreover, the n-step transition probabilities are determined by the 1-step transition probabilities $p_{i,j}$ and this is known as the *Chapman-Kolmogorov equations*:

$$P^{n+m} = P^n P^m \qquad \text{or} \qquad p^{n+m}_{i,j} = \sum_{k\in\mathcal{S}} p^n_{i,k} p^m_{k,j}, \tag{2.3}$$

where P^n is the n-step transition probability matrix whose (i, j)th entry is $p^n_{i,j} = \mathbb{P}(X_n = j \mid X_0 = i)$.

Equation (2.3) says that the probability of going the state j from the state i in $n + m$ steps is the sum over all k of the probability of going to the state k from the state i in n steps, then to the state j from the state k in n steps.

We note that P^0 is the identity matrix and notice that $P^n = PP^{n-1} = P$ for $n \geq 1$.

Now we define a row vector μ which is called a *probability vector* if all coordinates of μ are nonnegative and their sum is 1. If the ith entry of μ satisfies $(\mu)_i = \mathbb{P}(X_0 = i)$ then μ is called the *initial distribution* of the Markov chain $\{X_n : n \geq 0\}$. By (2.1) and (2.3) we have that for $n \geq 1$;

$$\mathbb{P}(X_0 = i_0, \ldots, X_{n-1} = i_{n-1}, X_n = j) = (\mu)_{i_0}(P)_{i_0 i_1} \ldots (P)_{i_{n-1} j}.$$

If we sum with respect to $(i_0, \ldots, i_{n-1})$ we obtain

$$(\mu P^n)_j = \mathbb{P}(X_n = j), \quad n \geq 0 \text{ and } j \in \mathcal{S}.$$

It means that the row vector μP^n is the distribution of the Markov chain at time n if μ is the distribution at time 0. For a given row vector ρ we define

$$\|\rho\|_R = \sum_{i \in \mathcal{S}} |(\rho)_i| \tag{2.4}$$

as a measure for row vectors. It corresponds to *total variation norm* on the space of measures.

Lemma 2.1.1. *For a given row vector ρ and a transition probability matrix P we have*

$$\|\rho P\|_R \leq \|\rho\|_R.$$

Proof. It follows from

$$\|\rho P\|_R = \sum_{j \in \mathcal{S}} \left| \sum_{i \in \mathcal{S}} (\rho)_i (P)_{ij} \right| \leq \sum_{i \in \mathcal{S}} \left(\sum_{j \in \mathcal{S}} | (\rho)_i | (P)_{ij} \right) = \|\rho\|_R.$$

□

Now, we consider a column vector φ whose j^{th} coordinate is the value of a either nonnegative or a bounded function f on the state space $\mathcal{S}$. Then $\mu\varphi = \sum_{i \in \mathcal{S}} f(i)\mu(\{i\})$ is the expected value of f with respect to μ and $\mu(\{i\})$ denotes the ith value of μ. The column vector $P\varphi$ represents that whose value at i is the conditional expectation value of $f(X_n)$ given that $X_0 = i$ since

$$\mathbb{E}\left[f(X_n) \mid X_0 = i\right] = \sum_{j \in \mathcal{S}} f(j)\mathbb{P}(X_n = j \mid X_0 = i) = \sum_{j \in \mathcal{S}} (P^n)_{ij} (\varphi)_j = (P^n \varphi)_i.$$

If μ is the initial distribution of $\{X_n \colon n \geq 0\}$, then $\mathbb{E}\left[f(x_n)\right] = \mu P^n \varphi$. We define the uniform norm $\|\cdot\|_C$ for the column vectors defined as

$$\|\varphi\|_C = \sup_{j \in \mathcal{S}} |(\varphi)_j|. \tag{2.5}$$

Lemma 2.1.2. *For a given column vector ξ, a row vector φ and a transition probability matrix P we have*

$$|\mu\varphi| \leq \|\mu\|_R \|\varphi\|_C \quad \textit{and} \quad \|P\varphi\|_C \leq \|\varphi\|_C.$$

Proof. We have

$$|\mu\varphi| \leq \sum_{i\in\mathcal{S}} |(\mu)_i||(\varphi)_i| \leq \sup_{i\in\mathcal{S}} |(\varphi)_j| \sum_{i\in\mathcal{S}} |(\mu)_i| = \|\varphi\|_C \|\mu\|_R.$$

Moreover,

$$\|P\varphi\|_C = \sup_{j\in\mathcal{S}} |(P\varphi)_j| \leq \sup_{j\in\mathcal{S}} |(\varphi)_j| \sum_{i\in\mathcal{S}} (P)_{ij} = \|\varphi\|_C.$$

This brief introduction will be used in Section 2.2.1. □

If we want to know if the chain will stabilize after a sufficiently long time, then we are interested in the limiting probabilities,

$$\lim_{n\to\infty} p_{i,j}^n = \lim_{n\to\infty} \mathbb{P}(X_n = j \mid X_0 = i).$$

It is natural to look at the limit of P^n which is easier to determine for a finite dimensional state space $\mathcal{S}$. If the above limit exists and finite then the Markov process is called *ergodic*. In fact for any finite stochastic matrix P, there is exactly one eigenvalue which is equal to 1 and all the other eigenvalues are less than 1 in terms of the distance to 0, so that P^n converges to a matrix where each row is the eigenvector of P corresponding to the eigenvalue 1 (The proof can be found in [99] and in many other probability books.) But this argument fails when $\mathcal{S}$; thus P is infinite dimensional.

General state space

In this section we consider $\mathcal{S}$ as a general state space which may not necessarily be finite. In this case, the time-homogeneity assumption (2.2) for a discrete-time Markov process $\mathbf{X}$ corresponds to having a measurable map P from $\mathcal{S}$ into $\mathcal{P}(\mathcal{S})$, the space of probability measures on $\mathcal{S}$, such that

$$\mathbb{P}(X_n \in A \mid X_{n-1} = y) = P(y, A),$$

for every $A \in \mathcal{B}(\mathcal{S})$, σ-field of all subsets of $\mathcal{S}$, almost every $y \in \mathcal{S}$, and every $n > 0$. In this case $P : \Omega \times \mathcal{S} \mapsto \mathbb{R}$ is called *transition probabilities* of $\mathbf{X}$ or the *transition probability* *function.* We note that $P(x,\cdot)$ is a probability measure for every x and $x \mapsto P(x,A)$ is a measurable function for every $A \in \mathcal{S}$. Similarly the Chapman-Kolmogorov equations (2.3) are given by the following theorem;

Theorem 2.1.1. *Suppose that $\boldsymbol{X}$ is a time-homogenous Markov process and P is the transition probabilities of $\boldsymbol{X}$. Then we have*

$$\mathbb{P}(X_n \in A \,|\, X_0 = y) = P^n(y, A),$$

where P^n is defined by

$$P^0 = I, \qquad P^1 = P, \qquad P^n(y,A) = \int_{\mathcal{S}} P(x,A)P^{n-1}(y,\,\mathrm{d}x). \tag{2.6}$$

Moreover we also have

$$P^{n+m}(y,A) = \int_{\mathcal{S}} P^n(x,A)P^m(y,\,\mathrm{d}x),$$

for every $n, m \geq 1$.

Associated *Markov operator* M on $\mathcal{P}(\mathcal{S})$ is defined by means of transition probabilities P through;

$$(M\mu)(A) = \int_{\mathcal{S}} P(x,A)\mu(\,\mathrm{d}x). \tag{2.7}$$

We similarly define M^* the dual of M acting on the space of bounded measurable functions from $\mathcal{S}$ to $\mathbb{R}$ by

$$(M^*\varphi)(x) = \int_{\mathcal{S}} \varphi(y)P(x,\,\mathrm{d}y) = \mathbb{E}\left[\varphi(y) \mid x_0 = x\right], \tag{2.8}$$

so that we have for every $\mu \in \mathcal{P}(\mathcal{S})$ and for every bounded function φ the following holds true:

$$\int_{\mathcal{S}} (M^*\varphi)(x)\mu(\,\mathrm{d}x) = \int_{\mathcal{S}} \varphi(x)(M\mu)(\,\mathrm{d}x).$$

2.1.2 Continuous-time Markov processes

Now we define the analogous concepts in the continuous-time setting beginning with the Markov property:

For any $j, i, i_1, i_2, \cdots, i_{n-1} \in \mathcal{S}$ and for any integer $n \geq 1$,

$$\mathbb{P}(X(t) = j \mid X(s) = i, X_{t_{n-1}} = i_{n-1}, \cdots, X(t_1) = i_1) = \mathbb{P}(X(t) = j \mid X(s) = i), \tag{2.9}$$

where $0 \leq t_1 \leq t_2 \leq \cdots \leq t_{n-1} \leq s \leq t$ any nondecreasing sequence of $n+1$ times. That is, given state of the process at time s, the distribution of the process at any time after s is independent of the entire past of the process before time s, i.e. it depends only on the process at the most recent time prior to time t. Similar to the discrete case, a continuous-time Markov chain is time-homogeneous if for any $s \leq t$ and any states $i, j \in \mathcal{S}$,

$$\mathbb{P}(X(t) = j \mid X(s) = i) = \mathbb{P}(X(t-s) = j \mid X(0) = i).$$

We also define the *transition probability function* $P_{i,j}(t)$ for a time-homogeneous, continuous time Markov chain, as a counterpart to the n-step transition probabilities pi, j^n in the discrete-time version by

$$P_{i,j}(t) = \mathbb{P}(X(t) = j \mid X(0) = i).$$

For each $i, j \in \mathcal{S}$, the transition probability function $P_{i,j}(t)$ is a continuous function of t.

For a continuous-time Markov chain $\{X(t) : t \geq 0\}$ with the state space $\mathcal{S}$ and the transition probability functions $(P_{i,j}(t))_{\{i,j \in \mathcal{S}\}}$, the Chapman-Kolmogorov equations are given by

$$P_{i,j}(t+s) = \sum_{k \in \mathcal{S}} P_{i,k}(t) P_{k,j}(s) \tag{2.10}$$

for any $t, s \geq 0$.

For a given t, if we construct a matrix $\mathbf{P}_t$ whose (i,j)th entry is the transition probability function $P_{i,j}(t)$ then (2.10) is equivalent to

$$\mathbf{P}_{t+s} = \mathbf{P}_t \mathbf{P}_s.$$

We note that in the discrete-time case P^n is called the n-step transition probability matrix. Since there is no time step in the continuous case we call $\mathbf{P}_t$ the matrix transition probability function, which is a matrix-valued function of the continuous variable t.

General state space

Similar to the discrete-time Markov process, now we consider $\mathcal{S}$ a general state space. In this case, a continuous-time Markov process is described by a family of transition operators P_t instead of transition probabilities P. This family of transition operators P_t satisfies the Chapman-Kolmogorov equations such that

- $P_0(x, \cdot) = \delta_x$.
- $P_{s+t} = P_s P_t$.

We also have that

1. The map $t \mapsto P_t(x, A)$ is measurable for every x.
2. For every x the process is right-continuous with left limits (càdlàg).

Since we are looking at a process we have a transition kernel P_t for each $t > 0$. We also define $M_t : \mathcal{M}(\Omega) \to \mathcal{M}(\Omega)$ the Markov semigroup and its dual M_t^* through P_t as in (2.7) and (2.8).

In the following chapters $M_t\mu$ will be the weak solution to the evolution equation with initial data μ. If we define $\mathcal{M}(\mathcal{S})$ as the space of finite measures on $(\Omega, \mathcal{F})$ then we have that M_t is a *linear* map

$$M_t : \mathcal{M}(\mathcal{S}) \to \mathcal{M}(\mathcal{S}).$$

From the conditions on P_t we see that M_t will be *linear, mass preserving* and *positivity preserving.*

We also define the *forwards operator* L, associated to P_t as the operator which satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} M_t^* \varphi \bigg|_{t=0} = L\varphi, \tag{2.11}$$

for all $\varphi \in C_c^\infty(\mathcal{S})$, whenever this is well defined.

A time homogeneous Markov process satisfies the *Feller property* if its transition operator $M_t^*\varphi$ is continuous whenever φ is continuous and bounded. Similarly, it is *strong Feller* if $M_t^*\varphi$ is continuous whenever φ is measurable and bounded.

2.2 Convergence of Markov processes

Here we will give some notions about the long-time behavior of Markov processes. Finally we will give the statements of the main theorems which will be the core of this work.

We start with the definition of *an invariant measure* which is the fundamental tool in the study of long-time behaviour of Markov processes;

Definition 2.2.1. *A positive measure μ on $\mathcal{S}$ is invariant for the Markov process x if $P\mu = \mu$. In the case of continuous time process, a positive measure μ is invariant if $P_t\mu = \mu$ for every $t \geq 0$.*

We denote τ_A as the first time a Markov chain reaches the set A and it is defined as

$$\tau_A := \inf\{n \geq 1 \colon x_n \in A\}.$$

If τ_A is infinite, then the set A is never reached by the associated chain.

Next, we define the concept of *irreducibility*.

Definition 2.2.2. *A Markov chain is* irreducible *if in finite time every state is reachable starting from any state.*

This concept is the main idea behind Doeblin's Theorem. It is very natural to think that if the state space of a Markov process is finite and if the process run long enough, the initial distribution of the process is going to get forgotten and the process will stabilize.

2.2.1 Doeblin's Theorem

Doeblin Theorem says that if a Markov process has a positive probability of visiting some fixed state independently from the starting point, then this process stabilizes. It dates back to [52]. Let us give the following theorem first:

Theorem 2.2.1. *Let P be a transition probability of a Markov process on a state space $\mathcal{S}$. Assume that there exists a constant $\alpha > 0$ and a probability measure ν on $\mathcal{S}$ such that*

$$P(x,\cdot) \geq \alpha\nu(\cdot) \quad \textit{for every } x \in \mathcal{S}, \tag{2.12}$$

then, P has a unique invariant measure μ_*.

Proof. The assumption (2.12) implies that $M\mu \geq \alpha\nu$ for every probability measure μ on $\mathcal{S}$. Therefore we define probability measures $\bar{M}\mu$ by

$$M\mu = \alpha\nu + (1-\alpha)\bar{M}\mu. \tag{2.13}$$

Let μ_1 and μ_2 be any two probability measures on $\mathcal{S}$. Using the inequality

$$\|\mu_1 - \mu_2\|_{\mathrm{TV}} \leq 2 - 2\min\{\mu_1,\mu_2\}(\mathcal{S})$$

where $\bar{\mu_1}$ and $\bar{\mu_2}$ are probability measures such that

$$\mu_1 = \min\{\mu_1, \mu_2\} + \frac{1}{2}\|\mu_1 - \mu_2\|_{\mathrm{TV}}\bar{\mu_1}, \quad \mu_2 = \min\{\mu_1, \mu_2\} + \frac{1}{2}\|\mu_1 - \mu_2\|_{\mathrm{TV}}\bar{\mu_2}.$$

Therefore we obtain

$$\|M\mu_1 - M\mu_2\|_{\mathrm{TV}} = \frac{1}{2}\|M\bar{\mu_1} - M\bar{\mu_2}\|_{\mathrm{TV}}\|\mu_1 - \mu_2\|_{\mathrm{TV}}.$$

Also by (2.13),

$$\begin{aligned}\|M\bar{\mu_1} - M\bar{\mu_2}\|_{\mathrm{TV}} = \|\alpha\nu + (1-\alpha)\bar{M}\bar{\mu_1} - \alpha\nu - (1-\alpha)\bar{M}\bar{\mu_2}\|_{\mathrm{TV}}\\ = (1-\alpha)\|M\bar{\mu_1} - M\bar{\mu_2}\|_{\mathrm{TV}} \le 2(1-\alpha),\end{aligned}$$

since the total variation distance between two probability measures cannot exceed 2. Finally we obtain

$$\|M\mu_1 - M\mu_2\|_{\mathrm{TV}} \le (1-\alpha)\|\mu_1 - \mu_2\|_{\mathrm{TV}}, \tag{2.14}$$

which shows that M is a contraction. Then by Banach fixed point theorem, the result follows. □

Hypothesis 2.2.1 (Doeblin's Condition). *We assume that $(M_t)_{t\ge0}$ is a Markov semigroup, defined through a transition probability functions, and that there exists $t_0 > 0$, a probability distribution ν and a constant $\alpha \in (0,1)$ such that for any x in the state space we have*

$$M_{t_0}\delta_x \ge \alpha\nu.$$

By using Theorem 2.2.1 and Hypothesis 2.2.1 we prove;

Theorem 2.2.2 (Doeblin's Theorem). *If we have a Markov (transition) semigroup $(M_t)_{t\ge0}$ satisfying Doeblin's condition (Hypothesis 2.2.1) then for any two measures μ_1 and μ_2 and any integer $n \ge 0$ we have*

$$\|M_{t_0}^n\mu_1 - M_{t_0}^n\mu_2\|_{\mathrm{TV}} \le (1-\alpha)^n\|\mu_1 - \mu_2\|_{\mathrm{TV}}. \tag{2.15}$$

Moreover, there exists a unique equilibrium probability measure μ_* for the semigroup, and for all μ we have

$$\|M_t(\mu - \mu_*)\|_{\mathrm{TV}} \le \frac{1}{1-\alpha}e^{-\lambda t}\|\mu - \mu_*\|_{\mathrm{TV}}, \qquad t \ge 0, \tag{2.16}$$

where

$$\lambda := \frac{\log(1-\alpha)}{t_0} > 0.$$

Proof. This proof is classical and can be found in [70] and various versions in many other places.

First, we show that if $M_t\delta_x \geq \alpha\nu$ for every x, then we also have $M_t\mu \geq \alpha\nu$ for every probability measure μ. Here since M_t comes from a Markov transition kernel, Hypothesis 2.2.1 implies that

$$P_t(x,\cdot) \geq \alpha\nu(\cdot)$$

for every x. This is the semigroup version of the condition of Theorem 2.2.1. Therefore,

$$M_t\mu(\cdot) = \int_{\mathcal{S}} P_t(x,\cdot)\mu(\,\mathrm{d}x) \geq \alpha \int_{\mathcal{S}} \nu(\cdot)\mu(\,\mathrm{d}x) = \alpha\nu(\cdot).$$

By the triangle inequality we have

$$\|M_{t_0}\mu_1 - M_{t_0}\mu_2\|_{\mathrm{TV}} \leq \|M_{t_0}\mu_1 - \alpha\nu\|_{\mathrm{TV}} + \|M_{t_0}\mu_2 - \alpha\nu\|_{\mathrm{TV}}.$$

Now, since $M_{t_0}\mu_1 \geq \alpha\nu$, due to mass conservation we can write

$$\|M_{t_0}\mu_1 - \alpha\nu\|_{\mathrm{TV}} = \int_{\mathcal{S}} (M_{t_0}\mu_1 - \alpha\nu) = \int_{\mathcal{S}} \mu_1 - \alpha = 1 - \alpha.$$

The same holds for the term $\|M_{t_0}\mu_2 - \alpha\nu\|_{\mathrm{TV}}$. This gives

$$\|M_{t_0}\mu_1 - M_{t_0}\mu_2\|_{\mathrm{TV}} \leq 2(1-\alpha) = (1-\alpha)\|\mu_1 - \mu_2\|_{\mathrm{TV}}$$

if μ_1, μ_2 have disjoint support. By homogeneity, this inequality is obviously also true for any nonnegative μ_1, μ_2 having disjoint support with $\int \mu_1 = \int \mu_2$ so that this proves

$$\|M_{t_0}\mu_1 - M_{t_0}\mu_2\|_{\mathrm{TV}} \leq (1-\alpha)\|\mu_1 - \mu_2\|_{\mathrm{TV}}. \tag{2.17}$$

If we iterate this we obtain (2.15). Notice that (2.17) is the fixed time version of (2.14).

The contractivity (2.17) gives that the operator M_{t_0} has a unique fixed point, which we call μ_*. In fact, μ_* is a stationary state of the whole semigroup since for all $t \geq 0$ we have

$$M_{t_0}T_t\mu_* = M_tM_{t_0}\mu_* = M_t\mu_*,$$

which shows that $M_t\mu_*$ (which is again a probability measure) is also a stationary state of M_{t_0}; due to uniqueness,

$$M_t\mu_* = \mu_*.$$

Hence the only stationary state of M_t must be μ_*, since any stationary state of M_t is in particular a stationary state of M_{t_0}.

In order to show (2.16), for any probability measure μ and any $t \geq 0$ we write

$$k := \left\lfloor \frac{t}{t_0} \right\rfloor,$$

(where $\lfloor \cdot \rfloor$ denotes the integer part) so that

$$\frac{t}{t_0} - 1 < k \leq \frac{t}{t_0}.$$

Then,

$$\begin{aligned} \|M_t(\mu - \mu_*)\|_{\mathrm{TV}} &= \|M_{t-kt_0} T_{kt_0}(\mu - \mu_*)\|_{\mathrm{TV}} \leq \|M_{kt_0}(\mu - \mu_*)\|_{\mathrm{TV}} \\ &\leq (1-\alpha)^k \|\mu - \mu_*\|_{\mathrm{TV}} \leq \frac{1}{1-\alpha} \exp\left(t \log \frac{1-\alpha}{t_0} \right) \|\mu - \mu_*\|_{\mathrm{TV}}. \quad \square \end{aligned}$$

Doeblin's Theorem in the discrete setting

In this section we present Doeblin's Theorem and its prrof in the discrete-time setting with a finite and bounded state space based on [99]. The whole aim of this part is that this version of the theorem is a good starting point to show analogous convergence properties when both the time and the state space is discrete, such as a numerical approximation of a PDE.

Theorem 2.2.3 (Doeblin's Theorem as in [99])**.** *Let P be a transition probability matrix with the property that, for some state $j_0 \in \mathcal{S}$ and $\epsilon > 0$, $(P)_{ij_0} > \epsilon$ for all $i \in \mathcal{S}$. Then P has a unique stationary probability vector π, $(\pi)_{j_0} > \epsilon$, and, for all initial distributions μ,*

$$\|\mu P^n - \pi\|_R \leq 2(1-\epsilon)^n, \quad n \geq 0.$$

Proof. Let $\rho \in \mathbb{R}^{\mathcal{S}}$ be a row vector with $\|\rho\|_R < \infty$, then since by Fubini's theorem,

$$\sum_{j \in \mathcal{S}} (\rho P)_j = \sum_{j \in \mathcal{S}} \left(\sum_{i \in \mathcal{S}} (\rho)_i (\rho P)_{ij} \right) = \sum_{i \in \mathcal{S}} \left(\sum_{i \in \mathcal{S}} (\rho)_i (\rho P)_{ij} \right) = \sum_{i \in \mathcal{S}} (\rho)_i,$$

we obtain

$$\sum_{j \in \mathcal{S}} (\rho P)_j = \sum_{i \in \mathcal{S}} (\rho)_i.$$

Next we suppose that, $\sum_i (\rho)_i = 0$ and observe that

$$|(\rho P)_j| = \left|\sum_{i\in\mathcal{S}} (\rho)_i (\rho P)_{ij}\right| = \left|\sum_{i\in\mathcal{S}} (\rho)_i \left((\rho P)_{ij} - \epsilon\delta_{j,j_0}\right)\right| \leq \sum_{i\in\mathcal{S}} |(\rho)_i| \left((\rho P)_{ij} - \epsilon\delta_{j,j_0}\right).$$

Therefore,

$$\begin{aligned}\|\rho P\|_R \leq \sum_{j\in\mathcal{S}} \left(\sum_{i\in\mathcal{S}} |(\rho)_i| \left((\rho P)_{ij} - \epsilon\delta_{j,j_0}\right)\right) = \sum_{i\in\mathcal{S}} |(\rho)_i| \left(\sum_{j\in\mathcal{S}} \left((\rho P)_{ij} - \epsilon\delta_{j,j_0}\right)\right) \\ = (1-\epsilon)\|\rho\|_R.\end{aligned}$$

Now, let μ be a probability vector such that $\mu_n = \mu P^n$ holds. Notice also that $\mu_n = \mu_{n-m} P^m$ holds true. Then we have for $n > m \geq 1$:

$$\|\mu_n - \mu_m\|_R \leq (1-\epsilon)^m \|\mu_{n-m} - \mu\|_R \leq 2(1-\epsilon)^m,$$

since $\|\mu_{n-m}\|_R = \sum_i (\mu_{n-m})_i = 1 = \sum_i (\mu)_i = \|\mu\|_R$. Since $\{\mu^n\}_1^\infty$ is a Cauchy convergent sequence, then there exists a probability vector π such that

$$\lim_{n\to\infty} \|\mu_n - \pi\|_R = 0.$$

Moreover, since we also have

$$\pi = \lim_{n\to\infty} \mu P^{n+1} = \lim_{n\to\infty} (\mu P^n) P = \pi P,$$

then π is an equilibrium and we obtain

$$(\pi)_{j_0} = \sum_{i\in\mathcal{S}} (\pi)_i (P)_{ij_0} \geq \epsilon \sum_{i\in\mathcal{S}} (\pi)_i.$$

Thus, for any probability vector ν we obtain

$$\|\nu P^m - \pi\|_R \leq \|(\nu - \pi) P^m\|_R \leq 2(1-\epsilon)^m.$$

□

Corollary 2.2.1. *For any $M \geq 0$ and $\epsilon > 0$;*

$$\sup_j \inf_i (P^M)_{ij} \geq \epsilon \implies \|\mu P - \pi\|_R \leq 2(1-\epsilon)^{\lfloor \frac{n}{M} \rfloor} \tag{2.18}$$

for all probability vectors μ and a unique stationary probability vector π.

Proof. Let π be the stationary probability vector for P^M. For any probability vector μ, for any $m \in \mathbb{N}$ and any $0 \leq r < M$,

$$\|\mu P^{mM+r} - \pi\|_R = \|(\mu P^r - \pi)P^{mM}\|_R \leq 2(1-\epsilon)^m.$$

□

Doeblin's Threorem corresponds to irreducibility property in the bounded state space. But when the state space is unbounded, often we cannot prove this *minorisation condition* on the whole state space but in a set at the "center" of the state space. Then we need an extra condition, which can be thought as a *geometric drift condition* which will ensure that this set at the center will be visited infinitely often. We verify this condition by using a Lyapunov structure. This constructive argument is given by Harris's Theorem:

2.2.2 Harris's Theorem

For some fixed time t_0 we make two assumptions on the behaviour of M_{t_0}:

Hypothesis 2.2.2 (Lyapunov condition)**.** *There exists some function $V : \mathcal{S} \to [0, \infty)$ and constants $D \geq 0, \gamma \in (0,1)$ such that*

$$(M^*_{t_0} V)(x) \leq \gamma V(x) + D.$$

Remark 2.2.4. *We remark that the name* Lyapunov condition *is the standard name used for this type of condition in probability literature. It is nothing to do with proving monotonicity of a functional to obtain convergence to equilibrium.*

In the continuous time setting this condition (2.2.2) is equivalent to prove

$$\int_{\mathcal{S}} f(t_0, x) V(x) \, \mathrm{d}x \leq \gamma \int_{\mathcal{S}} f(0, x) V(x) \, \mathrm{d}x + D. \tag{2.19}$$

We verify this by showing

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{S}} f(t, x) V(x) \, \mathrm{d}x \leq -\lambda \int_{\mathcal{S}} f(t, x) V(x) \, \mathrm{d}x + K, \tag{2.20}$$

for some positive constants K and λ. We multiply left hand side of (2.20) by $e^{\lambda t}$ to obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\lambda t}\int_{\mathcal{S}} f(t,x)V(x)\,\mathrm{d}x = \lambda e^{\lambda t}\int_{\mathcal{S}} f(t,x)V(x)\,\mathrm{d}x + e^{\lambda t}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathcal{S}} f(t,x)V(x)\,\mathrm{d}x\right)\right)\\ \leq \lambda e^{\lambda t}\int_{\mathcal{S}} f(t,x)V(x)\,\mathrm{d}x - \lambda e^{\lambda t}\int_{\mathcal{S}} f(t,x)V(x)\,\mathrm{d}x + Ke^{\lambda t}.\end{aligned}$$

Then, we integrate from 0 to t_0 in time

$$\begin{aligned}\int_0^{t_0}\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\lambda t}\int_{\mathcal{S}} f(t,x)V(x)\,\mathrm{d}x\right)\,\mathrm{d}t\right) = e^{\lambda t_0}\int_{\mathcal{S}} f(t_0,x)V(x)\,\mathrm{d}x + \int_{\mathcal{S}} f(0,x)V(x)\,\mathrm{d}x\\ \leq K\int_0^{t_0} e^{\lambda t}\,\mathrm{d}t \leq \frac{K}{\lambda}(e^{\lambda t_0}-1).\end{aligned}$$

Therefore

$$\int_{\mathcal{S}} f(t_0,x)V(x)\,\mathrm{d}x \leq e^{-\lambda t_0}\int_{\mathcal{S}} f(t_0,x)V(x)\,\mathrm{d}x + \frac{K}{\lambda}(1-e^{-\lambda t_0}).$$

which then implies (2.19) for $\gamma = e^{-\lambda t_0}$ and $D = \frac{K}{\lambda}(1-e^{-\lambda t_0}) \leq Kt_0$.

The next assumption is a local minorisation condition as in Doeblin's Theorem;

Hypothesis 2.2.3. *There exists a probability measure ν and a constant $\alpha \in (0,1)$ such that*

$$\inf_{x\in\mathcal{C}} M_{t_0}\delta_x \geq \alpha\nu,$$

where

$$\mathcal{C} = \{x\colon V(x) \leq R\}$$

for some $R > \frac{2D}{1-\gamma} := \frac{2K}{\lambda}$, where the constants D,γ are obtained by Hypothesis 2.2.2.

Harris's Theorem extends the ideas of Doeblin to the unbounded state space setting by finding a Lyapunov functional with small level sets. If the Lyapunov functional is strong enough, then we can prove a spectral gap property in a weighted supremum norm. Harris's Theorem states that under a geometric drift condition and if T admits sufficiently large "small" level sets, then its transition probabilities converge towards a unique invariant measure at an exponential rate. The proof of the theorem relies on existence of a Lyapunov functional and irreducibility; thus it is based on providing a combination of a minorisation and a geometric drift conditions. The minorisation condition can be thought as finding a bound on the probability of transitioning in one step from any initial state to some specified region small level set in the state space.

Contrary to Doeblin's argument, if the state space is unbounded and the process may drift arbitrarily far away; that is why we need a drift condition as well.

Following the notes [69] we define a weighted supremum norm:

$$\|\phi\| = \sup_x \frac{|\phi(x)|}{1+V(x)}, \tag{2.21}$$

for ϕ measurable function. We will give the statement of the theorem in the norm (2.21) but proof will reply on considering a family of weighted supremum norms for every $\beta > 0$;

$$\|\phi\|_\beta = \sup_x \frac{|\phi(x)|}{1+\beta V(x)} \tag{2.22}$$

We also define the dual metric of (2.22) on the space of probability measures;

$$\rho_\beta(\mu_1,\mu_2) = \sup_{\|\phi\|_\beta \le 1} \int_{\mathcal{S}} \phi(x)(\mu_1-\mu_2)(\,\mathrm{d}x).$$

Moreover, it is equivalent to the total variation norm given by

$$\|\mu_1-\mu_2\|_{V,\beta} = \int (1+\beta V(x))|\mu_1-\mu_2|(\,\mathrm{d}x). \tag{2.23}$$

Now we are ready to give the main statement:

Theorem 2.2.5 (Harris's Theorem as in [70])**.** *If Hypotheses 2.2.2 and 2.2.3 hold then there exist $\bar{\alpha} \in (0,1)$ and $\beta > 0$ so that*

$$\|M_{t_0}\mu_1 - M_{t_0}\mu_2\|_{V,\beta} \le \bar{\alpha}\|\mu_1-\mu_2\|_{V,\beta}, \tag{2.24}$$

for any probability measures μ_1 and μ_2 in $\mathcal{S}$.

Explicitly if we choose $\epsilon \in (0,\alpha)$ and $\delta \in (\gamma + 2D/R, 1)$ then we can set $\beta = \epsilon/D$ where all the constants are coming from Hypotheses 2.2.3 and 2.2.2. Then we have $\bar{\alpha} = \max\{1-(\alpha-\epsilon), (2+R\delta\beta)/(2+R\beta)\}$.

The trick done in [70] for the proof of this theorem is to adjust β in a way that M is a strict contraction for the total variation distance $\|\cdot\|_{V,\beta}$. This does not imply that the same result holds for ρ_1. But the equivalence of norms $\|\cdot\|$ and $\|\cdot\|_\beta$ implies the existence of $n > 0$ such that M^n is such a contraction.

TV to Lipschitz seminorm

In [70], the authors reformulate the total variation distance between two probability measure ρ_β as a Lipschitz seminorm. A metric d_β between the points of $\mathcal{S}$ is defined by

$$d_\beta(x_1,x_2)=\begin{cases}0, & \text{if } x_1=x_2,\\ 2+\beta(V(x_1)+V(x_2)), & \text{if } x_1\neq x_2.\end{cases} \tag{2.25}$$

By using d_β, we define a Lipschitz seminorm with its dual for the probability measures by

$$\begin{aligned}\|\phi\|_{d_\beta}&=\sup_{x_1\neq x_2}\frac{|\phi(x_1)-\phi(x_2)|}{d_\beta(x_1,x_2)},\\ d_\beta(\mu_1,\mu_2)&=\sup_{\||\phi\||_{d_\beta}\le 1}\int_{\mathcal{S}}\phi(x)(\mu_1-\mu_2)(\,\mathrm{d}x).\end{aligned}$$

Next lemma gives a proof of equivalence between the total variation norm ρ_β as in (2.23) and the Lipschitz seminorm d_β as given by (2.25).

Lemma 2.2.1. *We have the identity* $\|\phi\|_{d_\beta}=\inf\limits_{a\in\mathbb{R}}\|\phi+a\|_\beta$ *and* $d_\beta=\rho_\beta$.

Proof. Since $\|\phi\|_{d_\beta}\le\|\phi(x)\|_\beta$, we have $\|\phi\|_{d_\beta}\le\inf\limits_{a\in\mathbb{R}}\|\phi+a\|_\beta$.

To prove the reverse inequality, for a given ϕ with $\|\phi\|_{d_\beta}\le 1$ we set $a=\inf\limits_{x}(1+\beta V(x)-\phi(x))$. Since for any x_1 and x_2 we have

$$\phi(x_1)\le|\phi(x_2)|+|\phi(x_1)-\phi(x_2)|\le|\phi(x_2)|+2(V(x_1)+V(x_2)),$$

we obtain

$$\begin{aligned}1+\beta V(x_1)-\phi(x)\ge 1+\beta V(x_1)-(|\phi(x_2)|+2(V(x_1)+V(x_2))\\ \ge -1-\beta V(x_2)-|\phi(x_2)|.\end{aligned}$$

There exists at least one point such that $V(x_2)<\infty$, which provided a lower bound for a, thus $|a|<\infty$. Also

$$\phi(x_1)+a\le\phi(x_1)+1+\beta V(x_1)=1+\beta V(x_1),$$

and

$$\phi(x_1) + a = \inf_{x_2}(\phi(x_1) + 1 + \beta V(x_2) - \phi(x_2) = \inf_{x_2}(1 + \beta V(x_2) + (\phi(x_1) - \phi(x_2)))$$
$$\geq \inf_{x_2}(1 + \beta V(x_2) - \|\phi\|_{d_\beta} \cdot d_\beta(x_1, x_2) \geq -(1 + \beta V(x_1)).$$

Therefore, $|\phi(x) + a| \leq 1 + \beta V(x)$ for any x which gives the result. □

Proof of Theorem 2.2.5. Since we have that

$$\min\{1, \beta\}\|\mu_1 - \mu_2\|_{V,1} \leq \|\mu_1 - \mu_2\|_{V,\beta} \leq \max\{1, \beta\}\|\mu_1 - \mu_2\|_{V,1}.$$

The result follows if we can find a $\gamma_0 < 1$ such that

$$\|M_{t_0}\mu_1 - M_{t_0}\mu_2\|_{V,1} \leq \gamma_0\|\mu_1 - \mu_2\|_{V,\beta}.$$

Assuming that μ_1 and μ_2 have disjoint support and that $V(x) \geq R$. Then, by choosing any $\gamma_1 \in (\gamma, 1)$ and by Hypotheses 2.2.2 and 2.2.3 we obtain

$$\|M_{t_0}\mu_1 - M_{t_0}\mu_2\|_{V,1} \leq 2 + \beta(MV)(x) \leq 2 + \beta\alpha(MV)(x) + 2\beta D$$
$$\leq 2 + \beta\alpha_1(MV)(x) + \beta(2D - (\gamma_1 - \gamma)R).$$

If R is sufficiently large so that $(\gamma_1 - \gamma)R > 2D$, then there exists some $\alpha_1 < 1$ (depending on β) such that we have

$$\|M_{t_0}\mu_1 - M_{t_0}\mu_2\|_{V,1} \leq \alpha_1\|\mu_1 - \mu_2\|_{V,\beta}.$$

Now, we choose an appropriate β.

We consider the case $V(x) \leq R$. To treat this case, we split the measure μ_1 as

$$\mu_1 = \mu_1^{(1)} + \mu_1^{(2)} \text{ where } |\mu_1^{(1)}| \leq 1, \quad |\mu_1^{(2)}| \leq \beta V(x), \text{ for all } x \in \mathcal{X}.$$

Then we obtain

$$\|M_{t_0}\mu_1 - M_{t_0}\mu_2\|_{V,1} \leq \|M_{t_0}\mu_1^{(1)} - M_{t_0}\mu_1^{(2)}\|_{V,1} + \|M_{t_0}\mu_2\|_{V,1} \leq 2(1-\gamma) + \beta\alpha V(x) + 2\beta D$$
$$\leq 2 - 2\gamma + \beta(\gamma R + 2D).$$

Hence fixing for example $\beta = \alpha/(\gamma R + 2D)$ we obtain

$$\|M_{t_0}\mu_1 - M_{t_0}\mu_2\|_{V,1} \leq 2 - \alpha \leq \left(1 - \frac{1}{2}\alpha\right) \|\mu_1 - \mu_2\|_{V,\beta},$$

since $\|\mu_1 - \mu_2\|_{V,\beta} \geq 2$. Setting $\bar{\alpha} = \max\left\{1 - \frac{1}{2}\alpha, \alpha_1\right\}$ concludes the proof. □

2.2.3 Subgeometric Harris's Theorem

There are versions of Harris's Theorem adapted to weaker Lyapunov conditions which give subgeometric convergence [54]. We use the following theorem which can be found in Section 4 of [69].

Theorem 2.2.6 (Subgeometric Harris's Theorem). *Given the forwards operator, L, of the Markov semigroup $(M_t)_{t\geq 0}$, suppose that there exists a continuous function $V : \mathcal{S} \to [1, \infty)$ with precompact level sets such that*

$$LV \leq K - \phi(V),$$

for some constant K and some strictly concave function $\phi : \mathbb{R}_+ \to \mathbb{R}$ with $\phi(0) = 0$ and ϕ is increasing to infinity. Assume that for every $C > 0$ we have the minorisation condition like Hypothesis 2.2.3. i.e. for some t_0 a time and ν a probability distribution and $\alpha \in (0, 1)$, then for all x with $V(x) \leq C$:

$$M_{t_0}\delta_x \geq \alpha\nu.$$

With these conditions we have that

- *There exists a unique invariant measure μ for the Markov process and it satisfies*

$$\int \phi(V(x))\, \mathrm{d}\mu \leq K.$$

- *Let H_ϕ be the function defined by*

$$H_\phi = \int_1^u \frac{\mathrm{d}s}{\phi(s)}.$$

Then there exists a constant C such that for every $x_1, x_2 \in \mathcal{S}$ we have

$$\int_0^\infty (\phi \circ H_\phi^{-1})(t)\|M_t(x_1, \cdot) - M_t(x_2, \cdot)\|_{\mathrm{TV}} \leq C(V(x_1) + V(x_2))$$

so that

$$\|M_t(x_1,\cdot) - M_t(x_2,\cdot)\|_{\mathrm{TV}} \leq C\frac{V(x_1)+V(x_2)}{H_\phi^{-1}(t)}.$$

- *There exists a constant C such that*

$$\|M_t(x,\cdot) - \mu\|_{\mathrm{TV}} \leq \frac{CV(x)}{H_\phi^{-1}(t)} + \frac{C}{(\phi\circ H_\phi^{-1})(t)}$$

holds for every initial condition $x \in \mathcal{S}$.

The proof is based on that fact that for two Markov processes x_t and y_t with the transition semigroup $(M_t)_{t\geq 0}$ then we have

$$\|M_t(x_0,\cdot) - M_t(y_0,\cdot)\|_{\mathrm{TV}} \leq 2\mathbb{P}(x_t \neq y_t).$$

For the full proof and more details can be found in [69].

Chapter 3

On the asymptotic behaviour of elapsed-time structured neuron populations

"The human brain has 100 billion neurons, each neuron connected to 10 thousand other neurons. Sitting on your shoulders is the most complicated object in the known universe."

— Michio Kaku

Contents

3.1 Overview

In this chapter we present some results on asymptotic behaviour of neuron population models introduced in Chapter 1. If the network of neurons consists of large enough number of members which are connected and interacting with each other then the underlying global dynamics can be described by partial differential equations based on electrical activity of a single neuron and averaging some relevant structures on the network. We need to make certain modelling assumptions in order to achieve this. Main assumption is that neurons randomly create spikes which can be thought of a sudden discharge followed by a charging process. The rate of neurons producing spikes depends on the global activity of other neurons as well as the time passed since the last discharge, called the *elapsed time*. Therefore, structuring the integro-PDE with a variable representing the time elapsed since last discharge comes very natural for modelling.

Moreover, due to this random firing behaviour, interacting neuron population models in the linear setting can be considered as piecewise deterministic Markov processes (PDMP) from probability theory. PDMPs represent the processes whose behaviour is regulated by stochastic jumps at some points in time but the evolution of the process can be defined deterministically between these points. Continuous time Markov chains are examples of PDMPs. This is the reason we can easily apply Doeblin's Theorem to obtain some results on the long time behaviour of solutions.

This chapter is organized as follows: In the next section we recall the two nonlinear integro-PDEs (3.1) and (3.2) which are modelling the dynamics of interacting neuron populations. We also state the modelling assumptions and the main Theorem 3.1.1 concerning the exponential convergence rate to equilibrium in the low connectivity regime. Next two sections are dedicated to the age-structured neuron population model (3.1) and the second model (3.2) with a property of displaying adaptation and fatigue properties respectively. We give proofs for well-posedness, existence and uniqueness of stationary solutions, always in the weak nonlinearity regime. For the exponential relaxation to equilibrium we first consider the linear equations (when p does not depend on N) and we prove that solutions have positive lower bounds which ensures that the associated stochastic semigroups satisfy the *Doeblin condition*. Therefore we obtain the exponential relaxation results in the linear setting. Finally, we prove exponential relaxation to the steady state for the nonlinear models (3.1)-(3.2) by a perturbation argument based on the linear theory.

3.1.1 Assumptions and the main theorem

Now, we recall these two nonlinear evolution equations. First one is nonlinear version of the renewal equation and describes the time evolution of a neuron population density depending on the time elapsed since last discharge x;

$$\begin{aligned} \frac{\partial}{\partial t}n(t,x) + \frac{\partial}{\partial x}n(t,x) &= -p(N(t),x)n(t,x), && t,x>0,\\ N(t) = n(t,0) &= \int_0^{+\infty} p(N(t),x)n(t,x)\,\mathrm{d}x, && t>0,\\ n(0,x) &= n_0(x), && x\geq 0. \end{aligned} \tag{3.1}$$

where p is the firing rate depending on current state x and the global activity of neurons at time t via $N(t)$. We also remark that connectivity of the network which is related to level of nonlinearity is enclosed in dependence of p to $N(t)$. Since the boundary condition ensures neurons re-enter the cycle from $x = 0$ immediately after they fire with a rate p, in this model the structuring variable x can also be considered as *age*. This model was first proposed in [89]. For the second model we consider a modification by defining the structuring variable x as a generic "state" of the neuron, not necessarily the time elapsed since the last discharge. We also assume that neurons in a state y return to a certain state $x < y$ after firing, with a certain probability distribution $\kappa(x, y)$. Thus the model is given by;

$$\begin{aligned} \frac{\partial}{\partial t}n(t,x) + \frac{\partial}{\partial x}n(t,x) + p(N(t),x)n(t,x) &= \int_0^{+\infty} \kappa(x,y)p(N(t),y)n(t,y)\,\mathrm{d}y, && x,y,t>0,\\ n(t,x=0) &= 0, && t>0,\\ n(t=0,x) &= n_0(x), && x\geq 0, \end{aligned} \tag{3.2}$$

where the flux of discharging neurons at time t is defined by

$$N(t) := \int_0^{+\infty} p(N(t),x)n(t,x)\,\mathrm{d}x.$$

This model was introduced in [91]. Our contribution in this work is a simplified study of the low-connectivity case (corresponding to a weak nonlinearity) which gives improved results by using a promising new probabilistic method for these type of models.

Now we give a detailed notation and assumptions we are going to consider throughout the chapter.

Assumption 3.1.1 (regularity of p)**.** *We assume that p is a bounded, Lipschitz and nonnegative function such that $p \in W^{1,\infty}([0,+\infty) \times [0,+\infty))$ satisfying*

$$p(N,x) \geq 0 \quad \textit{for all } N, x \in [0,+\infty). \tag{3.3}$$

We denote L the Lipschitz constant of p with respect to N; that is, L is the smallest number such that

$$|p(N_1,x) - p(N_2,x)| \leq L|N_1 - N_2| \quad \textit{for all } N_1, N_2, x \geq 0. \tag{3.4}$$

Assumption 3.1.2 (bounds on p)**.** *We assume that there exist some constants $x_*, p_{\min}, p_{\max} > 0$ such that*

$$p_{\min} \mathbb{1}_{[x_*,\infty)} \leq p(N,x) \leq p_{\max} \quad \textit{for all } N, x \geq 0, \tag{3.5}$$

where $\mathbb{1}_A$ denotes the characteristic function of a set A.

Assumption 3.1.3. *We assume that for a fixed global activity N the firing rate increases as time passes; more precisely,*

$$\frac{\partial}{\partial x} p(N,x) > 0, \quad \textit{for all } N, x \geq 0. \tag{3.6}$$

where the derivative is well-defined.

