Scuola Normale Superiore

Classe di Scienze

Corso di perfezionamento in Matematica

XXXIII Ciclo

Universidad de Granada

Programa de doctorado en Física y Matemáticas

(B04.56.1)

CURVATURE PRESCRIPTION PROBLEMS ON MANIFOLDS WITH BOUNDARY

Tesi di perfezionamento in Analisi Matematica

Candidato

dr. Sergio Cruz Blázquez

Direttore di tesi Scuola Normale Superiore
prof. Andrea Malchiodi

Direttore di tesi Universidad de Granada
prof. David Ruiz

Progetto finanziato dalla borsa di studio Marie Sklodowska-Curie dell'Istituto Nazionale di Alta Matematica n. 713485.

Anno Accademico 2020/21

Editor: Universidad de Granada. Tesis Doctorales
Autor: Sergio Cruz Blázquez
ISBN: 978-84-1306-931-9
URI: http://hdl.handle.net/10481/69647

A mis padres, Ana María y Juan,
y al resto de personas que creyeron en mí
cuando ni siquiera yo lo hacía.

Contents

Chapter 1

Introduction and objectives

This thesis addresses the study of two semilinear elliptic problems that arise in Riemannian Geometry. More precisely, we are interested in the prescription of certain geometric quantities on Riemannian manifolds with boundary under conformal changes of the metric, namely, the Gaussian and geodesic curvatures on a compact surface and its boundary, and the scalar and mean curvatures on a manifold of higher dimension.

Most of the results available in the literature concern closed manifolds, whereas the boundary cases have been less considered. In that regard, we highlight that the presence of the boundary leads to a wider variety of phenomena, many of which find no counterpart on the closed versions of these problems. In particular, the variational approach in Chapter 4, and the compactness and existence arguments of Chapter 5 are strictly related to the presence of boundary.

Furthermore, the focus of our research concerns the case in which *both curvatures are nonconstant*, for which there are only a few known results.

These problems admit a variational structure, so we will discuss the existence of solutions from the point of view of the Calculus of Variations. Sometimes the energy functionals considered here are bounded from below and a minimizer can be found; in other cases, though, this is not possible, and the use of min-max theory is needed. In the latter situation we are led to the blow-up analysis of solutions of approximated problems.

The work developed in this thesis has given rise to two research papers, [31] and [32].

1.1 Motivation of the problems

The main purpose of this thesis is to contribute in the deepening of our understanding of the properties of conformal classes of metrics in Riemannian manifolds. We refer to [6] for the basics on this topic.

The most classical example in this field is the *Uniformization Theorem*, which finds its origin in the search for *canonical metrics* on a given surface via conformal transformations. It was conjectured by Klein and Poincaré in [62, 85], and states that every simply connected Riemannian surface is conformally equivalent to one of three model spaces: $\mathbb{R}^2$, $\mathbb{S}^2$ or $\mathbb{H}^2$. The result was proved by Koebe and Poincaré using complex analysis in the works [63, 64, 65] and as a consequence we have that every compact and oriented Riemannian surface admits a conformal metric with constant Gaussian curvature.

At this point, one may ask the following question: *given a compact closed Riemannian surface (Σ, g) and a function $K(x)$ defined on Σ, can we find a conformal metric $\tilde{g} \in [g]$, such that its Gaussian curvature is equal to K?* This problem is known as the *prescribed Gauss curvature problem*, and was proposed by Kazdan and Warner in [60]. If we let $\tilde{g}$ be given by $\tilde{g} = e^u g$, it is known that the Gaussian curvatures K_g and $K_{\tilde{g}}$ satisfy the relation

$$-\Delta_g u + 2K_g = 2K_{\tilde{g}} e^u \quad \text{on } \Sigma, \tag{1.1}$$

where Δ_g stands for the Laplace-Beltrami operator on (Σ, g). Therefore, prescribing Gaussian curvature $K(x)$ reduces to solve (1.1) for $K = K_{\tilde{g}}$:

$$-\Delta_g u + 2K_g = 2K e^u \quad \text{on } \Sigma. \tag{1.2}$$

Integrating (1.2) on Σ and applying the Gauss-Bonnet theorem, we realise that this is not always possible, since there is a topological constraint: the sign of K is conditioned by that of the Euler characteristic of the surface, $\chi(\Sigma)$;

$$\int_\Sigma K e^u = \int_\Sigma K_g = 2\pi\chi(\Sigma). \tag{1.3}$$

So far, only the cases $\chi(\Sigma) = 0$ and $\chi(\Sigma) = 1$ have been completely understood, see [60, 78, 79]. An especially delicate case is the so-called *Nirenberg's problem*, $\Sigma = \mathbb{S}^2$, because of the effect of the non-compact group of conformal transformations of the sphere. In this case, there are other well-known obstructions to the existence of solutions besides (1.3), such as the Kazdan-Warner conditions detailed in [60]. To be precise, if u solves (1.2) in $\mathbb{S}^2$ and $x_i : \mathbb{S}^2 \to \mathbb{R}$ is the restriction to the sphere of a coordinate function in $\mathbb{R}^3$, then

$$\int_{\mathbb{S}^2} \nabla K \cdot \nabla x_i e^{2u} dV_{g_{\mathbb{S}^2}} = 0. \tag{1.4}$$

This forbids, for example, the prescription of curvature functions that are affine or generally functions that are monotone in one Euclidean direction. More examples were given in [18].

There is a large amount of literature about the Nirenberg's problem, and many sufficient conditions for the existence of solutions are available. For example, in [79], Moser showed that it is possible to prescribe antipodally symmetric curvatures on $\mathbb{S}^2$, and motivated the study of (1.2) under symmetry assumptions. Existence results without symmetries were also obtained in [23]. In this work, the authors proved the following existence result:

Theorem. *[23, Th. II] Let K be a positive, smooth function with only nondegenerate critical points. Suppose that there are at least two local maxima of K, and suppose that at all saddle points q of K, we have $\Delta_{g_{\mathbb{S}^2}} K(q) > 0$. Then K is the Gauss curvature of a conformal metric $g = e^{2u} g_{\mathbb{S}^2}$ on $\mathbb{S}^2$.*

Struwe demonstrated in [89] that the previous result is sharp to a certain extent, giving examples of functions K, having exactly two local maxima and a saddle point q_0 with $\Delta_{g_{\mathbb{S}^2}} K(q_0) < 0$, that cannot be the Gaussian curvature of a conformal metric in $\mathbb{S}^2$. Moreover, the author recovers the existence result by Chang and Yang by means of the prescribed curvature flow. This geometric flow is determined by the parabolic evolution equation:

$$\frac{du}{dt} = \alpha K - K_g, \tag{1.5}$$

where g is the family of conformal metrics $g(t) = e^{2u(t)} g_{\mathbb{S}^2}$, $g(0) = g_0$ is a metric satisfying $\mathrm{Vol}(\mathbb{S}^2, g_0) = 4\pi$, and α is chosen in such a way that the conformal volume is preserved, that is,

$$\alpha \int_{\mathbb{S}^2} K e^{2u} = 4\pi,$$

for all $t \geq 0$. In the paper, it is proved the long-existence of the flow (1.5), and that it evolves g_0 towards a conformal metric with Gaussian curvature proportional to K.

In addition, the results in [23] were extended in [24] to the situation of many local maxima, using Morse and degree theory.

Theorem. *[24] Let $K(x)$ be a positive function on $\mathbb{S}^2$, with nondegenerate critical points, and such that $|\nabla K| + |\Delta K| \neq 0$ everywhere on $\mathbb{S}^2$. Then there is a solution of (1.2) provided*

$$\sum_{x \in \mathcal{K}_-} (-1)^{\mathrm{ind}(x)} \neq 1, \tag{1.6}$$

where $\mathcal{K}_- = \{x \in \mathbb{S}^2 : \nabla K(x) = 0,\ \Delta K(x) < 0\}$, and $\mathrm{ind}(x)$ denotes the Morse index of K at the point x.

In higher dimensions, an equivalent result for the Uniformization Theorem is not expectable because of the tensorial nature of the curvature, which has a number of linearly independent components of order $\frac{1}{12}n^2(n^2-1)$. Thus, if $n > 2$, it is natural to consider contractions of the curvature which still provide us some information. For example, on a compact and closed Riemannian manifold (M, g) of dimension $n \geq 3$, if we consider a conformal metric $\tilde{g} = u^{\frac{4}{n-2}}g$, with $u > 0$, the scalar curvatures S_g and $S_{\tilde{g}}$ satisfy the following relation:

$$-\frac{4(n-1)}{n-2}\Delta_g u + S_g u = S_{\tilde{g}} u^{\frac{n+2}{n-2}} \text{ on } M \tag{1.7}$$

(see [6, Ch. 5, §1]). The question of finding conformal metrics with constant scalar curvature was first proposed by Yamabe in [94], and completely solved thanks to the works of Trudinger, Aubin and Schoen [91, 4, 86]. When the prescribed scalar curvature is an arbitrary function $K(x)$, this is known as the *prescribed scalar curvature problem.*

In this case, we also have a restriction for the sign of $S_{\tilde{g}}$ depending on the conformal class of M, even though it is not a topological one as in the two-dimensional case. It can be proved that the *Yamable class* of M, defined as the infimum

$$Y(M, g) = \inf \left\{ \frac{\int_M c_n |\nabla u|^2 + S_g u^2}{\left(\int_M u^{2^*} dV_g\right)^{2/2^*}} : \ u \in H^1(M), \ u \neq 0 \right\},$$

only depends on the conformal class of (M, g), that is, $Y(M, g) = Y(M, [g])$.

When we consider (1.7) with $S_{\tilde{g}} = K$ zero or negative (in which case (M, g) has to be of zero or negative Yamabe class respectively), the nonlinear term in (1.7) makes the associated energy functional coercive and solutions always exist, as proved in [61] using the method of sub and super solutions. However, in the same paper, the authors proved that there are obstructions to existence in the positive case. In particular, they proved the analogue of (1.4) in $\mathbb{S}^n$, namely,

$$\int_{\mathbb{S}^n} \nabla K \cdot \nabla x_i u^{\frac{2n}{n-2}} = 0,$$

for all solutions u of (1.7), where $x_i : \mathbb{S}^n \to \mathbb{R}$ are the restriction to the sphere of the coordinate functions in $\mathbb{R}^{n+1}$.

Existence results for K positive in positive Yamabe class manifolds were found later. Inspired by the pioneering work [79], in [43] the authors gave existence results when K is invariant under a group of isometries without fixed points and satisfies suitable flatness conditions. Other results with symmetries were obtained in [54, 56]. Theorems for more general functions K were found in [8] and [9] (see also [88]) for the case of $\mathbb{S}^3$, and they can be understood as an adaptation of the results of [24] to dimension $n = 3$.

As a final remark, let us point out that equations (1.1) and (1.7) are of critical type from the point of view of Partial Differential Equations theory; the exponent $\frac{n+2}{n-2}$ is the critical Sobolev exponent for the equation (1.7), coming from its geometrical meaning, while the non-linearity $u \to e^u$ in (1.1) is, somehow, the analogue of the critical growth in dimension $n = 2$.

1.2 The Gaussian-Geodesic prescription problem

The first study subject of this thesis will be equation (1.1) in a surface with boundary, hence boundary conditions are in order. Homogeneous Dirichlet and Neumann boundary conditions have already been considered in the literature. However, motivated by its geometric meaning, we will consider a nonlinear boundary condition.

Indeed, our aim is to prescribe not only the Gaussian curvature in Σ, but also the geodesic curvature on $\partial\Sigma$. More precisely, given a metric $\tilde{g} = e^u g$, if $K_g, K_{\tilde{g}} = K$ are the Gaussian curvatures and $h_g, h_{\tilde{g}} = h$ the geodesic curvatures of $\partial\Sigma$, relative to these metrics, then u satisfies the boundary value problem

$$\begin{cases} -\Delta_g u + 2K_g = 2Ke^u & \text{in } \Sigma, \\ \frac{\partial u}{\partial \eta} + 2h_g = 2he^{u/2} & \text{on } \partial\Sigma, \end{cases} \tag{1.8}$$

where η denotes the exterior normal vector to $\partial\Sigma$. As in the closed version of the problem, a topological condition links K and h: integrating (1.8) in Σ, by the Gauss-Bonnet theorem we obtain

$$\int_\Sigma Ke^v + \int_{\partial\Sigma} he^{v/2} = 2\pi\chi(\Sigma). \tag{1.9}$$

Some versions of this problem have been studied in the literature. The case $h = 0$ has been treated in [21] by A. Chang and P. Yang, while the case $K = 0$ in [19, 70, 72], see also the work of Da Lio, Martinazzi and Riviere ([33]) for more recent development under the perspective of nonlocal operators.

The case of constants K, h has also been considered. For instance, Brendle ([15]) studied the long-existence and behaviour of a parabolic flow, and showed its convergence towards a metric with constant curvatures. More precisely, the author considers the evolution equation

$$\begin{cases} \frac{d}{dt} g(t) = -\frac{2}{\alpha}(K_g - \alpha\lambda)g & \text{in } \Sigma, \\ \frac{d}{dt} g(t) = -\frac{2}{\beta}(h_g - \beta\lambda)g & \text{on } \partial\Sigma, \\ g(0) = g_0, \end{cases} \tag{1.10}$$

where K_g is the Gaussian curvature of Σ, h_g is the geodesic curvature of $\partial\Sigma$, α and β are positive real numbers and $\lambda(t)$ is given by

$$\lambda(t) = \frac{2\pi\chi(\Sigma)}{\alpha \mathrm{Vol}_g(M) + \beta \mathrm{Vol}_g(\partial M)}.$$

Applying the Gauss-Bonnet theorem, one can also assume that

$$\alpha \mathrm{Vol}_g(M) + 2\beta \mathrm{Vol}_g(\partial M) = 2\pi \tag{1.11}$$

for all $t \geq 0$, and consequently $\chi(\Sigma) < \lambda < 2\chi(\Sigma)$. The following convergence result was achieved:

Theorem. *[15, Th. 1.1] The evolution equation* (1.10) *admits a unique solution which is defined for all times and converges exponentially to a metric g_∞ satisfying $K_{g_\infty} = \alpha\lambda_\infty$ and $h_{g_\infty} = \beta\lambda_\infty$, being*

$$\lambda_\infty = \lim_{t\to+\infty} \lambda(t).$$

Moreover, if $\chi(\Sigma) = 1$ and g_0 satisfies (1.11), *λ_∞ satisfies the equation*

$$(\lambda_\infty - 1)^2(\alpha + \beta^2\lambda_\infty) = \beta^2\lambda.$$

By using complex analysis techniques, explicit expressions for the solutions of (1.8) with constants K and h and the exact values of the constants are determined if Σ is a disk or an annulus, see [53, 58]. The case of the half-plane has also been studied, see [71, 46, 95].

However, the case in which both curvatures are nonconstant has not been much considered. In [29], some partial existence results are given, but they include a Lagrange multiplier which is out of control. Moreover, a Kazdan-Warner type of obstruction to existence has been found in [50] for the case of the disk. In the work [74], the case of $K < 0$ in domains different from the disk is treated, and also a blow-up analysis is performed (see also [7]). Moreover, the authors obtained existence of solutions using variational techniques, that we now briefly describe.

In the first place, it is shown that the problem (1.8) always admits a solution for $h = 0$ and $K = \mathrm{sgn}(\chi(\Sigma))$. Therefore, without loss of generality it can be assumed that the starting metric is the analogous of the Escobar's metric, with $h_g = 0$ and K_g constant. Assume also that $K < 0$, and consider the energy functional

$$I(u) = \int_\Sigma \left(\frac{1}{2} |\nabla u|^2 + 2K_g u + 2|K| e^u \right) - 4 \int_{\partial\Sigma} h e^{u/2},$$

defined for $u \in H^1(\Sigma)$. The authors showed that the behaviour of the functional I was determined by the scale-invariant quotient $\mathfrak{D} : \mathbb{S}^1 \to \mathbb{R}$ given by the formula

$$\mathfrak{D}(x) = \frac{h(x)}{\sqrt{|K(x)|}},$$

via a trace inequality. The first obtained result concerns the case $\chi(\Sigma) < 0$.

Theorem. *[74, Th. 1.1] Assume that $K_g < 0$. Let K, h be continuous functions such that $K < 0$ and $\mathfrak{D} < 1$ everywhere on $\partial\Sigma$. Then* (1.8) *admits a solution as minimum of I. If moreover $h \leq 0$, then the solution is unique.*

The next theorems apply for the case $\chi(\Sigma) = 0$.

Theorem. *[74, Th. 1.2] Assume that $K_g = 0$. Let K, h be continuous functions such that $K < 0$ and*

(1) $\mathfrak{D} < 1$ everywhere on $\partial\Sigma$,
(2) $\int_{\partial\Sigma} h > 0$.

Then (1.8) *admits a solution as minimum of I.*

Theorem. *[74, Th. 1.3] Assume that $K_g = 0$. Let K, h be C^1 functions such that $K < 0$ and*

(1) $\mathfrak{D} > 1$ somewhere in $\partial\Sigma$,
(2) $\int_{\partial\Sigma} h < 0$,
(3) $\partial^T\mathfrak{D}(p) \neq 0$ for any $p \in \partial\Sigma$ with $\mathfrak{D}(p) = 1$,

where $\partial^T\mathfrak{D}$ denotes the tangential derivative of $\mathfrak{D}$ along $\partial\Sigma$. Then (1.8) *admits a min-max solution.*

1.2.1 The case of the disk

In this thesis we shall consider the case in which $\chi(\Sigma) = 1$. By the Uniformization Theorem, we can pass via a conformal map to a disk, obtaining $K_g = 0$, $h_g = 1$. Taking this into account we can consider the problem:

$$\begin{cases} -\Delta u = 2Ke^u & \text{in } \mathbb{D}^2, \\ \frac{\partial u}{\partial \eta} + 2 = 2he^{u/2} & \text{on } \mathbb{S}^1, \end{cases} \tag{1.12}$$

where now K and h are the curvatures to be prescribed. In this particular case, it is relatively easy to prove existence of solutions for certain constant values of K and h using purely geometrical arguments.

Proposition 1.1. *Let $(\mathbb{D}^2, g_0)$ be the unit disk of $\mathbb{R}^2$ with the standard metric. For all constants $h \in \mathbb{R}$ and $K > 0$, there exists $\tilde{g} \in [g_0]$ with $K_{\tilde{g}} = K$ and $h_{\tilde{g}} = h$.*

The idea behind this result is to deform the disk into a spherical cap via the stereographic projection, which is a conformal map. By applying suitable dilations before and after, one can adjust the radius of the spherical cap and its height, which determine the Gaussian and geodesic curvature of the model.

In order to formalize these ideas, let us denote by δ_λ and $\overline{\delta}_\mu$ the dilations in $\mathbb{R}^2$ and $\mathbb{R}^3$ of factors $\lambda > 0$ and $\mu > 0$, respectively. If $\pi^{-1} : \mathbb{R}^2 \to \mathbb{S}^2 \backslash \{S\}$ is the inverse of the stereographic projection, then the conformal map $\Phi_{\lambda,\mu} = \overline{\delta}_\mu \pi^{-1} \delta_\lambda$ satisfies

$$K_{\Phi^* g_{\mathbb{S}^2}} = \frac{1}{\mu^2}.$$

Thus, it is possible to choose μ in such a way that $K_{\Phi^* g_{\mathbb{S}^2}} = K$. Moreover, the image of $\mathbb{D}^2$ via Φ is a spherical cap with height depending on λ. $\Phi(\mathbb{D}^2)$ covers $\mathbb{S}^2 \backslash S$ when $\lambda \to +\infty$, sending $h_{\Phi^* g_{\mathbb{S}^2}}$ towards $-\infty$, while it concentrates around $\{N\}$ when $\lambda \to 0$, making the geodesic curvature diverge positively. By continuity, there exists a value $\lambda > 0$ such that $h_{\Phi^* g_{\mathbb{S}^2}} = h$.

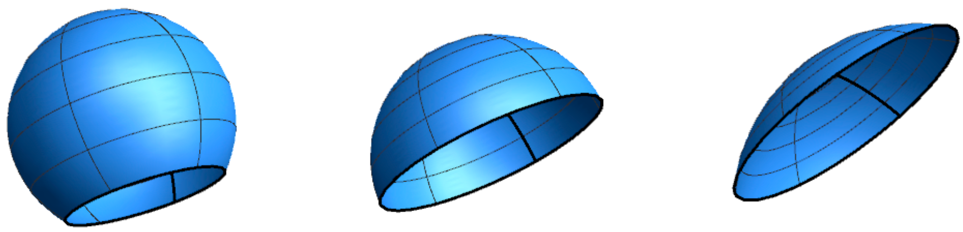

Figure 1.1: $\Phi(\mathbb{D}^2)$ for different values of $\lambda > 0$.

The case $K = 0$ and $h = 1$ is worth discussing too. It admits $u = 0$ as trivial solution, that is, to leave the Euclidean metric unchanged. However, any other conformal transformation of the disk onto itself will produce a conformal pullback metric with identical curvatures, and therefore a new solution for the problem. It is known that the group of conformal transformations of the disk coincides with the group of Möbius transformations preserving $\mathbb{D}^2$, namely,

$$Aut(\mathbb{D}^2) = \left\{ T : \mathbb{D}^2 \to \mathbb{D}^2,\ T(z) = \lambda \frac{z - a}{\bar{a} z - 1} \text{with } |\lambda| = 1,\ |a| < 1 \right\} \tag{1.13}$$

(see for instance [26]). Since $Aut(\mathbb{D}^2)$ is non-compact, neither is the set of solutions of the problem

$$\begin{cases} -\Delta u = 0 & \text{in } \mathbb{D}^2, \\ \frac{\partial u}{\partial \eta} + 2 = 2e^{u/2} & \text{on } \mathbb{S}^1. \end{cases}$$

When K and h are nonconstant functions, some partial positive results are available: for the case $K = 0$, in [19] the following results were found.

Theorem. *[19, Th. 1] Let us assume that $K = 0$ and $h > 0$ is of C^2 class in $\mathbb{S}^1$, and that all its critical points are isolated. Let $\hat{h}$ be the conjugate function of h, and suppose that $\hat{h}'(z) \neq 0$ for every $z \in \mathbb{S}^1$ such that $k'(z) = 0$. Then* (1.8) *admits a solution if $\mu_0 \neq \mu_1 + 1$, where μ_0 and μ_1 are the values of local maximum and minimum of h, respectively, in the region $\Omega = \{z \in \mathbb{S}^1 : h(z) > 0,\ \hat{h}'(z) > 0\}$.*

Theorem. *[19, Th. 1'] Assume that $K = 0$ and $h > 0$ is of C^2 class in $\mathbb{S}^1$, it only has isolated critical points and $\hat{h}'(z) \neq 0$ on points $z \in \mathbb{S}^1$ with $h'(z) = 0$. If the harmonic extension of h has saddle points, then* (1.8) *has a solution.*

In [72, 73] other existence results are given for the case $K = 0$ under symmetry assumptions on h. The case $h = 0$ is treated in [21], for instance.

Generally speaking, the non-compact action of the group of conformal maps of the disk is what renders the case of a disk especially challenging, as it happens in the Nirenberg problem for $\Sigma = \mathbb{S}^2$. This issue has been treated in [19] for $K = 0$ (see also [33]). A blow-up analysis in this case for nonconstant K, h was performed in [59]. Moreover, the existence of bubbling type solutions in the disk was proved in [11] using singular perturbation methods. To that aim, the authors first considered the function

$$\Phi(x) = H(x) + \sqrt{H(x)^2 + K(x)},$$

where H denotes the harmonic extension of h to $\mathbb{D}^2$, and the perturbed problem

$$\begin{cases} -\Delta_g u + 2 = 2K_\varepsilon e^u & \text{in } \mathbb{D}^2, \\ \frac{\partial u}{\partial \eta} = 2h_\varepsilon e^{u/2} & \text{on } \mathbb{S}^1, \end{cases} \tag{1.14}$$

with $K_\varepsilon = K + \varepsilon\, G$ and $h_\varepsilon = h + \varepsilon\, I$, being G and I perturbation functions satisfying certain transversality conditions. Then, if $\nabla\Phi$ vanishes at some point $p \in \mathbb{S}^1$, the authors showed that, for small values of $\varepsilon > 0$, there exist solutions u_ε of (1.14) which blow-up around p when $\varepsilon \to 0$.

1.2.2 Objectives and strategies of the proofs

Our principal objective with respect to the problem (1.12) is that of *presenting new existence results for the case of nonconstant K and h.*

From the point of view of the Calculus of Variations, one of the main difficulties in dealing with this problem is that, a priori, there is no clear variational approach. Therefore, *to find a reasonable energy functional and to study its properties* becomes an equally important target of this thesis.

Integrating (1.12), we find:

$$\int_{\mathbb{D}^2} K e^u + \int_{\mathbb{S}^1} h e^{u/2} = 2\pi,$$

from which it should be clear that K and h cannot be arbitrarily chosen: for instance, they cannot be simultaneously non-positive. We will define the parameter ρ as

$\rho = \int_{\mathbb{D}^2} Ke^u = 2\pi - \int_{\mathbb{S}^1} he^{u/2}$. In order to fix the ideas, assume that $0 < \rho < 2\pi$ and that K and h are positive functions. We will prove that (1.12) is equivalent to:

$$\begin{cases} -\Delta u = 2\rho \frac{Ke^u}{\int_{\mathbb{D}^2} Ke^u} & \text{in } \mathbb{D}^2, \\ \frac{\partial u}{\partial \eta} + 2 = 2(2\pi - \rho)\frac{he^{u/2}}{\int_{\mathbb{S}^1} he^{u/2}} & \text{on } \mathbb{S}, \\ \frac{(2\pi-\rho)^2}{\rho} = \frac{\left(\int_{\mathbb{S}} he^{u/2}\right)^2}{\int_{\mathbb{D}^2} Ke^u} & \text{for } 0 < \rho < 2\pi. \end{cases} \tag{1.15}$$

Compared to the problem of prescribing Gaussian curvature on closed surfaces, here the quantity ρ is not quantized, and (1.15) cannot be read as a mean-field equation. Instead, ρ must be treated as a second unknown.

The problem (1.15) is now invariant under the addition of constants to u, and even if this formulation may seem rather artificial, it has the advantage of being related to the critical points of a nice energy functional, which we now define:

Definition 1.2. *Let $K : \mathbb{D}^2 \to \mathbb{R}$ and $h : \mathbb{S}^1 \to \mathbb{R}$ be Hölder continuous functions that are positive somewhere. We define the space of functions*

$$\mathbb{X} = \left\{ u \in H^1(\mathbb{D}^2) : \int_{\mathbb{D}^2} Ke^u > 0, \int_{\mathbb{S}^1} he^{u/2} > 0 \right\},$$

which is nonempty because of the assumptions on K and h, and the Lagrangian $I : \mathbb{X} \times \ (0, 2\pi) \to \mathbb{R}$ given by

$$\begin{aligned} I(u,\rho) = & \frac{1}{2}\int_{\mathbb{D}^2} |\nabla u|^2 - 2\rho \log \int_{\mathbb{D}^2} Ke^u + 2\int_{\mathbb{S}^1} u - 4(2\pi - \rho)\log \int_{\mathbb{S}^1} he^{u/2} \\ & + 4(2\pi-\rho)\log(2\pi-\rho) + 2\rho + 2\rho\log\rho. \end{aligned} \tag{1.16}$$

We highlight the fact that the above functional depends on the couple (u, ρ), where $u \in H^1(\mathbb{D}^2)$ and ρ is a positive real number. As far as we know, *a functional with this geometry has not been considered before in the literature of this problem.* In order to simplify the notation, for a fixed $\rho \in (0, 2)$, we denote by I_ρ the functional $u \to I(u, \rho)$ defined for every function $u \in \mathbb{X}$.

We obtain existence results by minimizing the functional I. In order to derive its analytical propierties, we first study the *Moser-Trudinger type inequalities (or Onofri type inequalities)*, and give generalized versions of them. The interpolation of these inequalities permits us to show that, for each ρ fixed, the functional I_ρ is bounded from below on $\mathbb{X}$. Since the lower bound does not depend on ρ, we also obtain one for I.

However, in the case of the disk, the constants in the inequalities are sharp and we do not get coercivity. This can be understood as a consequence of the non-compact action of the group of Möbius transformations of the disk. Indeed, a bounded sequence of solutions could concentrate around a point of the boundary, sending the energy to $+\infty$. This behaviour is studied in [59] and the presence of *bubbling* solutions is confirmed in [11].

As a first step in the understanding of the problem, we shall impose symmetry conditions on K, h in order to rule out this phenomenon, in the spirit of Moser [79] (see also [72, 73]). Being more specific, we fix a symmetry group as follows:

Definition 1.3. *We denote by G one of the following groups of symmetries of the disk:*

> *G is the dihedral group $\mathbb{D}_k$ with $k \geq 3$, or*
> *G is the group of rotations with minimal angle $2\pi/k$, $k \geq 2$, or*
> *G is the whole group of symmetries $O(2)$.*

Notice that none of the above groups has fixed points on $\mathbb{S}^1$, that is, for each $x \in \mathbb{S}^1$ there exists $\phi \in G$ such that $\phi(x) \neq x$. We say that a function f is $G-$symmetric if $f(x) = f(\phi(x))$ for all $\phi \in G$ and for all x in the domain of f.

Under those symmetry assumptions, we can guarantee the distribution of masses around different points of the boundary, and then we can use the improved *Chen-Li* type inequalities to gain coercivity. Finally, in order to find a global minimizer for I, we need to verify that the minimum of the function $\rho \to \min I(\cdot, \rho)$ is not achieved at the extrema of the inverval. To that aim, our main tools are *the analysis of the limiting cases $\rho = 0$ and $\rho = 2\pi$*, and *energy estimates involving the minimizers of those functionals.*

Our main result for the disk case is the following:

Theorem 1.4. *Let G be as in Definition 1.3, and let $K : \mathbb{D}^2 \to \mathbb{R}$, $h : \mathbb{S}^1 \to \mathbb{R}$ be $G-$symmetric, Hölder continuous and nonnegative functions, not both of them identically equal to 0. Then problem (1.12) admits a solution.*

Thanks to *compactness of solutions*, we can also deal with changing sign curvatures K, h, as long as their negative part is small:

Theorem 1.5. *Let G be as in Definition 1.3, and let $K_0 : \mathbb{D}^2 \to \mathbb{R}$, $h_0 : \mathbb{S}^1 \to \mathbb{R}$ be $G-$symmetric, Hölder continuous and nonnegative functions, none of them identically equal to 0. Then there exists $\varepsilon > 0$ such that problem (1.12) admits a solution for any Hölder continuous and $G-$symmetric functions K, h with*

$$\|K - K_0\|_{L^\infty} + \|h - h_0\|_{L^\infty} < \varepsilon.$$

1.3 The Scalar-Mean prescription problem

The second problem that we study in this thesis concerns Equation (1.7) on a compact Riemannian manifold with boundary, under geometric boundary conditions. More precisely, if (M,g) is a compact Riemannian manifold of dimension $n \geq 3$ with boundary ∂M, we are interested in the transformation of the scalar curvature S_g and the mean curvature on ∂M under conformal charges of the metric.

