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NUMERICAL SEMIGROUPS BOUNDED BY A CYCLIC MONOID

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Abstract. We will say that a numerical semigroup S is bounded by a cyclic monoid if there exist integer numbers $0 \leqslant \alpha < \beta$ such that $S = \{x \in \mathbb{N} \mid k\alpha < x < k\beta \text{ for some } k \in \mathbb{N}\} \cup \{0\}$. The goal of this work is to study this kind of numerical semigroups. In particular, we will determine important invariants of them such as multiplicity, embedding dimension, Frobenius number and genus.

1. Introduction

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ be the set of nonnegative integers. A *numerical semigroup* is a subset S of $\mathbb{N}$ which is closed by sum, $0 \in S$ and $\mathbb{N} \setminus S$ is finite.

Let M be a submonoid of $(\mathbb{N}^2, +)$. Following the notation introduced in [5], we will say that a positive integer n is *bounded by* M if there exists $(x, y) \in M$ such that $x < n < y$. We will denote by $\mathrm{A}(M) = \{n \in \mathbb{N} \mid n \text{ is bounded by } M\}$. The following result appears in [5, Proposition 2.2].

PROPOSITION 1. *If M is a submonoid of $(\mathbb{N}^2, +)$ and $\mathrm{A}(M)$ is a non-empty set, then $\mathrm{A}(M) \cup \{0\}$ is a numerical semigroup.*

The above result induces us to give the following definition. We will say that a numerical semigroup S is *bounded by* a submonoid of $\mathbb{N}^2$, if there exists M a submonoid of $(\mathbb{N}^2, +)$ such that $S = \mathrm{A}(M) \cup \{0\}$. These numerical semigroups will be called $\mathscr{A}$*-semigroups*.

The concept of $\mathscr{A}$-semigroup was introduced in [5]. The following result is a characterization of these numerical semigroups and it appears in [5, Theorem 2.4].

THEOREM 2. *Let S be a numerical semigroup. Then S is $\mathscr{A}$-semigroup if and only if $\{x + y - 1, x + y + 1\} \subseteq S$ for all $\{x, y\} \subseteq S \setminus \{0\}$.*

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We will say that a submonoid M of $(\mathbb{N}^2,+)$ is *cyclic* if there exists $(\alpha,\beta)\in M$ such that $M=\{(k\alpha,k\beta)\mid k\in\mathbb{N}\}$.

Our aim in this work is to study the numerical semigroups bounded by a cyclic submonoid of $(\mathbb{N}^2,+)$. This kind of numerical semigroups will be called $\mathscr{AC}$*-semigroups*. Accordingly, a numerical semigroup S is an $\mathscr{AC}$-semigroup if only if there exists $(\alpha,\beta)\in\mathbb{N}^2\setminus\{(0,0)\}$ such that $S=\{x\in\mathbb{N}\mid k\alpha<x<k\beta \text{ for some } k\in\mathbb{N}\cup\{0\}\}$.

Following the notation introduced in [8], a *proportionally modular Diophantine inequation* is an expression of the form $ax \bmod b\leqslant cx$ where a, b and c are positive integers, and $ax \bmod b$ is the remainder of the division of ax by b. If we denote by $S(a,b,c)$ the set of integer solutions of the previous inequality, then in [8] it is shown that $S(a,b,c)$ is a numerical semigroup. We say that a numerical semigroup is a *proportionally modular numerical semigroup* (PM-semigroup, for short) if there exist positive integers a, b and c such that $S=S(a,b,c)$.

We will denote by $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{R}_0^+$ the set of real, positive real and non-negative real numbers respectively. If X is a subset non empty of $\mathbb{R}_0^+$, then we will denote by $\langle X\rangle$ the submonoid of $(\mathbb{R}_0^+,+)$ generated by X. That is, $\langle X\rangle=\{\lambda_1x_1+\ldots+\lambda_nx_n\mid n\in\mathbb{N}\setminus\{0\},x_1,\ldots,x_n\in X \text{ and } \lambda_1,\ldots,\lambda_n\in\mathbb{N}\}$. In [8] it is proved that $S(X)=\langle X\rangle\cap\mathbb{N}$ is a submonoid of $(\mathbb{N},+)$. Moreover, it is shown that a numerical semigroup is a PM-semigroup if and only if it has the form $S([p,q])$ where $p,q\in\mathbb{R}_0^+$, $p<q$ and $[p,q]=\{r\in\mathbb{R}\mid p\leqslant r\leqslant q\}$. We will refer to $[p,q]$ as the closed interval with ends p and q. We will call open interval, semi-open interval and semi-closed interval to the sets $]p,q[=\{r\in\mathbb{R}\mid p<r<q\}$, $[p,q[=\{r\in\mathbb{R}\mid p\leqslant r<q\}$ and $]p,q]=\{r\in\mathbb{R}\mid p<r\leqslant q\}$ respectively. In [7] it is shown that if I is an interval (closed, open, semi-closed or semi-open) then $S(I)$ is also a PM-semigroup. In Section 2, we will see that every $\mathscr{AC}$-semigroup is a PM-semigroup. More concretely, we prove that if $(\alpha,\beta)\in\mathbb{N}^2\setminus\{(0,0)\}$ where $\alpha<\beta$ and $M=\{(k\alpha,k\beta)\mid k\in\mathbb{N}\}$, then $\mathrm{A}(M)\cup\{0\}=S(]\alpha,\beta[)$. So, the $\mathscr{AC}$-semigroups are the PM-semigroups associated to the open intervals which have nonnegative integers ends.

If S is a numerical semigroup, we denote by $F(S)$ the maximum integer number that does not belong to S and by $g(S)$ the cardinality of $\mathbb{N}\setminus S$. We will call to $F(S)$ and $g(S)$ the *Frobenius number* and *the genus* of S respectively. In Section 2 we will give formulas to calculate $F(S(]\alpha,\beta[))$ and $g(S(]\alpha,\beta[))$ in function of α and β.