Assumption 3.1.4 (support of κ)**.** *We assume that for each $y \geq 0$,*

$$\kappa(\cdot, y) \textit{ is a probability measure supported on } [0,y]. \tag{3.7}$$

Assumption 3.1.5 (positivity of κ)**.** *We assume that there exists $\epsilon > 0$, $0 < \delta < x_*$ such that*

$$\kappa(\cdot, y) \geq \epsilon \mathbb{1}_{[0,\delta]} \quad \textit{for all } y \geq x_*. \tag{3.8}$$

The last assumption 3.1.5 regarding the positivity of κ guarantees that after firing there is always a sizeable probability of jumping to a state with x close to 0.

Now we give the main result for both equation (3.1) and (3.2):

Theorem 3.1.1. *We make the assumptions 3.1.1-3.1.3 for the equation (3.1); so that (3.3)-(3.5) hold true for (3.1). We additionally assume 3.1.4 and 3.1.5 for the equation (3.2) so that (3.3)-(3.8) hold true for (3.2). We further assume that L is small enough* *depending on p and κ (with an explicit estimate; see remarks after the statement). Let n_0 be a probability measure on $[0, +\infty)$. There exists a unique probability measure n_* which is a stationary solution to (3.1) or (3.2), and there exist constants $C \geq 1$, $\lambda > 0$ depending only on p and κ such that the (mild or weak) measure solution $n = n(t)$ to (3.1)-(3.2) satisfies*

$$\|n(t) - n_*\|_{\mathrm{TV}} \leq C e^{-\lambda t} \|n_0 - n_*\|_{\mathrm{TV}}, \quad \textit{for all } t \geq 0. \tag{3.9}$$

Remark 3.1.2. *The constants in Theorem 3.1.1 are all constructive. Precisely, we can take*

$$\lambda = \lambda_1 - \tilde{C}, \qquad C = C_1 \quad \textit{for } (3.1),$$
$$\lambda = \lambda_2 - \tilde{C}, \qquad C = C_2 \quad \textit{for } (3.2),$$

where

$$C_1 := \frac{1}{1 - x_* \beta}, \qquad \lambda_1 = -\frac{\log(1 - x_* \beta)}{2x_*}$$
$$C_2 := \frac{1}{1 - \epsilon\delta(x_* - \delta)\beta}, \qquad \lambda_2 = -\frac{\log(1 - \epsilon\delta(x_* - \delta)\beta)}{2x_*}$$

and with

$$\beta = p_{\min} e^{-2p_{\max} x_*} \quad \textit{and } \tilde{C} = \frac{2p_{\max} L}{1 - L}.$$

Remark 3.1.3. *Smallness condition on L regarding the network connectivity and the degree of nonlinearity can be written as*

$$L < \min\left\{ \frac{p_{\min}^2}{p_{\max}^2 \left(x_* p_{\min}(x_* p_{\min} + 2) + 2\right)}, \frac{\log(1 - x_*\beta)}{\log(1 - x_*\beta) - 4p_{\max} x_*} \right\} \quad \textit{for } (3.1)$$

or

$$L < \min\left\{ \frac{p_{\min}\epsilon\delta(x_* - \delta)\beta}{p_{\min}\epsilon\delta(x_* - \delta)\beta + p_{\max} e^{4p_{\max} x_*}}, \frac{\log(1 - \epsilon\delta(s_* - \delta))}{\log(1 - \epsilon\delta(x_* - \delta)) - 4p_{\max} x_*} \right\} \quad \textit{for } (3.2).$$

3.2 An age-structured neuron population model

In this section we consider equation (3.1) for an age-structured neuron population. We first develop a well-posedness theory in the sense of measures, and then we use Doeblin's Theorem for the linear problem (when time dependence of p is neglected) to show exponential convergence to the equilibrium. After giving conditions for existence and uniqueness of a stationary solution to equation (1.2), we use a perturbation argument in order to obtain a result on its asymptotic behaviour in the nonlinear setting.

3.2.1 Well posedness

In order to develop our well-posedness theory in measures we need to introduce our notation and the norms we will be considering. We denote $\mathbb{R}_0^+ := [0, +\infty)$, and $\mathcal{M}(\mathbb{R}_0^+)$ is the set of finite, signed Borel measures on $\mathbb{R}_0^+$. We denote a subset $\mathcal{M}_+(\mathbb{R}_0^+) \subset \mathcal{M}(\mathbb{R}_0^+)$ formed by the nonnegative measures. Since we will always work in $\mathbb{R}_0^+$, for simplicity we will often write $\mathcal{M}$ and $\mathcal{M}_+$ to denote these sets, respectively.

We often identify a measure $\mu \in \mathcal{M}(\mathbb{R}_0^+)$ with its density with respect to Lebesgue measure, denoting the latter by the function $\mu = \mu(x)$. We abuse notation by writing $\mu(x)$ even for measures that may not have a density with respect to Lebesgue measure. Similarly, for a function $n\colon [0,T) \to \mathcal{M}(\mathbb{R}_0^+)$ we may often write $n(t,x)$ even if $n(t)$ does not have a density with respect to Lebesgue measure. In these cases any identities involved should be understood as identities between measures.

We denote by $\mathcal{C}_0(\mathbb{R}_0^+) \equiv \mathcal{C}_0$ the set of continuous functions ϕ on $\mathbb{R}_0^+$ with

$$\lim_{x\to+\infty} \phi(x) = 0$$

endowed with the supremum norm

$$\|\phi\|_\infty := \sup_{x\geq 0} |\phi(x)|,$$

With this setup, $\mathcal{C}_0(\mathbb{R}_0^+)$ is a Banach space. Similarly, $\mathcal{C}_c(\mathbb{R}_0^+) \equiv \mathcal{C}_c$ denotes the set of compactly supported continuous functions on $[0, +\infty)$.

In $\mathcal{M}$ one can define the usual total variation norm, which we will denote by $\|\cdot\|_{TV}$. We recall that $(\mathcal{M}, \|\cdot\|_{TV})$ is a Banach space, and is the topological dual of $\mathcal{C}_0([0,+\infty))$ with the supremum norm, as stated by the Riesz representation theorem.

The weak-$*$ topology on $\mathcal{M}$ is the weakest topology that makes all functionals $T\colon \mathcal{M} \to \mathbb{R}$, $\mu \mapsto \int_{\mathbb{R}_0^+} \phi\mu$ continuous, for all $\phi \in \mathcal{C}_0$. In the associated topology, a sequence $\{\mu_k\}_{k\geq 1}$ in $\mathcal{M}$ converges in the weak-$*$ sense to $\mu \in \mathcal{M}$ when

$$\lim_{k\to+\infty} \int_{\mathbb{R}_0^+} \phi\mu_k = \int_{\mathbb{R}_0^+} \phi\mu, \text{ for all } \phi \in \mathcal{C}_0.$$

We will also use the bounded Lipschitz norm $\|\cdot\|_{\mathrm{BL}}$ on $\mathcal{M}$, sometimes known as the flat metric or the $W^{1,\infty}$ dual metric, defined by

$$\|\mu\|_{\mathrm{BL}} := \sup_{\psi \in \mathcal{L}} \int_{\mathbb{R}_0^+} \psi \mu, \quad \mu \in \mathcal{M}$$

where

$$\mathcal{L} := \left\{ \psi \in \mathcal{C}(\mathbb{R}_0^+) \mid \psi \text{ bounded and Lipschitz with } \|\psi\|_\infty + \|\psi'\|_\infty \leq 1 \right\}.$$

One sees from this definition that the bounded Lipschitz norm is dual to the norm

$$\|\psi\|_{1,\infty} := \|\psi\|_\infty + \|\psi'\|_\infty, \qquad \psi \in W^{1,\infty}(\mathbb{R}_0^+)$$

defined on $W^{1,\infty}(\mathbb{R}_0^+) = \{\psi \in \mathcal{C}(\mathbb{R}_0^+) \mid \psi \text{ is bounded and Lipschitz}\}$ (but $(\mathcal{M}, \|\cdot\|_{\mathrm{BL}})$ is *not* the topological dual of $W^{1,\infty}$). An important property of this norm is that it metrises the weak-$*$ topology on any tight set with bounded total variation. We recall that a set $B \subseteq \mathcal{M}$ is *tight* if for every $\epsilon > 0$ there exists $R > 0$ such that $|\mu|((R, +\infty)) < \epsilon$ for all $\mu \in B$.

Lemma 3.2.1 ([77], 2.5.1, Proposition 43). *If $B \subseteq \mathcal{M}$ is tight and is bounded in total variation norm, then the topology associated to $\|\cdot\|_{\mathrm{BL}}$ on B is equal to the weak-$*$ topology on B.*

If $I \subseteq \mathbb{R}$ is an interval we denote by $\mathcal{C}(I, \mathcal{M}_+(\mathbb{R}_0^+))$ the set of functions $n \colon I \to \mathcal{M}_+(\mathbb{R}_0^+)$ which are continuous with respect to the bounded Lipschitz norm on $\mathcal{M}_+(\mathbb{R}_0^+)$. We define mild measure solutions to equation (1.2) by the usual procedure of rewriting it using Duhamel's formula. We denote by $(T_t)_{t\geq 0}$ the translation semigroup generated on $(\mathcal{M}, \|\cdot\|_{\mathrm{BL}})$ by the operator $-\partial_s$. That is: for $t \geq 0$, any measure $n \in \mathcal{M}(\mathbb{R}_0^+)$ and any $\phi \in \mathcal{C}_0(\mathbb{R}_0^+)$,

$$\int_{\mathbb{R}_0^+} \phi(s) T_t n(x)\, \mathrm{d}x := \int_{\mathbb{R}_0^+} \phi(x+t) n(x)\, \mathrm{d}x. \tag{3.10}$$

In other words, using the notation we follow in this chapter,

$$T_t n(x) := n(x-t),$$

with the understanding that n is zero on $(-\infty, 0)$.

Definition 3.2.1. *Assume p satisfies (3.3) and is nonnegative. A couple of functions $n \in \mathcal{C}([0,T), \mathcal{M}_+(\mathbb{R}_0^+))$ and $N \in \mathcal{C}([0,T), [0,+\infty))$, defined on an interval $[0,T)$*

for some $T \in (0, +\infty)$*, is called a* mild measure solution *to* (1.2) *with initial data* $n_0 \in \mathcal{M}(\mathbb{R}_0^+)$ *and* $N_0 \in \mathbb{R}$ *if it satisfies* $n(0) = n_0$*,* $N(0) = N_0$*,*

$$n(t,x) = T_t n_0(x) - \int_0^t T_{t-\tau}\Big(p(N(\tau),\cdot)n(\tau,\cdot)\Big)(x)\,\mathrm{d}\tau + \int_0^t T_{t-\tau}\Big(N(\tau)\delta_0\Big)(x)\,\mathrm{d}\tau \quad (3.11)$$

for all $t \in [0,T)$*, and*

$$N(t) = \int_0^\infty p(N(t),x)n(t,x)\,\mathrm{d}x, \qquad t \in [0,T).$$

Remark 3.2.1. *We notice that the second term in* (3.11) *can be rewritten as*

$$\int_0^t T_{t-\tau}\Big(N(\tau)\delta_0\Big)(x)\,\mathrm{d}\tau = \int_0^t N(\tau)\delta_{t-\tau}(x)\,\mathrm{d}\tau$$
$$= N(t-x)\mathbb{1}_{[0,t]}(x) = N(t-x)\mathbb{1}_{[0,\infty)}(t-x). \quad (3.12)$$

This will sometimes be a more convenient form.

By integrating in $\mathbb{R}_0^+$, Definition 3.2.1 directly implies mass conservation:

Lemma 3.2.2 (Mass conservation for measure solutions)**.** *Let* $T \in (0, +\infty]$*. Any mild measure solution* (n, N) *to* (1.2) *defined on* $[0,T)$ *satisfies*

$$\int_{\mathbb{R}_0^+} n(t,x)\,\mathrm{d}x = \int_{\mathbb{R}_0^+} n_0(x)\,\mathrm{d}x, \qquad \textit{for all } t \in [0,T), \quad (3.13)$$

or in other words (since solutions are nonnegative measures by definition),

$$\|n(t)\|_{\mathrm{TV}} = \|n_0\|_{\mathrm{TV}} \textit{ for all } t \in [0,T).$$

Lemma 3.2.3. *Assume that* p *satisfies* (3.3) *and the Lipschitz constant* L *in* (3.4) *satisfies* $L < 1/\|n\|_{\mathrm{TV}}$*, and let* $n \in \mathcal{M}(\mathbb{R}_0^+)$*. There exists a unique* $N \in \mathbb{R}$ *satisfying*

$$N = \int_0^\infty p(N,x)n(x)\,\mathrm{d}x. \quad (3.14)$$

Under these conditions, if $n_1, n_2 \in \mathcal{M}(\mathbb{R}_0^+)$ *are two measures and* $N_1, N_2 \in \mathbb{R}$ *are the corresponding solutions to* (3.14)*, then*

$$|N_1 - N_2| \le \frac{\|p\|_\infty}{1 - L\|n_1\|_{\mathrm{TV}}}\|n_1 - n_2\|_{\mathrm{TV}}. \quad (3.15)$$

Proof. We define the map $\Phi \colon \mathbb{R} \to \mathbb{R}$ by

$$\Phi(N) := \int_0^\infty p(N,x) n(x) \, \mathrm{d}x,$$

and we notice that for any $N_1, N_2 \in \mathbb{R}$,

$$|\Phi(N_1) - \Phi(N_2)| \leq \|p(N_1,\cdot) - p(N_2,\cdot)\|_\infty \|n\|_{\mathrm{TV}} \leq L|N_1 - N_2| \|n\|_{\mathrm{TV}}.$$

Since $L < 1/\|n\|_{\mathrm{TV}}$, the map Φ is contractive and has a unique fixed point, which is a solution to (3.14). For the second part of the lemma, consider $n_1, n_2 \in \mathcal{M}(\mathbb{R}_0^+)$ and N_1, N_2 the corresponding solutions to (3.14). Then

$$\begin{aligned} |N_1 - N_2| \leq \int_0^\infty |p(N_1,x) - p(N_2,x)| n_1(x) \, \mathrm{d}x + \left| \int_0^\infty p(N_2,x)(n_1(x) - n_2(x)) \, \mathrm{d}x \right| \\ \leq L|N_1 - N_2| \|n_1\|_{\mathrm{TV}} + \|p\|_\infty \|n_1 - n_2\|_{\mathrm{TV}}, \end{aligned}$$

which shows (3.15). □

Theorem 3.2.2 (Well-posedness of (1.2) in measures)**.** *Assume that p satisfies (3.3) and the Lipschitz constant L in (3.4) satisfies $L \leq 1/(4\|n_0\|_{\mathrm{TV}})$. For any given initial data $n_0 \in \mathcal{M}_+(\mathbb{R}_0^+)$ there exists a unique measure solution $n \in \mathcal{C}([0,+\infty); \mathcal{M}_+(\mathbb{R}_0^+))$ of (1.2) in the sense of Definition 3.2.1. In addition, if n_1, n_2 are any two mild measure solutions to (1.2) (with possibly different initial data) defined on any interval $[0,T)$ then*

$$\|n_1(t) - n_2(t)\|_{\mathrm{TV}} \leq \|n_1(0) - n_2(0)\|_{\mathrm{TV}} \, e^{4\|p\|_\infty t} \qquad \text{for all } t \in [0,T). \tag{3.16}$$

Remark 3.2.3. *We notice that the condition that L is small is already needed here, since otherwise the problem is* not *well-posed: consider for example the case $p(N,x) := N$, for which a solution should satisfy*

$$N(t) = N(t) \int_0^\infty n(t,x) \, \mathrm{d}x,$$

which only allows two options: either $N(t) = 0$ or $\int_0^\infty n(t,x) \, \mathrm{d}x = 1$. If $\int_0^\infty n_0(x) \, \mathrm{d}x = 1$ then the second option holds and there are infinitely many solutions (since the choice of $N = N(t)$ is free). If $\int_0^\infty n_0(x) \, \mathrm{d}x \neq 1$ then $N(t)$ must be 0 for all $t > 0$. In this latter case, either $N_0 = 0$ (and then the only solution is just pure transport: $n(t,x) = n_0(x-t)$ for $x > t$, $n(t,x) = 0$ otherwise) or $N_0 \neq 0$ (and there there are no solutions).

Similar ill-posed examples can be easily designed with firing rates of the form $p(N,x) = f(N)g(x)$.

Proof of Theorem 3.2.2. The proof of this result is a standard fixed-point argument as followed for example in [29], or in [89] for L^1 solutions.

Let us first show existence of a solution for a nonnegative initial measure $n_0 \in \mathcal{M}_+(\mathbb{R}_0^+)$. If $n_0 = 0$ it is clear that setting $n(t)$ equal to the zero measure on $\mathbb{R}_0^+$ for all t defines a solution, so we assume $n_0 \neq 0$. Fix $C, T > 0$, to be chosen later. Consider the complete metric space

$$\mathcal{X} = \{n \in \mathcal{C}([0,T], \mathcal{M}_+(\mathbb{R}_0^+)) \mid n(0) = n_0,\ \|n(t)\|_{\mathrm{TV}} \leq C \text{ for all } t \in [0,T]\},$$

endowed with the norm

$$\|n\|_{\mathcal{X}} := \sup_{t\in[0,T]} \|n(t)\|_{\mathrm{TV}}.$$

We remark that $\mathcal{C}([0,T], \mathcal{M}(\mathbb{R}_0^+))$ refers to functions which are continuous in the bounded Lipschitz topology, *not* in the total variation one. Define an operator $\Psi\colon \mathcal{X} \to \mathcal{X}$ by

$$\Psi[n](t) := T_t n_0 - \int_0^t T_{t-\tau}\Big(p(N(\tau),\cdot)n(\tau)\Big)\,\mathrm{d}\tau + \int_0^t T_{t-\tau}\Big(N(\tau)\delta_0\Big)\,\mathrm{d}\tau \tag{3.17}$$

for all $n \in \mathcal{X}$, where $N(t)$ is defined implicitly (see Lemma 3.2.3) as

$$N(t) = \int_0^\infty p(N(t),x)n(t,x)\,\mathrm{d}x \text{ for } t \in [0,T]. \tag{3.18}$$

The definition of $\Psi[n]$ indeed makes sense, since both $\tau \mapsto T_{t-\tau}\Big(p(N(\tau),\cdot)n(\tau)\Big)$ and $\tau \mapsto T_{t-\tau}\Big(N(\tau)\delta_0\Big)$ are continuous functions from $[0,T]$ to $\mathcal{M}(\mathbb{R}_0^+)$, hence integrable (in the sense of the Bochner integral).

We first check that $\Psi[n]$ is indeed in $\mathcal{X}$. It is easy to see that $t \mapsto \Psi[n](t)$ is continuous in the bounded Lipschitz topology, and it is a nonnegative measure for each $t \in [0,T]$. We also have

$$\begin{aligned}\|\Psi[n](t)\|_{\mathrm{TV}} &\leq \|n_0\|_{\mathrm{TV}} + \int_0^t \|p(N(\tau),\cdot)n(\tau,\cdot)\|_{\mathrm{TV}}\,\mathrm{d}\tau + \int_0^t \|N(\tau)\delta_0\|_{\mathrm{TV}}\,\mathrm{d}\tau \\ &\leq \|n_0\|_{\mathrm{TV}} + TC\|p\|_\infty + T\|N\|_{L^\infty([0,T])} \leq \|n_0\|_{\mathrm{TV}} + 2TC\|p\|_\infty.\end{aligned}$$

We choose

$$T \leq \frac{1}{4\|p\|_\infty} \quad \text{and} \quad C := 2\|n_0\|_{\mathrm{TV}}, \tag{3.19}$$

so that

$$\|n_0\|_{\mathrm{TV}} + 2TC\|p\|_\infty \le \|n_0\|_{\mathrm{TV}} + \frac{C}{2} \le C.$$

Hence with these conditions on T and C we have $\Psi[n] \in \mathcal{X}$.

Let us show that Ψ is a contraction mapping. Take $n_1, n_2 \in \mathcal{X}$ and let N_1, N_2 be defined by (3.18) corresponding to n_1 and n_2, respectively. We have

$$\begin{aligned}\|\Psi[n_1](t) - \Psi[n_2](t)\|_{\mathrm{TV}} &\le \int_0^t \|(p(N_1(\tau),\cdot) - p(N_2(\tau),\cdot))n_1(\tau,\cdot)\|_{\mathrm{TV}}\,\mathrm{d}\tau \\ &+ \int_0^t \|p(N_2(\tau),\cdot)(n_1(\tau) - n_2(\tau))\|_{\mathrm{TV}}\,\mathrm{d}\tau + \int_0^t \|(N_1(\tau) - N_2(\tau))\delta_0\|_{\mathrm{TV}}\,\mathrm{d}\tau \\ &=: T_1 + T_2 + T_3.\end{aligned}$$

We bound each term separately. For T_1, since $L \le 1/(4\|n_0\|_{\mathrm{TV}})$, using Lemma 3.2.3 we have

$$\begin{aligned}T_1 &\le TCL \sup_{\tau\in[0,T]} |N_1(\tau) - N_2(\tau)| \\ &\le 2TCL\|p\|_\infty \|n_1 - n_2\|_{\mathcal{X}} \le T\|p\|_\infty\|n_1 - n_2\|_{\mathcal{X}}. \end{aligned} \tag{3.20}$$

For T_2,

$$T_2 \le T\|p\|_\infty \|n_1 - n_2\|_{\mathcal{X}}, \tag{3.21}$$

and for T_3, using again Lemma 3.2.3,

$$T_3 \le T \sup_{\tau\in[0,T]} |N_1(\tau) - N_2(\tau)| \le 2T\|p\|_\infty \|n_1 - n_2\|_{\mathcal{X}}. \tag{3.22}$$

Putting equations (3.20)–(3.22) together and taking the supremum over $0 \le t \le T$,

$$\|\Psi[n_1] - \Psi[n_2]\|_{\mathcal{X}} \le 4T\|p\|_\infty \|n_1 - n_2\|_{\mathcal{X}}.$$

Taking now $T \le 1/(8\|p\|_\infty)$ ensures that Ψ is contractive, so it has a unique fixed point in $\mathcal{X}$, which is a mild measure solution on $[0, T]$. If we call n this fixed point, since $\|n(T)\|_{\mathrm{TV}} = \|n_0\|_{\mathrm{TV}}$ by mass conservation (see Lemma 3.2.2), we may repeat this argument to continue the solution on $[T, 2T]$, $[2T, 3T]$, showing that there is a solution defined on $[0, +\infty)$.

In order to show stability of solutions with respect to the initial data (which implies uniqueness of solutions), take two measures $n_0^1, n_0^2 \in \mathcal{M}_+(\mathbb{R}_0^+)$, and consider two solutions n_1, n_2 with initial data n_0^1, n_0^2 respectively. We have

$$\|n_1(t)-n_2(t)\|_{\mathrm{TV}} \le \|n_0^1-n_0^2\|_{\mathrm{TV}} + \int_0^t \|(p(N_1(\tau),\cdot)-p(N_2(\tau),\cdot))n_1(\tau,\cdot)\|_{\mathrm{TV}}\,\mathrm{d}\tau \\ + \int_0^t \|p(N_2(\tau),\cdot)(n_1(\tau)-n_2(\tau))\|_{\mathrm{TV}}\,\mathrm{d}\tau + \int_0^t \|(N_1(\tau)-N_2(\tau))\delta_0\|_{\mathrm{TV}}\,\mathrm{d}\tau, \quad (3.23)$$

and with very similar arguments as before we obtain that

$$\|n_1(t)-n_2(t)\|_{\mathrm{TV}} \le \|n_0^1-n_0^2\|_{\mathrm{TV}} + 2L\|n_0\|_{\mathrm{TV}}\|p\|_\infty \int_0^t \|n_1(\tau)-n_2(\tau)\|_{\mathrm{TV}}\,\mathrm{d}\tau \\ + \|p\|_\infty \int_0^t \|n_1(\tau)-n_2(\tau)\|_{\mathrm{TV}}\,\mathrm{d}\tau + 2\|p\|_\infty \int_0^t \|n_1(\tau)-n_2(\tau)\|_{\mathrm{TV}}\,\mathrm{d}\tau \\ \le \|n_0^1-n_0^2\|_{\mathrm{TV}} + 4\|p\|_\infty \int_0^t \|n_1(\tau)-n_2(\tau)\|_{\mathrm{TV}}\,\mathrm{d}\tau. \quad (3.24)$$

Gronwall's inequality then implies (3.16). □

Weak solutions Definition 3.2.1 is convenient for finding solutions but later we will need a more manageable form:

Definition 3.2.2. *(Weak solution to (1.2)) Assume p satisfies (3.3) and is nonnegative. A couple of functions $n \in \mathcal{C}([0,T),\mathcal{M}_+(\mathbb{R}_0^+))$ and $N \in \mathcal{C}([0,T),[0,+\infty))$, defined on an interval $[0,T)$ for some $T \in (0,+\infty]$, is called a* weak measure solution *to (1.2) with initial data $n_0 \in \mathcal{M}(\mathbb{R}_0^+)$ and $N_0 \in \mathbb{R}$ if it satisfies $n(0)=n_0$, $N(0)=N_0$, and for each $\varphi \in \mathcal{C}_c^\infty(0,+\infty)$ the function $t \mapsto \int_0^\infty \varphi(x)n(t,x)\,\mathrm{d}x$ is absolutely continuous and*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty \varphi(x)n(t,x)\,\mathrm{d}x \\ = \int_0^\infty \partial_x\varphi(s)n(t,x)\,\mathrm{d}x - \int_0^\infty p(N(t),x)n(t,x)\varphi(x)\,\mathrm{d}x + \int_0^\infty N(t)\delta_0(s)\varphi(x)\,\mathrm{d}x. \quad (3.25)$$

for almost all $t \in [0,T)$, and

$$N(t) = \int_0^\infty p(N(t),x)n(t,x)\,\mathrm{d}x, \ \textit{for all } t \in [0,T).$$

Equivalence results between definitions based on the Duhamel formula and definitions of weak solutions based on integration against a test function are fairly common. Here we use the main theorem in [5] with $f(t,\cdot) = -p(N(t),\cdot)n(t,\cdot) - N(t)\delta_0(\cdot)$, which implies that mild solutions of our equation are are equivalent to weak solutions:

Theorem 3.2.4 ([5]). *Assume p satisfies* (3.3) *and is nonnegative, and take $T \in (0, +\infty]$. A function $n\colon [0,T) \to \mathcal{M}_+(\mathbb{R}_0^+)$ is a weak measure solution (cf. Definition 3.2.2) to* (3.1) *if and only if it is a mild measure solution (cf. Definition 3.2.1).*

3.2.2 The linear equation

When $p = p(N, s)$ does not depend on N, equation (1.2) becomes linear:

$$\begin{aligned} \frac{\partial}{\partial t} n(t,x) + \frac{\partial}{\partial x} n(t,x) &= -p(x) n(t,x), && t, x > 0, \\ N(t) := n(t, x=0) &= \int_0^{+\infty} p(x) n(t,x)\,\mathrm{d}x, && t > 0, \\ n(t=0, x) &= n_0(x), && x \geq 0. \end{aligned} \tag{3.26}$$

Note that this is referred to as the “no-connectivity” case or the “$J = 0$ case” in [89].

Well posedness

We give a similar definition for *mild measure solutions*:

Definition 3.2.3. *Assume $p\colon [0,+\infty) \to [0,+\infty)$ is a bounded measurable function. A function $n \in \mathcal{C}([0,T), \mathcal{M}_+(\mathbb{R}_0^+))$, defined on an interval $[0,T)$ for some $T \in (0,+\infty]$, is called a* mild measure solution *to* (3.26) *with initial data $n_0 \in \mathcal{M}(\mathbb{R}_0^+)$ if it satisfies $n(0) = n_0$ and*

$$n(t,x) = T_t n_0(x) - \int_0^t T_{t-\tau}\Big(p(\cdot) n(\tau,\cdot)\Big)(x)\,\mathrm{d}\tau + \int_0^t T_{t-\tau}\Big(N(\tau)\delta_0\Big)(x)\,\mathrm{d}\tau \tag{3.27}$$

for all $t \in [0,T)$, with

$$N(t) := \int_0^\infty p(x) n(t,x)\,\mathrm{d}x, \quad t \in [0,T). \tag{3.28}$$

Our existence result stated in 3.2.2 easily gives the following as a consequence:

Theorem 3.2.5 (Well-posedness of (3.26) in measures)**.** *Assume that $p\colon [0,+\infty) \to [0,+\infty)$ is bounded, Lipschitz and nonnegative. For any given initial data $n_0 \in \mathcal{M}(\mathbb{R}^+)$ there exists a unique measure solution $n \in \mathcal{C}([0,+\infty); \mathcal{M}(\mathbb{R}_0^+))$ of the linear equation* (3.26) *in the sense of Definition 3.2.3. In addition, if n is any mild measure solution to* (3.26) *defined on any interval $[0,T)$ then*

$$\|n(t)\|_{\mathrm{TV}} \leq \|n(0)\|_{\mathrm{TV}} \text{ for all } t \in [0,T). \tag{3.29}$$

Proof. This result can be mostly deduced from Theorem 3.2.2. For the existence part, split n_0 into its positive and negative parts as $n_0 = n_0^+ - n_0^-$. Theorem 3.2.2 gives the existence of two solutions n^+ and n^- with initial data n_0^+ and n_0^-, respectively; then $n := n^+ - n^-$ is a mild measure solution with initial data n_0.

For uniqueness, if n is any mild solution on $[0, T)$, the same argument as it (3.23)–(3.24) shows that

$$\|n(t)\|_{\mathrm{TV}} \leq \|n_0\|_{\mathrm{TV}} + 4\|p\|_\infty \int_0^t \|n_1(\tau)\|_{\mathrm{TV}} \, \mathrm{d}\tau,$$

which gives by Gronwall's inequality that

$$\|n(t)\|_{\mathrm{TV}} \leq \|n_0\|_{\mathrm{TV}} \, e^{4\|p\|_\infty t} \text{ for all } t \in [0, T).$$

In particular, by linearity this implies solutions are unique. Finally, with the same argument as in Lemma 3.2.2 one sees that for any solution n defined on $[0, T)$ it holds

$$\int_{\mathbb{R}_0^+} n(t, x) \, \mathrm{d}x = \int_{\mathbb{R}_0^+} n_0(x) \, \mathrm{d}x$$

for all $t \in [0, T)$. Due to uniqueness, with the same splitting we used at the beginning of the proof we have $n(t) = n^+(t) - n^-(t)$, so

$$\|n(t)\|_{\mathrm{TV}} \leq \|n^+(t)\|_{\mathrm{TV}} + \|n^-(t)\|_{\mathrm{TV}} = \|n_0^+\|_{\mathrm{TV}} + \|n_0^-\|_{\mathrm{TV}} = \|n_0\|_{\mathrm{TV}},$$

which finishes the proof. □

The above result allows us to define an evolution semigroup $(S_t)_{t \geq 0}$ (in fact it is a C_0-semigroup on $\mathcal{M}$ with the bounded Lipschitz topology) by setting

$$S_t \colon \mathcal{M} \to \mathcal{M}, \qquad S_t(n_0) := n(t)$$

for any $n_0 \in \mathcal{M}$, where $n(t)$ is the mild measure solution to (3.26) with initial data n_0.

Stationary solutions for the linear equation We remark that Theorem 3.2.7 below implies that the linear equation (3.26) has a unique stationary solution in the space of probabilities on $[0, +\infty)$ (for p bounded, Lipschitz, and satisfying (3.5)); of course in this case this solution is explicit, given by

$$n_*(x) := N_* e^{-\int_0^x p(\tau) \, \mathrm{d}\tau}, \quad x \geq 0,$$

where N_* is the appropriate normalisation constant that makes this a probability density. Although Theorem 3.2.7 does not rule out the existence of other stationary solutions which may not be probabilities, since the solution is explicit it is not difficult to see that, up to a constant factor, n_* is the only stationary solution within the set of all finite measures. This is also a consequence of Doeblin's theorem below.

Uniform minorisation condition

Our main result on the spectral gap for the linear operator is based on the fact that for any initial probability distribution, solutions have a universal lower bound after a fixed time. We give the following lemma:

Lemma 3.2.4. *Let $p\colon [0,+\infty) \to [0,+\infty)$ be bounded, Lipschitz function satisfying (3.6) and (3.5), and consider the semigroup $(S_t)_{t\geq 0}$ given by the existence Theorem 3.2.5. Then S_{t_0} satisfies Doeblin's condition (2.2.1) for $t_0 = 2x_*$ and $\alpha = p_{\min} x_* e^{-2p_{\max} x_*}$. More precisely, for $t_0 = 2x_*$ we have*

$$S_{2x_*} n_0(x) \geq p_{\min} e^{-2p_{\max} x_*} \mathbb{1}_{\{0<x<x_*\}}$$

for all probability measures n_0 on $[0,+\infty)$.

Proof. We define a semigroup $\tilde{S}_t$ associated to the linear problem

$$\begin{aligned} \frac{\partial}{\partial t}\tilde{n}(t,x) + \frac{\partial}{\partial x}\tilde{n}(t,x) = -p(x)\tilde{n}(t,x), \quad t,x>0,\\ \tilde{n}(t,0) = 0, \quad t>0 \\ \tilde{n}(0,x) = n_0(x), \quad x \geq 0, \end{aligned} \tag{3.30}$$

which has the explicit solution

$$\tilde{n}(t,x) = \begin{cases} n_0(x-t)e^{-\int_0^t p(x-t+\tau)\,\mathrm{d}\tau}, & x>t,\\ 0, & t>x. \end{cases}$$

Then we write the solution to the linear equation (3.26) as

$$n(t,x) = \tilde{S}_t n_0(x) + \int_0^t \tilde{S}_{t-\tau}(N(\tau)\delta_0)(x)\,\mathrm{d}\tau.$$

For $x > t$ we have

$$n(t,x) \geq \tilde{S}_t n_0(x) = n_0(x-t)e^{-\int_0^t p(x-t+\tau)\,\mathrm{d}\tau} \geq n_0(x-t)e^{-p_{\max}t}$$
$$\tilde{S}_{t-\tau} n_0(x) \geq n_0(x-t+\tau)e^{-p_{\max}(t-\tau)}.$$

Then for $t > x_*$ it holds that

$$N(t) = \int_0^{+\infty} p(x)n(t,x)\,\mathrm{d}x \geq p_{\min}e^{-p_{\max}t}\int_{x_*}^{\infty} n_0(x-t)\,\mathrm{d}x$$
$$\geq p_{\min}e^{-p_{\max}t}\int_0^{+\infty} n_0(x)\,\mathrm{d}x = p_{\min}e^{-p_{\max}t}.$$

Therefore, for any $x > 0$ and $t > x + x_*$ we have:

$$n(t,x) \geq \int_0^t \tilde{S}_{t-\tau}(N(\tau)\delta_0)(x)\,\mathrm{d}\tau \geq \int_{s_*}^t \tilde{S}_{t-\tau}(p_{\min}e^{-p_{\max}\tau}\delta_0)(x)\,\mathrm{d}\tau$$
$$\geq p_{\min}\int_{x_*}^t e^{-p_{\max}\tau}e^{-p_{\max}(t-\tau)}\delta_0(x-t+\tau)\,\mathrm{d}\tau = p_{\min}e^{-p_{\max}t}\mathbb{1}_{\{0<x<t-x_*\}}.$$

Hence for $t = 2x_*$ and all $0 < x < x_*$ we obtain the result. □

Spectral gap

Exponential convergence to the equilibrium for the linear equation is an immediate consequence of Theorem 2.2.2. We give the following proposition based on that:

Proposition 3.2.6. *For a given initial data $n_0 \in \mathcal{M}(\mathbb{R}^+)$, let $p\colon [0,\infty) \to [0,+\infty)$ be bounded, Lipschitz function satisfying (3.6) and (3.5). Then, there exists a unique probability measure $n_* \in \mathcal{P}([0,+\infty))$ which is a stationary solution to (3.26), and any other stationary solution is a multiple of it. Also, for*

$$C = \frac{1}{1-\alpha} > 1 \text{ and } \lambda := -\frac{\log(1-\alpha)}{t_0}$$

we have

$$\|S_t(n_0 - n_*)\|_{\mathrm{TV}} \leq Ce^{-\lambda t}\|n_0 - n_*\|_{\mathrm{TV}}, \text{ for all } t \geq 0. \tag{3.31}$$

In addition, for $t_0 := 2x_*$ we have

$$\|S_{t_0}(n_1 - n_2)\|_{\mathrm{TV}} \leq (1-\alpha)\|n_1 - n_2\|_{\mathrm{TV}} \tag{3.32}$$

for any probability distributions n_1, n_2, and with

$$\alpha := p_{\min} x_* e^{-2p_{\max} x_*}.$$

Proof. We apply Theorem 2.2.2, since Lemma 3.2.4 shows that S_{t_0} satisfies the Doeblin condition for $t_0 = 2x_*$. Moreover,

$$C = \frac{1}{1-\alpha} = e^{\lambda t_0} = e^{-\log(1-p_{min} x_* e^{-2p_{\max} x_*})} = \frac{1}{1 - p_{min} x_* e^{-2p_{\max} x_*}} > 1.$$

□

3.2.3 Steady state for the nonlinear equation

Definition 3.2.4. *We say that a nonnegative function $n_* \in \mathcal{C}([0,+\infty)) \cap \mathcal{C}^1(0,+\infty)$ is a stationary solution to (1.2) if it satisfies*

$$\begin{aligned} &\frac{\partial}{\partial x} n_*(x) + p(N_*, x) n_*(x) = 0, \quad x > 0, \\ &n_*(0) =: N_* = \int_0^{+\infty} p(N_*, x) n_*(x)\, \mathrm{d}x. \end{aligned} \tag{3.33}$$

The following result is essentially the same as that given in [89]. There it is proved for a particular form of p, i.e. for $p(N,s) = \mathbb{1}_{x > x^*(N)}$, for some nonnegative $x^* \in C^1([0,+\infty))$ such that $\frac{d}{dN} x^*(N) \leq 0$ with $x^*(0) < 1$. So we prove it here for completeness, and to adapt it to our precise assumptions:

Theorem 3.2.7. *Assume (3.3), (3.4), (3.5) for p and also that*

$$L < (p_{\max})^{-2} \left(\frac{x_*^2}{2} + \frac{x_*}{p_{\min}} + \frac{1}{p_{\min}^2} \right)^{-1}.$$

Then there exists a unique probability measure n_* which is a stationary solution to (1.2).

Proof. If there is a stationary solution n_* then (since the first equation of (3.33) is an ordinary differential equation) it must satisfy

$$n_*(x) = n_*(0) e^{-\int_0^x p(N_*,\tau)\, \mathrm{d}\tau} = N_* e^{-\int_0^x p(N_*,\tau)\, \mathrm{d}\tau}, \qquad s \geq 0. \tag{3.34}$$

If n_* is a probability, by integrating we see that

$$N_* = \left(\int_0^{+\infty} e^{-\int_0^x p(N_*,\tau)\,\mathrm{d}\tau} \,\mathrm{d}x \right)^{-1}. \tag{3.35}$$

In particular, N_* must be strictly positive. Conversely, if $N_* > 0$ is such that (3.35) is satisfied then we may define $n_* = n_*(x)$ by (3.34) and it is straightforward to check that it is a probability, and it is a stationary solution to (1.2). Hence the problem is reduced to showing that there exists a unique solution $N_* > 0$ to (3.35); this is ensured by a simple fixed point argument, since

$$\begin{aligned} &\frac{\partial}{\partial N} \left(\int_0^{+\infty} e^{-\int_0^x p(N,\tau)\,\mathrm{d}\tau} \,\mathrm{d}x \right)^{-1} \\ &\qquad = \int_0^{+\infty} \Big(\int_0^x \partial_N p(N,\tau) d\tau \Big) \Big(e^{-\int_0^x p(N,\tau)\,\mathrm{d}\tau} \Big) \,\mathrm{d}x \left(\int_0^{+\infty} e^{-\int_0^x p(N,\tau)\,\mathrm{d}\tau} \,\mathrm{d}x \right)^{-2} \\ &\qquad\qquad \leq L \int_0^{+\infty} x e^{-\int_0^x p(N,\tau)\,\mathrm{d}\tau} \,\mathrm{d}x \left(\int_0^{+\infty} e^{-p_{\max} x} \,\mathrm{d}x \right)^{-2} \end{aligned}$$

where we have used (3.4) and (3.5). Note that these calculation is rigorous due to (3.3) and the fact that the integrals in s converge uniformly for all N. □

Similarly to our main results, the condition on L in the above theorem can be understood as a condition of weak nonlinearity.

3.2.4 Asymptotic behaviour

In this section we prove Theorem 3.1.1 for equation (3.1). Formally, the proof is based on rewriting it as

$$\frac{\partial}{\partial t} n = \mathcal{L}_N(n) = \mathcal{L}_{N_*}(n) + (\mathcal{L}_N(n) - \mathcal{L}_{N_*}(n)) =: \mathcal{L}_{N_*}(n) + h, \tag{3.36}$$

where we define

$$\mathcal{L}_N(n)(t,x) := -\frac{\partial}{\partial x} n(t,x) - p(N(t),x) n(t,x) + \delta_0(x) \int_0^{\infty} p(N(t),y) n(t,y) \,\mathrm{d}y,$$

and

$$\begin{aligned} h(t,x) := {}& [p(N_*,x) - p(N(t),x)]\, n(t,x) \\ &+ \delta_0(x) \int_0^{+\infty} [p(N(t),y) - p(N_*,y)]\, n(t,y) \,\mathrm{d}y. \end{aligned} \tag{3.37}$$

We treat the term h as a perturbation. In order to do this rigorously, notice that h contains a multiple of δ_0, so it is necessary to use a concept of solution in a space of measures. Then, since the solutions we are using do not allow us to write (3.36) rigorously, we need to use a concept of solution that allows for the same formal computation; this is the reason why weak solutions were introduced earlier.

Before proving the Theorem 3.1.1 for equation (3.1) we need the following lemma:

Lemma 3.2.5. *Assume the conditions in Theorem 3.1.1 for equation (3.1). Then h, defined by (3.37), satisfies*

$$\|h(t)\|_{\mathrm{TV}} \leq \tilde{C}\|n(t) - n_*\|_{\mathrm{TV}} \qquad \text{for all } t \geq 0, \tag{3.38}$$

where $\tilde{C} := 2p_{\max}\frac{L}{1-L}$. It also satisfies

$$\int_0^\infty h(t,x)\,\mathrm{d}x = 0 \qquad \text{for all } t \geq 0.$$

Proof. We notice that the stationary solution n_* exists due to Theorem 3.2.7, and the solution $n(t) \equiv n(t,x)$ with initial data n_0 was obtained in Theorem 3.2.2. Call N_* the total firing rate corresponding to the stationary solution n_*. We estimate directly each of the terms in the expression of h:

$$\begin{aligned}
&\|h(t)\|_{\mathrm{TV}} \\
&\leq \|(p(N_*,x) - p(N(t),x))n(t,x)\|_{\mathrm{TV}} + \left\|\delta_0 \int_0^{+\infty} (p(N(t),x) - p(N_*,x))n(t,x)\,\mathrm{d}x\right\|_{\mathrm{TV}} \\
&\quad \leq \|p(N_*,x) - p(N(t),x)\|_\infty \|n(t)\|_{\mathrm{TV}} + \left| \int_0^{+\infty} (p(N(t),x) - p(N_*,x))n(t,x)\,\mathrm{d}x \right| \\
&\qquad \leq L|N_* - N(t)| + \|p(N_*,x) - p(N(t),x)\|_\infty \|n(t)\|_{\mathrm{TV}} \\
&\qquad \leq \frac{Lp_{\max}}{1-L}\|n(t) - n_*\|_{\mathrm{TV}} + L|N_* - N(t)| \leq 2p_{\max}\frac{L}{1-L}\|n(t) - n_*\|_{\mathrm{TV}},
\end{aligned}$$

where the last inequality is due to Lemma 3.2.3 and the fact that $\|n_*\|_{\mathrm{TV}} = \|n(t)\|_{\mathrm{TV}} = 1$, which imply

$$|N_* - N(t)| \leq \frac{p_{\max}}{1-L}\|n(t) - n_*\|_{\mathrm{TV}}.$$

Regarding the integral of h in x we have

$$\begin{aligned}\int_0^{+\infty} h(t,x)\,\mathrm{d}x &= \int_0^{+\infty}[p(N_*,x)-p(N(t),x)]n(t,x)\,\mathrm{d}x \\ &\quad + \int_0^{+\infty}\delta_0(y)\int_0^{+\infty}[p(N(t),x)-p(N_*,x)]n(t,x)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_0^{+\infty}[p(N_*,x)-p(N(t),x)]n(t,x)\,\mathrm{d}x + \int_0^{+\infty}[p(N(t),x)-p(N_*,x)]n(t,x)\,\mathrm{d}x \\ &= 0,\end{aligned}$$

which gives the result. □

Proof of Theorem 3.1.1 for eq. (3.1). Call N_* the value of the total firing rate at equilibrium. The solution n to equation (1.2) is in particular a weak solution (see Theorem 3.2.4). Then one sees it is also a weak solution (in the sense of [5]) to the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}n(t,\cdot) = \mathcal{L}_{N_*}n(t,\cdot) + h(t,\cdot),$$

where $\mathcal{L}_{N_*}$ is the linear operator corresponding to $p = p(N_*,x)$ for N_* fixed,

$$\mathcal{L}_{N_*}n(t,x) := -\frac{\partial}{\partial x}n(t,x) - p(N_*,x)n(t,x) + \delta_0\int_0^{+\infty}p(N_*,y)n(t,y)\,\mathrm{d}y.$$

Then by [5] we may use Duhamel's formula and write the solution as

$$n(t,x) = S_t n_0(x) + \int_0^t S_{t-\tau}h(\tau,x)\,\mathrm{d}\tau, \tag{3.39}$$

where S_t is the linear semigroup defined in Section 3.2.2. We subtract the stationary solution from both sides;

$$n(t,x) - n_*(x) = S_t n_0(x) - n_*(x) + \int_0^t S_{t-\tau}h(\tau,x)\,\mathrm{d}\tau.$$

Then we take the TV norm;

$$\|n(t)-n_*\|_{\mathrm{TV}} \le \|S_t n_0 - n_*\|_{\mathrm{TV}} + \left\|\int_0^t S_{t-\tau}h(\tau,x)\,\mathrm{d}\tau\right\|_{\mathrm{TV}}. \tag{3.40}$$

By using Lemma 3.2.5 and Proposition 3.2.6, Equation (3.40) becomes:

$$\begin{aligned}\|n(t)-n_*\|_{\mathrm{TV}} &\leq \|S_t(n_0-n_*)\|_{\mathrm{TV}} + \int_0^t \|S_{t-\tau}h(\tau,x)\|_{\mathrm{TV}}\,\mathrm{d}\tau \\ &\leq Ce^{-\lambda t}\|n_0-n_*\|_{\mathrm{TV}} + \tilde{C}\int_0^t e^{-\lambda(t-\tau)}\|n(\tau)-n_*\|_{\mathrm{TV}}\,\mathrm{d}\tau.\end{aligned}$$

Therefore, by Gronwall's inequality we obtain

$$\|n(t)-n_*\|_{\mathrm{TV}} \leq Ce^{-(\lambda-\tilde{C})t}\|n_0-n_*\|_{\mathrm{TV}}. \qquad \square$$

3.3 A neuron population model with a fatigue property

We now consider the equation (3.2) for a neuron population model. We follow the same order as in Section 3.2.