If $\tilde{g} = u^{\frac{4}{n-2}} g$ is a conformal metric and we write $S_{\tilde{g}} = K$, and $h_{\tilde{g}} = H$, then the following transformation rule holds (see [29]):

$$\begin{cases} -c_n \Delta_g u + S_g u = K u^{\frac{n+2}{n-2}} & \text{in } M, \\ \frac{2}{n-2} \frac{\partial u}{\partial \eta} + h_g u = H u^{\frac{n}{n-2}} & \text{on } \partial M. \end{cases} \tag{1.17}$$

In the literature, we can find several problems associated to that equation. They have been less investigated than the closed case, but still there are results worth commenting.

The first one is the analogue of the Yamabe problem, that is, to study if it is possible to deform g conformally in such a way that the new scalar curvature of M and the mean curvature of its boundary are constant. Analytically, this is equivalent to solve (1.17) with $K = K_0$ and $H = h_0$ constants. A first criterion for existence of solutions was given in [29], though they depend on unknown Lagrange multipliers.

Escobar worked on the case $h_0 = 0$ and K equal to a positive constant, now known as *Escobar's problem*, and proved existence of solutions except for non-locally conformally flat manifolds of dimension $n \geq 6$, with umbilical boundary and null Weyl tensor on ∂M, see [40, 41, 42]. Later on, more general results that complemented those by Escobar were given by Han and Li ([52, 51]). They proved existence of solutions for any constant $h_0 \in \mathbb{R}$ when either M is locally conformally flat with umbilic boundary (but not conformally equivalent to the Euclidean half-sphere), or $n \geq 5$ and the boundary is non-umbilic. We refer to [77] and the references therein for the most recent progress in the topic.

The case of variable functions has been studied in specific situations.
Existence results for the case $H = 0$ are given in [67, 12, 13], while the works [1, 36, 25, 93] concern the scalar-flat version of the problem.

For the complete problem with variable curvatures, we highlight the work [2], which contains perturbative results about nearly constant curvature functions on the unit ball of $\mathbb{R}^n$. In this article, the authors first considered (1.17) on the unit ball of $\mathbb{R}^n$ equipped with an arbitrary Riemannian metric g, and $K = 1$, $H = c \in \mathbb{R}$:

$$\begin{cases} -c_n \Delta_g u + S_g u = u^{\frac{n+2}{n-2}} & \text{in } B, \\ \frac{2}{n-2} \frac{\partial u}{\partial \eta} + h_g u = c u^{\frac{n}{n-2}} & \text{on } \mathbb{S}^{n-1}, \end{cases} \tag{1.18}$$

and proved existence of solutions for any metric g *close enough* to the standard metric g_0 on B, in the following sense:

Theorem. *[2, Th. 1] Given $M > 0$, there exists $\varepsilon_0 > 0$ such that, for every $0 < \varepsilon < \varepsilon_0$, every $c > -M$ and $g \in \mathcal{G}_\varepsilon$,* (1.18) *admits a positive solution, being $\mathcal{G}_\varepsilon$ the set of bilinear forms on B defined by*

$$\mathcal{G}_\varepsilon = \Big\{ g \in C^\infty(B) : \ \|g - g_0\|_{L^\infty(B)} \leq \varepsilon, \ \|\nabla g\|_{L^n(B)} \leq \varepsilon, \ \|\nabla g\|_{L^{n-1}(\mathbb{S}^{n-1})} \leq \varepsilon \Big\}.$$

After that, perturbed versions of (1.18) were studied, namely,

$$(P_\varepsilon) \quad \begin{cases} -c_n \Delta_g u + S_g u = (1 + \varepsilon K(x)) u^{\frac{n+2}{n-2}} & \text{in } B, \\ \frac{2}{n-2} \frac{\partial u}{\partial \eta} + h_g u = (c + \varepsilon H(x)) u^{\frac{n}{n-2}} & \text{on } \mathbb{S}^{n-1}. \end{cases} \tag{1.19}$$

Concerning (1.19), the following existence results were obtained when either $H = 0$ or $K = 0$:

Theorem. *[2, Th. 2] Assume $H = 0$ and that there exists a point of global maximum (resp. minimum) p of $K|_{\mathbb{S}^{n-1}}$ such that*

$$\nabla^T K(p) \cdot p < 0 \ \ \big(\textit{resp. } \nabla^T K(p) \cdot p > 0\big).$$

Then, for $|\varepsilon|$ small enough, (1.19) *admits a positive solution.*

Theorem. *[2, Th. 3] Assume that $H = 0$ and that K is a Morse function satisfying*

$$\nabla^T K(x) \cdot x \neq 0 \quad \textit{for every critical point } x \textit{ of } K|_{\mathbb{S}^{n-1}},$$

and

$$\sum_{\substack{x \in \mathrm{crit}\left(K|_{\mathbb{S}^{n-1}}\right) \\ \nabla^T K(x) \cdot x < 0}} (-1)^{\mathrm{ind}\, x} \neq 1.$$

Then, for $|\varepsilon|$ small enough, (1.19) *admits a positive solution.*

Theorem. *[2, Th. 4] Assume that $K = 0$ and that $H \in C^\infty(\mathbb{S}^{n-1})$ is a Morse function satisfying*

$$\Delta^T H(x) \neq 0 \quad \textit{for every } x \in \mathrm{crit}\, H,$$

and

$$\sum_{\substack{x \in \mathrm{crit}(H) \\ \Delta^T H(x) < 0}} (-1)^{\mathrm{ind}\, x} \neq 1.$$

Then, for $|\varepsilon|$ small enough, (1.19) *admits a positive solution.*

Moreover, in the work [35], the case of variable K and H, with $K > 0$, was treated in the half sphere $\mathbb{S}^3_+$, and a blow-up analysis is performed. Finally, the work [28] addresses the problem for negative curvatures, but solutions are obtained up to Lagrange multipliers. The setting is that of a compact Riemannian manifold (M, g_0) of dimension $n \geq 3$, and as in [40], the authors consider the Sobolev quotient

$$Q(M, \partial M) = \inf_{g \in [g_0]} \frac{\int_M S_g + 2(n-1)\int_{\partial M} h_g}{\mathrm{Vol}_g(\partial M)^{\frac{n-2}{n-1}}}.$$

Inspired by Brendle's original work [15], in [28] the authors defined the following geometric flow $g(t) = u(t)^{\frac{4}{n-2}} g_0$ by:

$$\begin{cases} \frac{dg}{dt} = \left(\alpha(t)\frac{S_g}{K} - \lambda(t)\right) g & \text{in } M, \\ \frac{dg}{dt} = \left(\beta(t)\frac{h_g}{H} - \lambda(t)\right) g & \text{on } \partial M, \\ u(0) = u_0 \in C^\infty(\overline{M}), \end{cases} \tag{1.20}$$

where K and H are negative smooth functions defined on M and ∂M respectively, and, given two constants $a, b > 0$,

$$\alpha(t) = \frac{1}{a}\left(\int_M |K|\, dV_g\right)^{\frac{2}{n}}, \quad \beta(t) = \frac{1}{b}\left(\int_{\partial M} |h|\, ds_g\right)^{\frac{1}{n-1}}$$

$$\lambda(t) = \frac{\int_M \left(c_n |\nabla u|^2_{g_0} + S_{g_0} u^2\right) dV_{g_0} + 2(n-1)\int_{\partial M} h_{g_0} u^2 ds_{g_0}}{a\left(\int_M |K|\, dV_g\right)^{\frac{n-2}{n}} + 2(n-1)b\left(\int_{\partial M} |h|\, ds_g\right)^{\frac{n-2}{n-1}}}.$$

The following result was obtained:

Theorem. *[28][Th. 1.1] Let (M, g_0) be a compact manifold of dimension $n \geq 3$ such that*

$$-\infty < Q(M, \partial M) < 0.$$

Then, if K is a smooth negative function defined on M and H is a smooth negative function on ∂M, for any given initial data the flow (1.20) *exists for all time and converges towards a smooth metric $g_\infty \in [g_0]$ satisfying*

$$S_{g_\infty} = \frac{\lambda_\infty}{\alpha_\infty} K, \quad h_{g_\infty} = \frac{\lambda_\infty}{\beta_\infty} H,$$

being

$$\alpha_\infty = \lim_{t \to +\infty} \alpha(t), \quad \beta_\infty = \lim_{t \to +\infty} \beta(t), \text{ and } \lambda_\infty = \lim_{t \to +\infty} \lambda(t).$$

1.3.1 Objectives and strategies of the proofs

Our goal here is to consider (1.17) with variable functions $K < 0$ and H of arbitrary sign, and to give results about existence of solutions and bubbling behaviour. In fact we obtain some counterparts of the results in [74], which was dealing with the two-dimensional case for domains with positive genus. As we will see, there are many differences when the dimension is higher than two.

To state our theorems, we first reduce the problem to a simpler situation using a result by Escobar, which states that every compact Riemannian manifold of dimension $n \geq 3$ with boundary admits a conformal metric whose scalar curvature does not change sign and its boundary is minimal (see [41]). This implies that, without losing generality, via an initial conformal change one can start with $h_g = 0$ and $S_g = S$ not changing sign. In what follows, we will assume that **the starting metric is the one given by Escobar**, as well as the fact that $n \geq 3$.

In view of (1.17), we are led to find *positive* solutions of the boundary value problem:

$$\begin{cases} -\frac{4(n-1)}{n-2}\Delta_g u + Su = Ku^{\frac{n+2}{n-2}} & \text{on } M, \\ \frac{2}{n-2}\frac{\partial u}{\partial \eta} = Hu^{\frac{n}{n-2}} & \text{on } \partial M. \end{cases} \tag{1.21}$$

The variational formulation of (1.21) is classical; weak solutions can be obtained as critical points of the following energy functional, defined on $H^1(M)$:

$$I(u) = \frac{2(n-1)}{n-2}\int_M |\nabla u|^2 + \frac{1}{2}\int_M Su^2 - \frac{1}{2^*}\int_M K\,|u|^{2^*} - (n-2)\int_{\partial M} H\,|u|^{2^\sharp}, \tag{1.22}$$

where $2^* = \frac{2n}{n-2}$ and $2^\sharp = \frac{2(n-1)}{n-2}$ are the critical Sobolev exponents for M and ∂M, respectively. As written before, we will assume that $K < 0$, so that the third term in the right-hand side of (1.22) is positive. The interaction between this term and the boundary critical term will be crucial for the behaviour of the energy functional.

Via a *trace inequality* we show that, in fact, the nature of the functional is ruled by a quotient of the prescribed curvatures at the boundary, which also allows us to compare both terms. For convenience, we define the scaling invariant function $\mathfrak{D}_n : \partial M \to \mathbb{R}$ by

$$\mathfrak{D}_n(x) = \sqrt{n(n-1)}\frac{H(x)}{\sqrt{|K(x)|}}. \tag{1.23}$$

Depending on whether $\mathfrak{D}_n$ is strictly less than 1 or not, we find ourselves in two completely different scenarios. We notice that boundaries of geodesic spheres in hyperbolic spaces satisfy $\mathfrak{D}_n > 1$, while $\mathfrak{D}_n = 1$ at boundaries of *horospheres*. Therefore, when $\mathfrak{D}_n \geq 1$, there might be blow-ups for (1.21) with such profiles.

Assuming that $\mathfrak{D}_n(x) < 1$ for every $x \in \partial M$, it turns out that K *shadows* H, and the corresponding positive term in I dominates the one at the boundary involving H. The result is that the functional becomes coercive and a global minimizer can be found.

Our first result concerns the case when the Escobar metric satisfies $S < 0$, and compared to [29, 28], we can solve the original geometric problem without any extra Lagrange multiplier.

Theorem 1.6. *Suppose $K < 0$, and that $\mathfrak{D}_n$ as in* (1.23) *verifies $\mathfrak{D}_n < 1$ everywhere on ∂M. Then, if $S < 0$,* (1.21) *admits a solution.*

If $S = 0$, extra hypotheses are needed to rule out the possibility of the minimizer beging identically zero, so the solution we obtain is geometrically admisible.

Theorem 1.7. *Suppose $K < 0$ on M and $\mathfrak{D}_n < 1$ on ∂M. Then, if $S = 0$ and $\int_{\partial M} H > 0$,* (1.21) *admits a solution.*

On the other hand, if there exists $p \in \partial M$ such that $\mathfrak{D}_n(p) > 1$, we can construct a sequence of functions u_i, with masses concentrated around p, such that the energy $I(u_i)$ tends to $-\infty$. While this prevents the existence of minimizers, in three dimension we can use the *Mountain-pass Theorem* to obtain a solution for (1.21).

Theorem 1.8. *Let $n = 3$, assume that $S = 0$, $K < 0$ and that H is such that*

(1) $\int_{\partial M} H < 0$,
(2) $\mathfrak{D}_n(\overline{p}) > 1$ *for some* $\overline{p} \in \partial M$, *and*
(3) 1 *is a regular value for* $\mathfrak{D}_n$.

Then, (1.21) *admits a positive solution.*

We will explain below why we have a dimensional restriction in Theorem 1.8, while giving an outline of the proof. To prove the existence of Min-Max solutions it is necessary to show that Palais-Smale sequences of approximate solutions converge. For doing this, two main difficulties occur: first, one needs to show their boundedness in norm, which is unclear in our case due to the triple homogeneity of the Euler-Lagrange functional. Second, because of the presence of critical exponents in (1.21), even bounded Palais-Smale sequences may not converge.

To deal with the first issue we make use of *Struwe's monotonicity trick*, see §3.3, consisting in perturbing the problem by a parameter for which the energy is monotone. Moreover, we will use a *subcritical approximation* to guarantee compactness of Palais-Smale sequences. Hence, we will consider the following situation.

Let $(K_i)_i$ be a sequence of regular functions defined on M such that $K_i \to K$ in $C^2(\overline{M})$ and let $(H_i)_i$ be a sequence of smooth functions on ∂M such that $H_i \to H$ in

$C^2(\partial M)$. Assuming that $K<0$, we consider positive solutions $(u_i)_i$ to the perturbed problem

$$\begin{cases} -4\frac{n-1}{n-2}\Delta_g u_i + S u_i = K_i {u_i}^{p_i} & \text{on } M, \\ \frac{2}{n-2}\frac{\partial u_i}{\partial \eta} = H_i {u_i}^{\frac{p_i+1}{2}} & \text{on } \partial M, \end{cases} \tag{1.24}$$

namely critical points of the energy functional:

$$I_i(u) = \frac{2(n-1)}{n-2}\int_M |\nabla u|^2 + \frac{1}{2}\int_M S u^2 - \frac{1}{p_i+1}\int_M K_i |u|^{p_i+1} - 4\frac{n-1}{p_i+3}\int_{\partial M} H_i |u|^{\frac{p_i+3}{2}}, \tag{1.25}$$

with $p_i \nearrow \frac{n+2}{n-2}$. The question is then whether such solutions could be uniformly bounded from above, in which case they would converge to a solution of the original problem (1.21). Assuming the contrary, take (u_i) as detailed above, and define its *singular set* as

$$\mathscr{S} = \{p \in \overline{M} : \exists x_i \to p \text{ such that } u_i(x_i) \text{ is unbounded}\}.$$

In this regard, we have the following compactness result.

Theorem 1.9. *Let (u_i) be a sequence of solutions of* (1.24)*, and $\mathscr{S}$ the associated singular set. Then*

(1) $\mathscr{S} \subset \{p \in \partial M : \mathfrak{D}_n(p) \geq 1\}$.

Therefore, we can write $\mathscr{S} = \mathscr{S}_0 \sqcup \mathscr{S}_1$, with $\mathscr{S}_1 = \mathscr{S} \cap \{\mathfrak{D}_n > 1\}$ and $\mathscr{S}_0 = \mathscr{S} \cap \{\mathfrak{D}_n = 1\}$. In dimension $n=3$, we have further:

(2.1) $\mathscr{S}_1$ is a finite collection of points.

(2.2) If $S \leq 0$, then $\mathscr{S}_1 = \emptyset$.

(2.3) If $I_i(u_i)$ is uniformly bounded and 1 is a regular value of $\mathfrak{D}_n$, then $\mathscr{S}_0 = \emptyset$.

The above result gives a description of two types blow-up points, gathered in the sets $\mathscr{S}_0$ and $\mathscr{S}_1$. The different blow-up profiles are in correspondence with the different type of solutions for the following problem in the half-space

$$\begin{cases} \frac{-4(n-1)}{n-2}\Delta v = K(p) v^{\frac{n+2}{n-2}} & \text{on } \mathbb{R}^n_+, \\ \frac{2}{n-2}\frac{\partial v}{\partial \eta} = H(p) v^{\frac{n}{n-2}} & \text{on } \partial\mathbb{R}^n_+, \end{cases} \tag{1.26}$$

where $p \in \mathscr{S}$. Solutions to (1.26) were classified in [30] (see also [71]) as follows:

- $\star$ If $\mathfrak{D}_n(p) < 1$, then (1.26) admits no solutions.

⋆ If $\mathfrak{D}_n(p) = 1$, the only solutions are $1-$*dimensional* and given by:

$$v(x) = v_\alpha(x) := \left(\frac{2}{\sqrt{n(n-2)}} x_n + \alpha \right)^{-\frac{n-2}{2}}, \tag{1.27}$$

for any $\alpha > 0$.

⋆ If $\mathfrak{D}_n(p) > 1$, the solutions are called *bubbles* and given by

$$v(x) = b_\beta(x) := \frac{(n(n-2))^{\frac{n-2}{4}} \beta^{\frac{n-2}{2}}}{\left(|x - x_0(\beta)|^2 - \beta^2\right)^{\frac{n-2}{2}}}, \tag{1.28}$$

with $x_0(\beta) = -\mathfrak{D}_n(p)\beta, e_n \in \mathbb{R}^n$, for $\beta > 0$ arbitrary.

We would like to emphasize that blow-up profiles can have infinite volume, contrarily to what happens in the case without boundary, at least in low dimensions. *The development of a blow-up analysis in a situation of an infinite number of blow-up points or blow-up profiles with infinite volume* is one of the main goals of this thesis. Moreover, both types of blow-up behaviour are indeed possible; the one around points of $\mathscr{S}_1$ can be understood by the invariance of the problem under conformal maps of the disk, in analogy with what happens in the closed case. But in this framework we can have blow-up around infinite sets $\mathscr{S}_0$. An explicit example of this phenomenon will be given later on in this thesis.

Compared to the two-dimensional case studied in [74] we have more rigidity in the classification, since for the half-plane other solutions are generated by meromorphic functions, see [46]. On the other hand, in the two-dimensional case one can make use complex-analytic tools which are not availabe in higher dimensions.

To deal with loss of compactness at points where $\mathfrak{D}_n > 1$, we perform *a precise study on the behaviour of blow-ups*, showing that, in dimension $n = 3$ they are isolated and simple and therefore they form a finite collection (see also as for [35] in this regard). Once this is proved, we are able to determine the behaviour of solutions also away from such points, dismissing this kind of blow-up by some *integral estimates* which hold true when $S \leq 0$.

On the other hand, around blow-up points with $\mathfrak{D}_n = 1$, the terms $\int_M |\nabla u_i|^2$, $\int_M |K_i| \, u_i^{p_i+1}$ and $\int_{\partial M} H_i u_i^{\frac{p_i+3}{2}}$ diverge. Assuming boundedness of the energies $I_i(u_i)$, (which is natural for min-max sequences) we can show that they converge weakly towards the same measure after proper normalization. Using then a *domain-variation technique* we show that at such blow-up points the gradient of $\mathfrak{D}_n$ along ∂M at $\{\mathfrak{D}_n = 1\}$ must vanish, contradicting our assumption on the regularity of this level. Compared to a similar step in [74] for the two-dimensional case, we have to choose arbitrary deformations tangent to ∂M.

Chapter 2

Methodology

As can be seen, the realization of our objectives requires extensive knowledge of various branches of mathematics, such as Elliptic Partial Differential Equations, Functional Analysis, Calculus of Variations or Geometry. For the development of this thesis, we applied the following methodology:

i. *A cotutelle regime.* Through the codirection agreement between the Scuola Normale Superiore and the University of Granada, we have been able to take advantage of the wide range of Ph.D courses offered by the former, as well as to attend the seminars and workshops held at both institutions.

   Among the courses attended, the following have had a direct impact on the academic training required for the preparation of this thesis: *Riemannian Geometry*, *Elliptic Partial Differential Equations*, *Variational Methods*, *Conformal Geometry.* Moreover, others such as *Initial Data Sets in General Relativity* and *Geometric Flows* have enhanced the education in other disciplines.

ii. *Communication of the results obtained in this thesis.* In addition to their publication in high-level mathematical journals, we have been concerned with presenting our results in the form of posters and seminars at various scientific events, such as the *Bienal congress of the Royal Spanish Mathematical Society*, the *Seminar of Young Researchers of the University of Granada*, the *Séminare Jeunes Chercheurs of the University of Cergy-Pointoise* and the *5th. Congress of young researchers of the Royal Spanish Mathematical Society.*

iii. *Continuous mobility and permanent contact with the mathematical community.* The international framework in which the thesis is placed has facilitated the interaction with other research groups, such as the *Calculus of Variations and Geometric Measure Theory research group* of Pisa, as well as the participation in congresses in different parts of Europe.

   Our program has been decisively affected by the outbreak of the COVID-19 sanitary crisis. The research work carried out since the beginning of the pandemic has been done telematically, as well as the follow-up of seminars and conferences. In addition, it has cancelled the research stays planned for the year 2020/21.

iv. *Use of online resources.* The online libraries of both institutions, as well as their agreements with electronic publishing portals such as MathSciNet and Springer Link, have provided, free of charge, all the bibliographic resources necessary for the completion of the project. In addition, the licenses for mathematical software like Wolfram Mathematica provided by the Scuola Normale Superiore have been useful for the computation of explicit solutions of some problems of this thesis.

Finally, the financial support has been ideal for a project of these characteristics. The Marie Sklodowska-Curie fellowships of the Istituto Nazionale di Alta Matematica were proposed specifically for the attainment of a Ph.D in Mathematics at one of the most notorious institutions in Italy, thus setting us in an excellent framework for the production of high quality doctoral thesis.

Chapter 3

Notation and preliminaries

In this chapter we set the notation that we will be using during in this thesis, we remind basic definitions in Differential Geometry and Calculus of Variations, and develop some analytical tools needed for the study of the functionals that we will consider.

3.1 Notation

Given a set $A \subset X$ in a metric space, we will denote

$$(A)^r = \{x \in X : \operatorname{dist}(x, A) < r\}.$$

If Ω is a domain in a Riemannian manifold M, or in the half space $\mathbb{R}^n_+$, we will use different notation for each portion of its boundary: $\partial_0\Omega$ stands for $\overline{\Omega} \cap M$, while $\partial^+\Omega$ represents $\partial\Omega \cap M$. The unit normal vector to $\partial\Omega$ pointing outwards will be denoted by η.

For functions defined on ∂M, we will add the superscript T for their derivatives. For instance, if $\nabla\mathfrak{f}$ is the gradient of a function $\mathfrak{f}$, $\nabla^T\mathfrak{f}$ will be use to denote its tangential gradient on ∂M.

When we consider the domain of definition of our functionals, the superscript G will denote the restriction to $G-$symmetric functions, for instance

$$\mathbb{X}^G = \{f \in \mathbb{X} : f \text{ is } G-\text{symmetric}\}.$$

Concerning the integrals, we shall only consider the Lebesgue measure and, unless it is necessary, we will omit the volume element. The symbol $⨍ \mathfrak{f}$ will be used to denote the mean value of $\mathfrak{f}$, that is

$$⨍_\Omega \mathfrak{f} = \frac{1}{|\Omega|}\int_\Omega \mathfrak{f}.$$

We will often work in *geodesic coordinates* centered at some point of $\overline{M}$. In that situation, $|x-y|$ will denote the Riemannian distance in $\overline{M}$ between the points with coordinates x and y, respectively.

In our estimates we sometimes write C to denote a positive constant, independent of the variables considered, that can vary from line to line, or also within the same one. For the sake of simplicity, we sometimes use the notation

$$C_n = \frac{4(n-2)}{n-2}.$$

3.2 Function spaces

We will deal with compact Riemannian manifolds (M, g), and mainly consider the following spaces of functions $u : M \to \mathbb{R}$:

- $L^p(M)$, the spaces of Lebesgue integrable functions with the norm $\|\cdot\|_{L^p}$.
- $H^1(M)$, the Sobolev space with the norm

$$\|u\|^2_{H^1(M)} = \int_M |\nabla u|^2 + \int_M u^2.$$

- $C_0^\infty(M)$, the space of C^∞ functions with compact support on M.
- $H_0^1(M)$, the closure of $C_0^\infty(M)$ in $H^1(M)$.

We recall the fact that the well-known Sobolev embeddings and the Rellich-Kondrachov theorem still work in compact manifolds (see [55, §3.3]).

Theorem. *Let (M, g) be a compact Riemannian manifold of dimension n. If we write $2^* = 2n/(n-2)$, then*

(i) If $n > 2$, $H^1(M) \subset L^p(M)$ for all $1 \le p \le 2^*$. The embedding is continuous, and compact provided $1 \le p < 2^*$.

(ii) If $n = 2$, $H^1(M) \subset L^p(M)$ for all $1 \le p < \infty$, and the embedding is continuous and compact.

By combining the Poincaré inequalities with the Sobolev embeddings, we get the Sobolev-Poincaré inequalities. Namely, one has the following.

Theorem. *Let (M, g) be a compact Riemannian manifold of dimension n. Then, for every p with $1 \le p \le 2^*$, there exists a positive constant C, that depends on M and p, such that for any $u \in H^1(M)$,*

$$\left(\int_M |u-\overline{u}|^p\right)^{1/p} \le C\left(\int_M |\nabla u|^2\right)^{1/2}.$$

If $n = 2$, then the previous inequality holds for every $1 \le p < +\infty$.

If M is a manifold with boundary ∂M, then we have the following trace embedding, that can be consulted in [82, Th. 6.2]

Theorem. *Let (M, g) be a compact Riemannian manifold of dimension n with boundary ∂M. If we write $2^\sharp = 2(n-1)/(n-2)$, then*

(i) If $n > 2$, the trace operator $u \to u|_{\partial M}$ is continuous from $H^1(M)$ to $L^p(\partial M)$ for every $1 \leq p \leq 2^\sharp$, and compact provided $1 \leq p < 2^\sharp$.

(ii) If $n = 2$, the trace operator $u \to u|_{\partial M}$ is continuous and compact from $H^1(M)$ to $L^p(\partial M)$ for every $1 \leq p < \infty$.

3.2.1 Existence of trial functions

In the following lemma we prove the existence of certain trial functions that will be used for adapting some inequalities to our framework. It also serves to illustrate the so-called *Tonelli's direct method in Calculus of Variations,* used to prove the existence of a minimizer for a given functional.

Lemma 3.1. *Let M be a compact manifold of dimension $n \geq 2$ with C^1 boundary ∂M. Let p_1, p_2 be two real numbers satisfying*

$$\begin{aligned}&2n/(n+2) \leq p_1 \leq \infty \text{ and } 2(n-1)/n \leq p_2 \leq \infty \quad \text{if } n > 2, \text{ or}\\ &1 < p_1 \leq \infty \text{ and } 1 < p_2 \leq \infty \quad \text{if } n = 2.\end{aligned}$$

Given $\mathfrak{f} \in L^{p_1}(M)$ y $\mathfrak{h} \in L^{p_2}(\partial M)$, the contour problem

$$\begin{cases} -\Delta u = \mathfrak{f} & \text{in } M, \\ \frac{\partial u}{\partial \eta} = \mathfrak{h} & \text{on } \partial M, \end{cases} \tag{3.1}$$

admits a weak solution in $H^1(M)$ if and only if $\int_M \mathfrak{f} + \int_{\partial M} \mathfrak{h} = 0$.

Proof. Integrating by parts the first equation:

$$\int_M \mathfrak{f} = \int_M -\Delta u = -\int_{\partial M} \frac{\partial u}{\partial \eta} = -\int_{\partial M} \mathfrak{h}.$$

Conversely, it is clear that the Lagrangian associated to (3.1) is the following:

$$I(u) = \frac{1}{2} \int_M |\nabla u|^2 - \int_{\partial M} \mathfrak{h} u - \int_M \mathfrak{f} u,$$

defined for $u \in H^1(M)$. The continuity of I follows from the embeddings of the preceding section and Hölder's inequality. Moreover, we have the following bounds:

$$\int_M \mathfrak{f} u \leq C_1 \|\mathfrak{f}\|_{L^{p_1}(M)} \|u\|_{H^1(M)}, \tag{3.2}$$

$$\int_{\partial M} \mathfrak{h} u \leq C_2 \|\mathfrak{h}\|_{L^{p_2}(\partial M)} \|u\|_{H^1(M)}. \tag{3.3}$$

We notice that, for every $c \in \mathbb{R}$:

$$I(u+c) = I(u) - c\left(\int_M \mathfrak{f} + \int_{\partial M} \mathfrak{h}\right) = I(u),$$

which means that the energy functional I is invariant under the addition of constants. Therefore, we can reduce ourselves to functions $u \in H^1(M)$ with $\int_M u = 0$. For these functions, the Poincaré-Sobolev inequality grants the existence of a constant $C > 0$ such that

$$\|u\|_{H^1(M)} \leq C\|\nabla u\|_{L^2(M)}, \ \ \forall u \in H^1(M). \tag{3.4}$$

Combining (3.2), (3.3) and (3.4), we get a lower bound for I:

$$I(u) \geq C_3\|u\|^2_{H^1(M)} - C_1\|\mathfrak{f}\|_{L^{p_1}(M)}\|u\|_{H^1(M)} - C_2\|\mathfrak{h}\|_{L^{p_2}(\partial M)}\|u\|_{H^1(M)},$$

where $C_1, C_2, C_3 > 0$. In particular

$$\lim_{\|u\|_{H^1(M)} \to +\infty} I(u) = +\infty \tag{3.5}$$

and its infimum exists. Consider (u_k) a sequence in $H^1(M)$ with $\int_M u_k = 0$ for every $k \in \mathbb{N}$ and such that

$$u_k \to \alpha = \inf\left\{I(u) : u \in H^1(M)\right\}.$$

By (3.5), (u_k) must be bounded, so up to taking a subsequence we can assume that $u_k \rightharpoonup u_0 \in H^1(M)$, since $H^1(M)$ is a reflexive space. Since the second and third terms of the functional are linear and continuous in u, we can pass to the limit and obtain:

$$\int_{\partial M} \mathfrak{h} u_k \to \int_{\partial M} \mathfrak{h} u_0, \quad \int_M \mathfrak{f} u_k \to \int_M \mathfrak{f} u_0.$$

Finally, using the fact that the function $u \to \int_M |\nabla u|^2$ is weakly lower-semicontinuous:

$$\liminf_{n \to +\infty} \int_M |\nabla u_k|^2 \geq \int_M |\nabla u_0|^2.$$

Consequently, $I(u_0) \leq \alpha$ and u_0 is a minimizer. ■

3.3 The Mountain-pass Theorem and Struwe's monotonicity trick

Let E be a Banach space, and denote by E^{-1} its dual. By saying that a functional $I \in C^1(E, \mathbb{R})$ has the *mountain pass geometry*, we mean that there exist two points, $e_1 = 0$, e_2 in E such that:

(MP-1) $I(0) = 0,$ and there exist $\varepsilon > 0$ and $\rho > 0$ such that $I(x) > \varepsilon$ if $\|x\| = \rho$.

