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EL PROCESO DE MEDIDA EN FÍSICA CUÁNTICA

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Resumen

Esta memoria se centra en el estudio de uno de los problemas que aparece en la Mecánica Cuántica: la medida. Primeramente se ha hecho una revisión bibliográfica sobre los postulados de la teoría cuántica, los tipos de medidas actuales, el entrelazamiento y su relación con la desigualdad de Bell y por último, el experimento del amigo de Wigner. Todo esto ha servido de base para implementar un código que simule uno de estos tipos de medidas y permita estudiar la evolución temporal de los valores esperados de diferentes sistemas con sus respectivos hamiltonianos en función del periodo con el que se mide sobre él.

Abstract

This memory focuses on the study of one of the problems that appears in Quantum Mechanics: measurement. First, a bibliographic review has been made on the Quantum Mechanics postulates, the current types of measurements, entanglement and its relationship with Bell's inequality and finally, the Wigner's friend experiment. All this has settled the basis to personally implement a source code that simulates one of these types of measurements and allows us to study the temporal evolution of the expectation values of different systems with their respective Hamiltonians based on the period with which it is measured.

Contents

1 Introduction.

All sciences have their origin on the observation of nature. We acquire knowledge about reality, which is thought to be objective, measuring it using an active process. In order to do that we make an apparatus interacts with the physical system that we are studying, so that a property of that system can affect a property of the apparatus. Since both interact, measuring one property of a system necessarily causes a disturbance to some of its properties. Classical physics assumes that the physical property which is measured objectively exits prior to the interaction while quantum physics is incompatible with measuring some unknown but pre-existing reality [1].

The measurement problem in quantum physics is describing how a superposition of many possible values becomes a single measured value, that is the system collapse. This is illustrated by a though experiment called the Schrödinger's cat experiment.

A cat is placed inside a box with a poison which is released if a radioactive atom decays. This is a quantum event that determines the state of the cat, which is a macroscopic object used as a measuring device. Thus, if the cat is alive when opening the box, the atom will not have disintegrated and if it is dead it will have. Nevertheless, before opening the box, the atom is in a quantum superposition of states: its excited and fundamental states, which evolve over time according to the Schrödinger equation. Therefore, the cat that interacts with the atom must also be in a superposition of states: alive and dead. Each of these possibilities will be associated to a specific probability amplitude. However, a single particular observation of the cat does not find a superposition state: the cat is always either alive or a dead. After opening the box, i.e., measurement, the cat is definitely dead or alive. So the point is interpreting how this probabilities turn into a real well-defined classic outcome.

Quantum mechanics has different interpretations and each one deals with the measurement problem its own way. The most popular one are the Copenhagen interpretation, the many-worlds interpretation, Bohm interpretation and the objective-collapse theories.

In this TFG we will deal with the measurement problem in quantum physics. First, in Sec. 2, a base is introduced in quantum mechanics key concepts that allow us to understand the rest of the reading. In Sec. 3 we continue to examine the main characteristics of the different types of measurement. In Sec. 4 we focus on another key point in quantum physics: entanglement, which manifests in the violation of the famous Bell inequality. Actually, it leads to one of the counter-intuitive behaviours that has still no solution and of which several interpretations are developed such as the many-worlds interpretation. In Sec. 5 we explain the type of measurement that we have chosen to be simulated and the scenarios to be measured. In Sec. 6 we show the results of the simulation when the system has not been measured and in Sec. 7 when it has. For the latter case, a study has also been made based on the measurement-time. In Sec. 8 all the simulation results are summed up. Finally, in Sec. 9 we try to summarize all the conclusions obtained from the study carried out in this TFG.

2 Quantum mechanics general framework.

Quantum theory combines physical reality and mathematical formalism. It provides us with a set of laws so that we can identify the different states of a quantum system, their evolution, and their measurement probabilities and outcomes over a set of macroscopic tests which have been prepared in a specific way. It is important to note that those probabilities are computed over an ensemble of preparations but not over a unique one. Before getting into the topic, we will refresh some key concepts such as the quantum mechanics postulates and Heisenberg's uncertainty principle.

Postulate 1: Quantum system states. A quantum system is defined by a Hilbert space, $\mathcal{H}$, which contains all the possible system states represented as $\mid \psi >\in \mathcal{H}$ with $< \psi \mid \psi >= 1$.

The simplest quantum system is the qubit which is defined by a two-dimensional vector space, $\{\mid 0 >, \mid 1 >\}$, with state vectors of the form:

$$\mid \psi >= \alpha_0 \mid 0 > + \alpha_1 \mid 1 > ; \quad |\alpha_0|^2 + |\alpha_1|^2 = 1, \tag{2.1}$$

where $\alpha_0, \alpha_1 \in \mathbb{C}$ are the transition amplitudes and let us compute the probability distribution of the state, $|\alpha_j|^2 = \mid < j|\psi > \mid^2$.

Note that two parallel state vectors that differ just on their phase represent the same physical state but are not equivalents. Experimentally, each vector state is a well-defined system preparation. When it comes to measurements what it is used is its probability distribution, $|\alpha_j|^2$, so a phase shift does not affect them. However, if we take a look at the transition amplitude from $|\alpha_j >$ to any other vector state $|\phi >$, that is $< \phi|\alpha_j >$, a phase shift does make the difference.

Postulate 2: System evolution. A closed quantum system evolves according to a unitary transformation, i.e., the same state vector, $\mid \psi >$, in two different times, t, t_0, is related by a unitary operator, $\mathcal{U}$, which depends only on those times.

$$\mid \psi(t) >= \mathcal{U}(t - t_0)\mid \psi(t_0) >; \quad \mathcal{U}\mathcal{U}^\dagger = \mathcal{U}^\dagger\mathcal{U} = \mathbb{1}. \tag{2.2}$$

An alternative formulation of this postulate is given by Schrödinger's equation, where $H = H^\dagger$ is the system's Hamiltonian:

$$i\hbar \frac{d \mid \psi >}{dt} = H \mid \psi > . \tag{2.3}$$

Solving that differential equation, we can connect both of them:

$$\mid \psi(t_1) >= \exp\left\{\frac{-iH(t_1 - t_0)}{\hbar}\right\} \mid \psi(t_0) >= U(t_1, t_0) \mid \psi(t_0) > . \tag{2.4}$$

Postulate 3: Quantum measurements. In general, the measurement process is represented by a set of measurement operators, $\{M_m\}$, which has m outcomes and satisfies the law of probability conservation,

$$\sum_m M_m^\dagger M_m = \mathbb{1}. \tag{2.5}$$

The probability of measuring m is given by Eq. (2.7) and its final state associated would be:

$$| \psi_m >= \frac{M_m \mid \psi >}{\sqrt{p(m)}}, \tag{2.6}$$

$$p(m) =< \psi \mid M_m^\dagger M_m \mid \psi > . \tag{2.7}$$

Although measurements are performed by classical devices, the results have a probabilistic intrinsic nature. That has to do with the intrinsic quantum system behaviour and not with imperfections on the preparations, nor the measurement process or the measurement apparatus themselves.

When we are measuring it is important to know whether the vector states that we want to distinguish are orthogonal or not. If they are then a set of observables can be defined in order to determine the state measuring with perfect reliability. However, if they are not then there is no way to achieve this purpose with a general measurement. For example in quantum computing information is codified using a quantum system by an emitter. It defines the system's initial quantum state which will be the message transmitted to a receptor. Eventually, the receptor will need to measure the system's state in order to decode it. If the transmitted messages are not orthogonal to each other, then the receptor will have limitations on determining the initial quantum state without ambiguity and here it is where error in communication arises.

