REGULARITY OF LIPSCHITZ BOUNDARIES WITH PRESCRIBED SUB-FINSLER MEAN CURVATURE IN THE HEISENBERG GROUP $\mathbb{H}^1$

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Abstract. For a strictly convex set $K \subset \mathbb{R}^2$ of class C^2 we consider its associated sub-Finsler K-perimeter $|\partial E|_K$ in $\mathbb{H}^1$ and the prescribed mean curvature functional $|\partial E|_K - \int_E f$ associated to a function f. Given a critical set for this functional with Euclidean Lipschitz and intrinsic regular boundary, we prove that their characteristic curves are of class C^2 and that this regularity is optimal. The result holds in particular when the boundary of E is of class C^1.

1. Introduction

The aim of this paper is to study the regularity of the characteristic curves of the boundary of a set with continuous prescribed mean curvature in the first Heisenberg group $\mathbb{H}^1$ with a sub-Finsler structure. Such a structure is defined by means of an asymmetric left-invariant norm $||\cdot||_K$ in $\mathbb{H}^1$ associated to a convex set $K \subset \mathbb{R}^2$ containing 0 in its interior, see [38]. We assume in this paper that K has C^2 boundary with positive geodesic curvature.

Following De Giorgi [14], the authors of [38] defined a notion of sub-Finsler K-perimeter, see also [18]. Given a measurable set $E \subset \mathbb{H}^1$ and an open subset $\Omega \subset \mathbb{H}^1$, it is said that E has locally finite K-perimeter in Ω if for any relatively compact open set $V \subset \Omega$ we have

$$|\partial E|_K(V) = \sup \left\{ \int_E \operatorname{div}(U)\, d\mathbb{H}^1 : U \in \mathcal{H}_0^1(V), ||U||_{K,\infty} \leqslant 1 \right\} < +\infty,$$

where $\mathcal{H}_0^1(V)$ is the space of horizontal vector fields of class C^1 with compact support in V, and $||U||_{K,\infty} = \sup_{p \in V} ||U_p||_K$. Both the divergence and the integral are computed with respect to a fixed left-invariant Riemannian metric g on $\mathbb{H}^1$. When $S = \partial E \cap \Omega$ is a Euclidean Lipschitz surface the K-perimeter coincides with the area functional

$$A_K(S) = \int_S ||N_h||_{K,*}\, d\mathcal{H}^2,$$

where $\mathcal{H}^2$ is the 2-dimensional Hausdorff measure associated to the left-invariant Riemannian metric g, N is the outer unit normal to S, defined $\mathcal{H}^2$-a.e on S, N_h is

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the horizontal projection of N to the horizontal distribution in $\mathbb{H}^1$ and $||\cdot||_{K,*}$ is the dual norm of $||\cdot||_K$.

We say that a set E with Euclidean Lipschitz boundary has prescribed K-mean curvature $f \in C^0(\Omega)$ if, for any bounded open subset $V \subset \Omega$, E is a critical point of the functional

$$A_K(S \cap B) - \int_{E \cap B} f \, d\mathbb{H}^1.$$

This notion extends the classical one in Euclidean space and the one introduced in [21] for the sub-Riemannian area. We refer the reader to the introduction of [21] for a brief historical account and references.

We say that a set E has *constant* prescribed K-mean curvature if there exists $\lambda \in \mathbb{R}$ such that E has prescribed K-mean curvature λ. In Proposition 2.2 we consider a set E with Euclidean Lipschitz boundary and positive K-perimeter. We show that if E is a critical point of the K-perimeter for variations preserving the volume up to first order then E has constant prescribed K-mean curvature on any open set Ω avoiding the singular set S_0 and where $|\partial E|_K(\Omega) > 0$. This result can be applied to isoperimetric regions in $\mathbb{H}^1$ with Euclidean Lipschitz boundary.

The main result of this paper is Theorem 3.1, where we prove that the boundary S of a set E with prescribed continuous K-mean curvature is foliated by horizontal characteristic curves of class C^2 in its regular part. The minimal assumptions we require for the boundary S of E are to be Euclidean Lipschitz and $\mathbb{H}$-regular. The result holds in particular when the boundary of E is of class C^1. As we point out in Remark 3.5, C^2 regularity is optimal since the Pansu-Wulff shapes obtained in [38] have prescribed constant mean curvature and their boundaries are foliated by characteristic curves with the same regularity as that of ∂K, that may be just C^2. In the proof of the Theorem 3.1 we exploit the first variation formula of the area following the arguments developed in [20, 21] and make use of the biLipschitz homeomorphism considered in [35]. One of the main differences in our setting is that the area functional strongly depends on the inverse π_K of the Gauss map of ∂K. Therefore the first variation of the area depends on the derivative of the map that describes the boundary ∂K. In order to use the bootstrap regularity argument in [20, 21] we need to invert this map on the boundary ∂K, that is possible since the geodesic curvature of ∂K is strictly positive, see Lemma 3.2. Moreover, the C^2 regularity of the characteristic curves implies that, on characteristic curves of a boundary with prescribed continuous K-mean curvature f, the ordinary differential equation

$$\langle D_Z \, \pi_K(\nu_h), Z \rangle = f, \tag{*}$$

is satisfied. In this equation $\nu_h = N_h/|N_h|$ is the classical sub-Riemannian horizontal unit normal, Z is the unit characteristic vector field tangent to the characteristic curves and D the Levi-Civita connection associated to the left-invariant Riemannian metric g on $\mathbb{H}^1$. Equation (*) was proved to hold for C^2 surfaces in [38]. For regularity assumptions below $\mathbb{H}$-regular and Euclidean Lipschitz, equation (*) holds in a suitable weak sense, a result proved in [1] for the sub-Riemannian area, when K coincides with the unit disk centered at 0.

Moreover, in Proposition 4.2 we stress that equation (*) is equivalent to

$$H_D = \kappa(\pi_K(\nu_h)) f, \tag{**}$$

where $H_D = \langle D_Z \nu_h, Z\rangle$ is the classical sub-Riemannian mean curvature introduced in [1] and κ is the strictly positive Euclidean curvature of the boundary ∂K. A key ingredient to obtain equation (**) is Lemma 4.3, that exploits the ideas of Lemma 3.2 in an intrinsic setting.

This manuscript is a natural continuation of the many recent papers concerning sub-Riemannian area minimizers [22, 11, 8, 7, 5, 13, 2, 1, 26, 27, 28, 41, 29, 17, 4, 10, 30, 32, 24, 23, 6]. The sub-Riemannian perimeter functional is a particular case of the sub-Finsler functionals considered in this paper where the convex set is the unit disk D centered at 0. In the pioneering paper [22] N. Garofalo and D.M. Nhieu showed the existence of sets of minimal perimeter in Carnot-Carathéodory spaces satisfying the doubling property and a Poincaré inequality. In [31] Leonardi and Rigot showed the existence of isoperimetric sets in Carnot groups. However the optimal regularity of the critical points of these variational problems involving the sub-Riemannian area is not completely understood. Indeed, even in the sub-Riemannian Heisenberg group $\mathbb{H}^1$ there are several examples of non-smooth area minimizers: S. D. Pauls in [37] exhibited a solution of low regularity for the Plateau problem with smooth boundary datum; on the other hand in [8, 39, 34] the authors provided solutions of Bernstein's problem in $\mathbb{H}^1$ that are only Euclidean Lipschitz.

In [36] P. Pansu conjectured that the boundaries of isoperimetric sets in $\mathbb{H}^1$ are given by the surfaces now called Pansu's spheres, union of all sub-Riemannian geodesics of a fixed curvature joining two point in the same vertical line. This conjecture has been solved only assuming *a priori* some regularity of the minimizers of the area with constant prescribed mean curvature. In [41] the authors solved the conjecture assuming that the minimizers of the area are of class C^2, using the description of the singular set, the characterization of area-stationary surfaces, and the ruling property of constant mean curvature surfaces developed in [7]. Hence the *a priori* regularity hypothesis are central to study the sub-Riemannian isoperimetric problem. Motivated by this issue, it was shown in [9] that a C^1 boundary of a set with continuous prescribed mean curvature is foliated by C^2 characteristic curves. Regularity results for Lipschitz viscosity solutions of the minimal surface equation were obtained in [4]. Furthermore, in [21] the authors generalized the previous result when the boundary S is immersed in a three-dimensional contact sub-Riemannian manifold. Finally M. Galli in [20] improved the result in [21] only assuming that the boundary S is Euclidean Lipschitz and $\mathbb{H}$-regular in the sense of [19]. The Bernstein problem in $\mathbb{H}^1$ with Euclidean Lipschitz regularity was treated by S. Nicolussi and F. Serra-Cassano [35]. Partial solutions of the sub-Riemannian isoperimetric problem have been obtained assuming Euclidean convexity [33], or symmetry properties [12, 40, 32, 17]. An analogous sub-Finsler isoperimetric problem might be considered. Candidate solutions would be the Pansu-Wulff shapes considered in [38]. See [38, 18] for partial results in the sub-Finsler isoperimetric problem and [42] for earlier work.