(MP-2) $\|e_2\| > \rho$ and $I(e_2) \leq 0$.

In that case, it is known that setting $\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0, \gamma(1) = e_2\}$, the real value $c \in \mathbb{R}$ given by

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)) > 0, \tag{3.6}$$

called the *mountain pass level*, is a good candidate for being a critical level of I. Indeed, the celebrated Mountain pass theorem from [3] claims that:

Theorem. *Suppose that $I \in C^1(E, \mathbb{R})$ satisfies (MP-1) and (MP-2). Let $c \in \mathbb{R}$ be as in* (3.6)*, and assume that every Palais-Smale sequence at the level c possess a convergent subsequence. Then, there exists $e \in E$ such that $I(e) = c$ and $I'(e) = 0$.*

By looking at the proof given in [3], one sees that the mountain pass geometry implies the existence of a Palais-Smale sequence (u_i) for I at level c, that is to say, $I(u_i) \to c$ and $I'(u_i) \to 0$ in E^{-1}. Therefore, it is enough to prove that this particular subsequence admits a convergent subsequence. This is usually done in two steps: first one proves that (u_i) is bounded and, assuming E is reflexive, this yields to the existence of $u \in E$ such that $u_i \rightharpoonup u$. The second step is to prove that $u_i \to u$ strongly in E, and use the continuity of I and I' to conclude that $I(u) = c$ and $I'(u) = 0$. However, in general, it is unclear that the Palais-Smale sequences at the level c are bounded.

Struwe's monotonicity trick is probably the most relevant contribution to the problem of finding conditions on I ensuring the existence of a bounded Palais-Smale sequence. It first appeared in [90], and here we give a version adapted for its use in our problems, inspired by [57].

Theorem. *Let E be a Banach space, and let $J \subset \mathbb{R}^+$ be an interval. We consider a family $(I_\lambda)_{\lambda\in J}$ of C^1 functionals on E of the form*

$$I_\lambda(u) = A(u) - \lambda B(u), \ \ \forall \lambda \in J,$$

where either $A(u) \geq 0$ or $B(u) \geq 0$ for all $u \in E$ and such that either $A(u) \to +\infty$ or $B(u) \to +\infty$ as $\|u\|_E \to +\infty$. We assume that (MP-1) and (MP-2) hold for every $\lambda \in J$, and we call c_λ the associated mountain pass level to I_λ, defined in (3.6)*. Then, for almost every $\lambda \in J$, there is a sequence u_i in E such that*

(i) $I_\lambda(u_i) \to c_\lambda$,
(ii) $I'_\lambda(u_i) \to 0$ *in* E^{-1}, *and*
(iii) (u_i) *is bounded.*

The conclusion is that, for λ in a dense subset of J, we have bounded Palais-Smale sequences. If we can pass to the limit, we will obtain solutions for approximated problems.

In our work, we are interested a specific value $\lambda_0 \in J$. Therefore, the drawback is that we need to consider a sequence of solutions of approximated problems, u_λ, and try to pass to the limit in λ. There are several advantages in this method, compared to considering arbitrary Palais-Smale sequences. In fact, u_λ solves an approximate problem, and then it is suitable for the application of tools such as regularity of solutions, blow-up analysis or Pohozaev-type identities.

3.4 Conformal maps

Definition 3.2. *Let (M, g) be a Riemannian manifold. Another metric $\tilde{g}$ on M is said to be conformal to g if it can be written in the form $\tilde{g} = \rho(x)g$, where ρ is a positive, differentiable function on M called the conformal factor. We will denote by $[g]$ the family of conformal metrics to g on M.*

Conformal metrics can be induced from conformal maps, which have the property of preserving angles between tangent vectors.

Definition 3.3. *We say that a diffeomorfism $\varphi : (M_1, g_1) \to (M_2, g_2)$ is conformal if the pullback metric $\varphi^* g_2 \in [g_1]$.*

An example of a conformal map that we will use in this thesis is the inverse of the stereographic projection, $\pi^{-1} : (\mathbb{C}, |dz|^2) \to (\mathbb{S}^2 \backslash \{N\}, g_0)$, which satisfies the equality

$$(\pi^{-1})^* g_0 = \frac{4}{\left(1 + |z|^2\right)^2} \, |dz|^2 .$$

We highlight a result by Escobar that concerns the existence of certain conformal metric on a Riemannian manifold with boundary. Its usefulness is to give a simpler starting metric for the study of our problems.

Lemma. *[41, Lemma 1.1]. If (M^n, g_0) is a compact Riemannian manifold with boundary and $n \geq 3$, there exists a conformal metric to g_0 whose scalar curvature does not change sign and the boundary is minimal.*

3.5 Moser-Trudinger inequalities

The Moser-Trudinger inequalities (see [21, 78, 79, 91]) can be understood as limiting versions of the Sobolev inequalities for compact surfaces, and they are powerful tools for the study of the energy functional (4.1), since they allow us to control the non-linearities of exponential type with linear terms.

Our starting point is the following well-known result by Moser:

Theorem 3.4. *Let (Σ, g) be a compact Riemannian surface with C^1 boundary. Then there exists a constant $C > 0$, depending only on the geometry of Σ, such that* *(i)* $\int_\Sigma e^{4\pi u^2} \leq C, \;\; \forall u \in H_0^1(\Sigma)$ *with* $\int_\Sigma |\nabla u|^2 \leq 1$,

(ii) $\int_\Sigma e^{2\pi u^2} \leq C, \;\; \forall u \in H^1(\Sigma)$ *with* $\int_\Sigma u = 0$ *and* $\int_\Sigma |\nabla u|^2 \leq 1$.

Notice that, in (i), the constant 4π is sharp. Moser himself gave an explicit example of a sequence (u_k) in $H_0^1(\Sigma)$ with $\int_\Sigma |\nabla u|^2 \leq 1$ such that

$$\int_\Sigma e^{\alpha {u_k}^2} \to \infty, \;\; \alpha > 1.$$

Moreover, the statement in (ii) is false if we remove the assumption $\int_\Sigma u = 0$, since we could apply it to $u + a$ for every $a > 0$ and contradict the theorem for large values of a.

Moser-Trudinger inequalities were generalized for higher dimensions by Fontana, see [45]. We are specially interested in weaker versions of them, also called Onofri type inequalities.

Corollary 3.5. *Let (Σ, g) be a compact surface with C^1 boundary. Then there exists a constant $C \in \mathbb{R}$, depending only on Σ, such that*

$$\log \int_\Sigma e^u \leq \frac{1}{16\pi} \int_\Sigma |\nabla u|^2 + C \quad \forall u \in H_0^1(\Sigma), \tag{3.7}$$

and

$$\log \int_\Sigma e^u \leq \frac{1}{8\pi} \int_\Sigma |\nabla u|^2 + \fint_\Sigma u + C \quad \forall u \in H^1(\Sigma). \tag{3.8}$$

The first inequality is classical, whereas the second is given in [21]. We give their proofs below to show how they are derived from Theorem 3.4. In both cases, the constant is optimal.

Proof. (i) Let $u \in H_0^1(\mathbb{D}^2)$. By Cauchy's inequality,

$$u \leq \frac{4\pi u^2}{\int_\Sigma |\nabla u|^2} + \frac{\int_\Sigma |\nabla u|^2}{16\pi}.$$

Then,

$$e^u \leq exp\left(\frac{4\pi u^2}{\int_\Sigma |\nabla u|^2}\right) exp\left(\frac{\int_\Sigma |\nabla u|^2}{16\pi}\right).$$

Now we integrate on Σ, take logarithms and apply Theorem 3.4, (i).

(ii) Analogously, take $u \in H^1(\Sigma)$ and consider the function $v = u - \fint_\Sigma u$, which satisfies $\int_\Sigma v = 0$. Again by Cauchy's inequality,

$$e^u \leq exp\left(\fint_\Sigma u\right) exp\left(\frac{2\pi v^2}{\int_\Sigma |\nabla v|^2}\right) exp\left(\frac{\int_\Sigma |\nabla v|^2}{8\pi}\right).$$

Finally, we integrate on Σ, take logarithms and apply (ii) of Theorem 3.4. ∎

For the non-linear boundary terms of the functional I, we will use an analogous version of Theorem 3.4 for the boundary of a compact surface that can be found in [69], for instance.

Theorem 3.6. *Let (Σ, g) be a compact surface with C^1 boundary. Then there exists a constant $C > 0$, depending only on Σ, such that*

$$\log \int_{\partial\Sigma} e^{\pi u^2} \leq C, \ \ \forall u \in H^1(\Sigma) \ \textit{with} \int_\Sigma |\nabla u|^2 \leq 1 \ \textit{and} \int_{\partial\Sigma} u = 0. \tag{3.9}$$

With the same proof of Corollary 3.7, we obtain a weaker version of (3.9):

Corollary 3.7. *Let (Σ, g) be a compact surface with C^1 boundary. Then there exists a constant $C > 0$, depending only on Σ, such that*

$$\log \int_{\partial\Sigma} e^{u} \leq \frac{1}{4\pi} \int_\Sigma |\nabla u|^2 + \fint_{\partial\Sigma} u + C, \quad \forall u \in H^1(\Sigma). \tag{3.10}$$

In the case of the disk, the above inequality with $C = 0$ is the so-called *Lebedev-Milin inequality*, see [83]. Our intention is to interpolate the previous inequalities to get a lower bound for the functional I. To do that, it is convenient to manipulate inequality (3.8) so that the mean value of u in Σ is replace by its average in $\partial\Sigma$. This is possible thanks to certain *trial functions*, whose existence is granted in Lemma 3.1.

Proposition 3.8. *Let (Σ, g) be a compact Riemannian manifold with C^1 boundary. Given $f \in L^p(\Sigma)$ and $h \in L^q(\partial\Sigma)$, with $1 < p, q \leq \infty$ satisfying $\int_\Sigma f \neq 0$ and $\int_{\partial\Sigma} h \neq 0$, there exist constants $C_i \in \mathbb{R}$ such that*

(i) $\log \int_\Sigma e^v \leq \frac{1}{8\pi} \int_\Sigma |\nabla v|^2 + \frac{\int_\Sigma fv}{\int_\Sigma f} + C_1,$

(ii) $\log \int_\Sigma e^v \leq \frac{1}{8\pi} \int_\Sigma |\nabla v|^2 + \frac{\int_{\partial\Sigma} hv}{\int_{\partial\Sigma} h} + C_2,$

(iii) $\log \int_{\partial\Sigma} e^v \leq \frac{1}{4\pi} \int_\Sigma |\nabla v|^2 + \frac{\int_\Sigma fv}{\int_\Sigma f} + C_3,$ *and*

(iv) $\log \int_{\partial\Sigma} e^v \leq \frac{1}{4\pi} \int_\Sigma |\nabla v|^2 + \frac{\int_{\partial\Sigma} hv}{\int_{\partial\Sigma} h} + C_4, \ \forall v \in H^1(\Sigma).$

In particular, taking $h = 1$ in (ii):

$$\log \int_\Sigma e^v \leq \frac{1}{8\pi} \int_\Sigma |\nabla v|^2 + \fint_{\partial\Sigma} v + C, \ \forall v \in H^1(\Sigma). \tag{3.11}$$

Proof. We only show the proof for (i), as the subsequent claims can be obtained with analogous reasoning. However, for each case we point out the trial function required to achieve the conclusion.

The problem

$$\begin{cases} -\Delta w = 4\pi \left(\frac{f}{\int_\Sigma f} - \frac{1}{|\Sigma|} \right) & \text{in } \Sigma \\ \frac{\partial w}{\partial \eta} = 0 & \text{on } \partial\Sigma \end{cases}$$

admits a solution in $H^1(\Sigma)$ by Lemma 3.1. Therefore, we can fix a solution w and apply (3.8) to $v+w$, obtaining:

$$\log \int_\Sigma e^{v+w} \le \frac{1}{8\pi} \int_\Sigma |\nabla v|^2 + \frac{1}{4\pi} \int_\Sigma \langle \nabla v, \nabla w \rangle + \fint_\Sigma v + C.$$

Using the fact that

$$\log \int_\Sigma e^{v+w} \ge C + \log \int_\Sigma e^v,$$

after integrating by parts we are left with:

$$\log \int_\Sigma e^v \le \frac{1}{8\pi} \int_\Sigma |\nabla v|^2 + \frac{1}{4\pi} \int_{\partial\Sigma} \frac{\partial w}{\partial \eta} v - \frac{1}{4\pi} \int_\Sigma (\Delta w) v + \fint_\Sigma v + C.$$

Finally, we use the equation satisfied by w to conclude:

$$\log \int_\Sigma e^v \le \frac{1}{8\pi} \int_\Sigma |\nabla v|^2 + \frac{\int_\Sigma f v}{\int_\Sigma f} + C.$$

For (ii), (iii) and (iv), we repeat the above steps using, respectively, solutions of the following problems:

$$\begin{cases} -\Delta w = -\frac{4\pi}{|\Sigma|} & \text{in } \Sigma, \\ \frac{\partial w}{\partial \eta} = 4\pi \frac{h}{\int_{\partial\Sigma} h} & \text{on } \partial\Sigma, \end{cases} \qquad \begin{cases} -\Delta w = 2\pi \frac{f}{\int_\Sigma f} & \text{in } \Sigma, \\ \frac{\partial w}{\partial \eta} = -\frac{2\pi}{|\partial\Sigma|} & \text{on } \partial\Sigma, \end{cases}$$

$$\text{and} \begin{cases} -\Delta w = 0 & \text{in } \Sigma, \\ \frac{\partial w}{\partial \eta} = 2\pi \left(\frac{h}{\int_{\partial\Sigma} h} - \frac{1}{|\partial\Sigma|} \right) & \text{on } \partial\Sigma. \end{cases}$$

■

3.5.1 Localized versions

As we will see, the direct application of the above inequalities guarantees the existence of a finite infimum for the functional I, but it is not enough to have coercivity because a minimizing sequence could concentrate and explode in a single point on the boundary of the disk. Intuitively, imposing a symmetry condition like in Definition (1.3) dismisses that possibility, but in order to give a rigurous analytical argument, we must introduce local versions of the inequalities above. This idea dates back to [5], and are known as Chen-Li type inequalities (see [27] for more details).

The next one is a localized version for Corollary 3.5, but we follow the proof given in [75]:

Proposition 3.9. *Let (Σ, g) be a compact surface with C^1 boundary, $\Sigma_1 \subset \Sigma$ and $\delta > 0$ such that $(\Sigma_1)^\delta \cap \partial\Sigma = \emptyset$. Then, for every $\varepsilon > 0$, there exists a constant $C \in \mathbb{R}$, depending on ε and δ, such that*

$$16\pi \log \int_{\Sigma_1} e^u \leq \int_{(\Sigma_1)^\delta} |\nabla u|^2 + \varepsilon \int_\Sigma |\nabla u|^2 + C, \tag{3.12}$$

for all $u \in H^1(\Sigma)$ with $\int_\Sigma u = 0$. If we allow $(\Sigma_1)^\delta \cap \partial\Sigma \neq \emptyset$, then (3.12) holds with coefficient 8π:

$$8\pi \log \int_{\Sigma_1} e^u \leq \int_{(\Sigma_1)^\delta} |\nabla u|^2 + \varepsilon \int_\Sigma |\nabla u|^2 + C. \tag{3.13}$$

Proof. We define the cut-off function $f_\delta : \Sigma \to [0,1]$, given by

$$f_\delta = \begin{cases} 1 & \text{if } x \in \Sigma_1, \\ 0 & \text{if } x \in \Sigma \backslash (\Sigma_1)^{\delta/2}. \end{cases}$$

The condition $(\Sigma_1)^\delta \cap \partial\Sigma = \emptyset$ implies that $f_\delta u \in H_0^1(\Sigma)$, and we can apply the first statement in Corollary 3.5:

$$16\pi \log \int_{\Sigma_1} e^u = 16\pi \log \int_{\Sigma_1} e^{f_\delta u} \leq 16\pi \log \int_\Sigma e^{f_\delta u} \leq \int_\Sigma |\nabla (f_\delta u)|^2 + C.$$

By Leibniz's rule:

$$\begin{aligned} 16\pi \log \int_{\Sigma_1} e^u &\leq \int_\Sigma u^2 |\nabla f_\delta|^2 + 2\int_\Sigma f_\delta u \langle \nabla u, \nabla f_\delta \rangle + \int_\Sigma {f_\delta}^2 |\nabla u|^2 \\ &\leq C_\delta \int_\Sigma u^2 + 2\int_\Sigma f_\delta u |\nabla u||\nabla f_\delta| + \int_{(\Sigma_1)^\delta} |\nabla u|^2 . \end{aligned} \tag{3.14}$$

The central term can be bounded using Cauchy's inequality, obtaining:

$$\int_\Sigma f_\delta u |\nabla u||\nabla f_\delta| \leq C_\delta \int_\Sigma u|\nabla u| \leq C_{\varepsilon,\delta} \int_\Sigma u^2 + \varepsilon \int_\Sigma |\nabla u|^2 . \tag{3.15}$$

Combining (3.14) and (3.15):

$$16\pi \log \int_{\Sigma_1} e^u \leq \int_{(\Sigma_1)^\delta} |\nabla u|^2 + \varepsilon \int_\Sigma |\nabla u|^2 + C_{\varepsilon,\delta} \int_\Sigma u^2. \tag{3.16}$$

Finally we address the term $\int_\Sigma u^2$. Let $a \in \mathbb{R}$, $\eta = |\{x \in \Sigma : u(x) \geq a\}|$ and $(u-a)^+ = \max\{0, u-a\}$. Clearly we have $u \leq (u-a)^+ + a$. By direct application of (3.16) to $(u-a)^+$, we obtain:

$$\begin{aligned} 16\pi \log \int_{\Sigma_1} e^u &\leq 16\pi \log \left(e^a \int_{\Sigma_1} e^{(u-a)^+} \right) \leq 16\pi a + \log \int_{\Sigma_1} e^{(u-a)^+} \\ &\leq 16\pi a + \int_{(\Sigma_1)^\delta} |\nabla u|^2 + \varepsilon \int_\Sigma |\nabla u|^2 + C_{\varepsilon,\delta} \int_\Sigma \left((u-a)^+\right)^2 \end{aligned} \tag{3.17}$$

By the Sobolev, Hölder and Poincaré-Sobolev's inequalities:

$$\int_\Sigma \left((u-a)^+\right)^2 = \int_{\{u\geq a\}} \left((u-a)^+\right)^2 \leq \eta^{1/2} \left(\int_\Sigma \left((u-a)^+\right)^4\right)^{1/2}$$
$$\leq \eta^{1/2} \|(u-a)^+\|^2_{H^1(\Sigma)} \leq C\eta^{1/2} \int_\Sigma |\nabla u|^2 . \tag{3.18}$$

Again, by Poincaré-Sobolev:

$$a\eta \leq \int_{\{u\geq a\}} u \leq \int_\Sigma |u| \leq C \left(\int_\Sigma |u|^2\right)^{1/2} \leq C \left(\int_\Sigma |\nabla u|^2\right)^{1/2} . \tag{3.19}$$

If we apply Cauchy's inequality in (3.19), we get:

$$a \leq \theta \int_\Sigma |\nabla u|^2 + \frac{C^2}{\eta^2 \theta}, \;\; \forall \theta > 0. \tag{3.20}$$

Thus, mixing (3.17), (3.18) and (3.20):

$$16\pi \log \int_{\Sigma_1} e^u \leq 16\pi\theta \int_\Sigma |\nabla u|^2 + \int_{(\Sigma_1)^\delta} |\nabla u|^2$$
$$+ \varepsilon \int_\Sigma |\nabla u|^2 + C_{\varepsilon,\delta}\eta^{1/2} \int_\Sigma |\nabla u|^2 + C,$$

and it is enough to take $\theta = \frac{1}{16\pi}$ y $\eta^{1/2} \leq \frac{\varepsilon}{C_{\varepsilon,\delta}}$ to have (3.12).

If we allow $\Sigma_1 \cap \partial\Sigma \neq \emptyset$, $f_\delta u \in H^1(\Sigma)$, and we need to use (3.8) in Corollary 3.5:

$$8\pi \log \int_{\Sigma_1} e^u \leq 8\pi \log \int_\Sigma e^{f_\delta u} \leq \int_\Sigma |\nabla (f_\delta u)|^2 + 8\pi \fint_\Sigma f_\delta u + C. \tag{3.21}$$

We have the bound:

$$\fint_\Sigma f_\delta u \leq \frac{1}{2} \fint_\Sigma {f_\delta}^2 + \frac{1}{2|\Sigma|} \int_\Sigma u^2 \leq C_\delta + C \int_\Sigma u^2.$$

Repeating the above steps, we are led to:

$$\int_\Sigma |\nabla (f_\delta u)|^2 \leq \int_{(\Sigma_1)^\delta} |\nabla u|^2 + \varepsilon \int_\Sigma |\nabla u|^2 + C_{\varepsilon,\delta} \int_\Sigma u^2 + C.$$

Putting together the previous two inequalities and (3.21), and bounding $\int_\Sigma u^2$ as before, we finish the proof for (3.13):

$$8\pi \log \int_{\Sigma_1} e^u \leq \int_{(\Sigma_1)^\delta} |\nabla u|^2 + \varepsilon \int_\Sigma |\nabla u|^2 + C,$$

as desired. ∎

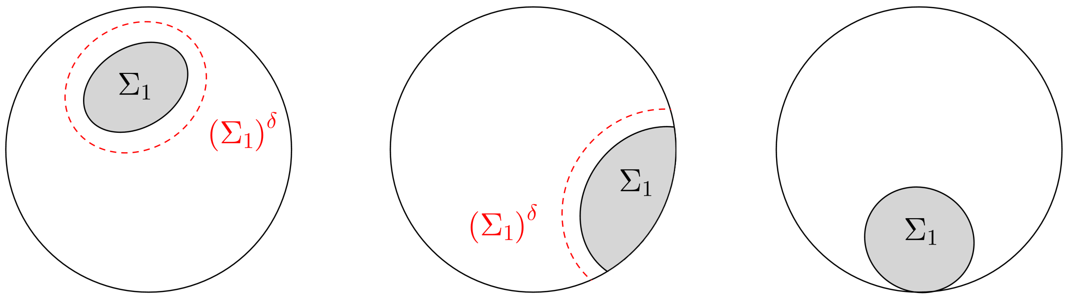

Figure 3.1: For (3.12), the set Σ_1 must be away from the boundary (left), while it can contain a portion of it in (3.13) (center). In fact, Chang and Yang proved in [21] that (3.12) is false if $\delta = 0$ (right).

If the function u has mass in several separated regions satisfying the hypothesis of Proposition 3.9, the bounds improve with the number of such regions. This information is collected in the following corollary (see for instance [75] for the case $l = 2$; the case of general l is analogous).

Corollary 3.10. *Let (Σ, g) be a compact manifold with C^1 boundary, $l \in \mathbb{N}$ and $\Sigma_1, \ldots, \Sigma_l \subset \Sigma$ for which there exits $\delta > 0$ such that $(\Sigma_i)^\delta \cap (\Sigma_j)^\delta = \emptyset$ if $i \neq j$ and $(\Sigma_i)^\delta \cap \partial\Sigma = \emptyset$ for every $i = 1, \ldots, l$. Moreover, assume that there exists a $\gamma \in (0, \frac{1}{l})$ in such a way that*

$$\frac{\int_{\Sigma_i} e^u}{\int_\Sigma e^u} \geq \gamma, \quad \forall i = 1, \ldots, l \tag{3.22}$$

Then, for every $\varepsilon > 0$, there exists a constant $C \in \mathbb{R}$, depending on ε, δ and γ, such that

$$16 l\pi \log \int_\Sigma e^u \leq (1+\varepsilon) \int_\Sigma |\nabla u|^2 + C, \quad \forall u \in H^1(\Sigma) \text{ with } \int_\Sigma u = 0, \tag{3.23}$$

and, if we allow $(\Sigma_i)^\delta \cap \partial\Sigma \neq \emptyset$,

$$8 l\pi \log \int_\Sigma e^u \leq (1+\varepsilon) \int_\Sigma |\nabla u|^2 + C, \;\; \forall u \in H^1(\Sigma) \text{ with } \int_\Sigma u = 0. \tag{3.24}$$

Proof. We will only do the proof for (3.23), since it can be easily adapted to prove (3.24). We start applying (3.12) to each Σ_i, obtaining

$$16\pi \log \int_{\Sigma_i} e^u \leq \int_{(\Sigma_i)^\delta} |\nabla u|^2 + \varepsilon \int_\Sigma |\nabla u|^2 + C.$$

By (3.22):

$$13\pi \log \int_{\Sigma_i} e^u \geq C + 16\pi \log \int_\Sigma e^u.$$

Therefore,

$$16\pi \log \int_\Sigma e^u \le \int_{(\Sigma_i)^\delta} |\nabla u|^2 + \varepsilon \int_\Sigma |\nabla u|^2 + C.$$

Finally, summing on $i \in \{1, \dots, l\}$:

$$16\pi l \log \int_\Sigma e^u \le \int_\Sigma |\nabla u|^2 + l\varepsilon \int_\Sigma |\nabla u|^2 + C.$$

■

We also need localized versions of Corollary 3.7. We just follow the steps of Proposition 3.9 and the subsequent corollary.

Proposition 3.11. *Let (Σ, g) be a compact manifold with C^1 boundary, and $\Gamma_1 \subset \partial\Sigma$. Then, for every $\varepsilon, \delta > 0$, there exists a constant $C \in \mathbb{R}$, depending on ε and δ, such that*

$$4\pi \log \int_{\Gamma_1} e^u \le \int_{(\Gamma_1)^\delta} |\nabla u|^2 + \varepsilon \int_\Sigma |\nabla u|^2 + C, \ \ \forall u \in H^1(\Sigma) \text{ with } \int_\Sigma u = 0.$$

Proof. As we did for 3.9, we consider a cut-off function $f_\delta : \Sigma \to [0, 1]$ defined by:

$$f_\delta = \begin{cases} 1 & \text{if } x \in \Gamma_1, \\ 0 & \text{if } x \in \Sigma \backslash (\Gamma_1)^{\delta/2}. \end{cases}$$

We have $f_\delta u \in H^1(\Sigma)$, and we can apply Corollary 3.7:

$$\begin{aligned} 4\pi \log \int_{\Gamma_1} e^u &= \log \int_{\Gamma_1} e^{f_\delta u} \le \log \int_{\partial\Sigma} e^{f_\delta u} \\ &\le \int_\Sigma |\nabla (f_\delta u)|^2 + 4\pi \fint_{\partial\Sigma} f_\delta u \, + C. \end{aligned} \tag{3.25}$$

As in the proof of Corollary (3.9), we have the following estimates:

$$\begin{aligned} \fint_{\partial\Sigma} f_\delta u \, &\le C_\delta + C \int_\Sigma u^2, \\ \int_\Sigma |\nabla (f_\delta u)|^2 &\le \int_{(\Sigma_1)^\delta} |\nabla u|^2 + \varepsilon \int_\Sigma |\nabla u|^2 + C_{\varepsilon,\delta} \int_\Sigma u^2 + C. \end{aligned}$$

Combining the inequalities above with (3.25), and bounding $\int_\Sigma u^2$ as in Corollary 3.9:

$$4\pi \log \int_{\Gamma_1} e^u \le \int_{(\Gamma_1)^\delta} |\nabla u|^2 + \varepsilon \int_\Sigma |\nabla u|^2 + C.$$

■

Corollary 3.12. *Let (Σ, g) be a compact manifold with C^1 boundary, $l \in \mathbb{N}$ and $\Gamma_1, \ldots, \Gamma_l \subset \partial\Sigma$ for which there exists $\delta > 0$ such that $(\Gamma_i)^\delta \cap (\Gamma_j)^\delta = \emptyset$ if $i \neq j$. Furthermore, assume that there is $\gamma \in (0, \frac{1}{l})$ such that*

$$\frac{\int_{\Gamma_i} e^u}{\int_{\partial\Sigma} e^u} \geq \gamma, \quad \forall i = 1, \ldots, l. \tag{3.26}$$

Then, for every $\varepsilon > 0$ there exists a constant $C \in \mathbb{R}$, depending on ε, δ and γ, in such a way that

$$4l\pi \log \int_{\partial\Sigma} e^u \leq (1+\varepsilon) \int_\Sigma |\nabla u|^2 + C, \quad \forall u \in H^1(\Sigma) \text{ with } \int_\Sigma u = 0. \tag{3.27}$$

Proof. Following the proof of Corollary 3.10, we apply Proposition 3.7 to each Γ_i, obtaining:

$$4\pi \log \int_{\Gamma_i} e^u \leq \int_{(\Gamma_i)^\delta} |\nabla u|^2 + \varepsilon \int_\Sigma |\nabla u|^2 + C.$$

Using hypothesis (3.26),

$$4\pi \log \int_{\Gamma_i} e^u \geq 4\pi \log \int_{\partial\Sigma} e^u + C.$$

Hence,

$$4\pi \log \int_{\partial\Sigma} e^u \leq \int_{(\Gamma_i)^\delta} |\nabla u|^2 + \varepsilon \int_\Sigma |\nabla u|^2 + C.$$

Finally, summing on $i \in \{1, \ldots, l\}$:

$$\begin{aligned} 4l\pi \log \int_{\partial\Sigma} e^u &\leq \int_{\bigsqcup_i (\Gamma_i)^\delta} |\nabla u|^2 + \varepsilon l \int_\Sigma |\nabla u|^2 + C \\ &\leq \int_\Sigma |\nabla u|^2 + \varepsilon l \int_\Sigma |\nabla u|^2 + C, \end{aligned}$$

as desired. ∎

In order to show that our symmetry assumptions together with the localized versions of the Moser-Trudinger inequalities described above transform into coercivity for the functional, we will need to use a particular covering for the unit disk $\mathbb{D}^2$. We do the construction in the following Lemma:

Lemma 3.13. *For every $\delta > 0$, there exists a finite open covering of $\mathbb{D}^2$, $R_\delta = \{\Sigma_i : i = 1, \ldots, n_0(\delta)\}$, such that, either*

(i) $(\Sigma_i)^\delta \cap \mathbb{S}^1 = \emptyset$, *or*

(ii) $\Sigma_i = (\{x_i\})^\delta$ *for some* $x_i \in \mathbb{S}^1$.