Let S be a numerical semigroup and $A\subseteq\mathbb{N}$. If $S=\langle A\rangle$, we will say that A is a *system of generators* of S. Besides, if $S\neq\langle B\rangle$ for all $B\subsetneq A$, then we will say A is a *minimal system of generators* of S. In [6, Corollary 2.8] it is shown that every numerical semigroup has a unique minimal system of generators which, in addition, is finite. We denote by $\mathrm{msg}(S)$ the minimal system of generators of S. The cardinality of $\mathrm{msg}(S)$ is called the *embedding dimension* of S and will be denoted by $\mathrm{e}(S)$. The smallest positive integer that belongs to S is called the *multiplicity* of S and it is denoted by $\mathrm{m}(S)$. It is well known (see, for instance [6, Proposition 2.10]) that $\mathrm{e}(S)\leqslant\mathrm{m}(S)$. A numerical semigroup is called a MED*-semigroup* (numerical semigroup with maximal embedding dimension) if $\mathrm{e}(S)=\mathrm{m}(S)$.

In Section 3 we will study the PM-semigroups $S(]\alpha,\alpha+1[)$, $S(]\alpha,\alpha+1])$, $S([\alpha,\alpha+1[)$ and $S(]\alpha,\alpha+2[)$, where α is a positive integer. We will prove that they are MED-semigroups and we will give explicitly their minimal systems of generators.

In Section 4, we will study $S(]\alpha,\beta[)$ where α, $\beta \in \mathbb{N}\setminus\{0\}$ and $\beta-\alpha \geqslant 3$. In this case, we will see that $S(]\alpha,\beta[)$ is not a MED-semigroup. We will give explicitly a finite system of generators of $S(]\alpha,\beta[)$. Finally, we will calculate positive integers a, b and c such that $S(]\alpha,\beta[) = \{x\in\mathbb{N} \mid ax \bmod b \leqslant cx\}$.

2. $\mathscr{AC}$-semigroups and PM-semigroups

If α and β are integer numbers, we will denote by $A(\alpha,\beta) = \{n\in\mathbb{N} \mid \alpha < n < \beta\}$.

PROPOSITION 3. *Let S be a numerical semigroup. Then S is an $\mathscr{AC}$-semigroup if and only if there exist $\alpha,\beta\in\mathbb{N}$ such that $\alpha<\beta$ and $S = (\bigcup_{k\in\mathbb{N}} A(k\alpha,k\beta))\cup\{0\}$.*

Proof. If S is an $\mathscr{AC}$-semigroup, then there exists M a cyclic submonoid of $(\mathbb{N}^2,+)$ such that $S = \mathrm{A}(M)\cup\{0\}$. But, if M is a cyclic submonoid of $(\mathbb{N}^2,+)$ then there exits $(\alpha,\beta)\in\mathbb{N}^2$ such that $M = \{(k\alpha,k\beta)\mid k\in\mathbb{N}\}$. Therefore, $S = \mathrm{A}(M)\cup\{0\} = (\bigcup_{k\in\mathbb{N}} A(k\alpha,k\beta))\cup\{0\}$. Finally, applying that $\mathrm{A}(M)\neq\emptyset$, we deduce that $\alpha<\beta$.

Conversely, if $M = \{(k\alpha,k\beta)\mid k\in\mathbb{N}\}$ then M is a cyclic submonoid of $(\mathbb{N}^2,+)$. Moreover, $S = \mathrm{A}(M)\cup\{0\}$ and $\mathrm{A}(M)\neq\emptyset$. Thus, S is an $\mathscr{AC}$-semigroup. $\square$

The next result appears in [7, Lemma 2].

LEMMA 4. *Let I be an interval (open, closed, semi-open, semi-closed) of $\mathbb{R}_0^+$ and $x\in\mathbb{R}^+$. Then $x\in\langle I\rangle$ if and only if there exists $k\in\mathbb{N}\setminus\{0\}$ such that $\frac{x}{k}\in I$.*

The following result shows the relationship between the $\mathscr{AC}$-semigroups and the PM-semigroups.

PROPOSITION 5. *Let α and β be integer numbers such that $0\leqslant\alpha<\beta$ and $M = \{(k\alpha,k\beta)\mid k\in\mathbb{N}\}$. Then $\mathrm{A}(M)\cup\{0\} = S(]\alpha,\beta[)$.*

Proof. If $x\in\mathrm{A}(M)\setminus\{0\}$ then there exists $k\in\mathbb{N}\setminus\{0\}$ such that $x\in A(k\alpha,k\beta)$. This is, $k\alpha<x<k\beta$. So, $\alpha<\frac{x}{k}<\beta$. Thus $\frac{x}{k}\in]\alpha,\beta[$. By applying Lemma 4, we have that $x\in S(]\alpha,\beta[)$. For the other inclusion, if $x\in S(]\alpha,\beta[)\setminus\{0\}$, then applying Lemma 4, we have that there exists $k\in\mathbb{N}\setminus\{0\}$ such that $\frac{x}{k}\in]\alpha,\beta[$. Therefore, $x\in A(k\alpha,k\beta)$. Thus, $x\in\mathrm{A}(M)$. $\square$

As an immediate consequence of Proposition 5, we have the following result.

COROLLARY 6. *Every $\mathscr{AC}$-semigroup is a PM-semigroup. Moreover, the set formed by all the $\mathscr{AC}$-semigroups is $\{S(]\alpha,\beta[)\mid(\alpha,\beta)\in\mathbb{N}^2$ and $\alpha<\beta\}$.*

Following the notation introduced in [10], a sequence of fractions $\frac{a_1}{b_1}<\frac{a_2}{b_2}<\dots\frac{a_p}{b_p}$ is a *Bézout sequence* if $\{a_1,b_1,a_2,b_2,\dots,a_p,b_p\}\subseteq\mathbb{N}\setminus\{0\}$ and $a_{i+1}b_i-a_ib_{i+1}=1$ for all $i \in \{1,\dots,p-1\}$.

The following result follows of [10, Lemma 11, Corollary 27 and Theorem 12].

