RIEMANNIAN APPARENT HORIZON

JOSE TORRES SANTAELLA

Abstract. These notes introduce Riemannian apparent horizons based on the book[Lee19]. The Riemannian viewpoint might be extended in future works.

1. Introduction

Apparent horizon can be seen as the intersection of General Relativity, Minimal Surface and Constant Mean Curvature theory. When one has a trapped surface in a asymptotically flat space, it follows from theorems of Penrose and Hawking, that the resulting space-time will be singular, and so such surface is referred as a black hole initial data. In these notes we restrict to the riemannian cases. More precisely, a riemannian asymptotically flat manifold (M^3, g) is a 3-manifold diffeomorphic to $\mathbb{R}^3 \setminus K$ where $K \subset \mathbb{R}^3$ is compact and the metric g satisfies

$$|g_{ij} - \delta_{ij}| \leq \frac{C}{|x|}, |g_{ij,k}| \leq \frac{C}{|x|^2},$$

as $|x| \to \infty$. The derivatives are taken with respect the euclidean metric in $\mathbb{R}^3 \setminus K$. In addition, it is requiere that the Ricci curvature of M satisfies

$$\mathrm{Ric} \geq -C\frac{g}{|x|^2}.$$

Roughly speaking a asymptotically flat manifold behaves like the euclidean flat space from far away the black hole.
One objectives of these note is to show that when a asymptotically flat manifolds contains a trapped surface, there is alway bes an outermost trapped surface, which is often called an apparent horizon, and can be seen as a stable minimal hypersurface.

On of the main reason to study apparent horizon, comes from the Penrose conjecture which state

Conjecture 1.1 ((Riemannian) Penrose inequality)**.** *Let (M^{n+1}, g) be a complete asymptotically flat manifold with non-negative scalar curvature, and let Σ be an apparent horizon with respect to some end M_k.*

Then

$$m_{ADM}(M_k, g) \geq \frac{1}{2}\left(\frac{|\Sigma|}{\omega_n}\right)^{\frac{n-1}{n}}$$

Moreover, if equality holds, then the part of M outside Σ is isometric to half of the Schwarzschild space of mass $m_{ADM}(M_k, g)$.

The ADM mass of a asymptotically flat manifold (M^{n+1}, g) is defined by

$$m_{\mathrm{ADM}}(M_k, g) = \lim_{\rho\to\infty} \frac{1}{2n\omega_n} \int_{S_\rho} (\mathrm{div}(g) - d\mathrm{tr}(g))(\nu) d\mu_{S_\rho},$$

where ω_n denote the euclidean volumen of the n-ball, the divergence, trace, volume form $d\mu_{S_\rho}$ are taking with respect the euclidean metric δ, and S_ρ denote a geodesic sphere of radius ρ.

Here, we follow a purely Riemannian viewpoint even if these notions come from a Lorentzian setting. For the reader interested in relativistic interpretations from a mathematical viewpoint, we recommend [MS14]. It is worth pointing out that the recently introduced notion of wind Riemannian structure [CJS14] allows one to give a spacelike description beyond the horizon [[AGM15], Sect. 8], [JS16], which might be taken into account in further works.

The structure of the notes is summarized as follows: In Section 2 we study some fundamental aspect from the Mean Curvature equation. In Section 3 we study some definition of set of finite perimeter in $\mathbb{R}^{n+1}$. In Section 4 we study riemannian apparent horizon.

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2. Differential Geometry Background

In this section we study a fundamental aspect from the Mean Curvature equation will be important in the rest of these notes.

First we start by showing a Tangential Principle which state that if two hypersurface meet in one point, and the Mean Curvature of each hypersurface satisfy an order relation, then each hypersurface must coincide.