Proof. For every $r > 0$, the family of open sets $\{(\{x\})^r \subset \mathbb{D}^2 : \; x \in \mathbb{S}^1\}$ forms a covering of $\mathbb{S}^1$. By compactness, we can extract a finite covering

$$\left\{\overline{\Sigma}_i = (\{x_i\})^r : \; x_i \in \mathbb{S}^1, \;\; i = 1, \dots, k_0(r)\right\} \tag{3.28}$$

such that $\mathbb{S}^1 \subset \bigcup_{i=1}^{k_0} \overline{\Sigma}_i$. Up to taking a smaller $r > 0$, we can assume that the compact set $K = \mathbb{D}^2 \backslash \bigcup_{i=1}^{k_0} \overline{\Sigma}_i$ is nonempty. Since K and $\mathbb{S}^1$ are disjoint compact sets, it is possible to take $0 < s(r) < \frac{1}{2}\operatorname{dist}(K, \mathbb{S}^1)$, and a finite open covering of K,

$$\left\{\tilde{\Sigma}_i : \; i = 1, \dots, k_1(s)\right\}, \tag{3.29}$$

in such a way that $(\tilde{\Sigma}_i)^s \cap \mathbb{S} = \emptyset$. The union of (3.28) and (3.29) is the covering we are looking for. ∎

To conclude the section, we see an important application of Theorems 3.4 and 3.6, which states the compactness of the embedding $\exp : H^1(\Sigma) \to L^p(\Sigma)$ and its trace $\exp : H^1(\Sigma) \to L^p(\partial\Sigma)$, for $1 \leq p < \infty$. This will be relevant when applying Tonelli's direct method to the funcional (1.16). We start with some observations on the condition $\fint_\Sigma u = 0$:

Proposition 3.14. *Let* (Σ, g) *be a compact surface with* C^1 *boundary, and* $a \in \mathbb{R}$. *For each* $\varepsilon > 0$ *there exists a constant* $C \in \mathbb{R}$, *depending on* a *and* ε *such that*

$$\int_\Sigma e^{2\pi(1-\varepsilon)u^2} \leq C, \;\; \forall u \in H^1(\Sigma) \textit{ with } \int_\Sigma |\nabla u|^2 \leq 1 \textit{ and } \fint_\Sigma u = a.$$

Proof. The function $u - a$ satisfies the conditions of Theorem 3.4. Its application gives:

$$\int_\Sigma e^{2\pi u^2 - 4\pi u a} \leq C e^{-2\pi a^2}.$$

By Cauchy's inequality, for every $\varepsilon > 0$ we have

$$ua \leq \frac{\varepsilon u^2}{2} + \frac{a^2}{2\varepsilon},$$

and consequently

$$\int_\Sigma e^{2\pi(1-\varepsilon)u^2} \leq C e^{-2\pi a^2(1-\frac{1}{\varepsilon})} = C_{a,\varepsilon},$$

as we wanted to prove. ∎

If we pay attention to the definition of $C_{a,\varepsilon}$, it is clear that, when $\varepsilon \to 0$, the inequality does not provide any information unless $a = 0$. This remarks the importance of such condition in Theorem 3.4. Analogously:

Proposition 3.15. *Let (Σ, g) be a compact manifold with C^1 boundary, and $a \in \mathbb{R}$. For every $\varepsilon > 0$ there exists a constant $C \in \mathbb{R}$, depending on a and ε such that*

$$\int_{\partial\Sigma} e^{\pi(1-\varepsilon)u^2} \leq C, \ \forall u \in H^1(\Sigma) \text{ with } \int_\Sigma |\nabla u|^2 \leq 1 \text{ and } \fint_{\partial\Sigma} u = a$$

Proof. We apply Theorem 3.6 to $u - a$ and proceed as in Proposition 3.14. ∎

Proposition 3.16. *Let (Σ, g) be a compact surfaces and u_k be a sequence in $H^1(\mathbb{D}^2)$ such that $u_k \rightharpoonup u_0 \in H^1(\Sigma)$. Then, up to considering a subsequence,*

(i) $e^{u_k} \to e^{u_0}$ *for* $L^p(\Sigma)$, *con* $1 \leq p < \infty$,

(ii) $e^{u_k} \to e^{u_0}$ *for* $L^p(\partial\Sigma)$, *con* $1 \leq p < \infty$.

Proof. Given $a, b \in \mathbb{R}$, by the Mean Value Theorem we have the inequality

$$|e^a - e^b| \leq e^{|a|+|b|}|a - b|.$$

In particular,

$$|e^{u_k} - e^{u_0}|^p \leq e^{p(|u_k|+|u_0|)}|u_k - u_0|^p. \tag{3.30}$$

To prove (i), we integrate (3.30) on Σ and apply Hölder's inequality:

$$\int_\Sigma |e^{u_k} - e^{u_0}|^p \leq \left(\int_\Sigma e^{2p(|u_k|+|u_0|)}\right)^{\frac{1}{2}} \left(\int_\Sigma |u_k - u_0|^{2p}\right)^{\frac{1}{2}}.$$

By Cauchy's inequality, for every $\varepsilon > 0$:

$$e^u \leq e^{\frac{1}{2\varepsilon}} e^{\frac{u^2\varepsilon}{2}} = C_\varepsilon e^{\frac{u^2\varepsilon}{2}}.$$

We choose ε in such a way that

$$0 \leq \frac{\varepsilon}{2} < \frac{(1-\delta)2\pi}{\int_\Sigma 2p|\nabla(|u_k| + |u_0|)|^2},$$

where δ is an arbitrarily small positive number. Substituting and applying Proposition 3.14:

$$\begin{aligned}\int_\Sigma e^{2p(|u_k|+|u_0|)} &\leq C \int_\Sigma e^{\frac{\varepsilon}{2}2p(|u_k|+|u_0|)^2} \\ &\leq C \int_\Sigma \exp\left(\frac{2\pi(1-\delta)(|u_k| + |u_0|)^2}{\int_\Sigma |\nabla(|u_k| + |u_0|)|^2}\right) \leq C.\end{aligned}$$

Finally,

$$\int_\Sigma |e^{u_k} - e^{u_0}|^p \leq C \left(\int_\Sigma |u_k - u_0|^{2p}\right)^{\frac{1}{2}}.$$

Claim (i) follows from Rellich-Kondrachov Theorem.

To prove (ii), we integrate 3.30 on $\partial\Sigma$ and repeat the previous steps to get:

$$\int_{\partial\Sigma} |e^{u_k} - e^{u_0}|^p \leq \left(\int_{\partial\Sigma} e^{2p(|u_k|+|u_0|)} \right)^{\frac{1}{2}} \left(\int_{\partial\Sigma} |u_k - u_0|^{2p} \right)^{\frac{1}{2}}. \tag{3.31}$$

We take $\varepsilon > 0$ such that

$$0 \leq \frac{\varepsilon}{2} < \frac{(1-\delta)\pi}{\int_\Sigma 2p|\nabla(|u_k| + |u_0|)|^2},$$

with $\delta > 0$ small enough. Using Cauchy's inequality, Proposition 3.15 and operating as before, we get to:

$$\int_{\partial\Sigma} e^{2p(|u_k|+|u_0|)} \leq C \int_{\partial\Sigma} \exp\left(\frac{\pi(1-\delta)(|u_k| + |u_0|)^2}{\int_\Sigma |\nabla(|u_k| + |u_0|)|^2} \right) \leq C,$$

We plug the previous inequality in (3.31), obtaining

$$\int_{\partial\Sigma} |e^{u_k} - e^{u_0}|^p \leq C \left(\int_{\partial\Sigma} |u_k - u_0|^{2p} \right)^{\frac{1}{2}}.$$

The trace embedding $T : H^1(\Sigma) \to L^q(\partial\Sigma)$ is compact for every $q \geq 1$, because it is the composition of a compact embedding and a continuous function. Therefore, up to taking a subsequence, we can assume that $\|u_k - u_0\|_{L^{2p}(\partial\Sigma)} \to 0$, giving in turn

$$\int_{\partial\Sigma} |e^{u_k} - e^{u_0}|^p \to 0.$$

■

3.6 Solutions of the limit problem in $\mathbb{R}^n_+$

Throughout the rest of the chapter, we will develop analytical tools that specifically concern (1.21). Therefore, *our framework will be that of a Riemannian manifold of dimension* $n \geq 3$.

When performing blow-up analysis, one is usually concerned with certain limit problems after a proper rescaling of solutions. In this case, we are interested in solutions with constant curvatures in the half-space:

$$\begin{cases} \frac{-4(n-1)}{n-2}\Delta v = K(p)v^{\frac{n+2}{n-2}} & \text{on } \mathbb{R}^n_+ \\ \frac{2}{n-2}\frac{\partial v}{\partial \eta} = H(p)v^{\frac{n}{n-2}} & \text{on } \partial\mathbb{R}^n_+ \end{cases}, \tag{3.32}$$

The following result appears in [30]:

Proposition 3.17. *The following assertions hold true:*

(0) If $\mathfrak{D}_n(p) < 1$, then (1.26) *admits no solutions.*

(1) If $\mathfrak{D}_n(p) = 1$, the only solutions are 1−dimensional *and given by:*

$$v(x) = v_\alpha(x) := \left(\frac{2}{\sqrt{n(n-2)}} x_n + \alpha \right)^{-\frac{n-2}{2}}, \tag{3.33}$$

for any $\alpha > 0$.

(2) If $\mathfrak{D}_n(p) > 1$, the solutions are called bubbles *and given by*

$$v(x) = b_\beta(x) := \frac{(n(n-2))^{\frac{n-2}{4}} \beta^{\frac{n-2}{2}}}{\left(|x - x_0(\beta)|^2 - \beta^2\right)^{\frac{n-2}{2}}}, \tag{3.34}$$

with $x_0(\beta) = -\mathfrak{D}_n(p)\beta, e_n \in \mathbb{R}^n$, for $\beta > 0$ arbitrary. In this case, we highlight the following assymptotic behaviour:

$$\lim_{|x| \to +\infty} |x|^{n-2} b_{\beta_0}(x) = (n(n-2))^{\frac{n-2}{2}} \beta_0^{\frac{n-2}{2}} \tag{3.35}$$

for any fixed $\beta_0 > 0$.

3.7 Domain-variations

In order to derive global properties of blowing-up solutions to our problem, we will make use of domain-variations, with calculations that are gathered in this subsection. For variation vector fields of radial type these coincide with the classical Pohozaev identity, but we will need more general ones in Section 5.3. We start with some definitions that will be useful here and in further sections of the thesis.

Definition 3.18. *Given a point $p \in \partial M$, a function $u : M \to \mathbb{R}$ and a vector field $F : M \to TM$, we define $B_p(u, F) : \partial M \to \mathbb{R}$,*

$$B_p(u, F) = C_n |\cdot - p| \Big((F \cdot \eta)^2 - \frac{1}{2}|F|^2 \Big) + 2(n-1)uF \cdot \eta. \tag{3.36}$$

If the point p is clear from the notation we will drop the subscript and write simply $B(u, F)$.

Lemma 3.19. *Given regular functions $\mathfrak{f}, \mathfrak{g} : M \to \mathbb{R}$ and $\mathfrak{h} : \partial M \to \mathbb{R}$, and exponents $0 \leq p \leq \frac{n+2}{n-2}$, $0 \leq q \leq \frac{n}{n-2}$, consider a positive solution $u \in C^2(\overline{M})$ of*

$$-C_n \Delta u + \mathfrak{g} u = \mathfrak{f} u^p \qquad \text{on } M, \tag{3.37}$$

with $C_n = \frac{4(n-1)}{n-2}$*, and* $F : M \to TM$ *any smooth vector field. Then,*

$$C_n \int_M DF(\nabla u, \nabla u) - \frac{C_n}{2} \int_M |\nabla u|^2 \operatorname{div} F - \frac{1}{p+1} \int_M \mathfrak{f} F \cdot \nabla(u^{p+1}) + \frac{1}{2} \int_M \mathfrak{g} F \cdot \nabla(u^2) = C_n \int_{\partial M} (\nabla u \cdot F) \frac{\partial u}{\partial \eta} - \frac{C_n}{2} \int_{\partial M} |\nabla u|^2 F \cdot \eta, \tag{3.38}$$

where $DF(\nabla u, \nabla u) := \sum_{k,j=1}^n \nabla_j F^k u_k u^j$*.*

Proof. Let us introduce the vector field $\mathbb{Y} = (\nabla u \cdot F) \nabla u - \frac{1}{2} |\nabla u|^2 F$: a direct computation shows that

$$\operatorname{div} \mathbb{Y} = \sum_{j=1}^n \nabla_j \mathbb{Y}^j = \sum_{k,j=1}^n \nabla_j F^k u_k u^j + (\nabla u \cdot F) \Delta u - \frac{1}{2} |\nabla u|^2 \operatorname{div} F. \tag{3.39}$$

Finally, multiply (3.37) by $\nabla u \cdot F$ and integrate by parts, using (3.39). ■

Corollary 3.20. *Consider the domain* $\Omega = B(0,r)^+ \subset \mathbb{R}^n_+$*, and let* u *be as in Lemma 3.19. Define:*

$$\mathfrak{P}_\Omega(u) = \frac{1}{p+1} \int_\Omega u^{p+1} X \cdot \nabla \mathfrak{f} + \left(\frac{n}{p+1} - \frac{n-2}{2} \right) \int_\Omega \mathfrak{f} u^{p+1} - \frac{1}{2} \int_\Omega u^2 X \cdot \nabla \mathfrak{g} - \int_\Omega \mathfrak{g} u^2 + \frac{r}{2} \int_{\partial^+\Omega} \mathfrak{g} u^2 - \frac{r}{p+1} \int_{\partial^+\Omega} \mathfrak{f} u^{p+1}. \tag{3.40}$$

Then, if u *solves also* $\frac{2}{n-2} \frac{\partial u}{\partial \eta} = \mathfrak{h} u^q$ *on* ∂M*, we have that*

$$\mathfrak{P}_\Omega(u) = \int_{\partial^+\Omega} B(u, \nabla u) + 2(n-1) \left(\frac{n-2}{2} - \frac{n-1}{q+1} \right) \int_{\partial_0\Omega} \mathfrak{h} u^{q+1} + \frac{2(n-1)}{q+1} \int_{\partial(\partial_0\Omega)} \mathfrak{h} u^{q+1} (X \cdot \nu) - \frac{2(n-1)}{q+1} \int_{\partial_0\Omega} u^{q+1} (\nabla \mathfrak{h} \cdot X), \tag{3.41}$$

Proof. We first apply Lemma 3.19 with $F = X$, taking into account that $\operatorname{div} X = n$ and $DX(\nabla u, \nabla u) = |\nabla u|^2$. We obtain:

$$-2(n-1) \int_\Omega |\nabla u|^2 - \frac{1}{p+1} \int_\Omega \mathfrak{f} X \cdot \nabla(u^{p+1}) + \frac{1}{2} \int_\Omega \mathfrak{g} X \cdot \nabla(u^2) = C_n \int_{\partial\Omega} (\nabla u \cdot X)(\nabla u \cdot \eta) - \frac{C_n}{2} \int_{\partial\Omega} |\nabla u|^2 (X \cdot \eta). \tag{3.42}$$

Moreover, using the Divergence Theorem,

$$\int_\Omega \mathfrak{f} X \cdot \nabla(u^{p+1}) = \int_{\partial\Omega} \mathfrak{f} u^{p+1} X \cdot \eta - \int_\Omega u^{p+1} X \cdot \nabla \mathfrak{f} - n \int_\Omega \mathfrak{f} u^{p+1}; \tag{3.43}$$

$$\int_\Omega g X \cdot \nabla(u^2) = \int_{\partial\Omega} \mathfrak{g} u^2 X \cdot \eta - \int_\Omega u^2 X \cdot \nabla \mathfrak{g} - n \int_\Omega \mathfrak{g} u^{p+1}. \tag{3.44}$$

Multiplying (3.37) by u and integrating by parts we can relate the last two terms, namely

$$\frac{n-2}{2}\int_{\Omega}\mathfrak{f}u^{p+1}-\frac{n-2}{2}\int_{\Omega}\mathfrak{g}u^{2}=2(n-1)\int_{\Omega}|\nabla u|^{2}-2(n-1)\int_{\partial\Omega}u\frac{\partial u}{\partial\eta}. \tag{3.45}$$

Combining (3.43) and (3.44) with (3.45) and pugging them into (3.42), we obtain:

$$\begin{aligned}\frac{-1}{p+1}\int_{\partial\Omega}\mathfrak{f}u^{p+1}X\cdot\eta+\frac{1}{p+1}\int_{\Omega}u^{p+1}X\cdot\nabla\mathfrak{f}-\frac{1}{2}\int_{\Omega}u^{2}X\cdot\nabla\mathfrak{g}&\\ +\frac{1}{2}\int_{\partial\Omega}\mathfrak{g}u^{2}X\cdot\eta-\int_{\Omega}\mathfrak{g}u^{2}+\left(\frac{n}{p+1}-\frac{n-2}{2}\right)\int_{\Omega}\mathfrak{f}u^{p+1}&\\ =C_n\int_{\partial\Omega}(\nabla u\cdot X)\frac{\partial u}{\partial\eta}-\frac{C_n}{2}\int_{\partial\Omega}|\nabla u|^{2}(X\cdot\eta)+2(n-1)\int_{\partial\Omega}u\frac{\partial u}{\partial\eta}.&\end{aligned} \tag{3.46}$$

Taking into account that the exterior normal vector to $\partial\Omega$ satisfies

$$\eta(x)=\begin{cases}\frac{x}{r} & \text{if } x\in\partial^{+}\Omega,\\ -e_n & \text{if } x\in\partial_0\Omega,\end{cases}$$

we proceed to study the right-hand side of (3.46). Integrating by parts we have

$$\begin{aligned}&\int_{\partial_0\Omega}\frac{\partial u}{\partial\eta}(X\cdot\nabla u)=\frac{n-2}{2(q+1)}\int_{\partial_0\Omega}\mathfrak{h}X\cdot\nabla(u^{q+1})\\ &=\frac{n-2}{2(q+1)}\left(\int_{\partial(\partial_0\Omega)}\mathfrak{h}u^{q+1}(X\cdot\nu)-\int_{\partial_0\Omega}u^{q+1}(X\cdot\nabla\mathfrak{h})-(n-1)\int_{\partial_0\Omega}\mathfrak{h}u^{q+1}\right).\end{aligned}$$

Therefore,

$$\begin{aligned}C_n\int_{\partial\Omega}(X\cdot\nabla u)\frac{\partial u}{\partial\eta}&=C_n\int_{\partial^{+}\Omega}r\left(\frac{\partial u}{\partial\eta}\right)^{2}+C_n\int_{\partial_0\Omega}(X\cdot\nabla u)\frac{\partial u}{\partial\eta}\\ &=C_n\int_{\partial^{+}\Omega}r\left(\frac{\partial u}{\partial\eta}\right)^{2}+\frac{2(n-1)}{q+1}\int_{\partial(\partial_0\Omega)}\mathfrak{h}u^{q+1}(X\cdot\nu)\\ &\quad-2(n-1)\int_{\partial_0\Omega}u^{q+1}(X\cdot\nabla\mathfrak{h})-\frac{2(n-1)^{2}}{q+1}\int_{\partial_0\Omega}\mathfrak{h}u^{q+1}.\end{aligned} \tag{3.47}$$

In addition,

$$\frac{-C_n}{2}\int_{\partial\Omega}|\nabla u|^{2}(X\cdot\eta)=-\frac{C_n}{2}r\int_{\partial^{+}\Omega}|\nabla u|^{2} \tag{3.48}$$

$$2(n-1)\int_{\partial\Omega}u\frac{\partial u}{\partial\eta}=2(n-1)\int_{\partial^{+}\Omega}u\frac{\partial u}{\partial\eta}+(n-1)(n-2)\int_{\partial_0\Omega}\mathfrak{h}u^{q+1}. \tag{3.49}$$

Finally, notice also that, by (3.37)

$$\frac{n-2}{2}\int_{\partial_0\Omega}\mathfrak{h}u^{q+1}=\int_{\partial_0\Omega}u\frac{\partial u}{\partial\eta}.$$

Identity (3.41) is a consequence of (3.46), (3.47), (3.48) and (3.49). ∎

As a final goal of this section, we study a particular case of Corollary 3.20 that will be useful later on.

For any fixed constant $a > 0$, define the function $G(x) = a\,|x|^{2-n}$: direct computations show that

$$B(G, \nabla G) = -2(n-1)\left(\frac{(n-2)a^2}{|x|^{2n-3}} - 2\frac{(n-2)a^2}{|x|^{2n-3}} + \frac{(n-2)a^2}{|x|^{2n-3}}\right) = 0.$$

Furthermore, we have the following result:

Proposition 3.21. *Define the function $h : B(r)_+ \to \mathbb{R}$ as*

$$h(x) = G(x) + b(x),$$

for any function $b \in C^1(\overline{B(r)_+})$. Then,

$$\lim_{r\to 0}\int_{\partial^+\Omega} B(h, \nabla h) = -(n-1)(n-2)\omega_{n-1} a\, b(0). \tag{3.50}$$

Proof. The fact that $B(G, \nabla G) = 0$ implies

$$B(h, \nabla h) = B_r(b, \nabla b) - \frac{6a(n-1)}{r^{n-1}}(X \cdot \nabla b) - \frac{2a(n-1)(n-2)}{r^{n-1}} b(x).$$

Integrating on $\partial^+ B_r$, and taking into account that $|\partial^+ B_r| = \frac{1}{2}\omega_{n-1} r^{n-1}$, we obtain:

$$\begin{aligned}\int_{\partial^+ B_r} B(h, \nabla h) = \int_{\partial^+ B_r} B(b, \nabla b) - 3a(n-1)\omega_{n-1} \fint_{\partial^+ B_r} X \cdot \nabla b \\ - a(n-1)(n-2)\omega_{n-1} \fint_{\partial^+ B_r} b.\end{aligned}$$

We conclude by taking the limit $r \to 0$. ■

3.8 An explicit example of blow-up with infinite singular set for (1.21)

Here we show that the cardinality of the singular set $\mathscr{S}_0 = \mathscr{S} \cap \{\mathfrak{D}_n = 1\}$ can be infinite. Indeed, consider $\rho > 1$ and the function $u_\rho : B(0,1) \subset \mathbb{R}^n \to \mathbb{R}$ as:

$$u_\rho(x) = \left(\frac{2\rho}{\rho^2 - |x|^2}\right)^{\frac{n-2}{2}}.$$

It is easy to check that u_ρ solves:

$$\begin{cases} -\frac{4(n-1)}{n-2}\Delta u = -n(n-1)u^{\frac{n+2}{n-2}} & \text{on } B(0,1), \\ \frac{2}{n-2}\frac{\partial u}{\partial \eta} + u = H_\rho u^{\frac{n}{n-2}} & \text{on } \partial B(0,1), \end{cases} \tag{3.51}$$

where $H_\rho = \frac{\rho^2+1}{2\rho}$. As $\rho \to 1$, $H_\rho \to 1$ but $K = -n(n-1)$. In this case, the function u_ρ diverges on the whole boundary $\partial B(0,1) = \mathscr{S}_0$.

Obseve that the function u gives rise to a model of the Hyperbolic space in a ball of radius $\rho > 1$, and hence H_ρ is nothing but the mean curvature of the sphere of Euclidean radius 1 in such a model.

Chapter 4

The Gaussian-Geodesic prescription problem on $\mathbb{D}^2$

This chapter is devoted to the study of (1.12). Firstly, we formalise our variational approach and derive properties of the associated Energy Functional using the Moser-Trudinger type inequalities developed in Chapter 3. Under the symmetry assumptions described in Definition (1.3), the functional is coercive and a minimizer can be found. As a first consequence, we obtain existence results for the limiting cases $\rho = 0$ and $\rho = 2\pi$, associated to the cases of prescribing h with $K = 0$ and K with $h = 0$, respectively. As we will see, their analysis represent a first step in the proof of Theorem 1.4.

4.1 Variational study of the problem

As commented in the introduction, we will be working with the functional I given in (1.16), defined on the space

$$\mathbb{X} \times (0, 2\pi) = \left\{ u \in H^1(\mathbb{D}^2) : \int_{\mathbb{D}^2} K e^u > 0, \; \int_{\mathbb{S}^1} h e^{u/2} > 0 \right\}.$$

Lemma 4.1. *$\mathbb{X}$ is non-empty if and only if K and h are positive somewhere.*

Proof. We reduce ourselves to prove that if K and h are positive somewhere then $\mathbb{X}$ is non-empty, as the reciprocal is immediate. By continuity, since there exists $x_0 \in \mathrm{Int}(\mathbb{D}^2)$ such that $K(x_0) > 0$, then there exists $r > 0$ such that $(\{x_0\})^r \cap \mathbb{S} = \emptyset$ and $K(x) > 0$ for all $x \in (\{x_0\})^r$.

Moreover, there is $x_1 \in \mathbb{S}$ satisfying $h(x_1) > 0$, and again by continuity we get $s > 0$ such that $h(x) > 0$ for all $x \in (\{x_1\})^s \cap \mathbb{S}$. Without loss of generality we can assume $(\{x_0\})^r \cap (\{x_1\})^s = \emptyset$. Now, call $\Omega_0^r := (\{x_0\})^r$ and $\Omega_1^s := (\{x_1\})^s$, and consider a cutoff function $\varphi \in H^1(\mathbb{D}^2)$ satisfying

$$\varphi(x) = \begin{cases} a & \text{if } x \in \Omega_0^{r/2}, \\ b & \text{if } x \in \Omega_1^{s/2}, \\ 0 & \text{if } \mathbb{D}^2 \setminus \left(\Omega_0^r \cup \Omega_1^s\right), \end{cases}$$

where a and b are real constants yet to be determined. We see that:

$$\begin{aligned}\int_{\mathbb{S}^1} he^{\varphi/2} &= \int_{\Omega_1^{s/2}\cap\mathbb{S}^1} he^{\varphi/2} + \int_{\left(\Omega_1^s\setminus\Omega_1^{s/2}\right)\cap\mathbb{S}^1} he^{\varphi/2} + \int_{\mathbb{S}^1\setminus\Omega_1^s} he^{\varphi/2} \\ &\geq e^{b/2}\int_{\Omega_1^{s/2}\cap\mathbb{S}^1} h + \int_{\mathbb{S}^1\setminus\Omega_1^{s/2}} h = C_1 e^{b/2} + C,\end{aligned}$$

being $C_1 > 0$ and $C \in \mathbb{R}$. We can choose b large enough so that

$$\int_{\mathbb{S}^1} he^{\varphi/2} > 0.$$

Furthermore,

$$\begin{aligned}\int_{\mathbb{D}^2} Ke^{\varphi} = \int_{\Omega_0^{r/2}} Ke^{\varphi} + \int_{\Omega_1^{s/2}} Ke^{\varphi} + \int_{\mathbb{D}^2\setminus\left(\Omega_0^r\cup\Omega_1^r\right)} Ke^{\varphi} + \int_{\Omega_1^s\setminus\Omega_1^{s/2}} Ke^{\varphi} \\ + \int_{\Omega_0^r\setminus\Omega_0^{r/2}} Ke^{\varphi} \geq e^a \int_{\Omega_0^{r/2}} K - e^b C_2\|K\|_\infty + C = C_1' e^a - C_2' e^b + C.\end{aligned}$$

Thus, a can also be chosen in such a way that

$$\int_{\mathbb{D}^2} Ke^{\varphi} > 0.$$

■

Critical points of I are in correspondence with weak solutions of the problem (1.15). We now can demonstrate that this formulation is equivalent to the original one.

Lemma 4.2. *Problems* (1.12) *and* (1.15) *are equivalent.*

Proof. Proving that every solution of (1.12) is a solution of (1.15) is immediate, as we just need to take $\rho = \int_{\mathbb{D}^2} Ke^u = 2\pi - \int_{\mathbb{S}^1} he^{u/2} > 0$. Reciprocally, if $u \in \mathbb{X}$ solves (1.15), we can apply the invariance of (1.15) under addition of constants and get, for any $C \in \mathbb{R}$:

$$\begin{aligned}-\Delta(u+C) &= 2\rho\frac{Ke^{u+C}}{e^C\int_{\mathbb{D}^2} Ke^u}\,, \\ \frac{\partial(u+C)}{\partial\eta} + 2 &= 2(2\pi-\rho)\frac{he^{u/2}}{e^{\frac{C}{2}}\int_{\mathbb{S}^1} he^{u/2}}\,.\end{aligned}$$

If we want $u + C$ to be a solution for (1.12), we need $C \in \mathbb{R}$ such that

$$e^C = \frac{\rho}{\int_{\mathbb{D}^2} Ke^u}\,, \quad e^{\frac{C}{2}} = \frac{(2\pi-\rho)}{\int_{\mathbb{S}^1} he^{u/2}}\,.$$

The third equation of (1.15) tells us that both conditions are actually the same. Thus, it is enough to choose $C = \log\rho - \log\int_{\mathbb{D}^2} Ke^u$. ■

As it happens with problem (1.15), the functional I is invariant under the addition of constants. This information allows us to consider functions with null mean in Σ when we work on energy estimates, as we did for the proof of Lemma 3.1.

Proposition 4.3. *Let I be the energy functional defined on* (1.16), *and $C \in \mathbb{R}$. Then $I(u+C,\rho) = I(u,\rho)$ for every $u \in \mathbb{X}$ and $\rho \in (0, 2\pi)$.*

Proof. Clearly, for every $u \in \mathbb{X}$, $\rho \in (0, 2\pi)$ and $C \in \mathbb{R}$:

$$I(u+C,\rho) = I(u,\rho) - 2\rho C + 2\left|\mathbb{S}^1\right| C - 2(2\pi - \rho)C = I(u,\rho),$$

as desired. ■

As we anticipated, the combined use of the inequalities (3.10) and (3.11) allows us to prove that I is bounded from below in $H^1(\mathbb{D}^2)$.