PROPOSITION 7. *If $\frac{a_1}{b_1} < \frac{a_2}{b_2} < \dots < \frac{a_p}{b_p}$ is a Bézout sequence, then:*

1. $S\left(\left[\frac{a_1}{b_1}, \frac{a_2}{b_2}\right]\right) = \langle \{a_1, a_2\}\rangle$.
2. $S\left(\left[\frac{a_1}{b_1}, \frac{a_p}{b_p}\right]\right) = \langle \{a_1, a_2\}\rangle \cup \langle \{a_2, a_3\}\rangle \cup \dots \cup \langle \{a_{p-1}, a_p\}\rangle$.
3. $S\left(\left[\frac{a_1}{b_1}, \frac{a_p}{b_p}\right]\right) = \langle \{a_1, a_2, \dots, a_p\}\rangle$.

By applying Lemma 4, we easily deduce that if $\beta \in \mathbb{N}\setminus\{0\}$, then $S(]0,\beta[) = \mathbb{N}$. Therefore, from now on, and unless otherwise stated, we will assume that α, $\beta \in \mathbb{N}\setminus\{0\}$ and $\alpha < \beta$. Note that $\frac{\alpha}{1} < \frac{\alpha+1}{1} < \dots < \frac{\beta}{1}$ is a Bézout sequence. So, by applying Proposition 7 (3), we obtain the following result.

COROLLARY 8. *If α, $\beta \in \mathbb{N}\setminus\{0\}$ and $\alpha < \beta$ then $S([\alpha,\beta]) = \langle\{\alpha, \alpha+1, \dots, \beta\}\rangle$.*

If $r \in \mathbb{R}$, we denote by $\lfloor r \rfloor = \max\{z \in \mathbb{Z} \mid z \leqslant r\}$. The following result is deduced from [12, Theorem 4] which is, in turn, attributed by its author to [4], [2] and [3].

PROPOSITION 9. *If $\{a,k\} \subseteq \mathbb{N}\setminus\{0\}$, $a \geqslant 2$ and $S = \langle\{a, a+1, \dots, a+k\}\rangle$, then*

1. $F(S) = \left(\lfloor \frac{a-2}{k} \rfloor + 1\right) a - 1$.
2. $g(S) = \left(\lfloor \frac{a-2}{k} \rfloor + 1\right) \dfrac{a + (a-2) \bmod k}{2}$.

Note that as $]\alpha,\beta[\subseteq [\alpha,\beta]$ then, applying Lemma 4, we deduce easily that $S(]\alpha,\beta[) \subseteq S([\alpha,\beta])$.

LEMMA 10. *Let $\{\alpha,\beta\} \subseteq \mathbb{N}\setminus\{0\}$ where $\alpha < \beta$. Then $x \in S([\alpha,\beta]) \setminus S(]\alpha,\beta[)$ if and only if $x \in \{t\alpha \mid t \in \{1, \dots, \lfloor \frac{\beta}{\beta-\alpha} \rfloor\}\} \cup \{t\beta \mid t \in \{1, \dots, \lfloor \frac{\alpha}{\beta-\alpha} \rfloor\}\}$.*

Proof. If $x \in S([\alpha,\beta]) \setminus S(]\alpha,\beta[)$ then, applying Lemma 4, we deduce that there exists $t \in \mathbb{N}\setminus\{0\}$ such that $\frac{x}{t} = \alpha$ or $\frac{x}{t} = \beta$. In addition, if $\frac{x}{t} = \alpha$, then $\beta \leqslant \frac{x}{t-1}$ and so $t \leqslant \frac{\beta}{\beta-\alpha}$. If $\frac{x}{t} = \beta$, then $\frac{x}{t+1} \leqslant \alpha$ and so $t \leqslant \frac{\alpha}{\beta-\alpha}$. The another inclusion is trivial. $\square$

We illustrate the above result with an example.

EXAMPLE 11. By applying Lemma 10, we obtain the following.

1. $S([5,7]) \setminus S(]5,7[) = \{t.5 \mid t \in \{1, \dots, \lfloor \frac{7}{2} \rfloor\}\} \cup \{t.7 \mid t \in \{1, \dots, \lfloor \frac{5}{2} \rfloor\}\} = \{5, 10, 15\} \cup \{7, 14\} = \{5, 7, 10, 14, 15\}$.

2. $S([4,6]) \setminus S(]4,6[) = \{t.4 \mid t \in \{1,\dots,\lfloor \frac{6}{2} \rfloor\}\} \cup \{t.6 \mid t \in \{1,\dots,\lfloor \frac{4}{2} \rfloor\}\} = \{4,8,12\} \cup \{6,12\} = \{4,6,8,12\}$.

Given a set X, we denote by $\#X$, the cardinality of X. If α and β are integer numbers, we denote by $d = \gcd\{\alpha,\beta\}$ the greater common divisor of α and β.

LEMMA 12. *Let $\{\alpha,\beta\} \subseteq \mathbb{N} \setminus \{0\}$ such that $\alpha < \beta$. Then we have*

$$\#(S([\alpha,\beta]) \setminus S(]\alpha,\beta[)) = \left\lfloor \frac{\beta}{\beta-\alpha} \right\rfloor + \left\lfloor \frac{\alpha}{\beta-\alpha} \right\rfloor - \left\lfloor \frac{d}{\beta-\alpha} \right\rfloor.$$

Proof. By Lemma 10, we know that

$$\#(S([\alpha,\beta]) \setminus S(]\alpha,\beta[))$$
$$= \#\left(\left\{t\alpha \mid t \in \left\{1,\dots,\left\lfloor \frac{\beta}{\beta-\alpha} \right\rfloor\right\}\right\} \cup \left\{t\beta \mid t \in \left\{1,\dots,\left\lfloor \frac{\alpha}{\beta-\alpha} \right\rfloor\right\}\right\}\right).$$

If $t\alpha = t'\beta$ then $t\alpha = t'\beta = l\frac{\alpha\beta}{d}$ for some $l \in \mathbb{N} \setminus \{0\}$. Therefore, $\frac{\beta}{d} \leqslant t$ and $\frac{\alpha}{d} \leqslant t'$. So, $\frac{\beta}{d} \leqslant \lfloor \frac{\beta}{\beta-\alpha} \rfloor$ and $\frac{\alpha}{d} \leqslant \lfloor \frac{\alpha}{\beta-\alpha} \rfloor$. Thus $\frac{\beta}{d} \leqslant \frac{\beta}{\beta-\alpha}$ and $\frac{\alpha}{d} \leqslant \frac{\alpha}{\beta-\alpha}$. Then $\beta-\alpha \leqslant d$. On the other hand, it is clear that $d \leqslant \beta-\alpha$. Hence, $d = \beta-\alpha$.