Proof.

Let's take two vector states $|\psi_1 >$, $|\psi_2 >$ with $< \psi_1|\psi_2 >\neq 0$ and $< \psi_j|\psi_j >= 1$ for j = 1, 2. Now, we define a set of observable satisfying $< \psi_j \mid M_i^\dagger M_i \mid \psi_j >= \delta_{ij}$ for i,j = $\{1,2\}$ with $\sum_m = M_m^\dagger M_m = \mathbb{1}$. Next, we decompose $|\psi_2 >$ on a linear combination of a parallel state to $|\psi_1 >$ and an orthogonal one to it, $|\phi >$ (Eq. (2.8)) and compute explicitly $< \psi_2 \mid M_2^\dagger M_2 \mid \psi_2 >$.

$$|\psi_2 >= \alpha|\psi_1 > +\beta|\phi >; \quad |\alpha|^2 + |\beta|^2 = 1, \tag{2.8}$$

$$\begin{aligned} < \psi_2 \mid M_2^\dagger M_2 \mid \psi_2 > &=< (\alpha^* < \psi_1| + \beta^* < \phi|) \mid M_2^\dagger M_2 \mid (\alpha|\psi_1 > +\beta|\phi >) > \\ &= |\beta|^2 < \phi \mid M_2^\dagger M_2 \mid \phi >\leq |\beta|^2 \sum_i < \phi \mid M_i^\dagger M_i \mid \phi > \\ &= |\beta|^2 < \phi|\phi > \\ &= |\beta|^2 \\ &< 1. \end{aligned} \tag{2.9}$$

This way we have arrived to a contradiction of the initial hypothesis.

End of proof.

Heisenberg uncertainty principle. Associated to the measurement of two different observables, represented by their operators A and B, over an ensemble of identical states, $|\psi >$, there is an uncertainty that would depend on their commutator.

- If $[A, B] = 0$ then exits a common eigenstate basis with associated eigenvalues, that is, if we measure A on some particles and B on the others and compute their standard deviation we would get that both can be measured with total precision if the measurement device are good enough.

$$A|a_i, b_j >= a_i|a_i, b_j > . \tag{2.10}$$

$$B|a_i, b_j >= b_j|a_i, b_j > . \tag{2.11}$$

- If $[A, B] \neq 0$ and we measure A on some particles and B on the others and compute their standard deviation we would get that their product satisfies the following inequality called Heisenberg's uncertainty principle:

$$(\Delta A)(\Delta B) \geq \frac{| < \psi|[A, B]|\psi > |}{2}. \tag{2.12}$$

Postulate 4: Composite quantum system. The Hilbert space that holds the state vectors of a quantum system, which is composed of N systems with hamiltonians $\mathcal{H}_i$, is given by its tensor product. Thus, $\mathcal{H}_T = \mathcal{H}_1 \otimes ... \otimes \mathcal{H}_j \otimes ... \otimes \mathcal{H}_N$ and its state vector must be $| \psi_T >=| \psi_1 > \otimes ... \otimes | \psi_j > \otimes ... \otimes | \psi_N >$ [1], [2].

3 Type of measurements.

At postulate 3 we have introduced general measurements on quantum mechanics. From now on, we will focus on giving some light to their scopes and on how to perform them. We will start from projective measurements which are quite widespread and after that, we will introduce the mathematical formalism of positive operator-value measurements. Next, we will introduce the interaction with the measurement device by the weak measurements and the measurement time by the nondemolition measurements. Finally, we will explain the protective measurements which are not performed over an ensemble of identical particles but on a single one.

3.1 Projective measurements.

Projective measurements, quite often called strong measurements, are the most popular ones and require the definition of observables as hermitic operators, $M^\dagger = M$, which according to the spectral theorem can be discomposed on a set of orthonormal projectors, $\{P_m\}$, as follows:

$$M = \sum_m m P_m \tag{3.1}$$

where $P_m^2 = P_m = P_m^\dagger$. Besides, we can get a projective measurement from a general one just asking for the additional condition of:

$$M_{m'}^\dagger M_m = \delta_{m'm} M_m, \tag{3.2}$$

Then, using the properties Eq. (3.1) and Eq. (3.2) is easy to compute the expectation value of an observable as:

$$< M >= \sum_m m p(m) = \sum_m m < \psi \mid P_m^\dagger P_m \mid \psi >=< \psi \mid \sum_m m P_m \mid \psi >=< \psi \mid M \mid \psi > . \tag{3.3}$$

This gives us the following standard deviation formula associated to observations of M:

$$(\Delta M)^2 =< M \ - < M >>^2 =< M^2 > - < M >^2 . \tag{3.4}$$

On the other hand, a general measurement can be expressed in terms of projective measurements if (1) it is possible to add an auxiliary system to our current one, i.e., the Hilbert space can increase its dimension; and (2) any arbitrary unitary operation can be done on the total system space [2].

3.2 POVM, Positive Operator-Value Measurements.

Since the vast majority of the measurements performed in labs are destructive, POVMs focus not on the final system state but on getting information of it. POVMs are defined by a set of positive operators $\{E_m\}$ that satisfies $\sum E_m = \mathbb{1}$ and where $p(m) =< \psi \mid E_m \mid \psi >$.

The set $\{E_m\}$ is just a mathematical formalism that has no physical meaning and it may be discomposed, but not in a univocal way, on the form $E_m = M_m^\dagger M_m$ where $\{M_m\}$ is a general measurement and has its physical meaning.

This mathematical method is specially used when we are looking forward to distinguishing non-orthogonal states. It allows us to measure and depending on the outcome to determine the state with total reliability or not extract information at all [2].

Example.

Given the states $|\psi_1 >= |1>$, $|\psi_2 >= (|0> - |1>)/\sqrt{2}$ with $< \psi_1|\psi_2 > \neq 0$ and $< \psi_j|\psi_j >= 1$ for j = 1, 2. As we have concluded earlier it is impossible to distinguish them with total reliability, (Eq. (2.9)).

Now, we define a set of observable E_1, E_2, E_3 satisfying $\sum_m = E_m = \mathbb{1}$ that describes a POVM measure.

$$E_1 = \frac{\sqrt{2}}{1+\sqrt{2}} |0 >< 0|, \tag{3.5}$$

$$E_2 = \frac{\sqrt{2}}{1+\sqrt{2}} \frac{(|0 > + |1 >)(< 0| + < 1|)}{2}, \tag{3.6}$$

$$E_3 = \mathbb{1} - E_1 - E_2. \tag{3.7}$$

It is straight noticed that $< \psi_1 \, |E_1| \, \psi_1 >= 0$ and $< \psi_1 \, |E_2| \, \psi_1 >= 0$, so there is no probability of getting E_1 if we have the state $| \, \psi_1 >$ and the same can be said to E_2 and $| \, \psi_2 >$. Therefore, if we measure the outcome E_1, we are totally sure that the system state is $| \, \psi_2 >$. The identical reasoning is applied to measuring E_2 and knowing that the state will be $| \, \psi_1 >$. However, if E_3 is measured, then nothing about the system state can be said.

End of example.

3.3 Weak measurements.

Weak measurements reveal us information about the amplitudes of a quantum state. Instead of collapsing the state into eigenvectors of the observable operator and measuring a clear eigenvalue, the state is biased by a small angle and the measurement device shows a superposition of several eigenvalues.