Proposition 4.4. *There exists a constant $C \in \mathbb{R}$ such that $I_\rho(u) \geq C$ for every $u \in \mathbb{X}$ and every $\rho \in\ [0, 2\pi]$.*

Proof. Let us denote by $L : (0, 2\pi) \to \mathbb{R}$ as the correction term in (1.16), that is

$$L(\rho) = 4(2\pi - \rho)\log(2\pi - \rho) + 2\rho + 2\rho \log \rho.$$

It is clear that

$$\lim_{\rho \to 0} L(\rho) = 8\pi \log(2\pi), \quad \lim_{\rho \to 2\pi} L(\rho) = 4\pi + 4\pi \log(2\pi).$$

Then, L can be continuously extended to the compact interval $[0, 2\pi]$. Thus, there exists a constant $M > 0$ such that $|L(\rho)| \leq M$ for all $\rho \in [0, 2\pi]$. Moreover, by the continuity of K and h, there are constants $M_1, M_2 \in \mathbb{R}$ such that

$$\log \int_{\mathbb{D}^2} Ke^u \leq \log \int_{\mathbb{D}^2} e^u + C, \;\; \log \int_{\mathbb{S}^1} he^{u/2} \leq \int_{\mathbb{S}^1} e^{u/2} + C.$$

Then, for every $a, b \in \mathbb{R}$:

$$\begin{aligned} I_\rho(u) &\geq \frac{1}{2}\int_{\mathbb{D}^2} |\nabla u|^2 - 2\rho \log \int_{\mathbb{D}^2} e^u - 4(2\pi - \rho)\log \int_{\mathbb{S}^1} e^{u/2} + 2\int_{\mathbb{S}^1} u + C \\ &= \frac{8\pi - 2a - b}{16\pi}\int_{\mathbb{D}^2} |\nabla u|^2 + \frac{a}{8\pi}\int_{\mathbb{D}^2} |\nabla u|^2 + \frac{b}{16\pi}\int_{\mathbb{D}^2} |\nabla u|^2 \\ &\quad - 2\rho \log \int_{\mathbb{D}^2} e^u - 4(2\pi - \rho)\log \int_{\mathbb{S}^1} e^{u/2} + 2\int_{\mathbb{S}^1} u + C. \end{aligned}$$

The functional I is invariant under the addition of constants, so we can assume that $\fint_{\mathbb{D}^2} u = 0$ and apply inequalities 3.10 and 3.11, taking $a = 2\rho$ and $b = 4(2\pi - \rho)$, obtaining:

$$I_\rho(u) \geq -2\rho \fint_{\mathbb{S}^1} u - 2(2\pi - \rho)\fint_{\mathbb{S}^1} u + 2\int_{\mathbb{S}^1} u + C = C.$$

We highlight that the constant C does not depend on ρ. ■

Proposition 4.4 states that the functional I is bounded from below, but we do not achieve coercivity. The reason behind this is the non-compact action of the conformal group of the disk. This effect appears also in the Nirenberg problem in the sphere, for instance, and makes the problem rather difficult.

In Chapter 3 we have seen that, the more regions the mass of a function is separated in, the better bounds we obtain using the previous local versions of the Moser-Trudinger inequalities. If a function is concentrated in an interior point of the disk, equation (3.23) of Proposition 3.10 with $l = 1$ gives us a lower bound which is sufficient to achieve coercivity, but that is not the case when a function concentrates around a boundary point. Restricting ourselves to spaces of $G-$symmetric functions, with G being a symmetry group without fixed points in $\mathbb{S}^1$, excludes this possibility, since we can always find a second point in the boundary where it concentrates. Hence we will obtain coercivity by interpolating (3.23) with $l = 1$ and (3.24) and (3.27) with $l = 2$.

Lemma 4.5. *Let G be as in Definition* (1.3). *There exist $\delta > 0$ and $\gamma \in (0, 1)$ such that, for every $u \in H^1_G(\mathbb{D}^2)$, one of the following holds:*

(i) Either there exists $\Sigma_0 \subset \mathbb{D}^2$ such that $(\Sigma_0)^\delta \cap \mathbb{S}^1 = \emptyset$ and

$$\frac{\int_{\Sigma_0} e^u}{\int_{\mathbb{D}^2} e^u} \geq \gamma, \quad \textit{or}$$

(ii) There exist $\Sigma_1 = (\{x_0\})^r$ and $\Sigma_2 = (\{x_1\})^r$, with $x_0, x_1 \in \mathbb{S}^1$, such that $(\Sigma_1)^\delta \cap (\Sigma_2)^\delta = \emptyset$ and

$$\frac{\int_{\Sigma_i} e^u}{\int_{\mathbb{D}^2} e^u} \geq \gamma, \quad i = 1, 2.$$

Proof. We start proving the following weaker result: there exist $\gamma \in (0, 1)$ and $\delta > 0$ such that, for every $u \in H^1(\mathbb{D}^2)$, one of the following claims holds:

(i) There exists $\Sigma_1 \subset \mathbb{D}^2$ with $(\Sigma_1)^\delta \cap \mathbb{S}^1 = \emptyset$ and $\frac{\int_{\Sigma_1} e^u}{\int_{\mathbb{D}^2} e^u} \geq \gamma$, or

(ii*) There exists $\Sigma_1 = (\{x_0\})^r$ such that $\frac{\int_{\Sigma_1} e^u}{\int_{\mathbb{D}^2} e^u} \geq \gamma$.

Reasoning by contradiction, we deny both (i) and (ii*). Let R_δ be the open covering for $\mathbb{D}^2$ constructed in Lemma 3.13, which is compound only of open sets as in (i) and (ii*), and take $0 < \gamma < \frac{1}{k}$. There exists $u_\gamma \in H^1(\mathbb{D}^2)$ such that:

$$\int_{\mathbb{D}^2} e^{u_\gamma} \leq \sum_{i=1}^{n} \int_{\Sigma_i} e^{u_\gamma} < \gamma \sum_{i=1}^{n} \int_{\mathbb{D}^2} e^{u_\gamma} = k\gamma \int_{\mathbb{D}^2} e^{u_\gamma}.$$

Therefore, $1 \leq k\gamma$, a contradiction. Assume (ii*) holds, and fix $\Sigma_{i_0} \in R_\delta$, $\Sigma_{i_0} = (\{x_0\})^r$ for some $x_0 \in \mathbb{S}^1$. Since G does not have fixed points in $\mathbb{S}^1$, there exists $\phi \in G$ such that $\phi(x_0) \neq x_0$. By continuity, up to taking a smaller $r > 0$ (and thus, a bigger covering), we can assume that

$$(\phi(\Sigma_{i_0}))^\delta \cap (\Sigma_{i_0})^\delta = \emptyset.$$

Finally, by the area formula:

$$\frac{\int_{\phi(\Sigma_{i_0})} e^u}{\int_{\mathbb{D}^2} e^u} = \frac{\int_{\Sigma_{i_0}} e^u}{\int_{\mathbb{D}^2} e^u} \geq \gamma, \quad \forall u \in H^1_G(\mathbb{D}^2),$$

being $\phi(\Sigma_{i_0}) = (\{\phi(x_0)\})^r$ because of the nature of ϕ. ∎

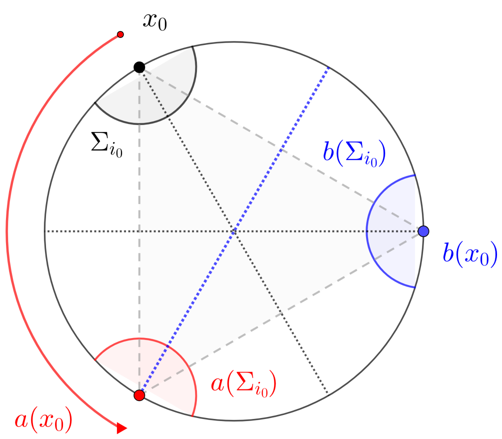

Figure 4.1: Here $G = D_3 = \langle a, b : \ a^3 = b^2 = Id, \ ab = ba^{-1} \rangle$.
The dots represent $G \cdot x_0$.

Lemma 4.6. *Let G be as in Definition* (1.3)*, and $u \in H^1_G(\mathbb{D}^2)$. There exist $\delta > 0$, $\gamma \in (0, \frac{1}{2})$, and $\Gamma_1, \Gamma_2 \subset \mathbb{S}^1$ satisfying $(\Gamma_1)^\delta \cap (\Gamma_2)^\delta = \emptyset$ and*

$$\frac{\int_{\Gamma_i} e^u}{\int_{\mathbb{S}^1} e^u} \geq \gamma, \quad i = 1, 2.$$

Proof. Reasoning as before, it is easy to prove the existence of $0 < \gamma < 1$ and $\Gamma_1 \subset \mathbb{S}^1$ such that

$$\frac{\int_{\Gamma_1} e^u}{\int_{\mathbb{S}^1} e^u} \geq \gamma, \quad \forall u \in H^1(\mathbb{D}^2).$$

To that aim, we consider a finite covering of $\mathbb{S}^1$ of the form

$$R' = \left\{\Gamma_i = B_{\mathbb{S}^1}(x_i, r) : \ x_i \in \mathbb{S}^1, \, i = 1, \dots, k_0(r)\right\}$$

and reason by contradiction: for a fixed $0 < \gamma < \frac{1}{k}$, there exists $u_\gamma \in H^1(\mathbb{D}^2)$ satisfying

$$\int_{\mathbb{S}^1} e^{u_\gamma} \leq \sum_{i=1}^{n} \int_{\Gamma_i} e^{u_\gamma} < \gamma \sum_{i=1}^{n} \int_{\mathbb{S}^1} e^{u_\gamma} = \gamma k \int_{\mathbb{S}^1} e^{u_\gamma}.$$

Thus, $1 < \gamma k$, which is a contradiction for large enough values of k. Let $\Gamma_{i_0} = B_{\mathbb{S}^1}(x_0, r)$ be such subset. Following the proof of Lemma 4.5, given $\delta > 0$ small enough, up to taking a smaller $r > 0$, we can assume that there exists $\phi \in G$, with $\phi(x_0) \neq x_0$, that satisfies $\phi(\Gamma_{i_o}) = B_{\mathbb{S}^1}(\phi(x_0), r)$ and

$$(\phi(\Gamma_{i_0}))^\delta \cap (\Gamma_{i_0})^\delta = \emptyset.$$

Finally, by the area formula:

$$\frac{\int_{g(\Gamma_{i_0})} e^u}{\int_{\mathbb{S}^1} e^u} = \frac{\int_{\Gamma_{i_0}} e^u}{\int_{\mathbb{S}^1} e^u} \geq \gamma,$$

as desired. ■

Corollary 4.7. *Let G be as in Definition* (1.3). *For every $\varepsilon > 0$, there exists a constant $C \in \mathbb{R}$, depending on ε, such that*

(a) $16\pi \log \int_{\mathbb{D}^2} e^u \leq \int_{\mathbb{D}^2} |\nabla u|^2 + \varepsilon \int_{\mathbb{D}^2} |\nabla u|^2 + C \;\; \forall u \in H^1_G(\mathbb{D}^2)$ *with* $\int_{\mathbb{D}^2} u = 0,$

(b) $32\pi \log \int_{\mathbb{S}^1} e^{\frac{u}{2}} \leq \int_{\mathbb{D}^2} |\nabla u|^2 + \varepsilon \int_{\mathbb{D}^2} |\nabla u|^2 + C \;\; \forall u \in H^1_G(\mathbb{D}^2)$ *with* $\int_{\mathbb{D}^2} u = 0.$

Proof. Let $u \in H^1_G(\mathbb{D}^2)$, and apply Lemma 4.5. If (i) holds, then we are under the hypothesis of Corollary 3.10, and we can apply equation (3.23) with $l = 1$, obtaining (a). The same happens if (ii) is true, via (3.24) with $l = 2$. (b) follows from Lemma 4.6 and (3.27) with $l = 2$. ■

The previous estimates provide us the coercivity of the functional I in a space of symmetric functions.

Lemma 4.8. *Let G be as in Definition* (1.3). *If K and h are $G-$symmetric functions, positive somewhere, then $\mathbb{X}^G$ is non-empty.*

Proof. We will follow the proof of Lemma 4.1, constructing a $G-$symmetric test function φ for each type of group in 1.3.

Take $x_0 \in \operatorname{Int} \mathbb{D}^2$ such that $K(x_0) > 0$, and $x_1 \in \mathbb{S}^1$ with $h(x_1) > 0$. If G has finite order, then the orbits $G \cdot x_0$ and $G \cdot x_1$ are discrete and finite, hence it is possible to consider a small radius $s > 0$ such that

i. $(\{x\})^s \cap (\{y\})^s = \emptyset$ for every $x, y \in G \cdot x_i$ with $x \neq y$, $i = 0, 1$.

ii. $(\{x\})^s \cap (\{z\})^s = \emptyset$ for every $x \in G \cdot x_1$ and $z \in G \cdot x_0$.

iii. $(\{x\})^s \cap \mathbb{S}^1 = \emptyset$ for every $z \in G \cdot x_0$.

It is enough to consider the disconnected domains

$$\Omega_i^s = \bigcup_{x \in G \cdot x_i} (\{x\})^s, \;\; i = 0, 1.$$

and repeat the steps of the proof for Lemma 4.1. On the other hand, if the order of G is infinite, then by continuity K is a radial function and h is constant and positive. For this case, we can consider a test function of the form $\varphi(x) = \psi(|x|)$, where $\psi : [0,1] \to \mathbb{R}$ is given by

$$\psi(r) = \begin{cases} a & \text{if } r = r_0 \\ b & \text{if } r = 1 \\ 0 & \text{for } r \text{ outside small neighbourhoods of } r_0 \text{ and } 1, \end{cases}$$

and $r_0 \in [0,1)$ is such that $K(x) > 0$ for $|x| = r_0$. ■

Proposition 4.9. *Let G be as in Definition* (1.3)*, and K, h Hölder-continuous and $G-$symmetric functions. Given $\rho \in (0, 2\pi)$, the functional I_ρ defined on 1.16 is coercive on $\mathbb{X}^G$, that is to say,*

$$\lim_{\substack{\|u\|_{H^1(\mathfrak{D})} \to +\infty \\ u \in \mathbb{X}^G}} I_\rho(u) = +\infty.$$

Proof. Let (u_k) be a sequence in $\mathbb{X}^G$, which is not empty due to Lemma 4.8. Since I_ρ is invariant under the addition of constants by Proposition 4.3, we can assume without loss of generality that $\int_{\mathbb{D}^2} u_k = 0$ for every $k \in \mathbb{N}$. As in the proof of Proposition 4.4, we get the following lower bound.

$$I_\rho(u_k) \geq \frac{1}{2} \int_{\mathbb{D}^2} |\nabla u_k|^2 - 2\rho \log \int_{\mathbb{D}^2} e^{u_k} - 4(2\pi - \rho) \log \int_{\mathbb{S}^1} e^{\frac{u_k}{2}} + 2 \int_{\mathbb{S}^1} u_k + C.$$

We rewrite the previous inequality: for every $a, b \in \mathbb{R}$ we have:

$$I(u_k, \rho) \geq \frac{16\pi - 2a - b}{32\pi} \int_{\mathbb{D}^2} |\nabla u_k|^2 + \frac{a}{16\pi} \int_{\mathbb{D}^2} |\nabla u_k|^2 + \frac{b}{32\pi} \int_{\mathbb{D}^2} |\nabla u_k|^2 + 2 \int_{\mathbb{S}^1} u_k$$
$$- 2\rho \log \int_{\mathbb{D}^2} e^{u_k} - 4(2\pi - \rho) \log \int_{\mathbb{S}^1} e^{\frac{u_k}{2}}.$$

Now we apply Proposition 4.7, passing to a subsequence if necessary:

$$I(u_k, \rho) \geq \frac{16\pi - 2a - b}{32\pi} \int_{\mathbb{D}^2} |\nabla u_k|^2 + a \log \int_{\mathbb{D}^2} e^{u_k} - a\varepsilon \int_{\mathbb{D}^2} |\nabla u_k|^2 + b \log \int_{\mathbb{S}^1} e^{\frac{u_k}{2}}$$
$$- b\varepsilon \int_{\mathbb{D}^2} |\nabla u_k|^2 - 2\rho \log \int_{\mathbb{D}^2} e^{u_k} - 4(2\pi - \rho) \log \int_{\mathbb{S}^1} e^{\frac{u_k}{2}} + 2 \int_{\mathbb{S}^1} u_k + C.$$

Choosing $a = 2\rho$ and $b = 4(2\pi - \rho)$, and using the trace inequality:

$$I(u_k, \rho) \geq \left(\frac{1}{4} - \varepsilon\right) \int_{\mathbb{D}^2} |\nabla u_k|^2 - 2C_2 \|u_k\|_{H^1(\mathbb{D}^2)} + C, \quad C_2 > 0.$$

Finally, we choose ε small enough and apply Poincaré-Sobolev inequality:

$$I(u_k, \rho) \geq C_1 \|u_k\|^2_{H^1(\mathbb{D}^2)} - C_2 \|u_k\|_{H^1(\mathbb{D}^2)} + C, \quad C_1, C_2 > 0$$

We highlight the fact that C_1 is independent of ρ. ∎

4.2 Proof of Theorem 1.4 and its generalizations

We begin this section addressing the limiting cases $\rho = 0$ and $\rho = 2\pi$. These cases are interesting in their own, as will be shown, but their study is also necessary for the proof of Theorems 1.4 and 1.5.

In the limiting case $\rho = 0$, the energy functional I_0 is given by:

$$I_0(u) = I(u, 0) = \frac{1}{2} \int_{\mathbb{D}^2} |\nabla u|^2 + 2 \int_{\mathbb{S}^1} u - 8\pi \log \int_{\mathbb{S}^1} h e^{\frac{u}{2}} + 8\pi \log(2\pi), \tag{4.1}$$

and since K does not play any role, it can be defined on the bigger set

$$\mathbb{X}_0 = \left\{ u \in H^1(\mathbb{D}^2) : \int_{\mathbb{S}^1} h e^{u/2} > 0 \right\} \supset \mathbb{X}.$$

The critical points of I_0 on $\mathbb{X}_0$ are weak solutions of the problem

$$\begin{cases} -\Delta u = 0 & \text{in } \mathbb{D}^2, \\ \frac{\partial u}{\partial \eta} + 2 = 4\pi \frac{h e^{u/2}}{\int_{\mathbb{S}^1} h e^{u/2}} & \text{on } \mathbb{S}^1, \end{cases}$$

which is clearly equivalent to the problem of prescribing Gaussian curvature $K = 0$ and geodesic curvature h, that is,

$$\begin{cases} -\Delta u = 0 & \text{in } \mathbb{D}^2, \\ \frac{\partial u}{\partial \eta} + 2 = 2h e^{u/2} & \text{on } \mathbb{S}^1. \end{cases} \tag{4.2}$$

For this problem, we obtain the following result:

Theorem 4.10. *Let G be as in Definition 1.3, and $h : \mathbb{S}^1 \to \mathbb{R}$ a $G-$symmetric, Hölder continuous and somewhere positive function. Then Problem* (4.2) *admits a solution as a minimum of I_0 on $\mathbb{X}_0^G$.*

Proof. We recall the definition of the Energy Functional associated to the problem, described in (4.1):

$$I_0(u) = I(u,0) = \frac{1}{2}\int_{\mathbb{D}^2} |\nabla u|^2 + 2\int_{\mathbb{S}^1} u - 8\pi \log \int_{\mathbb{S}^1} he^{\frac{u}{2}} + 8\pi \log(2\pi).$$

By Proposition 4.4, I_0 is bounded from below in $H^1(\mathbb{D}^2) \supset \mathbb{X}_0^G$, so there exists

$$\alpha = \inf_{u\in\mathbb{X}_0^G} I_0(u).$$

Our intention is to show that I_0 admits a minimizer in the space of $G-$symmetric functions X_0^G. Let (u_k) be a minimizing sequence in $\mathbb{X}_0^G$, that is, $I_0(u_k) \to \alpha$. By Proposition 4.9 we know that I_0 is coercive, so u_k is bounded in the $H^1(\mathbb{D}^2)$ norm. Therefore, since $H^1(\mathbb{D}^2)$ is reflexive, we can assume that there exists u_0 in $H^1(\mathbb{D}^2)$ such that, up to a subsequence, $u_k \rightharpoonup u_0$. Then, by compactness we also have

$$\int_{\mathbb{S}^1} u_k \to \int_{\mathbb{S}^1} u_0,$$

and by Lemma 3.16

$$\int_{\mathbb{S}^1} he^{u_k/2} \to \int_{\mathbb{S}^1} he^{u_0/2}.$$

Combining this information with the fact that the function $u \to \int_{\mathbb{D}^2} |\nabla u|^2$ is weakly lower semicontinuous, we have $I_0(u_0) \leq \alpha$. It is easy to check that $\int_{\mathbb{S}^1} he^{\frac{u_0}{2}} > 0$, because if we had $\int_{\mathbb{S}} he^{u_k/2} \to 0$ then $I_0(u_k) \to +\infty$, which contradicts that u_k is minimizing. Also, notice that weak convergence respect symmetry, so u_0 is a $G-$symmetric function. ∎

Analogously, we can consider the functional related to the limiting case $\rho = 2\pi$,

$$I_{2\pi}(u) = I(u, 2\pi) = \frac{1}{2}\int_{\mathbb{D}^2} |\nabla u|^2 + 2\int_{\mathbb{S}^1} u - 4\pi \log \int_{\mathbb{D}^2} Ke^u + 4\pi + 4\pi\log(2\pi), \quad (4.3)$$

defined on

$$\mathbb{X}_{2\pi} = \left\{ u \in H^1(\mathbb{D}^2) : \int_{\mathbb{D}^2} Ke^u > 0 \right\} \supset \mathbb{X}.$$

One can check that its variation with respect to u produces weak solutions of the problem

$$\begin{cases} -\Delta u = 4\pi \frac{Ke^u}{\int_{\mathbb{D}^2} Ke^u} & \text{in } \mathbb{D}^2, \\ \frac{\partial u}{\partial \eta} + 2 = 0 & \text{on } \mathbb{S}^1, \end{cases}$$

which is equivalent to the problem of prescribing geodesic curvature $h = 0$ and Gaussian curvature K:

$$\begin{cases} -\Delta u = 2Ke^u & \text{in } \mathbb{D}^2, \\ \frac{\partial u}{\partial \eta} + 2 = 0 & \text{on } \mathbb{S}^1. \end{cases} \quad (4.4)$$

Theorem 4.11. *Let G be as in Definition 1.3, and $K : \mathbb{D}^2 \to \mathbb{R}$ a $G-$symmetric, Hölder continuous and somewhere positive function. Then Problem* (4.4) *admits a solution as a minimum of $I_{2\pi}$ on $\mathbb{X}_G^{2\pi}$.*

Proof. This proof is a trivial adaptation of the previous one: as before, we will find a minimizer of the Energy Functional (4.3),

$$I_{2\pi}(u) = \frac{1}{2}\int_{\mathbb{D}^2} |\nabla u|^2 + 2\int_{\mathbb{S}^1} u - 4\pi \log \int_{\mathbb{D}^2} Ke^u + 4\pi + 4\pi \log(2\pi),$$

in the space of $G-$symmetric functions $\mathbb{X}_{2\pi}^G$. By Proposition 4.4, we can consider the infimum

$$\beta = \inf_{u \in \mathbb{X}_{2\pi}^G} I_{2\pi}(u),$$

and a minimizing sequence (u_k) in $\mathbb{X}_{2\pi}^G$, such that $I(u_k) \to \beta$. Coercivity, granted by Proposition 4.9, gives us the boundedness of (u_k), so it is not restrictive to assume that $u_k \rightharpoonup u_0 \in H^1(\mathbb{D}^2)$. Compactness plus Lemma 3.16 make

$$\int_{\mathbb{S}^1} u_k \to \int_{\mathbb{S}^1} u_0, \quad \int_{\mathbb{D}^2} Ke^{u_k} \to \int_{\mathbb{D}^2} Ke^{u_0}.$$

Then, $I_{2\pi}$ is weak lower semi-continuous and $I_{2\pi}(u_0) \leq \beta$. To prove that $u_0 \in \mathbb{X}^G$, it is enough to notice that if $\int_{\mathbb{D}^2} Ke^{u_k} \to 0$, then $I(u_k) \to +\infty$. ∎

The existence result of Theorem 4.10 is known, see for instance [72]. We have not found an explicit statement of the existence result of Theorem 4.11, but we guess that it must be also known. However we have reinterpreted those solutions as minimizers of I_0 and $I_{2\pi}$, respectively, needed in what follows.

The minimization of the functional I will be done in two steps: first, we freeze the real variable ρ, and find a minimizer for each functional $u \to I(u, \rho)$. After that, we minimize the function $\rho \to \min I(\cdot, \rho)$, and see that its minimum cannot be achieved at the extrema of the interval. The following result guarantees that this approach produces a global minimum for $I(u, \rho)$.

Lemma 4.12. *Let $I : H^1 \times (0, 2\pi) \to \mathbb{R}$ a real functional. Then,*

$$\inf_{(u,\rho) \in H^1 \times (0,2\pi)} I(u, \rho) = \inf_{\rho \in (0,2\pi)} \left(\inf_{u \in H^1} I(u, \rho) \right).$$

Proof. Let us call

$$\alpha = \inf_{(u,\rho) \in H^1 \times (0,2\pi)} I(u, \rho), \quad \text{and } \beta = \inf_{\rho \in (0,2\pi)} \left(\inf_{u \in H^1} I(u, \rho) \right).$$

We start proving that $\alpha \leq \beta$. By the characterization of the infimum, given $\varepsilon > 0$ there exists $\rho' \in (0, 2\pi)$ such that

$$\inf_{u \in H^1} I(u, \rho') < \beta + \varepsilon.$$

Clearly,

$$\alpha = \inf_{(u,\rho) \in H^1 \times (0,2\pi)} I(u, \rho) \leq \inf_{u \in H^1} I(u, \rho') < \beta + \varepsilon.$$

Thus, $\alpha < \beta + \varepsilon$, and it is enough to take limits when $\varepsilon \to 0$ to get $\alpha \leq \beta$.
Conversely, for a fixed $\varepsilon > 0$ we find $(u', \rho') \in H^1 \times (0, 2\pi)$ such that

$$I(u', \rho') < \alpha + \varepsilon.$$

Taking the infimum, firstly in $\rho' \in (0, 2\pi)$, and in $u' \in H^1$ afterwards, we are left with

$$\beta = \inf_{\rho' \in (0,2\pi)} \left(\inf_{u' \in H^1} I(u', \rho') \right) < \alpha + \varepsilon,$$

and it is enough to send $\varepsilon \to 0$ to obtain $\beta \leq \alpha$, concluding the proof. ∎

Let us now conclude the proof of Theorem 1.4.

Proof of Theorem 1.4. If $K = 0$ or $h = 0$, then we are under the assumptions of Theorem 4.10 or Theorem 4.11. Then, we can assume that both K and h are positive in some point and non-negative. In this particular case, $\mathbb{X}^G = \mathbb{X}^G_0 = \mathbb{X}^G_{2\pi} = H^1_G(\mathbb{D}^2)$. By Proposition 4.9, using Tonelli's direct method as we did in the proofs of Theorems 4.10 and 4.11, it is easy to prove that, for each $\rho \in (0, 2\pi)$, there exists u_ρ in $\mathbb{X}^G$ such that

$$I_\rho(u_\rho) = \min_{u \in \mathbb{X}^G} I_\rho(u).$$

Denote by $\hat{\rho}$ the minimum of the function $\rho \to I(u_\rho)$ in $[0, 2\pi]$. By Lemma 4.12, we know $(u_{\hat{\rho}}, \hat{\rho}) = (\hat{u}, \hat{\rho}) \in H^1_G(\mathbb{D}^2) \times [0, 2\pi]$ is a global minimizer for I. We conclude if we exclude the possibilities $\hat{\rho} = 0$ or $\hat{\rho} = 2\pi$.

Assume that $\hat{\rho} = 0$. Observe that in this case, $\hat{u}$ is a minimizer for $I(\cdot, 0)$. Then,

$$\begin{aligned} I(\hat{u}, 0) \leq I(\hat{u}, \rho) = I(\hat{u}, 0) - 2\rho \log\left(\int_{\mathbb{D}^2} K e^{\hat{u}}\right) + 4\rho \log\left(\int_{\mathbb{S}^1} h e^{\hat{u}/2}\right) \\ +8\pi \log\left(\frac{2\pi - \rho}{2\pi}\right) - 4\rho \log(2\pi - \rho) + 2\rho + 2\rho \log \rho. \end{aligned} \tag{4.5}$$

But observe that, as $\rho \to 0$, the main term above is $2\rho \log \rho$, which is negative. This gives a contradiction that excludes the case $\hat{\rho} = 0$. One can exclude the case $\hat{\rho} = 2\pi$ in an analogous way:

$$\begin{aligned} I(\hat{u}, 2\pi) \leq I(\hat{u}, \rho) = I(\hat{u}, \rho) + (4\pi - \rho) \log \int_{\mathbb{D}^2} K e^{\hat{u}} - 4(2\pi - \rho) \log \int_{\mathbb{S}^1} h e^{\hat{u}/2} \\ + 4(2\pi - \rho) \log(2\pi - \rho) + 2\rho + 2\rho \log \rho - 4\pi - 4\pi \log 2\pi. \end{aligned} \tag{4.6}$$

In this case, the leading term is $4(2\pi-\rho)\log(2\pi-\rho)$, which is again negative as ρ approaches 2π. ■

The proof of Theorem 1.4 can be adapted to a more general setting as follows:

Theorem 4.13. *Let G be as in 1.3, and let K, h be $G-$symmetric, Hölder continuous functions that are positive somewhere. We define*

$$S_0=\left\{u\in\mathbb{X}^1_G:\ I_0(u)=\min_{\mathbb{X}^1_G} I_0\right\},\quad S_{2\pi}=\left\{u\in\mathbb{X}^2_G:\ I_{2\pi}(u)=\min_{\mathbb{X}^2_G} I_{2\pi}\right\}.$$

If $S_0\cap\mathbb{X}_G$ and $S_{2\pi}\cap\mathbb{X}_G$ are nonempty, then (1.12) *admits a solution.*

Firstly, observe that Theorem 1.4 is an immediate consequence of Theorem 4.13. Notice also that the sets S_0 and $S_{2\pi}$ of the hypotheses are nonempty because of Theorems 4.10 and 4.11.

Proof. The proof relies on the same energy comparison argument than above, but a couple of details are worth to be written down. First, the existence of a minimizer in $\mathbb{X}^G\times[0,2\pi]$ is not clear a priori.

By Proposition 4.4, we can consider a minimizing sequence $(u_k,\rho_k)\in\mathbb{X}_G\times(0,2\pi)$, that is, $I(u_k,\rho_k)\to\inf I$. Clearly u_k is bounded in $H^1_G(\mathbb{D}^2)$ by Proposition 4.9, but in this case the weak limit $\hat{u}$ could fall in $\partial\mathbb{X}_G$, which was empty in Theorem 1.4. If either $\int_{\mathbb{D}^2}Ke^{\hat{u}}=0$, or $\int_{\mathbb{S}^1}he^{\hat{u}/2}=0$, then knowing the leading term in (4.5) and (4.6) requires a deeper analysis.