The proof is concluded observing that

$$\left\{t\alpha \mid t \in \left\{1,\dots,\frac{\beta}{d}\right\}\right\} \cap \left\{t\beta \mid t \in \left\{1,\dots,\frac{\alpha}{d}\right\}\right\} = \left\{\frac{\alpha\beta}{d}\right\}. \quad \square$$

THEOREM 13. *Let $\{\alpha,\beta\} \subseteq \mathbb{N} \setminus \{0\}$ such that $\alpha < \beta$. Then*

1. $F(S(]\alpha,\beta[)) = \max\left\{\left\lfloor \frac{\beta}{\beta-\alpha} \right\rfloor \alpha, \left\lfloor \frac{\alpha}{\beta-\alpha} \right\rfloor \beta\right\}$.

2. $g(S(]\alpha,\beta[)) = \left(\left\lfloor \frac{\alpha-2}{\beta-\alpha} \right\rfloor + 1\right) \dfrac{\alpha+(\alpha-2)\bmod(\beta-\alpha)}{2} + \left\lfloor \dfrac{\beta}{\beta-\alpha} \right\rfloor + \left\lfloor \dfrac{\alpha}{\beta-\alpha} \right\rfloor - \left\lfloor \dfrac{\gcd\{\alpha,\beta\}}{\beta-\alpha} \right\rfloor$.

Proof.

1. By applying Corollary 8, Proposition 9 and Lemma 10, we deduce that $F(S(]\alpha,\beta[)) = \max\left\{\left(\left\lfloor \frac{\alpha-2}{\beta-\alpha} \right\rfloor + 1\right)\alpha - 1, \left\lfloor \frac{\beta}{\beta-\alpha} \right\rfloor \alpha, \left\lfloor \frac{\alpha}{\beta-\alpha} \right\rfloor \beta\right\}$. The proof of this point is concluded observing that $\left(\left\lfloor \frac{\alpha-2}{\beta-\alpha} \right\rfloor + 1\right)\alpha - 1 \leqslant \left\lfloor \frac{\beta}{\beta-\alpha} \right\rfloor \alpha$.

2. This result is an immediate consequence of Corollary 8, Proposition 9 and Lemmas 10 and 12. $\square$

We will illustrate the above theorem with an example.

EXAMPLE 14. By using Theorem 13, we can easily deduce the following.

1. $F(S(]5,7[) = \max\{\lfloor\frac{7}{2}\rfloor 5, \lfloor\frac{5}{2}\rfloor 7\} = \max\{15,14\} = 15$ and
$g(S(]5,7[) = (\lfloor\frac{3}{2}\rfloor)+1)\frac{5+3 \bmod 2}{2} + \lfloor\frac{7}{2}\rfloor + \lfloor\frac{5}{2}\rfloor - \lfloor\frac{1}{2}\rfloor = 11.$
2. $F(S(]6,8[) = \max\{\lfloor\frac{8}{2}\rfloor 6, \lfloor\frac{6}{2}\rfloor 8\} = \max\{24,24\} = 24$ and
$g(S(]6,8[) = (\lfloor\frac{4}{2}\rfloor)+1)\frac{6+4 \bmod 2}{2} + \lfloor\frac{8}{2}\rfloor + \lfloor\frac{6}{2}\rfloor - \lfloor\frac{2}{2}\rfloor = 15.$

3. Particular cases

Let $\alpha \in \mathbb{N}\setminus\{0\}$. In this section we are especially interested in calculating the minimal system of generators of the numerical semigroups $S(]\alpha,\alpha+1[)$, $S(]\alpha,\alpha+1])$, $S([\alpha,\alpha+1[)$ and $S(]\alpha,\alpha+2[)$.

3.1. The PM-semigroup $S(]\alpha,\alpha+1[)$

As an immediate consequence of Theorem 13, we have the following result.

PROPOSITION 15. *If α is a positive integer, then $F(S(]\alpha,\alpha+1[)) = \alpha(\alpha+1)$ and $g(S(]\alpha,\alpha+1[)) = \frac{\alpha(\alpha+3)}{2}$.*

The following result gives us explicitly how we can calculate the elements of $S(]\alpha,\alpha+1[)$.

PROPOSITION 16. *If α is a positive integer, then $S(]\alpha,\alpha+1[) = \{k\alpha+i \mid k \in \mathbb{N}\setminus\{0,1\}$ and $i \in \{1,\dots,k-1\}\} \cup \{0\}$.*

Proof. By applying the Propositions 3 and 5, we obtain

$$S(]\alpha,\alpha+1[) = \left(\bigcup_{k\in\mathbb{N}} \mathrm{A}(k\alpha,k(\alpha+1)\right) \cup \{0\} = \{k\alpha+i \mid k\in\mathbb{N}\setminus\{0,1\}$$

and $i \in \{1,\dots,k-1\}\} \cup \{0\}$. $\square$

If S is a numerical semigroup and $n \in S\setminus\{0\}$, the Apéry set of n in S (called so by [1]) is $\mathrm{Ap}(S,n) = \{s \in S \mid s-n \notin S\}$. The following result appears in [6, Lemma 2.4].

LEMMA 17. *Let S be a numerical semigroup and let n be a nonzero element of S. Then $\mathrm{Ap}(S,n) = \{0 = w(0), w(1), \dots, w(n-1)\}$, where $w(i)$ is the least element of S congruent with i modulo n for all $i \in \{0,\dots,n-1\}$.*

Notice that as a consequence of previous lemma, we have $\#\mathrm{Ap}(S,n) = n$. Observe that an integer x belongs to S if and only if there exists $(k,w) \in \mathbb{N}\times \mathrm{Ap}(S,n)$ such that $x = kn+w$. Therefore, $(\mathrm{Ap}(S,n)\setminus\{0\})\cup\{n\}$ is a finite system of generators of S.