First of all, we weakly couple the quantum system, $| \, \psi >$ (Eq. (3.8)), and the quantum measurement device, $| \, \phi_d >$ (Eq. (3.9)), using an interaction Hamiltonian , H_{int} (Eq. (3.10)), where $g(t)$ is the coupling impulse function between systems and T its coupling time.

The observable is given by $A = A^\dagger$ where $A \mid a_j >= a_j \mid a_j >$. The position and momentum operators associated to the detector are X_d, P_d where $X_d| \, x >= x| \, x >$, $[X_d, P_d] = i\hbar$ and the variable x represents the measuring needle position whose wave function distribution is a Gaussian around 0 with variance σ^2.

$$| \psi >= \sum_j \alpha_j | a_j >; \quad \alpha_j =< a_j | \psi > . \tag{3.8}$$

$$| \phi_d >= \int_x | x >< x | \phi > dx = \int_x \phi(x) | x > dx; \quad \phi(x) = (2\pi\sigma^2)^{-1/4} e^{-x^2/4\sigma^2}. \tag{3.9}$$

$$H_{int} = g(t) A \otimes P_d; \quad \int_0^T g(t) dt = 1, \tag{3.10}$$

with $g(t)$ any function that satisfies the above condition. Therefore, the evolution of the total system $| \Phi(t) >$ would be:

$$| \Phi(t) >= \exp\{-i/\hbar H_{int} t\} | \Phi(0) >= \exp\left\{-i/\hbar \int_0^t g(t) A \otimes P_d \;\; dt\right\} | \psi > \otimes | \phi_d >, \tag{3.11}$$

$$| \Phi(t) >= \sum_j \alpha_j \exp\left\{-i/\hbar \int_0^t g(t) a_j P_d \;\; dt\right\} | a_j > \otimes | \phi_d > . \tag{3.12}$$

Since the exponential term is a translational operator itself, it just translates the detector's distribution without deformation and generates an entanglement.

$$| \Phi(t) >= \sum_j \alpha_j | a_j > \otimes | \phi_d(x - a_j) > . \tag{3.13}$$

It is important to note that the variance, σ, controls the $| \phi_d(x - a_j) >$ overlapping and therefore, whether the measurement process will be weak or strong (see Fig. 1). The higher the variance, the weaker the measurement. After that, we measure the device. That will collapse the needle's device position, i.e., give an outcome vale, and bias the system's vector in the direction that corresponds to the needle's outcome value. For a measurement to be weak, the standard deviation of the measurement outcome should be larger than the difference between the eigenvalues of the system.

For computing the average of all eigenvalues, since the probability density to get the needle's position x is given by Eq. (3.14) and $|a_i - a_j| << \sigma$, we can approximate $p(x)$ to a normal distribution centered around the average of all eigenvalues (Eq. (3.15)).

$$p(x) = (2\pi\sigma^2)^{-1/2} \sum_j |\alpha_j|^2 e^{-(x-a_j)^2/2\sigma^2}, \tag{3.14}$$

$$p(x) \approx (2\pi\sigma^2)^{-1/2} e^{-(x - \sum_j |\alpha_j|^2 a_j)^2/2\sigma^2}. \tag{3.15}$$

Finally, if we prepare an ensemble of identical particles and we weakly measure part of them by its position operator, X, and the rest by its conjugate operator, P, we will obtain a lower bound by the variance of the detector's operator:

$$\Delta X \geq \Delta P_d, \tag{3.16}$$

$$\Delta P \geq \Delta X_d. \tag{3.17}$$

Since the detector performs a strong measurement, it satisfies the uncertainty principle (Eq. 2.12). Therefore, the variances of the particle's operators satisfy it too [3].

$$\Delta X \Delta P \geq \Delta X_d \, \Delta P_d \geq \hbar. \tag{3.18}$$

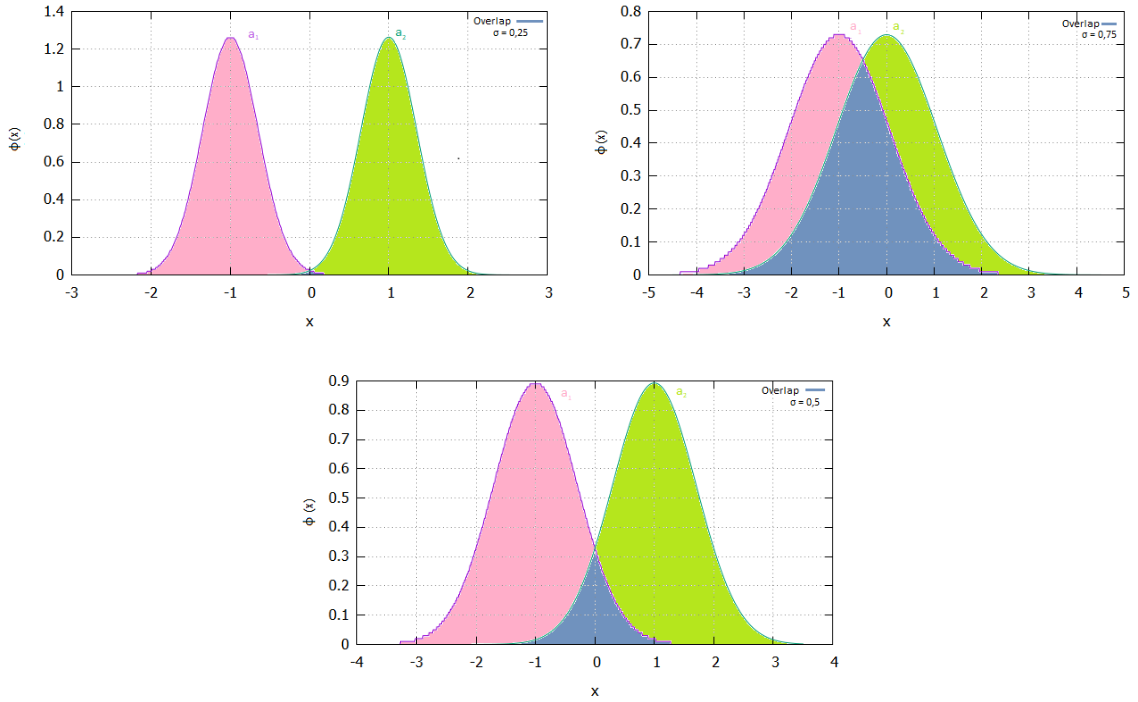

Figure 1: Overlapping depending on σ for fixed eigenvalues $a_1 = -1$ and $a_2 = 1$. Top-left: Strong measurement with $\sigma = 0.25$. System's vector collapses. Top-right: Weak measurement with over overlapping $\sigma = 0.75$. Vague information on the needle's outcome value. Bottom: Weak measurement with overlapping, $\sigma = 0.50$. Optimal weak measurement.

3.4 Nondemolition measurements.

As it is stated on Heisenberg's uncertainty principle (Eq. (2.12)) the more accurate is the measurement of one observable, A, the more unpredictable will be the measurement of the observable B if $[A, B] \neq 0$ and that is something intrinsic to the quantum system and inevitable that has to do with extracting information out of the particle. If all the information is reinserted to the particle with no trace of it left anywhere, that is, the backward process is perfect then the disturbance in the particle state can be undone.