If $\rho_k\to\hat{\rho}\in(0,2\pi)$, from the fact that $I(u_k,\rho_k)$ is bounded we obtain:

$$0<\varepsilon<\int_{\mathbb{D}^2}Ke^{u_k}<C,\quad 0<\varepsilon<\int_{\mathbb{S}^1}he^{u_k/2}<C,$$

for some $\varepsilon>0$, $C>0$. Therefore, $\hat{u}\in\mathbb{X}_G$ and we conclude as in the proof of Theorem 1.4.

Assume now that $\rho_k\to 0$. For large values of k, the following estimate holds:

$$I(u_k,\rho_k)\geq -2\rho_k\log\left(\int_{\mathbb{D}^2}Ke^{u_k}\right)-4(2\pi-\rho_k)\log\left(\int_{\mathbb{S}^1}he^{u_k/2}\right)+C. \tag{4.7}$$

Notice that

$$\liminf_{k\to\infty}-2\rho_k\log\left(\int_{\mathbb{D}^2}Ke^{u_k}\right)\geq 0.$$

Substituting in (4.7), we see that $-\log\left(\int_{\mathbb{S}}he^{u_k/2}\right)$ must be bounded from above, which means that

$$0<\varepsilon<\int_{\mathbb{S}^1}he^{u_k/2}.$$

Now, we write:

$$I(u_k, \rho_k) = I(u_k, 0) - 2\rho_k \log\left(\int_{\mathbb{D}^2} K e^{u_k}\right) + 4\rho_k \log\left(\int_{\mathbb{S}^1} h e^{u_k/2}\right)$$
$$+ 8\pi \log\left(\frac{2\pi - \rho_k}{2\pi}\right) - 4\rho_k \log(2\pi - \rho_k) + 2\rho_k + 2\rho_k \log \rho_k.$$

From this we deduce that:

$$\inf I = \lim_{k\to\infty} I(u_k, \rho_k) \geq \liminf_{k\to\infty} I(u_k, 0) \geq I(u_0, 0),$$

where $u_0 \in S_0 \cap \mathbb{X}_G$. But, as in the proof of Theorem 1.4,

$$I(u_0, 0) > I(u_0, \rho),$$

for small values of ρ. This contradiction shows that ρ_k cannot converge to 0. In a similar fashion we can exclude the case $\rho_k \to 2\pi$. ■

4.2.1 A perturbation result

In this section it is necessary to specify the dependence of I on the curvature functions K and h, so we are writting $I(u, \rho) = I[K, h](u, \rho)$. We begin with a compactness result:

Lemma 4.14. *Let (K_k) and (h_k) be sequences of Hölder continuous $G-$symmetric functions, defined on $\mathbb{D}^2$ and $\mathbb{S}^1$ respectively, such that*

$$K_k \to K \text{ uniformly in } \mathbb{D}^2 \text{ and } K \in C^{0,\alpha}(\mathbb{D}^2),$$
$$h_k \to h \text{ uniformly on } \mathbb{S}^1 \text{ and } h \in C^{0,\alpha}(\mathbb{S}^1).$$

Let us consider a sequence (u_k), where each u_k is a solution of the problem

$$\begin{cases} -\Delta u = 2K_k e^u & \text{in } \mathbb{D}^2, \\ \frac{\partial u}{\partial \eta} + 2 = 2h_k e^{u/2} & \text{on } \mathbb{S}^1, \end{cases} \tag{4.8}$$

satisfying

$$\rho_k = \int_{\mathbb{D}^2} K_k e^{u_k} > 0, \quad \int_{\mathbb{S}^1} h_k e^{u_k/2} > 0, \quad \forall k \in \mathbb{N}. \tag{4.9}$$

Assume that $I[K_k, h_k](u_k, \rho_k)$ is uniformly bounded from above. Then $u_k \rightharpoonup u_\infty$ on $H^1(\mathbb{D}^2)$, being u_∞ a solution of the problem

$$\begin{cases} -\Delta u = 2K e^u & \text{in } \mathbb{D}^2, \\ \frac{\partial u}{\partial \eta} + 2 = 2h e^{u/2} & \text{on } \mathbb{S}^1. \end{cases} \tag{4.10}$$

Proof. First, by definition of convergence, for every $\varepsilon > 0$ we find $n_0 \in \mathbb{N}$ such that, for $n \geq n_0$:

$$\|K_k\|_\infty < \|K\|_\infty + \varepsilon, \;\; \|h_k\|_\infty < \|h\|_\infty + \varepsilon.$$

Condition (4.9), together with the Gauss-Bonnet theorem, give us $0 < \rho_k < 2\pi$ for all $k \in \mathbb{N}$. Then, for $n \geq n_0$ we have the following bound:

$$I[K_k, h_k](u_k, \rho_k) \geq I(\|K\|_\infty + \varepsilon, \|h\|_\infty + \varepsilon)(u_k, \rho_k).$$

Then, by Proposition 4.9, there exist constants $C_1, C_2 > 0$, independent of k, such that

$$I[K_k, h_k](u_k, \rho_k) \geq C_1 \|u_k\|^2_{H^1} - C_2 \|u_k\|_{H^1} + C_\varepsilon.$$

Since $I[K_k, h_k](u_k, \rho_k)$ is uniformly bounded from above by hypothesis, we have that u_k is bounded in the $H^1(\mathbb{D}^2)$ norm. Hence, up to a subsequence we can assume that there exists $u_\infty \in H^1(\mathbb{D}^2)$ such that $u_k \rightharpoonup u_\infty$.

By Proposition 3.16, $2K_k e^{u_k} \to 2Ke^{u_\infty}$ and $2h_k e^{u_k/2} \to 2he^{u_\infty/2}$ on L^p for $1 \leq p < +\infty$. Since weak convergence agrees with continuous functions, we also have $\langle \nabla u_k, w \rangle \to \langle \nabla u_\infty, w \rangle$ for all $w \in H^1(\mathbb{D}^2)$. Moreover, by the trace inequality

$$u_k|_{\mathbb{S}^1} \to u_\infty|_{\mathbb{S}^1} \;\text{ in } L^2(\mathbb{S}^1).$$

Passing to the limit in the weak formulation of (4.8):

$$\int_{\mathbb{D}^2} \langle \nabla u_k, \nabla v \rangle - 2\int_{\mathfrak{D}} K_k e^{u_k} v + 2\int_{\mathbb{S}} v - \int_{\mathbb{S}} h_k e^{u_k/2} v = 0, \tag{4.11}$$

for all $v \in H^1(\mathbb{D}^2)$. As a consequence u_∞ is a weak solution of (4.10). By standard regularity estimates u_∞ is indeed a classical solution.

∎

In the following step we prove that, when (u_k) is a sequence of minimum type solutions, the hypothesis of Lemma 4.14 are automatically satisfied.

Observe that in our framework, $I[K_k, h_k](\cdot, \cdot) \to I[K, h](\cdot, \cdot)$ pointwise in $\mathbb{X} \times (0, 2\pi)$. Then, if (u_k, ρ_k) is a sequence of minimum type solutions of (4.8),

$$\limsup_{n \to +\infty} I[K_k, h_k](u_k, \rho_k) = \limsup_{n \to +\infty} \min_{\mathbb{X} \times (0,2\pi)} I[K_k, h_k](\cdot, \cdot) \leq \min_{\mathbb{X} \times (0,2\pi)} I.$$

The previous inequality is due to the fact that (f_n) converging pointwise to f implies $\lim_{n \to +\infty} \inf f_n(y) \leq \inf f(y)$.

Proof of Theorem 1.5. Our intention is to apply Theorem 4.13 to the problems

$$\begin{cases} -\Delta u = 2Ke^u & \text{in } \mathbb{D}^2, \\ \frac{\partial u}{\partial \eta} + 2 = 2he^{u/2} & \text{on } \mathbb{S}^1 \end{cases}$$

for which we need that the limiting problems

$$(P^1_K)\begin{cases} -\Delta u = 2Ke^u & \text{in } \mathbb{D}^2, \\ \frac{\partial u}{\partial \eta} + 2 = 0 & \text{on } \mathbb{S}^1, \end{cases} \quad (P^2_h)\begin{cases} -\Delta u = 0 & \text{in } \mathbb{D}^2, \\ \frac{\partial u}{\partial \eta} + 2 = 2he^{u/2} & \text{on } \mathbb{S}^1, \end{cases}$$

admit minimum type solutions, u_1 and u_2 respectively, verifying

$$\int_{\mathbb{D}^2} Ke^{u_2} > 0, \quad \int_{\mathbb{S}^1} he^{u_1/2} > 0.$$

Reasoning by contradiction, take K_k and h_k Hölder continuous functions converging uniformly to K_0 and h_0. We can assume that k is large enough so that K_k and h_k are somewhere positive, and therefore u_1 and u_2 can be obtained via Theorems 4.10 and 4.11. Now, take $(\tilde{u}_k)$ a sequence of minimum type solutions of the problems $(P^1_{K_k})$, and $(\hat{u}_n)$ a sequence of minimum type solutions of the problems $(P^2_{h_k})$ such that

$$\text{either} \int_{\mathbb{D}^2} K_k e^{\hat{u}_k} \leq 0, \quad \text{or} \quad \int_{\mathbb{S}^1} h_k e^{\tilde{u}_n/2} \leq 0, \ \ \forall k \in \mathbb{N}. \tag{4.12}$$

By Lemma 4.14 we know that $\hat{u}_n \rightharpoonup \hat{u}$ and $\tilde{u}_n \rightharpoonup \tilde{u}$, solutions for the limiting problems $(P^1_{K_0})$ and $(P^2_{h_0})$. But in that case we can taking limit when $k \to +\infty$ in (4.12) and obtain:

$$\text{either} \int_{\mathbb{D}^2} K_0\, e^{\tilde{u}} \leq 0, \quad \text{or} \quad \int_{\mathbb{S}^1} h_0\, e^{\hat{u}/2} \leq 0,$$

which is a contradiction since both K_0 and h_0 are nonnegative functions somewhere positive. ■

Chapter 5

The Scalar-Mean curvature prescription problem on a manifold with boundary

The framework during this chapter will be that of a Riemannian manifold (M, g) of dimension $n \geq 3$, equiped with the Escobar metric detailed in §3.4. We deal with (1.21), where $K < 0$ and H has arbitrary sign.

5.1 The variational study of the energy functional

Firstly, we analyze the geometric properties of the energy functional in (1.22). This study will readily imply the proof of Theorems 1.6 and 1.7. Moreover, we will show that under the assumptions of Theorem 1.8, I satisfies the hypotheses of the mountain pass lemma. However, the proof of Theorem 1.8 will require the compactness result of Theorem 1.9, which will be proved further on.

We begin with the following inequality, showing that the nature of the functional is ruled by the interaction between its critical terms. This will allow us to prove Theorems 1.6 and 1.7.

Proposition 5.1. *For every $\varepsilon > 0$, there exists $C > 0$ such that*

$$\int_{\partial M} H|u|^{2^\sharp} \leq (\bar{\mathfrak{D}} + \varepsilon) \left(\frac{2(n-1)}{(n-2)^2} \int_M |\nabla u|^2 + \frac{1}{2n} \int_M |K||u|^{2^*} \right) + C \int_M |u|^{2^\sharp}, \quad (5.1)$$

where $\bar{\mathfrak{D}} = \max_{x \in \partial M}\{0, \mathfrak{D}_n(x)\}$.

Proof. Take a partition of unity $\{\phi_j\}_{j=1}^m$ on M, a vector field $N \in \mathfrak{X}(M)$ on M with $|N| \leq 1$ and such that $N = \eta$ on ∂M. Firstly we see that, for every $1 \leq j \leq m$ and $u \in H^1(M)$

$$\int_{\partial M} \phi_j |u|^{2^\sharp} = \int_{\partial M} \phi_j |u|^{2^\sharp} N \cdot \eta.$$

Calling $X_j = \phi_j u^{2^\sharp} N$, a direct computation shows that

$$\operatorname{div} X_j = \left(\phi_j \operatorname{div} N + \nabla \phi_j \cdot N + \frac{2(n-1)}{n-2} \frac{\phi_j}{u} N \cdot \nabla u \right) u^{2^\sharp}.$$

Thus, by the Divergence Theorem

$$\begin{aligned} \int_{\partial M} \phi_j |u|^{2^\sharp} &= \int_M \{\phi_j \operatorname{div} N + \nabla \phi_j N\} |u|^{2^\sharp} + \frac{2(n-1)}{n-2} \int_M \phi_j \nabla u \cdot N |u|^{\frac{2}{n-2}} u \\ &\le C \int_M |u|^{2^\sharp} + \frac{2(n-1)}{n-2} \int_M \phi_j |\nabla u| \cdot |u|^{\frac{n}{n-2}}. \end{aligned} \tag{5.2}$$

Consider $\mathfrak{D}_n$ as defined in (1.23), and let

$$H_j = \max\{H(x), x \in \text{supp } \phi_j\}, \ \ |K|_j = \min\{|K(x)|, x \in \text{supp } \phi_j\}.$$

Then, summing (5.2) on j and assuming that the supports of the ϕ_j's are sufficiently small, we have:

$$\begin{aligned} (n-2) \int_{\partial M} H |u|^{2^\sharp} &= (n-2) \sum_{j=1}^m \int_{\partial M} \phi_j H |u|^{2^\sharp} \le (n-2) \sum_{j=1}^m H_j \int_{\partial M} \phi_j |u|^{2^\sharp} \\ &\le C(n-2) \int_M |u|^{2^\sharp} + 2(n-1) \left(\sum_{j=1}^m \frac{H_j}{\sqrt{|K|_j}} \right) \int_M |\nabla u| |u|^{\frac{n}{n-2}} \sqrt{|K|} \\ &\le C' \int_M |u|^{2^\sharp} + 2\sqrt{\frac{n-1}{n}} (\bar{\mathfrak{D}} + \varepsilon) \int_M |\nabla u| |u|^{\frac{n}{n-2}} \sqrt{|K|} \\ &\le C' \int_M |u|^{2^\sharp} + 2\sqrt{\frac{n-1}{n}} (\bar{\mathfrak{D}} + \varepsilon) \left(\frac{\lambda}{2} \int_M |\nabla u|^2 + \frac{1}{2\lambda} \int_M |u|^{2^*} |K| \right), \end{aligned}$$

for every $\lambda > 0$. Choosing $\lambda = \frac{2\sqrt{n(n-1)}}{n-2}$, and renaming ε properly, we conclude. ∎

5.1.1 Proof of Theorems 1.6 and 1.7

For this part, we will rely on the inequality proven in Proposition 5.1 with $\bar{\mathfrak{D}} < 1$. In this case, the positive term of the Energy functional (1.22), $\int_M |K| \, |u|^{2^\sharp}$, dominates over the boundary term, and the functional becomes coercive.

Proposition 5.2. *Suppose $K < 0$, and that $\mathfrak{D}_n$ as in 1.23 satisfies $\mathfrak{D}_n < 1$ everywhere on ∂M. Then, the energy functional I defined on (1.22) is coercive, that is to say,*

$$I(u) \to +\infty \quad \textit{as} \quad \|u\|_{H^1(M)} \to +\infty.$$

Proof. The main idea is to use (5.1) with $\bar{D} < 1$, taking into account that by Hölder's inequality

$$\int_M |u|^{2^\sharp} \leq \delta \int_M |u|^{2^*} + C.$$

Then, for sufficiently small $\delta > 0$,

$$I(u) \geq \delta \int_M |\nabla u|^2 + \frac{1}{2} \int_M Su^2 + \delta \int_M |K|\, |u|^{2^*} - C. \tag{5.3}$$

Hence, if we take a sequence (u_i) in $H^1(M)$ such that $\|u_i\|_{H^1} \to +\infty$, then either $\|\nabla u_i\|_{L^2}$ or $\|u_i\|_{L^2}$ must tend to $+\infty$, which implies by (5.3) that $I(u_i) \to +\infty$. ∎

Proof of Theorems 1.6 and 1.7. First, let us show that a global minimizer for I can always be found. Let us consider

$$\alpha = \inf \left\{ I(u) : u \in H^1(M) \right\},$$

which is finite by inequality (5.3), and a minimizing sequence (u_i) in $H^1(M)$ such that $I(u_i) \to \alpha$. Proposition 5.2 implies that the sequence (u_i) is bounded in $H^1(M)$. Thus, up to a subsequence, $(u_i) \rightharpoonup u$ in $H^1(M)$.

Using Brezis-Lieb's result in [17], we decompose $I(u_i)$ as $I(u_i) = I(u) + I(u_i - u) + o_i(1)$ and study the second term in the right-hand side. Using the trace inequality (5.1) for $u_i - u$ and the compactness of the embedding $H^1(M) \hookrightarrow L^{2^\sharp}(M)$, we obtain:

$$I(u_i - u) \geq \delta \left\{ \frac{2(n-1)}{n-2} \int_M |\nabla(u_i - u)|^2 + \frac{n-2}{2n} \int_M |K|\, |u_i - u|^{2^*} \right\} + o_i(1).$$

Therefore,

$$I(u_i) \geq I(u) + o_i(1),$$

and it suffices to take limits to see that u is a minimizer for I in $H^1(M)$. The relation $I(u) = I(|u|)$ permits us to assume that the minimizer is non-negative. In order to conclude, we neeed to ensure that $u > 0$.

Case 1: $S = 0$ and $\int_{\partial M} H > 0$, corresponding to Theorem 1.7. Firstly, we check that $0 \in H^1(M)$ is not a global minimum for I, so u must be positive somewhere. Take $\varepsilon > 0$ and consider

$$I(\varepsilon) = \frac{n-2}{2n} \varepsilon^{2^*} \int_M |K| - (n-2)\varepsilon^{2^\sharp} \int_{\partial M} H,$$

being the second the leading term as ε approaches zero since $2^\sharp < 2^*$. Thus, there exists $\varepsilon_0 > 0$ such that $I(\varepsilon) < 0$ for all $0 < \varepsilon < \varepsilon_0$, and consequently $\inf I < 0$. We can conclude that the minimizer u is not identically zero. Observe now that $|u|$ is also a minimizer, so we can assume that $u \geq 0$.

In order to show that $u > 0$ in M, we start recalling that it solves the problem

$$\begin{cases} -4\frac{n-1}{n-2}\Delta_g u = K u^{\frac{4}{n-2}} u & \text{on } M, \\ \frac{2}{n-2}\frac{\partial u}{\partial \eta} = H u^{\frac{2}{n-2}} u & \text{on } \partial M. \end{cases} \tag{5.4}$$

Apply now the maximum principle ([48, Ch. 3]) to the elliptic operator

$$\mathfrak{L}(\phi) = \frac{4(n-1)}{n-2}\Delta_g \phi + K|u|^{\frac{4}{n-2}}\phi.$$

By Hopf's Strong Maximum Principle ([48, Th. 3.5]), if there exists a point $x \in \operatorname{Int} M$ with $u(x) = 0$, then u achieves its minimum in $\operatorname{Int} M$ and thus $u \equiv 0$, which is impossible.

Moreover, if there would exist $x_0 \in \partial M$ with $u(x_0) = 0$, using the second equation in (5.4) we would get that $\frac{\partial u}{\partial \eta}(x_0) = 0$, which is a contradiction to Hopf's Lemma.

Case 2: $S < 0$, corresponding to Theorem 1.6. Analogously, we can prove that $u \not\equiv 0$ by showing that 0 is not a global minimum. To see this, take $\varepsilon > 0$ and consider

$$I(\varepsilon) = \frac{\varepsilon^{2^*}}{2^*}\int_M |K| - \frac{\varepsilon^2}{2}\int_M |S| - \varepsilon^{2^\sharp}(n-2)\int_{\partial M} H.$$

In this case, the leading term as $\varepsilon \to 0$ is $-\frac{\varepsilon^2}{2}\int_M |S| < 0$, so, again, there exists $\varepsilon_0 > 0$ such that $I(\varepsilon) < I(0) = 0$ for all $0 < \varepsilon < \varepsilon_0$. To see that $u > 0$ one can follow the same strategy as in the first case. ∎

5.1.2 A first step for proving Theorem 1.8

This subsection is devoted to the proof of the following result:

Proposition 5.3. *Assume $S = 0$, $K < 0$ and that H is taken so that*

1. $\int_{\partial M} H < 0$,
2. $\mathfrak{D}_n(\overline{p}) > 1$ *for some* $\overline{p} \in \partial M$.

Consider a sequence of exponents $p_i \nearrow \frac{n+2}{n-2}$. Then there exist $\kappa_i \to 1$ and solutions u_i of the perturbed problem:

$$\begin{cases} \frac{-4(n-1)}{n-2}\Delta u = K u^{p_i} & \text{in } M, \\ \frac{2}{n-2}\frac{\partial u}{\partial \eta} = \kappa_i H u^{\frac{p_i+1}{2}} & \text{on } \partial M. \end{cases} \tag{5.5}$$

Moreover the solutions have bounded energy, that is, $I_i(u_i)$ is bounded, with

$$I_i(u) = \frac{2(n-1)}{n-2}\int_M |\nabla u|^2 + \frac{1}{p_i+1}\int_M |K|\,|u|^{p_i+1} - \kappa_i \frac{4(n-1)}{p_i+3}\int_{\partial M} H|u|^{\frac{p_i+3}{2}}. \tag{5.6}$$

Proposition 5.3 is a first step in the proof of Theorem 1.8 and gives existence of solutions of aproximating problems. The proof of this result consists in applying the classical *Mountain-pass theorem* to the perturbed functional I_i, defined in (5.6).

The following lemmas show that under the assumptions of Proposition 5.3, the energy functional I has a mountain pass geometry. We prove them for I, since they are preserved by continuity if κ_i is close enough to 1 and p_i to $\frac{n+2}{n-2}$.

Lemma 5.4. *Assume that $K < 0$, $S = 0$ and $\int_{\partial M} H < 0$. Then there exists $\varepsilon > 0$ such that for any $u \in H^1(M)$, $\|u\| = \varepsilon$, we have that $I(u) > \delta > 0$, where δ is independent of u.*

Proof. We write $u = \tilde{u} + \bar{u}$, where

$$\bar{u} \in \mathbb{R}, \qquad \int_M \tilde{u} = 0.$$

Then, we can write:

$$\begin{aligned} I(u) &\geq \frac{2(n-1)}{n-2}\int_M |\nabla \tilde{u}|^2 - (n-2)\int_{\partial M} H|u|^{2^\sharp} \\ &= \frac{2(n-1)}{n-2}\int_M |\nabla \tilde{u}|^2 - (n-2)|\bar{u}|^{2^\sharp}\int_{\partial M} H - (n-2)\int_{\partial M} H(|u|^{2^\sharp} - |\bar{u}|^{2^\sharp}). \end{aligned} \tag{5.7}$$

Now we estimate the last term in the right hand side as:

$$\begin{aligned} \left|\int_{\partial M} H(|\bar{u}+\tilde{u}|^{2^\sharp} - |\bar{u}|^{2^\sharp})\right| &\leq \|H\|_{L^\infty}\int_{\partial M} 2^\sharp|\tilde{u}|(|\tilde{u}|^{2^\sharp-1} + |\bar{u}|^{2^\sharp-1}) \\ &\leq C\int_{\partial M}|\tilde{u}|^{2^\sharp} + \int_{\partial M}|\bar{u}|\,|\tilde{u}|^{2^\sharp-1}. \end{aligned}$$

By the trace and Sobolev's inequalities,

$$\int_{\partial M}|\tilde{u}|^{2^\sharp} \leq C\left(\int_M |\nabla\tilde{u}|^2\right)^{2^\sharp/2}.$$

Moreover, by combining Young's inequality with the trace and Sobolev's inequality, we have:

$$\int_{\partial M} |\bar{u}|\, |\tilde{u}|^{2^\sharp -1} \leq \gamma \int_{\partial M} |\bar{u}|^{2^\sharp} + C|\tilde{u}|^{2^\sharp} \leq \gamma |\partial M| |\bar{u}|^{2^\sharp} + C_\gamma \left(\int_M |\nabla \tilde{u}|^2 \right)^{2^\sharp/2},$$

for some $\gamma > 0$ to be chosen. Coming back to (5.7), we obtain that for certain constants $c_0 > 0$, $C > 0$, one has

$$I(u) \geq c_0 \|\tilde{u}\|^2 + c_0 |\bar{u}|^{2^\sharp} - \gamma |\partial M| |\bar{u}|^{2^\sharp} - C \|\tilde{u}\|^{2^\sharp}.$$

By choosing γ sufficiently small, we conclude the proof. ■

In next lemma we show that under the hypotheses of Theorem 1.8, the energy functional I is not bounded from below.

Lemma 5.5. *If there exists $p \in \partial M$ such that $\mathfrak{D}_n(p) > 1$, then one can find a sequence of functions (φ_k) in $H^1(M)$ such that $I(\varphi_k) \to -\infty$ as $k \to +\infty$.*

The proof of this lemma is postponed to the appendix.

Proof of Proposition 5.3. As it can be seen, the previous two lemmas imply that the functional I has a mountain-pass geometry (see §3.3). However, there are several obstacles in order to prove the existence of a solution. The first one is that we do not know whether Palais-Smale's sequences are bounded or not. But, even if they are, we cannot guarantee compactness because of the lack of a compact embedding in the critical Sobolev inequalities. We can bypass these difficulties by considering the perturbed problems (5.5).

As we mentioned, if κ_i is close to 1 and p_i is close enough to $\frac{n+2}{n-2}$, the functional I_i also satisfies the geometric assumptions of the mountain-pass lemma. Let us fix $i \in \mathbb{N}$: by Struwe's Monotonicity trick, there exists a bounded Palais-Smale sequence u_k^i for I_i for some κ_i, with $|\kappa_i - 1| < \frac{1}{i}$.

In order to conclude the proof of Proposition 5.3, we intend to find a positive solution as a strong limit of u_k^i as $k \to \infty$. We summarize the available information about the sequence $(u_k^i)_k$:

1. $\|u_k^i\|_{H^1(M)}$ is uniformly bounded in k. Therefore, there exists a weak limit $u_k^i \rightharpoonup u^i$ in $H^1(M)$.

2. $I_i(u_k^i) \to c_i$ as $k \to +\infty$, for some positive constant c_i (which is, actually, bounded also in i). Then,

$$\frac{2(n-1)}{n-2} \int_M |\nabla u_k^i|^2 + \frac{1}{p_i+1} \int_M |K|\, |u_k^i|^{p_i+1} - \kappa_i \frac{4(n-1)}{p_i+3} \int_{\partial M} H |u_k^i|^{\frac{p_i+3}{2}} = c_i + o_k(1). \tag{5.8}$$

3. $I(u_k^i)' \to 0$ in $H^{-1}(M)$, so $I'(u_k^i)[v] = o_k(1)$ for every $v \in H^1(M)$.

Using the compactness of the embedding $H^1(M) \hookrightarrow L^p(M)$ for $1 \leq p < 2^*$ as well as that of the trace inequality $H^1(M) \hookrightarrow L^q(\partial M)$ for $1 \leq q < 2^\sharp$, we obtain

$$\int_M |K|\,|u_k^i|^{p_i+1} \to \int_M |K|\,|u^i|^{p_i+1}, \quad \int_{\partial M} H|u_k^i|^{\frac{p_i+1}{2}} \to \int_{\partial M} H|u^i|^{\frac{p_i+1}{2}}, \tag{5.9}$$

as $k \to +\infty$. By testing I_i' on $u_k^i - u^i$, we also find that:

$$I_i'(u_k^i)(u_k^i - u^i) = \frac{4(n-1)}{n-2}\Big(\int_M |\nabla u_k^i|^2 - \nabla u_k^i \cdot \nabla u^i\Big) + o_k(1).$$

This implies the strong convergence $u_k^i \to u^i$ as $k \to +\infty$. Hence u^i is a nontrivial solution of the problem:

$$\begin{cases} \frac{-4(n-1)}{n-2}\Delta u^i = K|u^i|^{p_i-1}u^i & \text{in } M. \\ \frac{2}{n-2}\frac{\partial u^i}{\partial \eta} = \kappa_i H|u^i|^{\frac{p_i-1}{2}}u^i & \text{on } \partial M. \end{cases}$$

As is well-known, the (PS) sequence can be taken very close to the family of curves given by deformations of γ under the gradient flow of I^i (see for instance [49]). We can now use the fact that the gradient flow of I^i leaves invariant the cone of nonnegative functions to conclude that the limit of the (PS) sequence u^i is nonnegative. This is rather standard, see for instance [10, Lemma 4.1, (a)]. By using the maximum principle as previously we conclude that $u_i > 0$. ■

5.2 Blow-up analysis

5.2.1 Generalities on the singular set

In this section we prove (1) in Theorem 1.9. The tricky point here is that we cannot assume that around points in $\mathscr{S}_0$ there are local maxima of u_i. We perform then a blow-up analysis around a suitably defined sequence of points, which are chosen thanks to *Ekeland's variational principle*.

Let $K_i : M \to \mathbb{R}$ and $H_i : \partial M \to \mathbb{R}$ be sequences of regular functions such that $K_i \to K$ and $H_i \to H$ in the C^2 sense. Assume that $K < 0$ and that S has constant sign. For a sequence $(u_i)_i$ of positive solutions to

$$\begin{cases} -4\frac{n-1}{n-2}\Delta_g u_i + Su_i = K_i {u_i}^{p_i} & \text{on } M, \\ \frac{2}{n-2}\frac{\partial u_i}{\partial \eta} = H_i {u_i}^{\frac{p_i+1}{2}} & \text{on } \partial M, \end{cases} \tag{5.10}$$

with $p_i \nearrow \frac{n+2}{n-2}$, we recall the definition of the singular set

$$\mathscr{S} = \left\{p \in \overline{M} :\ \exists (x_i) \to p \text{ such that } u_i(x_i) \to +\infty\right\}. \tag{5.11}$$

Proposition 5.6. $\mathscr{S} \subset \{p \in \partial M : \mathfrak{D}_n \geq 1\}$.

The idea is to make a suitable rescaling and pass to a problem in a half-space, whose solutions have been classified ([30]). In general, this analysis is made around points of local maximum for the sequence u_i. However, in our situation we cannot guarantee their existence, as for example u_i might be monotone in some direction when restricted to a portion of the boundary. We bypass this obstacle by choosing good sequences, even if they are not local maxima, by means of Ekeland's variational principle. For convenience we state it below, see e.g. Chapter I in [90].