If α is a positive integer, then $\alpha < \frac{2\alpha+1}{2} < \alpha+1$. Thus, applying Lemma 4 we have that $2\alpha+1 \in S(]\alpha,\alpha+1[)$.

THEOREM 18. *If α is a positive integer, then* $\mathrm{Ap}(S(]\alpha,\alpha+1[),2\alpha+1)=\{k\alpha+i \mid k\in\{3,\dots,\alpha+2\}$ *and* $i\in\{1,k-1\}\}\cup\{0\}$.

Proof. It is clear from Proposition 16 that

$$B=\{k\alpha+i \mid k\in\{3,\dots,\alpha+2\} \text{ and } i\in\{1,k-1\}\}\cup\{0\}$$

is a subset of $S(]\alpha,\alpha+1[)$. Moreover, $\#B=2\alpha+1$ because if k and $\overline{k}$ are integers greater or equal than 3 and $k\alpha+1=\overline{k}\alpha+\overline{k}-1$, then $(k-\overline{k})\alpha=\overline{k}-2$, so $\overline{k}-2\geqslant\alpha$ whence $k\geqslant\alpha+3$. As $B\subseteq S(]\alpha,\alpha+1[)$ and $\#B=2\alpha+1$, then applying Lemma 17, to prove $B=\mathrm{Ap}(S(]\alpha,\alpha+1[),2\alpha+1)$, it is enough to see that if $x\in B$ then $x-(2\alpha+1)\notin S(]\alpha,\alpha+1[)$. For this purpose, we distinguish two cases.

1. We will see that if $k\in\{3,\dots,\alpha+2\}$ then $(k\alpha+1)-(2\alpha+1)\notin S(]\alpha,\alpha+1[)$. But, $(k\alpha+1)-(2\alpha+1)=(k-2)\alpha$ and $(k-2)\alpha\notin S(]\alpha,\alpha+1[)$ as a consequence of Lemma 10.

2. We will see that if $k\in\{3,\dots,\alpha+2\}$ then $(k\alpha+k-1)-(2\alpha+1)\notin S(]\alpha,\alpha+1[)$. But, $(k\alpha+k-1)-(2\alpha+1)=(k-2)(\alpha+1)$ and $(k-2)(\alpha+1)\notin S(]\alpha,\alpha+1[)$ as a consequence from Lemma 10. $\square$

The following result appears in [9, Proposition 1].

PROPOSITION 19. *Let S be a numerical semigroup. Then S is a* MED*-semigroup if and only if $x+y-\mathrm{m}(S)\in S$ for all $\{x,y\}\subseteq S\setminus\{0\}$.*

By applying the Propositions 16, 19 and Theorem 18, we obtain easily the following result.

COROLLARY 20. *If α is a positive integer, then $S(]\alpha,\alpha+1[)$ is a* MED*-semigroup with multiplicity $2\alpha+1$. Moreover, $\{k\alpha+i \mid k\in\{2,\dots,\alpha+2\}$ and $i\in\{1,k-1\}\}$ is its minimal system of generators.*

We illustrate the above result with an example.

EXAMPLE 21. By applying Corollary 20, we know that $S(]4,5[)$ is a MED-semigroup and $\{2.4+1,3.4+1,3.4+2,4.4+1,4.4+3,5.4+1,5.4+4,6.4+1,6.4+5\}=\{9,13,14,17,19,21,24,25,29\}$ is its minimal system of generators.

3.2. The PM-semigroup $S(]\alpha,\alpha+1])$

Let α be a positive integer. In this subsection we will study the PM-semigroup $S(]\alpha,\alpha+1])$.

THEOREM 22. *If α is a positive integer, then $S(]\alpha,\alpha+1])=\{k\alpha+i \mid k\in\mathbb{N}\setminus\{0\}$ and $i\in\{1,\dots,k\}\}\cup\{0\}$. Moreover,* $\mathrm{Ap}(S(]\alpha,\alpha+1]),\alpha+1)=\{k\alpha+1 \mid k\in\{2,\dots,\alpha+1\}\}\cup\{0\}$.

Proof. By Lemma 4, we know that a positive integer x belongs to $S(]\alpha,\alpha+1])$ if and only if there exists $k\in\mathbb{N}\setminus\{0\}$ such that $k\alpha < x \leqslant k(\alpha+1)$. Therefore, $x = k\alpha+i$ for some $k\in\mathbb{N}\setminus\{0\}$ and some $i\in\{1,\dots,k\}$. Let $B=\{0,2\alpha+1,3\alpha+1,\dots,(\alpha+1)\alpha+1\}$. It is clear that $B\subseteq S(]\alpha,\alpha+1])$ and $\#B=\alpha+1$. So, using Lemma 17, to prove that $\mathrm{Ap}(S(]\alpha,\alpha+1]),\alpha+1)=B$ it is enough to see that if $k\in\{2,\dots,\alpha+1\}$ then $k\alpha+1-(\alpha+1)=(k-1)\alpha\notin S(]\alpha,\alpha+1])$. By applying Lemma 4 and that $\dfrac{(k-1)\alpha}{k-1}=\alpha<\alpha+1<\dfrac{(k-1)\alpha}{k-2}$, we deduce that $(k-1)\alpha\notin S(]\alpha,\alpha+1])$. □

The following result appears in [11].

LEMMA 23. *If S is a numerical semigroup and $n\in S\setminus\{0\}$, then:*

1. $F(S)=\max(\mathrm{Ap}(S,n))-n$.
2. $g(S)=\frac{1}{n}\left(\sum_{w\in \mathrm{Ap}(S,n)} w\right)-\frac{n-1}{2}$.

By applying Theorem 22 and Lemma 23, we obtain the following result.

COROLLARY 24. *If α is a positive integer, then $F(S(]\alpha,\alpha+1]))=\alpha^2$ and $g(S(]\alpha,\alpha+1]))=\dfrac{\alpha(\alpha+1)}{2}$.*

By using Proposition 19 and Theorem 22, we easily obtain the following result.