We have organized this paper into several sections. In Section 2 we introduce sub-Finsler norms in the first Heisenberg group $\mathbb{H}^1$ and their associated sub-Finsler perimeter, the notion of $\mathbb{H}$-regular surfaces, intrinsic Euclidean Lipschitz graphs and the definition of sets with prescribed mean curvature. Moreover, at the end of this section we prove Proposition 2.2. Section 3 is dedicated to the proof of the main Theorem 3.1, that ensures that the characteristic curves are C^2. Finally in Section 4 we deal with the K-mean curvature equation, see Proposition 4.1 and Proposition 4.2.

2. Preliminaries

2.1. **The Heisenberg group.** We denote by $\mathbb{H}^1$ the first Heisenberg group, defined as the 3-dimensional Euclidean space $\mathbb{R}^3$ endowed with a product $*$ defined by

$$(x,y,t)*(\bar{x},\bar{y},\bar{t}) = (x+\bar{x}, y+\bar{y}, t+\bar{t}+\bar{x}y-x\bar{y}).$$

A basis of left invariant vector fields is given by

$$X = \frac{\partial}{\partial x} + y\frac{\partial}{\partial t}, \qquad Y = \frac{\partial}{\partial y} - x\frac{\partial}{\partial t}, \qquad T = \frac{\partial}{\partial t}.$$

For $p \in \mathbb{H}^1$, the left translation by p is the diffeomorphism $L_p(q) = p * q$. The horizontal distribution $\mathcal{H}$ is the planar distribution generated by X and Y, that coincides with the kernel of the (contact) one-form $\omega = dt - ydx + xdy$.

We shall consider on $\mathbb{H}^1$ the left invariant Riemannian metric $g = \langle \cdot, \cdot \rangle$, so that $\{X, Y, T\}$ is an orthonormal basis at every point, and let D be the Levi-Civita connection associated to the Riemannian metric g. The following relations can be easily computed

$$\begin{aligned} &D_X X = 0, \qquad && D_Y Y = 0, \qquad && D_T T = 0 \\ &D_X Y = -T, && D_X T = Y, && D_Y T = -X \\ &D_Y X = T, && D_T X = Y, && D_T Y = -X. \end{aligned} \tag{2.1}$$

Setting $J(U) = D_U T$ for any vector field U in $\mathbb{H}^1$ we get $J(X) = Y$, $J(Y) = -X$ and $J(T) = 0$. Therefore $-J^2$ coincides with the identity when restricted to the horizontal distribution. The Riemannian volume of a set E is, up to a constant, the Haar measure of the group and is denoted by $|E|$. The integral of a function f with respect to the Riemannian measure by $\int f \, d\mathbb{H}^1$.

2.2. **The pseudo-hermitian connection.** The pseudo-hermitian connection ∇ is the only affine connection satisfying the following properties:

1. ∇ is a metric connection,
2. $\mathrm{Tor}(U,V) = 2\langle J(U), V\rangle T$ for all vector fields U, V.

In the previous line the torsion tensor $\mathrm{Tor}(U,V)$ is given by $\nabla_U V - \nabla_V U - [U,V]$. From the above definition and the Koszul formula it follows easily that $\nabla X = \nabla Y = 0$ and $\nabla J = 0$. For a general discussion about the pseudo-hermitian connection see for instance [15, § 1.2]. Given a curve $\gamma : I \to \mathbb{H}^1$ we denote by ∇/ds the covariant derivatives induced by the pseudo-hermitian connection along γ.

2.3. **Sub-Finsler norms.** Given a convex set $K \subset \mathbb{R}^2$ with $0 \in \mathrm{int}(K)$ an associated asymmetric norm $||\cdot||$ in $\mathbb{R}^2$, we define on $\mathbb{H}^1$ a left-invariant norm $||\cdot||_K$ on the horizontal distribution by means of the equality

$$(||fX + gY||_K)(p) = ||(f(p), g(p))||,$$

for any $p \in \mathbb{H}^1$. The dual norm is denoted by $||\cdot||_{K,*}$.

If the boundary of K is of class C^ℓ, $\ell \geqslant 2$, and the geodesic curvature of ∂K is strictly positive, we say that K is of class C^ℓ_+. When K is of class C^2_+, the outer Gauss map N_K is a diffeomorphism from ∂K to $\mathbb{S}^1$ and the map

$$\pi_K(fX + gY) = N_K^{-1}\left(\frac{(f,g)}{\sqrt{f^2+g^2}}\right),$$

defined for non-vanishing horizontal vector fields $U = fX + gY$, satisfies

$$||U||_{K,*} = \langle U, \pi_K(U)\rangle.$$

See § 2.3 in [38].

2.4. **Sub-Finsler perimeter.** Here we summarize some of the results contained in subsection 2.4 in [38].

Given a convex set $K \subset \mathbb{R}^2$ with $0 \in \text{int}(K)$, the norm $||\cdot||_K$ defines a perimeter functional: given a measurable set $E \subset \mathbb{H}^1$ and an open subset $\Omega \subset \mathbb{H}^1$, we say that E has locally finite K-perimeter in Ω if for any relatively compact open set $V \subset \Omega$ we have

$$|\partial E|_K(V) = \sup\left\{ \int_E \text{div}(U)\, d\mathbb{H}^1 : U \in \mathcal{H}_0^1(V), ||U||_{K,\infty} \leqslant 1\right\} < +\infty,$$

where $\mathcal{H}_0^1(V)$ is the space of horizontal vector fields of class C^1 with compact support in V, and $||U||_{K,\infty} = \sup_{p\in V} ||U_p||_K$. The integral is computed with respect to the Riemannian measure $d\mathbb{H}^1$ of the left-invariant Riemannian metric g. When $K = D$, the closed unit disk centered at the origin of $\mathbb{R}^2$, the K-perimeter coincides with classical sub-Riemannian perimeter.

If K, K' are bounded convex bodies containing 0 in its interior, there exist constants $\alpha, \beta > 0$ such that

$$\alpha||x||_{K'} \leqslant ||x||_K \leqslant \beta||x||_{K'}, \quad \text{for all } x \in \mathbb{R}^2,$$

and it is not difficult to prove that

$$\tfrac{1}{\beta}|\partial E|_{K'}(V) \leqslant |\partial E_K|(V) \leqslant \tfrac{1}{\alpha}|\partial E|_{K'}(V).$$

As a consequence, E has locally finite K-perimeter if and only if it has locally finite K'-perimeter. In particular, any set with locally finite K-perimeter has locally finite sub-Riemannian perimeter.

Riesz Representation Theorem implies the existence of a $|\partial E|_K$ measurable vector field ν_K so that for any horizontal vector field U with compact support of class C^1 we have

$$\int_\Omega \text{div}(U)\, d\mathbb{H}^1 = \int_\Omega \langle U, \nu_K\rangle\, d|\partial E|_K.$$

In addition, ν_K satisfies $|\partial E|_K$-a.e. the equality $||\nu_K||_{K,*} = 1$, where $||\cdot||_{K,*}$ is the dual norm of $||\cdot||_K$.

Given two convex sets $K, K' \subset \mathbb{R}^2$ containing 0 in their interiors, we have the following representation formula for the sub-Finsler perimeter measure $|\partial E|_K$ and the vector field ν_K

$$|\partial E|_K = ||\nu_{K'}||_{K,*}|\partial E|_{K'}, \quad \nu_K = \frac{\nu_{K'}}{||\nu_{K'}||_{K,*}}.$$

Indeed, for the closed unit disk $D \subset \mathbb{R}^2$ centered at 0 we know that in the Euclidean Lipschitz case $\nu_D = \nu_h$ and $|N_h| = ||N_h||_{D,*}$ where N is the *outer* unit normal. Hence we have

$$|\partial E|_K = ||\nu_h||_{K,*} d|\partial E|_D, \quad \nu_K = \frac{\nu_h}{||\nu_h||_{K,*}}.$$

Here $|\partial E|_D$ is the standard sub-Riemannian measure. Moreover, $\nu_h = N_h/|N_h|$ and $|N_h|^{-1}d|\partial E|_D = dS$, where dS is the standard Riemannian measure on S. Hence we get, for a set E with Euclidean Lipschitz boundary S

$$|\partial E|_K(\Omega) = \int_{S\cap\Omega} ||N_h||_{K,*}\, dS, \tag{2.2}$$

where dS is the Riemannian measure on S, obtained from the area formula using a local Lipschitz parameterization of S, see Proposition 2.14 in [19]. It coincides with the 2-dimensional Hausdorff measure associated to the Riemannian distance induced by g. We stress that here N is the *outer* unit normal. This choice is important because of the lack of symmetry of $||\cdot||_K$ and $||\cdot||_{K,*}$.