3.3.1 Well-posedness

We refer the reader to Section 3.2.1 for preliminary notation and useful results. We define mild measure solutions in a similar way. Still denoting by $(T_t)_{t\geq 0}$ the translation semigroup generated on $(\mathcal{M}, \|\cdot\|_{\mathrm{BL}})$ by the operator $-\frac{\partial}{\partial x}$, we rewrite (1.4) as

$$\frac{\partial}{\partial t}n(t,x) - \mathcal{L}n(t,x) = \mathcal{A}[n](t,x),$$

where

$$\mathcal{L} = -\frac{\partial}{\partial x} \text{ and } \mathcal{A}[n](t,x) := -p(N(t),x)n(t,x) + \int_0^{+\infty} \kappa(x,y)p(N(t),y)n(t,y)\,\mathrm{d}y. \tag{3.41}$$

Definition 3.3.1. *Assume that p satisfies (3.3),(3.4) and κ satisfies (3.7). A couple of functions $n \in \mathcal{C}([0,T), \mathcal{M}_+(\mathbb{R}_0^+))$ and $N \in \mathcal{C}([0,T),[0,+\infty))$, defined on an interval $[0,T)$ for some $T \in (0,+\infty)$, is called a* mild measure solution *to (1.4) with initial data $n_0(x) \in \mathcal{M}(\mathbb{R}_0^+)$, $n(0) = n_0$ if it satisfies*

$$n(t,x) = T_t n_0(s) + \int_0^t T_{t-\tau}(\mathcal{A}[n(\tau,\cdot)])(x)d\tau, \text{ for all } t \in [0,T), \tag{3.42}$$

where $\mathcal{A}[n](t,x)$ *is defined as in* (3.41) *and*

$$N(t) = \int_0^{+\infty} p(N(t),x)n(t,x)dx, \quad t \in [0,T).$$

By integrating in $\mathbb{R}_0^+$, Definition 3.3.1 directly implies mass conservation. Therefore Lemma 3.2.2 holds true for this equation as well. Moreover we have the Lemma 3.2.3 satisfied with the same constants.

Theorem 3.3.1 (Well-posedness of (3.2) in measures)**.** *Assume that* p *satisfies* (3.3), (3.4) *and the Lipschitz constant* L *in* (3.4) *satisfies* $L \leq 1/(4\|n_0\|_{\mathrm{TV}})$. *Assume also* (3.7) *for* κ. *For any given initial data* $n_0 \in \mathcal{M}(\mathbb{R}^+)$ *there exists a unique measure solution* $n \in \mathcal{C}([0,+\infty);\mathcal{M}(\mathbb{R}_0^+))$ *of* (1.4) *in the sense of Definition 3.3.1. In addition, if* n_1, n_2 *are any two mild measure solutions to* (1.2) *(with possibly different initial data) defined on any interval* $[0,T)$ *then*

$$\|n_1(t) - n_2(t)\|_{\mathrm{TV}} \leq \|n_1(0) - n_2(0)\|_{\mathrm{TV}}\, e^{4\|p\|_\infty t} \text{ for all } t \in [0,T). \tag{3.43}$$

Proof of Theorem 3.3.1. Let us first show existence of a solution for a nonnegative initial measure $n_0 \in \mathcal{M}_+(\mathbb{R}_0^+)$. If $n_0 = 0$ it is clear that setting $n(t)$ equal to the zero measure on $\mathbb{R}_0^+$ for all t defines a solution, so we assume $n_0 \neq 0$. Fix $C, T > 0$, to be chosen later. Consider the complete metric space

$$\mathcal{Y} = \left\{ n \in \mathcal{C}([0,T],\mathcal{M}_+(\mathbb{R}_0^+)) \;\middle|\; n(0) = n_0,\ \|n(t)\|_{\mathrm{TV}} \leq C \text{ for all } t \in [0,T] \right\},$$

endowed with the norm

$$\|n\|_{\mathcal{Y}} := \sup_{t\in[0,T]} \|n(t)\|_{\mathrm{TV}}.$$

We remark that $\mathcal{C}([0,T],\mathcal{M}(\mathbb{R}_0^+))$ refers to functions which are continuous in the bounded Lipschitz topology, *not* in the total variation one. We define an operator $\Gamma : \mathcal{Y} \to \mathcal{Y}$ by

$$\Gamma[n](t) := T_t n_0 + \int_0^t T_{t-\tau}(A[n](\tau,\cdot))(x)d\tau \tag{3.44}$$

for all $n \in \mathcal{Y}$.

The definition of $\Gamma[n]$ indeed makes sense, since $\tau \mapsto T_{t-\tau}(A[n](\tau,\cdot))$ is a continuous function from $[0,T]$ to $\mathcal{M}(\mathbb{R}_0^+)$, hence integrable.

We first check that $\Gamma[n]$ is indeed in $\mathcal{Y}$. It is easy to see that $t \mapsto \Gamma[n](t)$ is continuous in the bounded Lipschitz topology, and it is a nonnegative measure for each $t \in [0, T]$. We also have

$$\begin{aligned}\|\Gamma[n](t)\|_{\mathrm{TV}} &\leq \|n_0\|_{\mathrm{TV}} + \int_0^t \|p(N(\tau), \cdot)n(\tau, \cdot)\|_{\mathrm{TV}} \,\mathrm{d}\tau \\ &\quad + \int_0^t \left\| \int_0^{+\infty} \kappa(., y)p(N(\tau), y)n(\tau, y)\,\mathrm{d}y \right\|_{\mathrm{TV}} \mathrm{d}\tau \\ &\leq \|n_0\|_{\mathrm{TV}} + TC\|p\|_\infty + \int_0^t \int_0^{+\infty} \left| \left(\int_0^{+\infty} \kappa(x, y)\,\mathrm{d}x\right) p(N(\tau), y)n(\tau, y) \right| \mathrm{d}y\,\mathrm{d}\tau \\ &\leq \|n_0\|_{\mathrm{TV}} + TC\|p\|_\infty + \int_0^t \|p(N(\tau), \cdot)n(\tau, \cdot)\|_{\mathrm{TV}} \,\mathrm{d}\tau \leq \|n_0\|_{\mathrm{TV}} + 2TC\|p\|_\infty.\end{aligned}$$

Here we used the assumption (3.7) on κ. We choose

$$T \leq \frac{1}{4\|p\|_\infty}, \quad \text{and} \quad C := 2\|n_0\|_{\mathrm{TV}}, \tag{3.45}$$

so that

$$\|n_0\|_{\mathrm{TV}} + 2TC\|p\|_\infty \leq \|n_0\|_{\mathrm{TV}} + \frac{C}{2} \leq C.$$

Hence with these conditions on T and C we have $\Gamma[n] \in \mathcal{Y}$.

Let us show that Γ is a contraction mapping. Take $n_1, n_2 \in \mathcal{Y}$ and let N_1, N_2 be defined by (3.18) corresponding to n_1 and n_2, respectively. We have

$$\begin{aligned}\|\Gamma[n_1](t) - \Gamma[n_2](t)\|_{\mathrm{TV}} &\leq \int_0^t \|(p(N_1(\tau), \cdot) - p(N_2(\tau), \cdot))n_1(\tau, \cdot)\|_{\mathrm{TV}} \,\mathrm{d}\tau \\ &\quad + \int_0^t \|p(N_2(\tau), \cdot)(n_1(\tau, \cdot) - n_2(\tau, \cdot))\|_{\mathrm{TV}} \,\mathrm{d}\tau \\ &\quad + \int_0^t \left\| \int_0^{+\infty} \kappa(\cdot, y)(p(N_1(\tau), y) - p(N_2(\tau), y))n_1(\tau, y)\,\mathrm{d}y \right\|_{\mathrm{TV}} \mathrm{d}\tau \\ &\quad + \int_0^t \left\| \int_0^{+\infty} \kappa(\cdot, y)p(N_2(\tau), y)(n_1(\tau, y) - n_2(\tau, y))\,\mathrm{d}y \right\|_{\mathrm{TV}} \mathrm{d}\tau \\ &=: T_1 + T_2 + T_3 + T_4.\end{aligned}$$

We bound each term separately. We can bound T_3 and T_4 in the following way;

$$\begin{aligned}T_3 &\leq \int_0^t \int_0^{+\infty} \left| \left(\int_0^{+\infty} \kappa(x, y)\,\mathrm{d}x\right) (p(N_1(\tau), y) - p(N_2(\tau), y))n_1(\tau, y) \right| \mathrm{d}y\,\mathrm{d}\tau = T_1, \\ T_4 &\leq \int_0^t \int_0^{+\infty} \left| \left(\int_0^{+\infty} \kappa(x, y)\,\mathrm{d}x\right) p(N_2(\tau), \cdot)(n_1(\tau) - n_2(\tau)) \right| \mathrm{d}y\,\mathrm{d}\tau = T_2\end{aligned}$$

For T_1, since $L \leq 1/(4\|n_0\|_{\mathrm{TV}})$, using Lemma 3.2.3 as previously calculated we have

$$T_1 \leq T\|p\|_\infty \|n_1 - n_2\|_{\mathcal{Y}}, \tag{3.46}$$

and for T_2,

$$T_2 \leq T\|p\|_\infty \|n_1 - n_2\|_{\mathcal{Y}}. \tag{3.47}$$

Putting equations (3.46)–(3.47) together and taking the supremum over $0 \leq t \leq T$,

$$\|\Gamma[n_1] - \Gamma[n_2]\|_{\mathcal{Y}} \leq 4T\|p\|_\infty \|n_1 - n_2\|_{\mathcal{Y}}.$$

Taking now $T \leq 1/(4\|p\|_\infty)$ ensures that Γ is contractive, so it has a unique fixed point in $\mathcal{Y}$, which is a mild measure solution on $[0, T]$. If we call n this fixed point, since $\|n(T)\|_{\mathrm{TV}} = \|n_0\|_{\mathrm{TV}}$ by mass conservation (see Lemma 3.2.2), we may repeat this argument to continue the solution on $[T, 2T]$, $[2T, 3T]$, showing that there is a solution defined on $[0, +\infty)$.

In order to show stability of solutions with respect to the initial data (which implies uniqueness of solutions), take two measures $n_0^1, n_0^2 \in \mathcal{M}_+(\mathbb{R}_0^+)$, and consider two solutions n_1, n_2 with initial data n_0^1, n_0^2 respectively. We have

$$\begin{aligned}\|n_1(t) - n_2(t)\|_{\mathrm{TV}} \leq \|n_0^1 - n_0^2\|_{\mathrm{TV}} &+ \int_0^t \|(p(N_1(\tau), \cdot) - p(N_2(\tau), \cdot))n_1(\tau, \cdot)\|_{\mathrm{TV}} \, \mathrm{d}\tau \\ &+ \int_0^t \|p(N_2(\tau), \cdot)(n_1(\tau) - n_2(\tau))\|_{\mathrm{TV}} \, \mathrm{d}\tau \\ &+ \int_0^t \left\| \int_0^{+\infty} \kappa(x, y)(p(N_1(\tau), \cdot) - p(N_2(\tau), \cdot))n_1(\tau, \cdot) \, \mathrm{d}y \right\|_{\mathrm{TV}} \mathrm{d}\tau \\ &+ \int_0^t \left\| \int_0^{+\infty} \kappa(x, y)p(N_2(\tau), \cdot)(n_1(\tau) - n_2(\tau)) \, \mathrm{d}y \right\|_{\mathrm{TV}} \mathrm{d}\tau,\end{aligned}$$

and with very similar arguments as before we obtain that

$$\|n_1(t) - n_2(t)\|_{\mathrm{TV}} \leq \|n_0^1 - n_0^2\|_{\mathrm{TV}} + 4\|p\|_\infty \int_0^t \|n_1(\tau) - n_2(\tau)\|_{\mathrm{TV}} \, \mathrm{d}\tau.$$

Gronwall's inequality then implies (3.43). □

3.3.2 The linear equation

The linear version of equation (1.4) obtained when $p = p(N, x)$ does not depend on N:

$$\begin{aligned} \frac{\partial}{\partial t}n(t,x) + \frac{\partial}{\partial x}n(t,x) + p(x)n(t,x) &= \int_0^{+\infty} \kappa(x,y)p(y)n(t,y)\,\mathrm{d}y, && x,y,t>0, \\ n(t,x=0) = 0, \quad N &= \int_0^{+\infty} p(x)n(t,x)\,\mathrm{d}x, && t>0, \\ n(t=0,x) &= n_0(x), && x \geq 0. \end{aligned} \tag{3.48}$$

Well-posedness

Similarly to Section 3.2.2, we can generalise slightly our concept of solution to include measures which are not necessarily nonnegative:

Definition 3.3.2. *Assume that* $p\colon [0,+\infty) \to [0,+\infty)$ *is a bounded, nonnegative function satisfying* (3.4) *and* κ *satisfies* (3.7)*. A couple of functions* $n \in \mathcal{C}([0,T), \mathcal{M}_+(\mathbb{R}_0^+))$ *and* $N \in \mathcal{C}([0,T), [0,+\infty))$*, defined on an interval* $[0,T)$ *for some* $T \in (0,+\infty]$*, are called a* mild measure solution *to* (1.4) *with initial data* $n_0(s) \in \mathcal{M}(\mathbb{R}_0^+)$ *if it satisfies* $n(0) = n_0$

$$n(t,x) = T_t n_0(x) + \int_0^t T_{t-\tau}\mathcal{A}[n(\tau,\cdot)](x)\,\mathrm{d}\tau \tag{3.49}$$

for all $t \in [0,T)$ *where*

$$\mathcal{A}[n](t,x) := -p(x)n(t,x) + \int_0^{+\infty} \kappa(x,y)p(y)n(t,y)\,\mathrm{d}y$$

and

$$N(t) = \int_0^{+\infty} p(x)n(t,x)\,\mathrm{d}x, \quad t \in [0,T).$$

By the existence result for (1.4) we have:

Theorem 3.3.2 (Well-posedness of (3.48) in measures)**.** *Assume that* $p\colon [0,+\infty) \to [0,+\infty)$ *is a bounded, nonnegative, Lipschitz function. Assume also that* κ *satisfies* (3.7)*. For any given initial data* $n_0 \in \mathcal{M}(\mathbb{R}^+)$ *there exists a unique measure solution* $n \in \mathcal{C}([0,+\infty); \mathcal{M}(\mathbb{R}_0^+))$ *of the linear equation* (3.48) *in the sense of Definition 3.3.2. In addition, if* n *is a mild measure solution to* (3.48) *defined on any interval* $[0,T)$ *then*

$$\|n(t)\|_{\mathrm{TV}} \leq \|n(0)\|_{\mathrm{TV}}, \ \textit{for all } t \in [0,T). \tag{3.50}$$

For the proof of this result one can follow the same procedure as in the proof of Theorem 3.2.5, so we omit it here.

Theorem 3.3.2 allows us to define a C_0-semigroup $(S_t)_{t\geq 0}$ on $\mathcal{M}$, such that $S_t(n_0) := n(t)$ for any $n_0 \in \mathcal{M}$ where $n(t)$ is the mild solution to (3.48) similarly as in Section 3.2.2.

Given p, we define $\mathcal{L}$ as the generator of the corresponding semigroup S_t, defined on its domain $\mathcal{D}(\mathcal{L})$. One can of course see that for sufficiently regular measures n,

$$\mathcal{L}n(x) = \frac{\partial}{\partial x} n(x) + p(x)n(x) - \int_0^{+\infty} \kappa(x,y)p(y)n(y)\,\mathrm{d}y. \tag{3.51}$$

Since the only unbounded operator involved in this expression is $\frac{\partial}{\partial s} n$, one sees that the domain $\mathcal{D}(\mathcal{L})$ can be described explicitly as

$$\mathcal{D}(\mathcal{L}) := \left\{ n \in \mathcal{M}(\mathbb{R}_0^+) \,\middle|\, \frac{\partial}{\partial x} n \in \mathcal{M}(\mathbb{R}_0^+) \right\},$$

where the derivative is taken in the sense of distributions on $\mathbb{R}$. Expression (3.51) is valid for all $n \in \mathcal{D}(\mathcal{L})$, again understanding the derivative in distributional sense.

Finally, for the arguments regarding the nonlinear equation (1.4) we will need a result on continuous dependence of the solutions of the linear equation (3.48) on the firing rate p:

Theorem 3.3.3 (Continuous dependence with respect to p for the linear equation)**.** *Let p_1, p_2 be bounded, nonnegative, Lipschitz functions. Assume also that κ satisfies (3.7). For any given initial data $n_0 \in \mathcal{M}(\mathbb{R}^+)$ consider n_1, n_2 the two solutions to the linear equation (3.48) on $[0,+\infty)$ with firing rate p_1, p_2 respectively and initial data n_0. Assuming $\|p_1\|_\infty \neq 0$, it holds that*

$$\|n_1(t) - n_2(t)\|_{\mathrm{TV}} \leq \frac{\|n_0\|_{\mathrm{TV}} \|p_1 - p_2\|_\infty}{\|p_1\|_\infty} \left(e^{2\|p_1\|_\infty t} - 1 \right) \qquad \text{for all } t \geq 0. \tag{3.52}$$

Proof. With the obvious changes in notation, from (3.49) we have

$$\begin{aligned} \|n_1(t) - n_2(t)\|_{\mathrm{TV}} &\leq \int_0^t \|T_{t-\tau}\mathcal{A}_1[n_1(\tau,\cdot)] - T_{t-\tau}\mathcal{A}_2[n_2(\tau,\cdot)]\|_{\mathrm{TV}}\,\mathrm{d}\tau \\ &= \int_0^t \|\mathcal{A}_1[n_1(\tau)] - \mathcal{A}_2[n_2(\tau)]\|_{\mathrm{TV}}\,\mathrm{d}\tau. \end{aligned}$$

In a very similar way as the estimate we carried out for Theorem 3.3.1, this last term can be estimated as

$$\begin{aligned}\|\mathcal{A}_1[n_1(\tau)] - \mathcal{A}_2[n_2(\tau)]\|_{\mathrm{TV}} \\ &\leq 2\|p_1(x)(n_1(\tau,x) - n_2(\tau,x))\|_{\mathrm{TV}} + 2\|n_2(x)(p_1(x) - p_2(x))\|_{\mathrm{TV}} \\ &\leq 2\|p_1\|_\infty \|n_1(\tau,\cdot) - n_2(\tau,\cdot))\|_{\mathrm{TV}} + 2\|n_0\|_{\mathrm{TV}}\|p_1 - p_2\|_\infty.\end{aligned}$$

Hence, calling $m(t) \equiv m(t) := \|n_1(t,\cdot) - n_2(t,\cdot)\|_{\mathrm{TV}}$ and $K := \|n_0\|_{\mathrm{TV}}\|p_1 - p_2\|_\infty$, we have

$$m(t) \leq 2\|p_1\|_\infty \int_0^t m(\tau)\,\mathrm{d}\tau + 2tK.$$

Gronwall's Lemma then shows that

$$m(t) \leq \frac{K}{\|p_1\|_\infty}\left(e^{2\|p_1\|_\infty t} - 1\right). \qquad \square$$

Steady state for the linear equation

Definition 3.3.3. *A stationary solution to (3.48) $n_* \in \mathcal{M}$ is defined as such that $n_* \in \mathcal{D}(\mathcal{L})$ and*

$$\mathcal{L}n_* = 0.$$

We remark that Proposition 3.3.4 below implies that the linear equation (3.48) has a unique stationary solution in the space of probabilities on $[0,+\infty)$ (for p bounded, Lipschitz, satisfying (3.5) and κ satisfying (3.7)); n_* is the only stationary solution up to a constant factor within the set of all finite measures.

Uniform minorisation condition

Analogous to Section 3.2.2, we want to show that for a given positive initial distribution, solutions of (3.48) after some time have a positive lower bound, so that the semigroup $S_t(n_0)$ satisfies the *Doeblin's condition.*

Lemma 3.3.1. *Let $p : [0,+\infty) \longrightarrow [0,+\infty)$ be a bounded, Lipschitz function satisfying (3.6) and (3.5). We assume also that κ satisfies (3.7) and (3.8). Consider the semigroup defined as $S_t(n_0) := n(t)$ for any $n_0 \in \mathcal{M}$. Then S_{t_0} satisfy the Doeblin condition (??) for $t_0 = 2x_*$ and $\alpha = \epsilon\delta p_{\min}(x_* - \delta)e^{-p_{\max}t_0}$. More precisely, for $t_0 = 2x_*$ we have*

$$S_{2x_*}n_0(s) \geq \epsilon\delta p_{\min} e^{-2p_{\max}x_*}\mathbb{1}_{\{\delta < x < x_*\}}$$

for all probability measures n_0 on $[0,+\infty)$.

Proof. Since for $x, t > 0$, it holds true for solutions of (3.48) that

$$\frac{\partial}{\partial t} n(t,x) + \frac{\partial}{\partial x} n(t,x) \geq -p(x)n(t,x).$$

Moreover, solutions of (3.48) satisfy $n(t,x) \geq \tilde{n}(t,x)$ where the equation on $\tilde{n}(t,x)$ was defined in (3.30) of Lemma 3.2.4. By the same argument we have for $t > x_*$, $N(t) \geq p_{\min} e^{-p_{\max} t}$.

We consider the same semigroup $\tilde{S}_t$ associated to (3.30). Then, solutions of (3.48) satisfy

$$\begin{aligned} n(t,s) = \tilde{S}_t n_0(x) + \int_0^t \tilde{S}_{t-\tau} \left(\int_0^{+\infty} \kappa(\cdot, y) p(y) n(t,y) \, \mathrm{d}y \right)(x) \, \mathrm{d}\tau \\ \geq \tilde{S}_t n_0(x) + \int_0^t \tilde{S}_{t-\tau} (\epsilon N(\tau) \mathbb{1}_{\{x \leq \delta\}}) \, \mathrm{d}\tau \end{aligned}$$

since

$$\begin{aligned} \int_0^{+\infty} \kappa(x,y) p(y) n(t,y) \, \mathrm{d}y \geq \int_0^{+\infty} \phi(x) p(y) n(t,y) \, \mathrm{d}y \geq \epsilon \mathbb{1}_{\{x \leq \delta\}} \int_0^{+\infty} p(y) n(t,y) \, \mathrm{d}y \\ = \epsilon \mathbb{1}_{\{x \leq \delta\}} N(t). \end{aligned}$$

Then for $t > x + x_*$ and $x > \delta$ we have

$$\begin{aligned} n(t,x) &\geq \int_0^t \tilde{S}_{t-\tau} (\epsilon N(\tau) \mathbb{1}_{\{x \leq \delta\}}) \, \mathrm{d}\tau \geq \int_{x_*}^t \tilde{S}_{t-\tau} (\epsilon \mathbb{1}_{\{x \leq \delta\}} p_{\min} e^{-p_{\max}\tau}) \, \mathrm{d}\tau \\ &\geq \epsilon p_{\min} \int_{x_*}^t e^{-p_{\max}\tau} e^{-p_{\max}(t-\tau)} \mathbb{1}_{\{0 < x - t + \tau \leq \delta\}} \, \mathrm{d}\tau = \epsilon p_{\min} e^{-p_{\max} t} \int_{x_*}^t \mathbb{1}_{\{0 < x - t + \tau \leq \delta\}} \, \mathrm{d}\tau \\ &= \epsilon \delta p_{\min} e^{-p_{\max} t} \mathbb{1}_{\{\delta < x < t_0 - x_*\}}. \end{aligned}$$

Hence for $t = 2x_*$ and $\delta < x < x_*$ we obtain the result. □

Spectral gap

We again obtain a spectral gap property as a consequence of Theorem 2.2.2:

Proposition 3.3.4. *Let $n_0 \in \mathcal{M}(\mathbb{R}^+)$ be the initial data given for (3.48). We assume that p is a nonnegative, Lipschitz function satisfying (3.4)–(3.5) and κ satisfies (3.7), (3.8). Then, there exists a unique probability measure $n_* \in \mathcal{P}([0, +\infty))$ which is a stationary solution to (3.48), and any other stationary solution is a multiple of it. Also* *for*

$$C = \frac{1}{1-\alpha} > 1, \text{ and } \lambda = -\frac{\log(1-\alpha)}{t_0},$$

we have

$$\|S_t(n_0 - n_*)\|_{TV} \leq Ce^{-\lambda t}\|n_0 - n_*\|_{\mathrm{TV}}, \text{ for all } t \geq 0.$$

In addition, for $t_0 := 2x_*$ *we have*

$$\|S_{t_0}(n_1 - n_2)\|_{\mathrm{TV}} \leq (1-\alpha)\|n_1 - n_2\|_{\mathrm{TV}} \tag{3.53}$$

for any probability distributions n_1, n_2*, and with*

$$\alpha := \epsilon\delta p_{\min}(x_* - \delta)e^{-2p_{\max}x_*}.$$

Proof. Lemma 3.3.1 ensures the operator S_{t_0} satisfies the *Doeblin condition* (2.2.1) for $t_0 = 2x_*$. We obtain the result by applying Theorem 2.2.2. □

3.3.3 Steady state for the nonlinear equation

Definition 3.3.4. *We say that a pair* (n_*, N_*)*, where* $n_* \in \mathcal{M}_+(\mathbb{R}_0^+)$ *and* $N_* \geq 0$*, is a* stationary solution *to* (1.4) *if* $n_* \in \mathcal{D}(\mathcal{L})$ *and*

$$\mathcal{L}_{N_*}n_* = 0, \qquad N_* = \int_0^{+\infty} p(N_*, x)n_*(x)\,\mathrm{d}x,$$

where $\mathcal{L}_{N_*}$ *is the semigroup generator associated to* $p(x) \equiv p(N_*, x)$ *(see Theorem 3.3.2 and the following remarks; observe that the domain* $\mathcal{D}(\mathcal{L})$ *does not depend on the value of* N_**). We say that* N_* *is the* global neural activity *associated to the stationary solution.*

We give the following theorem for existence and uniqueness of stationary solutions:

Theorem 3.3.5. *Assume* (3.3)*,* (3.4)*,* (3.5)*,* (3.7)*, and also that*

$$L < \left(1 + \frac{Cp_{\max}}{\alpha p_{\min}}\right)^{-1},$$

where $C := e^{4p_{\max}x_*}$ *and* α *is given by Proposition 3.3.4. Then there exists a unique stationary solution* (n_*, N_*) *of* (1.4) *such that* n_* *is a probability measure.*

Proof. Proposition 3.3.4 ensures that for a fixed N_*, there exists a unique probability stationary solution of the corresponding linear problem. We prove the existence of a stationary solution by recovering N_* from n_* and carrying out a fixed-point argument. We define a map $\Upsilon : [0,+\infty) \longrightarrow [0,+\infty)$, by

$$\Upsilon(N) := \int_0^{+\infty} p(N,x)n(x)\,\mathrm{d}x,$$

where n is the unique probability measure which is an equilibrium of the linear problem associated to $p(x) \equiv p(N,x)$. We notice that the statement we wish to prove is equivalent to the fact that Υ has a unique fixed point.

Let us show that this map is contractive. For any $N_1, N_2 \geq 0$,

$$\begin{aligned}|\Upsilon(N_1) - \Upsilon(N_2)| &= \left|\int_0^{+\infty} \left(p(N_1,x)n_1(x) - p(N_2,x)n_2(x)\right)\,\mathrm{d}x\right| \\ &\leq \int_0^{+\infty} \Big|(p(N_1,x) - p(N_2,x))n_2(x)\Big|\,\mathrm{d}x + \int_0^{+\infty} \Big|p(N_1,x)(n_1(x) - n_2(x))\Big|\,\mathrm{d}x \\ &\leq L|N_1 - N_2| + p_{\max}\|n_1 - n_2\|_{\mathrm{TV}}.\end{aligned}$$

Now, we will prove later that

$$\|n_1 - n_2\|_{\mathrm{TV}} \leq \frac{LC}{\alpha p_{\min}}|N_1 - N_2|, \tag{3.54}$$

where $C := e^{4p_{\max}x_*}$ and α is the one from Proposition 3.3.4. This implies that

$$|\Upsilon(N_1) - \Upsilon(N_2)| \leq L\left(1 + \frac{Cp_{\max}}{\alpha p_{\min}}\right)|N_1 - N_2|,$$

which makes Υ a contraction operator if L satisfies the inequality in the statement. So in order to complete the proof we only need to show (3.54). For this we define the two operators

$$P_1(n) := S^1_{t_0}n - n, \qquad P_2(n) := S^2_{t_0}n - n,$$

where for $i = 1, 2$, $(S^i_t)_{t\geq 0}$ is the linear semigroup given by Theorem 3.3.2, associated to the firing rate $p_i(x) := p(N_i, x)$, and $t_0 := 2x_*$ is the time mentioned in Proposition 3.3.4. We use that, since n_1, n_2 are equilibria for the linear equations with p_1, p_2,

$$0 = P_1(n_1) = P_2(n_2)$$

so that

$$0 = \|P_1(n_1) - P_2(n_2)\|_{\mathrm{TV}} = \|P_1(n_1 - n_2) + (P_1 - P_2)(n_2)\|_{\mathrm{TV}} \\ \geq \|P_1(n_1 - n_2)\|_{\mathrm{TV}} - \|(P_1 - P_2)n_2\|_{\mathrm{TV}},$$

which implies

$$\|P_1(n_1 - n_2)\|_{\mathrm{TV}} \leq \|(P_1 - P_2)n_2\|_{\mathrm{TV}}. \tag{3.55}$$

Then by Proposition 3.3.4 we have

$$\|P_1(n_1 - n_2)\|_{\mathrm{TV}} \geq \|n_1 - n_2\|_{\mathrm{TV}} - \|S^1_{t_0}(n_1 - n_2)\|_{\mathrm{TV}} \geq \alpha \|n_1 - n_2\|_{\mathrm{TV}},$$

since $\int n_1 \, \mathrm{d}x = \int n_2 \, \mathrm{d}x = 1$, where α is the one in Proposition 3.3.4. On the other hand, by (3.52),

$$\|(P_1 - P_2)n_2\|_{\mathrm{TV}} = \|(S^1_{t_0} - S^2_{t_0})n_2\|_{\mathrm{TV}} \leq \frac{\|n_2\|_{\mathrm{TV}} \|p_1 - p_2\|_\infty}{\|p_1\|_\infty} \left(e^{2\|p_1\|_\infty t_0} - 1\right) \\ \leq \frac{L|N_1 - N_2|}{p_{\min}} e^{2p_{\max} t_0}.$$

Using the last two equations in (3.55),

$$\|n_1 - n_2\|_{\mathrm{TV}} \leq \frac{1}{\alpha} \|(P_1 - P_2)n_2\|_{\mathrm{TV}} \leq \frac{L|N_1 - N_2|}{\alpha p_{\min}} e^{2p_{\max} t_0},$$

which proves (3.54). Therefore Υ has a unique fixed point, and hence (1.4) has a unique stationary solution. □

3.3.4 Asymptotic behaviour

In this section we prove Theorem 3.1.1 for equation (1.4). We define two operators in the following way:

$$\mathcal{L}_{N(t)} n(t,x) := \partial_t n(t,x) = -\partial_x n(t,x) - p(N(t),x)n(t,x) + \int \kappa(x,y) p(N(t),y) n(t,y) \, \mathrm{d}y, \\ \mathcal{L}_{N_*} \bar{n}(x) := -\partial_x \bar{n}(s) - p(N_*,x)\bar{n}(x) + \int \kappa(x,y) p(N_*,y) \bar{n}(y) \, \mathrm{d}y.$$

We rewrite (1.4) as

$$\frac{\partial}{\partial t} n(t,x) = \mathcal{L}_{N(t)} n(t,x) = \mathcal{L}_{N_*} n(t,x) - (\mathcal{L}_{N_*} - \mathcal{L}_{N(t)}) n(t,x). \tag{3.56}$$

Then, similarly as in Section 3.2.4 by [5] we may use Duhamel's formula and write the solution as

$$n(t,x) = S_t n_0(x) + \int_0^t S_{t-\tau} h(\tau,x)\,\mathrm{d}\tau, \tag{3.57}$$

where $S_t n_0(x) := e^{\mathcal{L}_{N_*}} n_0(x)$ and $\bar{n}$ is the solution to linear problem, $\mathcal{L}_{N_*}$ is acting on $n(t,x)$. Also,

$$\begin{aligned} h(t,x) &:= (\mathcal{L}_{N_*} - \mathcal{L}_{N(t)})n(t,x) \\ &= (p(N(t),x) - p(N_*,x))n(t,x) + \int_0^{+\infty} \kappa(x,y)(p(N_*,y) - p(N(t),y))n(t,y)\,\mathrm{d}y. \end{aligned} \tag{3.58}$$

Then we give the following lemma:

Lemma 3.3.2. *Assume that (3.4) and (3.5) hold true for a Lipschitz function p and κ satisfies (3.7). Then h, which is defined by (3.58), satisfies*

$$\|h(t)\|_{\mathrm{TV}} \leq \tilde{C}\|n(t) - n_*\|_{\mathrm{TV}}, \tag{3.59}$$

where $\tilde{C} = 2p_{\max}\frac{L}{1-L}$. Moreover $\int_0^{+\infty} h(t,x)\,\mathrm{d}x = 0$.

Proof.

$$\begin{aligned} &\|h(t)\|_{\mathrm{TV}} = \|(\mathcal{L}_{N_*} - \mathcal{L}_{N(t)})n(t,x)\|_{\mathrm{TV}} \\ &\leq \|(p(N(t),x)-p(N_*,x))n(t,x)\|_{\mathrm{TV}} + \left\| \int_0^{+\infty} \kappa(x,y)(p(N_*,y)-p(N(t),y))n(t,y)\,\mathrm{d}y \right\|_{TV} \\ &\qquad \leq L\|n(t)\|_{\mathrm{TV}}|N_* - N(t)| + L\|n(t)\|_{\mathrm{TV}}|N_* - N(t)| \\ &\qquad \leq 2p_{\max}\frac{L\|n(t)\|_{\mathrm{TV}}}{1 - L\|n(t)\|_{\mathrm{TV}}}\|n(t) - n_*\|_{\mathrm{TV}} = 2p_{\max}\frac{L}{1-L}\|n(t) - n_*\|_{\mathrm{TV}} \end{aligned}$$

Since

$$\begin{aligned} &|N_* - N(t)| = \left| \int_0^{+\infty} p(N_*,x)n_*(x)\,\mathrm{d}x - \int_0^{+\infty} p(N(t),x)n(t,x)\,\mathrm{d}x \right| \\ &\leq \left| \int_0^{+\infty} (p(N_*,x)n_*(x) + (p(N_*,x)n(t,x) - p(N_*,x)n(t,x))p(N(t),x)n(t,x))\,\mathrm{d}x \right| \\ &\leq \left| \int_0^{+\infty} p(N_*,x)(n_*(x) - n(t,x))\,\mathrm{d}x \right| + \left| \int_0^{+\infty} (p(N_*,x) - p(N(t),x)n(t,x)\,\mathrm{d}x \right| \\ &\qquad\qquad \leq p_{\max}\|n(t) - n_*\|_{\mathrm{TV}} + L|N_* - N(t)|\|n\|_{\mathrm{TV}} \end{aligned}$$

implies that

$$|N_* - N(t)| \leq \frac{p_{\max}}{1 - L\|n(t)\|_{TV}} \|n(t) - n_*\|_{\mathrm{TV}} = \frac{p_{\max}}{1-L} \|n(t) - n_*\|_{\mathrm{TV}} \tag{3.60}$$

since $\|n(t)\|_{\mathrm{TV}} = \|n_*\|_{\mathrm{TV}} = 1$. Moreover we have

$$\begin{aligned}
&\int_0^{+\infty} h(t,x)\,\mathrm{d}x = \int_0^{+\infty} p(N(t),x)\bar{n}(t,x)\,\mathrm{d}x - \int_0^{+\infty} p(N_*,x)\bar{n}(t,x)\,\mathrm{d}x \\
&+ \int_0^{+\infty}\int_0^{+\infty} \kappa(x,y)p(N_*,y)\bar{n}(t,y)\,\mathrm{d}y\,\mathrm{d}x - \int_0^{+\infty}\int_0^{+\infty} \kappa(x,y)p(N(t),y)\bar{n}(t,y)\,\mathrm{d}y\,\mathrm{d}x \\
&\qquad = N(t) - \int_0^{+\infty} p(N_*,x)\bar{n}(t,x)\,\mathrm{d}x \\
&+ \int_0^{+\infty} \left(\int_0^y \kappa(x,y)\,\mathrm{d}x\right) p(N_*,y)\bar{n}(t,y)\,\mathrm{d}y + \int_0^{+\infty}\int_0^y \kappa(x,y)\,\mathrm{d}y p(N(t),y)\bar{n}(t,y)\,\mathrm{d}y \\
&\qquad = N(t) - \int_0^{+\infty} p(N_*,x)\bar{n}(t,x)\,\mathrm{d}x + \int_0^{+\infty} p(N_*,y)\bar{n}(t,y)\,\mathrm{d}y - N(t) = 0.
\end{aligned}$$

□

Proof of Theorem 3.1.1 for (3.2). We subtract the unique probability stationary solution from both sides of (3.57):

$$n(t,x) - n_*(x) = S_t n_0(x) - n_*(x) + \int_0^t S_{t-\tau} h(\tau,x)\,\mathrm{d}\tau.$$

We take the total variation norm and obtain

$$\|n(t) - n_*\|_{\mathrm{TV}} \leq \|S_t n_0 - n_*\|_{\mathrm{TV}} + \left\| \int_0^t S_{t-\tau} h(\tau,x)\,\mathrm{d}\tau \right\|_{\mathrm{TV}}.$$

Then by Proposition 3.3.4 and Lemma 3.3.2 we have

$$\begin{aligned}
\|n(t) - n_*\|_{\mathrm{TV}} &\leq Ce^{-\lambda t}\|n_0 - n_*\|_{\mathrm{TV}} + \int_0^t \|S_{t-\tau}h(\tau,x)\|_{\mathrm{TV}}\,\mathrm{d}\tau \\
&\leq Ce^{-\lambda t}\|n_0 - n_*\|_{\mathrm{TV}} + \int_0^t e^{-\lambda(t-\tau)}\|h(\tau,x)\|_{\mathrm{TV}}\,\mathrm{d}\tau \\
&\leq Ce^{-\lambda t}\|n_0 - n_*\|_{\mathrm{TV}} + \tilde{C}\int_0^t e^{-\lambda(t-\tau)}\|n(\tau) - n_*\|_{\mathrm{TV}}\,\mathrm{d}\tau.
\end{aligned}$$

Therefore, by Gronwall's lemma we obtain

$$\|n(t) - n_*\|_{\mathrm{TV}} \leq Ce^{-(\lambda - \tilde{C})t}\|n_0 - n_*\|_{\mathrm{TV}}.$$

□

3.4 Summary and conclusion

We studied the long time behaviour of population models describing the dynamics of interacting neurons, initially proposed by [89, 91]. In the first model, the structuring variable x represents the time elapsed since its last discharge, while in the second one neurons exhibit a fatigue property and the structuring variable is a generic "state". We prove existence of solutions and steady states in the space of finite, nonnegative measures. Furthermore, we show that solutions converge to the equilibrium exponentially in time in the case of weak nonlinearity (i.e., weak connectivity). The main innovation is the use of Doeblin's theorem from probability in order to show the existence of a spectral gap property in the linear (no-connectivity) setting. Relaxation to the steady state for the nonlinear models is then proved by a constructive perturbation argument. The results presented in this chapter are based on [28].

The closest results in the literature are those of [89, 91]. Equation (3.1) is essentially the model in [89], written in a slightly different formulation that does not include time delay and does not highlight the connectivity as a separate parameter (the connectivity of neurons in our case is measured in the size of $\partial_N p$). The results in [89] use entropy methods and show exponential convergence to equilibrium (a similar statement to Theorem 3.1.1) in a weighted L^1 space, for the case with delay and for a particular form of the firing rate p. As compared to that, these results work in a space of measures and can be easily written for general firing rates p; however, we have not considered the large-connectivity case (which would correspond to large $\partial_N p$ in our case) or the effects of time delay.

Similar remarks apply to the results for equation (3.2) contained in [91]. In this case our strategy gives in general conditions which are simpler to state, and provide a general framework which may be applied to similar models. Again, we have not considered a time delay in the equation, which is a difference with the above work. There are numerical simulations and further results on regimes with a stronger nonlinearity in [89–91].

Here we use an alternative approach to prove convergence to equilibrium that is based on some results in the theory of Markov processes known as *Doeblin's theory*, with some extensions such as *Harris's Theorem*; see [71], or [70, 63] which we explained in detail in the previous Chapter 2. Applying Doeblin's Theorem for the nonlinear models which are introduced is based on first studying the linear case and then carrying out a perturbation argument. We study the spectral properties of the linear operator by Doeblin's theory, which is quite flexible and later simplifies the proofs. We obtain a spectral gap property of the linear equation in a set of measures, and this leads to a perturbation argument which naturally takes care of the boundary conditions in (3.1)-(3.2). Similar ideas are reviewed in [63] for the renewal equation, and have been recently used in [8] for neuron population models structured by voltage.

Due to this strategy, studying solutions to (3.1) and (3.2) in the sense of measures comes as a natural setting for two important reasons: first, it fits well with the linear theory; and second, it allows us to treat the weakly nonlinear case as a perturbation of the linear one. Note that one difference between the weakly nonlinear case and the linear case for equation (3.1) is in the boundary condition, and this is conveniently encoded as a difference in a measure source term; see the proof of Theorem 3.1.1 for details on this. Measure solutions are also natural since a Delta function represents an initial population whose age (or structuring variable) is known precisely. There exist also recent works on numerical schemes for structured population models in the space of nonnegative measures [44, 67]. Entropy methods have also been extended to measure initial data by [66] for the renewal equation.

Chapter 4

On the asymptotic behaviour of the growth-fragmentation equation

"There are no impossible obstacles; there are just stronger and weaker wills, that's all!"

— Jules Verne, The Adventures of Captain Hatteras

Contents

4.1 Overview

In this chapter we study the long-time behaviour of the growth-fragmentation equation which is a linear evolution equation describing a wide range of phenomena in structured population dynamics. We apply Harris's Theorem to the associated semigroup in order to obtain spectral gap. This part of the chapter is based on a collaboration [40]. In the last section 4.4 we also give a numerical approximation of the growth-fragmentation equation in a particular case. This part is an incomplete project started earlier this year jointly with José A. Carrillo and aimed at using Harris type arguments for numerical approximation, in the discrete setting, to show convergence to equilibrium. We give some part of it here just for the sake of completeness.

We already give a brief introduction for the growth-fragmentation equation in Chapter 1. Here we recall the equation, state the main assumptions and present the main result. General form of the growth-fragmentation equation is given by:

$$\begin{aligned} \frac{\partial}{\partial t}n(t,x) + \frac{\partial}{\partial x}(g(x)n(t,x)) &= \int_x^\infty \kappa(y,x)n(t,y)\,\mathrm{d}y - B(x)n(t,x), && t,x>0,\\ n(t,0) &= 0, && t\geq 0,\\ n(0,x) &= n_0(x), && x>0, \end{aligned} \tag{4.1}$$

where $n(t,x)$ represents the population density of individuals structured by a variable $x>0$ at a time $t\geq 0$. The structuring variable x could be *age, size, length, weight, DNA content, biochemical composition etc.* depending on the context but we consider it as *size* from now on. The boundary condition implies that no individuals are newly created at size 0. The function g is called the *growth rate* and B the *total division/fragmentation rate* of individuals with size $x\geq 0$. The kernel $\kappa(y,x)$ is the rate at which individuals of size x are obtained as the result of a fragmentation event of an individual of size y.

After a long time, the total population is expected to grow exponentially at a rate $e^{\lambda t}$, that for some $\lambda>0$, and the normalised population distribution tends to approach a *universal profile* for large times, independently of the initial condition under suitable conditions on the coefficients κ and g. Our goal is to prove this behaviour and determine the rate of convergence to the universal profile by carrying out totally constructive arguments via Harris's Theorem. However, studying the asymptotic behaviour of solutions to (4.1) is not a new problem and there are already good amount of previous works in the literature, most of which are mentioned in Chapter 1. If we briefly recall here, the milestone papers in the mathematical study of (4.1) date back to [51, 83, 93, 92]. There are also some results achieved by providing explicit solutions, using semigroup and probabilistic approaches. Our approach is also a probabilistic one. We use Harris's Theorem to prove the spectral gap property. Applying this type of argument into biological and kinetic models which can be defined as *Markov processes* is a subject of many recent works. Recently in [25], the authors used the growth-fragmentation equation with bounded fragmentation rate to model the dynamics of the carbon content of a forest whose deterministic growth is interrupted by natural disasters. The authors used Harris's Theorem to obtain quantitative convergence rates. This might be the closest work to the one we present here but we note that our result works for unbounded total fragmentation rate as well.

We continue by giving the modelling assumptions and the main result in the following section. Later we will dedicate the subsequent sections to showing the existence of solutions to the associated Perron eigenvalue problem (1.10)-(1.11) which was introduced in Chapter 1 and verifying Hypotheses 2.2.2 and 2.2.3 for the Harris's Theorem 2.2.5 for the growth-fragmentation equation.

4.1.1 Assumptions and the main theorem

In this section we list the modelling assumptions some of which are standard and used in the previous literature as well and the assumptions we need in order our method to work.