COROLLARY 25. *If α is a positive integer, then $S(]\alpha,\alpha+1])$ is a MED-semigroup with multiplicity $\alpha+1$. Further, $\{\alpha+1,2\alpha+1,\dots,(\alpha+1)\alpha+1\}$ is its minimal system of generators.*

EXAMPLE 26. By Corollary 25, we know that $S(]4,5])$ is a MED-semigroup with multiplicity 5 and $\{5,2.4+1,3.4+1,4.4+1,5.4+1\}=\{5,9,13,17,21\}$ is its minimal system of generators. Further, by Corollary 24, we know that $F(S(]4,5]))=4^2=16$ and $g(S(]4,5]))=\frac{4.5}{2}=10$.

3.3. The PM-semigroup $S([\alpha,\alpha+1[)$

Let α be a positive integer. In this subsection, we will study the PM-semigroup $S([\alpha,\alpha+1[)$.

THEOREM 27. *If α is a positive integer, then $S([\alpha,\alpha+1[)=\{k\alpha+i \mid k\in\mathbb{N}\setminus\{0\}$ and $i\in\{0,\dots,k-1\}\}\cup\{0\}$. Moreover, $\mathrm{Ap}(S([\alpha,\alpha+1[),\alpha)=\{k\alpha+k-1\mid k\in\{2,\dots,\alpha\}\}\cup\{0\}$.*

Proof. By Lemma 4, we know that a positive integer x belongs to $S([\alpha,\alpha+1[)$ if and only if there exists $k\in\mathbb{N}\setminus\{0\}$ such that $k\alpha\leqslant x<k(\alpha+1)$. Therefore, $x=k\alpha+i$ with $k\in\mathbb{N}\setminus\{0\}$ and $i\in\{0,\dots,k-1\}$. Let $B=\{k\alpha+k-1\mid k\in\{2,\dots,\alpha\}\}\cup\{0\}$. It is clear that $B\subseteq S([\alpha,\alpha+1[)$ and $\#B=\alpha$. Hence, using Lemma 17, to prove that

$B = \mathrm{Ap}(S([\alpha,\alpha+1[),\alpha)$ it will be enough to see that if $k \in \{2,\ldots,\alpha\}$ then $k\alpha+k-1-\alpha = (k-1)(\alpha+1) \notin (S([\alpha,\alpha+1[)$. But this is easily deduced from Lemma 4 and the fact that $\dfrac{(k-1)(\alpha+1)}{k} < \alpha < \alpha+1 = \dfrac{(k-1)(\alpha+1)}{k-1}$. $\square$

By applying Lemma 23 and Theorem 27, we obtain the following result.

COROLLARY 28. *If α is a positive integer, then $F(S([\alpha,\alpha+1[)) = \alpha^2-1$ and $g(S([\alpha,\alpha+1[)) = \dfrac{(\alpha+2)(\alpha-1)}{2}$.*

By using Proposition 19 and Theorem 27, we easily obtain the following result.

COROLLARY 29. *If α is a positive integer, then $S([\alpha,\alpha+1[)$ is a* MED*-semigroup with multiplicity α. Moreover, $\{k\alpha+k-1 \mid k \in \{1,\ldots,\alpha\}\}$ is its minimal system of generators.*

EXAMPLE 30. By Corollary 29, we know that $S([4,5[)$ is a MED-semigroup and $\{1.4+0, 2.4+1, 3.4+2, 4.4+3\} = \{4,9,14,19\}$ is its minimal system of generators. Besides, from Corollary 28, we know that $F(S([4,5[)) = 4^2-1 = 15$ and $g(S([4,5[)) = \frac{6.3}{2} = 9$.

3.4. The PM-semigroup $S(]\alpha,\alpha+2[)$

Let α be a positive integer. Our aim in this subsection is to study the numerical semigroup $S(]\alpha,\alpha+2[)$. As an immediate consequence from Lemma 4, we have $S(]\alpha,\alpha+2[) = S(]\alpha,\alpha+1]) \cup S([\alpha+1,\alpha+2[)$. Then, applying Lemma 17, we deduce that if $\mathrm{Ap}(S(]\alpha,\alpha+1]),\alpha+1) = \{0 = w(0), w(1),\ldots,w(\alpha)\}$ and $\mathrm{Ap}(S([\alpha+1,\alpha+2[),\alpha+1) = \{0 = w'(0), w'(1),\ldots,w'(\alpha)\}$, then $\mathrm{Ap}(S(]\alpha,\alpha+2[),\alpha+1) = \{0 = \overline{w}(0), \overline{w}(1),\ldots,\overline{w}(\alpha)\}$ where $\overline{w}(i) = \min\{w(i), w'(i)\}$ for all $i \in \{1,\ldots,\alpha\}$. By Theorems 22 and 27, we know that $w(i) = (\alpha+2-i)\alpha+1 = (\alpha+1-i)(\alpha+1)+i$ and $w'(i) = (i+1)(\alpha+1)+i$ for all $i \in \{1,\ldots,\alpha\}$. Hence, $\overline{w}(i) = \min\{(i+1)(\alpha+1)+i, (\alpha+1-i)(\alpha+1)+1\} = \begin{cases} (i+1)(\alpha+1)+i & \text{if } i \leqslant \frac{\alpha}{2}, \\ (\alpha+1-i)(\alpha+1)+i & \text{otherwise.} \end{cases}$

Now we can enunciate the following result.

THEOREM 31. *If α is a positive integer, then* $\mathrm{Ap}(S(]\alpha,\alpha+2[),\alpha+1) = \{0, 2(\alpha+1)+1, 3(\alpha+1)+2, \ldots, (\lfloor\frac{\alpha}{2}\rfloor+1)(\alpha+1)+\lfloor\frac{\alpha}{2}\rfloor, 1.(\alpha+1)+\alpha, 2(\alpha+1)+\alpha-1, \ldots, (\alpha+1-\lfloor\frac{\alpha}{2}\rfloor-1)(\alpha+1)+\lfloor\frac{\alpha}{2}\rfloor+1\}$.

The following result is obtained as an immediate consequence of Theorem 13.