2.5. **Immersed surfaces in $\mathbb{H}^1$.** Following [1, 19] we provide the following definition.

Definition 2.1 ($\mathbb{H}$-regular surfaces)**.** A real measurable function f defined on an open set $\Omega \subset \mathbb{H}^1$ is of class $C^1_{\mathbb{H}}(\Omega)$ if the distributional derivative $\nabla_{\mathbb{H}} f = (Xf, Yf)$ is represented by a continuous function.

We say that $S \subset \mathbb{H}^1$ is an $\mathbb{H}$-regular surface if for each $p \in \mathbb{H}^1$ there exist a neighborhood U and a function $f \in C^1_{\mathbb{H}}(U)$ such that $\nabla_{\mathbb{H}} f \neq 0$ and $S \cap U = \{f = 0\}$. Then the continuous horizontal unit normal is given by

$$\nu_h = \frac{\nabla_{\mathbb{H}} f}{|\nabla_{\mathbb{H}} f|}.$$

Given an oriented Euclidean Lipschitz surface S immersed in $\mathbb{H}^1$, its unit normal N is defined $\mathcal{H}^2$-a.e. in S, where $\mathcal{H}^2$ is the 2-dimensional Hausdorff measure associated to the Riemannian distance induced by g. In case S is the boundary of a set $E \subset \mathbb{H}^1$, we always choose the outer unit normal. We say that a point p belongs to the *singular set* S_0 of S if $p \in S$ is a differentiable point and the tangent space T_pS coincides with the horizontal distribution $\mathcal{H}_p$. Therefore the horizontal projection of the normal N_h at singular points vanishes. In $S \smallsetminus S_0$ the horizontal unit normal ν_h is defined $\mathcal{H}^2$-a.e. by

$$\nu_h = \frac{N_h}{|N_h|},$$

where N_h is the horizontal projection of the normal N. The vector field Z is defined $\mathcal{H}^2$-a.e. on $S \smallsetminus S_0'$ by $Z = J(\nu_h)$, and it is tangent to S and horizontal.

$\mathbb{H}$-regularity plays an important role in the regularity theory of sets of finite sub-Riemannian perimeter. In [19], B. Franchi, R. Serapioni and F. Serra-Cassano proved that the boundary of such a set is composed of $\mathbb{H}$-regular surfaces and a singular set of small measure.

2.6. **Sets with prescribed mean curvature.** Consider an open set $\Omega \subset M$, and an integrable function $f \in L^1_{loc}(\Omega)$. We say that a set of locally finite K-perimeter $E \subset \Omega$ has *prescribed K-mean curvature f in* Ω if, for any bounded open set $B \subset \Omega$, E is a critical point of the functional

$$|\partial E|_K(B) - \int_{E\cap B} f\, d\mathbb{H}^1. \tag{2.3}$$

If $S = \partial E \cap \Omega$ is a Euclidean Lipschitz surface then S has prescribed K-mean curvature f if it is a critical point of the functional

$$A_K(S\cap B) - \int_{E\cap B} f\, d\mathbb{H}^1, \tag{2.4}$$

for any bounded open set $B \subset \Omega$.

If E has boundary $S = \partial E \cap \Omega$ of class C^2, standard arguments imply that E has prescribed K-mean curvature f in Ω if and only if $H_K = f$, where H_K is the K-mean curvature

$$H_K = \langle D_Z \pi_K(\nu_h), Z \rangle,$$

and ν_h is the *outer* horizontal unit normal, see [38]. Since by [38, Lemma 2.1] the Levi-Civita connection D and the pseudo-hermitian connection ∇ coincide for horizontal vector fields, we obtain that

$$H_K = \langle D_Z \pi_K(\nu_h), Z \rangle = \langle \nabla_Z \pi_K(\nu_h), Z \rangle.$$

It is important to remark that the mean curvature H_K strongly depends on the choice of ν_h. When K is centrally symmetric, $\pi_K(-u) = -\pi_K(u)$ and so the mean curvature changes its sign when we take $-\nu_h$ instead of ν_h. When K is not centrally symmetric, there is no relation between the mean curvatures associated to ν_h and $-\nu_h$.

A set $E \subset \mathbb{H}^1$ with Euclidean Lischiptz boundary has locally finite K-perimeter: we know that it has locally bounded sub-Riemannian perimeter by Proposition 2.14 in [19] and we can apply the perimeter estimates in § 2.3. Letting $\mathcal{H}^2$ be the Riemannian 2-dimensional Hausdorff measure, the Riemannian outer unit normal N is defined $\mathcal{H}^2$-a.e. in ∂E, and it can be proven that

$$|\partial E|_K(V) = \int_{\partial E \cap V} ||N_h||_{K,*} \, d\mathcal{H}^2. \tag{2.5}$$

We say that a set E of locally finite K-perimeter in an open set $\Omega \subset \mathbb{H}^1$ has *constant* prescribed K-mean curvature if there exists $\lambda \in \mathbb{R}$ such that E has prescribed K-mean curvature λ. This means that E is a critical point of the functional $E \mapsto |\partial E|_K(B) - \lambda |E \cap B|$ for any bounded open set $B \subset \Omega$.

Our next result implies that Euclidean Lipschitz isoperimetric boundaries (for the K-perimeter) have constant prescribed K-mean curvature.

Proposition 2.2. *Let $E \subset \mathbb{H}^1$ be a bounded set with Euclidean Lipschitz boundary. Assume that E a critical point of the K-perimeter for variations preserving the volume of E up to first order. Let $\Omega \subset \mathbb{H}^1$ be an open set so that $\Omega \cap S_0 = \emptyset$ and $|\partial E|_K(\Omega) > 0$. Then E has constant prescribed K-mean curvature in Ω.*

Proof. Since the $K-$perimeter of E in Ω is positive there exists a horizontal vector field U_0 with compact support in Ω so that $\int_E \operatorname{div} U_0 \, d\mathbb{H}^1 > 0$. Let $\{\psi_s\}_{s \in \mathbb{R}}$ be the flow associated to U_0 and define

$$H_0 = \frac{\frac{d}{ds}|_{s=0} \, A_K(\psi_s(S))}{\frac{d}{ds}|_{s=0} |\psi_s(E)|}. \tag{2.6}$$

Let W any vector field with compact support in Ω and associated flow $\{\varphi_s\}_{s \in \mathbb{R}}$. Choose $\lambda \in \mathbb{R}$ so that $W - \lambda U_0$ satisfies

$$\frac{d}{ds}\bigg|_{s=0} |\varphi_s(E)| - \lambda \frac{d}{ds}\bigg|_{s=0} |\psi_s(E)| = 0.$$

This means that the flow of $W - \lambda U_0$ preserves the volume of E up to first order.

By our assumption on E we get

$$Q(W - \lambda U_0) = 0,$$

where Q is defined in (2.7). Now Lemma 2.3 implies $Q(W) = \lambda Q(U_0)$ and, from the definition of H_0, we get

$$Q(W) = \lambda Q(U_0) = \lambda H_0 \frac{d}{ds}\bigg|_{s=0} |\psi_s(E)| = H_0 \frac{d}{ds}\bigg|_{s=0} |\varphi_s(E)|.$$

This implies that E is a critical point of the functional $E \mapsto |\partial E|_K - H_0|E|$ and so it has prescribed K-mean curvature equal to the constant H_0. ☐

Lemma 2.3. *Let $E \subset \mathbb{H}^1$ be a bounded set with Euclidean Lipschitz boundary S. Let $\Omega \subset \mathbb{H}^1$ be an open set such that $\Omega \cap S_0 = \emptyset$. Let U be a vector field with compact support Ω and $\{\varphi_s\}_{s\in\mathbb{R}}$ the associated flow. Then the derivative*

$$Q(U) = \frac{d}{ds}\bigg|_{s=0} A_K(\varphi_s(S)) \tag{2.7}$$

exists and is a linear function of U.

Proof. For every $s \in \mathbb{R}$, the set $\varphi_s(E)$ has Euclidean Lipschitz boundary and so it has finite K-perimeter. By Rademacher's Theorem, the set

$$B = \{p \in S : S \text{ is not differentiable at } p\}$$

has $\mathcal{H}^2$-measure equal to 0.