Lemma 2.1. *Let $B \subset \mathbb{R}^n$ an open ball and consider $(\overline{B} \times \mathbb{R}, g)$, where g is a riemannian metric. In addition, let $u_i : \overline{B} \to \mathbb{R}$ and consider the*

graph $\Sigma_i = \left\{(x, u_i(x)) \in \mathbb{R}^{n+1} : x \in \overline{B}\right\}$.
Further assume that

(1) *The Mean Curvature of each* Σ_i *satisfies* $H_2 \leq 0 \leq H_1$.
(2) $u_2 \leq u_1$ *in* $\overline{B}$.

Then, if there exist a point $p \in \Sigma_1 \cap \Sigma_2$ *such that is an interior point or is tangent at the boundary of* Σ_i, $u_1 = u_2$ *in* $\overline{B}$.

Remark 2.2. *The hypothesis of Lemma 2.1 do not require any smooth assumption other than* $\mathcal{C}^{1,1}$, *which the Mean Curvature can be weakly define in the viscosity sense. The main reason is that the Mean Curvature Equation is quasilinear and enjoy suitable properties from comparison result in the viscosity theory. We refer the reader to* [AGH98] *for further details.*

Proof. We assume the g is the euclidean metric and each Σ_i is $\mathcal{C}^2$. First we mention that the hypothesis of the Lemma 2.1 can be describe as:

- Interior case: There exist $p \in \Sigma_1 \cap \Sigma_2$ such tha,t after a rotation or translation, the tangent space is $T_p\Sigma_1 = T_p\Sigma_2 = \{x_{n+1} = 0\}$, $p = (0, u_i(0))$ where $0 \in B(0, r) \subset B$, and $u_2 \leq u_1$ at $B(0, r)$.
- Boundary case: There exist $p \in \partial\Sigma_1 \cap \partial\Sigma_2$ such that $T_p\Sigma_1 = T_p\Sigma_2 = \{x_{n+1=0}\}$ and $T_p\partial\Sigma_1 = T_p\partial\Sigma_2$. In addition, form the interior case, we only need to change $B(0, r)$ by $B(0, r) \cap \{x_{n+1} \geq 0\}$.

Recall that the Mean Curvature of Σ_i in the local frame $(x, u_i(x))$ can be written by

$$H_i = \left(-\delta_{ab} + \frac{\partial_a u_i \partial_b u_i}{\sqrt{1 + |\nabla u_i|^2}}\right) \frac{\partial^2_{ab} u_i}{1 + |\nabla u_i|^2} = -\frac{\Delta u_i}{1 + |\nabla u_i|^2} + \text{ first order terms.}$$

Therefore, by taking $v = u_2 - u_1$, it follows that $v \leq 0$ in $B(0, r)$ an reach its maximum at 0. Moreover, v satisfies the inequality

$$0 \leq H_2 - H_1 = \underbrace{\Delta v}_{\leq 0 \text{ at } p} + \underbrace{\text{f.o.t.}}_{=0 \text{ at } p}.$$

In consequence, by the interior Hopf Maximum Principle, $v = v(0) = 0$ in $B(0, r)$ and, by connectedness, in all B.
For the boundary case, we would obtain that $\frac{\partial v}{\partial N}(p) > 0$, where N is unit normal of Σ_i at p. But, the boundary Hopf Maximum Principle, implies $\nabla v(p) = 0$, a contradiction. □

A direct consequence of Lemma 2.1 is

Corollary 2.3 (Tangential Principle for Mean Curvature)**.** *Suppose we have open sets* $\Omega_1 \subset \Omega_2$ *in a Riemannian manifold* (M, g) *and* *smooth hypersurfaces Σ_1 and Σ_2 (possibly with boundary) lie on $\partial\Omega_1$ and $\partial Omega_2$, re-spectively, with $H_{\Sigma_1} \leq 0 \leq H_{\Sigma_2}$, where these are computed with respect to the outward-pointing unit normal. If Σ_1 touches Σ_2 anywhere in their interiors, or if they are tangent to each other at a common boundary point, then they must be identically equal in a neighborhood of that point.*