COROLLARY 32. *If α is a positive integer, then*

$$F(S(]\alpha,\alpha+2[)) = \begin{cases} \dfrac{\alpha(\alpha+2)}{2} & \textit{if } \alpha \textit{ is even,} \\[2ex] \dfrac{\alpha(\alpha+1)}{2} & \textit{if } \alpha \textit{ is odd} \end{cases}$$

and

$$g(S(]\alpha,\alpha+2[)) = \begin{cases} \dfrac{\alpha^2}{4}+\alpha & \text{if } \alpha \text{ is even,} \\ \dfrac{\alpha^2-1}{4}+\alpha & \text{if } \alpha \text{ is odd.} \end{cases}$$

PROPOSITION 33. *If α is a positive integer, then $S(]\alpha,\alpha+2[)$ is a* MED*-semigroup with multiplicity $\alpha+1$. Moreover, $\{\alpha+1, 2(\alpha+1)+1, \dots, (\lfloor\frac{\alpha}{2}\rfloor+1)(\alpha+1)+\lfloor\frac{\alpha}{2}\rfloor, 1(\alpha+1)+\alpha, 2(\alpha+1)+\alpha-1, \dots, (\alpha-\lfloor\frac{\alpha}{2}\rfloor)(\alpha+1)+\lfloor\frac{\alpha}{2}\rfloor+1\}$ is its minimal system of generators.*

Proof. We know that $S(]\alpha,\alpha+2[) = S(]\alpha,\alpha+1]) \cup S([\alpha+1,\alpha+2[)$. Besides, from Corollaries 25 and 29, we know that $S(]\alpha,\alpha+1])$ and $S([\alpha+1,\alpha+2[)$ are MED-semigroups with multiplicity $\alpha+1$. To prove that $S(]\alpha,\alpha+2[)$ is a MED-semigroup, using Proposition 19, it will be enough to see that if $x \in S(]\alpha,\alpha+1]) \setminus S([\alpha+1,\alpha+2[)$ and $y \in S([\alpha+1,\alpha+2[) \setminus S(]\alpha,\alpha+1])$, then $x+y-(\alpha+1) \in S(]\alpha+1,\alpha+2[)$. To this effect, if we use the result at the beginning of this subsection, it will be enough to prove that if $i \in \{1,\dots,\lfloor\frac{\alpha}{2}\rfloor\}$ and $j \in \{\lfloor\frac{\alpha}{2}\rfloor+1,\dots,\alpha\}$ then $((i+1)(\alpha+1)+i)+((\alpha+1-j)(\alpha+1)+j)-(\alpha+1) = (\alpha+1+i-j)(\alpha+1)+i+j \in S(]\alpha,\alpha+2[)$. For this, we distinguish two cases.

1. If $i+j \leqslant \alpha+1$ then $(\alpha+1+i-j)(\alpha+1)+i+j = (\alpha+1-i-j)(\alpha+1)+i+j+2i(\alpha+1) \in S(]\alpha,\alpha+1]) \subseteq S(]\alpha,\alpha+2[)$.

2. If $i+j > \alpha+1$ then $(\alpha+1+i-j)(\alpha+1)+i+j = (\alpha+1+i-j+1)(\alpha+1)+(i+j-(\alpha+1)) = (i+j+1-(\alpha+1))(\alpha+1)+(i+j-(\alpha+1))+(2\alpha+2-2j)(\alpha+1) \in S([\alpha+1,\alpha+2[) \subseteq S(]\alpha,\alpha+2[)$. □

EXAMPLE 34. From Proposition 33, we know that $S(]5,7[)$ is a MED-semigroup and $\{6, 2.6+1, 3.6+2, 1.6+5, 2.6+4, 3.6+3\} = \{6,13,20,11,16,21\}$ is its minimal system of generators. Further, by Corollary 32, we know that $F(S(]5,7[)) = \frac{5.6}{2} = 15$ and $g(S(]5,7[)) = \frac{5^2-1}{4}+5 = 11$.

4. The general case

Let α and β be positive integers such that $\beta-\alpha \geqslant 3$. In this section we are interested in studying the PM-semigroup $S(]\alpha,\beta[)$.

THEOREM 35. *Let α and β be positive integers such that $\beta-\alpha \geqslant 3$. Then $\{\alpha+1, 2\alpha+1, 3\alpha+1, \dots, (\alpha+1)\alpha+1\} \cup \{\alpha+1, \alpha+2, \dots, \beta-1\} \cup \{\beta-1, 2(\beta-1)+1, 3(\beta-1)+2, \dots, (\beta-1)(\beta-1)+(\beta-2)\}$ is a system of generators of $S(]\alpha,\beta[)$.*

Proof. By Corollary 25, we know that $\{\alpha+1, 2\alpha+1, \dots, (\alpha+1)\alpha+1\}$ is a system of generators of $S(]\alpha,\alpha+1])$. By Corollary 8, $\{\alpha+1, \alpha+2, \dots, \beta-1\}$ is a system of generators of $S([\alpha+1,\beta-1])$. By Corollary 29, $\{k(\beta-1)+k-1 \mid k \in$

$\{1,\ldots,\beta-1\}\} = \{\beta-1, 2(\beta-1)+1, 3(\beta-1)+2,\ldots,(\beta-1)(\beta-1)+\beta-2\}$ is a system of generators of $S([\beta-1,\beta[)$. The proof ends noting that $S(]\alpha,\beta[) = S(]\alpha,\alpha+1]) \cup S([\alpha+1,\beta-1]) \cup S([\beta-1,\beta[)$. $\square$

COROLLARY 36. *Let α and β positive integers such that $\beta-\alpha \geqslant 3$ and $\beta \leqslant 2\alpha$. Then $S(]\alpha,\beta[)$ is a numerical semigroup with multiplicity $\alpha+1$. Moreover, $S(]\alpha,\beta[)$ is not a* MED*-semigroup.*

Proof. By Theorem 35, we easily deduce that $\alpha+1$ is the multiplicity of $S(]\alpha,\beta[)$. Moreover, using Proposition 19, we have that $S(]\alpha,\beta[)$ is not a MED-semigroup because $\{\alpha+2,\beta-1\} \subseteq S(]\alpha,\beta[)\setminus\{0\}$ and $\alpha+2+\beta-1-(\alpha+1) = \beta \notin S(]\alpha,\beta[)$. Let us observe that $\beta \notin S(]\alpha,\beta[)$ as a consequence of Lemma 4 and the fact that $\frac{\beta}{2} \leqslant \alpha < \beta = \frac{\beta}{1}$. $\square$

EXAMPLE 37. By Theorem 35, we know that a system of generators of $S(]4,7[)$ is $\{4+1, 2.4+1, 3.4+1, 4.4+1, 5.4+1\} \cup \{5,6\} \cup \{6, 2.6+1, 3.6+2, 4.6+3, 5.6+4, 6.6+5\} = \{5,9,13,17,21\} \cup \{5,6\} \cup \{6,13,20,27,34,41\} = \{5,6,9,13,17,20,21,27,34,41\}$.