For any $p \in S \setminus B$ we take the curve $\sigma(s) = \varphi_s(p)$. For every $s \in \mathbb{R}$ the surface $\varphi_s(S)$ is differentiable at $\sigma(p)$ and the vector field $W(s) = ((N_s)_h)_{\sigma(s)}$, where N_s is the outer unit normal to $\varphi_s(\partial E)$, is differentiable along the curve σ. Let us estimate the quotient

$$\frac{||W(s+h)||_{K,*} - ||W(s)||_{K_*}}{h}. \tag{2.8}$$

Writing $W(s) = f(s)X_{\sigma(s)} + g(s)Y_{\sigma(s)}$ we have $||W(s)||_{K,*} = ||(f(s), g(s))||$, where $||\cdot||$ is the planar asymmetric norm associated to the convex set K. We have

$$\begin{aligned} \big| ||W(s+h)||_{K,*} - ||W(s)||_{K_*} \big| &\leqslant ||(f(s+h) - f(s), g(s+h) - g(s))|| \\ &\leqslant C\left(|f(s+h) - f(s)| + |g(s+h) - g(s)|\right), \end{aligned}$$

for a constant $C > 0$ that only depends on K. The derivates of f and g can be estimated in terms of the covariant derivative $\frac{D}{ds}W = \frac{D}{ds}(N_s)_h$ along σ. Since

$$\left|\frac{D}{ds}(N_s)_h\right| \leqslant \left|\operatorname{div}_{\varphi_s(S)}(U)\right|$$

we get an uniform estimate on the derivatives of f and g independent of p. So the quotient (2.8) is uniformly bounded above by a constant independent of p.

To compute the derivative of $A_K(\varphi_s(S))$ at $s = 0$ we write

$$A_K(\varphi_s(S)) = \int_S \left(||(N_s)_h||_{K,*} \circ \varphi_s\right) \operatorname{Jac}(\varphi_s)\, d\mathcal{H}^2$$

The uniform estimate of the quotient (2.8) allows us to apply Lebesgue's dominated convergence theorem and Leibniz's rule to compute the derivative of $A_K(\varphi_s(S))$, given by

$$\int_S \frac{d}{ds}\bigg|_{s=0} \Big(\left(||(N_s)_h|| \circ \varphi_s\right) \operatorname{Jac}(\varphi_s)\Big)\, d\mathcal{H}^2.$$

Given a point $p \in (S \smallsetminus B) \cap \operatorname{supp}(U)$, since $\operatorname{supp}(U) \subset \Omega$ and $\Omega \cap S_0 = \emptyset$ we get $(N_h)_p \neq 0$ and so

$$\begin{aligned}\frac{D}{ds}\bigg|_{s=0} ||(N_s)_h||_{K,*}(\sigma(s)) &= \frac{D}{ds}\bigg|_{s=0} \langle (N_s)_h, \pi_K((N_s)_h)\rangle(\sigma(s)) \\ &= \langle \frac{D}{ds}\bigg|_{s=0}(N_s)_h, (N_h)_p\rangle + \langle (N_h)_p, (d\pi_K)\bigg(\frac{D}{ds}\bigg|_{s=0}(N_s)_h\bigg)\rangle.\end{aligned}$$

Since

$$\frac{D}{ds}\bigg|_{s=0}(N_s)_h = \frac{D}{ds}\bigg|_{s=0} N - \langle \frac{D}{ds}\bigg|_{s=0} N, T\rangle\, T,$$

and

$$\frac{D}{ds}\bigg|_{s=0} N = \sum_{i=1}^{2} \langle N_p, \nabla_{e_i} U\rangle\, e_i,$$

where e_i is an orthonormal basis of $T_p(\partial E)$, we get that

$$\frac{D}{ds}\bigg|_{s=0} ||N_s||_{K,*}$$

is a linear function $L(U)$ of U. □

Remark 2.4. Proposition 2.2 can be applied to isoperimetric regions in $\mathbb{H}^1$ with Euclidean Lipschitz boundary. Of course, the regularity of isoperimetric regions in $\mathbb{H}^1$ is still an open problem.

2.7. **Intrinsic Euclidean Lipschitz graphs on a vertical plane in $\mathbb{H}^1$.** We denote by $\operatorname{Gr}(u)$ the *intrinsic* graph (Riemannian normal graph) of the Lipschitz function $u : D \to \mathbb{R}$, where D is a domain in a vertical plane. Using Euclidean rotations about the vertical axis $x = y = 0$, that are isometries of the Riemannian metric g, we may assume that D is contained in the plane $y = 0$. Since the vector field Y is a unit normal to this plane, the intrinsic graph $\operatorname{Gr}(u)$ is given by $\{\exp_p(u(p)Y_p) : p \in D\}$, where exp is the exponential map of g, and can be parameterized by the map

$$\Phi^u(x,t) = (x, u(x,t), t - xu(x,t)).$$

The tangent plane to any point in $S = \operatorname{Gr}(u)$ is generated by the vectors

$$\begin{aligned}\Phi^u_x &= (1, u_x, -u - xu_x) = X + u_x Y - 2uT,\\ \Phi^u_t &= (0, u_t, 1 - xu_t) = u_t Y + T\end{aligned}$$

and the characteristic direction is given by $Z = \tilde{Z}/|\tilde{Z}|$ where

$$\tilde{Z} = X + (u_x + 2uu_t)Y. \tag{2.9}$$

A unit normal to S is given by $N = \tilde{N}/|\tilde{N}|$ where

$$\tilde{N} = \Phi^u_x \times \Phi^u_t = (u_x + 2uu_t)X - Y + u_t T$$

and $\operatorname{Jac}(\Phi^u) = |\Phi^u_x \times \Phi^u_t| = |\tilde{N}|$. Therefore the horizontal projection of the unit normal to S is given by $N_h = \tilde{N}_h/|\tilde{N}|$, where $\tilde{N}_h = (u_x + 2uu_t)X - Y$. Observe that $J(Z) = -\nu_h$.

We also assume that $S = \operatorname{Gr}(u)$ is an $\mathbb{H}$-regular surface, meaning that $\tilde{N}_h$ and $\tilde{Z}$ in (2.9) and are continuous. Hence also $(u_x + 2uu_t)$ is continuous.

Remark 2.5. Let $\gamma(s)=(x,t)(s)$ be a C^1 curve in D then

$$\Gamma(s)=(x,u(x,t),t-xu(x,t))(s)\subset \mathrm{Gr}(u)$$

is also C^1 and

$$\Gamma'(s)=x'X+(x'u_x+t'u_t)Y+(t'-2ux')T.$$

In particular horizontal curves in $\mathrm{Gr}(u)$ satisfy the ordinary differential equation

$$t'=2u(x,t)x'. \tag{2.10}$$

From (2.2), the sub-Finsler K-area for a Euclidean Lipschitz surface S is

$$A_K(S)=\int_S \|N_h\|_{K,*}dS,$$

where $\|N_h\|_{K,*}=\langle N_h,\pi(N_h)\rangle$ with $\pi=(\pi_1,\pi_2)=\pi_K$ and dS is the Riemannian area measure. Therefore when we consider the intrinsic graph $S=\mathrm{Gr}(u)$ we obtain

$$\begin{aligned} A(\mathrm{Gr}(u)) &= \int_D \langle \tilde{N}_h,\pi(\tilde{N}_h)\rangle\ dxdt \\ &= \int_D (u_x+2uu_t)\pi_1(u_x+2uu_t,-1)-\pi_2(u_x+2uu_t,-1)\,dxdt. \end{aligned}$$

Observe that the K-perimeter of a set was defined in terms of the *outer* unit normal. Hence we are assuming that S is the boundary of the *epigraph* of u.

Given $v\in C_0^\infty(D)$, a straightforward computation shows that

$$\frac{d}{ds}\Big|_{s=0}A(\mathrm{Gr}(u+sv))=\int_D (v_x+2vu_t+2uv_t)M\,dxdt, \tag{2.11}$$

where

$$M=F(u_x+2uu_t), \tag{2.12}$$

and F is the function

$$F(x)=\pi_1(x,-1)+x\frac{\partial \pi_1}{\partial x}(x,-1)-\frac{\partial \pi_2}{\partial x}(x,-1). \tag{2.13}$$

Since (u_x+2uu_t) is continuous and π is at least C^1 the function M is continuous.

3. Characteristic curves are C^2

Here we prove our main result, that characteristic curves in an intrinsic Euclidean Lipschitz $\mathbb{H}$-regular surface with continuous prescribed K-mean curvature are of class C^2. The reader is referred to Theorem 4.1 in [21] for a proof of the the sub-Riemannian case. The proof of Theorem 3.1 depends on Lemmas 3.2 and 3.3.