Proof. Write locally Σ_i as a graph around p and apply Lemma 2.1. □

Definition 2.4. *Let M be a smooth manifold with an atlas given by the charts*

$$\{\varphi_\beta : B \times (0,1) \to U_\alpha\},$$

where $U_\alpha \subset M$ is an open set, $B = B(0,1) \subset \mathbb{R}^n$ and φ is smooth. We say that M is foliated by the atlas if for each α, there exist $C_\alpha \subset (0,1)$ such that $\varphi^{-1}(U_\alpha) = B \times C_\alpha$.
Moreover, the sets $\varphi_\alpha(B \times t)$ are called the leafs of the foliation.
Finally, we say the a maniolfd is foliated by hypersurface with non-positive Mean Curvature if each leaf of the foliation have non-positive Mean Curvature.

Example 2.5. *An easy example of a surface being foliated by circles is the a Torus of revolution. Roghly speaking, each straight line in the domain of the parametrization correspond to a circle in the Torus.*

Another important consequence of the Tangential Principle is that a closed minimal hypersurface cannot touch in an interior point of a foliation of surfaces with $H \leq 0$.

Corollary 2.6. *Let $n < 8$, and let (M^n, g) be a compact riemannian manifold with boundary such that the boundary ∂M has non-negative mean curvature with respect to the outward pointing normal. For each nonzero homology class $\alpha \in H^{n-1}(M, \mathbb{Z})$, there exists an integral sum of smooth oriented minimal hypersurfaces $\Sigma \in \alpha$ that minimizes volume among all smooth cycles in α, and each of these minimal hypersurfaces must either be disjoint from ∂M, or else be equal to a component of ∂M (which of course is only possible if that component is minimal).*

Sketch of the proof. We are going to omit the part of the existence of an area minimizer in the homology class α. We refer the reader to [Lee19] Thm 2.22 for proof.
The other part is about to foliate M suitably. First we consider the cylinder $\partial M \times [0,1]$, note that each leaf posses non-positive Mean Curvature. Then, by identifying ∂M with $\partial M \times \{0\}$, we can find minimal hypersurface $\Sigma \in \alpha$ such that cannot touch $\partial M \times \{t\}$ unless coincide with in. □

3. Finite Perimeter sets

In this section we introduce some definition and properties of finite perimeter or Caccioppoli sets in $\mathbb{R}^{n+1}$. This type of sets are important to understand what it is aa apparent horizon.

Definition 3.1. *Let $U \subset \mathbb{R}^{n+1}$ be an open set and consider $f \in L^1(U)$. We say that f posses bounded variation, $f \in BV(U)$, if*

$$\int_U f\, div(X) dx < \infty,\ \forall\, X \in \mathcal{C}_c^{1,1}(U, \mathbb{R}^{n+1})\ s.t.\ \ |X| \leq 1.$$

In addition, we say that f posses locally bounded variation, $f \in BV_{loc}(U)$, if for all open set $V \subset\subset U$, $f \in BV(V)$.

Remark 3.2. *Recall that if $f \in \mathcal{C}^1(U)$, then the Divergence theorem implies*

$$\int_U f\, div(X) dx = \int_U div(fX) - \langle \nabla f, X \rangle\, dx = -\int_U \langle \nabla f, X \rangle\, dx.$$

Therefore, this relation permits to define weak derivatives of functions in a different manner as the Sobolev space.

The next theorem is the fundamental tool in the theory of functions of bounded variation.