Lemma 5.7. *Let (X, d) be a complete metric space and $\varphi : X \to (-\infty, +\infty]$ lower semicontinuous, bounded from below and not identically equal to $+\infty$. Let $\varepsilon > 0$ and $\lambda > 0$ be given and let $x \in X$ be such that $\varphi(x) \leq \inf_X \varphi + \varepsilon$. Then, there exists $x_\varepsilon \in X$ such that*

1. $\varphi(x_\varepsilon) \leq \varphi(x)$,
2. $d(x_\varepsilon, x) \leq \lambda$, *and*
3. $\varphi(x_\varepsilon) < \varphi(z) + \frac{\varepsilon}{\lambda} d(x_\varepsilon, z)$ *for every* $z \neq x_\varepsilon$.

Proof of Proposition 5.6. Let $p \in \mathscr{S}$. Take geodesic normal coordinates around p, valid in a small geodesic ball B, and $(y_i) \to 0$ a sequence in B such that $u_i(y_i) \to +\infty$. We define

$$\varepsilon_i := u_i(y_i)^{-\frac{p_i - 1}{2}} \to 0. \tag{5.12}$$

We apply Ekeland's variational principle taking $\varphi = u_i^{-\frac{p_i-1}{2}}$ and $\lambda = \sqrt{\varepsilon_i}$. Then, there exists a sequence (z_i) such that

1. $u_i(z_i)^{-\frac{p_i-1}{2}} \leq u_i(y_i)^{-\frac{p_i-1}{2}}$. Hence, $u_i(y_i) \leq u_i(z_i)$ and, consequently, $u_i(z_i) \to +\infty$.
2. $|z_i - y_i| \leq \sqrt{\varepsilon_i}$. In particular $(z_i) \to 0$.
3. $u_i(z_i)^{-\frac{p_i-1}{2}} < u_i(y_i)^{-\frac{p_i-1}{2}} + \sqrt{\varepsilon_i}\, |z_i - z|$ for every $z \neq z_i$.

The idea is that the new sequence (z_i) is more adequate to rescale and pass to the limit. Now, we set $\delta_i = u_i(z_i)^{-\frac{p_i-1}{2}} \to 0$, $B_i = B(z_i, \frac{r}{2}) \cap B$ and define the sequence of rescaled functions

$$v_i(x) = \delta_i^{\frac{2}{p_i - 1}} u_i(\delta_i x + z_i), \tag{5.13}$$

for every $x \in \tilde{B}_i = \frac{1}{\delta_i} B_i$. Clearly

$$v_i(0) = \frac{u_i(z_i)}{u_i(z_i)} = 1. \tag{5.14}$$

We claim that, for every given $\varepsilon > 0$ and $R > 0$, v_i satisfies the following uniform boundedness property:

$$v_i(x) \leq 1 + \varepsilon, \ \forall x \in \tilde{B}_i \text{ with } |x| < R. \tag{5.15}$$

Indeed, from (3), if $|z - z_i| < R\delta_i$, then:

$$u_i(z) < \left(\frac{1}{1 - \sqrt{\varepsilon_i}R}\right)^{\frac{2}{p_i-1}} u_i(z_i). \tag{5.16}$$

Taking $z = \delta_i x + z_i$ in (5.16) and using (5.13) we obtain:

$$v_i(x) < \frac{1}{(1 - \sqrt{\varepsilon_i}R)^{\frac{2}{p_i-1}}}, \tag{5.17}$$

which proves (5.15). This allows us to make a scaling argument in the spirit of [47]. We distinguish between two cases:

Case 1: $p \in \partial M$ and, up to a subsequence, $\frac{d(z_i, \partial_0 B)}{\delta_i} \to t_0 \geq 0$.

Straightforward computations show that the function v_i satisfies

$$\begin{aligned} &- (\sqrt{|g|}g^{jk})(\delta_i x + z_i)\frac{\partial^2 v_i(x)}{\partial x_j \partial x_k} + \delta_i \frac{\partial(\sqrt{|g|}g^{jk})}{\partial x_j}(\delta_i x + z_i)\frac{\partial v_i(x)}{\partial x_k} + \\ &+ \delta_i^2 \frac{1}{C_n}(\sqrt{|g|}R_i)(\delta_i x + z_i)v_i(x) - \frac{1}{C_n}(\sqrt{|g|}K_i)(\delta_i x + z_i)v_i(x)^{p_i} = 0 \end{aligned}$$

for $x \in \tilde{B}_i$, and

$$g_{jj}\frac{\partial v_i(x)}{\partial x_j}\eta^j - \frac{n-2}{2}H_i(\delta_i x + z_i)v_i(x)^{\frac{p_i+1}{2}} = 0$$

on the straight portion of its boundary. Using (5.15) and local regularity estimates we obtain that, up to a subsequence, $v_i \to v$ in $C^2_{loc}(\mathbb{R}^n_+)$. Also, $g_{jk} = \delta_{jk} + o(|x|^2)$ so, if we let $i \to +\infty$, we get a solution to the limit problem (1.26).
The results in [30] (recalled in Proposition 3.17 above) establish that this problem admits a solution whenever $\mathfrak{D}_n(p) \geq 1$.

Case 2: $\frac{d(x_k, \partial_0 B)}{\delta_k} \to +\infty$.

In this situation, the domains $\tilde{B}_i$ invade all of $\mathbb{R}^n$: reasoning as before, we have $v_i \to v$ in $C^2_{loc}(\mathbb{R}^n)$. Moreover, v solves the following equation:

$$-\Delta v = \frac{n-2}{4(n-1)}K(0)v^{\frac{n+2}{n-2}} \text{ on } \mathbb{R}^n. \tag{5.18}$$

Consider $R > 0$ and the domain $\Omega = B^n(0, R)$. Since $K < 0$, we have $\Delta v > 0$ in Ω so, by the Maximum Principle, it cannot achieve its maximum unless it is constant. However, passing (5.17) to the limit we obtain

$$v(x) \leq 1 \ \forall x \in \Omega,$$

while $v(0) = 1$. Thus, this second case can be dismissed and the proof of the proposition is completed.

■

As remarked previously, we cannot guarantee that around singular points we have a sequence of local maxima. We now show that, a posteriori, this is the case for singular points in $\{\mathfrak{D}_n > 1\}$.

Lemma 5.8. *Let $p \in \mathscr{S}_1$. Then there exists a sequence $x_i \in \partial M$, $x_i \to p$, such that $u_i(x_i) \to +\infty$ is local maximum of u_i. Moreover, there exist $\overline{R}_i \to +\infty$ and such that, up to a subsequence,*

$$r_i := \overline{R}_i u_i(x_i)^{-\frac{p_i-1}{2}} \to 0 \quad \textit{and}$$
$$\left\| u_i(x_i)^{-1} u_i\left(u_i(x_i)^{-\frac{p_i-1}{2}} x + x_i\right) - b_{\beta_0}(x) \right\|_{C^2\left(B\left(x_i, \overline{R}_i\right)\right)} \to 0,$$

where b_{β_0} is given by (3.34).

Proof. Given $p \in \mathscr{S}_1$, we come back to the proof of Proposition 5.6 (case 1). Hence we obtain that the rescaled functions v_i converge locally in the C^1 sense to the function in (3.34). Observe now that the limit solution (3.34) has a global maximum. As a consequence, the sequence v_i attains also its maximum at a certain point $\tilde{x}_i$. Rescaling back, we obtain points x_i which are local maxima of u_i.

Let us now show that if i is large enough, $x_i \in \partial M$. This is a consequence of the subharmonicity, given in turn by the hypothesis $K < 0$, since

$$-\Delta u_i(x_i) = K(x_i) u_i(x_i)^{p_i} - S\, u_i(x_i) < 0.$$

The convergence is due to the convergence of the newly defined sequence of rescaled functions v_i, as in the proof of Case 1 of Proposition 5.6 ■

5.2.2 Properties of isolated simple blow-up points

Along this section we analyze blow-up points $x \in \mathscr{S}_1$ and the sequence of maxima $(x_i) \to x$ given by Lemma 5.8, where locally the blow-up has profile given by (1.26). Below, we define the notions of isolated and simple blow-up points, first introduced in [87], adapted to our framework. Then we study the asymptotic behaviour of u_i around such points. This analysis is the counterpart of some results from [68]

and [35] (see also [44, 38, 39]), which we adapt here to the case of negative scalar curvature. From this study, one obtains that there are only finitely many isolated simple blow-up points. In next section we shall prove that if $n = 3$ all blow-up points of $\mathscr{S}_1$ are isolated and simple. In particular this implies (2.1) of Theorem 1.9.
In the asymptotic analysis that follow, we sometimes use the notation of the Euclidean space, which makes no difference in the computations.

Definition 5.9. *$\bar{x} \in \partial M$ is an isolated point of blow-up for (u_i) if there exist local maxima (x_i) such that $(x_i) \to x$, $u_i(x_i) \to +\infty$ and*

$$u_i(y) \leq \frac{C}{\operatorname{dist}(y, x_i)^{\frac{2}{p_i-1}}}, \quad \text{for every } y \in B(x_i, R)_+, \tag{5.19}$$

where C and R are positive constants independent of i.

Definition 5.10. *Let (u_i), (x_i), $\bar{x}$ be as in Definition 5.9, and consider the radial averages*

$$\overline{u}_i(r) = \fint_{\partial^+ B(x_i,r)_+} u_i.$$

$\bar{x}$ is called an isolated simple blow-up point of blow-up if there exists $\rho > 0$, independent of i, such that the 1-dimensional functions

$$\hat{u}_i(r) = r^{\frac{2}{p_i-1}} \overline{u}_i(r)$$

have exactly one critical point for $r \in (0, \rho)$.

Next proposition gives a version of the well-known Harnack inequality:

Proposition 5.11. *Let $\Omega = B(p, R_0) \subset M$. Assume that (u_i) is a sequence of solutions of (5.10) with $1 \leq p_i$, p is an isolated point of blow-up. Let $(x_i) \to p$ and $R < R_0$ be as in Definition 5.9. Then, for every $0 < r < \frac{R}{4}$, the following Harnack-type inequality holds*

$$\max_{B(x_i,2r)_+ \setminus B(x_i,\frac{r}{2})_+} u_i(y) \leq C \min_{B(x_i,2r)_+ \setminus B(x_i,\frac{r}{2})_+} u_i(y),$$

where C is a positive constant independent of i and r.

The details can be found in [51].
The main result of this section is the following description of the sequence u_i around isolated simple blow-up points:

Proposition 5.12. *Let us assume $\Omega = B_2 = B(0, 2)$ for simplicity. Assume $(K_i) \to K < 0$, in $C^1(\overline{\Omega_+})$ and $(H_i) \to H$ in $C^2(\overline{\partial_0 \Omega})$. Suppose that, for every $i \in \mathbb{N}$, u_i is a positive solution of (5.10) and that $x_i \to 0$ is an isolated simple blow-up point with*

$$u_i(x) \leq C_1 \left| x - x_i \right|^{\frac{2}{p_i-1}}, \quad \text{for all } x \in \Omega_+. \tag{5.20}$$

Then there exists a constant $C = C(K, S, H, n, C_1) > 0$ *such that*

$$u_i(x_i)u_i(x) \leq C\,|x - x_i|^{2-n}\,, \ \text{for all } x \in B(x_i, 1)_+. \tag{5.21}$$

Moreover, there exist $a > 0$ *and* $b : \overline{(B_1)_+} \to \mathbb{R}$ *satisfying*

$$\begin{cases} \Delta b = 0 & \text{in } (B_1)_+ \\ \frac{\partial b}{\partial x_n} = 0 & \text{on } \partial_0 (B_1)_+ \end{cases} \tag{5.22}$$

such that $u_i(x_i)u_i(x) \to a\,|x|^{2-n} + b$ *in* $C^2_{loc}\left(\overline{(B_1)_+}\backslash\{0\}\right)$. *Furthermore, for* $\rho < 1$, *in* $\overline{(B_\rho)_+}$ *one has that*

$$u_i = (1 + o_i(1) + o_\rho(1))b_{\beta_i}; \qquad \nabla u_i = (1 + o_i(1) + o_\rho(1))\nabla b_{\beta_i}, \tag{5.23}$$

with $\beta_i \to +\infty$, *where* b_β *is as in* (3.34).

We split the proof of Proposition 5.12 into some lemmas. As a preliminary result, we outline a version of the Maximum Principle that can be found in [51], pp. 539.

Lemma 5.13. *Let* Ω *be a bounded domain with piecewise smooth boundary* $\partial\Omega = \partial_1\Omega \cup \partial_2\Omega$, $V \in L^\infty(\Omega)$ *and* $w \in L^\infty(\Omega)$. *Suppose that* $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$, $u > 0$, *satisfies:*

$$\begin{cases} \Delta u + Vu \leq 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \eta} - wu \geq 0 & \text{on } \partial_1\Omega \end{cases}$$

Then, for every $v \in C^2(\Omega) \cap C^1(\overline{\Omega})$ *such that*

$$\begin{cases} \Delta v + Vv \leq 0 & \text{in } \Omega \\ \frac{\partial v}{\partial \eta} - wv \geq 0 & \text{on } \partial_1\Omega \\ v \geq 0 & \text{on } \partial_2\Omega \end{cases}$$

we have $v \geq 0$ *in* $\overline{\Omega}$.

Proof. Apply the standard Maximum Principle to the function $w = \frac{v}{u}$. ∎

In next lemma we give a less sharp version of (5.21).

Lemma 5.14. *Under the same hypotheses of Proposition 5.12, there exists* $\varepsilon_i > 0$, $\varepsilon_i = O\left(\overline{R}_i^{\,-2}\right)$, *such that*

$$u_i(x_i)^{\lambda_i}u_i(x) \leq C\,|x - x_i|^{2-n+\varepsilon_i}\,, \ \text{for all } r_i \leq |x - x_i| \leq 1,$$

with $\lambda_i = (n - 2 - \varepsilon_i)\frac{p_i - 1}{2} - 1$.

Proof. We follow some ideas in [68], together with the application of Lemma 5.13, as it is done in [51]. Recall first the definition of r_i in Lemma 5.8.

To begin, we claim that, for $|x - x_i| = r_i$, the following bound holds

$$u_i(x) \leq C u_i(x_i) \overline{R}_i^{\,2-n}. \tag{5.24}$$

Indeed, rescaling back the second statement of Lemma 5.8, we have

$$\begin{aligned} u_i(x) &\leq \frac{1}{u_i(x_i)} \frac{C}{|x - x_i|^{n-2}} = \frac{1}{u_i(x_i)} \frac{C}{r_i^{\,n-2}} \\ &= C\overline{R}_i^{\,2-n} u_i(x_i)^{(n-2)\left(\frac{p_i-1}{2}\right)-1} \leq C\overline{R}_i^{\,2-n} u_i(x_i). \end{aligned}$$

Consider the function $\tilde{u}_i : r \mapsto r^{\frac{1}{p_i-1}} \overline{u_i}(r)$. By our assumptions, it has a unique critical point in $(0,1)$. Moreover, by Lemma 5.8 we can always assume that i is so large that the only critical point stays in $(0, r_i))$. Therefore, we can suppose without loss of generality that $\tilde{u}_i$ is strictly decreasing in $r_i < r < 1$. Then,

$$|x - x_i|^{\frac{2}{p_i-1}} u_i(x) \leq C r_i^{\frac{2}{p_i-1}} \overline{u}_i(r_i) = C\overline{R}_i^{\frac{2}{p_i-1}} u_i(x_i)^{-1} \fint_{\partial^+ B(x_i, r_i)_+} u_i.$$

Using (5.24), we conclude that

$$|x - x_i|^{\frac{2}{p_i-1}} u_i(x) \leq C\overline{R}_i^{\,-\frac{2}{p_i-1}}.$$

Consequently,

$$u_i(x)^{p_i-1} \leq C\overline{R}_i^{\,-2+o_i(1)} \frac{1}{|x - x_i|^2}, \quad \text{for all } r_i \leq |x - x_i| \leq 1. \tag{5.25}$$

Next, we apply a comparison argument. Consider the second-order elliptic operator

$$\mathfrak{L}_i(\varphi) = \Delta\varphi + \frac{n-2}{4(n-1)} \left(K_i u_i^{p_i-1} + S\right) \varphi,$$

and the boundary operator

$$\mathfrak{B}_i(\varphi) = \frac{\partial \varphi}{\partial \eta} - \frac{n-2}{2} H_i u_i^{\frac{p_i-1}{2}} \varphi.$$

Since $u_i > 0$ solves (5.10), $\mathfrak{L}_i(u_i) = \mathfrak{B}_i(u_i) = 0$. Then, the pair $(\mathfrak{L}_i, \mathfrak{B}_i)$ satisfies the Maximum Principle stated in Lemma 5.13. We start constructing an adequate function to be compared with u_i. Considering $x \mapsto |x - x_i|^{-\mu}$, it is clear that

$$\Delta |x - x_i|^{-\mu} = -\mu(n - 2 - \mu) |x - x_i|^{-\mu-2}.$$

Then

$$\mathfrak{L}_i(|x-x_i|^{-\mu}) = -\mu(n-2-\mu)\frac{1}{|x-x_i|^{\mu+2}} + C_n^{-1}K_i u_i^{p_i-1}\frac{1}{|x-x_i|^{\mu}} - C_n^{-1}S\frac{1}{|x-x_i|^{\mu}}.$$

Using (5.25),

$$\mathfrak{L}_i(|x-x_i|^{-\mu}) = \left(-\mu(n-2-\mu) + C_n^{-1}K_i\overline{R}_i^{-2}\right)\frac{1}{|x-x_i|^{\mu+2}} - C_n^{-1}R_i\frac{1}{|x-x_i|^{\mu}}.$$

Then, we can choose $\varepsilon_i \searrow 0$, $\varepsilon_i = O\left(\overline{R}_i^{-2}\right)$ such that

$$\mathfrak{L}_i(|x-x_i|^{-\varepsilon_i}) \leq 0, \text{ and } \mathfrak{L}_i(|x-x_i|^{2-n+\varepsilon_i}) \leq 0. \tag{5.26}$$

Moreover, notice that

$$\begin{aligned}\mathfrak{B}_i(|x-x_i|^{-\mu}) &= \mu\frac{x^n}{|x-x_i|^{\mu+2}} - \frac{n-2}{2}H_i u_i^{\frac{p_i-1}{2}}\frac{1}{|x-x_i|^{\mu}} \\ &= \mu\frac{x^n}{|x-x_i|^{\mu+2}} + O\left(\overline{R}_i^{-2}\right)\frac{1}{|x-x_i|^{\mu}},\end{aligned}$$

again by (5.25). Reasoning as above, we also have

$$\mathfrak{B}_i(|x-x_i|^{-\varepsilon_i}) \geq 0, \text{ and } \mathfrak{B}_i(|x-x_i|^{2-n+\varepsilon_i}) \geq 0. \tag{5.27}$$

Set $M_i = \max_{\partial^+(B_1)_+} u_i$, $\lambda_i = (n-2+\varepsilon_i)\frac{p_i-1}{2} - 1$ and define

$$\varphi_i = M_i\,|x-x_i|^{-\varepsilon_i} + \alpha u_i(x_i)^{-\lambda_i}\,|x-x_i|^{2-n+\varepsilon_i}, \text{ for all } r_i \leq |x-x_i| \leq 1,$$

with $\alpha > 0$ yet to be determined. In order to apply the Maximum Principle and compare φ_i with u_i, we choose as domain the semi-annulus $A = B(x_i,1)_+ \backslash B(x_i,r_i)_+$. Observe that $\partial^+ A$ is composed by two semi-spheres, namely,

$$\partial^+ A = \{x \in A : |x-x_i| = 1\} \sqcup \{x \in A : |x-x_i| = r_i\}.$$

By (5.26) and (5.27), $\mathfrak{L}_i(\varphi_i - u_i) \leq 0$ in A and $\mathfrak{B}_i(\varphi_i - u_i) \geq 0$ on $\partial_0 A$. Then, we just need to prove that $\varphi_i \geq u_i$ on the circular boundaries.

If $|x-x_i| = 1$, then $\varphi_i = M_i + \alpha u_i(x_i)^{-\lambda_i} \geq u_i$ by the choice of M_i.

On the other hand, if $|x-x_i| = r_i$, then

$$\varphi_i = M_i\overline{R}_i^{-\varepsilon_i}u_i(x_i)^{\varepsilon_i\frac{p_i-1}{2}} + \alpha u_i(x_i)^{1+o_i(1)}\overline{R}_i^{2-n+o_i(1)}.$$

By (5.24), it is possible to choose $\alpha > 0$ large enough so that $\varphi_i \geq u_i$ for $|x - x_i| = r_i$. Then, the application of Lemma 5.13 to $\varphi_i - u_i$ gives:

$$u_i(x) \leq M_i \left|x - x_i\right|^{-\varepsilon_i} + \alpha u_i(x_i)^{-\lambda_i} \left|x - x_i\right|^{2-n+\varepsilon_i}.$$

Finally, using the monotonicity of the weighed radial averages, for every $r_i < \theta < 1$ we obtain

$$M_i \leq C\overline{u_i}(1) \leq C\theta^{\frac{p_i-1}{2}}\overline{u_i}(\theta) \leq C\theta^{\frac{p_i-1}{2}}\left(M_i\theta^{-\varepsilon_i} + \alpha u_i(x_i)^{-\lambda_i}\theta^{2-n+\varepsilon_i}\right). \tag{5.28}$$

We can choose θ small enough (up to taking a larger i) and derive that

$$M_i \leq C u_i(x_i)^{-\lambda_i}.$$

The proof can be concluded applying the Harnack inequality and absorbing the first term of the right-hand side of (5.28). ■

Lemma 5.15. *Under the assumptions of Proposition 5.12,*

$$\tau_i := \frac{n+2}{n-2} - p_i = O\left(u_i(x_i)^{\frac{-2}{n-2}+o_i(1)}\right), \quad \text{as } i \to +\infty.$$

Therefore,

$$u_i(x_i)^{\tau_i} = 1 + o_i(1).$$

Proof. We apply Corollary 3.20 for $u(x) = u_i(x + x_i)$ on $\Omega = B(0,1)$, and estimate all the terms using Lemmas 5.8 and 5.14:

$$\begin{aligned}\int_{\partial^+\Omega_+} |K_i|\, {u_i}^{p_i+1} &\leq C\int_{\partial^+\Omega_+} |K_i|\, u_i(x_i)^{\lambda_i(p_i+1)}\, |x|^{(2-n+\varepsilon_i)(p_i+1)} \\ &= O\left(u_i(x_i)^{-(p_i+1)+o_i(1)}\right).\end{aligned}$$

Analogously,

$$\begin{aligned}\int_{\partial^+\Omega_+} |S|\, {u_i}^2 &= O\left(u_i(x_i)^{-2+o_i(1)}\right), \\ \int_{\partial(\partial_0\Omega_+)} |H_i|\, {u_i}^{\frac{p_i+3}{2}} &= O\left(u_i(x_i)^{-\frac{p_i+3}{2}+o_i(1)}\right).\end{aligned}$$

In order to simplify the notation, let $\delta_i = u_i(x_i)^{-\frac{p_i-1}{2}}$. We now bound the gradient terms using a rescaling argument:

$$\begin{aligned}&\int_{\Omega_+} u_i(x + x_i)^{p_i+1}\, |X \cdot \nabla K_i(x + x_i)|\, dx \\ &= u_i(x_i)^{-\frac{2}{n-2}+o_i(1)} \int_{\left(B_{\delta_i^{-1}}\right)_+} v_i(y)^{p_i+1} Y \cdot \nabla K_i(\delta_i x + x_i) dy,\end{aligned}$$

where $v_i(y) = u_i(\delta_i y + x_i)$. By Lemma 5.8, $v_i \to b_\beta$ in $C^2_{loc}\left((B_{\delta_i^{-1}})_+\right)$ for some $\beta > 0$. This, together with the asymptotic behaviour of b_β described in (3.35) and the fact that $K_i(\delta_i x + x_i) \to K(0)$ uniformly, gives:

$$\int_{\Omega_+} u_i(x+x_i)^{p_i+1} \left| X \cdot \nabla K_i(x+x_i) \right| dx = O\left(u_i(x_i)^{-\frac{2}{n-2}+o_i(1)}\right).$$

In the same way, we obtain:

$$\int_{\Omega_+} u_i(x+x_i)^2 \left| X \cdot \nabla S(x+x_i) \right| dx = O\left(u_i(x_i)^{-\frac{6}{n-2}+o_i(1)}\right),$$

$$\int_{\partial_0\Omega_+} u_i(x+x_i)^{\frac{p_i+3}{2}} \left| X \cdot \nabla H_i(x+x_i) \right| dx = O\left(u_i(x_i)^{-\frac{2}{n-2}+o_i(1)}\right),$$

and, using the same argument,

$$\int_{\Omega_+} \left| S(x+x_i) \right| u_i(x+x_i)^2 dx = O\left(u_i(x_i)^{-\frac{4}{n-2}+o_i(1)}\right).$$

Moreover, by standard elliptic theory, Proposition 3.21 and Lemma 5.14, we have:

$$\int_{\partial^+\Omega_+} B_1(u_i, \nabla u_i) = O\left(u_i(x_i)^{-2+o_i(1)}\right).$$

Again, by a rescaling argument we see that

$$\begin{aligned} &\tau_i \left(\int_{B(x_i,1)_+} K_i u_i^{p_i+1} + 2(n-1) \int_{\partial_0 B(x_i,1)_+} H_i u_i^{\frac{p_i+3}{2}} \right) \\ &= \tau_i \left(\int_{\mathbb{R}^n_+} K(0) {b_\beta}^{2^*} + 2(n-1) \int_{\partial\mathbb{R}^n_+} H(0) {b_\beta}^{2^\sharp} + o_i(1) \right). \end{aligned} \tag{5.29}$$

We need to check that the coefficient of τ_i is positive. With this in mind, we recall that b_β solves the problem

$$\begin{cases} -\frac{4(n-1)}{n-2} \Delta b_\beta = K(0) {b_\beta}^{\frac{n+2}{n-2}} & \text{in } \mathbb{R}^n_+, \\ \frac{2}{n-2} \frac{\partial b_\beta}{\partial \eta} = H(0) {b_\beta}^{\frac{n}{n-2}} & \text{on } \partial\mathbb{R}^n_+. \end{cases}$$

Multiplying by b_β and integrating by parts, we obtain

$$\int_{\mathbb{R}^n_+} K(0) {b_\beta}^{2^*} + 2(n-1) \int_{\partial\mathbb{R}^n_+} H(0) {b_\beta}^{2^\sharp} = \frac{4(n-1)}{n-2} \int_{\mathbb{R}^n_+} |\nabla b_\beta|^2 > 0. \tag{5.30}$$

The combination of (5.29), (5.30) and Corollary 3.20 finally gives

$$\tau_i \left(\int_{\mathbb{R}^n_+} |\nabla b_\beta|^2 + o_i(1) \right) = O\left(u_i(x_i)^{-\frac{2}{n-2}+o_i(1)}\right). \tag{5.31}$$

This concludes the proof. ∎

Lemma 5.16. *Under the assumptions of Proposition 5.12, there holds:*

$$u_i(x_i)\int_{\partial^+(B_1)_+}\frac{\partial u_i}{\partial\eta}\to C<0,$$

for a negative constant $C=C(n,\beta)$.

Proof. Integrating (5.10) by parts, we get the relation

$$\begin{aligned} u_i(x_i)\int_{\partial^+(B_1)_+}\frac{\partial u_i}{\partial\eta} &= u_i(x_i)\frac{n-2}{4(n-1)}\int_{(B_1)_+}Su_i\\ &+u_i(x_i)\left(\frac{n-2}{4(n-1)}\int_{(B_1)_+}|K_i|\,u_i^{p_i}-\frac{n-2}{2}\int_{\partial_0(B_1)_+}H_iu_i^{\frac{p_i+1}{2}}\right). \end{aligned} \tag{5.32}$$

Using Lemma 5.14, we can bound the previous integrals for $r_i\le|x-x_i|\le 1$, as in [35]:

$$\begin{aligned} \int_{(B_1)_+\setminus(B_{r_i})_+}u_i^{p_i} &\le C\int_{(B_1)_+\setminus(B_{r_i})_+}u_i(x_i)^{-\lambda_i p_i}\,|x-x_i|^{p_i(2-n+\varepsilon_i)}\,dx\\ &\le Cu_i(x_i)^{-1+o_i(1)}\overline{R}_i^{\,-2+o_i(1)}=o_i(1)u_i(x_i)^{-1+o_i(1)}, \end{aligned}$$

and, analogously

$$\int_{(B_1)_+\setminus(B_{r_i})_+}u_i=O\left(\overline{R}_i^{\,2+o_i(1)}\right)u_i(x_i)^{-1+o_i(1)}$$

$$\int_{\partial_0(B_1)_+\setminus\partial_0(B_{r_i})_+}u_i^{\frac{p_i+1}{2}}=o_i(1)u_i(x_i)^{-1+o_i(1)}.$$

Then, (5.32) becomes:

$$\begin{aligned} u_i(x_i)\int_{\partial^+(B_{r_i})_+}\frac{\partial u_i}{\partial\eta} &= u_i(x_i)\frac{n-2}{4(n-1)}\int_{(B_{r_i})_+}Su_i\\ &+u_i(x_i)\left(\frac{n-2}{4(n-1)}\int_{(B_{r_i})_+}|K_i|\,u_i^{p_i}-\frac{n-2}{2}\int_{\partial_0(B_{r_i})_+}H_iu_i^{\frac{p_i+1}{2}}\right)\\ &+u_i(x_i)^{\tau_i}\left(o_i(1)+u_i(x_i)^{-\frac{4}{n-2}}\overline{R}_i^{\,2+o_i(1)}\right). \end{aligned}$$

By Lemmas 5.8 and 5.15 and a rescaling argument, we obtain:

$$\begin{aligned} u_i(x_i)\int_{\partial^+(B_1)_+}\frac{\partial u_i}{\partial\eta} &= \lim_{R\to+\infty}\int_{\partial^+(B_R)_+}\frac{\partial b_\beta}{\partial\eta}+o_i(1)\\ &= -\tilde{C}(\beta)\omega_{n-1}\frac{n-2}{2}+o_i(1)<0, \end{aligned}$$

with $\tilde{C}(\beta)>0$. ∎

Proof of Proposition 5.12. Inequality (5.21) for $0 \leq |x - x_i| \leq r_i$ is a consequence of Lemmas 5.8, 5.14 and 5.15, so let us address the inequality only for $r_i < |x - x_i| \leq 1$.