Quantum Non Demolition (QND) methods give us a chance to circumvent the effects of the Heisenberg's uncertainty principle, that is, from demolishing the possibility of a second accurate measurement. We define a QND measurement of an observable $\hat{A}$ as a sequence of precise measurements of $\hat{A}$ such that the result of each one is completely predictable from the result of the first one.

Besides, we define QND observables as the ones that do not contaminate the free evolution of the system so they allow us to measure with high-precision but evade the back-action effect on the system. $\hat{A}$ is a QND observable if and only if $[\hat{A}(t_i), \hat{A}(t_j)] = 0$.

If it is satisfies $[\hat{A}(t_i), \hat{A}(t_j)] = 0$ for all times then $\hat{A}$ is a continuous QND observable but if it just satisfies it for some special ones, then it is called a stroboscopic QND observable. For instance, in a free particle energy and momentum satisfies that condition so they are continuous QND observables and in a harmonic oscillator position and momentum are stroboscopic QND observables.

In order to let the information enter on the measuring apparatus, its interaction Hamiltonian with the system must depend on $\hat{A}$ and $[\hat{A}(t), \hat{H_{int}}(t)] = 0$ guarantees no direct instantaneous back action of the measuring apparatus on $\hat{A}$. If $\hat{A}$ is the only system observable in $\hat{H_{int}}$, then there is neither direct nor no direct back action.

We can couple the system to a classical external force F(t) that can be inferred from $\hat{A}$(t) and then it is called QNDF observable. For a harmonic oscillator Hamiltonian, $\hat{H} = \frac{1}{2}m\omega^2\hat{X}^2$, by a sequence of stroboscopic measurements one can monitor a classical force $F(t) = F_0 \cos(\omega t + \alpha)$ using the quantum system. For this Hamiltonian if $\alpha = \pi/2, 3\pi/2$, then the optimal times for the stroboscopic measurements are $t = 0, \pi/\omega, 2\pi/\omega, ...$ On the other hand, if $\alpha = 0, \pi$, then the optimal times for the stroboscopic measurements are $t = \pi/2\omega, 3\pi/2\omega, ...$ [4, 5, 6].

3.5 Protective measurements.

In general, expectation values are statistical properties obtained of a N identical particle ensemble. In this particular situation the expectation value of an observable can be found performing just a measurement over a single photon. A photon polarized with angle θ may be written as $|\psi_\theta >= \cos(\theta)|0 > + \sin(\theta)|1 >$ and the polarization operator is defined as $P = |0 >< 0| - |1 >< 1|$. In order to avoid the state collapse after measuring, we need to protect it. There are two protection methods: the active and the passive one. We will focus on the active one through the Zeno effect, that is totally effective only in the ideal case. Zeno effect let us freeze the unitary time evolution of an initial quantum state by for instance measuring it frequently enough.

If we measured the expectation value of the polarization of a photon polarized with angle $\pi/6$, $|\psi_{\pi/6} >= \frac{\sqrt{3}}{2}|0 > + \frac{1}{2}|1 >$, we would get $< P >= 0.5$ (Eq. (3.19)).

$$< \psi_{\pi/6}|P|\psi_{\pi/6} >= \frac{3}{4} - \frac{1}{4} = 0.5. \tag{3.19}$$

In figure 2 we represent the outcome that would give protective measurements (in red) versus projective ones (in blue) over the same ensemble of identical states $|\psi_{\pi/6}$. For the first one there is an unique outcome distribution around the observable expectation value, $< P >= 0.5$, for all the measurements over protected photons. For the second one there are two probability distributions around the eigenvalues ± 1 for non-protected photons [7].

4 Entangled systems.

The aim of measuring a system is clearly to get information of it. If the initial state is known then we will probably try to define its physics properties and vice versa. In Quantum Physics there are two kinds of states: separable and entangled. Separable states are pure states of composite systems that can be factorized as a product of states associated with each subsystem but entangled cannot. This intrinsic characteristic of entangled states shows surprising results [8].

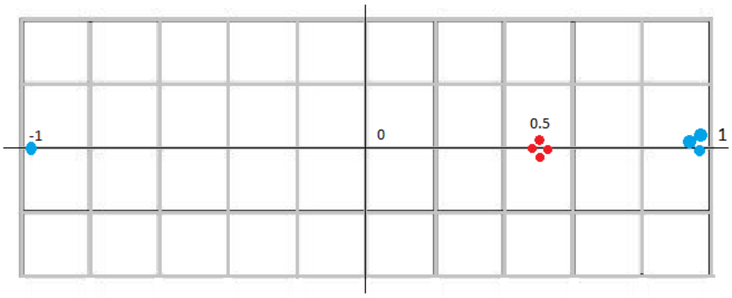

Figure 2: In red, protective measurement distribution about the polarization expectation value for $\psi_{\pi/6}$, $<P>= 0.5$. In blue, projective measurement distributions about the polarization eigenvalues ± 1.

4.1 Bell inequality.

In EPR article [9] it was first pointed out that entangled system properties were not described well enough by quantum mechanics and in 1964 Bell published its local hidden variable model (LHVM) based on the ideas of realism, locality and free will [10]. They led him to an inequality for the statistics correlations on measurements of a bipartite system.

For a bipartite system with dichotomic observables (A_1, A_2) over the subsystem A and (B_1, B_2) over the subsystem B, the inequality reads as follows:

$$\| < A_1 B_1 > + < A_1 B_2 > + < A_2 B_1 > - < A_2 B_2 > \| \leq 2. \tag{4.1}$$

For instance, we can choose $A_1 = \sigma_x^1$, $A_2 = \sigma_z^1$, $B_1 = -\frac{1}{\sqrt{2}}(\sigma_z^2 + \sigma_x^2)$ and $B_2 = \frac{1}{\sqrt{2}}(\sigma_z^2 - \sigma_x^2)$, where σ_x, σ_y, σ_z are the Pauli's matrices. Next, we will compute those quantities for a separable state, $\psi_{sep} = |01>$, and an entangled one, $\psi_{ent} = \frac{1}{\sqrt{2}}(|01> - |10>)$. So then, we will need to compute over the two different systems Eq. (4.2). In Eq. (4.3) an example is done in order to illustrate the operational method followed.

$$\frac{1}{\sqrt{2}} \left\| - < \sigma_x^1(\sigma_z^2 + \sigma_x^2) > + < \sigma_x^1(\sigma_z^2 - \sigma_x^2) > - < \sigma_z^1(\sigma_z^2 + \sigma_x^2) > - < \sigma_z^1(\sigma_z^2 - \sigma_x^2) > \right\|, \tag{4.2}$$

$$< 01| \frac{1}{\sqrt{2}} \sigma_x^1(\sigma_z^2 - \sigma_x^2)|01 >= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0. \tag{4.3}$$

For the separable system the inequality is satisfied, since:

$$\left\| 0 + 0 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right\| = 0 \leq 2. \tag{4.4}$$

For the entangled system the inequality is violated, since:

$$\left\| \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right\| = 2\sqrt{2} > 2. \tag{4.5}$$

The inequality violation is due to the entanglement. That is why computing this inequality can be used as a first filter for classifying quantum states once we have fixed the dichotomic observables [10].

4.2 Wigner's friend experiment.

Wigner's friend is a thought experiment related with the measurement problem. We consider an observer ("the friend") who performs measurements on a quantum system inside an isolated lab. The system is in a superposition state so when it is measured by the friend it collapses. Besides, there is an external observer, Wigner, who also performs measurements on the lab and everything that is inside of it and controls them coherently. When his friend has performed the measured for Wigner the superposition still remains. This seems to lead to a contradiction between observer's perspectives.