Theorem 3.1. *Let K be a C^2_+ convex set in $\mathbb{R}^2$ with $0\in \mathrm{int}(K)$ and $||\cdot||_K$ the associated left-invariant norm in $\mathbb{H}^1$. Let $\Omega\subset\mathbb{H}^1$ be an open set and $E\subset\Omega$ a set of prescribed K- mean curvature $f\in C^0(\Omega)$ with an Euclidean Lipschitz and $\mathbb{H}$-regular boundary S. Then the characteristic curves of $S\cap\Omega$ are of class C^2.*

Proof. By the Implicit Function Theorem for $\mathbb{H}$-regular surfaces, see Theorem 6.5 in [19], given a point $p\in S$, after a rotation about the vertical axis, there exists an open neighborhood $B\subset\mathbb{H}^1$ of p such that $B\cap S$ is the intrinsic graph $\mathrm{Gr}(u)$ of a function $u:D\to\mathbb{R}$, where D is a domain in the vertical plane $y=0$, and $B\cap E$ is the epigraph of u. The function u is Euclidean Lipschitz by our assumption. .Since $\mathrm{Gr}(u)$ has prescribed continuous mean curvature f, from equation (2.11) we get

$$\int_D (v_x + 2vu_t + 2uv_t)M + fv\,dxdt = 0, \tag{3.1}$$

for each $v \in C_0^\infty(D)$. The function M is defined in (2.12). By Remark 4.3 in [21] implies that (3.1) holds for each $v \in C_0^0(D)$ for which $v_x + 2uv_t$ exists and is continuous.

Let $\Gamma(s)$ be a characteristic horizontal curve passing through p whose velocity is the vector field $\tilde{Z}$ defined in (2.9), that only depends on $u_x + 2uu_t$. Since S is $\mathbb{H}$-regular the function $u_x + 2uu_t$ is continuous and $\Gamma(s)$ is of class C^1. Let us consider the function F defined in (2.13) and define

$$g(s) = (u_x + 2uu_t)_{\Gamma(s)}.$$

Hence $F(g(s)) = M(s)$. The function F is C^1 for any convex set K of class C^2_+ and, from Lemma 3.2, we obtain that $F'(x) > 0$ for each $x \in \mathbb{R}$. Therefore F^{-1} is also C^1 and $g(s) = F^{-1}(M(s))$. Thanks to Lemma 3.3 we obtain that M is C^1 along Γ and we conclude that also g is C^1 along Γ. So $\tilde{Z}$ is C^1 and the curve Γ is C^2. □

Lemma 3.2. *Let $K \subset \mathbb{R}^2$ be a convex body of class C^2_+ such that $0 \in \mathrm{int}(K)$. Then the function F defined in (2.13) is C^1 and $F'(x) > 0$ for each $x \in \mathbb{R}$.*

Proof. Parameterize the lower part of the boundary of the convex body K by a function ϕ defined on a closed interval $I \subset \mathbb{R}$. The function ϕ is of class C^2 in $\mathring{I}$ and the graph becomes vertical at the endpoints of I. As K is of class C^2_+ we have $\phi''(x) > 0$ for each $x \in \mathbb{R}$. Take $x \in \mathbb{R}$, then we have

$$\pi(x,-1) = N_K^{-1}\left(\frac{(x,-1)}{\sqrt{1+x^2}}\right),$$

where N_K is the outer unit normal to ∂K. Let $\varphi(x) \in \mathring{I}$ be the point where

$$(\varphi(x), \phi(\varphi(x))) = \pi(x,-1).$$

Therefore, if we consider the normal N_K of the previous equality we obtain

$$\frac{(\phi'(\varphi(x)),-1)}{\sqrt{1+(\phi'(\varphi(x)))^2}} = \frac{(x,-1)}{\sqrt{1+x^2}}.$$

Hence $\phi'(\varphi(x)) = x$ and so φ is the inverse of ϕ', that is invertible since $\phi''(x) > 0$ for each $x \in \mathbb{R}$. Notice that

$$\begin{aligned} F(x) &= \pi_1(x,-1) + x\frac{\partial \pi_1}{\partial x}(x,-1) - \frac{\partial \pi_2}{\partial x}(x,-1) \\ &= \varphi(x) + x\varphi'(x) - \phi'(\varphi(x))\varphi'(x) = \varphi(x), \end{aligned}$$

since $\phi'(\varphi(x)) = x$. Hence we obtain

$$F'(x) = \varphi'(x) = \frac{1}{\phi''(\varphi(x))} > 0$$

for each $x \in \mathbb{R}$. □

Lemma 3.3. *Let $\Omega \subset \mathbb{H}^1$ be an open set and $E \subset \Omega$ a set of prescribed K-mean curvature $f \in C^0(\Omega)$ with Euclidean Lipschitz and $\mathbb{H}$-regular boundary S. Then the* *function M defined in (2.12) is of class C^1 along characteristic curves. Moreover, the differential equation*

$$\frac{d}{ds}M(\gamma(s)) = f(\gamma(s))$$

is satisfied along any characteristic curve γ.

Proof. Let $\Gamma(s)$ be a characteristic curve passing through p in $\mathrm{Gr}(u)$. Let $\gamma(s)$ be the projection of $\Gamma(s)$ onto the xt-plane, and $(a,b) \in D$ the projection of p to the xt-plane. We parameterize γ by $s \to (s, t(s))$. By Remark 2.5 the curve $s \to (s, t(s))$ satisfies the ordinary differential equation $t' = 2u$. For ε small enough, Picard-Lindelöf's theorem implies the existence of $r > 0$ and a solution $t_\varepsilon :]a-r, a+r[\to \mathbb{R}$ of the Cauchy problem

$$\begin{cases} t_\varepsilon'(s) = 2u(s, t_\varepsilon(s)), \\ t_\varepsilon(a) = b + \varepsilon. \end{cases} \tag{3.2}$$

We define $\gamma_\varepsilon(s) = (s, t_\varepsilon(s))$ so that $\gamma_0 = \gamma$. Here we exploit an argument similar to the one developed in [35]. By Theorem 2.8 in [44] we gain that t_ε is Lipschitz with respect to ε with Lipschitz constant less than or equal to e^{Lr}. Fix $s \in]a-r, a+r[$, the inverse of the function $\varepsilon \to t_\varepsilon(s)$ is given by $\bar{\chi}_t(-s) = \chi_t(-s) - b$ where χ_t is the unique solution of the following Cauchy problem

$$\begin{cases} \chi_t'(\tau) = 2u(\tau, \chi_t(\tau)) \\ \chi_t(a+s) = t. \end{cases} \tag{3.3}$$

Again by Theorem 2.8 in [44] we have that $\bar{\chi}_t$ is Lipschitz continuous with respect to t, thus the function $\varepsilon \to t_\varepsilon$ is a locally biLipschitz homeomorphisms.

We consider the following Lipschitz coordinates

$$G(\xi, \varepsilon) = (\xi, t_\varepsilon(\xi)) = (s, t) \tag{3.4}$$

around the characteristic curve passing through (a,b). Notice that, by the uniqueness result for (3.2), G is injective. Given (s,t) in the image of G using the inverse function $\bar{\chi}_t$ defined in (3.3) we find ε such that $t_\varepsilon(s) = t$, therefore G is surjective. By the Invariance of Domain Theorem [3], is a homeomorphism. The Jacobian of G is defined by

$$\mathbf{J}_G = \det \begin{pmatrix} 1 & 0 \\ t_\varepsilon' & \frac{\partial t_\varepsilon}{\partial \varepsilon} \end{pmatrix} = \frac{\partial t_\varepsilon}{\partial \varepsilon}(s) \tag{3.5}$$

almost everywhere in ε. Any function φ defined on D can be considered as a function of the variables (ξ, ε) by making $\tilde{\varphi}(\xi, \varepsilon) = \varphi(\xi, t_\varepsilon(\xi))$. Since the function G is C^1 with respect to ξ we have

$$\frac{\partial \tilde{\varphi}}{\partial \xi} = \varphi_x + t_\varepsilon' \, \varphi_t = \varphi_x + 2u\varphi_t.$$