The total fragmentation rate B is obtained through κ as

$$B(x) = \int_0^y \frac{y}{x}\kappa(x,y)\,\mathrm{d}y.$$

We consider two types of fragmentation kernels:

1. *Mitosis kernel* describes mitosis process where individuals only break into two equal fragments, such as many in biological cells. It is given by

$$\kappa(x,y) = B(x)\frac{2}{x}\delta_{\{y=\frac{x}{2}\}}.$$

2. *Constant kernel* describes uniform fragment distribution where the fragmentation event gives fragments of any size less than the original one with equal probability and given by

$$\kappa(x,y) = B(x)\frac{2}{x}.$$

We make the following assumptions regarding the fragmentation kernel κ:

Assumption 4.1.1. *We consider $\kappa(x,y)$, the fragmentation kernel is of the form;*

$$\kappa(x,y) = \frac{1}{x} p\Big(\frac{y}{x}\Big) B(x), \quad \text{for } y > x > 0, \tag{4.2}$$

so that the total fragmentation rate of cells of size $x > 0$ satisfies

$$xB(x) = \int_0^x y\kappa(x,y)\,\mathrm{d}y. \tag{4.3}$$

Assumption 4.1.2. *We define p, distribution of fragments, by*

$$p_k := \int_0^1 z^k p(z)\,\mathrm{d}z, \text{ for all } k \geq 0, \tag{4.4}$$

satisfying

$$0 < p_k < p_1 = 1 < p_0 \text{ for } k \geq 2 \tag{4.5}$$

and p is a nonnegative finite measure on $[0,1]$.

Remark 4.1.1. *Uniform fragment distribution corresponds to taking $p(z) = 2$; whereas for the equal mitosis process we consider $p(z) = 2\delta_{1/2}(z)$. We also note that for both the cases $p_0 = \int_0^1 p(z)\,\mathrm{d}z = 2$.*

Next two assumptions are on growth and total division rates.

Assumption 4.1.3. *We assume that $g\colon (0,+\infty) \to (0,+\infty)$ is a locally Lipschitz function and there exists $C > 0$ such that $g(x) \leq Cx$ for all $x \geq 1$. Moreover we also assume*

$$\int_0^1 \frac{1}{g(x)}\,\mathrm{d}x < +\infty. \tag{4.6}$$

and on the behaviour close to 0 and $+\infty$,

$$\begin{aligned} g(x) &\sim g_0 x^{\alpha_0} \quad \text{as } x \to 0, \\ g(x) &\sim g_\infty x^{\alpha} \quad \text{as } x \to +\infty, \end{aligned} \tag{4.7}$$

where $0 < \alpha, \alpha_0 < 1$.

The total fragmentation rate $B\colon [0,+\infty) \to [0,+\infty)$ is a continuous function which is locally integrable in $[0,\infty)$. Moreover the following holds

$$\frac{xB(x)}{g(x)} \underset{x\to+\infty}{\longrightarrow} +\infty, \qquad \frac{xB(x)}{g(x)} \underset{x\to 0}{\longrightarrow} 0, \tag{4.8}$$

When the growth rate is linear, if we consider the mitosis kernel; it is known that there does not exist a unique equilibrium but a set of equilibria ([14] and the references therein). Therefore we only look at uniform fragmentation kernel if the growth rate is linear.

Main result of the paper is given in the following theorem:

Theorem 4.1.2. *Assume 4.1.1, 4.1.2 and 4.1.3. There exist $C, \lambda, \rho > 0$ and a* universal profile *n_* such that any solution $n = n(t,x) \equiv n_t(x)$ to equation (4.1) with initial data $n_0 \in \mathcal{M}(\mathbb{R}^+)$ satisfies*

$$\|e^{-\lambda t} n_t - M n_*\|_V \le C e^{-\rho t} \|n_0 - M n_*\|_V \qquad \textit{for all } t \ge 0, \tag{4.9}$$

for $M = \int \phi(x) n_0(x)\,\mathrm{d}x > 0$ a positive constant and $n_*(x) = N(x)$ where $N(x)$ and $\phi(x)$ are the eigenfunction and the dual eigenfunction, solution to (4.10)-(4.11). Moreover, the norm $\|\cdot\|_\beta$ is the weighted total variation norm defined by

$$\|\mu\|_V = \int_0^{+\infty} V(x)|\mu|(\,\mathrm{d}x),$$

4.2 Existence of Perron eigenelements

In Chapter 1, we introduced the Perron eigenvalue problem (1.10)-(1.11) which we recall the definition below:

$$\begin{aligned} \frac{\partial}{\partial x}(g(x)N(x)) + (B(x)+\lambda)N(x) &= \int_x^{+\infty} \frac{B(y)}{y} p\left(\frac{x}{y}\right) N(y)\,\mathrm{d}y, \\ g(0)N(0) = 0, \quad N(x) \ge 0, \quad & \int_0^{+\infty} N(x)\,\mathrm{d}x = 1. \end{aligned} \tag{4.10}$$

$$\begin{aligned} -g(x)\frac{\partial}{\partial x}\phi(x) + (B(x)+\lambda)\phi(x) &= \frac{B(x)}{x}\int_0^x p\left(\frac{y}{x}\right)\phi(y)\,\mathrm{d}y, \\ \phi(x) \ge 0, \quad & \int_0^{+\infty} \phi(x)N(x)\,\mathrm{d}x = 1. \end{aligned} \tag{4.11}$$

If there exists a solution to (4.10)-(4.11), that is a triple $(\lambda, N(x), \phi(x))$, which are called *eigenelements*, then the equation (4.1) converges to a universal profile whose shape is given by the eigenfunction $N(x)$ and growth rate of the population is given by the dominated eigenvalue $\lambda > 0$. If we scale the equation (4.1) by defining

$m(t,x) := n(t,x)e^{-\lambda t}$ and under Assumptions 4.1.1, 4.1.2 and 4.1.3 we obtain:

$$\begin{aligned} \frac{\partial}{\partial t}m(t,x) + \frac{\partial}{\partial x}(g(x)m(t,x)) + c(x)m(t,x) &= \mathcal{A}(t,x), \quad t,x \geq 0, \\ m(t,0) &= 0, \qquad\qquad t > 0, \\ m(0,x) &= n_0(x), \qquad x > 0. \end{aligned} \tag{4.12}$$

where

$$c(x) := B(x) + \lambda$$

and

$$\mathcal{A}(t,x) := \int_x^{+\infty} \frac{B(y)}{y} p\left(\frac{x}{y}\right) m(t,y)\,\mathrm{d}y,$$

with $p(z) = 2$ or $p(z) = 2\delta_{1/2}(z)$.

Notice that $N(x)$ is the stationary state for (4.12).

If the eigenelements exist, we can consider equation (4.12) instead of (4.1) and study the long-time behaviour of the former. We can easily recover the nature of (4.1) through (4.12) .

We define the *fragmentation operator* acting only on x:

$$\mathcal{F}[m](t,x) := \mathcal{F}_+[m](t,x) - c(x)m(t,x)$$

where

$$\mathcal{F}_+[m](t,x) := \mathcal{A}(t,x),$$

is the positive part. Then, we define a linear operator by

$$\mathcal{L}[m](t,x) := \frac{\partial}{\partial t}m(t,x) = -\frac{\partial}{\partial x}(g(x)m(t,x) + \mathcal{F}[m](t,x),$$

which describes the evolution of (4.12). We notice that λ is the first positive eigenvalue of $\mathcal{L}$ and $N(x)$ is the corresponding eigenvector. We also define the adjoint of $\mathcal{L}$ by

$$\mathcal{L}^*[\phi](x) := g(x)\frac{\partial}{\partial x}\phi(x) + \frac{B(x)}{x}\int_0^x p\left(\frac{y}{x}\right)\phi(y)\,\mathrm{d}y - c(x)\phi(x). \tag{4.13}$$

We also note that the quantity $f(t,x) := \phi(x)m(t,x)$ is conserve for the equation (4.12) such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \phi(x)m(t,x)\,\mathrm{d}x = 0.$$

4.2.1 Bound on the dual eigenfunction ϕ

Now, we prove a theorem that implies existence and boundedness of the dual eigenfunction ϕ which is a solution to the dual eigenproblem given by (4.11). First, we state the main theorem of this section and give the poof after several lemmas and a corollary;

Theorem 4.2.1 (bounds on the eigenfunction ϕ)**.** *We make Assumptions 4.1.1, 4.1.2 and 4.1.3. There exists a solution to* (4.11) *with* $0 < \phi(x) < 1 + x^k$ *for* $k > 1$.

We prove this theorem at the end of the section.

We begin with defining a truncated version of the Perron eigenproblem (4.10)-(4.11) in an interval $[0, R]$ for some $R > 0$:

$$\begin{gathered} \frac{\partial}{\partial x}\left(g(x)N_R(x)\right) + (B(x) + \lambda_R)N_R(x) = \int_x^R \frac{B(y)}{y} p\left(\frac{x}{y}\right) N_R(y)\,\mathrm{d}y, \\ g(0)N_R(0) = 0, \quad N_R(x) \geq 0, \quad \int_0^R N_R(x)\,\mathrm{d}x = 1. \end{gathered} \tag{4.14}$$

$$\begin{gathered} -g(x)\frac{\partial}{\partial x}\phi_R(x) + (B(x) + \lambda_R)\phi_R(x) = \frac{B(x)}{x}\int_0^R p\left(\frac{y}{x}\right)\phi_R(y)\,\mathrm{d}y, \\ \phi_R(x) \geq 0, \quad \int_0^R \phi_R(x)N_R(x)\,\mathrm{d}x = 1. \end{gathered} \tag{4.15}$$

Now we give some lemmas which will be used in the proof of Theorem 4.2.1. The existence of a weak solution to (4.14)-(4.15) is proved by the Krein-Rutman theorem in the Appendix of [55]. Moreover in [4], the authors proved that there exists $R_0 > 0$ large enough such that for all $R > R_0$ we have $\lambda_R > 0$. They used (4.14) to pass to the limit as $R \to +\infty$ which requires assuming polynomial growth for the total division rate B as $x \to +\infty$. Here we want to relax this assumption, that is why we pass to the limit as $R \to +\infty$ by using (4.15). In order to do that we need to prove some bounds on the truncated eigenfunction ϕ_R.

First we recall a maximum principle. We begin with defining an operator $\mathcal{L}_R$,

$$\mathcal{L}_R\varphi(x) := -g(x)\varphi'(x) + (\lambda_R + B(x))\,\varphi(x) - \frac{B(x)}{x}\int_0^x p\left(\frac{y}{x}\right)\varphi(y)\,\mathrm{d}y. \tag{4.16}$$

We have the following maximum principle like in Lemma C.1. in Appendix C of [55]:

Lemma 4.2.1. *Suppose that* $\varphi(x) \geq 0$ *for* $x \in [0, A]$ *for some* $A \in (0, R)$ *with* $\varphi(R) \geq 0$ *and* $\mathcal{L}_R\varphi(x) > 0$ *on* $[A, R]$. *Then* $\varphi(x) \geq 0$ *on* $[0, R]$.

Proof. Proof is the same as in in Lemma C.1. in Appendix C of [55] where 0 is the supersolution of (4.16). □

Next lemmas are on the boundedness of the truncated dual eigenfunction ϕ_R, the truncated eigenvalue λ_R and $| \phi_R' |$ respectively:

Lemma 4.2.2. *Under Assumptions 4.1.1, 4.1.2 and 4.1.3, for all $R > R_0$ and for all $x \in [0, R]$ we have*

$$0 \leq \phi_R(x) \leq 1 + x^k,$$

for some $k > 1$.

Proof. For the bound below we want to use the maximum principle in Lemma 4.2.1. Therefore we want to prove $\mathcal{L}_R\varphi(x) > 0$ for $x \in (A, R)$ for such $A \in (0, R)$ as in the Lemma 4.2.1.

We take $\varphi(x) = 1 + x^k$ for some $k > 1$. Then for $R \geq R_0$ we have

$$\begin{aligned} \mathcal{L}_R\varphi(x) &= \lambda_R(1 + x^k) - kg(x)x^{k-1} + B(x)\left((1 + x^k) - \frac{1}{x}\int_0^x (1 + y^k)p\left(\frac{y}{x}\right)\,\mathrm{d}y\right) \\ &= \lambda_R(1 + x^k) - kg(x)x^{k-1} + B(x)(1 + x^k - p_0 - x^k p_k) \\ &> x^{k-1}\left(-kg(x) - B(x)x^{1-k} + (1 - p_k)B(x)x\right) := \rho(x) \end{aligned} \tag{4.17}$$

since $p_0 = 2$ and $0 < p_k < 1 = p_1$ for $k > 1$. Moreover assuming (4.8) gives that behaviour of ρ will be dominated by the positive term $(1 - p_k)B(x)x^k > 0$. Therefore, we can find $A(k) > 0$ such that for all $A(k) < x < R$, we have $\mathcal{L}_R\varphi(x) > 0$. We fix $A > 0$ as above and normalize ϕ_R such that

$$\sup_{x\in[0,A]} \phi_R(x) = 1. \tag{4.18}$$

Then by the maximum principle (4.16) we obtain the result. □

Lemma 4.2.3. *Under Assumptions 4.1.1, 4.1.2 and 4.1.3, there exists a constant $C > 0$ such that $\lambda_R \leq C$ for all $R > R_0$.*

Proof. Since ϕ_R is continuous and by (4.18), there exists $x_R \in [0, A]$ such that $\phi_R(x_R) = 1$. Moreover, the equation $\mathcal{L}_R\phi_R = 0$ ensures that for all $x > 0$ we have

$$\begin{aligned} &\left(\phi_R(x)\exp\left(-\int_A^x \frac{\lambda_R + B(s)}{g(s)}\,\mathrm{d}s\right)\right)' \\ &\qquad = -\frac{B(x)}{xg(x)}\exp\left(-\int_A^x \frac{\lambda_R + B(s)}{g(s)}\,\mathrm{d}s\right)\int_0^x p\left(\frac{y}{x}\right)\phi_R(y)\,\mathrm{d}y \end{aligned}$$

By integrating this from x_R to $x \geq A$;

$$\begin{aligned}
&\phi_R(x)\exp\left(-\int_A^x \frac{\lambda_R + B(s)}{g(s)}\,\mathrm{d}s\right) - \phi_R(x_R)\exp\left(-\int_A^{x_R} \frac{\lambda_R + B(s)}{g(s)}\,\mathrm{d}s\right) \\
&\quad = \phi_R(x)\exp\left(-\int_A^x \frac{\lambda_R + B(s)}{g(s)}\,\mathrm{d}s\right) - \exp\left(\int_{x_R}^A \frac{\lambda_R + B(s)}{g(s)}\,\mathrm{d}s\right) \\
&\quad = -\int_{x_R}^x \frac{B(y)}{yg(y)}\exp\left(-\int_A^y \frac{\lambda_R + B(s)}{g(s)}\,\mathrm{d}s\right)\int_0^y p\left(\frac{z}{y}\right)\phi_R(z)\,\mathrm{d}z\,\mathrm{d}y
\end{aligned}$$

By using the upper bound on ϕ_R we obtain, for $R > R_0$,

$$\begin{aligned}
&\phi_R(x)\exp\left(-\int_A^x \frac{\lambda_R + B(s)}{g(s)}\,\mathrm{d}s\right) \\
&= \exp\left(\int_{x_R}^A \frac{\lambda_R + B(s)}{g(s)}\,\mathrm{d}s\right) - \int_{x_R}^x \frac{B(y)}{yg(y)}\exp\left(-\int_A^y \frac{\lambda_R + B(s)}{g(s)}\,\mathrm{d}s\right)\int_0^y p\left(\frac{z}{y}\right)\phi_R(z)\,\mathrm{d}z\,\mathrm{d}y \\
&\qquad \geq 1 - \int_{x_R}^x \frac{B(y)}{yg(y)}\exp\left(-\int_A^y \frac{\lambda_R + B(s)}{g(s)}\,\mathrm{d}s\right)\int_0^y p\left(\frac{z}{y}\right)(1 + z^k)\,\mathrm{d}z\,\mathrm{d}y \\
&\qquad\qquad \geq 1 - \int_{x_R}^x \frac{B(y)}{g(y)}\exp\left(-\int_A^y \frac{\lambda_R + B(s)}{g(s)}\,\mathrm{d}s\right)\left(p_0 + y^k p_k\right)\,\mathrm{d}y
\end{aligned}$$

Since $\phi_R(R) = 0$ we deduce that for all $R > R_0$, denoting

$$\Theta := \sup_{y \geq 0} \frac{B(y)}{g(y)}\exp\left(-\int_A^y \frac{B(s)}{g(s)}\,\mathrm{d}s\right),$$

$$\begin{aligned}
0 &\geq 1 - \int_{x_R}^R \frac{B(y)}{g(y)}\exp\left(-\int_A^y \frac{\lambda_R + B(s)}{g(s)}\,\mathrm{d}s\right)\left(p_0 + y^k p_k\right)\,\mathrm{d}y \\
&\geq 1 - \Theta\int_0^\infty (p_0 + y^k p_k)\exp\left(-\lambda_R\int_A^y \frac{1}{g(s)}\,\mathrm{d}s\right)\,\mathrm{d}y.
\end{aligned}$$

By monotone convergence theorem, we have

$$\Theta\int_0^\infty (p_0 + y^k p_k)\exp\left(-\lambda\int_A^y \frac{1}{g(s)}\,\mathrm{d}s\right)\,\mathrm{d}y \xrightarrow[\lambda \to +\infty]{} 0$$

so the inequality

$$\Theta\int_0^\infty (p_0 + y^k p_k)\exp\left(-\lambda_R\int_A^y \frac{1}{g(s)}\,\mathrm{d}s\right)\,\mathrm{d}y \geq 1$$

enforces λ_R to be bounded from above. □

We remark that in [4] the proof of the positivity of λ_R is done by using the equation on N_R and this requires an assumption of polynomial growth for B at infinity.

Lemma 4.2.4. *Under Assumptions 4.1.1, 4.1.2 and 4.1.3,* $|\ \phi_R'\ |$ *is bounded.*

Proof of Theorem 4.2.1. By the equation $\mathcal{L}_R\phi_R = 0$ and bounds on ϕ_R and λ_R we obtain

$$
\begin{aligned}
|\ \phi_R'\ | &= \frac{\lambda_R\phi_R}{g(x)} + \frac{B(x)}{g(x)}\left|\phi_R - \frac{1}{x}\int_0^x p\left(\frac{y}{x}\right)\phi_R(y)\,\mathrm{d}y\right| \\
&\leq \frac{\lambda_R}{g(x)}(1+x^k) + \frac{B(x)}{g(x)}\left|1+x^k - \frac{1}{x}(1+x^k)\int_0^x p\left(\frac{y}{x}\right)\,\mathrm{d}y\right| \\
&\leq \frac{\lambda_R}{g(x)}(1+x^k) + \frac{B(x)}{g(x)}\left|1+x^k - \frac{1}{x}(1+x^k)xp_0\right| \leq \frac{\lambda_R}{g(x)}(1+x^k) + \frac{B(x)}{g(x)}(1-p_0).
\end{aligned}
$$

□

Finally we end this section with the proof of Theorem 4.2.1:

Proof of Theorem 4.2.1. Lemmas 4.2.2, 4.2.3 and 4.2.4 give the proof. Since there exists a solution to the truncated Perron eigenproblem (4.14)-(4.15) for $R > 0$ by the Krein-Rutman theorem, it only remains to prove bounds in order to pass to the limit as $R \to +\infty$. We show the bounds on ϕ_R, λ_R and ϕ_R' by Lemmas 4.2.2, 4.2.3, 4.2.4 respectively.

These bounds ensure that we can extract a subsequence of (λ_R) which converges to $\lambda > 0$ and a subsequence of (ϕ_R) which converges locally uniformly to a limit ϕ which satisfies $0 \leq \phi(x) \leq 1 + x^k$. Clearly (λ, ϕ) is the solution to the dual Perron eigenproblem (1.11), and $\phi \not\equiv 0$ since $\sup_{x\in[0,A]} \phi(x) = 1$. □

4.3 Hypotheses for Harris's Theorem

We dedicate this section to verify Hypotheses 2.2.2 and 2.2.3 from Chapter 2 and then we give a proof for Theorem 4.1.2.

4.3.1 Lyapunov condition

In this part, we prove that Lyapunov condition 2.2.2 is satisfied for (4.12). Due to the special nature of the case with the linear growth, we treat it first, separately than the general case. Thus, in this particular case (4.12) takes the form

$$\frac{\partial}{\partial t}m(t,x)+\frac{\partial}{\partial x}(xm(t,x))=2\int_x^\infty\frac{B(y)}{y}m(t,y)\,\mathrm{d}y-(B(x)+1)m(t,x) \tag{4.19}$$

coupled with the usual initial and boundary conditions.

We remark that we consider only the constant fragmentation kernel in this case since with the mitosis kernel there is no convergence to a universal profile. Also note that for(4.19), the eigenvalue and the dual eigenfunction are know explicitly, which are $\lambda = 1$ and $\phi(x) = x$.

Lemma 4.3.1. *We consider* (4.19) *under the assumptions 4.1.1, 4.1.2 and 4.1.3. For* $V(x) = 1 + x^{k-1} + x^{K-1}$ *where* $k < 1$, $K > 1$ *and* $f(t,x) := xm(t,x)$ *there exist some time* $t_0 > 0$, $C_1, \tilde{C} > 0$ *such that for all* $t \geq t_0$*:*

$$\int_0^{+\infty}V(x)f(t_0,x)\,\mathrm{d}x\leq e^{-C_1t_0}\int_0^{+\infty}V(x)f_0(x)\,\mathrm{d}x+\tilde{C}\int_0^{+\infty}f_0(x)\,\mathrm{d}x. \tag{4.20}$$

Proof. We have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{+\infty}\left(x^k+x^K\right)m(t,x)\,\mathrm{d}x\\
&=-\int_0^{+\infty}\left(x^k+x^K\right)\frac{\partial}{\partial x}\left(xm(t,x)\right)\,\mathrm{d}x-\int_0^{+\infty}\left(x^k+x^K\right)(B(x)+1)m(t,x)\,\mathrm{d}x\\
&+2\int_0^{+\infty}\left(x^k+x^K\right)\int_x^{+\infty}\frac{B(y)}{y}m(t,y)\,\mathrm{d}y\,\mathrm{d}x\\
&=-\frac{1}{2}(1-k)\int_0^{+\infty}(x^{k-1}+x^{K-1})xm(t,x)\,\mathrm{d}x\\
&+\int_0^{+\infty}\left(c_1B(x)x^{K-1}+c_2x^{K-1}+c_3B(x)x^{k-1}+c_4x^{k-1}\right)xm(t,x)\,\mathrm{d}x
\end{aligned}$$

where

$$-1<c_1:=\frac{K-1}{K+1}<0,\; c_2:=K-\frac{k+1}{2}>0,\; 0<c_3:=\frac{1-k}{1+k}<1,\; c_4:=\frac{k-1}{2}<0.$$

We define $\varphi(x) := c_1B(x)x^{K-1} + c_2x^{K-1} + c_3B(x)x^{k-1} + c_4x^{k-1}$ and notice that when $x \to +\infty$, the first term will dominate the behaviour of φ; thus it will approach to $-\infty$. Similarly when $x \to 0$, the last term will dominate the behaviour of φ, which is negative as well. Since B is continuous and by the assumption 4.1.3 we can always bound $\sup_{x\geq 0}\varphi(x) \leq C_2$ with some positive quantity $C_2 > 0$. Therefore by denoting

$f(t,x) = xm(t,x)$ we obtain;

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{+\infty}(1+x^{k-1}+x^{K-1})f(t,x)\,\mathrm{d}x \leq -C_1\int_0^{+\infty}(1+x^{k-1}+x^{K-1})f(t,x)\,\mathrm{d}x + \tilde{C}\int_0^{+\infty} f_0(x)\,\mathrm{d}x \quad (4.21)$$

where $C_1 = \frac{1}{2}(1-k) > 0$, $\tilde{C} = C_1 + C_2 > 0$ and since $\int f(t,x)dx = \int f_0(x)dx$. This gives (4.20) with $\tilde{C} = 1 + \frac{C_2}{C_1}$. □

Now, we consider the general case (1.12):

Lemma 4.3.2. *We consider (4.12) under the assumptions 4.1.1, 4.1.2 and 4.1.3. We take $K > 1-\alpha_0$. Then the following holds true*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{+\infty} x^K m(t,x)\,\mathrm{d}x \leq -C_1\int_0^{+\infty} x^K m(t,x)\,\mathrm{d}x + C_2\int_0^{+\infty} m(t,x)\,\mathrm{d}x \quad (4.22)$$

where $C_1 = \lambda > 0$ and $C_2 > 0$.

Proof. We have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}&\int_0^{+\infty} x^K m(t,x)\,\mathrm{d}x\\ &= -\int_0^{+\infty} x^K \frac{\partial}{\partial x}\left(g(x)m(t,x)\right)\,\mathrm{d}x - \int_0^{+\infty} x^K (B(x)+\lambda)m(t,x)\,\mathrm{d}x\\ &+ \int_0^{+\infty} x^K \int_x^{+\infty}\frac{B(y)}{y}p\left(\frac{x}{y}\right)m(t,y)\,\mathrm{d}y\,\mathrm{d}x\\ &= -\lambda\int_0^{+\infty} x^K m(t,x)\,\mathrm{d}x + \int_0^{+\infty}\Big((p_K-1)x^K B(x) + Kx^{K-1}g(x)\Big)\,m(t,x)\,\mathrm{d}x\end{aligned}$$

We define $\varphi(x) := (p_K-1)x^K B(x) + Kx^{K-1}g(x)$ and notice that $\sup_{x\geq 0}\varphi(x) \leq C_2$ for some $C_2 > 0$ because of the assumption (4.8), concerning the behaviour of $\frac{xB(x)}{g(x)}$ when $x \to +\infty$ and $x \to 0$, where $\alpha_0 < 1$. □

Corollary 4.3.1. *We consider (4.12) under the assumptions 4.1.1, 4.1.2 and 4.1.3. For $V(x) = 1 + \frac{x^K}{\phi(x)}$ where $K > 1-\alpha_0$ and $f(t,x) := \phi(x)m(t,x)$ there exists some time $t_0 > 0$, $C_1, \tilde{C} > 0$ and for all $t \geq t_0$ such that*

$$\int_0^{+\infty} V(x)f(t_0,x)\,\mathrm{d}x \leq e^{-\frac{C_1}{2}t_0}\int_0^{+\infty} V(x)f_0(x)\,\mathrm{d}x + \tilde{C}\int_0^{+\infty} f_0(x)\,\mathrm{d}x. \quad (4.23)$$

Proof. By adding $\phi(x)$ of both sides of (4.22) we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{+\infty} x^K m(t,x)\,\mathrm{d}x &= \frac{\mathrm{d}}{\mathrm{d}t}\int_0^{+\infty}(x^K+\phi(x))m(t,x)\,\mathrm{d}x\\ &\leq -\frac{C_1}{2}\int_0^{+\infty}(x^K+\phi(x))m(t,x)\,\mathrm{d}x + \frac{C_1}{2}\int_0^{+\infty}\phi(x)m(t,x)\,\mathrm{d}x\\ &\qquad + \int_0^{+\infty}\Big(C_2 - \frac{C_1}{2}x^K\Big)m(t,x)\,\mathrm{d}x,\end{aligned}$$

that is

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{+\infty} x^K m(t,x)\,\mathrm{d}x\\ &\leq -\frac{C_1}{2}\int_0^{+\infty}(x^K+\phi(x))m(t,x)\,\mathrm{d}x+\frac{C_1}{2}\int_0^{+\infty}\phi(x)m(t,x)\,\mathrm{d}x+C_3\int_0^{+\infty}\phi(x)m(t,x)\,\mathrm{d}x.\end{aligned}$$

Therefore, we have for $f(t,x)=\phi(x)m(t,x)$;

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{+\infty}\left(1+\frac{x^K}{\phi(x)}\right)f(t,x)\,\mathrm{d}x\\ &\qquad\leq -\frac{C_1}{2}\int_0^{+\infty}\left(1+\frac{x^K}{\phi(x)}\right)f(t,x)\,\mathrm{d}x + \left(\frac{C_1}{2}+C_3\right)\int_0^{+\infty}f_0(x)\,\mathrm{d}x,\end{aligned}$$

since $\int f(t,x)dx = \int f_0(x)dx$. This gives (4.23) with $\tilde{C} = 1+\frac{2C_3}{C_1}$. □

Remark that we verified the Lyapunov condition 2.2.2;

Lemma 4.3.3 (Lyapunov condition). *Hypothesis 2.2.2 is verified for* (4.12).

Proof. We consider (4.19), and by Lemma 4.3.1 we have the Lyapunov condition satisfied for $\gamma = e^{-C_1 t_0}$ and $K = \tilde{C}\int f_0(x)\,\mathrm{d}x > 0$ with $V(x) = 1 + x^k + x^K$, where $k < 1$ and $K > 1$.

Similarly for (4.12), excluding the case above, by Corollary 4.3.1 we obtain the result $\gamma = e^{-\frac{C_1}{2}t_0}$ and $K = \tilde{C}\int f_0(x)\,\mathrm{d}x > 0$ with $V(x) = \phi(x) + x^K$ where $K > 1 - \alpha_0$. □

4.3.2 Minorisation condition

In this section, we show that (4.12) either with equal mitosis or uniformly distributed fragmentation process satisfies the minorisation condition 2.2.3 given in Chapter 2.

We start by recalling some known results on the solution of the transport part of (4.12). We consider the equation

$$
\begin{aligned}
\frac{\partial}{\partial t} m(t,x) + \frac{\partial}{\partial x}(g(x)m(t,x)) &= -c(x)m(t,x), && t,x>0,\\
m(t,0) &= 0, && t>0,\\
m(0,x) &= n_0(x), && x>0,
\end{aligned}
\tag{4.24}
$$

which is the same as (4.12) without the positive part of the fragmentation operator. We remark that Assumption 4.1.3 ensures that the characteristic ordinary differential equation

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} X_t(x_0) &= g(X_t(x_0)),\\
X_0(x_0) &= x_0,
\end{aligned}
\tag{4.25}
$$

has a unique solution, defined for $t \in [0,+\infty)$, for any initial condition $x_0 > 0$. In fact, it is defined in some interval $(t_*(x_0), +\infty)$, for some $t_*(x_0) < 0$. If $1/g(x)$ is locally integrable close to 0, the solution can be explicitly given in terms of H^{-1}, where

$$
H(x) := \int_0^x \frac{1}{g(y)}\,\mathrm{d}y, \qquad x \geq 0.
$$

We notice that H is strictly increasing with $H(0) = \lim_{x\to 0} H(x) = 0$ and $\lim_{x\to+\infty} H(x) = +\infty$ (since g grows sublinearly as $x \to +\infty$), so that it is invertible. It can easily be checked that

$$
X_t(x_0) = H^{-1}(t + H(x_0)) \qquad \text{for } x_0 > 0 \text{ and } t > -H(x_0), \tag{4.26}
$$

so that that the maximal interval where the solution of (4.25) is defined is precisely $(-H(x_0), +\infty)$.

For each $t \geq 0$, this defines the *flow map* $X_t \colon (0,+\infty) \to (H^{-1}(t), +\infty)$, which is strictly increasing. For negative times, we may consider $X_{-t} \colon (H^{-1}(t), +\infty) \to (0,+\infty)$ (where $t > 0$). Of course, $X_{-t} = (X_t)^{-1}$.

If n_0 is a nonnegative measure, it is well known that the unique measure solution to (4.24) is given by

$$
\begin{aligned}
m(t,x) = X_t\#n_0(x) \exp\left(-\int_0^t c(X_{-\tau}(x))\,\mathrm{d}\tau\right), &\qquad t \geq 0,\ x > H^{-t}(t),\\
m(t,x) = 0, &\qquad t \geq 0,\ x \leq H^{-t}(t),
\end{aligned}
\tag{4.27}
$$

where we abuse notation by evaluating the measures $m(t,\cdot)$ and $X_t \# n_0$ at a point $x>0$. For a Borel measurable map $X\colon (0,+\infty) \to (0,+\infty)$, the expression $X\#n_0$ denotes the *transport*, or *push forward*, of the measure n_0 by the map X, defined by duality through

$$\int_0^\infty \varphi(x) X\#n_0(x)\,\mathrm{d}x := \int_0^\infty \varphi(X(y)) n_0(y)\,\mathrm{d}y$$

for all continuous, compactly supported $\varphi\colon (0,+\infty) \to \mathbb{R}$. If additionally n_0 is a function and X has a left inverse $X^{-1}\colon (a,b) \to (0,+\infty)$, one has

$$X\#n_0(x) = \begin{cases} n_0(X^{-1}(x)) \left|\frac{\mathrm{d}}{\mathrm{d}x}(X^{-1})(x)\right| & \text{if } x \in (a,b), \\ 0 & \text{otherwise.} \end{cases}$$

Using this for the solution to (4.24), if n_0 is a function we may write m in the equivalent form

$$\begin{aligned} m(t,x) = n_0(X_{-t}(x)) \frac{\mathrm{d}}{\mathrm{d}x} X_{-t}(x) \exp\left(- \int_0^t c(X_{-\tau}(x))\,\mathrm{d}\tau \right), &\qquad t \geq 0,\ x > H^{-t}(t), \\ m(t,x) = 0, &\qquad t \geq 0,\ x \leq H^{-t}(t). \end{aligned} \tag{4.28}$$

Using that $Y_t(x) := \frac{\mathrm{d}}{\mathrm{d}x} X_t(x)$ satisfies $\frac{\mathrm{d}}{\mathrm{d}t} Y_t(x) = g'(X_t(x)) Y_t(x)$, we note for later that

$$\frac{\mathrm{d}}{\mathrm{d}x} X_{-t}(x) = \exp\left(- \int_0^t g'(X_{-\tau}(x))\,\mathrm{d}\tau, \right) \qquad t \geq 0,\ x > H^{-t}(t). \tag{4.29}$$

Uniform fragment distribution

In this section, we consider the case $p(z) = 2$, corresponding to $\kappa(x,y) = \frac{2}{x} B(x) \mathbb{1}_{\{0 \leq x \leq y\}}$. The growth-fragmentation equation (4.1) in this case is widely studied and depending on some assumptions made on growth and division rates existence (in some cases exact values) of eigenelements are known.

If we consider a linear growth $g(x) = g_0 x$ ($\alpha = 1$) and a power like total division $B(x) = b_0 x^\gamma$ with $\gamma > 0$, and $g_0, b_0 > 0$, the eigenvalue and the corresponding dual eigenfunction are given by

$$\lambda = 1 \text{ and } \phi(x) = \frac{x}{\int yN(y)}.$$

In this case, egenelements can be computed explicitly (depending on the value of γ) and given in Table 1 of [55].

$\gamma = 1$	$\lambda = g_0$	$N(x) = \frac{b_0}{g_0}\exp\left(-\frac{b_0}{g_0}x\right)$	$\phi(x) = \frac{b_0}{g_0}x$
$\gamma = 2$	$\lambda = g_0$	$N(x) = \sqrt{\frac{2b_0}{\pi g_0}}\exp\left(-\frac{1}{2}\frac{b_0}{g_0}x^2\right)$	$\phi(x) = \sqrt{\frac{\pi b_0}{2g_0}}x$
γ	$\lambda = g_0$	$N(x) = \left(\frac{b_0}{\eta g_0}\right)^{1/\gamma}\frac{\gamma}{\Gamma\left(\frac{1}{\gamma}\right)}\exp\left(-\frac{1}{\gamma}\frac{b_0}{g_0}x^\gamma\right)$	$\phi(x) = \left(\frac{b_0}{\gamma g_0}\right)^{1/\gamma}\frac{\Gamma\left(\frac{1}{\gamma}\right)}{\Gamma\left(\frac{2}{\gamma}\right)}x$

Table 4.1 Explicit eigenelements for the growth-fragmentation equation.

The case where $g(x) = g_0 x$ and $B(x) = b_0 x^\gamma$ with $\gamma > 0$ and for $p(z) = 2$.

Moreover, in [4], the authors gave the asymptotics of the profile N and provided accurate bounds on the dual eigenfunction ϕ in a more general form of the growth-fragmentation equation where growth and total division rates behave like a power law for large and small x. This also contains the case in Table 4.1 above. In this work we relax these assumptions a bit.

We consider the scaled growth-fragmentation equation with the uniform fragment distribution;

$$\begin{aligned}
\frac{\partial}{\partial t}m(t,x) + \frac{\partial}{\partial x}(x^\alpha m(t,x)) &= 2\int_x^\infty \frac{B(y)}{y}m(t,y)\,\mathrm{d}y - (B(x)+\lambda)m(t,x), \quad t,x \geq 0,\\
m(t,0) &= 0, \qquad\qquad\qquad\qquad\qquad\qquad t > 0,\\
m(0,x) &= n_0(x), \qquad\qquad\qquad\qquad\qquad x > 0.
\end{aligned} \tag{4.30}$$

Lemma 4.3.4 (Lower bound for constant fragment distribution)**.** *We make Assumptions 4.1.1, 4.1.2 and 4.1.3 with $p(z) = 2$. Let $(\mathcal{S}_t)_{t\geq 0}$ be the Markov semigroup associated to equation (4.30). For all $\beta > 0$ given, there exists $t_B > 0$ such that for all $t > t_B$ and $x_0 \in [0,\beta]$ it holds that*

$$\mathcal{S}_t\delta_{x_0}(x) \geq C(t,\beta)$$

for all x in an open interval I_t which depends on the time t, and for some quantity $C = C(t,\beta)$ depending only on t and β.

Proof. Calling $(\mathcal{T}_t)_{t\geq 0}$ the semigroup associated to the transport equation

$$\frac{\partial}{\partial t}m(t,x) + \frac{\partial}{\partial x}(g(x)m(t,x)) + c(x)m(t,x) = 0, \tag{4.31}$$

where $c(x) = B(x) + \lambda$, by Duhamel's formula we have

$$\mathcal{S}_t n_0(x) = m(t,x) = \mathcal{T}_t n_0(x) + \int_0^t \mathcal{T}_{t-\tau}(\mathcal{A}(\tau,.))(x)\,\mathrm{d}\tau, \tag{4.32}$$

where $\mathcal{A}(t,x) := 2\int_x^\infty \frac{B(y)}{y} m(t,y)\,\mathrm{d}y$. Taking $n_0 = \delta_{x_0}$ with $x_0 \in (0,\beta]$, the first bound gives

$$\begin{aligned}\mathcal{S}_t\delta_{x_0} \geq \mathcal{T}_t\delta_{x_0} &= X_t\#\delta_{x_0}\exp\left(-\int_0^t c(X_{-\tau}(x))\,\mathrm{d}\tau\right),\\ &= \delta_{X_t(x_0)}\exp\left(-\int_0^t c\left(X_{-\tau}\left(X_t(x_0)\right)\right)\,\mathrm{d}\tau\right),\end{aligned}$$

where we have used the expression of $\mathcal{T}_t$ given in (4.27). Since for some $C_1 = C_1(t,\beta) > 0$ (increasing in t and β) we have

$$c(X_{t-\tau}(x_0)) = B(X_{t-\tau}(x_0)) + \lambda \leq 1 + \sup_{y\leq X_t(\beta)} c(y) := C_1(t,\beta)$$

we deduce that

$$\mathcal{S}_t\delta_{x_0} \geq \delta_{X_t(x_0)}e^{-C_1 t}. \tag{4.33}$$

Using this we obtain

$$\mathcal{A}(t,x) \geq 2e^{-C_1 t}\frac{B(X_t(x_0))}{X_t(x_0)} \quad \text{for all } t>0 \text{ and } x < X_t(x_0).$$

We use that there is some $x_B > 0$ for which B is bounded below by a positive quantity on any interval of the form $[x_B, R]$. There is some $t_B > 0$ such that for $t > t_B$ we have $X_t(x_0) > x_B$ for all $x_0 > 0$. Hence, for some $C_2 = C_2(\beta)$,

$$\mathcal{A}(t,x) \geq C_2 e^{-C_1 t} \qquad \text{for all } t > t_B \text{ and } x < X_t(x_0).$$

As a consequence, using (4.27) and (4.29),

$$\mathcal{T}_{t-\tau}\mathcal{A}(\tau,x) \geq C_2 e^{-C_1\tau}\exp\left(-\int_0^{t-\tau} g'(X_{-s}(x))\,\mathrm{d}s\right),$$

for all $\tau > t_B$ and $H^{-1}(t-\tau) < x < X_t(x_0)$. In particular, the bound is true for all $\tau > t_B$ and $H^{-1}\left(t - \frac{t_B}{2}\right) < x < H^{-1}(t)$. In this range, and for $0 < s < t-\tau$ we have (using (4.25))

$$H^{-1}\left(\frac{t_B}{2}\right) \leq X_{\tau-t}(x) \leq X_{-s}(x) \leq x \leq H^{-1}(t),$$

Using that $g'(X) \leq C_3$ for all $X \in \left[H^{-1}\left(\frac{t_B}{2}\right), H^{-1}(t)\right]$ we have

$$\mathcal{T}_{t-\tau}\mathcal{A}(\tau, x) \geq C_2 e^{-C_1\tau} e^{-C_3(t-\tau)} \geq C_2 e^{-C_4 t}$$

for all $\tau > t_B$ and $H^{-1}\left(t - \frac{t_B}{2}\right) < x < H^{-1}(t)$. A final integration gives, for $H^{-1}\left(t - \frac{t_B}{2}\right) < x < H^{-1}(t)$,

$$\mathcal{S}_t\delta_{x_0}(x) \geq \int_0^t \mathcal{T}_{t-\tau}(\mathcal{A}(\tau, \cdot))(x)\,\mathrm{d}\tau \geq C_2 e^{-C_4 t}\int_{t_B}^t \mathrm{d}\tau = C_2 e^{-C_4 t}(t - t_B). \qquad \square$$

This gives the result.

Remark 4.3.1. *The above argument relies on defining*

$$H(x) := \int_0^x \frac{1}{g(y)}\,\mathrm{d}y, \qquad x \geq 0.$$

where H is strictly increasing with $H(0) = \lim_{x\to 0} H(x) = 0$ and $\lim_{x\to+\infty} H(x) = +\infty$. So it excludes the case with the linear growth $g(x) = x$. We treat this case separately below.

We give the following lemma concerning a lower bound for (4.19).

Lemma 4.3.5 (Lower bound for the linear growth rate)**.** *We make Assumptions 4.1.1 and 4.1.2 with $g(x) = x$, $p(z) = 2$. Let $(\mathcal{S}_t)_{t\geq 0}$ be the Markov semigroup associated to equation (4.19). We also assume that the total division rate B is continuous and bounded below on a compact interval, that is; there exists $C_2(\beta) > 0$ such that*

$$B(y) \geq C_2(\beta) \qquad \textit{for all } y \in [0, \beta]. \tag{4.34}$$

For all $\alpha, \beta > 0$ given, there exists $t_B > 0$ such that for all $t > t_B$ and $x_0 \in [\alpha, \beta]$ it holds that

$$\mathcal{S}_t\delta_{x_0}(x) \geq C(t, \beta) \qquad \textit{for all } x \in (0, \alpha e^t]$$

for some quantity $C = C(t, \beta)$ depending only on t and β.

Proof. We consider (4.24) where $c(x) = B(x) + 1$. As before we denote $(\mathcal{S})_{t\geq 0}$ and $(\mathcal{T})_{t\geq 0}$ semigroups associated to (4.19) and (4.24) respectively.

We rewrite the solution to (4.24) by the method of characteristics

$$\mathcal{T}_t n_0(x) = m(t, x) = n_0(xe^{-t}) \exp\left(-\int_0^t c(xe^{s-t})\,\mathrm{d}s\right), \quad \text{for all } x \geq 0. \tag{4.35}$$

Taking $n_0 = \delta_{x_0}$ we obtain the first bound:

$$\mathcal{S}_t\delta_{x_0} \geq \mathcal{T}_t n_0(x) = \delta_{x_0}(xe^{-t}) \exp\left(-\int_0^t c(xe^{s-t})\,\mathrm{d}s\right) = e^t\delta_{x_0e^t}(x)\exp\left(-\int_0^t c(x_0e^s)\,\mathrm{d}s\right)$$
$$\geq e^{(1-C_1)t}\delta_{x_0e^t}(x),$$

since $\delta_{x_0}(xe^{-t}) = e^t\delta_{x_0e^t}(x)$. We used also that there exists $C_1(t,\beta) > 0$ (decreasing in t) such that

$$c(x_0e^s) = 1 + B(x_0e^s) \leq 1 + \sup_{y\in[0,\beta e^t]} B(y) := C_1(t,\beta), \qquad \text{for all } 0 < s < t.$$

Therefore for $\mathcal{A}(t,x) := 2\int_x^{+\infty} \frac{B(y)}{y} m(t,y)\,\mathrm{d}y$ and using (4.34) we obtain

$$\mathcal{A}(t,x) \geq 2e^{(1-C_1)t}\int_x^{+\infty}\frac{B(y)}{y}\delta_{x_0e^t}(y)\,\mathrm{d}y = 2e^{(1-C_1)t}\frac{B(x_0e^t)}{x_0e^t} \geq \frac{2}{\beta}e^{-C_1t}B(x_0e^t)$$
$$\geq C_3(t,\beta) \quad \text{for all } x \leq x_0e^t.$$

Hence for some $C_4(t,\beta) > 0$ we obtain that

$$\mathcal{T}_{t-\tau}\mathcal{A}(\tau,x) \geq C_4(t,\beta)\mathbb{1}_{\{xe^{-(t-\tau)}\leq x_0e^\tau\}} \geq C_4(t,\beta)\mathbb{1}_{\{x\leq\alpha e^t\}}.$$

Integrating this gives the final bound:

$$\int_0^t \mathcal{T}_{t-\tau}\mathcal{A}(\tau,x)\,\mathrm{d}\tau \geq tC_4(t,\beta)\mathbb{1}_{\{x\leq\alpha e^t\}}.$$

This gives the result. □

Equal mitosis

In this section, we consider kernel $p(z) = 2\delta_{\frac{1}{2}}(z)$ which describes the process of cells of size x breaking down into two equal daughter cells of size $x/2$, so that in (4.12) $\mathcal{A}(t,x) := 4B(2x)m(t,2x)$. Particularly we have the growth-fragmentation equation of the form

$$\begin{aligned}
\frac{\partial}{\partial t}m(t,x) + \frac{\partial}{\partial x}(g(x)m(t,x)) &= 4B(2x)m(t,2x) - (B(x)+\lambda)m(t,x), && t,x\geq 0,\\
m(t,0) &= 0, && t>0,\\
m(0,x) &= n_0(x), && x>0.
\end{aligned} \tag{4.36}$$

The case where g and B are constant is a subject of numerous works in the past and we refer to Chapter 4 of [92] for a detailed description. For $g(x) = 1$ and $B(x) = 1$, eigenelements are explicit and given by

$$(\lambda, N(x), \phi(x)) = \left(1, \bar{N} \sum_{n=0}^{\infty} (-1)^n \alpha_n e^{-2^{n+1}x}, \phi \equiv 1\right),$$

which implies the existence of a unique stationary state and convergence to this stationary state with a rate e^{-t}. However, when a linear growth rate $g(x) = x$ is considered (4.12) exhibits oscillatory behaviour in long time. This is because instead of a dominant real eigenvalue, we observe nonzero imaginary part of the principal eigenvalue so that there exists a set of dominant eigenvalues. This type of periodic long time behaviour first explained in [51] and then it was proved in [64] by using theory of positive semigroups combined with spectral analysis to obtain the convergence to a semigroup of rotations. Since the method relies on some compactness arguments, the authors considered the equation in a compact subset of $(0, \infty)$. Recently in [14], the authors proved the oscillatory behaviour of the solution for a general division rate on $(0, +\infty)$. The proof in this case relies on a general relative entropy argument in a convenient weighted L^2 space. Moreover, they proposed a non-diffusive numerical scheme which captures the oscillatory behaviour. Here we consider a sublinear growth rate and a more general division rate than those so far considered in the literature, excluding the case above, and prove a localized minorisation condition to verify 2.2.3.