Theorem 3.3. *Let $f \in BV_{loc}(U)$. Then there exist a radon measure μ and a μ-medible vector field $\sigma : U \to \mathbb{R}^{n+1}$ such that*

1. *$|\sigma(x)| = 1$ μ-a.e.*
2. *For all $X \in \mathcal{C}_c^1(U, \mathbb{R}^{n+1})$, $\int_U f\, div(X) dx = -\int_U \langle X, \sigma \rangle\, dx$.*

Proof. For a proof we refer [EG18]. □

The next definition and notation correspond when a characteristic function belongs to $BV(U)$

Definition 3.4. *Let E be a Lebesgue measurable set in $\mathbb{R}^{n+1}$. We say that E poses (locally) finite perimeter if $\mathbb{1}_E \in BV(U)$ ($BV_{loc}(U)$). In addition, we denote the measure and the vector field from Theorem 3.3 by*

$$\mu(\cdot) = ||\partial E||\,(\cdot)\ and\ \sigma = \nu_E.$$

Example 3.5. *If E is an open bounded set of class $\mathcal{C}^1$, $||\partial E||\,(\cdot) = \mathcal{H}^n(\partial E \cap \cdot)$ and ν_E coincide $\mathcal{H}^n(\partial E \cap \cdot)$-a.e. with the unit normal vector of ∂E. Here $\mathcal{H}^n(\cdot)$ denote the n-dimensional Hausdorff measure.*

We end this section with the definition and properties of the reduced boundary of a set, which namely is the restriction of the above properties to the boundary of a set.

Definition 3.6. *The reduced boundary of a set of finite perimeter E, $x \in \partial^* E$, if*

(1) For all $r > 0$, $||\partial E|| \left(B(x, r)\right) > 0$.
(2) $\lim\limits_{r \to 0} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} \nu_E \, d\,||\partial E|| = \nu_E(x)$.
(3) $|\nu_E|(x) = 1$.

Example 3.7. *Let E be the unit square in the first quadrant glued with the line $[-1, 0] \times \{0\}$. It is not hard to see that the edges of the square and the line $[-1, 0] \times \{0\}$ would not satisfy property* (2)*. Therefore,*

$$\partial^* E = (0, 1) \times (0, 1),$$

is contained in $\partial E = [-1, 1] \times \{0\} \cup \{1\} \times [0, 1] \cup \{0\} \times [0, 1] \cup [0, 1] \times \{1\}$.

The next Theorem is the structural result about set of finite perimeter in $\mathbb{R}^{n+1}$.

Theorem 3.8. *Let E be a finite perimeter set in $\mathbb{R}^{n+1}$. Then there exist a collection of compact subset K_k of a $\mathcal{C}^1$ hypersurface S_k, a $||\partial E||$-negible set N such that*

$$\partial^* E = \bigcup_{k \geq 1} K_k \cup N.$$

In addition, $\nu_E\big|_{S_k} \perp S_k$ and $||\partial E|| = \mathcal{H}^n|_{\partial^* E}$.

Remark 3.9. *The sets of finite perimeter do not depend on the rieamannian structure of the manifold. See* [AGM15] *for further details in more general context.*

4. Riemannian Apparent Horizon

In this section we introduce the concept of rieamannian apparent an their properties.

Definition 4.1. *Let (M, g) a riemannian manifold with a non-compact distinguish end. From now on, each open set $\Omega \subset M$ will satisfies*

- *$\partial\Omega$ es a compact set.*
- *Ω posses finite perimeter.*
- *If there are more ends than the distinguish one, Ω contain them all.*

In addition, we will refer as enclosed regions to these open sets, their boundaries enclosing boundaries, or enclosing hypersurfaces if they happen to be smooth.

Definition 4.2 (Apparent Horizon)**.** *We say that an enclosed region Ω is a minimizing hull or that $\Sigma = \partial^*\Omega$ is outward-minimizing, if Ω has perimeter less than or equal to every other enclosed region containing it. If the perimeter is always strictly less, then we say that Ω is a strictly minimizing hull or that Σ is strictly outward- minimizing.*
Moreover, if $\Sigma = \partial\Omega$ is a smooth minimal enclosing hypersurface, then we say that Σ is an outermost minimal hypersurface if there are no other minimal hypersurfaces enclosing Σ (in the sense of being the boundary of an enclosed region containing Ω).
We will often refer to an outermost minimal hypersurface as an apparent horizon for the distinguished end.