Furthermore, by [16, Theorem 2 in Section 3.3.3] or [25, Theorem 3], we may apply the change of variables formula for Lipschitz maps. Assuming that the support of v is contained in a sufficiently small neighborhood of (a,b), we can express the integral (3.1) as

$$\int_I \Big(\int_{a-r}^{a+r} \big(\big(\frac{\partial \tilde{v}}{\partial \xi} + 2\tilde{v}\, \tilde{u}_t \big) \tilde{M} + \tilde{f} \tilde{v} \big) \frac{\partial t_\varepsilon}{\partial \varepsilon} \, d\xi \Big) d\varepsilon = 0, \tag{3.6}$$

where I is a small interval containing 0. Instead of $\tilde{v}$ in (3.6) we consider the function $\tilde{v}h/(t_{\varepsilon+h}-t_\varepsilon)$, where h is a small enough parameter. Then we obtain

$$\begin{aligned}\frac{\partial}{\partial\xi}\left(\frac{\tilde{v}h}{(t_{\varepsilon+h}-t_\varepsilon)}\right) &= \frac{\partial\tilde{v}}{\partial\xi}\frac{h}{(t_{\varepsilon+h}-t_\varepsilon)} - \tilde{v}h\frac{t'_{\varepsilon+h}-t'_\varepsilon}{(t_{\varepsilon+h}-t_\varepsilon)^2}\\ &= \frac{\partial\tilde{v}}{\partial\xi}\frac{h}{(t_{\varepsilon+h}-t_\varepsilon)} - 2\tilde{v}h\frac{u(\xi,t_{\varepsilon+h}(\xi))-u(\xi,t_\varepsilon(\xi))}{(t_{\varepsilon+h}-t_\varepsilon)^2},\end{aligned}$$

that tends to

$$\left(\frac{\partial t_\varepsilon}{\partial\varepsilon}\right)^{-1}\left(\frac{\partial\tilde{v}}{\partial\xi}-2\tilde{v}\tilde{u}_t\right)\qquad a.e.\text{ in }\varepsilon,$$

when h goes to 0. Putting $\tilde{v}h/(t_{\varepsilon+h}-t_\varepsilon)$ in (3.6) instead of $\tilde{v}$ we gain

$$\int_I\left(\int_{a-r}^{a+r}\frac{h\frac{\partial t_\varepsilon}{\partial\varepsilon}}{(t_{\varepsilon+h}-t_\varepsilon)}\left(\frac{\partial\tilde{v}}{\partial\xi}+2\tilde{v}\,(\tilde{u}_t-\frac{\tilde{u}(\xi,\varepsilon+h)-\tilde{u}(\xi,\varepsilon)}{(t_{\varepsilon+h}-t_\varepsilon)})\right)\tilde{M}+\tilde{f}\tilde{v}\,d\xi\right)d\varepsilon=0.$$

Using Lebesgue's dominated convergence theorem and letting $h\to 0$ we have

$$\int_I\left(\int_{a-r}^{a+r}\frac{\partial\tilde{v}}{\partial\xi}\tilde{M}+\tilde{f}\tilde{v}\,d\xi\right)d\varepsilon=0. \tag{3.7}$$

Let $\eta:\mathbb{R}\to\mathbb{R}$ be a positive function compactly supported in I and for $\rho>0$ we consider the family $\eta_\rho(x)=\rho^{-1}\eta(x/\rho)$, that weakly converge to the Dirac delta distribution. Putting the test functions $\eta_\rho(\varepsilon)\psi(\xi)$ in (3.7) and letting $\rho\to 0$ we get

$$\int_{a-r}^{a+r}\psi'(\xi)\tilde{M}(\xi,0)+\tilde{f}(\xi,0)\psi(\xi)\,d\xi=0, \tag{3.8}$$

for each $\psi\in C_0^\infty((a-r,a+r))$. Since u_x+2uu_t is continuous, M in (2.12) is continuous, thus also $\tilde{M}$. Hence thanks to Lemma 3.4 we conclude that M is C^1 along γ, thus by Remark 2.5 is also C^1 along Γ.

Since M is C^1 along the characteristic curve, we can integrate by parts in equation (3.8) to obtain

$$\int_{a-r}^{a+r}\left(-\tilde{M}'(0,\xi)+\tilde{f}(0,\xi)\right)\psi(\xi)\,d\xi=0,$$

for each $\psi\in C_0^\infty((a-r,a+r))$. That means that M satisfies the equation

$$\frac{d}{ds}M(\gamma(s))=f(\gamma(s))$$

along characteristic curves. □

Lemma 3.4 ([21, Lemma 4.2])**.** *Let $J\subset\mathbb{R}$ be an open interval and $g,h\in C^0(J)$. Let $H\in C^1(J)$ be a primitive of h. Assume that*

$$\int_J\psi'g+h\psi=0,$$

for each $\psi\in C_0^\infty(J)$. Then the function $g-H$ is a constant function in J. In particular $g\in C^1(J)$.

Remark 3.5. Let K be a convex body of class C^2_+ such that $0\in K$. Following [38] we consider a clockwise-oriented P-periodic parameterization $\gamma:\mathbb{R}\to\mathbb{R}^2$ of ∂K. For a fixed $v\in\mathbb{R}$ we take the translated curve $s\to\gamma(s+v)-\gamma(v)=(x(s),y(s))$

and we consider its horizontal lifting $\Gamma_v(s)$ to $\mathbb{H}^1$ starting at $(0,0,0) \in \mathbb{H}^1$ for $s = 0$, given by

$$\Gamma_v(s) = \left((x(s), y(s), \int_0^s y(\tau)x'(\tau) - x(\tau)y'(\tau)d\tau \right).$$

The Pansu-Wulff shape associated to K is defined by

$$\mathbb{S}_K = \bigcup_{v\in[0,P)} \Gamma_v([0,P]).$$

In [38, Theorem 3.14] it is shown that the horizontal liftings Γ_v, for each $v \in [0,P)$, are solutions for $H_K = 1$, therefore $\mathbb{S}_K$ has constant prescribed K-mean curvature equal to 1. Since the curves Γ_v have the same regularity as ∂K, the C^2 regularity result for horizontal curves obtained in Theorem 3.1 is optimal.

Corollary 3.6. *Let K be a C^2_+ convex set in $\mathbb{R}^2$ with $0 \in \text{int}(K)$ and $||\cdot||_K$ the associated left-invariant norm in $\mathbb{H}^1$. Let $\Omega \subset \mathbb{H}^1$ be an open set and $E \subset \Omega$ a set of prescribed K-mean curvature $f \in C^0(\Omega)$ with C^1 boundary S. Then the characteristic curves in $S \smallsetminus S_0$ are of class C^2.*

Proof. Since S is of class C^1, in the regular part $S \smallsetminus S_0$ the horizontal normal ν_h is a nowhere-vanishing continuous vector fields, thus $S \smallsetminus S_0$ is an $\mathbb{H}$-regular surface. In particular a C^1 surface is Lipschitz, thus $S \smallsetminus S_0$ verifies the hypotheses of Theorem 3.1 and the characteristic curves in $S \smallsetminus S_0$ are of class C^2. □

Remark 3.7. When S is of class C^1 the proof of Lemma 3.3 is is much easier. Indeed the solution t_ε of the Cauchy Problem (3.2) is differentiable in ε, thus the function $\partial t_\varepsilon/\partial\varepsilon$ satisfies the following ODE

$$\left(\frac{\partial t_\varepsilon}{\partial \varepsilon}\right)'(s) = 2u_t(s, t_\varepsilon(s))\frac{\partial t_\varepsilon}{\partial \varepsilon}, \qquad \frac{\partial t_\varepsilon}{\partial \varepsilon}(a) = 1.$$

That implies that

$$\frac{\partial t_\varepsilon}{\partial \varepsilon}(s) = e^{\int_a^s 2u_t(\tau, t_\varepsilon(\tau)))d\tau} > 0.$$

Since the Jacobian $\mathbf{J}_G$ defined in (3.5) is equal to $\partial t_\varepsilon/\partial\varepsilon > 0$ the change of variables $G(\xi,\varepsilon)$ is invertible. Hence the rest of the proof of Lemma 3.3 goes in the same way as before.

4. The sub-Finsler mean curvature equation

Given an Euclidean Lipschitz boundary S whose characteristic curves in $S \smallsetminus S_0$ are of class C^2, for each point $p \in S \smallsetminus S_0$ we can define the K-mean curvature H_K of S by

$$H_K = \langle D_Z \pi_K(\nu_h), Z\rangle = \langle \nabla_Z \pi_K(\nu_h), Z\rangle, \tag{4.1}$$

where ν_h is the outer horizontal unit normal to S. This definition was given in [38] for surfaces of class C^2.

Proposition 4.1. *Let $\Omega \subset \mathbb{H}^1$ be an open set and $E \subset \Omega$ a set of prescribed K-mean curvature $f \in C^0(\Omega)$ Euclidean Lipschitz and $\mathbb{H}$-regular boundary S. Then $H_K(p) = f(p)$ for each $p \in S \smallsetminus S_0$.*

Proof. By the Implicit Function Theorem for $\mathbb{H}$-regular surfaces, Theorem 6.5 in [19], given a point $p \in S$, after a rotation about the t-axis, there exists an open neighborhood $B \subset \mathbb{H}^1$ of p such that $B \cap S$ is the intrinsic graph of a function $u : D \to \mathbb{R}$ where D is a domain in the vertical plane $y = 0$. The function u is Euclidean Lipschitz by our assumption. We set $B \cap S = \mathrm{Gr}(u)$. We assume that E is locally the epigraph of u.