Now we give a lemma which will be used later:

Lemma 4.3.6. *We have the following equality for the time integration of a measure moving in time;*

$$\int_0^t \delta_{F(\tau)}(x)\, \mathrm{d}\tau = \left(F^{-1}\right)'(x) \mathbb{1}_{\{F(0) \le x \le F(t)\}}.$$

Proof. Integrating against a smooth test function $\varphi(x)$ we obtain

$$\int_0^{+\infty} \varphi(x) \int_0^t \delta_{F(\tau)}(x)\, \mathrm{d}\tau\, \mathrm{d}x = \int_0^t \int_0^{+\infty} \varphi(x) \delta_{F(\tau)}(x)\, \mathrm{d}x\, \mathrm{d}\tau$$
$$= \int_0^t \varphi(F(\tau))\, \mathrm{d}\tau = \int_{F(0)}^{F(t)} \varphi(y) \left(F^{-1}\right)'(y)\, \mathrm{d}y.$$

by using a change of variable $y = F(\tau)$. □

Lemma 4.3.7 (Lower bound for equal mitosis)**.** *We make Assumptions 4.1.1, 4.1.2, 4.1.3 with a mitosis kernel $p(z) = 2\delta_{\frac{1}{2}}(z)$. Let $(\mathcal{S}_t)_{t\ge 0}$ be the Markov semigroup associated to equation (4.36). For $\beta > 0$ given, there exists $t_C > 0$ such that for all $t > t_C$,* *and $x_0 \in [0,\beta]$ it holds that*

$$\mathcal{S}_t\delta_{x_0}(x) \geq C(t,\beta),$$

for all x in an open interval I_t which depends on time t, and for some quantity $C = C(t,\beta)$ depending only on t and β.

Proof. We follow the same strategy as in the proof of Lemma 4.3.4. Here the only different part is the term $\mathcal{A}(t,x)$. We consider the semigroup $(\mathcal{T}_t)_{t\geq 0}$ defined as in (4.31) and $(\mathcal{S}_t)_{t\geq 0}$ defined as the semigroup associated to (4.36) with $\mathcal{A}(t,x) = 4B(2x)m(t,2x)$. Using (4.33) and particularly

$$\mathcal{T}_t\delta_{x_0}(2x) = X_t\#\delta_{x_0}(2x)e^{-C_1 t} = \frac{1}{2}\delta_{\frac{1}{2}X_t(x_0)}(x)e^{-C_1 t}$$

we obtain

$$\mathcal{A}(t,x) \geq 2e^{-C_1 t}B\left(X_t\left(x_0\right)\right)\delta_{\frac{1}{2}X_t(x_0)}(x) \quad \text{for all } t > 0.$$

Similarly for some $x_B > 0$ for which B is bounded below by a positive quantity in $[x_B, R]$ and there exists a $t_B > 0$ such that for $t > t_B$ we have $X_t\left(x_0\right) > x_B$ for all $x_0 > 0$. Hence, for some $C_2 = C_2(\beta)$,

$$\mathcal{A}(t,x) \geq C_2 e^{-C_1 t}\delta_{\frac{1}{2}X_t(x_0)}(x) \qquad \text{for all } t > t_B.$$

By (4.27) and (4.29) we have

$$\begin{aligned}\mathcal{T}_{t-\tau}\mathcal{A}(\tau,x) &\geq C_2 e^{-C_1\tau}\delta_{X_{t-\tau}\left(\frac{1}{2}X_t(x_0)\right)}(x)\exp\left(-\int_0^{t-\tau} g'(X_{-s}(x))\,\mathrm{d}s\right)\\ &\geq C_2 e^{-C_1\tau}e^{-C_3(t-\tau)}\delta_{X_{t-\tau}\left(\frac{1}{2}X_t(x_0)\right)}(x) \geq C_2 e^{-C_4 t}\delta_{X_{t-\tau}\left(\frac{1}{2}X_t(x_0)\right)}(x),\end{aligned}$$

for all $\tau > t_B$. Here we used $g'(X) \leq C_3$ for all $X \in \left[H^{-1}\left(\frac{t_B}{2}\right), H^{-1}(t)\right]$.

By Lemma 4.3.6 we obtain

$$\int_0^t \mathcal{T}_{t-\tau}\mathcal{A}(\tau,x)\,\mathrm{d}\tau \;\geq\; C_2 e^{-C_4 t}\int_0^t \delta_{X_{t-\tau}\left(\frac{1}{2}X_t(x_0)\right)}(x)\,\mathrm{d}\tau \;\geq\; C_2 e^{-C_4 t}\left(F(\tau)\right)'(x)\mathbb{1}_{\mathcal{I}_{x_0}}$$

where $F(\tau) := X_{t-\tau}\left(\frac{1}{2}X_t(x_0)\right)$ and

$$\mathcal{I}_{x_0} := \left[\frac{1}{2}X_t(x_0), X_t\left(\frac{x_0}{2}\right)\right]. \tag{4.37}$$

Remark 4.3.2. *To be able to give a further bound to this integral and continue the similar argument as in the case of constant fragmentation we need to have (4.37) to* *be a nonempty interval for* x_0 *values we considered and* $\left(\left(X_{t-\tau}\left(\frac{1}{2}X_t(x_0)\right)\right)^{-1}\right)'(x)$ *to be a finite quantity. Therefore we need to consider a specific from of the growth rate satisfying these properties. From now on, we will take* $g(x) = x^\alpha$ *with* $0 < \alpha < 1$.

We take $g(x) = x^\alpha$ with $0 < \alpha < 1$, and then (4.25) implies that

$$X_t(x_0) = \left((1-\alpha)t + x_0^{1-\alpha}\right)^{1/(1-\alpha)}.$$

Therefore,

$$\int_0^t \mathcal{T}_{t-\tau}\mathcal{A}(\tau, x)\,\mathrm{d}\tau \geq C_2 e^{-C_4 t}\mathbb{1}_{\mathcal{I}_{x_0}}\int_0^t (F(\tau))'(x)\,\mathrm{d}\tau$$

Since $x_0 \in [0, \beta]$, we notice that

$$\mathcal{I}_0 = \left[\frac{1}{2}\left((1-\alpha)t\right)^{1/(1-\alpha)}, \left((1-\alpha)t\right)^{1/(1-\alpha)}\right],$$
$$\mathcal{I}_\beta = \left[\frac{1}{2}\left((1-\alpha)t + \beta^{1-\alpha}\right)^{1/(1-\alpha)}, \left((1-\alpha)t + (\beta/2)^{1-\alpha}\right)^{1/(1-\alpha)}\right]$$

Then we define

$$\mathcal{I} := \bigcap_{x_0\in[0,\beta]} \mathcal{I}_{x_0} = \left[\frac{1}{2}((1-\alpha)t + \beta^{1-\alpha})^{1/(1-\alpha)}, ((1-\alpha)t)^{1/(1-\alpha)}\right]$$

for all

$$t > \frac{\beta^{1-\alpha}}{(2^{1-\alpha}-1)(1-\alpha)} := t_A.$$

For all $t > \max\{t_A, t_B\} := t_C$ and there exists C_5 such that $x \leq x_M$ for all $x \in \mathcal{I}$ so that $\int_0^t (F(\tau))'(x)\,\mathrm{d}\tau \leq (F(\tau))'(C_5) := F_C > 0$ by taking $C_2 = C_2 F_C$

$$\mathcal{S}_t\delta_{x_0} \geq \int_0^t \mathcal{T}_{t-\tau}\mathcal{A}(\tau, x)\,\mathrm{d}\tau \geq C_2 e^{-C_4 t}\mathcal{I}_t.$$

for all $t > t_C$, $x_0 \in [0, \beta]$. □

Lemma 4.3.8 (Minorisation condition). *Let* $(\mathcal{S}_t)_{t\geq 0}$ *be the Markov semigroup associated to equation* (4.12). *Under the assumptions 4.1.1, 4.1.2 and 4.1.3. For* $\beta > 0$ *given, there exists* $t_0 > 0$ *such that for all* $t > t_0$*, and* $x_0 \in (0, \beta]$ *Hypothesis 2.2.3 is verified for* $(\mathcal{S}_t)_{t\geq 0}$.

Proof. Lemmas 4.3.4, 4.3.5 and 4.3.7 give the result. □

We conclude this part by giving the proof of Theorem 4.1.2:

Proof. (proof of Theorem 4.1.2) Lemmas 4.3.8, 4.3.5 and 4.3.3 verify the hypotheses of Harris's Theorem 2.2.5 for (4.12). This gives final result (4.9). □

4.4 A numerical scheme

Constructing a numerical scheme for the growth-fragmentation equation and study the asymptotic properties by using Harris's Theorem in the discrete setting can be a natural extension for this work. Here in this section, we present a brief introduction about how this could be done in the simplest case.

This part of the Chapter is a joint work in a collaboration with José Antonio Carrillo. I started working on this project earlier this year during my stay at Imperial College London. I would like to express that this section of the chapter is still in progress, subject to changes and improvements and can be considered as perspectives.

In the case where constant growth and fragmentation rates are considered in (4.12) ($g = 1$ and $B = 1$), we give an explicit numerical scheme to approximate the solutions of (4.12). We note that in this case, the eigenvalue is known: $\lambda = 1$.

We define a grid for a given N such that $x_k = k\Delta x$ for $k = 0, \dots, N$, where $\Delta x = x_N/N$ so that we denote approximate solution to (4.12) where $g = 1$, $B = 1$ and $\lambda = 1$ by $n_k^j = n(t = j, x = x_k)$.

We take $\Delta t = \frac{\Delta x}{2} = \frac{x_N}{2N}$ and consider the following explicit scheme

$$\frac{n_k^{j+1} - n_k^j}{\Delta t} + \frac{n_k^j - n_{k-1}^j}{\Delta x} + 2n_k^j = 2\sum_{i=k}^{N} \frac{n_i^j}{x_i}\Delta x \quad \text{for all } k = 1, \dots, N, \tag{4.38}$$

so that

$$n_k^{j+1} = (1 - 2\Delta t)n_k^j + \frac{\Delta t}{\Delta x}(n_{k-1}^j - n_k^j) + 2\Delta t \sum_{i=k}^{N} \frac{n_i^j}{i\Delta x}\Delta x.$$

Since mass is conserved, we also have

$$\sum_{k=1}^{N} \Delta x n_k^{j+1} = \sum_{k=1}^{N} \Delta x n_k^j. \tag{4.39}$$

In order to keep the mass conservation property at each time step we impose a boundary condition $n_N^j = n_0^j$. Moreover, the stationary solution is given by

$$n_k^j = \frac{k}{k + 2\Delta x(k-1)} \left(n_{k-1}^j + 2\Delta x \sum_{i=k+1}^{N} \frac{n_i^j}{i} \right) \tag{4.40}$$

for all $k = 1, \ldots, N$.

First, we prove a useful equality which will be used later:

Lemma 4.4.1. *Under the setting given in* (4.38)-(4.40) *we have*

$$\sum_{k=1}^{N} \sum_{i=k}^{N} \frac{n_i^j}{i} = \sum_{k=1}^{N} n_k^j$$

Proof. We calculate

$$\begin{aligned}
\sum_{k=1}^{N} \sum_{i=k}^{N} \frac{n_i^j}{i} &= \sum_{k=1}^{N} \left(\frac{n_k^j}{k} + \frac{n_{k+1}^j}{k+1} + \ldots + \frac{n_{N-1}^j}{N-1} + \frac{n_N^j}{N} \right) \\
(k=1) \quad &= \frac{n_1^j}{1} + \frac{n_2^j}{2} + \frac{n_3^j}{3} + \ldots + \frac{n_{N-1}^j}{N-1} + \frac{n_N^j}{N} \\
(k=2) \quad & \qquad\quad + \frac{n_2^j}{2} + \frac{n_3^j}{3} + \ldots + \frac{n_{N-1}^j}{N-1} + \frac{n_N^j}{N} \\
& \qquad\qquad\qquad\qquad \vdots \\
(k=N-1) \quad & \qquad\qquad\qquad\qquad + \frac{n_{N-1}^j}{N-1} + \frac{n_N^j}{N} \\
(k=N) \quad & \qquad\qquad\qquad\qquad\qquad + \frac{n_N^j}{N} \\
= \frac{n_1^j}{1} + 2\frac{n_2^j}{2} + 3\frac{n_3^j}{3} + \ldots + (N-1)\frac{n_{N-1}^j}{N-1} &+ N\frac{n_N^j}{N} = \sum_{k=1}^{N} n_k^j.
\end{aligned}$$

□

4.4.1 Some properties of the scheme

Lemma 4.4.2. *Scheme* (4.38) *is positivity preserving for* $\Delta t < \frac{\Delta x}{1-2\Delta x}$ *with* $\Delta x < \frac{1}{2}$.

Proof. It is clear since if for all k, n_k^j is positive, then we have

$$n_k^{j+1} = \left(1 - \frac{\Delta t}{\Delta x} - 2\Delta t\right) n_k^j + \frac{\Delta t}{\Delta x} n_{k-1}^j + 2\Delta t \sum_{i=k}^{N} \frac{n_i^j}{i\Delta x} \Delta x.$$

For $\Delta t < \frac{\Delta x}{1-2\Delta x}$ with $\Delta x < \frac{1}{2}$, makes the right-hand side terms positive. $\square$

Remark 4.4.1. *Discrete conservation law* (4.39) *and positivity implies the scheme is a contraction for the discrete L^1 norm $\|\cdot\|_1$ defined for a vector $n = (n_k)_{1\le k\le N}$ by*

$$\|n\|_1 = \sum_{k=1}^{N} \Delta x |n_k|.$$

Now we give an intuition about how to prove the convergence of the scheme (4.38); We write the scheme in the condensed form $n^{j+1} = Pn^j$ where P is the iteration matrix in the following form:

$$\begin{pmatrix} n_1^{j+1} \\ n_2^{j+1} \\ n_3^{j+1} \\ \vdots \\ n_N^{j+1} \end{pmatrix}_{N\times 1} = \begin{pmatrix} \frac{1}{2} & \frac{\Delta x}{2} & \frac{\Delta x}{3} & \dots & \dots & \frac{1}{2}+\frac{\Delta x}{N} \\ \frac{1}{2} & \frac{1}{2}-\frac{\Delta x}{2} & \frac{\Delta x}{3} & \dots & \dots & \frac{\Delta x}{N} \\ 0 & \frac{1}{2} & \frac{1}{2}-\frac{2\Delta x}{3} & \dots & \dots & \frac{\Delta x}{N} \\ \vdots & \vdots & \ddots & \dots & \dots & \frac{\Delta x}{N} \\ 0 & 0 & 0 & \dots & \frac{1}{2} & \frac{1}{2}-\frac{(N-1)\Delta x}{N} \end{pmatrix}_{N\times N} \times \begin{pmatrix} n_1^{j} \\ n_2^{j} \\ n_3^{j} \\ \vdots \\ n_N^{j} \end{pmatrix}_{N\times 1}$$

The contraction property reads $\|P\|_1 \le 1$ and this implies the stability of the scheme. Now we prove consistency. We have by Taylor expansion

$$\begin{aligned} n(t_{j+1}, x_k) &= n(t_j, x_k) + \Delta t \partial_t n + \frac{\Delta t^2}{2}\partial_{tt} + \mathcal{O}(\Delta t^3) \\ n(t_j, x_k) &= n(t_j, x_{k-1}) + \Delta x \partial_x n + \frac{\Delta x^2}{2}\partial_{xx} n + \mathcal{O}(\Delta x^3) \end{aligned}$$

which implies as $(\Delta t, \Delta x) \to 0$ we have

$$\begin{aligned} & n(t_{j+1}, x_k) - n_k^{j+1} \\ & \quad = \left(1 - \frac{\Delta t}{\Delta x}\right) n(t_j, x_k) + \frac{\Delta t}{\Delta x} n(t_j, x_{k-1}) + \Delta t \left(\partial_x n(t_j, x_k) + \partial_t n(t_j, x_k)\right) + \\ & \qquad \Delta t \left(\frac{\Delta x}{2}\partial_{xx} n(t_j, x_k) + \frac{\Delta t}{2}\partial_{tt} n(t_j, x_k)\right) + \mathcal{O}(\Delta x^3 + \Delta t^3) \\ & \qquad - \left(1 - \frac{\Delta t}{\Delta x} - 2\Delta t\right) n_k^j - \frac{\Delta t}{\Delta x} n_{k-1}^j - 2\Delta t \sum_{i=k}^{N} \frac{n_i^j}{i}. \end{aligned}$$

Therefore,

$$n(t_{j+1}, x_k) - n_k^{j+1} = 2\Delta t n_k^j - 2\Delta t \sum_{i=k}^{N} \frac{n_i^j}{i}$$
$$+ \Delta t \left(\partial_x n(t_j, x_k) + \partial_t n(t_j, x_k) + \frac{\Delta x}{2} \partial_{xx} n(t_j, x_k) + \frac{\Delta t}{2} \partial_{tt} n(t_j, x_k) \right) + \mathcal{O} \to 0$$

Having proved the stability and the consistency implies the convergence of the scheme (4.38).

Now, we give some intuition about how to prove a minorisation and Lyapunov conditions similar in the discrete setting for this case. We remind that this work is not complete there might be mistakes in the calculations of this part. However, we include them with the purpose of conveying the main idea.

4.4.2 Minorisation condition

We consider an initial data satisfying

$$n(0, x_k) = n_k^0 = \begin{cases} \delta_{x_p}(x_k) \text{ for some } p \in [0, N], \\ 0 \text{ elsewhere.} \end{cases} \tag{4.41}$$

Lemma 4.4.3 (local Doeblin condition)**.** *We consider the scheme* (4.38)*. Suppose that initial data satisfies* (4.41)*. Then we have for all* $j \geq 1$*:*

$$n_k^{j+1} \geq \gamma, \tag{4.42}$$

for some $\gamma \in (0, 1)$ *and for* $p \geq 1$*.*

Proof. We give the proof by induction. For $j = 1$ we have for $\Delta t = \frac{\Delta x}{2}$:

$$n_k^1 = (1 - \Delta x) n_k^0 + \frac{1}{2}(n_{k-1}^0 - n_k^0) + \Delta x \sum_{i=k}^{N} \frac{n_i^0}{i}.$$

We want to prove that n_k^{j+1} is positive for some j by induction. We assume (4.41) for n_k^0. Then we have 4 possible outcomes for n_k^1:

1. $p = k$, then $n_k^1 = (1 - \Delta x) - \frac{1}{2} + \frac{\Delta x}{p} = \frac{1}{2} - \Delta x + \frac{\Delta x}{p} = \frac{1}{2} + \Delta x \left(\frac{1-p}{p} \right) > 0$.
2. $p = k - 1$, then i.e. $n_k^1 = \frac{1}{2}$.
3. $p > k$, then $n_k^1 = \frac{1}{p}$.

4. $p < k-1$, then $n_k^1 = 0$.

Therefore, there exists $\alpha(p) = \min\left\{\frac{1}{2} + \Delta x\left(\frac{1-p}{p}\right), \frac{1}{p}\right\} \in (0,1)$ such that $n_k^1 \geq \alpha$ for $p \geq k-1$, $k > 1$.

We assume for some j, $n_k^j \geq \beta$ where $\beta \in (0,1)$ for $p \geq k-1$, $k > 1$. Then,

$$n_k^{j+1} = \frac{1}{2}n_{k-1}^j + \left(\frac{1}{2} - \Delta x\right)n_k^j + \Delta x \sum_{i=k}^{N} \frac{n_i^j}{i} = \frac{1}{2}\beta + \left(\frac{1}{2} - \Delta x\right)\beta + \beta\Delta x \sum_{i=k}^{N} \frac{1}{i}$$
$$\geq (1-\Delta x)\beta + \beta\Delta x \frac{N-k}{N} = \beta\left(1 - \frac{k\Delta x}{N}\right) \quad (4.43)$$

Setting $\gamma = \beta\left(1 - \frac{k\Delta x}{N}\right)$ gives the result. □

4.4.3 Lyapunov condition

Lemma 4.4.4 (Lyapunov condition)**.** *We consider the scheme* (4.38)*. Then we have for some positive constants C_1, C_2 that:*

$$\sum_{k=1}^{N}(1+x_k^2)\frac{n_k^{j+1} - n_k^j}{\Delta t}\Delta x \leq -C_1 \sum_{k=1}^{N}(1+x_k^2)n_k^j\Delta x + C_2 \sum_{k=1}^{N} n_k^j \Delta x. \quad (4.44)$$

for some k, N.

Proof. We have

$$\sum_{k=1}^{N}(1+x_k^2)\frac{n_k^{j+1} - n_k^j}{\Delta t}\Delta x$$
$$= + \sum_{k=1}^{N}(1+x_k^2)(n_{k-1}^j - n_k^j) - 2\sum_{k=1}^{N}(1+x_k^2)n_k^j\Delta x + 2\sum_{k=1}^{N}(1+x_k^2)\sum_{i=k}^{N}\frac{n_i^j}{i}\Delta x$$
$$= -\sum_{k=1}^{N}(1+x_k^2)n_k^j\Delta x + \sum_{k=1}^{N-1}(x_{k+1}^2 - x_k^2)n_k^j + x_1^2 n_0^j - x_N^2 n_N^j$$
$$- \sum_{k=1}^{N}(1+x_k^2)n_k^j\Delta x + 2\sum_{k=1}^{N} n_k^j\Delta x + 2\sum_{k=1}^{N} x_k^2 \sum_{i=k}^{N}\frac{n_i^j}{i}\Delta x$$
$$= -\sum_{k=1}^{N}(1+x_k^2)n_k^j\Delta x + \sum_{k=1}^{N-1}(2k+1)n_k^j\Delta x^2 - \sum_{k=1}^{N-1}(1+x_k^2)n_k^j\Delta x + 2\sum_{k=1}^{N-1} n_k^j\Delta x$$
$$+ (1-N^2)\Delta x^2 n_N^j - (1+N^2\Delta x^2)\Delta x n_N^j + 2\Delta x n_N^j + 2\sum_{k=1}^{N}\Delta x^2 k^2 \sum_{i=k}^{N}\frac{n_i^j}{i}\Delta x,$$

so that

$$\sum_{k=1}^{N}(1+x_k^2)\frac{n_k^{j+1}-n_k^j}{\Delta t}\Delta x$$
$$= -\sum_{k=1}^{N}(1+x_k^2)n_k^j\Delta x + \sum_{k=1}^{N-1}(-\Delta x^2k^2+2\Delta xk+\Delta x+1)n_k^j\Delta x$$
$$+ (\Delta x+1)(1-N^2\Delta x)n_N^j\Delta x + 2\Delta x^2\sum_{k=1}^{N}k^2\sum_{i=k}^{N}\frac{n_i^j}{i}\Delta x.$$

Here we define $\phi_1(k) := -\Delta x^2k^2 + 2\Delta xk + \Delta x + 1$ and notice that we have $\phi_1(k) \leq \Delta x + 2$ for $k \geq \frac{1}{\Delta x}$.

Moreover we decompose the sum $2\Delta x^2\sum_{k=1}^{N}k^2\sum_{i=k}^{N}\frac{n_i^j}{i}\Delta x$ and we obtain

$$\sum_{k=1}^{N}(1+x_k^2)\frac{n_k^{j+1}-n_k^j}{\Delta t}\Delta x$$
$$\leq -\sum_{k=1}^{N}(1+x_k^2)n_k^j\Delta x + \sum_{k=1}^{N-1}(\Delta x+2)n_k^j\Delta x + (\Delta x+1)(1-N^2\Delta x)n_N^j\Delta x$$
$$+2\Delta x^3\left(\sum_{k=1}^{N}kn_k^j + \sum_{k=1}^{N}\frac{k^2}{k+1}n_{k+1}^j + \cdots + \sum_{k=1}^{N}\frac{k^2}{N-1}n_{N-1}^j + \sum_{k=1}^{N}\frac{k^2}{N}n_N^j\right)$$

such that

$$\sum_{k=1}^{N}(1+x_k^2)\frac{n_k^{j+1}-n_k^j}{\Delta t}\Delta x \leq -\sum_{k=1}^{N}(1+x_k^2)n_k^j\Delta x + \sum_{k=1}^{N-1}(\Delta x+2)n_k^j\Delta x$$
$$+2\Delta x^3\left(\sum_{k=1}^{N}kn_k^j + \sum_{k=1}^{N}\frac{k^2}{k+1}n_{k+1}^j + \cdots + \sum_{k=1}^{N}\frac{k^2}{N-1}n_{N-1}^j\right)$$
$$+(\Delta x+1)(1-N^2\Delta x)n_N^j\Delta x + \frac{\Delta x^2}{3}(N+1)(2N+1)n_N^j\Delta x$$

We define $\phi_2(N) := -N^2\left(\frac{\Delta x^2}{3}+\Delta x\right) + \Delta x^2N + \frac{\Delta x^2}{2} + \Delta x + 1$ and we have $\phi(N) \leq C_3$ where $C_3 > 0$ for some $N > 0$. Therefore we obtain Inequality (4.44) for $C_1 = 1$ and for some $C_2 > 0$ could be calculated explicitly. □

4.4.4 Simulations

Stable stationary size distribution

In the last part of this chapter we present some simulations using the scheme (4.38)-(4.39) for $g(x) = 1$ and $B(x) = 1$ with a smooth initial data:

$$n_0(x) = x^2 \exp\left(-\frac{x}{2}\right). \tag{4.45}$$

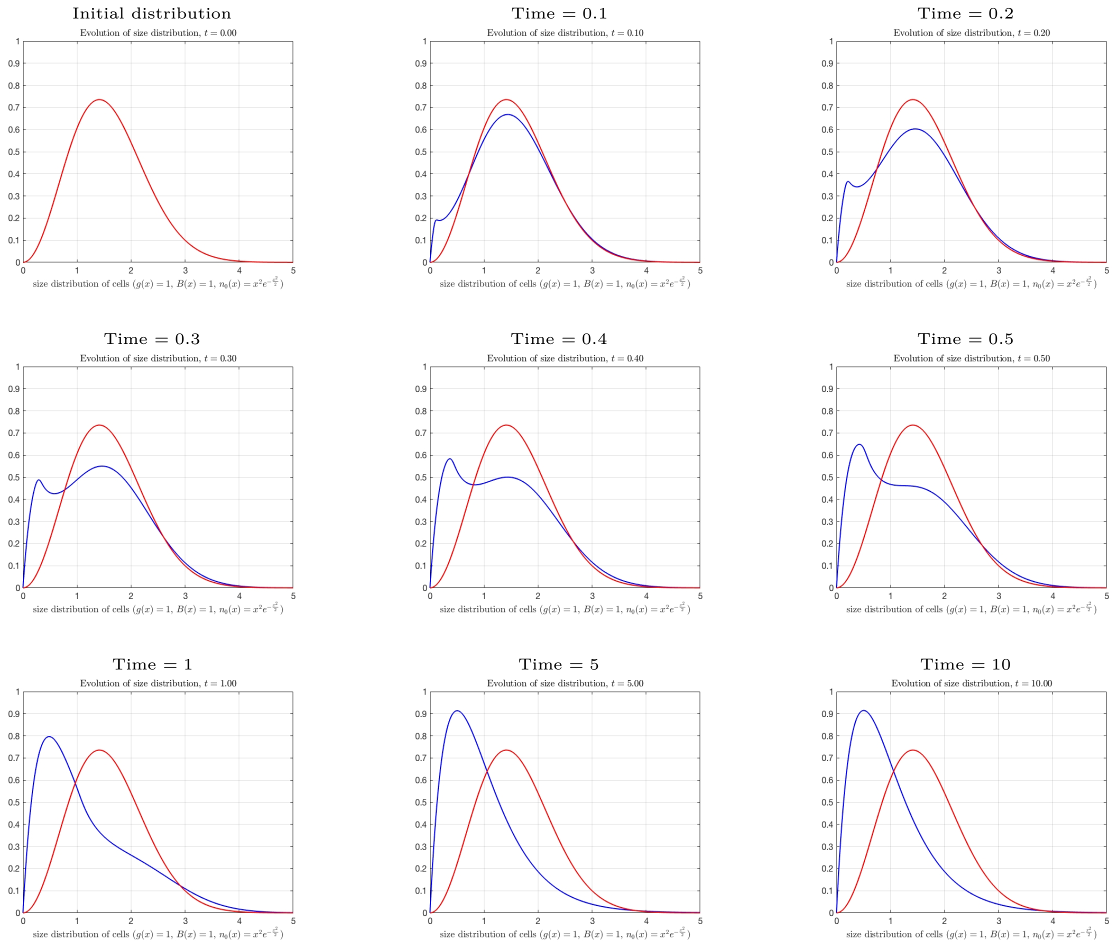
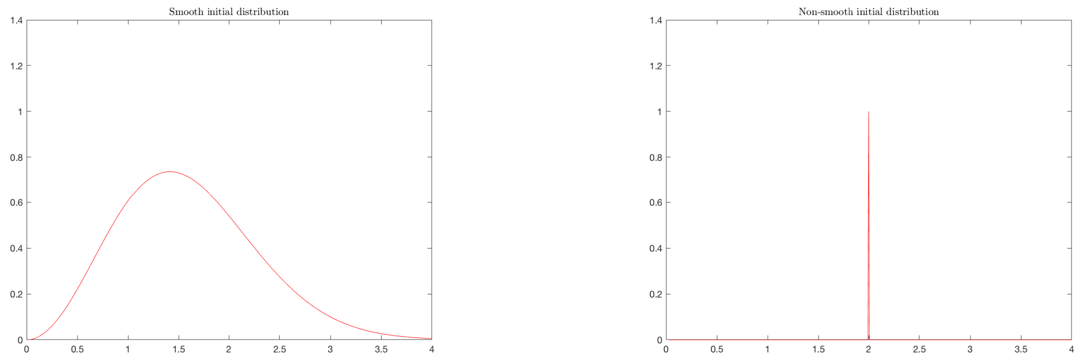


Fig. 4.1 Time evolution of size distribution of the growth-fragmentation equation.

In Figure 4.1, red line plots the initial size distribution and blue line is the size distribution at the time which is specified in the caption. Time difference between sub-figures is not uniform because the change in the distribution is fast in the earlier times and then it stabilizes and converge to the stationary distribution so that the change very less and slow at later times. In conclusion, Figure 4.1 shows that after some time the growth-fragmentation equation reaches a universal size distribution given in “time=10”.

Oscillatory size distribution

We also present results of some simulations for the case of linear growth rate and mitosis kernel. It is already pointed out that in this case instead of a unique real dominant eigenvalue there is a family of dominant eigenvalues whose imaginary part is nonzero. For the figures below we implemented the semi implicit flux splitting scheme given in [14].

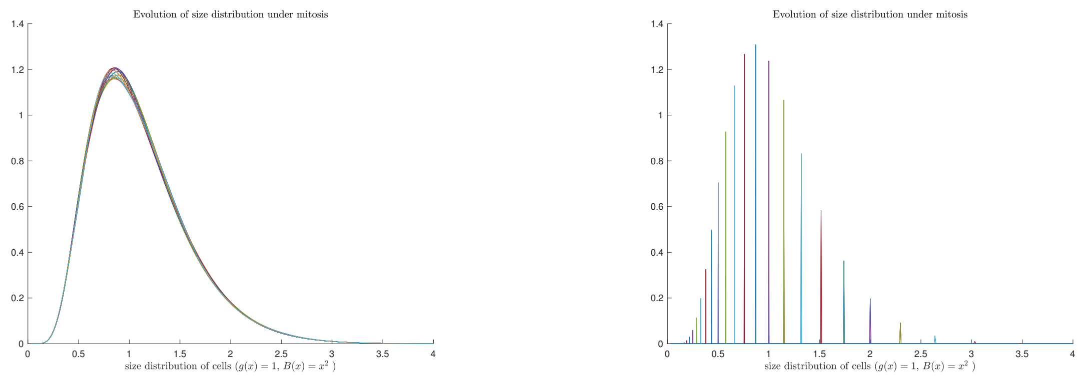


(a) Smooth initial data

(b) Non-smooth initial data

Fig. 4.2 Initial size distributions.

We considered two types of initial data given in Figure 4.2. The smooth initial data is given in (4.45) and the non-smooth one mimics the behaviour of the Dirac delta function concentrated at the middle of the domain. In Figure 4.3 we present size distributions at different times printed on the same figure. The color is different for each time. We see that no matter how long we run the simulations, there are several stationary size distributions, not a unique one, and the solutions will oscillate in between each of those.



(a) Evolution of smooth initial data

(b) Evolution of non-smooth initial data

Fig. 4.3 Time evolution of the size distribution at different times.

4.5 Summary and conclusion

In this chapter, we studied the asymptotic behaviour of the growth-fragmentation equation (4.1). This linear, nonlocal, evolution equation describes the dynamics of growing and dividing processes. This equation appears in modelling of wide range of real-life processes in cell biology, ecology, neurology or in telecommunication. Depending on the balance (or imbalance) between growth and division terms, size distribution of the population density may or may not converge to a stable distribution. In this work we were interested in the case where there is a relaxation towards a universal profile.

What differs in our results from the previous works on this type of models is that under fairly general assumptions we give a completely constructive proof for the existence of the spectral gap property in a weighted total variation distance. We use Harris's theorem for the main result; thus we obtain quantitative convergence rates to the universal profile. If the associated perron eigenvalue problem has a solution, then it is straightforward to show the spectral gap. Indeed, in some special cases for the growth and the total division rates, eigenelements can be obtained explicitly. However, in this work we considered quite general cases where eigenelements may not be known explicitly. In this case we give a rigorous proof for the existence of eigenelements.

In the last section, we also considered a numerical scheme for the growth-fragmentation equation where we aimed at proving asymptotic behaviour in the discrete setting as well via Harris's Theorem. This work is yet to be completed and it is expected to be useful and intuitive for wider range of models which generate a linear, positivity and mass preserving stochastic semigroup.

Chapter 5

Hypocoercivity of linear kinetic equations

"What we can do is to establish a bridge between the various levels in order to form a coherent picture; the whole of Boltzmann's work is a masterpiece of this procedure, i.e. how to construct, starting from atoms, a description that explains everyday life."

— Carlo Cercignani, Ludwig Boltzmann: The Man Who Trusted Atoms

Contents

5.1 Overview

In this chapter we show how to obtain quantitative rates of convergence to a stationary state for some linear kinetic equations, using Harris's Theorem presented in Chapter 2. If certain hypotheses concerning irreducibility are verified quantitatively, this theorem gives quantitative convergence rates to a unique stationary state for Markov processes and it is very well adapted to hypocoercive, nonlocal, dissipative equations. This chapter is based on a collaboration [39].

Studying rate of convergence to equilibrium for kinetic equations involves estimating dissipative effect on the space variable. Dissipation for kinetic equations is caused by the positive jump operator (later will be called as $\mathcal{L}$) which acts only on the velocity variable v; whereas transport takes place only in the space variable x and it mixes the dissipation in x. This is a celebrated problem for both linear and nonlinear equations. In [103, 72, 73], the authors developed a techique which is called *hypocoercivity* to tackle this problem for linear kinetic equations. In a landmark result [49], it is proved that the full nonlinear Boltzmann equation converges to equilibrium at least at an algebraic rate. Exponential convergence results for the (linear) Fokker-Planck equation were proven in [48], and a theory for a range of linear kinetic equations has been shown in [53]. All of these results give convergence in weigthed L^2 norms or H^1 norms. However, convergence to a stationary state in weigthed L^1 norms can be proven for several kinetic models by using the techniques in [65].

We consider linear kinetic equations of the type

$$\partial_t f + v \cdot \nabla_x f = \mathcal{L} f,$$

where $f = f(t, x, v)$ represents the probability density of having a particle that is located at around a point x in the space at time $t \geq 0$ and moving with a velocity v. We consider $v \in \mathbb{R}^d$ and $x \in \mathbb{T}^d$ (the d-dimensional unit torus) for the case above which implies periodic boundary conditions.

We also consider linear kinetic equations that are posed in $x \in \mathbb{R}^d$ and read as

$$\partial_t f + v \cdot \nabla_x f - (\nabla_x \Phi \cdot \nabla_v f) = \mathcal{L} f.$$

In this case we introduce a confining potential Φ as well. The operator $\mathcal{L}$ acts only on the v variable, and it is the generator of the associated stochastic semigroup we will work with later.

5.1.1 Assumptions and the main theorems

In this work, we take $\mathcal{L}$ as either the linear relaxation Boltzmann (linear BGK) or the linear Boltzmann operator. It is a well defined operator from $L^1(\mathbb{T}^d \times \mathbb{R}^d)$ to $L^1(\mathbb{T}^d \times \mathbb{R}^d)$ when the equations are posed in $\mathbb{T}^d$ or from $L^1(\mathbb{R}^d \times \mathbb{R}^d)$ to $L^1(\mathbb{R}^d \times \mathbb{R}^d)$ when the equations are posed on $\mathbb{R}^d$. Similarly it can also be defined as an operator from $\mathcal{M}(\mathbb{T}^d \times \mathbb{R}^d)$ to $\mathcal{M}(\mathbb{T}^d \times \mathbb{R}^d)$ or $\mathcal{M}(\mathbb{R}^d \times \mathbb{R}^d)$ to $\mathcal{M}(\mathbb{R}^d \times \mathbb{R}^d)$.

We denote the set of probability measures on a set $\Omega \subseteq \mathbb{R}^d$ by $\mathcal{P}(\Omega)$ (that is, the probability measures defined on the Borel σ-algebra of Ω).

Definition 5.1.1. *In the linear BGK case, we define $\mathcal{L}$ as*

$$\mathcal{L} := \mathcal{L}^+ f - f,$$

where

$$\mathcal{L}^+ f = \left(\int f(t,x,u)\,\mathrm{d}u \right) \mathcal{M}(v),$$

and

$$\mathcal{M}(v) := (2\pi)^{-d/2} \exp(-|v|^2/2)$$

the Maxwellian distribution which is known to be an equilibrium state for the equation.

In the linear Boltzmann case, the operator $\mathcal{L}$ is the Boltzmann operator Q which is given by

$$Q(f,g) = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} B(|v-v_*|,\sigma) \left(f(v')g(v'_*) - f(v)g(v_*) \right) \mathrm{d}\sigma\, \mathrm{d}v_*,$$

where

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \qquad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma,$$

and B is the collision kernel*.*

First assumption is on the collision kernel B:

Assumption 5.1.1. *We assume that B is a hard kernel and can be written as a product*

$$B(|v-v_*|,\sigma) = |v-v_*|^\gamma \; b\left(\sigma \cdot \frac{v-v_*}{|v-v_*|} \right), \tag{5.1}$$

for some $\gamma \geq 0$, meaning that we work in the hard spheres/Maxwell molecules regime. Moreover we will make a cutoff assumption on b so that b integrable in σ. We also *assume that b is uniformly positive on $[-1,1]$; that is, there exists $C_b > 0$ such that*

$$b(z) \geq C_b \qquad \text{for all } z \in [-1,1]. \tag{5.2}$$

Remark 5.1.1. *Assumption 5.1.1 includes the physical hard-spheres collision kernel, for which $b \equiv 1$. The* non-cutoff *kernels, where b is not integrable, are not considered.*

Remark 5.1.2. *Assumption 5.1.1 implies that we can write the linear Boltzmann equation on the torus*

$$\partial_t f + v \cdot \nabla_x f = \mathcal{L}^+ f - \kappa(v) f,$$

where $\kappa(v) \geq 0$ and $\kappa(v)$ behaves like $|v|^\gamma$ for large v; that is,

$$0 \leq \sigma(v) \leq (1 + |v|^2)^{\gamma/2}, \qquad v \in \mathbb{R}^d. \tag{5.3}$$

See [30] Lemma 2.1 for example.

Assumption 5.1.2. *We assume that the confining potential $\Phi\colon \mathbb{R}^d \to \mathbb{R}$ is bounded from below and $\Phi \in C^2(\mathbb{R}^d)$. We consider potentials growing at least quadratically at infinity and some weaker potentials. Later we will also make some spesific assumptions regarding the growth rate depending on each case we consider.*

Under these assumptions we present convergence results on the following equations:

- The linear relaxation Boltzmann equation

  either posed in $(x, v) \in \mathbb{T}^d \times \mathbb{R}^d$;

$$\partial_t f + v \cdot \nabla_x f = \mathcal{L}^+ f - f. \tag{5.4}$$

  or posed in $(x, v) \in \mathbb{R}^d \times \mathbb{R}^d$;

$$\partial_t f + v \cdot \nabla_x f - (\nabla_x \Phi \cdot \nabla_v f) = \mathcal{L}^+ f - f. \tag{5.5}$$

- The linear Boltzmann equation

  either posed in $(x, v) \in \mathbb{T}^d \times \mathbb{R}^d$;

$$\partial_t f + v \cdot \nabla_x f = Q(f, \mathcal{M}). \tag{5.6}$$

  or posed in $(x, v) \in \mathbb{R}^d \times \mathbb{R}^d$;

$$\partial_t f + v \cdot \nabla_x f - (\nabla_x \Phi \cdot \nabla_v f) = Q(f, \mathcal{M}). \tag{5.7}$$

Finally we state our main results on the torus $\mathbb{T}^d$, and then on $\mathbb{R}^d$ with a confining potential Φ:

Theorem 5.1.3 (Exponential convergence results on the torus)**.** *Suppose that $t \mapsto f_t$ is the solution to (5.4) or (5.6) with initial data $f_0 \in \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$. In the case of equation (5.6) we also assume (5.1) with $\gamma \geq 0$ and (5.2). Then there exist constants $C > 0, \lambda > 0$ (independent of f_0) such that*

$$\|f_t - \mu\|_* \leq Ce^{-\lambda t}\|f_0 - \mu\|_*,$$

where μ is the only equilibrium state of the corresponding equation in $\mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$ (that is, $\mu(x, v) = \mathcal{M}(v)$). The norm $\|\cdot\|_*$ is just the total variation norm $\|\cdot\|_{\mathrm{TV}}$ for equation (5.4),

$$\|f_0 - \mu\|_* = \|f_0 - \mu\|_{\mathrm{TV}} := \int_{\mathbb{R}^d}\int_{\mathbb{T}^d} |f_0 - \mu| \, \mathrm{d}x \, \mathrm{d}v \qquad \textit{for equation (5.4)},$$

and it is a weigthed total variation norm in the case of equation (5.6):

$$\|f_0 - \mu\|_* = \int_{\mathbb{R}^d}\int_{\mathbb{T}^d} (1 + |v|^2)|f_0 - \mu| \, \mathrm{d}x \, \mathrm{d}v \qquad \textit{for equation (5.6)}.$$

Theorem 5.1.4 (Exponential convergence results with a confining potential)**.** *Suppose that $t \mapsto f_t$ is the solution to (5.5) or (5.7) with initial data $f_0 \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ and a potential $\Phi \in \mathcal{C}^2(\mathbb{R}^d)$ which is bounded below, and satisfies*

$$x \cdot \nabla_x \Phi(x) \geq \gamma_1 |x|^2 + \gamma_2 \Phi(x) - A, \qquad x \in \mathbb{R}^d,$$

for some positive constants γ_1, γ_2, A. Define $\langle x \rangle = \sqrt{1 + |x|^2}$. In the case of equation (5.6) we also assume (5.1) with $\gamma \geq 0$, (5.2) and

$$x \cdot \nabla_x \Phi(x) \geq \gamma_1 \langle x \rangle^{\gamma+2} + \gamma_2 \Phi(x) - A,$$

for some positive constants γ_1, γ_2, A. Then there exist constants $C > 0, \lambda > 0$ (independent of f_0) such that

$$\|f_t - \mu\|_* \leq Ce^{-\lambda t}\|f_0 - \mu\|_*,$$

where μ is the only equilibrium state of the corresponding equation in $\mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$,

$$\mathrm{d}\mu = \mathcal{M}(v) e^{-\Phi(x)} \, \mathrm{d}v \, \mathrm{d}x.$$

The norm $\|\cdot\|_*$ *is a weighted total variation norm defined by*

$$\|f_t - \mu\|_* := \int \left(1 + \frac{1}{2}|v|^2 + \Phi(x) + |x|^2\right) |f_t - \mu| \,\mathrm{d}v \,\mathrm{d}x.$$

In all results above the constants C and λ can be explicitly estimated in terms of the parameters appearing in the equation by following the calculations in the proofs. We do not give them explicitly since we do not expect them to be optimal, but they are completely constructive.

We also consider weaker potentials by looking at Harris type theorems with weaker controls on moments to give analogues of all our theorems and give algebraic rates of convergence with rates depending on the assumption we make on the confining potential. Subgeometric convergence for kinetic Fokker-Planck equations with weak confinement has been shown in [54, 3, 41]> However for the linear BGK and the linear Boltzmann equations there is no other result giving a quantitative algebraic convergence rates up to our knowledge.