The extended Wigner's friend scenario (EWFS, Fig. 3) considers the bipartite of the Wigner's friend experiment, introducing two superobservers: Alice and Bob, and their respective friends: Charlie and Debbie with their own labs. Each friend has one particle from an entangled pair and makes a measurement on it. Assuming freedom of choice,

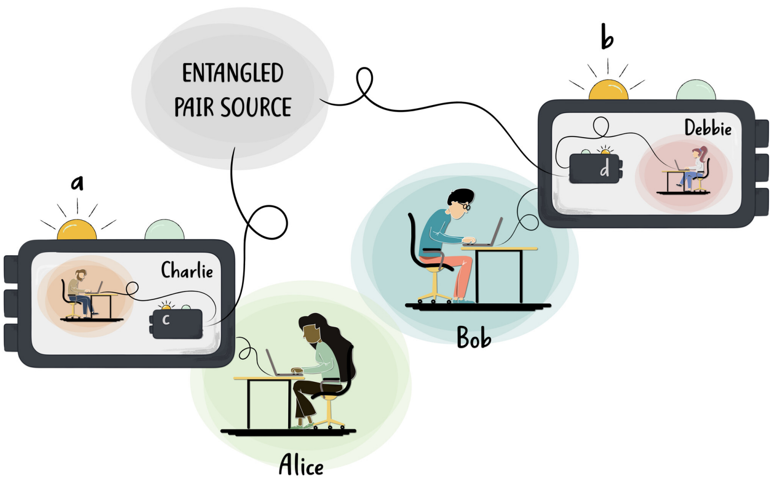

Figure 3: Extended Wigner's friend scenario [11].

locality and observer-independent facts, Brukner [11] derived a Bell inequality which could be violated in quantum mechanics and a recent six-photon experiment has proved it. Later, assuming Local Friendliness which includes No-Superdeterminism, Locality and Absoluteness of Observed Events (AOE) an experiment has been performed.

1. No-Superdeterminism: any set of events on a space-like hypersurface is uncorrelated with any set of freely chosen actions subsequent to that space-like hypersurface.

2. Locality: The probability of an observable event is unchanged by conditioning on a space-like-separated free choice.

3. Absoluteness of Observed Events: An observed event is a real single event, and not relative to anything or anyone.

Conclusions obtained in the experiment were independent of the quantum state and the measurements of it since the inequalities are independent of the measurement apparatus. The violation of the inequalities implies neglecting some of the LF assumptions and that is actually, the study line of some quantum theories including:

- QBism, relational interpretation and many-worlds interpretation neglect absoluteness of observed events.
- Bohmian Mechanics violates locality.
- Retrocausality, superdeterminism and other mechanics theories neglects no-superdeterminism [12].

5 Computational simulation algorithm.

In this section we will put into practice all the knowledge acquired so far with the goal of designing a simulation of a nondemolition measurement on a bipartite system. In particular, we have found them interesting because if we choose the measurement time wisely, we can obtain not only the eigenvalue of the observable measured over one of the systems, but also keep the expectation value temporal evolution of both systems invariant. We will study it for different initial system states and Hamiltonians in order to check how the system would evolve if the measurement were not a nondemolition one.

I have developed a code in order to study nondemolition measurements of a separable two qubit system, $|\psi_{sep}>$, and an entangled one, $|\psi_{ent}>$, evolving them according to Schrödinger's equation with a separable, H_{sep}, and an entangled Hamiltonian, H_{ent}, during fifty seconds. Besides, I have computed Bell inequalities in order to test that its implementation was right.

- Initial states:

$$|\psi_{sep}>=\frac{1}{2}(|0>+|1>)_1\otimes(|0>+|1>)_2=\frac{1}{2}(|00>+|01>+|10>+|11>), \tag{5.1}$$

$$|\psi_{ent}>=\frac{1}{\sqrt{3}}(|00>-|10>-|11>). \tag{5.2}$$

- Interaction Hamiltonians:

$$H_{sep}=\hbar\omega(\sigma_z^1+\sigma_z^2), \tag{5.3}$$

$$H_{ent}=\hbar\omega(\sigma_z^1\otimes\sigma_z^2). \tag{5.4}$$

To calculate the time evolution of the system, I have used Euler's algorithm (ec. 5.5) with $\Delta t = 0,0001$ s. I have measured periodically X_1 and computed the expectation values $<\sigma_x^1>, <\sigma_x^2>, <\sigma_z^1>$ and $<\sigma_z^2>$ every 0,001 s.

$$|\psi(t_0+\Delta t)>=\exp\left(-\frac{iH\Delta t}{h}\right)|\psi(t_0)>\approx(1+i/hH\Delta t)|\psi(t_0)>. \tag{5.5}$$

For the observables associated to the operators σ_x and σ_z we define its eigenstates as $|\sigma_x^{\uparrow}>= \frac{1}{\sqrt{2}}(|0>+|1>)$, $|\sigma_x^{\downarrow}>= \frac{1}{\sqrt{2}}(|0>-|1>)$, $|\sigma_z^{\uparrow}>=|0>$ and $|\sigma_z^{\downarrow}>=|1>$, having both of them eigenvalues ± 1 as specified:

$$\sigma_x|\sigma_x^{\uparrow}>=|\sigma_x^{\uparrow}>, \qquad \sigma_x|\sigma_x^{\downarrow}>=-|\sigma_x^{\downarrow}>\,. \tag{5.6}$$

$$\sigma_z|\sigma_z^{\uparrow}>=|\sigma_z^{\uparrow}>, \qquad \sigma_z|\sigma_z^{\downarrow}>=-|\sigma_z^{\downarrow}>\,. \tag{5.7}$$

Both Hamiltonians have an oscillating frequency, ω, that will determine the period of $<\sigma_x^1>,<\sigma_x^2>$ oscillations. Once we have estimated it, we fix this parameter on our simulation and run two different measurement processes: measuring the system every $T/4$ and every $T/2$. Additionally, we will fix a quantum system and an evolution Hamiltonian and will modify the measurement-time in the range $T \in [T/4, T/2]$. We will make special attention to their neighbourhood, e.g., $T/4 + T/30$ and $T/2 - T/30$. Each measurement process will be run for an ensemble of $N = 10000$ identical systems so that we can get the statistics with a standard deviation of $1/\sqrt{N}$.

The code developed in this TFG can be found at this link.