Therefore, $\{5,6,9,13\}$ is the minimal system of generators of $S(]4,7[)$. Note that $S(]4,7[)$ has multiplicity 5, embedding dimension 4 and so it is not a MED-semigroup.

Now, we are interested in calculating some positive integers a,b and c such that $S(]\alpha,\beta[) = \{x \in \mathbb{N} \mid ax \bmod b \leqslant cx\}$. The following result is a consequence of [10, Lemma 1].

LEMMA 38. *If a_1, b_1, a_2 and b_2 are positive integers and $\frac{a_1}{b_1} < \frac{a_2}{b_2}$, then* $S\left(\left[\frac{a_1}{b_1},\frac{a_2}{b_2}\right]\right) = \{x \in \mathbb{N} \mid a_2b_1x \bmod (a_1a_2) \leqslant (a_2b_1-a_1b_2)x\}$.

LEMMA 39. *Let α, β be positive integers such that $\beta-\alpha \geqslant 3$. Then*

$$S(]\alpha,\beta[) = S\left(\left[\frac{(\alpha+1)\alpha+1}{\alpha+1}, \frac{(\beta-1)(\beta-1)+\beta-2}{\beta-1}\right]\right).$$

Proof. It is clear that

$$\frac{(\alpha+1)\alpha+1}{\alpha+1} < \frac{\alpha.\alpha+1}{\alpha} < \ldots < \frac{\alpha+1}{1} < \frac{\alpha+2}{1} < \ldots$$

$$< \beta-1 < \frac{2(\beta-1)+1}{2} < \frac{3(\beta-1)+2}{3} < \ldots < \frac{(\beta-1)(\beta-1)+\beta-2}{\beta-1},$$

is a Bézout sequence. Hence, applying Proposition 7, we have that $\{\alpha+1, 2\alpha+1,\ldots,(\alpha+1)\alpha+1\} \cup \{\alpha+1,\alpha+2,\ldots,\beta-1\} \cup \{\beta-1, 2(\beta-1)+1,\ldots,(\beta-1)(\beta-$

$1)+\beta-2\}$ is a system of generators of $S\left(\left[\frac{(\alpha+1)\alpha+1}{\alpha+1},\frac{(\beta-1)(\beta-1)+\beta-2}{\beta-1}\right]\right)$. From Theorem 35, we obtain that

$$S(]\alpha,\beta[)=S\left(\left[\frac{(\alpha+1)\alpha+1}{\alpha+1},\frac{(\beta-1)(\beta-1)+\beta-2}{\beta-1}\right]\right). \qquad \square$$

As an immediate consequence of Lemmas 38 and 39, we have the following result.

THEOREM 40. *Let α, β be positive integers such that $\beta-\alpha\geqslant 3$. Then*

$$S(]\alpha,\beta[)=\big\{x\in\mathbb{N}\mid(\alpha+1)(\beta^2-\beta-1)x \bmod (\alpha^2+\alpha+1)(\beta^2-\beta-1) \leqslant \big((\alpha+1)(\beta^2-\beta-1)-(\beta-1)(\alpha^2+\alpha+1)\big)x\big\}.$$

EXAMPLE 41. By applying Theorem 40, we have

$$S(]4,7[)=\{x\in\mathbb{N}\mid 205x \bmod 861\}\leqslant 79x\}.$$

REMARK 42. 1. By applying Corollaries 20 and 8, and that

$$\frac{(\alpha+2)\alpha+1}{\alpha+2}<\frac{(\alpha+1)\alpha+1}{\alpha+1}<\cdots<\frac{2\alpha+1}{2}<\frac{3\alpha+2}{3}<\cdots<\frac{(\alpha+2)\alpha+\alpha+1}{\alpha+2},$$

is a Bézout sequence, we obtain

$$S(]\alpha,\alpha+1[)=S\left(\left[\frac{(\alpha+2)\alpha+1}{\alpha+2},\frac{(\alpha+2)\alpha+\alpha+1}{\alpha+2}\right]\right).$$

2. By applying Proposition 33, Corollary 8 and that

$$\frac{(\alpha-\lfloor\frac{\alpha}{2}\rfloor)(\alpha+1)+\lfloor\frac{\alpha}{2}\rfloor+1}{\alpha-\lfloor\frac{\alpha}{2}\rfloor+1}<\cdots<\frac{2(\alpha+1)+(\alpha-1)}{3}<\frac{(\alpha+1)+\alpha}{2}<\frac{\alpha+1}{1}<\frac{2(\alpha+1)+1}{2}<\cdots<\frac{(\lfloor\frac{\alpha}{2}\rfloor+1)(\alpha+1)+\lfloor\frac{\alpha}{2}\rfloor}{\lfloor\frac{\alpha}{2}\rfloor+1},$$

is a Bézout sequence, we obtain

$$S(]\alpha,\alpha+2[)=S\left(\left[\frac{(\alpha-\lfloor\frac{\alpha}{2}\rfloor)(\alpha+1)+\lfloor\frac{\alpha}{2}\rfloor+1}{\alpha-\lfloor\frac{\alpha}{2}\rfloor+1},\frac{(\lfloor\frac{\alpha}{2}\rfloor+1)(\alpha+1)+\lfloor\frac{\alpha}{2}\rfloor}{\lfloor\frac{\alpha}{2}\rfloor+1}\right]\right).$$

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