Let $\Gamma(s)$ be a characteristic curve passing through p in $\mathrm{Gr}(u)$ and $\gamma(s)$ its projection on the xt-plane. The characteristic vector Z defined in (2.9) is given by

$$Z = \frac{X + (u_x + 2uu_t)Y}{(1 + (u_x + 2uu_t)^2)^{\frac{1}{2}}}.$$

Since S is $\mathbb{H}$-regular, Z and the horizontal unit normal

$$\nu_h = \frac{(u_x + 2uu_t)X - Y}{(1 + (u_x + 2uu_t)^2)^{\frac{1}{2}}}$$

are continuous vector fields. By Lemma 3.3 we have that $M = F(u_x + 2uu_t)$ defined in (2.12) satisfies the differential equation

$$\frac{d}{ds}M(\gamma(s)) = f(\gamma(s))$$

along the characteristic curves. Therefore we obtain

$$\begin{aligned}\frac{d}{ds}M(\gamma(s)) &= F'(u_x + 2uu_t)\frac{d}{ds}\big[(u_x + 2uu_y)(\gamma(s))\big] \\ &= \frac{1}{\phi''(u_x + 2uu_t)}\frac{d}{ds}\big[(u_x + 2uu_y)(\gamma(s))\big],\end{aligned}$$

As in proof of Lemma 3.2, we parametrize the lower part of the boundary of the convex body K by a function ϕ defined on a closed interval $I \subset \mathbb{R}$. Again by Lemma 3.2 we have

$$\pi_K(x, -1) = (\varphi(x), \phi(\varphi(x)),$$

where φ is the inverse function of ϕ'. Furthermore the K-mean curvature defined (4.1) is equivalent to

$$\begin{aligned}H_K &= \langle D_Z \pi_K(u_x + 2uu_t, -1), Z\rangle \\ &= \frac{\langle \frac{D}{ds}\big[\varphi(u_x + 2uu_t)X_\gamma + \phi(\varphi(u_x + 2uu_t))Y_\gamma\big], Z\rangle}{1 + (u_x + 2uu_t)^2} \\ &= \frac{\varphi'(u_x + 2uu_t)\frac{d}{ds}(u_x + 2uu_t)\big(1 + \phi'(\varphi(u_x + 2uu_t))(u_x + 2uu_t)\big)}{1 + (u_x + 2uu_t)^2} \\ &= \frac{1}{\phi''(u_x + 2uu_t)}\frac{d}{ds}\big[(u_x + 2uu_t)(\gamma(s))\big].\end{aligned}$$

Hence we obtain $H_K = \frac{d}{ds}M(\gamma(s))$ and so $H_K(p) = f(p)$ for each $p \in S \smallsetminus S_0$. $\square$

The following result allows us to express the K-mean curvature H_K in terms of the sub-Riemannian mean curvature H_D.

Proposition 4.2. *Let $K \subset \mathbb{R}^2$ be a convex body of class C^2_+ such that $0 \in \mathrm{int}(K)$ and $\pi_K = N_K^{-1}$. Let κ be the strictly positive curvature of the boundary ∂K. Let*

$\Omega \subset \mathbb{H}^1$ be an open set and $E \subset \Omega$ a set of prescribed K-mean curvature $f \in C^0(\Omega)$ with Euclidean Lipschitz and $\mathbb{H}$-regular boundary S. Then, we have

$$H_D(p) = \kappa(\pi_K(\nu_h))f(p) \qquad \text{for each} \quad p \in S \setminus S_0,$$

where $H_D(p) = \langle D_Z \nu_h, Z\rangle$ is the sub-Riemannian mean curvature, ν_h be the horizontal unit normal at p to $S \setminus S_0$ and $Z = J(\nu_h)$ be the characteristic vector field.

Proof. By Proposition 4.1 we have $H_K(p) = f(p)$ for each $p \in S \setminus S_0$. We remark that Theorem 3.1 implies that H_K is well-defined.

Let $\gamma : (-\varepsilon, \varepsilon) \to S \setminus S_0$ be the integral curve of Z passing through p, namely $\gamma'(s) = Z_{\gamma(s)}$ and $\gamma(0) = p$. Let $\nu_h(s) = -J(Z_{\gamma(s)})$ be the horizontal unit normal along γ and let

$$\pi(\nu_h(s)) = \pi_1(\nu_h(s))X_{\gamma(s)} + \pi_2(\nu_h(s))Y_{\gamma(s)}.$$

Noticing that $\nabla X = \nabla Y = 0$ we gain

$$\frac{\nabla}{ds}\Big|_{s=0}\pi(\nu_h(s)) = \frac{d}{ds}\Big|_{s=0}\pi_1(\nu_h(s))X_{\gamma(0)} + \frac{d}{ds}\Big|_{s=0}\pi_2(\nu_h(s))Y_{\gamma(0)}.$$

Setting $\nu_h = aX + bY$ we obtain

$$\frac{\nabla}{ds}\Big|_{s=0}\pi(\nu_h(s)) = (d\pi)_{(a,b)}\left(\frac{\nabla}{ds}\Big|_{s=0}\nu_h(s)\right), \tag{4.2}$$

where

$$(d\pi)_{(a,b)} = \begin{pmatrix} \dfrac{\partial \pi_1}{\partial a}(a,b) & \dfrac{\partial \pi_1}{\partial b}(a,b) \\ \dfrac{\partial \pi_2}{\partial a}(a,b) & \dfrac{\partial \pi_2}{\partial b}(a,b) \end{pmatrix}.$$

Moreover, by Corollary 1.7.3 in [43] we get $\pi_K = \nabla h$, where h is a C^2 function. Thus by Schwarz's theorem the Hessian $\text{Hess}_{(a,b)}(h) = (d\pi)_{(a,b)}$ is symmetric, i.e. $(d\pi) = (d\pi)^*$. Equation (4.2) then implies

$$H_K = \langle \nabla_Z\, \pi_K(\nu_h), Z\rangle = \langle \nabla_Z \nu_h, (d\pi)^*_{\nu_h} Z\rangle = \langle \nabla_Z \nu_h, (d\pi)_{\nu_h} Z\rangle.$$

Finally, by Lemma 4.3 we get

$$H_K = \frac{1}{\kappa(\pi_K(\nu_h))}\langle \nabla_Z \nu_h, Z\rangle.$$

Hence we obtain $\langle D_Z \nu_h, Z\rangle = \kappa(\pi_K(\nu_h))$, since $D_Z \nu_h = \nabla_Z \nu_h$. □

Lemma 4.3. *Let $K \subset \mathbb{R}^2$ be a convex body of class C^2_+ such that $0 \in \text{int}(K)$ and N_K be the Gauss map of ∂K. Let κ be the strictly positive curvature of the boundary ∂K. Let S be an $\mathbb{H}$-regular surface with horizontal unit normal ν_h and characteristic vector field $Z = J(\nu_h)$. Then we have*

$$(d\pi)_{\nu_h} Z = \frac{1}{\kappa} Z \quad \text{and} \quad (d\pi)_{\nu_h} \nu_h = 0,$$

where $(d\pi)_{\nu_h}$ is the differential of $\pi_K = N_K^{-1}$.

Proof. Let $\alpha(t) = (x(t), y(t))$ be an arc-length parametrization of ∂K such that $\dot{x}^2(t) + \dot{y}^2(t) = 1$. Let $\nu_h = aX + bY$ be the horizontal unit normal to S, with $a = \cos(\theta)$ and $b = \sin(\theta)$ and $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Notice that $\theta = \arctan(\frac{b}{a})$. Then we have

$$\pi_K(a, b) = N_K^{-1}((a, b)).$$

Let $\varphi : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ be the function satisfying

$$\pi_K(\cos(\theta), \sin(\theta)) = (x(\varphi(\theta)), y(\varphi(\theta))).$$

If we consider the normal N_K of the previous equality we obtain

$$(\cos(\theta), \sin(\theta)) = (\dot{y}(\varphi(\theta)), -\dot{x}(\varphi(\theta))).$$

Therefore we have

$$\theta = \arctan\left(-\frac{\dot{x}}{\dot{y}}(\varphi(\theta))\right)$$

for each $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. That means that φ is the inverse of the function $\arctan(-\frac{\dot{x}}{\dot{y}}(t))$, that is invertible since

$$\frac{d}{dt}\arctan(-\tfrac{\dot{x}}{\dot{y}}(t)) = \dot{x}\ddot{y} - \dot{y}\ddot{x} = \kappa(t) > 0.$$

Let $Z = J(\nu_h) = -bX + aY$ be the characteristic vector field, then we have $(d\pi)_{(a,b)} = (d\pi)^*_{(a,b)}$ and

$$(d\pi)^*_{(a,b)} Z = \begin{pmatrix} -b\dfrac{\partial \pi_1}{\partial a} + a\dfrac{\partial \pi_2}{\partial a} \\ -b\dfrac{\partial \pi_1}{\partial b} + a\dfrac{\partial \pi_2}{\partial b} \end{pmatrix},$$

where

$$\pi_1(a,b) = x(\varphi(\arctan(\tfrac{b}{a}))), \qquad \pi_2(a,b) = y(\varphi(\arctan(\tfrac{b}{a}))).$$

Thus we get

$$(d\pi)^*_{(a,b)} Z = \varphi'(\arctan(\tfrac{b}{a}))Z = \frac{1}{\kappa(\varphi(\theta))} Z.$$

A similar straightforward computation shows that $(d\pi)_{\nu_h}\nu_h = 0$. □

References

[1] L. Ambrosio, F. Serra Cassano, and D. Vittone. Intrinsic regular hypersurfaces in Heisenberg groups. *J. Geom. Anal.*, 16(2):187–232, 2006.