Example 4.3. *Let consider the Schwarchild space $(\mathbb{R}^{n+1}, g_m)$ where $g_m = \left(1 - \frac{2m}{r^{n-1}}\right)^{-1} dr^2 + r d\theta^2$ is the rotationally symmetric metric with mass m.*
It is not hard to see that, by a conformal change, this space can be written by $((0,\infty) \times \mathbb{S}^n, g_m)$ where $g_m = u(\rho)^{\frac{4}{n-1}}(d\rho^2 + \rho^2 d\theta^2)$ and $u(\rho) = 1 + \frac{m}{2\rho^{n-1}}$.
Furthermore, the level set $\left\{\rho = \left(\frac{m}{2}\right)^{\frac{1}{n-1}}\right\}$ is totally geodesic, and in particular a minimal hypersurface. Therefore, by choosing $\Omega = \left\{\rho > \left(\frac{m}{2}\right)^{\frac{1}{n-1}}\right\}$, $\left\{\rho = \left(\frac{m}{2}\right)^{\frac{1}{n-1}}\right\}$ is an apparent horizon for the end $\rho \to \infty$.
Recall that to obtain this result we are using Corollary 2.3, since this space is foliated by non-negative Mean Curvature hypersurface, and therefore the unique minimal one is $\left\{\rho = \left(\frac{m}{2}\right)^{\frac{1}{n-1}}\right\}$.

Theorem 4.4 (Existence and regularity of strictly minimizing hulls)**.** *Let (M^n, g) be a complete Riemannian manifold (possibly with boundary) and a distinguished end. For each enclosed region $\Omega \subset M$, define Ω' to be the intersection of all strictly minimizing hulls that contain Ω. Then Ω' is itself a strictly minimizing hull.*
Now assume $n < 8$. If $\partial\Omega$ is $\mathcal{C}^2$, then $\partial\Omega'$ is $\mathcal{C}^{1,1}$ everywhere and is a smooth minimal hypersurface away from $\partial\Omega$.

Proof. For a proof see Theorem 1.3 in [HI01]. □

Theorem 4.5 (Existence and uniqueness of apparent horizons)**.** *Let $n < 8$, and let (M^n, g) be a complete asymptotically flat manifold (possibly with boundary).*

(1) *If M has nonempty boundary with nonpositive Mean Curvature (with respect to the outward normal pointing into M) and only one end, then there exists a smooth apparent horizon.*
(2) *If an end of M has an apparent horizon, then it is unique, and moreover both the horizon and the region outside the horizon are orientable.*
(3) *The apparent horizon encloses all enclosing minimal hypersurfaces.*
(4) *The apparent horizon is outward-minimizing.*

Remark 4.6. *The dimension assumption comes from the minimal surface theory. Actually, it can be show the existence of apparent horizon in any dimension admitting singular behavior.*

Sketch of the proof. First, we construct a single enclosing minimal hypersurface homologous to a large coordinate sphere.
Indeed, by the asymptotic flatness, the mean curvature of the coordinate sphere of radius ρ is approximately $\dfrac{n}{\rho}$, and thus the end is foliated by hypersurfaces with positive mean curvature.

Now we consider the region Ω_ρ enclosed by one of these large coordinate spheres S_ρ. Then, we minimize the area in the homology class of S_ρ in Ω_ρ. Consequently, by Corollary 2.6, we obtain a smooth minimal hypersurface Σ enclosing ∂M. Note that must be enclosing and have multiplicity 1 because it is homologous to S_ρ.
Secondly, we start a new minimization process in

$$\mathcal{F} = \{\Omega \subset M : \Omega \text{ is a enclosing region}, \partial\Omega \text{ is minimal in the homology class } [S_\rho]\},$$