6 System expectation value evolution with no measurements.

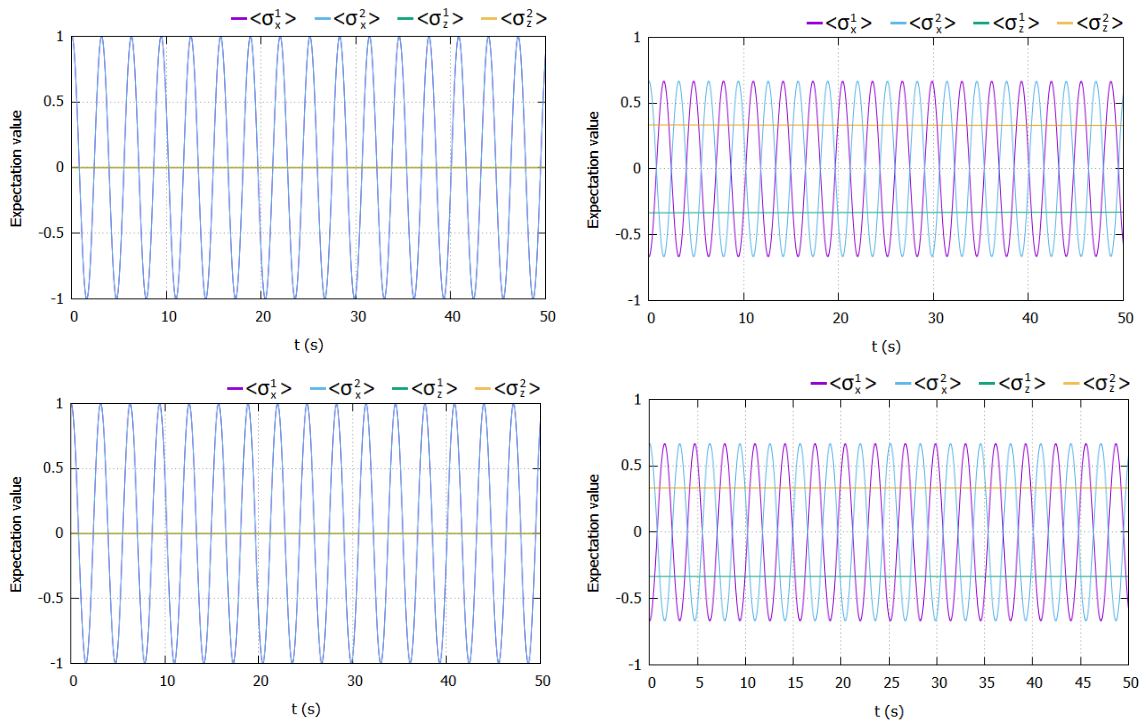

Figure 4: Expectation value evolution without measurement being performed for different systems and Hamiltonians. Top-left: H_{sep}, $|\phi>_{sep}$. Top-right: H_{sep}, $|\phi>_{ent}$. Bottom-left: H_{ent}, $|\phi>_{sep}$. Bottom-right: H_{ent}, $|\phi>_{ent}$.

In figure 4, the expectation values of $<\sigma_x^1>,<\sigma_z^1>,<\sigma_x^2>,<\sigma_z^2>$ are plotted for t $\in$ [0,50] s for the four system and evolution situations without measurement being performed.

If there is no initial interaction between qubits, that is, if the system is separable (Fig. 4 (top-left) and Fig. 4 (bottom-left)), then $<\sigma_x^1>$, $<\sigma_x^2>$ oscillate independently for each qubit between its eigenvalues -1 and 1. On the other hand, $<\sigma_z^1>$ and $<\sigma_z^2>$ values are constant with time and equal to zero. However, if qubits initially are linked since the system is entangled (Fig. 4 (top-right) and Fig. 4 (bottom-right)), then $<\sigma_x^1>$, $<\sigma_x^2>$ keep oscillating but have a different amplitude and phase. $<\sigma_z^1>$ and $<\sigma_z^2>$ are constant but take different values. Hence, when no measurement is performed, the evolution with time for a well-defined system is determined by its Hamiltonian and its initial state as it has been stated in Postulate 2 (Eq. (2.3)). Finally, we estimate the oscillation period that depends only on the Hamiltonian frequency which we have fixed to $\hbar\omega = 1$ in order to get a time reference interval to perform the measurements. Hence, for all the cases being analysed the period is $T = 3.142$ s.

7 System expectation value evolution measuring σ_x^1 periodically.

As it has been explained, we have measured the system with different measurement-times in order to determine the optimal one, i.e., the one that perturbs the system the less. This measurement method let us get information of the observables without dramatically modifying their expectation value time evolution so that we can keep measuring it.

7.1 No disturbance measurement-time.

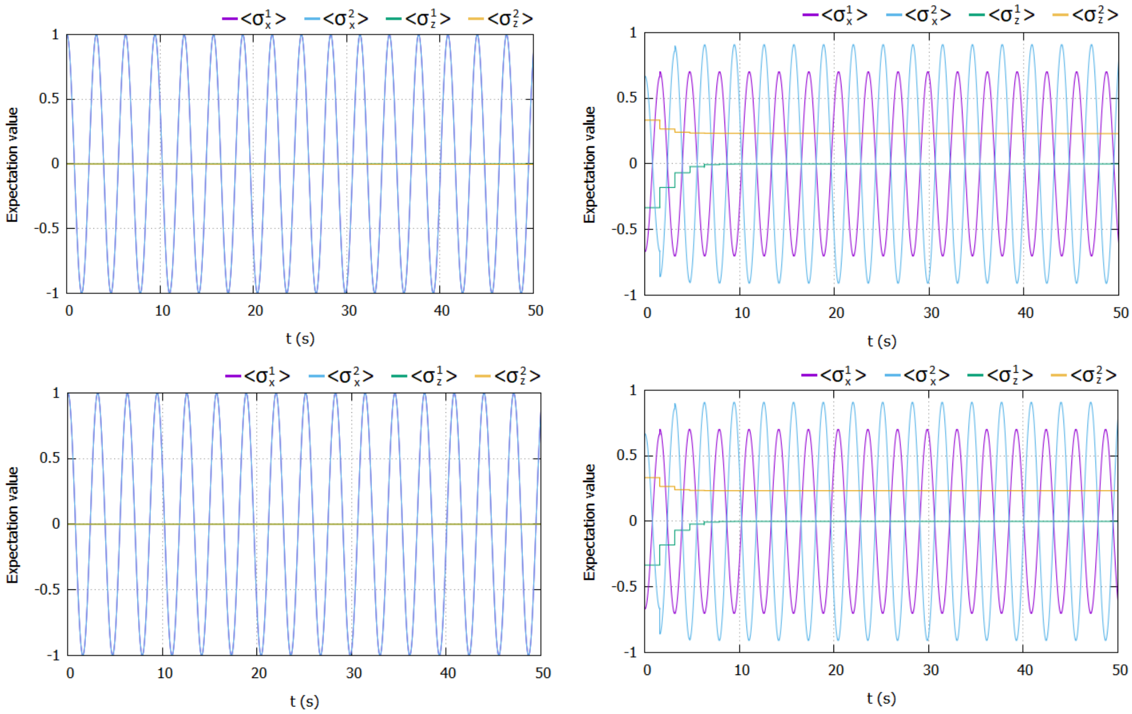

Figure 5: Expectation value evolution with measurement being performed each T/2 for different systems and Hamiltonians. Top-left: H_{sep}, $|\phi>_{sep}$. Top-right: H_{sep}, $|\phi>_{ent}$. Bottom-left: H_{ent}, $|\phi>_{sep}$. Bottom-right: H_{ent}, $|\phi>_{ent}$.

If we set the time measurement-time to T/2, and the initial state is separable then for both the separable Hamiltonian (Fig. 5 (top-left)) and the entangled one (Fig. 5 (bottom-left)) the system will be measured when $<\sigma_x^1>$, $<\sigma_x^2>= \pm 1$. As these are their operator eigenvalues, the system has a high probability of being on the eigenstate associated to that eigenvalue. That is why when we measure the system, they will collapse to the eigenstate associated to the eigenvalue that we have measured so we will expect the system disturbance to be minimal.