[2] V. Barone Adesi, F. Serra Cassano, and D. Vittone. The Bernstein problem for intrinsic graphs in Heisenberg groups and calibrations. *Calc. Var. Partial Differential Equations*, 30(1):17–49, 2007.

[3] L. E. J. Brouwer. Beweis des ebenen Translationssatzes. *Math. Ann.*, 72(1):37–54, 1912.

[4] L. Capogna, G. Citti, and M. Manfredini. Regularity of non-characteristic minimal graphs in the Heisenberg group $\mathbb{H}^1$. *Indiana Univ. Math. J.*, 58(5):2115–2160, 2009.

[5] L. Capogna, D. Danielli, S. D. Pauls, and J. T. Tyson. *An introduction to the Heisenberg group and the sub-Riemannian isoperimetric problem*, volume 259 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2007.

[6] J.-H. Cheng, H.-L. Chiu, J.-F. Hwang, and P. Yang. Umbilicity and characterization of Pansu spheres in the Heisenberg group. *J. Reine Angew. Math.*, 738:203–235, 2018.

[7] J.-H. Cheng, J.-F. Hwang, A. Malchiodi, and P. Yang. Minimal surfaces in pseudohermitian geometry. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 4(1):129–177, 2005.

[8] J.-H. Cheng, J.-F. Hwang, and P. Yang. Existence and uniqueness for p-area minimizers in the Heisenberg group. *Math. Ann.*, 337(2):253–293, 2007.

[9] J.-H. Cheng, J.-F. Hwang, and P. Yang. Regularity of C^1 smooth surfaces with prescribed p-mean curvature in the Heisenberg group. *Math. Ann.*, 344(1):1–35, 2009.

[10] G. Citti and A. Sarti. A cortical based model of perceptual completion in the roto-translation space. *J. Math. Imaging Vision*, 24(3):307–326, 2006.

[11] D. Danielli, N. Garofalo, and D. M. Nhieu. Sub-Riemannian calculus on hypersurfaces in Carnot groups. *Adv. Math.*, 215(1):292–378, 2007.

[12] D. Danielli, N. Garofalo, and D.-M. Nhieu. A partial solution of the isoperimetric problem for the Heisenberg group. *Forum Math.*, 20(1):99–143, 2008.

[13] D. Danielli, N. Garofalo, D. M. Nhieu, and S. D. Pauls. Instability of graphical strips and a positive answer to the Bernstein problem in the Heisenberg group $\mathbb{H}^1$. *J. Differential Geom.*, 81(2):251–295, 2009.

[14] E. De Giorgi. Su una teoria generale della misura $(r-1)$-dimensionale in uno spazio ad r dimensioni. *Ann. Mat. Pura Appl. (4)*, 36:191–213, 1954.

[15] S. Dragomir and G. Tomassini. *Differential geometry and analysis on CR manifolds*, volume 246 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 2006.

[16] L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions*. Textbooks in Mathematics. CRC Press, Boca Raton, FL, revised edition, 2015.

[17] V. Franceschi, G. P. Leonardi, and R. Monti. Quantitative isoperimetric inequalities in $\mathbb{H}^n$. *Calc. Var. Partial Differential Equations*, 54(3):3229–3239, 2015.

[18] V. Franceschi, R. Monti, A. Righini, and M. Sigalotti. The isoperimetric problem for regular and crystalline norms in $\mathbb{H}^1$. arXiv:2007.11384, 22 Jul 2020.

[19] B. Franchi, R. Serapioni, and F. Serra Cassano. Rectifiability and perimeter in the Heisenberg group. *Math. Ann.*, 321(3):479–531, 2001.

[20] M. Galli. The regularity of Euclidean Lipschitz boundaries with prescribed mean curvature in three-dimensional contact sub-Riemannian manifolds. *Nonlinear Anal.*, 136:40–50, 2016.

[21] M. Galli and M. Ritoré. Regularity of C^1 surfaces with prescribed mean curvature in three-dimensional contact sub-Riemannian manifolds. *Calc. Var. Partial Differential Equations*, 54(3):2503–2516, 2015.

[22] N. Garofalo and D.-M. Nhieu. Isoperimetric and Sobolev inequalities for Carnot-Carathéodory spaces and the existence of minimal surfaces. *Comm. Pure Appl. Math.*, 49(10):1081–1144, 1996.

[23] G. Giovannardi. Higher dimensional holonomy map for rules submanifolds in graded manifolds. *Anal. Geom. Metr. Spaces*, 8(1):68–91, 2020.

[24] G. Giovannardi, G. Citti, and M. Ritoré. Variational formulas for submanifolds of fixed degree. arXiv:1905.05131, 13 May 2019.

[25] P. Hajłasz. Change of variables formula under minimal assumptions. *Colloq. Math.*, 64(1):93–101, 1993.

[26] R. K. Hladky and S. D. Pauls. Constant mean curvature surfaces in sub-Riemannian geometry. *J. Differential Geom.*, 79(1):111–139, 2008.

[27] R. K. Hladky and S. D. Pauls. Variation of perimeter measure in sub-Riemannian geometry. *Int. Electron. J. Geom.*, 6(1):8–40, 2013.

[28] A. Hurtado, M. Ritoré, and C. Rosales. The classification of complete stable area-stationary surfaces in the Heisenberg group $\mathbb{H}^1$. *Adv. Math.*, 224(2):561–600, 2010.

[29] A. Hurtado and C. Rosales. Area-stationary surfaces inside the sub-Riemannian three-sphere. *Math. Ann.*, 340(3):675–708, 2008.

[30] G. P. Leonardi and S. Masnou. On the isoperimetric problem in the Heisenberg group $\mathbb{H}^n$. *Ann. Mat. Pura Appl. (4)*, 184(4):533–553, 2005.

[31] G. P. Leonardi and S. Rigot. Isoperimetric sets on Carnot groups. *Houston J. Math.*, 29(3):609–637, 2003.

[32] R. Monti. Heisenberg isoperimetric problem. The axial case. *Adv. Calc. Var.*, 1(1):93–121, 2008.

[33] R. Monti and M. Rickly. Convex isoperimetric sets in the Heisenberg group. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 8(2):391–415, 2009.

[34] S. Nicolussi and M. Ritoré. Area-minimizing cones in $\mathbb{H}^1$. arXiv:2008.04027, 10 Aug 2020.

[35] S. Nicolussi and F. Serra Cassano. The Bernstein problem for Lipschitz intrinsic graphs in the Heisenberg group. *Calc. Var. Partial Differential Equations*, 58(4):Paper No. 141, 28, 2019.

[36] P. Pansu. Une inégalité isopérimétrique sur le groupe de Heisenberg. *C. R. Acad. Sci. Paris Sér. I Math.*, 295(2):127–130, 1982.

[37] S. D. Pauls. H-minimal graphs of low regularity in $\mathbb{H}^1$. *Comment. Math. Helv.*, 81(2):337–381, 2006.

[38] J. Pozuelo and M. Ritoré. Pansu-Wulff shapes in $\mathbb{H}^1$. arXiv:2007:04683, 9 Jul 2020.

[39] M. Ritoré. Examples of area-minimizing surfaces in the sub-Riemannian Heisenberg group $\mathbb{H}^1$ with low regularity. *Calc. Var. Partial Differential Equations*, 34(2):179–192, 2009.

[40] M. Ritoré. A proof by calibration of an isoperimetric inequality in the Heisenberg group $\mathbb{H}^n$. *Calc. Var. Partial Differential Equations*, 44(1-2):47–60, 2012.

[41] M. Ritoré and C. Rosales. Area-stationary surfaces in the Heisenberg group $\mathbb{H}^1$. *Adv. Math.*, 219(2):633–671, 2008.

[42] A. P. Sánchez. Sub-finsler heisenberg perimeter measures, 2017.

[43] R. Schneider. *Convex bodies: the Brunn-Minkowski theory*, volume 151 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, expanded edition, 2014.

[44] G. Teschl. *Ordinary differential equations and dynamical systems*, volume 140 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2012.

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