note that by the previous part $\mathcal{F} \neq \emptyset$.
Moreover, since the end of M beyond S_ρ is foliated by positive mean curvature, Corollary 2.3 implies that every element of $\mathcal{F}$ lies in Ω.
For any $\Omega_1, \Omega_2 \in \mathcal{F}$, we claim that there exists $\Omega \in \mathcal{F}$ containing both Ω_i. Indeed, if $\partial(\Omega_1 \cup \Omega_2)$ is smooth, then it must be minimal, and thus $\Omega_1 \cup \Omega_2 \in \mathcal{F}$.
Therefore, we consider the case where $\partial(\Omega_1 \cup \Omega_2)$ contains a singular set, which must occur where $\partial\Omega_1$ touches $\partial\Omega_2$. The intuition here is that $\partial(\Omega_1 \cup \Omega_2)$ should have nonpositive mean curvature in a weak sense, and indeed a result from [KH97] implies that it can be smoothed out in such a way that it has nonpositive mean curvature and encloses $\partial(\Omega_1 \cup \Omega_2)$. Now we apply Corollary 2.6 to produce a new element of $\mathcal{F}$ that encloses $\Omega_1 \cup \Omega_2$.
Consequently, by the previous argument, $\bigcup_{\Omega \in \mathcal{F}} \Omega$ can be exhausted by a single increasing sequence Ω_i such that each $\Omega_i \subset \Omega$ and $|\partial\Omega_1| \leq |S_\rho|$.

Finally, this sequence of stable minimal hypersurfaces $\partial\Omega_i$ with bounded area must converge by estimates of [SS81]. The limit gives us an enclosing minimal hypersurface Σ_∞ homologous to S_ρ. By construction, Σ_∞ must enclose all elements of $\mathcal{F}$, which implies that it must be outermost and also property (3), which immediately implies the uniqueness in property (2). The orientability follows from the fact that Σ_∞ is homologous to S_ρ. Finally, if property (4) did not hold, then we could use Corollary 2.3 to construct a new element of $\mathcal{F}$ enclosing Σ_∞, which is impossible. □

The next corollary is about constructing an apparent horizon in a manifold with multiple ends

Corollary 4.7. *Let $n < 8$, and let (M^n, g) be a complete asymptotically flat manifold whose boundary is either empty or minimal. If M has more than one end, then there is an apparent horizon corresponding to each end. The result still holds if M has a boundary, as long as that boundary has non-positive mean curvature.*

Proof. A simple consequence of applying Theorem 4.5 in one and cutting off all of the other ends at large coordinate spheres. □

We end this section by a result in dimension $n = 3$ about the topology of an apparent horizon

Theorem 4.8. *Let (M^3, g) be an asymptotically flat manifold whose boundary is either empty or minimal. Assume that M contains no immersed minimal surfaces. Then M is diffeomorphic to $\mathbb{R}^3$ minus a finite number (possibly zero) of open balls.*

Proof. We first prove that M is simply connected. Suppose that it is not. A result from[Hem04], together with the geometrization of 3-manifolds, it follows that there exists a k-fold connected covering M' of M for some $k > 1$. Then M' is an asymptotically flat manifold with at least k ends (and minimal boundary, if any).
Then by Corollary 4.7, M' contains an embedded minimal surface, and hence M contains an immersed minimal surface, contrary to our hypothesis.
Thus M is simply connected and, in particular, it is orientable. Therefore ∂M is also orientable. Consequently, we can fill the boundary components by handlebodies (where we consider the ball to be a 0-handlebody) and compactify infinity at a point to obtain a closed manifold M^*.
This M^* is still simply connected since all of the fundamental group of a handlebody comes from its boundary surface, which lies in the simply

connected space M. Or more concretely, it is not hard to see that any curve in M^* that intersects one of the handlebodies is homotopic to one that does not.
Then, by the Poincare-Perelman Theorem [MT07], it follows that M^* is diffeomorphic to $\mathbb{S}^3$. Consequently, M is just $\mathbb{R}^3$ with a certain number of handlebodies removed. All of these handlebodies must be balls, since it is clear that $\mathbb{R}^3$ minus a higher genus handlebody is not simply connected. □

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