Following the same reasoning, if the initial state is entangled then the system will be measured when $<\sigma_x^1>$, $<\sigma_x^2>= \pm 2/3$ for both the separable Hamiltonian (Fig. 5 (top-right)) and the entangled one (Fig. 5 (bottom-right)). Since these are not the operator eigenvalues the system collapses to the eigenstate associated to the eigenvalue measured. This breaks the entanglement and the system becomes separable so from this moment it will have the same behaviour as explained before for the separable one. Once again we expect the system disturbance to be minimal.

As we can see on simulations, for a T/2 measurement-time the expectation value evolution has not changed in any case but for the first measurement on the entangled system, that is, measuring the system each T/2 seconds has exactly the same expectation value time evolution as no measuring it at all. In this particular case the collapse does not dramatically modify the qubit states.

For $<\sigma_z^1>$, $<\sigma_z^2>$ we see that for an initial separable system the value remains unalterable but for an initial entangled one it decreases in module, as we measure, until it reaches a constant value with time: both of them zero when the system is separable, and a negative and a positive one, respectively, when the system is entangled.

7.2 Maximal disturbance measurement-time.

On the other hand, if we set the measurement-time to T/4 then the disturbance is total in the first qubit and depending on the Hamiltonian and the initial state it will have effect on the second qubit or not.

Initially $<\sigma_x^1>$ oscillates as no measurements have been performed yet. The first measurement comes at T/4 when $<\sigma_x^1>= 0$ so at this time both σ_x^1 eigenstates are almost equally probable. On an identical system ensemble statistics $<\sigma_x^1>$ amplitude will decrease exponentially until it reaches zero. If we keep computing $<\sigma_x^1>$, no new information will be obtained.

Separable Hamiltonian and separable system. In figure 6 (top-left) we can see the $<\sigma_x^1>$ oscillations and its amplitude exponentially decreases as it has been predicted. Since the system and the Hamiltonian are separable, measuring qubit one does not affect qubit two. That is why $<\sigma_x^2>$ oscillates for all times and $<\sigma_z^2>$ remains constantly equal to zero as if the system was not have been measured. Finally, $<\sigma_z^1>$ decreases quicker than $<\sigma_x^1>$ until it reaches a negative constant value.

Separable Hamiltonian and entangled system. In figure 6 (top-right) we can see again the $<\sigma_x^1>$ oscillations and its amplitude exponentially decreases as it has been predicted. Since the system is entangled and the Hamiltonian is separable, measuring qubit one does not affect qubit two. As said for T/2 we just need to consider the first measurement that modifies the oscillation's amplitude but the evolution is analogous from this moment. The same results are applied to $<\sigma_z^2>$ so its behaviour is equal

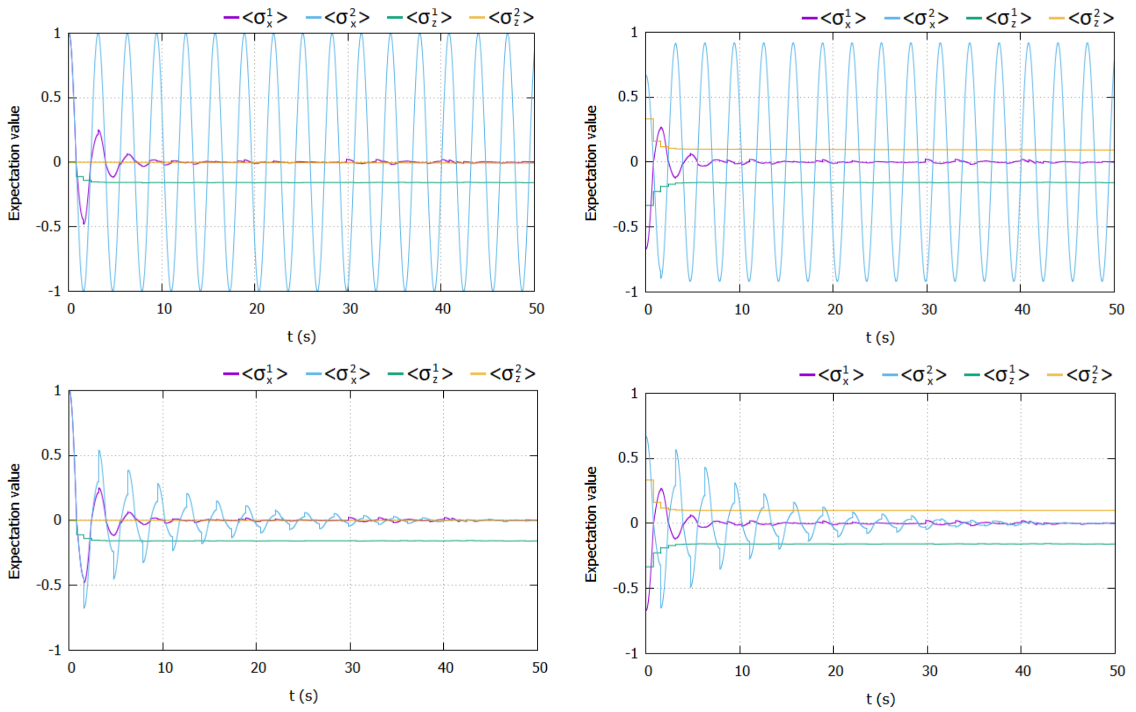

Figure 6: Expectation value evolution with measurement being performed each T/4 for different systems and Hamiltonians. Top-left: H_{sep}, $|\phi >_{sep}$. Top-right: H_{sep}, $|\phi >_{ent}$. Bottom-left: H_{ent}, $|\phi >_{sep}$. Bottom-right: H_{ent}, $|\phi >_{ent}$.

to the one described at $T/2$. Qubit two perceives no measure effect. Finally, $< \sigma_z^1 >$ increases quicker than $< \sigma_x^1 >$ decreases until it reaches a negative constant value.

Entangled Hamiltonian and separable system. In figure 6 (bottom-left) we can see the $< \sigma_x^1 >$ oscillations and its amplitude exponentially decreases as it has been predicted. Now, since the system is separable and the Hamiltonian is entangled, measuring qubit one affects qubit two. That is what makes $< \sigma_x^2 >$ amplitude decreases to zero. As we can see the decaiment is quicker for $< \sigma_x^1 >$ than for $< \sigma_x^2 >$. That is because qubit one is been directly measured and qubit two is the one that suffers the effect of the system being measured. Finally, $< \sigma_z^1 >$ decreases quicker than $< \sigma_x^1 >$ until it reaches a negative constant value and $< \sigma_z^2 >$ remains constantly equal to zero.

Entangled Hamiltonian and entangled system. In figure 6 (bottom-right) we can see the $< \sigma_x^1 >$ oscillations and its amplitude exponentially decreases as it has been predicted. Now, since the system is entangled and the Hamiltonian too, measuring qubit one affects qubit two evolution. That is what makes $< \sigma_x^2 >$ amplitude decrease to zero too. As we can see the decaiment is quicker for $< \sigma_x^1 >$, which is the qubit that is been directly measured, than for $< \sigma_x^2 >$, which is the one that suffers the effect of the measured.

7.3 Disturbance dependence on the measurement-time.

At this point, we choose to study the expectation value evolution for the entangled system with entangled Hamiltonian. We have discussed before the expectation value evolution for a measurement-time of T/2 and T/4 and now, we will focus on its evolution about T/4 and T/2.