Theorem 5.1.5 (Subgeometric convergence results with weak confining potentials)**.** *Suppose that* $t \mapsto f_t$ *is the solution to* (5.5) *with initial data* $f_0 \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ *in the whole space with a confining potential* $\Phi \in \mathcal{C}^2(\mathbb{R}^d)$ *which is bounded below. Define* $\langle x \rangle = \sqrt{1 + |x|^2}$*. Assume that for some* $\beta \in (0, 1)$ *the confining potential satisfies*

$$x \cdot \nabla_x \Phi(x) \geq \gamma_1 \langle x \rangle^{2\beta} + \gamma_2 \Phi(x) - A,$$

for some positive constants γ_1, γ_2, A*. Then we have that there exists a constant* $C > 0$ *such that*

$$\|f_t - \mu\|_{\mathrm{TV}} \leq \min\left\{\|f_0 - \mu\|_{\mathrm{TV}},\ C \int f_0(x, v) \left(1 + \frac{1}{2}|v|^2 + \Phi(x) + |x|^2\right) (1 + t)^{-\beta/(1-\beta)}\right\}.$$

Similarly if $t \mapsto f_t$ *is the solution to* (5.7) *in the whole space, satisfies* (5.1), (5.2) *and that*

$$x \cdot \nabla_x \Phi(x) \geq \gamma_1 \langle x \rangle^{\beta+1} + \gamma_2 \Phi(x) - A, \qquad \Phi(x) \leq \gamma_3 \langle x \rangle^{1+\beta},$$

for some positive constants $\gamma_1, \gamma_2, \gamma_3, A, \beta$*. Then we have that there exists a constant* $C > 0$ *such that*

$$\|f_t - \mu\|_{\mathrm{TV}} \leq \min\left\{\|f_0 - \mu\|_{\mathrm{TV}},\ C \int f_0(x, v) \left(1 + \frac{1}{2}|v|^2 + \Phi(x) + |x|\right) (1 + t)^{-\beta}\right\}.$$

We carry out all of our proofs using variations of Harris's Theorem from probability which was explained in detail in Chapter 2. Mainly it states that having a good confining property and some uniform mixing property in the centre of the state space is enough to obtain exponentially fast convergence to equilibrium in a weighted total variation norm for a Markov process.

These type of techniques have already been used to show convergence to equilibrium for some kinetic equations. In [78], the authors show convergence to equilibrium for the kinetic Fokker-Planck equation with non-quantitative rates. In [11], the authors use a strategy based on Doeblin's Theorem, which is a precursor to Harris's Theorem, to show non-quantitative rates for convergence to equilibrium for scattering equations with non-equilibrium steady states. In [43], the authors show quantitative exponential convergence to a non-equilibrium steady state for some non-linear kinetic equations on the torus using Doeblin's Theorem.

We dedicated the next two sections to presenting results on the linear relaxation Boltzmann and the linear Boltzmann equations respectively.

5.2 The linear relaxation Boltzmann equation

Recalling the definition for the *linear relaxation Boltzmann (or the linear BGK) equation*

$$\partial_t f + v \cdot \nabla_x f - (\nabla_x \Phi \cdot \nabla_v f) = \mathcal{L}^+ f - f,$$

where

$$\mathcal{L}^+ f = \left(\int f(t,x,u)\,\mathrm{d}u \right) \mathcal{M}(v),$$

and $\mathcal{M}(v) := (2\pi)^{-d/2} \exp(-|v|^2/2)$. The function $f = f(t,x,v)$ depends on time $t \geq 0$, space $x \in \mathbb{R}^d$, and velocity $v \in \mathbb{R}^d$, and the potential $\Phi \colon \mathbb{R}^d \to \mathbb{R}$ is a $\mathcal{C}^2$ function of x. We also consider this equation on the torus; that is, for $x \in \mathbb{T}^d$, $v \in \mathbb{R}^d$, assuming periodic boundary conditions. In that case we omit Φ (which corresponds to $\Phi = 0$ in the above equation):

$$\partial_t f + v \cdot \nabla_x f = \mathcal{L}^+ f - f.$$

This simple equation is well studied in kinetic theory and can be thought of as a toy model with similar properties to either the non-linear BGK equation or linear Boltzmann equation. It is also one of the simplest examples of a hypocoercive equation. Convergence to equilibrium in H^1 for this equation has been shown in [34], at a rate faster than any function of t. It was then shown to converge exponentially fast in both H^1 and L^2 using hypocoercivity techniques in [72, 87, 53].

5.2.1 On the torus

Since this is the simplest case we can use Doeblin's Theorem where we have a uniform minorisation condition. We consider

$$\partial_t f + v \cdot \nabla_x f = \mathcal{L} f, \tag{5.8}$$

posed for $(x, v) \in \mathbb{T}^d \times \mathbb{R}^d$, where $\mathbb{T}^d$ is the d-dimensional torus of side 1 and

$$\mathcal{L} f(x,v) := \mathcal{L}^+ f(x,v) - f(x,v) := \left(\int_{\mathbb{R}^d} f(x,u)\, \mathrm{d}u \right) \mathcal{M}(v) - f(x,v), \tag{5.9}$$

which is a well defined operator from $L^1(\mathbb{T}^d \times \mathbb{R}^d)$ to $L^1(\mathbb{T}^d \times \mathbb{R}^d)$, and can also be defined as an operator from $\mathcal{M}(\mathbb{T}^d \times \mathbb{R}^d)$ to $\mathcal{M}(\mathbb{T}^d \times \mathbb{R}^d)$ with the same expression (where $\int_{\mathbb{R}^d} f(x,u)\, \mathrm{d}u$ now denotes the marginal of the measure f with respect to u). We define $(T_t)_{t \geq 0}$ as the transport semigroup associated to the operator $-v \cdot \nabla_x f$ in the space of measures with the bounded Lipschitz topology (see for example [29]); that is, $t \mapsto T_t f_0$ solves the equation $\partial_t f + v \cdot \nabla_x f = 0$ with initial condition f_0. In this case one can write T_t explicitly as

$$T_t f_0(x,v) = f_0(x - tv, v). \tag{5.10}$$

Using Duhamel's formula repeatedly one can obtain that, if f is a solution of (5.8) with initial data f_0, then

$$e^t f_t \geq \int_0^t \int_0^s T_{t-s} \mathcal{L}^+ T_{s-r} \mathcal{L}^+ T_r f_0 \, \mathrm{d}r\, \mathrm{d}s. \tag{5.11}$$

Now we check two properties, which are listed as lemmas. The first one implies that the operator $\mathcal{L}$ always allows jumps to any small velocity.

Lemma 5.2.1. *For all $\delta_L > 0$ there exists $\alpha_L > 0$ such that for all nonnegative functions $g \in L^1(\mathbb{T}^d \times \mathbb{R}^d)$ we have*

$$\mathcal{L}^+ g(x,v) \geq \alpha_L \left(\int_{\mathbb{R}^d} g(x,u)\, \mathrm{d}u \right) \mathbb{1}_{\{|v| \leq \delta_L\}} \tag{5.12}$$

for almost all $(x,v) \in \mathbb{T}^d \times \mathbb{R}^d$.

Proof. Given any δ_L it is enough to choose $\alpha_L := \mathcal{M}(v)$ for any v with $|v| = \delta_L$. □

The second one is regarding to the behaviour of the transport part alone. It implies that if we start at any point inside a ball of radius R with any small velocity, then we can reach any point in the ball of radius R with a predetermined bound on the final velocity:

Lemma 5.2.2. *Given any time* $t_0 > 0$ *and radius* $R > 0$ *there exist* $\delta_L, R' > 0$ *such that for all* $t \geq t_0$ *it holds that*

$$\int_{B(R')} T_t\Big(\delta_{x_0}(x)\mathbb{1}_{\{|v|\leq\delta_L\}}\Big)\,\mathrm{d}v \geq \frac{1}{t^d}\mathbb{1}_{\{|x|\leq R\}} \qquad \text{for all } x_0 \text{ with } |x_0| < R. \tag{5.13}$$

In particular, if we take $R > \sqrt{d}$, *there exist* $\delta_L, R' > 0$ *such that*

$$\int_{B(R')} T_t\Big(\delta_{x_0}(x)\mathbb{1}_{\{|v|\leq\delta_L\}}\Big)\,\mathrm{d}v \geq \frac{1}{t^d} \qquad \text{for all } x_0 \in \mathbb{T}^d. \tag{5.14}$$

Proof. Take $t, R > 0$. We have

$$T_t\left(\delta_{x_0}(x)\mathbb{1}_{B(\delta_L)}(v)\right) = \delta_{x_0}(x - vt)\mathbb{1}_{B(\delta_L)}(v),$$

where $B(\delta)$ denotes the open ball $\{x \in \mathbb{R}^d \mid |x| < \delta\}$, and in general we will use the notation $B(z, \delta)$ to denote the open ball of radius δ centered at $z \in \mathbb{R}^d$. Integrating this and changing variables gives that

$$\int_{B(R')} T_t\left(\delta_{x_0}(x)\mathbb{1}_{B(\delta_L)}(v)\right)\,\mathrm{d}v = \frac{1}{t^d}\int_{B(x,tR')} \delta_{x_0}(y)\mathbb{1}_{B(\delta_L)}\left(\frac{x-y}{t}\right)\mathrm{d}y.$$

Since $|x - y| \leq |x| + |y|$ we have that

$$\mathbb{1}_{B(\delta_L)}\left(\frac{x-y}{t}\right) \geq \mathbb{1}_{B(\delta_L/2)}\left(\frac{x}{t}\right)\mathbb{1}_{B(\delta_L/2)}\left(\frac{y}{t}\right).$$

Therefore if we take $\delta_L > 2R/t$ we have

$$\mathbb{1}_{B(\delta_L)}\left(\frac{x-y}{t}\right) \geq \mathbb{1}_{B(R)}(x)\mathbb{1}_{B(R)}(y).$$

On the other hand, if we take $|x| < R$ and $R' > 2R/t$ then

$$B(x, tR') \supseteq B(x, 2R) \supseteq B(R).$$

Hence if $\delta_L > 2R/t$ and $R' > 2R/t$,

$$\int_{B(R')} T_t\left(\delta_{x_0}(x)\mathbb{1}_{B(\delta_L)}(v)\right)\,\mathrm{d}v \geq \frac{1}{t^d}\mathbb{1}_{B(R)}(x),$$

which proves the result. □

Lemma 5.2.3 (Doeblin condition for the linear relaxation Botzmann equation on the torus)**.** *For any $t_* > 0$ there exist constants $\alpha, \delta_L > 0$ (depending on t_*) such that any solution f to equation (5.8) with initial condition $f_0 \in \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$ satisfies*

$$f(t_*, x, v) \geq \alpha \mathbb{1}_{\{|v| \leq \delta_L\}}, \tag{5.15}$$

where the inequality is understood in the sense of measures.

Proof. It is enough to prove it for $f_0 := \delta_{(x_0,v_0)}$, where $(x_0, v_0) \in \mathbb{T}^d \times \mathbb{R}^d$ is an arbitrary point. From Lemma 5.2.2 (with $R > \sqrt{d}$ and $t_0 := t_*/3$) we will use that there exists $\delta_L > 0$ such that

$$\int_{\mathbb{R}^d} T_t\Big(\delta_{x_0}(x)\mathbb{1}_{\{|v|\leq\delta_L\}}\Big)\,\mathrm{d}v \geq \frac{1}{t^d} \qquad \text{for all } x_0 \in \mathbb{T}^d,\ t > t_0.$$

Also, Lemma 5.2.1 gives an $\alpha_L > 0$ such that

$$\mathcal{L}^+ g \geq \alpha_L \left(\int_{\mathbb{R}^d} g(x,u)\,\mathrm{d}u\right) \mathbb{1}_{\{|v|\leq\delta_L\}}.$$

Take any $r > 0$. Since $T_r f_0 = \delta_{(x_0 - v_0 r, v_0)}$, using this shows

$$\mathcal{L}^+ T_r f_0 \geq \alpha_L\, \delta_{x_0 - v_0 r}(x)\, \mathbb{1}_{\{|v|\leq\delta_L\}}.$$

Hence, whenever $s - r > t_0$ we have

$$\begin{aligned}
\mathcal{L}^+ T_{s-r} \mathcal{L}^+ T_r f_0 &\geq \alpha_L \left(\int_{\mathbb{R}^d} T_{s-r}\mathcal{L}^+ T_r f_0\,\mathrm{d}u\right) \mathbb{1}_{\{|v|\leq\delta_L\}} \\
&\geq \alpha_L^2 \left(\int_{\mathbb{R}^d} T_{s-r}\Big(\delta_{x_0 - v_0 r}(x)\,\mathbb{1}_{\{|u|\leq\delta_L\}}\Big)\,\mathrm{d}u\right) \mathbb{1}_{\{|v|\leq\delta_L\}} \\
&\geq \frac{1}{(s-r)^d}\alpha_L^2\, \mathbb{1}_{\{|v|\leq\delta_L\}}.
\end{aligned}$$

Finally, for the movement along the flow T_{t-s}, notice that

$$T_t\left(\mathbb{1}_{\mathbb{T}^d}(x)\mathbb{1}_{\{|v|<\delta_L\}}(v)\right) = \mathbb{1}_{\mathbb{T}^d}(x)\mathbb{1}_{\{|v|<\delta_L\}}(v) \qquad \text{for all } t \geq 0.$$

This means that for all $t > s > r > 0$ such that $s - r > t_0$ we have

$$T_{t-s}\mathcal{L}^+ T_{s-r}\mathcal{L}^+ T_r f_0 \geq \frac{1}{(s-r)^d}\alpha_L^2\, \mathbb{1}_{\{|v|\leq\delta_L\}}.$$

For any t_* we have then, recalling that $t_0 = t_*/3$,

$$\int_0^{t_*}\int_0^s T_{t_*-s}\mathcal{L}^+T_{s-r}\mathcal{L}^+T_r f_0 \,\mathrm{d}r\,\mathrm{d}s \geq \alpha_L^2 \,\mathbb{1}_{\{|v|\leq\delta_L\}} \int_{2t_0}^{t_*}\int_0^{t_0} \frac{1}{(s-r)^d}\,\mathrm{d}r\,\mathrm{d}s$$
$$\geq \frac{t_0^2}{t_*^d}\alpha_L^2 \mathbb{1}_{\{|v|\leq\delta_L\}} = \frac{1}{9}t_*^{2-d}\alpha_L^2\mathbb{1}_{\{|v|\leq\delta_L\}}.$$

Finally, from Duhamel's formula (5.11) we obtain

$$f(t_*,x,v) \geq \frac{1}{9}e^{-t_*}t_*^{2-d}\alpha_L^2\mathbb{1}_{\{|v|\leq\delta_L\}},$$

which gives the result. □

Proof of Theorem 5.1.3 in the case of the linear relaxation Boltzmann equation. We use Lemma 5.2.3 which allows to apply directly Doeblin's Theorem 2.2.2 to obtain fast exponential convergence to equilibrium in the total variation distance. This rate is also explicitly calculable. Therefore, the proof follows. □

5.2.2 On the whole space with a confining potential

Now we consider the equation

$$\partial_t f + v\cdot\nabla_x f - \nabla_x\Phi(x)\cdot\nabla_v f = \mathcal{L}f, \tag{5.16}$$

where $\mathcal{L}$ is defined as in the previous section and $x, v \in \mathbb{R}^d$. We want to use a slightly different strategy to show the minorisation condition based on the fact that we instantaneously produce large velocities. We first need a result on the trajectories of particles under the action of the potential Φ. Always assuming that Φ is a $\mathcal{C}^2$ function, we consider the characteristic ordinary differential equations associated to the transport part of (5.16):

$$\begin{aligned}\dot{x} &= v\\ \dot{v} &= -\nabla\Phi(x),\end{aligned} \tag{5.17}$$

and we denote by $(X_t(x_0,v_0), V_t(x_0,v_0))$ the solution at time t to (5.17) with initial data

$$\begin{aligned}x(0) &= x_0,\\ v(0) &= v_0.\end{aligned}$$

Performing time integration twice, it clearly satisfies

$$X_t(x_0, v_0) = x_0 + v_0 t + \int_0^t \int_0^s \nabla\Phi(X_u(x_0, v_0))\,\mathrm{d}u\,\mathrm{d}s \tag{5.18}$$

for any $x_0, v_0 \in \mathbb{R}^d$ and any t for which it is defined. Intuitively the idea is that for small times we can approximate (X_t, V_t) by $(X_t^{(0)}, V_t^{(0)})$ which is a solution to the ordinary differential equation

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= 0, \end{aligned} \tag{5.19}$$

whose explicit solution is

$$(X_t^{(0)}, V_t^{(0)}) = (x_0 + v_0 t, v_0).$$

If we want to hit a point x_1 in time t then if we travel with the trajectory $X^{(0)}$ we just need to choose $v_0 = (x_1 - x_0)/t$. Now we choose an interpolation between $(X^{(0)}, V^{(0)})$ and (X, V). We denote it by $(X^{(\epsilon)}, V^{(\epsilon)})$ which is a solution to the ordinary differential equation

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= -\epsilon^2 \nabla\Phi(x), \end{aligned} \tag{5.20}$$

still with initial data (x_0, v_0). We calculate that

$$X_t^{(\epsilon)}(x_0, v_0) = X_{\epsilon t}\left(x_0, \frac{v_0}{\epsilon}\right), \qquad V_t^{(\epsilon)}(x_0, v_0) = \epsilon V_{\epsilon t}\left(x_0, \frac{v_0}{\epsilon}\right).$$

Now we can see from the ODE representation (and we will make this more precise later) that (X, V) is a C^1 map of (t, ϵ, x, v). Therefore if we fix t and x_0 we can define a C^1 map

$$F\colon [0,1] \times \mathbb{R}^d \to \mathbb{R}^d,$$

by

$$F(\epsilon, v) = X_t^{(\epsilon)}(x_0, v).$$

Then for $\epsilon = 0$ we can find v_* such that $F(0, v_*) = x_1$ as given above. Furthermore $\nabla F(0, v_*) \neq 0$ so by the implicit function theorem for all ϵ less than some ϵ_* we have a C^1 function $v(\epsilon)$ such that $F(\epsilon, v(\epsilon)) = x_1$. This means that

$$X_{\epsilon t}\left(x_0, \frac{v(\epsilon)}{\epsilon}\right) = x_1.$$

So if we take $s < \epsilon_* t$ then we can choose v such that $X_s(x_0, v) = x_1$. We now need to get quantitative estimates on ϵ_*, and we do this by tracking the constants in the proof of the contraction mapping theorem.

In order to make these ideas quantitative and to check that the solution is in fact C^1 we need to get bounds on (X_t, V_t) and $\nabla\Phi(X_t)$ for t is some fixed intervals. For the potentials of interest we will have that the solutions to these ODEs will exist for infinite time. We prove bounds on the solutions and $\nabla\Phi(X_t)$ for any potential:

Lemma 5.2.4. *Assume that the potential Φ is $\mathcal{C}^2$ in $\mathbb{R}^d$. Take $\lambda > 1$, $R > 0$ and $x_0, v_0 \in \mathbb{R}^d$ with $|x_0| \le R$. The solution $t \mapsto X_t(x_0, v_0)$ to (5.17) is defined (at least) for $|t| \le T$, with*

$$T := \min\left\{ \frac{(\lambda-1)R}{2|v_0|}, \frac{\sqrt{(\lambda-1)R}}{\sqrt{2C_{\lambda R}}} \right\}, \qquad C_{\lambda R} := \max_{|x|\le\lambda R} |\nabla\phi(x)|.$$

(It is understood that any term in the above minimum is $+\infty$ if the denominator is 0.) Also, it holds that

$$|X_t(x_0, v_0)| \le \lambda R \qquad \textit{for } |t| \le T.$$

Proof. By standard ODE theory, the solution is defined in some maximal (open) time interval I containing 0; if this maximal interval has any finite endpoint t_*, then $X_t(x_0, v_0)$ has to blow up as t approaches t_*. Hence if the statement is not satisfied, there must exist $t \in I$ with $|t| \le T$ such that $|X_t(x_0, v_0)| \ge \lambda R$. By continuity, one may take $t_0 \in I$ to be the "smallest" time when this happens: that is, $|t_0| \le T$ and

$$X_{t_0}(x_0, v_0) = \lambda R,$$
$$|X_{t_0}(x_0, v_0)| \le \lambda R \qquad \text{for } |t| \le |t_0|.$$

By (5.18) and using that $|t_0| \le T$ we have

$$\begin{aligned}\lambda R = |X_{t_0}(x_0, v_0)| &\le |x_0| + |v_0 t_0| + \frac{t_0^2}{2} \max\{|\nabla\phi(X_t(x_0, v_0))| \colon t \le t_0\} \\ &\le R + \frac{(\lambda-1)R}{2} + \frac{C_{\lambda R}}{2} t_0^2 = \frac{(\lambda+1)R}{2} + \frac{C_{\lambda R}}{2} t_0^2,\end{aligned}$$

which implies that

$$(\lambda - 1)R \le C_{\lambda R} t_0^2.$$

If $C_{\lambda R} = 0$ this is false; if $C_{\lambda R} > 0$, then this contradicts with that $|t_0| \le T$. □

We now follow the intuition given at the beginning of this section. However we collapse the variables ϵ and t together and consequently look at $X_t\left(x, \frac{v}{t}\right)$ which is intuitively less clear but algebraically simpler.

Lemma 5.2.5. *Assume that* $\Phi \in \mathcal{C}^2(\mathbb{R}^d)$*, and take* $x_0, x_1 \in \mathbb{R}^d$*. Let* $R := \max\{|x_0|, |x_1|\}$*. There exists* $0 < T_1 = T_1(R)$ *such that for any* $t \le T_1$ *we can find a* $|v_0| \le 4R$ *such that*

$$X_t\left(x_0, \frac{v_0}{t}\right) = x_1.$$

In fact, it is enough to take $T_1 > 0$ *such that*

$$CT_1^2 e^{CT_1^2} \le \frac{1}{4}, \qquad T_1 \le \frac{\sqrt{R}}{\sqrt{2C_{2R}}}, \qquad T_1 \le \frac{2\sqrt{R}}{\sqrt{C_{9R}}},$$

where

$$C := \sup_{|x| \le 9R} |D^2\Phi(x)|,$$

and $C_{\lambda R}$ *is defined in Lemma 5.2.4. Here* $D^2\Phi$ *denotes the Hessian matrix of* Φ*.*

Proof. We define

$$f(t, v) = X_t\left(x_0, \frac{v}{t}\right) - x_1, \qquad t \ne 0,\, v \in \mathbb{R}^d,$$
$$f(0, v) := x_0 + v - x_1, \qquad v \in \mathbb{R}^d.$$

Notice that due to Lemma 5.2.4 with $\lambda = 9$, this is well-defined whenever

$$|t| \le \frac{2\sqrt{R}}{\sqrt{C_{9R}}} =: T_2, \;\; |v| \le 4R.$$

Our goal is to find a neighbourhood of $t = 0$ on which there exists $v = v(t)$ with $f(t, v(t)) = 0$, for which we will use the implicit function theorem.

Now, notice that we have

$$f(0, x_1 - x_0) = 0$$

and

$$\frac{\partial f}{\partial v_i}(0, x_1 - x_0) = 1, \;\; i = 1, \ldots, d.$$

We can apply the implicit function theorem to find a neighbourhood I of $t = 0$ and a function $v = v(t)$ such that $f(t, v(t)) = 0$ for $t \in I$. However, since we need to estimate the size of I and of $v(t)$, we carry out a constructive proof.

Take $v_0, v_1 \in \mathbb{R}^d$ with $|v_0|, |v_1| \leq 4R$, and denote

$$\tilde{v}_0 := \frac{v_0}{t}, \qquad \tilde{v}_1 := \frac{v_1}{t}.$$

By (5.18), for all $0 < t \leq T_2$ we have

$$X_t(x_0, \tilde{v}_1) - X_t(x_0, \tilde{v}_0) = (\tilde{v}_1 - \tilde{v}_0)t + \int_0^t \int_0^s \nabla\phi(X_u(x_0, \tilde{v}_1)) - \nabla\phi(X_u(x_0, \tilde{v}_0))\,\mathrm{d}u\,\mathrm{d}s. \tag{5.21}$$

Take any $T_1 \leq T_2$, to be fixed later. Then Lemma 5.2.4 implies, for all $0 \leq t \leq T_1$,

$$|X_t(x_0, \tilde{v}_1) - X_t(x_0, \tilde{v}_0)| \leq |\tilde{v}_1 - \tilde{v}_0|t + CT_1 \int_0^t |X_u(x_0, \tilde{v}_1) - X_u(x_0, \tilde{v}_0)|\,\mathrm{d}u.$$

by Gronwall's Lemma we have

$$|X_t(x_0, \tilde{v}_1) - X_t(x_0, \tilde{v}_0)| \leq |\tilde{v}_1 - \tilde{v}_0| t e^{CT_1 t} \qquad \text{for } 0 < t \leq T_1.$$

Using this again in (5.21) we have

$$\begin{aligned} |X_t(x_0, \tilde{v}_1) - X_t(x_0, \tilde{v}_0) - (\tilde{v}_1 - \tilde{v}_0)t| &\leq |\tilde{v}_1 - \tilde{v}_0| CT_1 \int_0^t u e^{CT_1 u}\,\mathrm{d}u \\ &\leq |\tilde{v}_1 - \tilde{v}_0| t\, CT_1^2 e^{CT_1^2}. \end{aligned}$$

Taking T_1 such that

$$CT_1^2 e^{CT_1^2} \leq \frac{1}{4} \tag{5.22}$$

we have

$$|X_t(x_0, \tilde{v}_1) - X_t(x_0, \tilde{v}_0) - (\tilde{v}_1 - \tilde{v}_0)t| \leq \frac{1}{4}|\tilde{v}_1 - \tilde{v}_0| t$$

which is the same as

$$\left| X_t\left(x_0, \frac{v_1}{t}\right) - X_t\left(x_0, \frac{v_0}{t}\right) - (v_1 - v_0)\right| \leq \frac{1}{4}|v_1 - v_0|, \tag{5.23}$$

for any $0 < t \leq T_1$ and any v_0, v_1 with $|v_0|, |v_1| \leq 4R$. Now, for any $0 \leq t \leq T_1$ and $|v| \leq 4R$ we define

$$A_t(v) = v - f(t, v).$$

A fixed point of $A_t(v)$ satisfies $f(t, v) = 0$, and by (5.23) $A_t(v)$ is contractive:

$$|A_t(v_1) - A_t(v_0)| \leq \frac{1}{4}|v_1 - v_0| \qquad \text{for } 0 \leq t \leq T_1,\ |v| \leq 4R.$$

(Equation (5.23) proves this for $0 < t \le T_1$, and for $t = 0$ it is obvious.) In order to use the Banach fixed-point theorem we still need to show that the image of A_t is inside the set with $|v| \le 4R$. Using (5.23) for $v_1 = 0$, $v_0 = v$ we also see that

$$\left| X_t(x_0, 0) - X_t\left(x_0, \frac{v}{t}\right) + v \right| \le \frac{1}{4}|v|,$$

which gives

$$|A_t(v) + x_1 - X_t(x_0, 0)| \le \frac{1}{4}|v|,$$

so

$$|A_t(v)| \le \frac{1}{4}|v| + |x_1| + |X_t(x_0, 0)| \le 2R + |X_t(x_0, 0)|. \tag{5.24}$$

If we take

$$T_1 \le \frac{\sqrt{R}}{\sqrt{2C_{2R}}} \tag{5.25}$$

then Lemma 5.2.4 (used for $\lambda = 2$) shows that

$$|X_t(x_0, 0)| \le 2R \qquad \text{for } 0 \le t \le T_1,$$

and from (5.24) we have

$$|A_t(v)| \le 4R \quad \text{for} \quad 0 < t \le T_1.$$

Hence, as long as T_1 satisfies (5.22) and (5.25), A_t has a fixed point $|v|$ for any $0 < t \le T_1$, and this fixed point satisfies $|v| \le 4R$. □

Lemma 5.2.6. *Assume that the potential $\Phi \in \mathcal{C}^2(\mathbb{R}^d)$ is bounded below, and let T_s denote the transport semigroup associated to the operator $f \mapsto -v \cdot \nabla_x f + \nabla_x \Phi(x) \cdot \nabla_v f$. Given any $R > 0$ there exists a time $T_1 > 0$ such that for any $0 < s < T_1$ one can find constants $\alpha, R', R_2 > 0$ (depending on s and R) such that*

$$\int_{B(R')} T_s(\delta_{x_0} \mathbb{1}_{\{|v| \le R_2\}})\, \mathrm{d}v \ge \alpha \mathbb{1}_{\{|x| \le R\}}, \tag{5.26}$$

for any x_0 with $|x_0| \le R$. The constants α, R', R_2 are uniformly bounded in bounded intervals of time; that is, for any closed interval $J \subseteq (0, T_1)$ one can find α, R', R_2 for which the inequality holds for all $s \in J$.

Proof. Since the statement is invariant if Φ changes by an additive constant, we may assume that $\Phi \ge 0$ for simplicity. Using Lemma 5.2.5 we find T_1 such that for any $s < T_1$ and every $x_1 \in B(R)$ there exists $v \in B(4R)$ (depending on x_0, x_1 and s) such that

$$X_s\left(x_0, \frac{v}{s}\right) = x_1.$$

Since $v/s \in B(4R/s)$, call $R_2 := 4R/s$. We see that for every $x_1 \in B(0,R)$ there is at least one $u \in \mathbb{R}^d$ such that

$$(x_1, u) \in T_s\left(\{x_0\} \times \{|v| \leq R_2\}\right).$$

In other words,

$$X_s(x_0, \{|v| \leq R_2\}) \supseteq B(0,R). \tag{5.27}$$

This essentially contains our result, and we just need to carry out a technical argument to complete it and estimate the constants α and R'. For any compactly supported, continuous and positive $\varphi\colon \mathbb{R}^d \to \mathbb{R}$ we have

$$\begin{aligned}\int_{\mathbb{R}^d} \varphi(x) \int_{B(R')} & T_s(\delta_{x_0} \mathbb{1}_{\{|v|\leq R_2\}}) \, \mathrm{d}v \, \mathrm{d}x \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbb{1}_{\{|V_s(x,v)|<R'\}} \, \varphi(X_s(x,v)) \delta_{x_0}(x) \mathbb{1}_{\{|v|\leq R_2\}}) \, \mathrm{d}v \, \mathrm{d}x \\ &= \int_{|v|\leq R_2} \mathbb{1}_{\{|V_s(x_0,v)|<R'\}} \, \varphi(X_s(x_0,v) \, \mathrm{d}v,\end{aligned} \tag{5.28}$$

since the characteristics map $(x,v) \mapsto (X_s(x,v), V_s(x,v))$ is measure-preserving. If we write the energy as $H(x,v) = |v|^2/2 + \Phi(x)$ and call

$$E_0 := \sup\{H(x,v) \ : \ |x| < R, |v| < R_2\}.$$

Then for all $s \geq 0$

$$E(X_s(x_0,v), V_s(x_0,v)) \leq E_0,$$

and in particular

$$|V_s(x_0,v)| \leq \sqrt{2E_0}.$$

If we take $R' > \sqrt{2E_0}$ then the term $\mathbb{1}_{\{|V_s(x_0,v)|<R'\}}$ is always 1 in (5.28) and we get

$$\int_{\mathbb{R}^d} \varphi(x) \int_{B(R')} T_s(\delta_{x_0} \mathbb{1}_{\{|v|\leq R_2\}}) \, \mathrm{d}v \, \mathrm{d}x = \int_{|v|\leq R_2} \varphi(X_s(x_0,v)) \, \mathrm{d}v.$$

Now, take an $M > 0$ such that $|\operatorname{Jac}_v X_s(x,v)| \leq M$ for all (x,v) with $|x| \leq R$ and $|v| \leq R_2$. (Notice that M depends only on Φ, R and R_2.) Then

$$\begin{aligned}\int_{|v|\leq R_2} \varphi(X_s(x_0,v))\,\mathrm{d}v &\geq \frac{1}{M}\int_{|v|\leq R_2} \varphi(X_s(x_0,v))|\operatorname{Jac}_v X_s(x_0,v)|\,\mathrm{d}v \\ &= \frac{1}{M}\int_{X_s(x_0,\{|v|\leq R_2\})} \varphi(x)\,\mathrm{d}x \geq \frac{1}{M}\int_{B(0,4R)} \varphi(x)\,\mathrm{d}x,\end{aligned}$$

where we have used (5.27) in the last step. In sum we find that

$$\int_{\mathbb{R}^d} \varphi(x) \int_{\mathbb{R}^d} T_s(\delta_{x_0} \mathbb{1}_{\{|v|\leq R_2\}})\,\mathrm{d}v\,\mathrm{d}x \geq \frac{1}{M}\int_{B(0,R)} \varphi(x)\,\mathrm{d}x$$

for all compactly supported, continuous and positive functions φ. This directly implies the result. □

Lemma 5.2.7 (Doeblin condition for linear relaxation Boltzmann equation with a confining potential)**.** *Let the potential* $\Phi\colon \mathbb{R}^d \to \mathbb{R}$ *be a* $\mathcal{C}^2$ *function with compact level sets. Given* $t > 0$ *and* $K > 0$ *there exist constants* $\alpha, \delta_X, \delta_V > 0$ *such that any solution* f *to equation* (5.16) *with initial condition* $f_0 \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ *supported on* $B(0,K) \times B(0,K)$ *satisfies*

$$f(t,x,v) \geq \alpha \mathbb{1}_{\{|x|<\delta_X\}}\, \mathbb{1}_{\{|v|<\delta_V\}}$$

in the sense of measures.

Proof. Fix any $t, K > 0$. Set

$$H_{\max}(K) = \max\left\{H(x,v) = |v|^2/2 + \Phi(x)\colon x \in B(0,K), v \in B(0,K)\right\},$$

and then define

$$R := \max\{|x| \;:\; \Phi(x) \leq H_{\max}(K)\}.$$

Since our conditions on Φ imply that its level sets are compact we know that R is finite. We use Lemma 5.2.6 to find constants $\alpha, R_2 > 0$ and an interval $[a,b] \subseteq (0,t)$ such that

$$\int_{\mathbb{R}^d} T_s(\delta_{x_0}\mathbb{1}_{\{|v|\leq R_2\}})\,\mathrm{d}v \geq \alpha \mathbb{1}_{\{|x|\leq R\}},$$

for any x_0 with $|x_0| \leq R$ and any $s \in [a,b]$. From Lemma 5.2.1 we will use that there exists a constant $\alpha_L > 0$ such that

$$\mathcal{L}^+ g(x,v) \geq \alpha_L \left(\int_{\mathbb{R}^d} g(x,u)\,\mathrm{d}u\right) \mathbb{1}_{\{|v|\leq R_2\}} \tag{5.29}$$

for all nonnegative measures g. We first notice that we can do the same estimate as in formula (5.11), where now $(T_t)_{t\geq 0}$ represents the semigroup generated by the operator $-v\cdot\nabla_x f+\nabla_x\Phi(x)\cdot\nabla_v f$; thus

$$e^t f_t \geq \int_0^t \int_0^s T_{t-s}\mathcal{L}^+ T_{s-r}\mathcal{L}^+ T_r f_0 \,\mathrm{d}r\,\mathrm{d}s. \tag{5.30}$$

Take $x_0, v_0 \in B(0,K)$, and call $f_0 := \delta_{(x_0,v_0)}$. For all r we have by the definition of R that

$$|X_r(x_0,v_0)| \leq R \qquad \text{for all } 0 \leq r. \tag{5.31}$$

For any $r>0$, since $T_r f_0 = \delta_{(X_r(x_0,v_0),V_r(x_0,v_0))}$, using (5.12) gives

$$\mathcal{L}^+ T_r f_0 \geq \alpha_L \delta_{X_r(x_0,v_0)}(x)\mathbb{1}_{\{|v|\leq R_2\}}.$$

Then, using (5.31) and our two lemmas, whenever $s-r\in[a,b]$ we have

$$\begin{aligned}
\mathcal{L}^+ T_{s-r}\mathcal{L}^+ T_r f_0 &\geq \alpha_L \left(\int_{\mathbb{R}^d} T_{s-r}\mathcal{L}^+ T_r f_0 \,\mathrm{d}u\right)\mathbb{1}_{\{|v|\leq R_2\}}\\
&\geq \alpha_L^2 \left(\int_{\mathbb{R}^d} T_{s-r}\Big(\delta_{X_r(x_0,v_0)}(x)\mathbb{1}_{\{|u|\leq R_2\}}\Big)\,\mathrm{d}u\right)\mathbb{1}_{\{|v|\leq R_2\}}\\
&\geq \alpha_L^2 \alpha\, \mathbb{1}_{\{|x|\leq R\}}\mathbb{1}_{\{|v|\leq R_2\}}.
\end{aligned}$$

We now need to allow for a final bit of movement along the flow T_{t-s}. By the continuity of the flow, there exist $\epsilon>0$ sufficiently small so that for all $0\leq\tau\leq\epsilon$ we have

$$T_\tau\Big(\mathbb{1}_{B(R)}(x)\mathbb{1}_{B(R_2)}(v)\Big) \geq \mathbb{1}_{B(R/2)}(x)\mathbb{1}_{B(R_2/2)}(v).$$

Then for all t,s,r such that $t-s\leq\epsilon$ and $s-r\in(a,b)$ we have

$$T_{t-s}\mathcal{L}^+ T_{s-r}\mathcal{L}^+ T_r f_0 \geq \alpha_L^2\alpha\, \mathbb{1}_{\{|x|\leq R/2\}}\, \mathbb{1}_{\{|v|\leq R_2/2\}}.$$

We have then

$$\begin{aligned}
\int_0^t\int_0^s T_{t-s}\mathcal{L}^+ T_{s-r}\mathcal{L}^+ T_r f_0 \,\mathrm{d}r\,\mathrm{d}s &\geq \alpha_L^2\alpha \int_{t-\epsilon}^t \int_{s-b}^{s-a} \mathbb{1}_{\{|x|\leq R/2\}}\mathbb{1}_{\{|v|\leq R_2/2\}}\,\mathrm{d}r\,\mathrm{d}s\\
&= \alpha_L^2\alpha\epsilon(b-a)\mathbb{1}_{\{|x|\leq R/2\}}\mathbb{1}_{\{|v|\leq R_2/2\}}.
\end{aligned}$$

Finally, from Duhamel's formula (5.30) we obtain

$$f(t,x,v) \geq e^{-t}\alpha_L^2\alpha\epsilon(b-a)\, \mathbb{1}_{\{|x|\leq R/2\}}\mathbb{1}_{\{|v|\leq R_2/2\}},$$

which gives the result. □

Lemma 5.2.8 (Lyapunov condition)**.** *Suppose that $\Phi(x)$ is a $\mathcal{C}^2$ function satisfying*

$$x \cdot \nabla \Phi(x) \geq \gamma_1 |x|^2 + \gamma_2 \Phi(x) - A$$

for positive constants γ_1 γ_2, A. Then we have that

$$V(x,v) = 1 + \Phi(x) + \frac{1}{2}|v|^2 + \frac{1}{4} x \cdot v + \frac{1}{8}|x|^2$$

is a function for which the semigroup satisfies Hypothesis 2.2.2.

Remark 5.2.1. *If Φ is superquadratic at infinity (which is implied by earlier assumptions) then V is equivalent to $1 + H(x,v)$ where the energy is defined as $H(x,v) = |v|^2/2 + \Phi(x)$. So the total variation distance weighted by V is equivalent to the total variation distance weighted by $1 + H(x,v)$.*

Proof. We look at the forwards operator acting on an observable ϕ,

$$\mathcal{U}\phi = v \cdot \nabla_x \phi - \nabla_x \Phi(x) \cdot \nabla_v \phi + \mathcal{L}^* \phi =: \mathcal{T}^* \phi + \mathcal{L}^* \phi,$$

where $\mathcal{L}^*$ is the adjoint of the linear relaxation Boltzmann operator $\mathcal{L}$, given by

$$\mathcal{L}^* \phi(x,v) = \int \phi(x,u) \mathcal{M}(u) \,\mathrm{d}u - \phi(x,v).$$

We want a function $V(x,v)$ such that

$$\mathcal{U} V \leq -\lambda V + K$$

for some constants $\lambda > 0, K \geq 0$. We need to make the assumption that

$$x \cdot \nabla_x \Phi(x) \geq \gamma_1 |x|^2 + \gamma_2 \Phi(x) - A, \tag{5.32}$$

for some positive constants γ_1, γ_2, A. We then try the function

$$V(x,v) = H(x,v) + \alpha x \cdot v + \beta |x|^2 = \Phi(x) + \frac{1}{2}|v|^2 + \alpha x \cdot v + \beta |x|^2,$$

with $\alpha, \beta > 0$ to be fixed later. We want this to be positive so we impose $\alpha^2 < 2\beta$. Using that

$$\mathcal{L}^*(|v|^2) = d - |v|^2, \qquad \mathcal{L}^*(x \cdot v) = -x \cdot v, \qquad \mathcal{L}^*(\Phi(x)) = \mathcal{L}^*(|x|^2) = 0$$

and that

$$\mathcal{T}^*(H(x,v)) = 0, \qquad \mathcal{T}^*(x\cdot v) = |v|^2 - x\cdot\nabla_x\Phi(x), \qquad \mathcal{T}^*(|x|^2) = 2x\cdot v,$$

we see that

$$\begin{aligned}\mathcal{U}V =& \frac{d}{2} - \frac{1}{2}|v|^2 - \alpha x\cdot v + \alpha|v|^2 - \alpha x\cdot\nabla_x\Phi(x) + 2\beta x\cdot v\\ \leq& C' - \left(\frac{1}{2}-\alpha\right)|v|^2 + (2\beta-\alpha)x\cdot v - \alpha\gamma_1|x|^2 - \alpha\gamma_2\Phi(x),\end{aligned}$$

where we have used (5.32), and $C' := \frac{d}{2} + \alpha A$. Now, taking $\alpha = 1/4, \beta = 1/8$,

$$\begin{aligned}\mathcal{U}V =& C' - \frac{1}{4}|v|^2 - \frac{\gamma_1}{4}|x|^2 - \frac{\gamma_2}{4}\Phi(x)\\ \leq& C' - \min\{\gamma_1, 1\}\frac{1}{4}(|x|^2+|v|^2) - \frac{\gamma_2}{4}\Phi(x)\\ \leq& C' - \min\{\gamma_1, 1\}\frac{1}{4}\left(\frac{1}{2}|v|^2 + \frac{1}{4}x\cdot v + \frac{1}{8}|x|^2\right) - \frac{\gamma_2}{4}\Phi(x).\end{aligned}$$

So $V(x,v)$ works with

$$\lambda = \frac{1}{4}\min\{\gamma_1,\gamma_2,1\}.$$

□

Proof of Theorem 5.1.4 in the case of the linear relaxation Boltzmann equation. We give the proof by applying Harris's Theorem since Lemmas 5.2.7 and 5.2.8 show that the equation satisfies the hypotheses of the theorem. □

5.2.3 Subgeometric convergence

When we do not have the superquadratic behaviour of the confining potential at infinity we can still use a Harris type theorem to show convergence to equilibrium. However since the confining potential is weaker we can only obtain subgeometric rates of convergence. We use the subgeometric Harris's Theorem given in Section 2.2.3 of Chapter 2 which can be found in Section 4 of [69]. Now instead of our earlier assumption on the confining potential Φ, we instead make a weaker assumption that Φ is a C^2 function satisfying

$$x\cdot\nabla_x\Phi(x) \geq \gamma_1\langle x\rangle^{2\beta} + \gamma_2\Phi(x) - A,$$

for some positive constant γ_1, γ_2, A where

$$\langle x\rangle = \sqrt{1+|x|^2},$$

and $\beta \in (0,1)$.

Proof of Theorem 5.1.5 in the case of the linear relaxation Boltzmann equation. We have already proved the minorisation condition. We can also replicate the calculations for the Lyapunov function to get that in this new situation, take the V in Lemma 5.2.8, we have that

$$\mathcal{U}V \le C' - \frac{1}{4}|v|^2 - \frac{\gamma_1}{4}\langle x\rangle^{2\beta} - \frac{\gamma_2}{4}\Phi(x).$$

for Since $x, y \ge 1$

$$(x+y)^\beta \le x^\beta + y^\beta.$$

We obtain

$$\begin{aligned}
\mathcal{U}V \le& C' - \min\{\gamma_1, 1\}\frac{1}{4}\left(\langle v\rangle^2 + \langle x\rangle^{2\beta}\right) - \frac{\gamma_2}{4}\Phi(x)\\
\le& C'' - \min\{\gamma_1, 1\}\frac{1}{4}\left(1 + |x|^2 + |v|^2\right)^\beta - \frac{\gamma_2}{4}\Phi(x)^\beta\\
\le& C'' - \lambda\left(1 + \frac{1}{2}|v|^2 + \frac{1}{4}x\cdot v + \frac{1}{8}|x|^2\right)^\beta - \lambda\Phi(x)^\beta\\
\le& C'' - \lambda\left(\Phi(x) + \frac{1}{2}|v|^2 + \frac{1}{4}x\cdot v + \frac{1}{8}|x|^2\right)^\beta,
\end{aligned}$$

for some constant $\lambda, C'' > 0$ that can be explicitly computed, so we have that

$$\mathcal{U}V \le -\lambda V^\beta + C''.$$

This means we can take $\phi(s) = 1 + s^\beta$. Therefore, for u large

$$H_\phi(u) = \int_1^u \frac{1}{1+t^\beta}\,\mathrm{d}t \sim 1 + u^{1-\beta},$$

and for t large

$$H_\phi^{-1}(t) \sim 1 + t^{1/(1-\beta)}$$

and

$$\phi \circ H_\phi^{-1}(t) \sim (1+t)^{\beta/(1-\beta)}.$$

□

5.3 The linear Boltzmann equation

We recall that the *linear Boltzmann equation* is given by

$$\partial_t f + v \cdot \nabla_x f - (\nabla_x \Phi \cdot \nabla_v f) = Q(f, \mathcal{M}),$$

where Φ is a $\mathcal{C}^2$ potential and $\mathcal{M}(v) := (2\pi)^{-d/2} \exp(-|v|^2/2)$, and Q is the Boltzmann operator and B is the *collision kernel*. We also consider the same equation posed for $x \in \mathbb{T}^d$, $v \in \mathbb{R}^d$, without any potential Φ:

$$\partial_t f + v \cdot \nabla_x f = Q(f, \mathcal{M}).$$

This equation models gas particles interacting with a backgroundmedium which is already in equilibrium. Moreover, it has been used in describing many other systems like radiative transfer, neutron transportation, cometary flow and dust particles. The spatially homogeneous case has been studied in [76, 21, 30]. The kinetic equations (5.7) or (5.6) fit into the general framework in [87, 53], so convergence to equilibrium in weighted L^2 norms may be proved by using the techniques described there. Here the interest is partly that this is a more complex and physically relevant operator. Also, it presents less globally uniform behaviour in v which means that we have to use a Lyapunov function even on the torus. Apart from this, the strategy is very similar to that from the linear relaxation Boltzmann equation. The full Boltzmann equation has been studied as a Markov process in [60], the linear case is similar and more simple. It is well known that this equation preserves positivity and mass, which follows from standard techniques both in the spatially homogeneous case and the case with transport. The Lyapunov condition on the torus and the bound below on the jump operator have to be verified in this situation.

When we consider the situation where the spatial variable is in $\mathbb{R}^d$ and we have a confining potential; in the hard sphere case ($\gamma > 0$), the operator $\mathcal{L}^+$ acting on $x \cdot v$ produces error terms which are difficult to deal with. We show that when we have hard spheres with $\gamma > 0$ we can still show exponential convergence when $\Phi(x)$ is growing at least as fast as $|x|^{\gamma+2}$. In the subgeometric case we suppose $\Phi(x)$ grows at least as fast as $|x|^{\epsilon+1}$, $\epsilon > 0$.

We begin by proving lemmas which are useful for proving the Doeblin condition in both situations. We want to reduce to a similar situation to the linear relaxation Boltzmann equation.

Lemma 5.3.1. *Let f be a solution to (1.24) or (1.26), and define the energy $H(x,v) := |v|^2/2$ on the torus for (1.24) or $H(x,v) := \Phi(x) + |v|^2/2$ in the whole space for (1.26), where Φ is a $\mathcal{C}^2$ potential bounded below. Take $E_0 > 0$ and assume that f has initial condition $f_0 = \delta_{(x_0,v_0)}$ with*

$$H(x_0, v_0) \le E_0.$$

Then there exists a constant $C_1 > 0$ such that

$$f(t,x,v) \ge e^{-tC_1} \int_0^t \int_0^s T_{t-s} \widetilde{\mathcal{L}}^+ T_{s-r} \widetilde{\mathcal{L}}^+ T_r (\mathbb{1}_E f_0(x,v)) \, \mathrm{d}r \, \mathrm{d}s,$$

where

$$\widetilde{\mathcal{L}}^+ g := \mathbb{1}_E \mathcal{L}^+ g, \qquad E := \left\{ (x,v) \in \mathbb{R}^d \times \mathbb{R}^d \,:\, H(x,v) \le E_0 \right\}.$$

Proof. Let $(X_t(x,v), V_t(x,v))$ be the solution to the backward characteristic equations obtained from the transport part of either (1.24) or (1.26). Let us call

$$\Sigma(s,t,x,v) = e^{\int_s^t \kappa(V_r(x,v)) \, \mathrm{d}r}.$$

By Duhamel's formula again we obtain

$$f(t,x,v) = \Sigma(0,t,x,v) T_t f_0 + \int_0^t \Sigma(0, t-s, x, v)(T_{t-s} \mathcal{L}^+ f_s)(x,v) \, \mathrm{d}s$$

If a function $g = g(x,v)$ has support on the set

$$E := \{(x,v) \,:\, H(x,v) \le E_0\},$$

then the same is true of $T_t g$ (since the transport part preserves energy). On the set E we have, using (5.3),

$$\int_s^t \kappa(V_r(x,v)) \, \mathrm{d}r \le (t-s) C \left(1 + 2E_0\right)^{\gamma/2} =: (t-s) C_1, \qquad \text{for all} (x,v) \in E.$$

Hence,

$$\begin{aligned} f(t,x,v) &\ge \Sigma(0,t,x,v) T_t(\mathbb{1}_E f_0) + \int_0^t \Sigma(0,t-s,x,v)(T_{t-s}(\mathbb{1}_E \mathcal{L}^+ f_s))(x,v) \, \mathrm{d}s \\ &\ge e^{-tC_1} T_t(\mathbb{1}_E f_0) + \int_0^t e^{-(t-s)C_1} (T_{t-s}(\mathbb{1}_E \mathcal{L}^+ f_s))(x,v) \, \mathrm{d}s \\ &= e^{-tC_1} T_t f_0 + \int_0^t e^{-(t-s)C_1} (T_{t-s}(\widetilde{\mathcal{L}}^+ f_s))(x,v) \, \mathrm{d}s, \end{aligned}$$

where we define

$$\widetilde{\mathcal{L}}^+ g := \mathbb{1}_E \mathcal{L}^+ g.$$

Iterating this formula we obtain the result. □

We have that

$$\mathcal{L}^+ f = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} b\left(\frac{v - v_*}{|v - v_*|} \cdot \sigma\right) |v - v_*|^\gamma f(v') \mathcal{M}(v'_*) \, \mathrm{d}\sigma \, \mathrm{d}v_*.$$

Using the Carleman representation we rewrite this as

$$\mathcal{L}^+ f = \int_{\mathbb{R}^d} \frac{f(v')}{|v - v'|^{d-1}} \int_{E_{(v,v')}} B(|u|, \xi) \mathcal{M}(v'_*) \, \mathrm{d}v'_* \, \mathrm{d}v,$$

where $E_{(v,v')}$ denotes the hyperplane $\left\{v'_* \in \mathbb{R}^d \mid (v - v') \cdot (v - v'_*) = 0\right\}$, and the integral in v'_* is understood to be with respect to $(n-1)$-dimensional measure on this hyperplane. We want to bound this in the manner of Lemma 5.2.1 from the first part. We consider hard spheres and no angular dependence, which means

$$B(|u|, \xi) = C|u|^\gamma \xi^{d-2}$$

with $\gamma \geq 0$. We also have that

$$\xi = \frac{|v - v'|}{|2v - v' - v'_*|}, \qquad |u| = |2v - v' - v'_*|.$$

So we have that

$$\mathcal{L}^+ f = \int_{\mathbb{R}^d} \frac{f(v')}{|v - v'|} \int_{E_{(v,v')}} |2v - v' - v'_*|^{\gamma - d - 2} \mathcal{M}(v'_*) \, \mathrm{d}v'_* \, \mathrm{d}v'.$$

Using this we can give the following lower bound of $\mathcal{L}^+$, which the reader can compare with the Lemma 5.2.1:

Lemma 5.3.2. *Consider the positive part $\mathcal{L}^+$ of the linear Boltzmann operator for hard spheres, assuming (5.1) with $\gamma \geq 0$, and (5.2). For all $R_L, r_L > 0$, there exists $\alpha > 0$ such that for all $g \in \mathcal{P}$ we have*

$$\mathcal{L}^+ g(v) \geq \alpha \int_{B(R_L)} g(u) \, \mathrm{d}u \qquad \textit{for all } v \in \mathbb{R}^d \textit{ with } |v| \leq r_L.$$

Proof. First we note that on $E_{(v,v')}$ we have

$$|2v - v' - v'_*|^{-d-2} \geq C_d \exp\left(-\frac{1}{2}|v - v'_*|^2 - \frac{1}{2}|v - v'|^2\right).$$

Then since $\gamma \geq 0$ we have

$$|2v - v' - v'_*|^{\gamma} = \left(|v - v'|^2 + |v - v'_*|^2\right)^{\gamma/2} \geq |v - v'_*|^{\gamma}.$$

So this means that

$$\begin{aligned}
\int_{E_{(v,v')}} & |2v - v' - v'_*|^{\gamma-d-2}\mathcal{M}(v'_*)\,\mathrm{d}v'_* \\
&\geq Ce^{-|v-v'|^2/2}\int_{E_{(v,v')}} |v - v'_*|^{\gamma} \exp\left(-\frac{1}{2}|v - v'_*|^2 - \frac{1}{2}|v'_*|^2\right)\,\mathrm{d}v'_* \\
&\geq Ce^{-|v-v'|^2/2-|v|^2/2}\int_{E_{(v,v')}} |v - v'_*|^{\gamma}e^{-|v-v'_*|^2}\,\mathrm{d}v'_* \\
&= C'e^{-|v-v'|^2/2-|v|^2/2}.
\end{aligned}$$

So we have that

$$\begin{aligned}
\mathcal{L}^+ f(v) &\geq C\int_{\mathbb{R}^d} f(v')|v - v'|^{-1}e^{-|v-v'|^2/2-|v|^2/2}\,\mathrm{d}v' \\
&\geq C\int_{\mathbb{R}^d} f(v')e^{-2|v'|^2-3|v|^2} \\
&\geq Ce^{-2R_L^2}e^{-3|v|^2}\int_{B(0,R_L)} f(v')\,\mathrm{d}v',
\end{aligned}$$

which is a similar bound to the one we found in Lemma 5.2.1. This gives the result by choosing $\alpha := C\exp(-2R_L^2 - 3|r_L|^2)$. □

5.3.1 On the torus

Now we consider the spatial variable on the torus. For the minorisation condition we can argue almost exactly as for the linear relaxation Boltzmann equation.

Lemma 5.3.3 (Doeblin condition)**.** *Assume* (5.1) *with* $\gamma \geq 0$*, and* (5.2)*. Given* $t_* > 0$ *and* $R > 0$ *there exist constants* $0 < \alpha < 1$*,* $\delta_L > 0$ *such that any solution* $f = f(t,x,v)$ *to the linear Boltzmann equation* (1.24) *on the torus with initial condition* $f_0 = \delta_{(x_0,v_0)}$ *with* $|v_0| \leq R$ *satisfies*

$$f(t_*, x, v) \geq \alpha \mathbb{1}_{\{|v|\leq\delta_L\}}$$

in the sense of measures.

Proof. Take $f_0 := \delta_{(x_0,v_0)}$, where $(x_0, v_0) \in \mathbb{T}^d \times \mathbb{R}^d$ is an arbitrary point with $|v_0| \le R$. From Lemma 5.2.2 (with $R > \sqrt{d}$ and $t_0 := t_*/3$) we will use that there exist $\delta_L, R' > 0$ such that

$$\int_{B(R')} T_t\Big(\delta_{x_0}(x)\mathbb{1}_{\{|v|\le\delta_L\}}\Big)\,\mathrm{d}v \ge \frac{1}{t^d} \qquad \text{for all } x_0 \in \mathbb{T}^d,\ t > t_0. \tag{5.33}$$

Also, Lemma 5.3.2 gives an $\alpha > 0$ such that

$$\mathcal{L}^+ g \ge \alpha \left(\int_{B(R_L)} g(x,u)\,\mathrm{d}u\right) \mathbb{1}_{\{|v|\le\delta_L\}}, \tag{5.34}$$

where $R_L := \max\{R', R\}$. Finally, from Lemma 5.3.1 we can find $C_1 > 0$ (depending on R) such that

$$f(t,x,v) \ge e^{-tC_1} \int_0^t \int_0^s T_{t-s}\widetilde{\mathcal{L}}^+ T_{s-r}\widetilde{\mathcal{L}}^+ T_r(\mathbb{1}_E \delta_{(x_0,v_0)})\,\mathrm{d}r\,\mathrm{d}s,$$

where E is the set of points with energy less than E_0, with

$$E_0 := \max\left\{\frac{R^2}{2}, \frac{\delta_L^2}{2}\right\},$$

and we recall that $\widetilde{\mathcal{L}}^+ f := \mathbb{1}_E \mathcal{L}^+ f$. Due to the choice of E_0, we see that equation (5.33) also holds with $\widetilde{\mathcal{L}}^+$ in the place of $\mathcal{L}^+$. One can then carry out the same proof as in Lemma 5.2.3, using estimates (5.33) and (5.34) instead of the corresponding ones there. □

Since our Doeblin condition holds only on sets which are bounded in $|v|$, we do need a Lyapunov functional in this case (as opposed to the linear relaxation Boltzmann equation, where Lemma 5.2.3 gives lower bound for all starting conditions (x, v)). Testing with $V = v^2$ involves proving a result similar to the moment control result from [21]. Instead of the σ representation we use the n-representation for the collisions:

$$v' = v - n(u \cdot n), \qquad v'_* = v_* + n(u \cdot n).$$

By our earlier assumption, the collision kernel can be written as

$$\tilde{B}(|v - v_*|, |\xi|) = |v - v_*|^\gamma \tilde{b}(|\xi|),$$

where
$$\xi := \frac{u \cdot n}{|u|}, \qquad u := v - v_*.$$
Here the $\tilde{B}, \tilde{b}$ are different from those in the σ representation because of the change of variables. We also have by assumption that $\tilde{b}$ is normalised, that is,
$$\int_{\mathbb{S}^d} \tilde{b}(|w \cdot n|) \, \mathrm{d}n = 1$$
for all unit vectors $w \in \mathbb{S}^{d-1}$.

Lemma 5.3.4. *The function $V(x, v) = |v|^2$ is a Lyapunov function for the linear Boltzmann equation on the torus in the sense that it is a function for which the associated semigroup satisfies Hypothesis 2.2.2.*

Proof. Let $\mathcal{L}$ be the linear Boltzmann operator. Using the weak formulation of $\mathcal{L}$ we have
$$\int_{\mathbb{R}^d} \mathcal{L}(f)|v|^2 \, \mathrm{d}v = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} f(v)\mathcal{M}(v_*)|v - v_*|^\gamma \tilde{b}(|\xi|)(|v'|^2 - |v|^2) \, \mathrm{d}n \, \mathrm{d}v \, \mathrm{d}v_*.$$
In other words,
$$\mathcal{L}^*(|v|^2) = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} \mathcal{M}(v_*)|v - v_*|^\gamma \tilde{b}(|\xi|)(|v'|^2 - |v|^2) \, \mathrm{d}n \, \mathrm{d}v_*.$$
We are going to prove the Lyapunov condition by showing that
$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (\mathcal{L}(f) + \mathcal{T}(f))|v|^2 \, \mathrm{d}x \, \mathrm{d}v \leq -\lambda \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f|v|^2 \, \mathrm{d}x \, \mathrm{d}v + K \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f \, \mathrm{d}x \, \mathrm{d}v,$$
where $\mathcal{T} f = -v\nabla_x f$ is the transport operator. The transport part plays no role, since
$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathcal{T}(f)|v|^2 \, \mathrm{d}x \, \mathrm{d}v = 0.$$
For the collisional part, we notice that
$$\begin{aligned} |v'|^2 - |v|^2 = |v_*|^2 - |v'_*|^2 &= -(u \cdot n)^2 - 2(v_* \cdot n)(u \cdot n) \\ &= -|u|^2\xi^2 - 2(v_* \cdot n)(v \cdot n) + 2(v_* \cdot n)^2 \\ &= -|v|^2\xi^2 - |v_*|^2\xi^2 + 2v \cdot v_*\xi^2 - 2(v_* \cdot n)(v \cdot n) + 2(v_* \cdot n)^2. \end{aligned}$$

Note that the first term is negative and quadratic in v, and the rest of the terms are of lower order in v. Hence, calling $\gamma_b := \int_{\mathbb{S}^{d-1}} \xi^2 \tilde{b}(|\xi|)\,\mathrm{d}\xi$ we have

$$\begin{aligned}
\int_{\mathbb{R}^d} \mathcal{L}(f)|v|^2\,\mathrm{d}v =& -\gamma_b \int_{\mathbb{R}^d} |v|^2 f(v) \int_{\mathbb{R}^d} \mathcal{M}(v_*)|v-v_*|^\gamma\,\mathrm{d}v_*\,\mathrm{d}v \\
&- \gamma_b \int_{\mathbb{R}^d} f(v) \int_{\mathbb{R}^d} |v_*|^2 \mathcal{M}(v_*)|v-v_*|^\gamma\,\mathrm{d}v_*\,\mathrm{d}v \\
&+ 2\gamma_b \int_{\mathbb{R}^d} v f(v) \int_{\mathbb{R}^d} v_* \mathcal{M}(v_*)|v-v_*|^\gamma\,\mathrm{d}v_*\,\mathrm{d}v \\
&- 2 \int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}^d} (v\cdot n) f(v) \int_{\mathbb{R}^d} (v_*\cdot n) \mathcal{M}(v_*)|v-v_*|^\gamma\,\mathrm{d}v_*\,\mathrm{d}v\,\mathrm{d}n \\
&+ \int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}^d} f(v) \int_{\mathbb{R}^d} (v_*\cdot n)^2 \mathcal{M}(v_*)|v-v_*|^\gamma\,\mathrm{d}v_*\,\mathrm{d}v\,\mathrm{d}n \\
\leq& -\gamma_b \int_{\mathbb{R}^d} |v|^2 f(v) \int_{\mathbb{R}^d} \mathcal{M}(v_*)|v-v_*|^\gamma\,\mathrm{d}v_*\,\mathrm{d}v \\
&+ (2+\gamma_b) \int_{\mathbb{R}^d} |v| f(v) \int_{\mathbb{R}^d} |v_*| \mathcal{M}(v_*)|v-v_*|^\gamma\,\mathrm{d}v_*\,\mathrm{d}v \\
&+ \int_{\mathbb{R}^d} f(v) \int_{\mathbb{R}^d} |v_*|^2 \mathcal{M}(v_*)|v-v_*|^\gamma\,\mathrm{d}v_*\,\mathrm{d}v.
\end{aligned}$$

We can now use the following bound, which holds for all $k \geq 0$ and some constants $0 < A_k \leq C_k$ depending on k:

$$A_k(1+|v|^\gamma) \leq \int_{\mathbb{R}^d} |v_*|^k \mathcal{M}(v_*)|v-v_*|^\gamma\,\mathrm{d}v_* \leq C_k(1+|v|^\gamma), \qquad v \in \mathbb{R}^d.$$

Choosing $\epsilon > 0$ we get

$$\begin{aligned}
\int_{\mathbb{R}^d} \mathcal{L}(f)|v|^2\,\mathrm{d}v \leq& -A_0\gamma_b \int_{\mathbb{R}^d} |v|^2(1+|v|^\gamma) f(v)\,\mathrm{d}v + C_1(2+\gamma_b) \int_{\mathbb{R}^d} |v|(1+|v|^\gamma) f(v)\,\mathrm{d}v \\
&+ C_2 \int_{\mathbb{R}^d} (1+|v|^\gamma) f(v)\,\mathrm{d}v \\
\leq& \int_{\mathbb{R}^d} \left(C_2 + \frac{C_1}{\epsilon}\left(1+\frac{\gamma_b}{2}\right)\right)(1+|v|^\gamma)\, f(v)\,\mathrm{d}v \\
&- \left(A_0\gamma_b - \epsilon C_1\left(1+\frac{\gamma_b}{2}\right)\right) \int_{\mathbb{R}^d} |v|^2(1+|v|^\gamma) f(v)\,\mathrm{d}v \\
\leq& \int_{\mathbb{R}^d} \left(C_2 + \frac{C_1}{\epsilon}\left(1+\frac{\gamma_b}{2}\right) + \left(\epsilon C_1\left(1+\frac{\gamma_b}{2}\right) - A_0\gamma_b\right)|v|^2\right)(1+|v|^\gamma) f(v)\,\mathrm{d}v \\
&- \left(A_0\gamma_b - \epsilon C_1\left(1+\frac{\gamma_b}{2}\right)\right) \int_{\mathbb{R}^d} |v|^2 f(v)\,\mathrm{d}v \\
\leq& \alpha_1 \int_{\mathbb{R}^d} f(v)\,\mathrm{d}v - \alpha_2 \int_{\mathbb{R}^d} |v|^2 f(v)\,\mathrm{d}v.
\end{aligned}$$

Here we choose ϵ sufficiently small to make the constant in front of the second moment negative. This also means that

$$\left(C_2 + \frac{C_1}{\epsilon}\left(1+\frac{\gamma_b}{2}\right) + \left(\epsilon C_1\left(1+\frac{\gamma_b}{2}\right) - A_0\gamma_b\right)|v|^2\right)(1+|v|^\gamma)$$

is bounded above. These things together give that

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(\mathcal{L}(f)+\mathcal{T}(f))|v|^2\,\mathrm{d}x\,\mathrm{d}v \leq -\alpha_2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|v|^2 f(v)\,\mathrm{d}x\,\mathrm{d}v + \alpha_1\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f * v\,\mathrm{d}x\,\mathrm{d}v,$$

which finishes the proof. □

Proof of Theorem 5.1.3 in the case of the linear Boltzmann equation. We have the Doeblin condition from Lemma 5.3.3 and the Lyapunov structure from Lemma 5.3.4. Therefore Harris's Theorem gives the result. □

5.3.2 On the whole space with a confining potential

Now we consider the spatial variable on the whole space with a confining potential. As we stated earlier, we cannot verify the Lyapunov condition in the hard spheres case. However, the proof for the Doeblin's condition is the same as in the hard sphere or Maxwell molecule case. We need to combine the Lemmas 5.2.6, 5.3.1 and 5.3.2.

Lemma 5.3.5. *Let the potential $\Phi\colon \mathbb{R}^d \to \mathbb{R}$ be a $\mathcal{C}^2$ function with compact level sets. Given $t > 0$ and $K > 0$ there exist constants $\alpha, \delta_X, \delta_V > 0$ such that for any (x_0, v_0) with $|x_0|, |v_0| < K$ the solution f to (1.26) with initial data $\delta_{(x_0,v_0)}$ satisfies*

$$f_t \geq \alpha \mathbb{1}_{\{|x|\leq \delta_X\}}\mathbb{1}_{\{|v|\leq\delta_V\}}.$$

Proof. We fix $R > 0$ as in Lemma 5.2.7. We use Lemma 5.2.6 to find constants $\alpha, R_2, R' > 0$ and an interval $[a, b] \subseteq (0, t)$ such that

$$\int_{B(R')} T_s(\delta_{x_0}\mathbb{1}_{\{|v|\leq R_2\}})\,\mathrm{d}v \geq \alpha\mathbb{1}_{\{|x|\leq R\}},$$

for any x_0 with $|x_0| \leq R$ and any $s \in [a, b]$. From Lemma 5.3.2 we will use that there exists a constant $\alpha_L > 0$ such that

$$\mathcal{L}^+ g(x, v) \geq \alpha_L\left(\int_{R_L} g(x, u)\,\mathrm{d}u\right)\mathbb{1}_{\{|v|\leq R_2\}} \tag{5.35}$$

for all nonnegative measures g, where $R_L := \max\{R, R'\}$. From Lemma 5.3.1 we can find $C_1 > 0$ (depending on R) such that

$$f(t,x,v) \geq e^{-tC_1} \int_0^t \int_0^s T_{t-s} \widetilde{\mathcal{L}}^+ T_{s-r} \widetilde{\mathcal{L}}^+ T_r (\mathbb{1}_E \delta_{(x_0,v_0)}) \, \mathrm{d}r \, \mathrm{d}s,$$

where E is the set of points where the energy is less than E_0, with

$$E_0 := \max\left\{H(x,v) \colon |x| \leq R, \, |v| \leq \max\{R_L, R_2\}\right\},$$

and we recall that $\widetilde{\mathcal{L}}^+ f := \mathbb{1}_E \mathcal{L}^+ f$. These three estimates allow us to carry out a proof which is completely analogous to that of Lemma 5.2.7; notice that the only difference is the appearance of R' here, and the need to use $\widetilde{\mathcal{L}}^+$ (which still satisfies a bound of the same type). □

Now we need to find a Lyapunov functional. As before we will look at V of the form

$$V(x,v) = \Phi(x) + \frac{1}{2}|v|^2 + \alpha x \cdot v + \beta |x|^2,$$

for some $\alpha, \beta > 0$. In this case we need a stronger bound for $\Phi(x)$; as stated in the following;

Lemma 5.3.6. *Assume that for some positive constants γ_1, γ_2, A we have,*

$$x \cdot \nabla_x \Phi(x) \geq \gamma_1 \langle x \rangle^{\gamma+2} + \gamma_2 \Phi(x) - A,$$

where γ is the exponent in (5.1). We can find α, β such that

$$V(x,v) = \Phi(x) + \frac{1}{2}|v|^2 + \alpha x \cdot v + \beta |x|^2$$

is a function for which the semigroup associated to (1.26) satisfies Hypothesis 2.2.2.

Proof. We are going to show that, for an appropriate choice of α, β it holds that

$$(\mathcal{T}^* + \mathcal{L}^*)(V) \leq -\lambda V + K,$$

for some $\lambda, K > 0$, where $\mathcal{L}^*$ is the dual of the Boltzmann collision operator, and $\mathcal{T}^* f = v \cdot \nabla_x f - \nabla_x \Phi \cdot \nabla_v f$ (the dual of the transport operator, which has the same expression as -$\mathcal{T}$). This will show Hypotheses 2.2.2. We first look at how the collision operator acts on the different terms

$$\int_{\mathbb{R}^d} \mathcal{L}(f)|v|^2 \, \mathrm{d}v = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} \mathcal{M}(v_*)\tilde{b}(|\xi|)|v-v_*|^\gamma f(v)\left(|v'|^2-|v|^2\right) \mathrm{d}n \, \mathrm{d}v \, \mathrm{d}v_*.$$

Repeating the same calculation as in the proof of Lemma 5.3.4 in this case, we see that

$$\int_{\mathbb{R}^d} \mathcal{L}(f)|v|^2 \mathrm{d}v \le -\alpha_1 \int_{\mathbb{R}^d} \langle v\rangle^{\gamma+2} f(v) \mathrm{d}v + \alpha_2 \int_{\mathbb{R}^d} f(v)\mathrm{d}v.$$

That is,

$$\mathcal{L}^*(|v|^2) \le -\alpha_1 \langle v\rangle^{\gamma+2} + \alpha_2. \tag{5.36}$$

Similarly we have

$$\int_{\mathbb{R}^d} \mathcal{L}(f) x\cdot v \, \mathrm{d}v = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} f(v)\mathcal{M}(v_*)\tilde{b}(|\xi|)|v-v_*|^\gamma \left(v'\cdot x - v\cdot x\right) \mathrm{d}n \, \mathrm{d}v \, \mathrm{d}v_*.$$

We can see that

$$v'\cdot x - v\cdot x = (v\cdot n)(x\cdot n) - (v_*\cdot n)(x\cdot n).$$

Integrating this gives that

$$\begin{aligned}\int_{\mathbb{R}^d} \mathcal{L}(f) x\cdot v \, \mathrm{d}v &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} f(v)\mathcal{M}(v_*)\tilde{b}(|\xi|)|v-v_*|^\gamma (v\cdot n)(x\cdot n) \, \mathrm{d}v_* \, \mathrm{d}v \, \mathrm{d}n \\ &\le \int_{\mathbb{R}^d} f(v)\langle v\rangle^{\gamma+1}|x| \, \mathrm{d}v,\end{aligned}$$

which is equivalent to

$$\mathcal{L}^*(x\cdot v) \le \langle v\rangle^{\gamma+1}|x|. \tag{5.37}$$

Now we look at the effect of $\mathcal{T}^*$; notice that

$$V(x,v) = \Phi(x) + |v|^2/2 + \alpha x\cdot v + \beta|x|^2 = H(x,v) + \alpha x\cdot v + \beta |x|^2,$$

where $H(x,v)$ denotes the energy. We have

$$\mathcal{T}^*(H(x,v)) = 0, \qquad \mathcal{T}^*(|x|^2) = 2x\cdot v, \qquad \mathcal{T}^*(x\cdot v) = |v|^2 - x\cdot\nabla_x\Phi(x)$$

Using this together with (5.36) and (5.37) we have

$$\begin{aligned}(\mathcal{L}^*+\mathcal{T}^*)(V) &\le -\frac{\alpha_1}{2}\langle v\rangle^{\gamma+2} + \frac{\alpha_2}{2} + \alpha\langle v\rangle^{\gamma+1}|x| + \alpha|v|^2 - \alpha x\cdot\nabla_x\Phi(x) + 2\beta x\cdot v \\ &\le \left(\alpha - \frac{\alpha_1}{2}\right)\langle v\rangle^{\gamma+2} + (\alpha+2\beta)|x|\langle v\rangle^{\gamma+1} - \alpha\gamma_1\langle x\rangle^{\gamma+2} - \alpha\gamma_2\Phi(x) + \frac{\alpha_2}{2} + \alpha A.\end{aligned}$$

Setting $\beta = \alpha$, $\alpha \leq \alpha_1/4$, and using Young's inequality $AB \leq \frac{1}{p}A^p + \frac{1}{q}B^q$ on $(|x|/\epsilon)\left(\langle v\rangle^{\gamma+1}\epsilon\right)$ with $p = (\gamma+2)/(\gamma+1)$ and $q = \gamma+2$, we get

$$(\mathcal{L}^* + \mathcal{T}^*)(V(x,v)) \leq \left(-\frac{\alpha_1}{4} + 3\epsilon^{\frac{\gamma+2}{\gamma+1}}\frac{\gamma+1}{\gamma+2}\right)\langle v\rangle^{\gamma+2} + \left(\frac{3\alpha^{\gamma+2}}{(\gamma+2)}\frac{1}{\epsilon^{\gamma+2}} - \alpha\gamma_1\right)\langle x\rangle^{\gamma+2} - \alpha\gamma_2\Phi(x) + \frac{\alpha_2}{2} + \alpha A.$$

Now we can choose ϵ small enough so that the $\langle v\rangle^{\gamma+2}$ term is negative and then for this ϵ choose α small enough so that the $\langle x\rangle^{\gamma+2}$ term is negative (since $\gamma+2 \geq 1$). Then, since $\langle z\rangle^{\gamma+2}$ grows faster than $|z|^2$ at infinity this gives

$$(\mathcal{L}^* + \mathcal{T}^*)(V(x,v)) \leq -\lambda_1(|x|^2 + |v|^2) - \lambda_2\Phi(x) + K.$$

Then using equivalence between the quadratic forms

$$|x|^2 + |v|^2 \quad \text{and} \quad \frac{1}{2}|v|^2 + \alpha x\cdot v + \alpha|x|^2,$$

when $\alpha < 1/2$ we obtain the result. □

Proof of Theorem 5.1.4 in the case of the linear Boltzmann equation. We have the minorisation condition in Lemma 5.3.5 and the Lyapunov condition from Lemma 5.3.6. Therefore we can apply Harris's Theorem. □

5.3.3 Subgeometric convergence

As with the linear relaxation Boltzmann equation, the minorisation results in Lemma 5.3.5 holds for Φ which are not sufficiently confining to prove the Lyapunov structure. However in this situation we can still prove subgeometric rates of convergence. Here in order to find a Lyapunov functional we need to be more precise about how $\mathcal{L}$ acts on the $x\cdot v$ moment.

We need $\Phi(x)$ to provide a stronger confinement if we consider hard potentials. We want

$$x\cdot\nabla_x\Phi(x) \geq \gamma_1\langle x\rangle^{1+\delta} + \gamma_2\Phi(x) - A, \quad \Phi(x) \leq \gamma_3\langle x\rangle^{1+\delta} \tag{5.38}$$

for some $\gamma_1, \gamma_2, \gamma_3.\delta > 0$. Then we have

Lemma 5.3.7. *Assume that* Φ *is a* C^2 *potential satisfying* (5.38)*. Then there exist some* $\alpha, \beta > 0$ *with* $4\alpha^2 < \beta$ *such that the function*

$$V(x,v) = \Phi(x) + \frac{1}{2}|v|^2 + \frac{\alpha x \cdot v}{\langle x \rangle} + \beta \langle x \rangle$$

satisfies

$$\mathcal{U}(V) \leq -\lambda V^{\delta/(1+\delta)} + K,$$

for some positive constants λ, K*.*

Remark 5.3.1. *Notice that*

$$V(x,v) \geq \Phi(x) + \frac{1}{4}|v|^2 + (\beta - 4\alpha^2)\langle x \rangle,$$

so that the sub level sets of V *are bounded.*

Proof. Using (5.36), (5.37) and that

$$\mathcal{T}^*(H(x,v)) = 0, \qquad \mathcal{T}^*(\langle x \rangle) = \frac{x \cdot v}{\langle x \rangle}, \qquad \mathcal{T}^*\left(\frac{x \cdot v}{\langle x \rangle}\right) = \frac{|v|^2}{\langle x \rangle} - \frac{(x \cdot v)^2}{\langle x \rangle^2} - \nabla_x \Phi \cdot \frac{x}{\langle x \rangle},$$

we have the following

$$(\mathcal{L}^* + \mathcal{T}^*)(V(x,v)) \leq -\alpha_1 \langle v \rangle^{\gamma+2} + \alpha_2 + \alpha \langle v \rangle^{\gamma+1} + \alpha |v|^2 - \frac{\alpha x \cdot \nabla_x \Phi(x)}{\langle x \rangle} + \frac{\beta x \cdot v}{\langle x \rangle}.$$

Using (5.38), $\frac{x \cdot v}{\langle x \rangle} \leq \langle v \rangle \leq \langle v \rangle^{2+\gamma}$ and $\Phi(x)^{\delta/(1+\delta)} \leq \gamma_3^{\delta/(1+\delta)} \langle x \rangle^{\delta}$ we get

$$\begin{aligned}
&(\mathcal{L}^* + \mathcal{T}^*)(V(x,v)) \\
&\qquad \leq (2\alpha - \alpha_1 + \beta)\langle v \rangle^{\gamma+2} - \alpha\gamma_1 \langle x \rangle^{\delta} - \alpha\gamma_2 \frac{\Phi(x)}{\langle x \rangle} + \alpha_2 + \alpha A \\
&\qquad\qquad \leq \lambda_1 \left(-|v|^2 - \langle x \rangle^{\delta} - \Phi(x)^{\delta/(1+\delta)} + C\right) \leq \lambda V(x,v)^{\delta/(1+\delta)} + K,
\end{aligned}$$

for some $\lambda_1, K > 0$. To make the last two inequalities valid we choose α and β satisfying $\alpha_1 > 2\alpha + \beta$ and $4\alpha^2 < \beta$ so that

$$V(x,v) \geq \Phi(x) + \frac{1}{4}|v|^2 + (\beta - 4\alpha^2)\langle x \rangle.$$

$\square$

Proof of Theorem 5.1.5 in the linear Boltzmann case. We have the minorisation condition in Lemma 5.3.5 and the Lyapunov condition from Lemma 5.3.7. Therefore we can apply Harris's Theorem. □

5.4 Summary and conclusion

We present explicit results for $\mathcal{L}$ is either equal to the linear relaxation Boltzmann (or linear BGK) operator, and for $\mathcal{L}$ equal to the linear Boltzmann operator.

We obtain exponential convergence results on the d-dimensional torus, or with confining potentials growing at least quadratically at ∞, in total variation or weighted total variation norms (alternatively, L^1 or weighted L^1 norms). For subquadratic potentials we give algebraic convergence rates, again in the same kind of weighted L^1 norms. Some results were already available for these equations [34, 87, 53, 72, 58].

Previous proofs of convergence to equilibrium relied strongly on weighted L^2 norms (typically with a weight which is the inverse of a Gaussian). One of the advantages of Harris's Theorem is that it directly provides convergence for a much wider range of initial conditions. In particular, the method works for initial conditions with slow decaying tails, and for measure initial conditions with very bad local regularity. Particularly it is only needed for an initial data f_0 to be a probability measure where $\|f_0 - \mu\|$ is finite. Moreover, the theorem gives existence of stationary solutions under quite general conditions; in some cases these are explicit and easy to find, but in other cases they can be nontrivial. Another advantage of this method is that the condition on the moments used here might be much easier to verify in the case where the equilibrium state cannot be made explicit. This is the motivation behind [11, 43].

We also note that the results in this work for subquadratic potentials are new up to our knowledge. Apart from these new results, we aimed at presenting a new application of a probabilistic method, using mostly PDE arguments, and which is probably useful for a wide range of models.

Harris's Theorem provides an alternative and very different strategy for proving quantitative exponential decay to equilibrium. By using this method we can observe hypocoercive effects on the level of stochastic processes. Moreover it allows us to produce quantitative theorems based on trajectorial intuition.

Another difference is that the confining behaviour is shown here by exploiting good behaviour of moments rather than a Poincaré inequality so that we study point wise bounds rather than integral controls on the operator. However these are often equivalent for time reversible processes [3, 45].

On the other hand, we can only consider Markov processes in order to use Harris's Theorem. Although many linear kinetic equations are Markov processes but this excludes the study of linearized non-linear equations which are not necessarily mass preserving.

Chapter 6

Conclusion and perspectives

ALICE: *"Where should I go?"*
THE CHESHIRE CAT: *"That depends on where you want to end up."*

— Lewis Carroll, Alice's Adventures in Wonderland

Contents

In this chapter we give a general conclusion and perspectives for some of the previous chapters. Moreover we list some of the possible future applications and extensions in the form of ideas and works in progress.

6.1 Conclusion

In this work, we presented some examples of how well adapted and suitable some probabilistic techniques such as Harris's Theorem are for studying the asymptotic behaviour of nonlocal linear partial differential equations coming from modelling of various context. Moreover, in one case we were able to obtain some results in the weakly nonlinear case. Predicting the long time behaviour of mathematical models is a well known modelling problem. There is a particular interest towards proving the existence of a stationary state (unique up to scaling in many cases) and estimating the rate of convergence to that state. We used a probabilistic approach to tackle these type of problems. The methods we use are presented in Chapter 2 in detail. These methods are particulary very useful for obtaining quantitative rates of relaxation to a unique equilibrium once the hypothesis for the theorems are verified quantitatively. Furthermore, existence of a unique steady state is also a conclusion of the theorems and rate of convergence does not require to know explicit form of the steady state.

Although these methods are very established and well known in the probability community, applications on partial differential equations has become more popular recently and is a relatively new field of research in the PDE community. We have given some references to earlier works in the previous chapters. We would like to add that it is a very promising research direction.

6.2 Perspectives

Most natural prospective of this work is to use Harris's and Doeblin's Theorems for obtaining convergence results for numerical schemes. It is natural because iterations in time can be described by discrete time Markov processes for which these theorems were first developed. Therefore it is expected to achieve convergence results similar to the continuous time case once the assumptions are satisfied in the discrete setting.

In addition to the summary and conclusions at the end of Chapters 3 - 5, here we will list some perspectives for each of these chapters.

In Chapter 3, we presented exponential convergence rates for two nonlinear and nonlocal evolution equations modelling the dynamics of interacting neuron populations. Doeblin's Theorem was used to prove spectral gap for the linear versions of the equations and then a perturbation argument carry the exponential convergence rate to the nonlinear setting. However, due to this perturbative argument the nonlinearity we considered was weak corresponding to the case of low connectivity of the underlying neural network. In [89] and [90], the authors provided some numerical results concerning the oscillatory behaviour of long time solutions depending on the network connectivity. It was shown that in some intermediate regime of the connectivity neurons will exhibit synchronous behaviour between two equilibria. On the other hand, they observed a unique stationary solution in the case of higher connectivities meaning full nonlinearities. One possible extension could be trying to prove this behaviour by using probabilistic methods. In the second model introduced in [91], there is an addition of an integral kernel modelling the fatigue property of neurons. Another prospective could be investigate this model numerically and check if the periodic behaviour could be observed in this case as well. Another difference from the previous works is that we did not take into account the effect of delay in [28]. This type of consideration enables us to construct more realistic models for real behaviour of nuron populations. Formulating these models in terms of delay differential equations and trying to achive results on the long time behaviour of solutions is another possible extension.

In Chapter 4, we presented exponential rates of relaxation to eqilibrium for the growth-fragmentation equation with two critical fragmentation kernels: mitosis and constant fragmentation. Moreover we proved existence of eigenelements and some bounds on the dual eigenfunction with considering less strong assumptions as compared to many previous works on these type of equations. Therefore, our work includes showing convergence results for the cases where mass conservation property may not achieved. This implies the violation of balance conditions. One extension of this work could be a consideration of different integral kernels, possibly the one combining the two cases we considered. We also gave some example of proving similar results for a numerical approximation to the growth-fragmentation equation in the case of constant growth rate and constant total division rate. Aim is to extend it to more general consideration of these rates and complete it.

In Chapter 5, we presented quantitative convergence rates to equilibrium for the inhomogeneous linear relaxation Boltzmann and linear Boltzmann equations either in the torus or on the whole space with a confining potential. We obtained exponential convergence results in the torus and on the whole space with potentials growing at quadratically at infinity. Moreover, apart from the techniques another innovation in this work is providing algebraic convergence rates for weaker potentials. One prospect could be to extend this to kinetic nonlocal diffusion equation which we could not achieve at the moment due to having to find a clear bound from below n number of jumps for the minorisation condition in Harris's theorem. But we believe this could be achieved. Another consideration is proving similar results for a numerical approximation of these type of equations. An inspiration for this direction is a recent work [19] where asymptotic preserving (AP) property of a finite volume scheme is presented for some linear kinetic equations.

6.2.1 Ongoing projects

Finally, I would like to present a brief introduction to ongoing works I started working on this year bu most likely they will not be completed during my doctoral period. The common feature of all the models that are presented below is that we can actually determine the long time behaviour of solutions by applying Doeblin's or Harris's theorems.

On the asymptotic behaviour of the run-and-tumble model for bacteria movement

This is a work in progress in collaboration with Josephine A. Evans at University Paris Dauphine and Angeliki Menegaki at the University of Cambridge.

Many microorganisms like bacteria undergo a biased random walk which is called *run and tumble* in response to a chemical substance. *Chemotaxis* is the mechanism for the microorganisms move towards or away from chemical stimuli to find food sources or avoid poisonous areas. The underlying biased random walk can be modelled mathematically by a kinetic equation describing the time evolution of the density of bacteria and it is called *run and tumble model.* This special movement is done by bacteria which have propeller like structure, called *flagella*, helping them swim in the substance. Bacteria move in a straight line with a velocity v when all the flagella incline towards the same direction rotating counterclockwise and propelling bacteria. This movement, *run*, followed by a random rotation, *tumble*, towards a direction depending

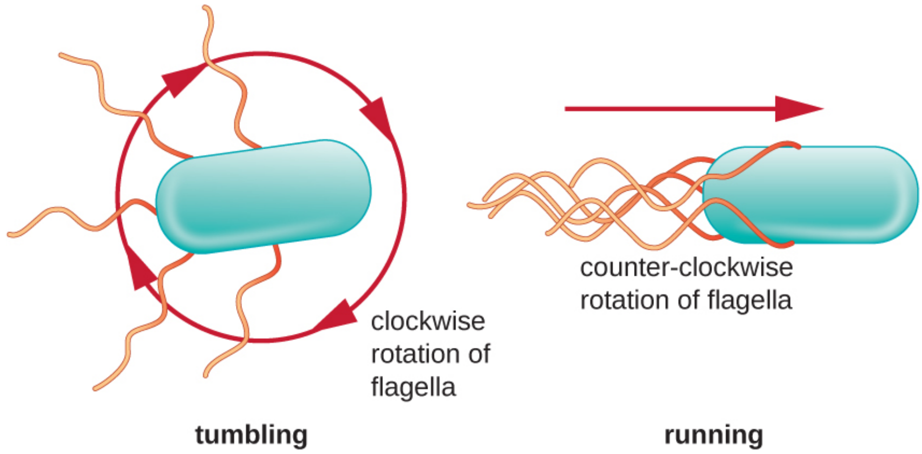


Fig. 6.1 Run and tumble movement of a bacterium.
Counter-clockwise rotation of the flagella results in a *run* and the clockwise rotation of the flagella results in a *tumble*.
Image: Figure 3.55 of Chapter 3, Section 3 of the book Microbiology: Canadian Edition by Wendy Keenleyside, CC by 4.0.

on a concentration gradient of a chemical and fulfilled by the rotation of the flagella in the clockwise direction (See Figure 6.1).

The original mathematical model is proposed in [1, 98] and given by

$$\begin{aligned} \partial_t f(t,x,v) + v\cdot\nabla_x f(t,x,v) &= \int \left(\mathcal{T}(t,x,v,v')f(t,x,v') - \mathcal{T}(t,x,v',v)f(t,x,v)\right)\,\mathrm{d}v', \\ f(0,x,v) &= f_0(x,v), \qquad x\in\mathbb{R}^d, v\in\mathcal{V}. \end{aligned} \tag{6.1}$$

where $f(t,x,v)\geq 0$ is the density distribution of bacteria at time $t\geq 0$ at a position $x\in\mathbb{R}^d$ moving with a velocity $v\in\mathcal{V}$. We start with taking $\mathcal{V}=B(0,V_0)$ with $|\mathcal{V}|=1$. Bacteria change their velocity from v' to v after an instantaneous tumbling event. The tumbling frequency $\mathcal{T}$ is defined as

$$\begin{aligned} \mathcal{T}(t,x,v,v') &:= \mathcal{T}(\partial_t M(t,x) + v'\cdot\nabla_x M(t,x), v, v'), \\ \mathcal{T}(t,x,v',v) &:= \mathcal{T}(\partial_t M(t,x) + v\cdot\nabla_x M(t,x), v', v), \end{aligned}$$

where M is related to the chemoattractant concentration, S by $M = m_0 + \log(S)$. The tumbling frequency $\mathcal{T}$ is assumed to be decomposed as

$$\mathcal{T}(t,x,v,v') = \lambda(m)K(v,v'), \tag{6.2}$$

where λ is the jump rate where $m(x,v) := \partial_t M(t,x) + v' \cdot \nabla_x M(t,x)$ and K is the conditional probability of jumping from velocity v' to velocity v. Therefore

$$\int K(v,v')\,\mathrm{d}v = 1.$$

Also we have that

$$\lambda : \mathbb{R} \times \mathcal{V} \to [0,\infty).$$

We assume that $\lambda(m)$ increases as $m \to -\infty$. We also consider a nonlinear version of this model by coupling it with a Poisson equation

$$\Delta_x S + S = \rho(t,x) = \int f(t,x,v)\,\mathrm{d}v.$$

In this case we assume that $S \to 0$ as $|x| \to \infty$. Various versions of this model is studied by many people in the past but since this is a brief introduction we will not mention all of them here. However we remark that in a recent work [85], the authors show global well posedness, existence, uniqueness and convergence to steady states of (6.1) in the case where K is uniform and

$$\lambda(m) = 1 - \chi\,\mathrm{sgn}(m),$$

where $\chi \in (0,1)$ a constant. This assumption implies some bounds on the jump rate such that

$$1 - \chi \leq \lambda(x,v) \leq 1 + \chi. \tag{6.3}$$

Goal of this work is to obtain converge rates in the linear case by using Harris's theorem and then exploring the nonlinear version of the model by making realistic assumptions, obtaining some properties of stationary solutions and finally treating long-time behaviour of solutions by considering perturbations similar to [28].

On the spectral gap for a biological model for genetic circuits

This is a work in progress in collaboration with José A. Cañizo at the University of Granada and José A. Carrillo at Imperial College London.

This work is based on obtaining quantitative rates for asymptotic behaviour of a kinetic model for genetic circuits by using Doeblin-Harris approach. Recently in [31], the authors used entropy methods to show exponentially fast convergence to equilibrium with explicit models. They also provided asymptotic equilibration results for the multidimensional case involving more than one gene through numerical simulations. Their results are however, given in weighted L^2 norms which is natural for the methods they use. Our aim in this work is to obtain exponential convergence rates in weighted L^1 distances which enables us to consider wider range of initial data.