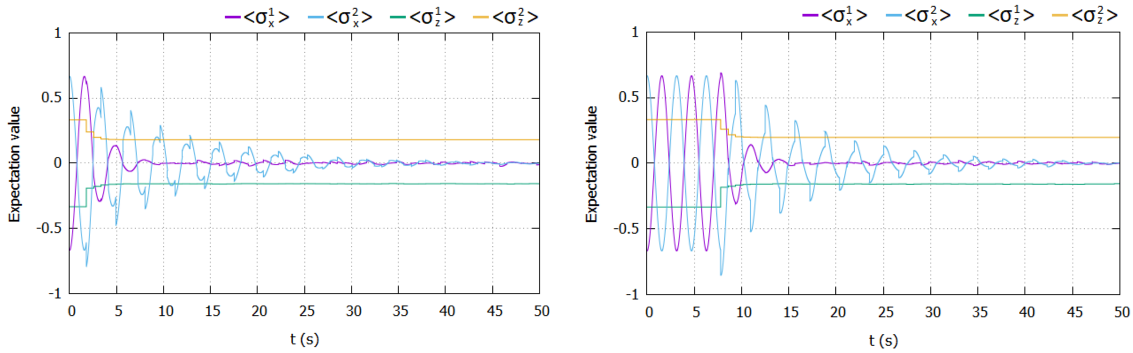

Figure 7: Expectation value evolution for H_{ent}, $|\phi>_{ent}$ being measured. Left: Measurement each T/4+T/30. Right: Measurement each T/2-T/30.

For T/4 (Fig. 6 (bottom-right)) the dumping is maximum and the amplitude decreases quicker than for any other measurement-time. At $t = 7$ s, $< \sigma_x^1 >$ oscillates no more. It takes a little more to $< \sigma_x^2 >$ in order to be totally dumped but at $t = 35$ s it is. As we increase the measurement-time, T/4 + T/30 (Fig. 7 (left)), their oscillating character last a second more before it gets totally dumped again, so at the end of the day the behaviour would be the same as for T/4. For a greater measurement-time, T/2-T/30 (Fig. 7 (right)), that is about T/2, we clearly distinguish two regions: the one which has the behaviour of the expectation value evolution at T/2 (Fig. 5 (bottom-right)) and the one which has the one of T/4. Its range will depend on how close we are to T/2. The closer we are, the oscillations will last the most. That's due to the fact that at $t \to \infty$ all the displacements from T/2, that is, all the T/30 will sum up and its contribution will be relevant for the system and will make it collapse to the T/4 expectation value evolution.

8 Simulation conclusions.

In a measurement-time of T/2 it is obtained that when measuring $< \sigma_x^1 >$ over a separable or entangled system with a separable or entangled Hamiltonian, the evolution of the expectation values $< \sigma_x^1 >, < \sigma_z^1 >, < \sigma_x^2 >$ and $< \sigma_z^2 >$ remains the same as if the system had not been measured. For T/4 $< \sigma_x^1 >$ oscillations get dumped and the behaviour of $< \sigma_x^2 >$ allows us to distinguish between an entangled or a separable Hamiltonian since if it is separable then it continues oscillating but if it is not then the amplitude decays to zero as quick as far we are from T/2. The expectation values of σ_z end up getting a constant value over time in all cases. The initial state and the Hamiltonian only determine whether this value will be greater than, less than, or equal to zero. Therefore, it gives us not much information about the evolution. Finally, if we fix a state and modifies the measurement-time in the range $T \in [T/4, T/2]$, as we increase T, the oscillations will dump later and for $T/2$ they are not dumped at all. Therefore, it can be concluded that in the case of wanting to extract information from a system without modifying its expectation values, we must take into account the frequency of oscillation that characterizes its Hamiltonian to set the measurement-time according to it to $T/2$. In the case of not achieving this, the $< \sigma_x^1 >$ dumping will appear earlier as we are closer to $T/4$, i.e., further from $T/2$, and will affect $< \sigma_x^2 >$ oscillations if the Hamiltonian is entangled.

9 Final conclusions.

This TFG focuses on the theoretical and practical analysis of the measurement problem in quantum physics.

In the theoretical field, we have begun by recalling concepts of quantum mechanics, such as its postulates and the uncertainty principle, which are crucial to the correct understanding of the problem. Next, an investigation was carried out on the types of measurements which focus on the computation of the expectation value of an observable, on the measurement of an observable or on the determination of the initial quantum state. Thus, projective measures, POVMs, weak measurements, nondemolition measurements, and protective measurements were found of special interest. In general lines, projective measurements are the most common and widespread in quantum physics. In them, when measuring an observable, an eigenvalue of its associated operator is obtained as a result and the system collapses to its associated eigenstate. Secondly, POVMs are an excellent mathematical tool in order to solve the problem of indistinguishable non-orthogonal states. Thus, in some cases it reports with absolute certainty the state of the system and in others it cannot provide any information about it, depending on the measured result. On the other hand, weak measurements take advantage of the interaction between the system and the measuring device to avoid the collapse that characterizes projective measurements and to keep a system state as close as possible to the initial one. They try to know the transition amplitudes of the quantum state and this has the cost of losing information about the value of the observable. They just obtain an average value of its eigenvalues. Next, nondemolition measurement's scope is to define an optimal measurement-time based on the Hamiltonian of the system. This time is defined as the one in which measuring disturbs the quantum state the less. Therefore, it allows us to make predictions about the result of the measurement of the observable along the time. Finally, protective measure goal is to protect the polarization state of a photon when it is measured. This way we are able to obtain the expectation value of its polarization with a single measurement on it.

Once we have known various types of measurements that could be simulated, we focused on the study of the different types of quantum states. According to their behaviour they are divided into: separable, the state of the total system can be expressed as the tensor product of several subsystems, and entangled states, it cannot. Due to this fact some of the theoretical predictions made before the incorporation of entanglement to quantum theory were violated, for example, Bell's inequality. Besides, it gave rise to counter-intuitive behaviours like the Wigner's friend experiment, which is still studied today, emerging new hypotheses about locality, superdeterminism and absoluteness of observed events. Extended versions of this one that incorporate relativity and the transmission of information between observers have been proposed recently and it has given rise to many physical currents such as QBism, the many-worlds interpretation and retrocausality.

Regarding the practical field, we have focused on developing an own code that implements the nondemolition measurements. This has allowed us to study different scenarios and draw conclusions based on the measurement-time on them. Before, the importance of entanglement in quantum theory was already highlighted. Therefore, four types of systems and evolutions were chosen combining the two possibilities that were available: separable or entangled initial system and separable or entangled evolution. The evolution was determined in all cases by the Hamiltonian to which a characteristic period could be associated. First, we studied it without carrying out measurements on the first subsystem. Then, for a measurement-time in the interval of a quarter of the period and half of the period, it was studied how the disturbance of the state of the system changed. It was obtained that for a measurement-time equals to half the Hamiltonian period, the dynamics of the first subsystem was not modified by the measurement and neither was that of the second subsystem, except for the collapse of the entangled system to the separable after the first measurement in both Hamiltonians. On the contrary, for a measurement-time equals to a quarter of the period, the disturbance was maximal and the information on the state of the first subsystem was lost exponentially if the measurement process continued. In the latter case, if the Hamiltonian was entangled, then information about the state of the second subsystem was also lost exponentially, although it had not been measured directly. Therefore, it was concluded that the nondemolition measurement for the four scenarios studied corresponded to a measurement-time of half the Hamiltonian period.

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