DNA molecules store the genetic code for living beings and the information encoded in the genes need to be read and processed for organisms to develop, survive, reproduce and function properly. This happens in two steps broadly;

- Some area of DNA containing the desired information is copied by cells inside the nucleid acid RNA. This process is called *transcription*.
- Then, RNA copies which carry the desired information now, are used for protein production. This process is called *translation*.

Resulting gene regulatory network depending on the signals perceived from DNA through some binding proteins is stochastic if the number of species involved in the network is not too large and it is described by the chemical master equation (CME). One way of obtaining a CME solution is assuming a bursting behaviour for protein production which eventually yields a partial integro-differential equation (PIDE) model. The PIDE is then a continuous approximation of the CME. If the gene regulatory network consists of only one gene the resulting kinetic equation, as first introduced in [61], has mathematically interesting analytical properties.

The time evolution of the probability density of the amount of proteins $p : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is given by the following partial integro-differential equation as in [31]:

$$\frac{\mathrm{d}p}{\partial t}(t,x) - \frac{\partial(xp)}{\partial x}(t,x) + ac(x)p(t,x) = a\int_0^x w(x-y)c(y)p(t,y)\,\mathrm{d}y, \tag{6.4}$$

$$p(0,x) = p_0(x), \quad x > 0. \tag{6.5}$$

This model is valid under the assumption that protein is produced in bursts. Therefore in (6.4)

- b represents *burst size* and typically modelled by an exponential distribution.
- The conditional probability for protein level to jump from a state y to a state $x > y$ after a burst is proportional to w which is defined as

$$w(x-y) = \frac{1}{b}\exp\left(-\frac{x-y}{b}\right), \qquad x > y > 0.$$

- $a > 0$ is a rate constant related to transcription.
- $c : \mathbb{R}^+ \to [\epsilon, 1]$ is the *input function* which represents the feedback mechanism, where ϵ is the *leakage constant*. DNA being in its its active state always corresponds to $c(x) = 1$.

Equation (6.4) has a unique solution after scaling such that $\int_0^{+\infty} p_\infty(x)\,\mathrm{d}x = 1$; and this solution is given by

$$p_\infty(x) = Z(\rho(x))^{\frac{a(1-\epsilon)}{H}} \frac{1}{x^{1-a\epsilon}} \exp\left(-\frac{x}{b}\right) \tag{6.6}$$

where Z is the normalising constant and $H \in \mathbb{Z} \setminus \{0\}$ is the *Hill coefficient*. *Negative feedback* ($H > 0$) means that proteins bound to the DNA inhibiting their production and $H > 0$ represents *positive feedback*, having the opposite effect. Therefore, the fraction of the parameter in the active or inactive state is described by *Hill function*. Then, probability that the promoter is in its inactive state in terms of amount of the protein x is defined by a function $\rho : \mathbb{R}^+ \mapsto [0, 1]$ and the input function defined in terms of ρ by $c(x) = (1 = \rho(x)) + \rho(x)\epsilon$. Finally we aim to present quantitative rates for relaxation to equilibrium for the generalised case as well via Harris's theorem.

Note also that, (6.4) conserves mass.

We define $(\mathcal{T})_{t\geq 0}$ as being the semigroup associated to the equation

$$\frac{\partial p}{\partial t}(t, x) - \frac{\partial (xp)}{\partial x}(t, x) + ac(x)p(t, x) = 0, \tag{6.7}$$

which can be written as

$$\frac{\partial p}{\partial t}(t, x) - x\frac{\partial p}{\partial x}(t, x) + d(x)p(t, x) = 0, \tag{6.8}$$

where $d(x) := ac(x) - 1$.

We remark that $\mathcal{T}_t p_0$ is the solution to (6.8) at time t with an initial data $p_0(x) > 0$. Moreover we define $(\mathcal{S})_{t\geq 0}$ as being the Markov semigroup associated to (6.4) similarly and call the nonlocal part

$$\mathcal{A}(t, x) := \frac{a}{b} \exp\left(-\frac{x}{b}\right) \int_0^x \exp\left(\frac{y}{b}\right) c(y)p(t, y)\,\mathrm{d}y.$$

Therefore we have by Duhamel's formula

$$\mathcal{S}_t p_0(x) = \mathcal{T}_t p_0(x) + \int_0^t \mathcal{T}_{t-\tau}\mathcal{A}(\tau, \cdot)\,\mathrm{d}\tau.$$

Eventually, verifying a Lyapunov condition and a minorisation condition we make use of Harris's theorem.

Moreover, in [31], the authors also considered a generalized n-dimensional PIDE model (which was proposed first in [88]) to handle with a network involving more than one gene. The network is composed of n genes:

$$\mathcal{G} = \{DNA_1, DNA_2, \cdots, DNA_n\}$$

which are transcribed into n messenger RNAs;

$$\mathcal{R} = \{mRNA_1, mRNA_2, \cdots, mRNA_n\}$$

and translated into n types of proteins;

$$\mathcal{P} = \{P_1, P_2, \cdots, P_n\}.$$

We define the amount of protein corresponding to each type protein by the n-vector $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in \mathbb{R}_+^n$. Under this set-up the generalised n-dimensional PIDE model is given by (as in [88])

$$\frac{\partial}{\partial t} p(t, \mathbf{x}) = \sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i} \left[\alpha_{x_i}(\mathbf{x}) x_i p(\mathbf{x})\right] + k_m^i \int_0^{x_i} \omega_i(x_i - y_i) c_i(\mathbf{y}_i) p(t, \mathbf{y}_i) \, \mathrm{d}y_i - k_m^i c_i(\mathbf{x}) p(\mathbf{x}) \right), \tag{6.9}$$

where

- α_{x_i} is the degragation rate of each protein.
- y_i represents the vector state of $\mathbf{x}$ with its i-th position is changed to $\mathbf{y}_i$ so that

$$\begin{cases} (\mathbf{y}_i)_j = x_j & \text{if } i \neq j, \\ (\mathbf{y}_i)_j = y_i & \text{if } i = j, \end{cases}$$

- The conditional probability for protein level to jump from a state y_i to a state x_i after a burst is given by

$$w_i(x_i - y_i) = \frac{1}{b_i} \exp\left(-\frac{x_i - y_i}{b_i} \right).$$

- b_i is the mean protein produced per bust size following an exponential distribution as a modelling assumption.
- $c_i : \mathbb{R}^n_+ \to [\epsilon_i, 1]$ is the input function modelling the regulation mechanism where ϵ_i is the leakage constant as before.
- k_m^i represents the transcription rate of mRNA for each gene.

Therefore, (6.9) describes the competition between protein degradation and protein production which takes place in bursts. Analytical expression of the stationary state of (6.9) is not known explicitly; however, the total mass is conserved as in the 1-dimensional case.

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Appendix A

An introduction to measure and probability theories

In this chapter we revise some definitions and theorems from the measure theory and probability.

A.1 Some measure theory

Definition A.1.1. *A* measurable space *is a pair* $(\mathcal{X}, \mathcal{F})$ *where* $\mathcal{X}$ *is a set, the* space *and* $\mathcal{B}(\mathcal{X})$ *is a* σ*-algebra of subsets of* $\mathcal{X}$ *and it satisfies;*

(i) $\mathcal{X} \in \mathcal{B}(\mathcal{X})$.

(ii) If $A \in \mathcal{B}(\mathcal{X})$ *then* $A^c \in \mathcal{B}(\mathcal{X})$.

(iii) If $A_k \in \mathcal{B}(\mathcal{X})$ *for* $k = 1, 2, 3, \cdots$ *then* $\bigcup_{k=1}^{\infty} A_k \in \mathcal{B}(\mathcal{X})$.

When we consider the state space $\mathbb{R}$, we always assume it is equipped with the Borel σ-algebra.

Definition A.1.2 (measurable function)**.** *Consider two measurable spaces* $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ *and* $(\mathcal{Y}, \mathcal{B}(\mathcal{Y}))$ *if the mapping* $f : \mathcal{X} \longrightarrow \mathcal{Y}$ *is a measurable function if*

$$f^{-1}(B) := \{x \, : \, f(x) \in B\} \in \mathcal{B}(\mathcal{Y})$$

for all sets $B \in \mathcal{B}(\mathcal{Y})$.

We always assume $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ is measurable.

Definition A.1.3 ((signed)measure)**.** *If a function* $\mu : \mathcal{B}(\mathcal{X}) \longrightarrow (-\infty, \infty)$ *on a measurable space* $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ *is called a (signed) measure if* $A_k \in \mathcal{B}(\mathcal{X})$ *for* $k = 1, 2, 3, \cdots$ *and* $A_i \cap A_j = \emptyset$ *for* $i \neq j$, *then*

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} A_i.$$

If $\mu(A) \geq 0$ for any set A then μ is a *positive measure*. If μ is positive and $\mu(\mathcal{X}) = 1$ then it is called a *probability measure*. Lebesgue measure on the real line $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is a positive measure defined for intervals.

Definition A.1.4 (total variation norm)**.** *The total variation norm of a signed measure is defined as*

$$\|\mu\|_{\mathrm{TV}} := \sup_{|f(x)| \leq 1} \int f \, \mathrm{d}\mu,$$

where the supremum is taken over all measurable function $f : (\mathcal{X}, \mathcal{B}(\mathcal{X})) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

Moreover for a signed measure μ, the total variation norm is defined by writing the state space $\mathcal{X}$ as the union of disjoint sets $\mathcal{X}_+$ and $\mathcal{X}_-$:

$$\|\mu\|_{\mathrm{TV}} = \mu(\mathcal{X}_+) - \mu(\mathcal{X}_-).$$

Next, we recall the definition of the *Lebesgue integral* for a non-negative and measurable function $f : (\mathcal{X}, \mathcal{B}(\mathcal{X})) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with respect to a positive measure μ. For $A \in \mathcal{B}(A)$ we define

$$\mu(A) := \int_{\mathcal{X}} \mathbb{1}_A(x) \mu(\mathrm{d}x),$$

where $\mathbb{1}_A$ is the *indicator function* defined as

$$\mathbb{1}_A(x) = \begin{cases} 1 \text{ if } x \in A, \\ 0 \text{ otherwise.} \end{cases}$$

If f is a *simple function* such that for sets $\{A+1, \cdots, A_N\} \subset \mathcal{B}(\mathcal{X})$ and for positive numbers $\{a_1, \cdots, a_N\} \subset \mathbb{R}_+$ we write $f = \sum_{k=1}^{N} a_k \mathbb{1}_{A_k}$ then we define the Lebesgue integral with respect to μ:

$$\int_{\mathcal{X}} f(x) \mu(\mathrm{d}x) := \sum_{k=1}^{N} a_k \mu\{A_k\}.$$

Since for a given any non-negative measurable function f, there exits a sequence of simple function $\{f_k\}_{k=1}^{\infty}$ such that for each $x \in \mathcal{X}$ we have $f_k(x) \longrightarrow f(x)$, the limit

$$\lim_k \int_{\mathcal{X}} f_k(x)\mu(\,dx) =: \int_{\mathcal{X}} f(x)\mu(\,dx)$$

always exists. If f is nonnegative we write $f = f^+ - f^-$ where both f^+, f^- are non-negative measurable functions and define

$$\int_{\mathcal{X}} f(x)\mu(\,dx) := \int_{\mathcal{X}} f^+(x)\mu(\,dx) - \int_{\mathcal{X}} f^-(x)\mu(\,dx),$$

if both terms on the right hand side are finite. The we call such function f as *μ-integrable* and mostly we denote the integral in the following way:

$$\int f\,d\mu := \int_{\mathcal{X}} f(x)\mu(\,dx).$$

Finally, we state three important theorems concerning the convergence of sequences of integrals that we are going to use in the future. Proofs are omitted and can be found in many books.

Theorem A.1.1 (Fatou's Lemma). *Suppose that μ is a finite, positive measure on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ and $f_i : (\mathcal{X}, \mathcal{B}(\mathcal{X})) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ for $i \in \mathbb{Z}_+$ are nonnegative measurable functions, then*

$$\int_{\mathcal{X}} \lim_i \inf f_i(x)\mu(\,dx) \leq \lim_i \inf \int_{\mathcal{X}} f_i(x)\mu(\,dx).$$

Theorem A.1.2 (Monotone Convergence Theorem). *Suppose that μ is a finite, positive measure on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ and $f_i : (\mathcal{X}, \mathcal{B}(\mathcal{X})) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ for $i \in \mathbb{Z}_+$ are measurable functions satisfying $0 \leq f_i(x) \to f(x)$ for μ-almost every $x \in \mathcal{X}$, then*

$$\int_{\mathcal{X}} f(x)\mu(\,dx) = \lim_i \int_{\mathcal{X}} f_i(x)\mu(\,dx).$$

Theorem A.1.3 (Dominated Convergence Theorem). *Suppose that μ is a finite, positive measure on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ and $h : (\mathcal{X}, \mathcal{B}(\mathcal{X})) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is a μ-integrable function such that $h \geq 0$. If f and $f_i : (\mathcal{X}, \mathcal{B}(\mathcal{X})) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ for $i \in \mathbb{Z}_+$ are measurable functions satisfying $|f_i(x)| \leq h(x)$ for μ-almost every $x \in \mathcal{X}$, and if $f_i(x) \underset{i\to\infty}{\longrightarrow} f(x)$ for μ-almost every $x \in \mathcal{X}$, then each f_i is μ-integrable and*

$$\int_{\mathcal{X}} f(x)\mu(\,dx) = \lim_i \int_{\mathcal{X}} f_i(x)\mu(\,dx).$$

A.2 Some probability theory

Now, we recall some concepts from probability theory.

Definition A.2.1 (probability space)**.** *A probability space is a triple* $(\Omega, \mathcal{F}, \mathcal{P})$ *where* Ω *is a set,* $\mathcal{F}$ *is a* σ*-algebra of subsets of* Ω *and* $\mathcal{P}$ *is a probability measure on* $\mathcal{F}$*, so that it consists of a measurable set* $(\Omega, \mathcal{F})$ *where* $\mathcal{F}$ *is a* σ*-agebra over* Ω *and a probability measure* $\mathcal{P}$*.*

Definition A.2.2 (probability measure)**.** *A probability measure* $\mathcal{P}$ *on the measurable space* $(\Omega, \mathcal{F})$ *is a map* $\mathcal{P} : \mathcal{F} \to [0, 1]$ *and satisfy the following properties;*

1. $\mathcal{P}(\emptyset) = 0$ *and* $\mathcal{P}(\Omega) = 1$*.*
2. *If* $\{A_n\}_{n>0}$ *is a countable collection of disjoint elements of* $\mathcal{F}$ *then*

$$\mathcal{P}\left(\bigcup_{n>0} A_n\right) = \sum_{n>0} \mathcal{P}(A_n).$$

Definition A.2.3 (random variable)**.** *Suppose* $(\Omega, \mathcal{F}, \mathcal{P})$ *is a probability space and* $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ *is a measurable space, then a mapping* $X : \Omega \to \mathcal{X}$ *is said to be* $\mathcal{X}$*-valued random variable if it is measurable., i.e. for any set* $B \in \mathcal{B}(\mathcal{X})$*:*

$$\{X \in A\} = \{X^{-1}(A)\} = \{\omega \in \Omega : X(\omega) \in A\} \in \mathcal{F}.$$

If $X : \Omega \to \mathcal{X}$ is a random variable and $f : (\mathcal{X}, \mathcal{B}(\mathcal{X})) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is a real-valued measurable function then $f(X)$ is a real-valued random variable on $(\Omega, \mathcal{F}, \mathcal{P})$. The *expectation* for $f(X)$ is defined as

$$\mathbb{E}[f(X)] = \int_\Omega f(X(\omega))\mathcal{P}(\,d\omega).$$

The set of real-valued random variables for which the expectation is well defined and finite is denoted by $L^1(\Omega, \mathcal{F}, \mathcal{P})$.

Now we recall another concept called *conditional expectation* which will be the key element of guessing the value of a random variable.

Definition A.2.4 (conditional expectation)**.** *Let* x *be a real valued random variable on some probability space* $(\Omega, \mathcal{F}, \mathcal{P})$ *such that* $\mathbb{E}[x] < \infty$ *and let* $\mathcal{F}'$ *be a sub* σ*-algebra of* $\mathcal{F}$*. Then the* conditional expectation of x *with respect to* $\mathcal{F}'$ *is the* $\mathcal{F}'$ *measurable random variable* x' *such that for all* $A \in \mathcal{F}'$*:*

$$\int_A x(\omega)\mathcal{P}(\,d\omega) = \int_A x'(\omega)\mathcal{P}(\,d\omega).$$

It is denoted as $x' = \mathbb{E}[x \,|\, \mathcal{F}']$.

We also recall Radon-Nikodym theorem from the measure theory:

Theorem A.2.1 (Radon-Nikodym)**.** *Let* μ *and* ν *be two finite measures on a measurable space* $(\Omega, \mathcal{F})$ *such that* μ *is absolutely continuous with respect to* ν *(i.e.* $\nu(\mathcal{X}) = 0$ *implies* $\mu(\mathcal{X}) = 0$ *for every measurable set* $\mathcal{X}$*) and* ν *is positive. Then, there is a unique measurable function* $f : \Omega \to \mathbb{R}$ *such that*

$$\mu(\mathcal{X}) = \int_{\mathcal{X}} f \, \mathrm{d}\nu,$$

for any measurable set $\mathcal{X} \subset \Omega$*.*

Let y be a $\mathcal{Y}$-valued random variable on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$. We denote by $\mathcal{F}_y \subset \mathcal{F}$ the σ-algebra consisting of all elements of the form $y^{-1}(A)$ with $A \in \mathcal{B}(\mathcal{Y})$ such that $\mathcal{F}_y = \{y^{-1}(A) \,|\, A \in \mathcal{B}(\mathcal{Y})\}$ and $\mathcal{F}_y$ is called the σ-algebra *generated by* y.

Appendix B

An introduction to semigroup theory

SHORT INTRO HERE

B.1 Operator semigroups

We give basic definitions and properties for operators as generators of semigroups. In the next chapter, we give statements of some theorems concerning covvergence of Markov processes and we work with stochastic semigroups in L^1 spaces.

Semigroups of linear operators are solutions of the initial value problem for the differential equation $u'(t) = Au(t)$, where A is a linear operator acting on a Banach space. They are particularly important when studying continuous-time Markov processes. We assume that $(\mathcal{K}, \|\cdot\|)$ is a real Banach space, that is $\mathcal{K}$ is a real vector space and the norm $\|\cdot\|$ is a non-negative function defined on $\mathcal{K}$ satisfying:

- $\|f\| = 0$ if and only if $f = 0$,
- $\|cf\| = |c|\|f\|$ for all $c \in \mathbb{R}$ and $f \in \mathcal{K}$,
- $\|f + g\| \leq \|f\| + \|g\|$ for all $f, g \in \mathcal{K}$,
- The metric space $(\mathcal{K}, \rho)$ with $\rho = \|f - g\|$ is complete.

A linear operator A on $\mathcal{K}$

- is a linear mapping $A : \mathcal{D}(A) \to \mathcal{K}$, where $\mathcal{D}(A)$ is a linear subspace of $\mathcal{K}$, called the domain of A.

- is said to be *bounded* if $\mathcal{D}(A) = \mathcal{K}$ and the norm of A, $\|A\| = \sup_{\|f\| \le 1} \|Af\|$, is finite. Note that a linear operator A with $\mathcal{D}(A) = \mathcal{K}$ is bounded if and only if it is continuous, i.e. the mapping $f \mapsto Af$ is continuous for all $f \in \mathcal{K}$.

- is a *contraction* if $\|A\| \le 1$.

- is said to be *densely defined* if its domain $\mathcal{D}(A)$ is dense in $\mathcal{K}$, so that every $f \in \mathcal{K}$ is a limit of a sequence of elements from $\mathcal{D}(A)$.

- is *closed* if its graph $\{(f, Af) : f \in \mathcal{D}(A)\}$ is a closed set in the product space $\mathcal{K} \times \mathcal{K}$ or equivalently if $f_n \in \mathcal{D}(A), n \ge 1$,

$$\lim_{n\to\infty} f_n = f, \text{ and } \lim_{n\to\infty} Af_n = g,$$

  then $f \in \mathcal{D}(A)$ and $g = Af$.

- is said to be *invertible* if there is a bounded operator A^{-1} on $\mathcal{K}$ such that $A^{-1}Af = f$ for all $f \in \mathcal{D}(A)$ and $A^{-1}g \in \mathcal{D}(A)$ and $A^{-1}Ag = g$ for all $g \in \mathcal{K}$.

We denote by $\mathcal{K}^*$ the space of all continuous linear functionals $\alpha : \mathcal{K} \mapsto \mathbb{R}$. It is a real Banach space with the norm

$$\|\alpha\| = \sup_{\|f\| \le 1} |\alpha(f)|, \quad \alpha \in \mathcal{K}^*,$$

and it is called the *dual space* of $\mathcal{K}$. We use the duality notation $\langle \alpha, f \rangle := \alpha(f)$ for $f \in \mathcal{K}, \alpha \in \mathcal{K}^*$. In particular, the Hahn–Banach theorem allows us to extend a nonzero continuous functional defined on a closed linear subspace of $\mathcal{K}$ to a continuous functional on the whole Banach space $\mathcal{K}$.

The adjoint operator A^* of a densely defined linear operator A is a linear operator from $\mathcal{D}(A^*) \subset \mathcal{K}^* \to \mathcal{K}^*$ defined as follows:

Let $\alpha \in \mathcal{D}(A^*)$ if there exists $\beta \in \mathcal{K}^*$ such that

$$\langle \alpha, Af \rangle = \langle \beta, f \rangle, \quad f \in \mathcal{D}(A),$$

where we set $\beta = A^*\alpha$.

Let $S(t) : \mathcal{K} \to \mathcal{K}$ be a bounded linear operator for each $t \ge 0$. The family $(S_t)_{t\ge 0}$ is called a *semigroup* if it satisfies the *semigroup properties*:

(i) $S_0 = I$, where I is the *identity operator*, i.e. $If = f$ for $f \in \mathcal{K}$,

(ii) $S_{s+t} = S_s S_t$, for all $s, t \geq 0$.

A semigroup $(S_t)_{t\geq 0}$ is said o be *strongly continuous* or a C_0*-semigroup* if for each $f \in \mathcal{K}$,

$$\|S_t f - f\| \to 0 \text{ as } t \to 0.$$

The *infinitesmall generator* (or shortly *generator*) of $(S_t)_{t\geq 0}$ is the operator $\mathcal{L}$ with domain $\mathcal{D}(\mathcal{L}) \subset \mathcal{K}$ defined as

$$\mathcal{D}(\mathcal{L}) = \left\{ f \in \mathcal{K} : \lim_{t\to 0} \frac{1}{t}(S_t f - f) \text{ exists in } \mathcal{K} \right\},$$
$$\mathcal{L} f = \lim_{t\to 0} \frac{1}{t}(S_t f - f), \quad f \in \mathcal{D}(\mathcal{L}).$$

If $(S_t)_{t\geq 0}$ is a strongly continuous semigrup with generator $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ then the following hold:

(i) There exist constants $\gamma \in \mathbb{R}$ and $M \geq 1$ such that

$$\|S_t\| \leq M e^{\gamma t}, \quad t \geq 0.$$

(ii) For each $f \in \mathcal{K}$ the mapping $[0, \infty) \ni t \mapsto S_t f \in \mathcal{K}$ is continuous.

(iii) If $f \in \mathcal{K}$ then

$$\int_0^t S_s f \, \mathrm{d}s \in \mathcal{D}(\mathcal{L}) \text{ and } S_t f - f = \mathcal{L} \int_0^t S_s f \, \mathrm{d}s, \quad t > 0.$$

(iv) If $f \in \mathcal{D}(\mathcal{L})$ then $S_t f \in \mathcal{D}(\mathcal{L})$,

$$\frac{\mathrm{d}}{\mathrm{d}t} S_t f = S_t \mathcal{L} f = \mathcal{L} S_t f \text{ and } S_t f - f = \int_0^t S_s \mathcal{L} f \, \mathrm{d}s, \quad t \geq 0.$$

We say that $\lambda \in R$ belongs to the *resolvent set* $\rho(A)$ of a linear operator A, if the operator $\lambda I - A : \mathcal{D}(A) \to \mathcal{K}$ is invertible. The operator $R(\lambda, A) := (\lambda I - A)^{-1}$ for $\lambda \in \rho(A)$ is called the *resolvent operator* of A at λ.

After some preliminary definition and introducing the notation; next, we give most commonly used examples of semigroups:

B.1.1 Uniformly continuous semigroups

We suppose that A is a bounded operator on a Banach space $\mathcal{K}$. For all $f \in \mathcal{K}$ and $t \geq 0$, we have

$$\lim_{n\to\infty} \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n f = e^{At} f,$$

for a Cauchy sequence. Then the family of operators

$$S_t = e^{At} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n$$

defines a semigroup.

Lemma B.1.1. *The semigroup given by* $S_t = e^{At} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n$ *is strongly and uniformly continuous.*

Proof. Since we have

$$\begin{aligned} \|S_t f - f\| = \left\| \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n - f \right\| = \left\| \sum_{n=1}^{\infty} \frac{t^n}{n!} A^n \right\| \leq \sum_{n=1}^{\infty} \frac{t^n}{n!} \|A\|^n \|f\| \\ = \left(\sum_{n=0}^{\infty} \frac{t^n}{n!} \|A\|^n - 1 \right) \|f\| = (e^{\|A\|t} - 1)\|f\|. \end{aligned}$$

We obtain

$$\|S_t - I\| \leq e^{\|A\|t} - 1,$$

and then

$$\lim_{t\to 0} \|S_t - I\| = 0,$$

which implies that the semigroup is uniformly continuous. $\square$

Lemma B.1.2. *The generator of the semigroup* $(S_t)_{t\geq 0}$ *is* A.

Proof. This is true since we have

$$\begin{aligned} \|S_t f - f - tAf\| = \left\| \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n f - f - tAf \right\| = \left\| \sum_{n=2}^{\infty} \frac{t^n}{n!} A^n f \right\| \leq \sum_{n=2}^{\infty} \frac{t^n}{n!} \|A\|^n \|f\| \\ = \left(\sum_{n=0}^{\infty} \frac{t^n}{n!} \|A\|^n - 1 - t\|A\| \right) \|f\| = (e^{\|A\|t} - 1 - t\|A\|)\|f\|. \end{aligned}$$

$\square$

Lemma B.1.3. *Uniformly continuous semigroups have bounded generators.*

Proof. Since $(S_t)_{t\geq 0}$ is a unifromly continuous semigroup, the function $[0,\infty) \ni t \mapsto S_t \in BL(\mathcal{K})$ is continuous, where $BL(\mathcal{K})$ is the space of all bounded linear operators on the Banach space $\mathcal{K}$. Moreover, $BL(\mathcal{K})$ is also a Banach space when equipped with the operator norm. We have

$$\lim_{t\to 0} T_t = I \text{ where } T_t = \frac{1}{t}\int_0^t S_s \,\mathrm{d}s, \tag{B.1}$$

since $S : [0,\infty) \to BL(\mathcal{K})$ is continuous. Then (B.1) implies that there is a $\delta > 0$ such that for $0 \leq t < \delta$ we have $\|T_t - I\| \leq 1/2$ so that the operator T_t is invertible for a sufficiently small $t > 0$. Since for any $f \in \mathcal{K}$ we have

$$\frac{1}{t}(S_t f - f) = \mathcal{L}T_t f,$$

and the linear operator $\mathcal{L}T_t$ is bounded and therefore

$$\|\mathcal{L}f\| = \|\mathcal{L}T_t T_t^{-1} f\| \leq \|\mathcal{L}T_t\|\|T_t^{-1}\|\|f\|.$$

Thus $\mathcal{D}(\mathcal{L}) = \mathcal{K}$ and $\mathcal{L}$ is bounded. □

However, not every semigroup is strongly continuous. We give an example of semigroups depending on the Banach space it is defined on, it might not be strongly continuous.

B.1.2 Translation semigroups

We suppose $\mathcal{K}$ to be either the space of Lebesgue integrable functions $L^1(\mathbb{R})$ on $\mathbb{R}$ or the space of bounded functions $B(\mathbb{R})$ on $\mathbb{R}$. We define a semigroup on $(S_t)_{t\geq 0}$ on $\mathcal{K}$ by

$$S_t f(x) = f(x - t), \quad x \in \mathbb{R},\, t \geq 0. \tag{B.2}$$

It is a semigroup of contractions on $L^1(\mathbb{R})$ since we have

$$\|S_t f\| = \int_{\mathbb{R}} |f(x-t)| \,\mathrm{d}x = \int_{\mathbb{R}} |f(x)| \,\mathrm{d}x = \|f\|.$$

Lemma B.1.4. *The translation semigroup given by* (B.2) *is strongly continuous in* $L^1(\mathbb{R})$.

Proof. We take $f \in C_c(\mathbb{R})$, the space of continuous functions with compact support and we have

$$\lim_{t\to 0} f(x-t) = f(x), \text{ for every } x \in \mathbb{R}.$$

Then we have

$$\lim_{t\to 0} \|S_t f - f\| = \int_{\mathbb{R}} \lim_{t\to 0} |f(x-t) - f(x)| \,\mathrm{d}x = 0, \text{ for every } f \in C_c(\mathbb{R}),$$

by the Lebesgue dominated convergence theorem. Since $C_c(\mathbb{R})$ is dense in $L^1(\mathbb{R})$, then the semigroup is strongly continuous on $L^1(\mathbb{R})$. □

Lemma B.1.5. *The generator of the translation semigroup* (B.2) *on* $L^1(\mathbb{R})$ *is* $\mathcal{L}f = -f'$ *with domain*

$$\mathcal{D}(\mathcal{L}) = \{f \in L^1(\mathbb{R}) : f \textit{ is absolutely continuous and } f' \in L^1(\mathbb{R})\}.$$

Proof. We denote the generator of the translation semigroup (B.2) by $\tilde{\mathcal{L}}$, with the domain $\mathcal{D}(\tilde{\mathcal{L}})$. Take $f \in \mathcal{D}(\tilde{\mathcal{L}})$ such that

$$\lim_{t\to 0} \frac{1}{t}(S_t f - f) = g \in L^1(\mathbb{R}).$$

For every compact interval $[a,b] \subset \mathbb{R}$ we have

$$\left| \int_a^b \frac{1}{t}(f(x-t) - f(x)) \,\mathrm{d}x - \int_a^b g(x) \,\mathrm{d}x \right| \le \left\| \frac{1}{t}(S_t f - f) - g \right\|,$$

which implies

$$\lim_{t\to 0} \int_a^b \frac{f(x-t) - f(x)}{t} \,\mathrm{d}x = \int_a^b g(x) \,\mathrm{d}x.$$

Also for all sufficiently small $t > 0$ we obtain

$$\int_a^b \frac{1}{t}(f(x-t) - f(x)) \,\mathrm{d}x = \frac{1}{t} \int_{a-t}^a \frac{1}{t} f(x) \,\mathrm{d}x - \frac{1}{t} \int_{b-t}^b f(x) \,\mathrm{d}x.$$

Morever, since

$$\lim_{t\to 0} \frac{1}{t} \int_{b-t}^b f(x) \,\mathrm{d}x = f(b), \text{ for a.e. } b \in \mathbb{R},$$

we obtain

$$\int_a^b g(x) \,\mathrm{d}x = f(a) - f(b) \text{ for a.e. } a, b \in \mathbb{R}.$$

Therefore f is absolutely continuous and its derivative is equal to $-g$ in $L^1(\mathbb{R})$ which is integrable. This implies that the operator $\mathcal{L}$ is an extension of the generator $\tilde{\mathcal{L}}$. We have $1 \in \rho(\tilde{\mathcal{L}})$. Furthermore, since the general solution to the differential equation $f(x) + f'(x) = 0$ is $f(x) = ce^{-t}$ for $x \in \mathbb{R}$ and a constant c, then $I - \mathcal{L}$ is one to one.

It only remains to show that $\tilde{\mathcal{L}} = \mathcal{L}$. For $k \in \mathcal{D}(\mathcal{L})$, take $h = k - \mathcal{L}k$. Since $1 \in \rho(\tilde{\mathcal{L}})$, the operator $(I - \tilde{\mathcal{L}})$ is invertible. Therefore we can find $d \in \mathcal{D}(\tilde{\mathcal{L}})$ such that $h = d - \tilde{\mathcal{L}}d$. Since $\tilde{\mathcal{L}} \subset \mathcal{L}$, we have $k - Ak = d - \tilde{A}d = d - Ad$. Being $I - A$ one to one makes $k = d$, showing that $\mathcal{D}(A) \subset \mathcal{D}(\tilde{\mathcal{L}})$. Thus $\tilde{\mathcal{L}} = \mathcal{L}$. □

Now, we consider the same semigroup given by (B.2) on the space $B(\mathbb{R})$. It is a semigroup of contractions on $B(\mathbb{R})$ since we have

$$\|S_t f\|_v = \sup_{x\in\mathbb{R}} |S_t f(x)| \leq \sup_{x\in\mathbb{R}} |f(x)| = \|f\|_v.$$

Lemma B.1.6. *The translation semigroup given by (B.2) is not strongly continuous on $B(\mathbb{R})$.*

Proof. We take $f(x) = \mathbb{1}_{[0,1)}(x)$, $x \in \mathbb{R}$ where $\mathbb{1}_A$ the indicator function defined on a set A:

$$\mathbb{1}_A(x) = \begin{cases} 1 \text{ if } x \in A, \\ 0 \text{ if } x \notin A. \end{cases}$$

We have

$$|S_t f(x) - f(x)| = |\mathbb{1}_{[t,1+t)}(x) - \mathbb{1}_{[0,1)}(x)| = |\mathbb{1}_{[1,1+t)}(x) - \mathbb{1}_{[0,1)}(x)|, \text{ for } t \in (0,1),\ x \in \mathbb{R},$$

so that

$$\|S_f - f\|_v = \sup_{x\in\mathbb{R}} |S_t f(x) - f(x)| = 1 \text{ for all } t \in (0,1).$$

□

Remark B.1.1. *Note that for a semigroup S_t on a Banach space $\mathcal{K}$, we define*

$$\mathcal{K}_0 = \{f \in \mathcal{K} : \lim_{t\to 0} \|S_t f - f\| = 0\}$$

is a closed linear subspace of $\mathcal{K}$ and $S_t(\mathcal{K}_0) \subseteq \mathcal{K}_0$ for all $t \geq 0$. This implies that S_t is a strongly continuous semigroup on the Banach space $\mathcal{K}$.

For the translation semigroup on $\mathcal{K} = B(\mathbb{R})$, the subspace $\mathcal{K}_0$ contains all uniformly continuous functions on $\mathbb{R}$.

Now, we state the first major result in abstract theory of contraction semigroups developped by Hille and Yosida independently:

Theorem B.1.2 (Hille-Yosida)**.** *A linear operator $\mathcal{L}$ with a domain $\mathcal{D}(\mathcal{L})$ on a Banach space $\mathcal{K}$ is the generator of a contraction semigroup if and only if $\mathcal{D}(\mathcal{L})$ is dense in $\mathcal{K}$, the resolvent set $\rho(\mathcal{L})$ of $\mathcal{L}$ contains $(0,\infty)$, and for every $\lambda > 0$*

$$\|\lambda R(\lambda, \mathcal{L})\| \leq 1.$$

In that case, the semigroup $(S_t)_{t\geq 0}$ with generator $\mathcal{L}$ is given by

$$S_t f = \lim_{\lambda\to\infty} e^{\lambda t} \sum_{n=0}^{\infty} \frac{\lambda^n t^n}{n!} (\lambda R(\lambda, \mathcal{L}))^n f, \quad f \in \mathcal{K}, t \geq 0.$$

Proof. For the proof we refer to [56]. □

Remark B.1.3. *By using Hille-Yosida Theorem, to check that an operator $\mathcal{L}$ with a dense domain is the generator, we need to show that for each $\lambda > 0$ and $g \in \mathcal{K}$ there exists a unique solution $f \in \mathcal{D}(\mathcal{L})$ of*

$$\lambda f - \mathcal{L} f = g \text{ and } \lambda\|f\| \leq \|g\|.$$

B.2 Stochastic semigroups

In this subsection, we introduce stochastic semigroups and provide characterizations of their generators. By definition, they are strongly continuous semigroups of stochastic operators on L^1 spaces.

We consider the Banach space $\mathcal{K} = L^1 = L^1(\Omega, \mathcal{F}, \mu)$ where $(\Omega, \mathcal{F}, \mu)$ is a $\sigma-$finite measure space equipped with the norm

$$\|f\| = \int_\Omega |f(x)|\mu(\,\mathrm{d}x), \quad f \in L^1.$$

Any f can be written as the difference of two non-negative functions $f = f^+ - f^-$, where the *positive part* of f is defined as

$$f^+ = \max\{0, f\},$$

whereas the *negative part* of f is defined as

$$f^- = (-f)^+ = \max\{0, -f\}.$$

We define the positive cone as the following;

$$L^1_+ := \{f \in L^1 : f \geq 0\}.$$

A linear operator $A : \mathcal{D}(A) \to L^1$ is said to be *positive* if $Af \geq 0$ for $f \in \mathcal{D}(A)_+$, where $\mathcal{D}(A)_+ = \mathcal{D}(A) \bigcap L^1_+$, and we write $A \geq 0$. Norm of the bounded (positive and everywhere defined) operator is determined by values on the positive cone:

Proposition B.2.1. *Let A be a linear operator with $\mathcal{D}(A) = L^1$. If*

$$\|(Af)^+\| \leq \|f^+\|, \quad f \in L^1,$$

then A is positive. If A is positive then A is a bounded operator and

$$\|A\| = \sup_{f\geq 0, \|f\|=1} \|Af\|.$$

Proof. We have $(Af)^- = (-Af)^+ = (A(-f))^+$. If $f \geq 0$ then since $f^- = 0$ we have

$$\|(Af)^-\| = \|(A(-f))^+\| \leq \|(-f)^+\| = \|f^-\| = 0.$$

Therefore $Af = (Af)^+ \geq 0$. Also, $|Af| \leq A|f|$ holds since for any $f \in L^1$, we have $-|f| \leq f \leq |f|$. Thus, we obtain $\|Af\| \leq \|A|f|\|$.

Let $\alpha = \sup \|A|f|\|$, then $\alpha \leq \|A\|$. Since $|f| \geq 0$, we see that

$$\|Af\| \leq \|A|f|\| \leq \alpha\|f\| \text{ for any } f \in L^1,$$

which proves that $\|A\| = \alpha$.

Suppose that A is not bounded. Then we can find a sequence $f_n \in L^1_+$ such that $\|f_n\| = 1$ and $\|Af_n\| \geq n^3$ for every n. Since non-negative and integrable f is defined by

$$f = \sum_{n=1}^{\infty} \frac{f_n}{n^2}$$

we have that $f \geq \frac{f_n}{n^2}$, which implies that $n^2 Af \geq Af_n$ for all $n \geq 1$. Thus we have for any n

$$n^3 \leq \|Af_n\| \leq n^2\|Af\|,$$

which is impossible, since $\|Af\| < \infty$. □

A family $(S_t)_{t\geq 0}$ of linear operators on L^1 is called a *substochastic semigroup (stochastic and positive semigroup)* if S_t is a substochastic (stochastic and positive) operator on L^1 for every t and $(S_t)_{t\geq 0}$ is a strongly continuous semigroup. Therefore, a substochastic semigroup is a positive contraction semigroup on L^1. Now we look at the generators of substochastic and stochastic semigroups. If a linear operator $\mathcal{L}$ with the domain $\mathcal{D}(\mathcal{L})$ is the generator of a substochastic semigroup $(S_t)_{\{t\geq 0\}}$ then the resolvent operator of $\mathcal{L}$ at each $\lambda > 0$ is positive (*resolvent positive*). It means that for all $\lambda > 0$ and $f \in L^1_+$ we have

$$R(\lambda, \mathcal{L})f = \int_0^\infty e^{-\lambda t} S_t f \,\mathrm{d}t \geq 0.$$

We recall that a *stochastic operator* is a *substochastic operator* which preserves the L^1 norm on the positive cone. Then we state the following theorems about the generators of substochastic and stochastic semigroups without the proofs:

Theorem B.2.2. *A linear operator $\mathcal{L}$ with the domain $\mathcal{D}(\mathcal{L})$ is the generator of a substochastic semigroup on L^1 if and only if $\mathcal{D}(\mathcal{L})$ is dense in L^1, $\mathcal{L}$ is resolvent positive and*

$$\int_\Omega \mathcal{L}f(x)\mu(\,\mathrm{d}x) \leq 0 \textit{ for all } f \in \mathcal{D}(\mathcal{L})_+.$$

Similarly,

Corollary B.2.1. *A linear operator $\mathcal{L}$ with the domain $\mathcal{D}(\mathcal{L})$ is the generator of a substochastic semigroup on L^1 if and only if $\mathcal{D}(\mathcal{L})$ is dense in L^1, $\mathcal{L}$ is resolvent positive and*

$$\int_\Omega \mathcal{L}f(x)\mu(\,\mathrm{d}x) = 0 \textit{ for all } f \in \mathcal{D}(\mathcal{L})_+.$$

B.2.1 Transition semigroups

A stochastic semigroup $(S_t)_{t\geq 0}$ corresponds to a *transition function* $P = \{P(t, \cdot) = t \geq 0\}$ if for each $t > 0$ the adjoint operator $P^*(t)$ of $P(t)$ is given by

$$P^*(t)g(x) = \int_\Omega g(y)P(t, x, \mathrm{d}y), \quad g \in L^\infty.$$

Each transition kernel $P(t, \cdot)$ satisfies

$$\int_B P(t)f(x)\mu(\,\mathrm{d}x) = \int_\Omega P(t, x, B)f(x)\mu(\,\mathrm{d}x), \quad B \in \mathcal{F},\ f \in L^1_+.$$

If a stochastic semigroup $(S_t)_{t\geq 0}$ corresponds to the transition function P induced by a Markov process $X = \{X_t : t \geq 0\}$, i.e.

$$P(t, x, B) = \mathcal{P}_{x_0}(X_t \in B), \quad x \in \Omega,\, B \in \mathcal{F}, \tag{B.3}$$

where $\mathcal{P}_x$ is the distribution of X_t starting at x_0, then $S_t f$ is the density of X_t if the distribution of X_0 has a density f.

We make a relation between a stochastic semigroup $(S_t)_{t\geq 0}$ to the transition semigroup $(T_t)_{t\geq 0}$ on $B(\Omega)$ associated to a homogeneous Markov process $X = \{X_t : t \geq 0\}$ with transition function $P = \{P(t, \cdot) : t \geq 0\}$. The transition semigroup $(T_t)_{t\geq 0}$ on the Banach space $B(\Omega)$ with supremum norm $\|\cdot\|_u$ associated to the process X is given by

$$T_t g(x) = \mathbb{E}_x(g(X_t)) = \int_\Omega g(y)P(t, x, \mathrm{d}y)), \quad g \in B(\Omega).$$

If the following holds for all $f \in L^1_+$ and all $g \in B(\Omega)$;

$$\int_\Omega T(t)g(x)f(x)\mu(\mathrm{d}x) = \int_\Omega g(x)P(t)\mu(\mathrm{d}x), \tag{B.4}$$

then (B.3) holds and $(S_t)_{t\geq 0}$ corresponds to the transition function P. The transition $(T_t)_{t\geq 0}$ semigroup on $B(\Omega)$ is strongly continuous on the closed subspace of $B(\Omega)$

$$B_0(\Omega) = \{g \in B(\Omega) : \lim_{t\to 0} \|T_t g - g\|_u = 0\}$$

and $T_t g \in B_0(\Omega)$ for $g \in B_0(\Omega)$. The generator $\mathcal{L}$ with the domain $\mathcal{D}(\mathcal{L})$ of the semigroup $(T_t)_{t\geq 0}$ is densely defined in $B_0(\Omega)$. Moreover, the adjoint $S^*(t)$ of the operator $S(t)$ is a contraction on L^∞ for each t.

$$\int_\Omega S_t^* g(x)f(x)\mu(\mathrm{d}x) = \int_\Omega g(x)S_t f(x)\mu(\mathrm{d}x), \quad g \in L^\infty,\, f \in L^1,$$

and $(S_t^*)_{t\geq 0}$ is a semigroup on L^∞, called the *adjoint semigroup*.

B.3 Markov semigroups

We consider the σ-finite measure space $(\Omega, \mathcal{F}, \mu)$ and a subset $\mathcal{D} \subset L^1(\Omega, \mathcal{F}, \mu)$ containing all densities

$$\mathcal{D} = \left\{f \in L^1 : f \geq 0, \|f\| = 1\right\}$$

Then a linear mapping $M : L^1 \to L^1$ is called as *Markov operator* if $M(\mathcal{D}) \subset \mathcal{D}$. Markov operators can be defined by means of *transition probability functions.* We recall that if $P(x, A)$ is a transition probability function on $(\Omega, \mathcal{F})$ if

- $P(x, \cdot)$ is a probability measure on $(\Omega, \mathcal{F})$ and
- $P(\cdot, B)$ is a measurable function for every $B \in \mathcal{F}$.

We also assume that if $\mu(B) = 0$ then $P(x, B) = 0$ for μ-almost every x so that for every $f \in \mathcal{D}$, the measure $\int_\Omega f(x)P(x, B)\mu(\,\mathrm{d}x)$ is absolutely continuous with respect to the measure μ. Then Mf where $M : L^1 \to L^1$ defines a *Markov operator.* We also define the adjoint of M as $M^* : L^\infty \mapsto L^\infty$ such that;

$$M^*g(x) = \int g(y)P(x,\,\mathrm{d}y).$$

Not every Markov operator has an adjoint. But if we take Ω to be a Polish space which is a complete and separable metric space, then $\mathcal{F} = \mathcal{B}(\Omega)$ is the σ-algebra of Borel subsets of Ω and μ corresponds to the probability measure on Ω. Then every Markov operator on $L^1(\Omega, \mathcal{B}(\Omega), \mathcal{P})$ is given by a *transition probability function.* A family of Markov operators $(M_t)_{t \geq 0}$ is called a *Markov semigroup* if

- $M_0 = Id$,
- $M_{t+s} = M_t M_s$ for $t, s \geq 0$,
- for every $f \in L^1$ the function $t \mapsto M_t f$ is continuous.