Let $\overline{u_i}$ be the radial average of u_i, and consider $\overline{u_i}(1)$. By (5.25),

$$u_i(x) = O\left(\overline{R}_i^{\,-\frac{2}{p_i-1}+o_i(1)}\right) |x - x_i|^{-\frac{2}{p_i-1}}, \quad \text{for } r_i \leq |x - x_i| \leq 1,$$

then $\overline{u_i}(1) \leq C\overline{R}_i^{\,-\frac{2}{p_i-1}} \to 0$, as $i \to +\infty$. Define the sequence $\xi_i = \overline{u_i}(1)^{-1} u_i$. It is easy to see that ξ_i satisfies

$$\begin{cases} -\Delta \xi_i = \frac{n-2}{4(n-1)} \overline{u_i}(1)^{p_i-1} K_i \xi_i^{p_i} - \frac{n-2}{4(n-1)} S \xi_i & \text{in } (B_2)_+ \backslash \{0\}, \\ \frac{\partial \xi_i}{\partial \eta} = \frac{n-2}{2} \overline{u_i}(1)^{\frac{p_i-1}{2}} H_i \xi_i^{\frac{p_i+1}{2}} & \text{on } \partial_0 (B_2)_+ \backslash \{0\}. \end{cases}$$

Harnack's Inequality holds in the annulus $(B_2)_+ \backslash \{0\}$ and we can pass to the limit to find a function h such that $\xi_i \to h$ in $C^2_{loc}\left((B_2)_+ \backslash \{0\}\right)$. This function verifies

$$\begin{cases} -\Delta h = 0 & \text{in } (B_2)_+ \backslash \{0\}, \\ \frac{\partial h}{\partial \eta} = 0 & \text{on } \partial_0 (B_2)_+ \backslash \{0\}. \end{cases} \tag{5.33}$$

Since the origin is an isolated simple blow-up point, h must be singular at 0. Equation (5.33) allows us to consider the symmetric extension of h to $B_2 \backslash \{0\}$, given by

$$\tilde{h}(x) = \begin{cases} h(x) & \text{if } x_n \geq 0, \\ h(x^1, \ldots, x^{n-1}, -x^n) & \text{otherwise,} \end{cases}$$

which is positive and harmonic in B_2. By uniqueness of the harmonic extension, it must coincide with the one given by Schwartz's Reflection Principle, so we can write

$$h(x) = a\,|x|^{2-n} + b(x),$$

where a is a positive constant and b satisfies

$$\begin{cases} -\Delta b = 0 & \text{in } (B_1)_+ \backslash \{0\}, \\ \frac{\partial b}{\partial \eta} = 0 & \text{on } \partial_0 (B_1)_+ \backslash \{0\}. \end{cases}$$

Let us prove (5.12) for $|x - x_i| = 1$, namely, $\overline{u_i}(1) \leq C u_i(x_i)^{-1}$. By the harmonicity of b,

$$0 = \int_{(B_1)_+} \Delta b = \int_{\partial^+ (B_1)_+} \frac{\partial b}{\partial \eta},$$

from which we deduce that

$$\lim_{i \to +\infty} \overline{u_i}(1)^{-1} \int_{\partial^+ (B_1)_+} \frac{\partial u_i}{\partial \eta} = \int_{\partial^+ (B_1)_+} \frac{\partial h}{\partial \eta} = a \int_{\partial^+ (B_1)_+} \frac{\partial\,|x|^{2-n}}{\partial \eta} < 0. \tag{5.34}$$

The result follows from Lemma 5.16. For the general case $r_i < |x - x_i| < 1$ we use a rescaling argument to reduce ourselves to the case $|x - x_i| = 1$, as in [68]. Assume by contradiction that there exists $\tilde{x}_i$, $r_i \le |\tilde{x}_i - x_i| \le 1$, such that

$$\lim_{i\to+\infty} u_i(\tilde{x}_i)u_i(x_i)\,|\tilde{x}_i - x_i|^{n-2} = +\infty. \tag{5.35}$$

Set $\tilde{r}_i = |\tilde{x}_i - x_i|$, and $\tilde{u}_i = \tilde{r}_i^{\frac{2}{p_i-1}} u_i(\tilde{r}_i x + x_i)$. As in Section 5.2, one can prove that $\tilde{u}_i$ solves the boundary value problem

$$\begin{cases} \frac{4(n-1)}{n-2}\Delta\tilde{u}_i(x) = K_i(\tilde{r}_i x + x_i)\tilde{u}_i(x)^{p_i} - \tilde{r}_i^2 S(\tilde{r}_i x + x_i)\tilde{u}_i(x) & \text{in } (B_{\tilde{r}_i^{-1}})_+, \\ \frac{2}{n-2}\frac{\partial\tilde{u}_i(x)}{\partial\eta} = H_i(\tilde{r}_i x + x_i)\tilde{u}_i(x)^{\frac{p_i+1}{2}} & \text{on } \partial_0(B_{\tilde{r}_i^{-1}})_+. \end{cases}$$

We claim that 0 is isolated simple for $\tilde{u}_i$. First, observe that

$$\tilde{u}_i(0) = \tilde{r}_i^{\frac{2}{p_i-1}} u_i(x_i) \ge r_i{}^{\frac{2}{p_i-1}} u_i(x_i) = \overline{R}_i{}^{\frac{2}{p_i-1}} \to +\infty.$$

Moreover, rescaling (5.20), we get:

$$\tilde{u}_i(x) = \tilde{r}_i^{\frac{2}{p_i-1}} u_i(\tilde{r}_i x + x_i) \le C_1\,|x|^{-\frac{2}{p_i-1}}, \text{ for } |x| \le \frac{2}{\tilde{r}_i}.$$

Finally, it is easy to check that the weighed radial averages of $\tilde{u}_i$ and u_i verify the relation

$$\begin{aligned} \widehat{(\tilde{u}_i)}(r) = r^{\frac{2}{p_i-1}}\overline{\tilde{u}_i}(r) &= r^{\frac{2}{p_i-1}}\frac{2}{\omega_{n-1}r^{n-1}}\int_{\partial^+(B_r)_+} \tilde{r}_i^{\frac{2}{p_i-1}} u_i(\tilde{r}_i x + x_i)dx \\ &= (r\tilde{r}_i)^{\frac{2}{p_i-1}} \fint_{\partial^+(B(x_i,r\tilde{r}_i))_+} u_i(y)dy = \hat{u}_i(r\tilde{r}_i), \end{aligned} \tag{5.36}$$

from which it follows that $\widehat{(\tilde{u}_i)}$ has a unique critical point in the interval $(0, \tilde{r}_i^{-1})$, concluding the proof of the claim. Therefore, the hypotheses of Proposition 5.12 hold for $\tilde{u}_i$, and in particular (5.21) in the unit sphere. This gives

$$\tilde{u}_i(0)\tilde{u}_i\left(\frac{\tilde{x}_i - x_i}{\tilde{r}_i}\right) = \tilde{r}_i^{\frac{4}{p_i-1}} u_i(\tilde{x}_i)u_i(x_i) \le C, \text{ for all } i.$$

Hence,

$$u_i(\tilde{x}_i)u_i(x_i)\,|\tilde{x}_i - x_i|^{n-2} \le C\tilde{r}_i^{\tau_i},$$

contradicting (5.35).

Concerning the final statement of the proposition, (5.23) clearly holds in a ball of size $R_i u_i(x_i)^{-\frac{p_i-1}{2}}$, by Lemma 5.8. Notice then that by (5.21) and Lemma 5.15, the measures $K_i u_i(x_i)u_i^{p_i}$ and $H_i u_i^{\frac{p_i+1}{2}}$ converge to Dirac masses at the origin on the closure of $(B_1)_+$ and on $\partial_0(B_1)_+$ respectively. This implies that $u_i(x_i)u_i(x) \to a|x|^{2-n} + b$, with a and b as in the statement, also uniformly on $(B_1(x_i))_+ \setminus (B_{R_i u_i(x_i)^{-\frac{p_i-1}{2}}})_+$, which proves the first estimate in (5.23). The gradient estimate follows by standard regularity results. ■

5.2.3 Blow-up points in $\mathscr{S}_1$ are isolated and simple for $n = 3$

In this section we prove the following result:

Theorem 5.17. *Suppose that $n = 3$. Then $\mathscr{S}_1$ consists of isolated simple points of blow-up. In particular, in view of Proposition 5.12, $\mathscr{S}_1$ is finite.*

In order to prove Theorem 5.17, some preliminary results are needed. The following proposition can be proved as in [51], with minor modifications.

Proposition 5.18. *Let (u_i) be a blowing-up sequence of solutions of (5.10). Given a large $\overline{R} > 0$ and a small $\varepsilon > 0$, for large enough i there exists a constant $C = C(\overline{R}, \varepsilon) > 0$ and a finite set of points $\left\{q_1^i, \dots, q_{m_i}^i\right\}$, with $m_i \geq 1$, such that each (q_j^i) is a local maximum for u_i and satisfies:*

1. $\left\{B\left(q_j^i, r_j^i\right)_+ : \ j = 1, \dots, m_i\right\}$ *is a disjoint collection for* $r_j^i = \overline{R} u_i(q_j^i)^{-\frac{p_i-1}{2}}$,

2. *If $y = (y_1, \dots, y_n)$ are geodesic coordinates centered at q_j^i, then*

$$\left\| u_i(q_j^i)^{-1} u_i\left(u_i(q_j^i)^{-\frac{p_i-1}{2}} y\right) - b_{\beta_j}(y) \right\|_{C^2(B_{\overline{R}})} < \varepsilon, \tag{5.37}$$

with b_{β_j} being a solution of (1.26) in $\mathbb{R}^n_+$ of the form (3.34).

3.

$$u_i(x) \leq \frac{C}{\operatorname{dist}(x, \{q_1^i, \dots, q_N^i\})^{\frac{2}{p_i-1}}} \quad \text{for every } x \in M. \tag{5.38}$$

Moreover, $u_i(q_j^i) \operatorname{dist}(q_j^i, q_k^i) \geq C^{-1}$ for every $j \neq k$.

We highlight what condition (3) locally means; if $y = (y_1, \dots, y_n)$ is a normal coordinate system centered at q_j^i, then

$$u_i(y) \leq \frac{C}{\operatorname{dist}(y, q_j^i)^{\frac{2}{p_i-1}}}, \quad \text{for } \ y \in B(q_j^i, r_j^i)_+. \tag{5.39}$$

Remark 5.19. *The essential difference between (5.39) and (5.19) is that, in the former, the radius depends on i and could colapse.*

In the sequel, we follow the steps in [66], using the assumption $n = 3$ as in [35]. We start proving that every isolated blow-up point in $\mathscr{S}_1$ is, in fact, isolated simple.

Proposition 5.20. *Let $n = 3$. If $q \in \mathscr{S}_1$ is an isolated point of blow-up, then it is isolated simple.*

Proof. We proceed by contradiction, recalling Definition 5.10. If $(q_i) \to q \in \mathscr{S}_1$ is not isolated simple, then there are at least two critical points of $\hat{u}_i$ in the interval $(0, \bar{t}_i)$, for some sequence $\bar{t}_i \to 0$.

We know by Lemma 5.8 that the sequence u_i, after rescaling it as in that statement, converges to a bubble b_β on arbitrarily large balls, so rescaling back we see that there is, at most, one critical point in the interval $\left(0, \overline{R}_i u_i(q_i)^{-\frac{p_i-1}{2}}\right)$ for the counterpart of the function $\hat{u}_i$ in Definition 5.10. Therefore the second critical point, that we call t_i, must verify

$$\frac{\overline{R}_i}{u_i(q_i)^{\frac{p_i-1}{2}}} \leq t_i \leq \bar{t}_i. \tag{5.40}$$

Now, take normal coordinates $y = (y_1, \ldots, y_n)$ centered at q_i, and rescale u_i in the following way:

$$w_i(y) = t_i^{\frac{2}{p_i-1}} u_i(t_i y), \text{ for } y \in B\left(0, t_i^{-1}\right)_+.$$

We claim that w_i has an isolated simple blow-up point at the origin according to Definition 5.10, after properly dilating the arguments of the functions K_i and H_i.

Since q is an isolated blow-up point for u_i, (5.19) holds in a ball of fixed radius $\rho > 0$. As this inequality is scale-invariant, the same stands for w_i for $|y| < \rho t_i^{-1}$. Moreover, inequality (5.40) implies that

$$w_i(0) = t_i^{\frac{2}{p_i-1}} u_i(0) \geq \overline{R}_i \to +\infty,$$

so $\{0\}$ is an isolated blow-up point for w_i. The next step is to show that the weighted radial averages $\hat{w}_i$ have a unique critical point in the interval $(0, 1)$. This follows from (5.36) together with the fact that, by definition of t_i, u_i has a unique critical point in $(0, t_i)$, proving the above claim.

We highlight that this also implies, via the chain rule, that

$$\left.\frac{d}{dr}\right|_{r=1} \hat{w}_i(r) = t_i \hat{u}_i'(t_i) = 0. \tag{5.41}$$

The sequence w_i satisfies the hypotheses of Proposition 5.12, so after passing to a subsequence we have

$$w_i(0) w_i(y) \to h(y) = a\,|y|^{2-n} + b(y), \text{ locally in } C^2(\mathbb{R}^n_+ \backslash \{0\}),$$

with $a > 0$ and b as in (5.22). We consider the function $b^* : \mathbb{R}^n \to \mathbb{R}$ defined by symmetrization as

$$b^*(x) = \begin{cases} b(x_1, \ldots, x_n) & \text{if } x \in \mathbb{R}^n_+, \\ b(x_1, \ldots, x_{n-1}, -x_n) & \text{if } x \in \mathbb{R}^n_-, \end{cases}, \tag{5.42}$$

and we notice that it is harmonic, by the fact that b satisfies Neumann boundary conditions. Also, since $h(y) > 0$, $\liminf_{|y|\to+\infty} b^*(y) \geq 0$, so b^* (and consequently b) is constant by Liouville's theorem, and using (5.41) we obtain:

$$0 = \frac{d}{dr}\bigg|_{r=1} \frac{\hat{h}(r)}{w_i(0)} = \frac{1}{w_i(0)} \frac{d}{dr}\bigg|_{r=1} \left(\frac{a}{r^{\frac{n-2}{2}}} + br^{\frac{n-2}{2}} \right) = a - b.$$

Then,

$$w_i(0)w_i(y) \to a\,|y|^{2-n} + a.$$

Proposition 3.21 then gives

$$\lim_{r\to 0} \int_{\partial^+(B_r)_+} B(w_i, \nabla w_i) = -(n-1)(n-2)\omega_{n-1} \frac{a^2}{w_i(0)^2}. \tag{5.43}$$

On the other hand, we can write w_i in terms of the limiting profile. Set $\delta_i = u_i(0)^{-\frac{p_i-1}{2}}$, and recall that $v_i(y) = \delta_i^{\frac{2}{p_i-1}} u_i(\delta_i y)$ satisfies $v_i(y) \to b_\beta(y)$ in the C^2 sense on balls of radius $\overline{R}_i$. With this in mind, we can define $\lambda_i := t_i\delta_i^{-1} \geq \overline{R}_i$ and write

$$w_i(y) = \lambda_i^{\frac{2}{p_i-1}} \delta_i^{\frac{2}{p_i-1}} u_i(\lambda_i\delta_i y) \to \lambda_i^{\frac{2}{p_i-1}} b_\beta(\lambda_i y).$$

With this notation, (5.43) becomes

$$\lim_{r\to 0} \int_{\partial^+(B_r)_+} B(w_i, \nabla w_i) = -(n-1)(n-2)\omega_{n-1} \frac{a^2}{\lambda_i^{n-2+O(\tau_i)}}. \tag{5.44}$$

Now, we apply Corollary 3.20 with $u(y) = w_i(y)$, $f(y) = K_i(t_i y)$, $g(y) = {t_i}^2 S(t_i y)$ and $h(y) = h_i(t_i y)$. We estimate all the terms involved using the asymptotic behaviour of b_β described in (3.35), Lemma 5.15 and a rescaling argument:

$$\begin{gathered}
\int_{(B_r)_+} w_i(x)^{p_i+1} \left| X \cdot \nabla K_i(t_i x) \right| dx = O\left(\lambda_i^{-1+O(\tau_i)}\right) o_i(1), \\
\int_{(B_r)_+} {t_i}^2 w_i(x)^2 \left| X \cdot \nabla S(t_i x) \right| dx = O\left({\lambda_i}^{-3+O(\tau_i)}\right) o_i(1), \\
\tau_i \int_{(B_r)_+} K_i(t_i x) w_i(x)^{p_i+1} dx = O\left(\lambda_i^{-1+O(\tau_i)}\right) o_i(1), \text{ and} \\
\int_{(B_r)_+} {t_i}^2 S(t_i x) w_i(x)^2 dx = O\left({\lambda_i}^{-2+O(\tau_i)}\right) o_i(1).
\end{gathered}$$

We remark that, by Proposition 5.15

$$\tau_i = O\left(u_i(0)^{-\frac{2}{n-2}}\right) = O\left({\delta_i}^{-1+o_i(1)}\right) = O\left({\lambda_i}^{-1+o_i(1)}\right) t_i. \tag{5.45}$$

In a similar way, we estimate

$$\int_{\partial^+(B_r)_+} t_i{}^2 S(t_i x) w_i(x)^2 dx = O\left(\lambda_i{}^{-2+O(\tau_i)}\right) o_i(1),$$

$$\int_{\partial^+(B_r)_+} K_i(t_i x) w_i(x)^{p_i+1} dx = O\left(\lambda_i{}^{-3+O(\tau_i)}\right),$$

$$\tau_i \int_{\partial_0(B_r)_+} H_i(t_i x) w_i^{\frac{p_i+3}{2}}(x) dx = O\left(\lambda_i{}^{-n+O(\tau_i)}\right) o_i(1),$$

$$\int_{\partial(\partial_0(B_r)_+)} H_i(t_i x) w_i(x)^{\frac{p_i+3}{2}} |X \cdot \nu|\, dx = O\left(\lambda_i{}^{-n+O(\tau_i)}\right), \text{ and}$$

$$\int_{\partial_0(B_r)_+} w_i(x)^{\frac{p_i+3}{2}} |X \cdot \nabla H_i(t_i x)|\, dx = O\left(\lambda_i{}^{1-n+O(\tau_i)}\right) o_i(1).$$

Then, taking $r > 0$ small enough, by Corollary 3.20, (5.44) and the previous estimates:

$$O\left(\lambda_i{}^{-1+O(\tau_i)}\right) o_i(1) + O\left(\lambda_i{}^{-2+O(\tau_i)}\right) = -(n-1)(n-2)\omega_{n-1} a^2 \lambda_i{}^{2-n+O(\tau_i)}. \quad (5.46)$$

Note that, by Lemma 5.15 and (5.45),

$$\lambda_i^{O(\tau_i)} = (1 + o_i(1)) t_i{}^{O(\tau_i)} = 1 + o_i(1).$$

Therefore, (5.46) leads to a contradiction when $n = 3$. ■

We now proceed to rule out bubble accumulations.

Proof of Theorem 5.17. Our goal is to prove that there exists $C > 0$, independent of i, such that $\operatorname{dist}(q_j^i, q_k^i) \geq C$ for every $j \neq k$ in $\{1, \dots, m_i\}$. Assume by contradiction that this is not the case. Then,

$$\lim_{i \to +\infty} \min_{j \neq k} \operatorname{dist}(q_j^i, q_k^i) = 0.$$

Since the blow-up points $\{q_1^i, \dots, q_{m_i}^i\}$ as in Proposition 5.18 are finitely-many for every i, without loss of generality we can assume that

$$\sigma_i := \min_{j \neq k} \operatorname{dist}(q_j^i, q_k^i) = \operatorname{dist}(q_1^i, q_2^i).$$

A direct application of item (3) in Proposition 5.18 gives that

$$u_i(q_j^i)\sigma_i \geq \frac{1}{C}, \text{ for } j = 1, 2.$$

Hence, $u_i(q_k^i) \to +\infty$ as $i \to +\infty$. Now, we take geodesic normal coordinates around q_1^i and rescale the functions u_i in the following way:

$$v_i(y) = \sigma_i^{\frac{2}{p_i - 1}} u_i(\sigma_i y), \text{ for } y \in B\left(0, \sigma_i{}^{-1}\right)_+.$$

Moreover, if $q_k^i \in B\left(0, \frac{1}{\sigma_i}\right)$ and if we set $y_k^i = \frac{q_k^i}{\sigma_i}$, then each y_k^i is a local maximum of v_i and $\operatorname{dist}(y_1^i, y_2^i) = |y_2^i| = 1$, so up to a subsequence we can assume that $y_2^i \to y_2$ with $|y_2| = 1$.

We claim that both $y_1 = 0$ and y_2 are isolated blow-up points for v_i. In first place, we check that $v_i(y_j) \to +\infty$ for $j = 1, 2$.

If $v_i(y_2)$ remains bounded but $v_i(0) \to +\infty$, then 0 is an isolated simple blow-up point for v_i by Proposition 5.20, while the sequence is bounded from above around y_2. Then, by Proposition 5.12, $v_i(y_2) \to 0$. However, by item (1) of Proposition 5.18 and the fact that the radii must be collapsing, for $\overline{R} > 0$,

$$\sigma_i \geq \max\left\{ \frac{\overline{R}}{u_i(q_1^i)^{\frac{p_i-1}{2}}}, \frac{\overline{R}}{u_i(q_2^i)^{\frac{p_i-1}{2}}} \right\}. \tag{5.47}$$

Rescaling back the previous inequality we obtain $\min\{v_i(0), v_i(y_2)\} \geq \overline{R}$, contradicting $v_i(y_2) \to 0$. On the other hand, if both $v_i(0)$ and $v_i(y_2)$ are bounded, we can apply Harnack's inequality and find a limiting function $v_i \to v$ in $C^2_{loc}(\mathbb{R}^n_+)$ such that

$$\begin{cases} -\Delta v = \frac{n-2}{4(n-1)} K(p) v^{\frac{n+2}{n-2}} & \text{on } \mathbb{R}^n_+. \\ \frac{\partial v}{\partial \eta} = \frac{n-2}{2} H(p) v^{\frac{n}{n-2}} & \text{on } \partial\mathbb{R}^n_+. \\ \nabla v(0) = \nabla(y_2) = 0. \end{cases}$$

However, the classification given by [30], recalled in Proposition 3.17, yields $v = 0$, which again contradicts (5.47). Now, observe that, by (1):

$$u_i(x) \leq \frac{C}{\left|x - q_j^i\right|^{\frac{2}{p_i-1}}}, \quad \text{for } |x| \leq \frac{\sigma_i}{2} \quad \text{and } j = 1, 2.$$

Then, v_i satisfies

$$v_i(y) = \sigma_i^{\frac{2}{p_i-1}} u_i(\sigma_i y) \leq \frac{C}{\left|y - y_j^i\right|^{\frac{2}{n-2}}}, \quad \text{for } |x| \leq \frac{1}{2} \text{ and } j = 1, 2,$$

and the claim is proved. Proposition 5.20 then guarantees that 0 and y_2 are isolated simple blow-up points of v_i (after a proper dilation of the arguments of K_i and H_i, as in the proof of Proposition 5.20) and we can apply Proposition 5.12 to obtain

$$v_k(0) v_i(y) \to h(y) = a_1 |y|^{2-n} + a_2 |y - y_2|^{2-n} + b(y), \text{ locally in } C^2(\mathbb{R}^n_+ \backslash \mathcal{S}),$$

with $\mathcal{S}$ being the blow-up set for v_i, and b a function satisfying

$$\begin{cases} \Delta b = 0 & \text{on } \mathbb{R}^n_+ \backslash \{\mathcal{S} \backslash \{0, y_2\}\}, \\ \frac{\partial b}{\partial \eta} = 0 & \text{on } \partial\mathbb{R}^n_+ \backslash \{\mathcal{S} \backslash \{0, y_2\}\}. \end{cases}$$

Define

$$f(y) = a_2 |y - y_2|^{2-n} + b(y).$$

It is clear that, for small $r > 0$, $f \in C^1(\overline{B_+(r)})$ and, by the maximum principle, $f(0) > 0$. Thus, we are under the assumptions of Proposition 3.21 and we can reason as in the last part of Proposition 5.20. ∎

5.2.4 Blow-up with finite Morse index

In this final section, we prove Theorem 1.9 (2.1) in arbitrary dimensions $n \geq 3$ under the additional hypothesis of Morse index equal to $\mathfrak{b}$. This is a natural hypothesis to add when working with Mountain-pass solutions, for instance. Furthermore, we give a first step for proving Theorem 1.9 (2.2) and (2.3) in dimensions higher than 3, namely, we prove that if $\mathrm{ind}(u_i) = \mathfrak{b}$, then $\mathscr{S}_1$ is a single and isolated blow-up point.

We begin defining and computing the Morse index of the solutions of (1.26). To that aim, we first demonstrate that a certain quantity is conformally invariant. This will be used for the study of the quadratic forms that define the Morse index.

Lemma 5.21. *Let (M^n, g) be a compact Riemannian manifold and $\tilde{g} = \varphi^{\frac{4}{n-2}} g$ a conformal metric with φ smooth and positive. If we denote by $\hat{f} = f\varphi^{-1}$, then*

$$\begin{aligned} &\frac{4(n-1)}{n-2} \int_M (\nabla_{\tilde{g}} \hat{u} \cdot \nabla_{\tilde{g}} \hat{v})\, dV_{\tilde{g}} + \int_M S_{\tilde{g}} \hat{u}\hat{v}\, dV_{\tilde{g}} + 2(n-1) \int_{\partial M} h_{\tilde{g}} \hat{u}\hat{v}\, ds_{\tilde{g}} \\ &= \frac{4(n-1)}{n-2} \int_M (\nabla_g u \cdot \nabla_g v)\, dV_g + \int_M S_g uv\, dV_g + 2(n-1) \int_{\partial M} h_g uv\, ds_g. \end{aligned} \tag{5.48}$$

Proof. We will use the following basic identities:

$$dV_{\tilde{g}} = \varphi^{2^*} dV_g, \quad ds_{\tilde{g}} = \varphi^{2^\sharp} ds_g, \quad \nabla_{\tilde{g}} = \varphi^{-\frac{4}{n-2}} \nabla_g,$$

and the relation between $S_{\tilde{g}}$, S_g, $h_{\tilde{g}}$ and h_g given by (1.17). The first term in the left-hand side of (5.48) can be decomposed using the previous identities:

$$\begin{aligned} &\int_M (\nabla_{\tilde{g}} \hat{u} \cdot \nabla_{\tilde{g}} \hat{v})\, dV_{\tilde{g}} = \int_M \varphi^2 (\nabla_g \hat{u} \cdot \nabla_g \hat{v})\, dV_g \\ &= \int_M (\nabla_g u \cdot \nabla_g v)\, dV_g - \int_M \left(\hat{v} \nabla_g \varphi \cdot \nabla_g u + \hat{u} \nabla_g \varphi \cdot \nabla_g v - \hat{u}\hat{v} |\nabla_g \varphi|^2\right) dV_g. \end{aligned} \tag{5.49}$$

On the other hand, integrating by parts on M and using (1.17):

$$\begin{aligned} &\int_M S_{\tilde{g}} \hat{u}\hat{v}\, dV_{\tilde{g}} = \int_M \hat{u}\hat{v} \left(S_g \varphi^2 - \frac{4(n-1)}{n-2} (\Delta_g \varphi)\varphi \right) dV_g \\ &= \int_M S_g uv\, dV_g - 2(n-1) \int_{\partial M} h_{\tilde{g}} \hat{u}\hat{v} \varphi^{2^\sharp} ds_g + 2(n-1) \int_{\partial M} h_g uv\, ds_g \\ &+ \frac{4(n-1)}{n-2} \int_M \left(\hat{v} \nabla_g \varphi \cdot \nabla_g u + \hat{u} \nabla_g \varphi \cdot \nabla_g v - \hat{u}\hat{v} |\nabla_g \varphi|^2\right) dV_g. \end{aligned} \tag{5.50}$$

Finally, (5.48) can be obtained from a linear combination of (5.49) and (5.50). ■

Up to renaming v as $\left(|K(p)|\,\frac{n-2}{4(n-1)}\right)^{\frac{n-2}{4}} v$, the problem (1.26) is equivalent to

$$\begin{cases} -\Delta v = -v^{\frac{n+2}{n-2}} & \text{in } \mathbb{R}^n_+, \\ \frac{\partial v}{\partial \eta} = -v_n = \sqrt{\frac{n-2}{2}}\mathfrak{D}_n(p) v^{\frac{n}{n-2}} & \text{on } \partial\mathbb{R}^n_+. \end{cases} \tag{5.51}$$

If v solves (5.51), we define the quadratic form associated as follows:

$$Q_v(\varphi) = \int_{\mathbb{R}^n_+} |\nabla\varphi|^2 + \frac{n+2}{n-2}\int_{\mathbb{R}^n_+} v^{\frac{4}{n-2}}\varphi^2 - \sqrt{\frac{n}{n-2}}\mathfrak{D}_n(p)\int_{\partial\mathbb{R}^n_+} v^{\frac{2}{n-2}}\varphi^2, \tag{5.52}$$

defined for $\varphi \in C_0^\infty(\mathbb{R}^n_+)$, the set of test functions with compact support (not necessarly zero on $\partial\mathbb{R}^n_+$). The **Morse index** of v is defined as the dimension of the biggest subespace of $C_0^\infty(\mathbb{R}^n_+)$ which Q_v is negative definite on, that is,

$$\operatorname{ind}(v) = \sup\left\{\dim(E) :\ E \leq C_0^\infty(\mathbb{R}^n_+),\ Q_v(\phi) < 0\ \forall\phi \in E\right\},$$

or $\operatorname{ind}(v) = +\infty$ if the previous set is unbounded from above.

Proposition 5.22. *Let v be a solution of* (1.26). *Then,*

(1) If $\mathfrak{D}_n(p) = 1$, then $\operatorname{ind}(v) = 0$. *In other words; $1-$dimensional solutions are stable.*

(2) If $\mathfrak{D}_n(p) > 1$, then $\operatorname{ind}(v) = \mathfrak{b} \geq 1$.

Proof. In the first case, v is given by (3.33). We consider the linearized problem

$$\begin{cases} -\Delta\varphi = -\frac{n+2}{n-2}\left(\frac{2}{\sqrt{n(n-2)}}x_n + 1\right)^{-2}\varphi & \text{on } \mathbb{R}^n_+, \\ -\frac{\partial\varphi}{\partial x_n} = \frac{n}{n-2}\left(\frac{2}{\sqrt{n(n-2)}}x_n + 1\right)^{-1}\varphi & \text{in } \partial\mathbb{R}^n_+. \end{cases} \tag{5.53}$$

It is easy to check that $\varphi(x) = \left(\sqrt{n(n-2)} + 2x_n\right)^{-\frac{n}{2}}$ is a positive solution of (5.53), which implies stability (see the book [37] for further information).

For the second case, let $v = b_{\beta_0}$ as in (3.34), for some $\beta_0 > 0$. If $\tilde{g} = v^{\frac{4}{n-2}}g$, then we know that $(\mathbb{R}^n_+, \tilde{g})$ is isometric to the hyperbolic space of curvature $S_{\tilde{g}} = -\frac{4(n-1)}{n-2}$. We consider the prescription problem

$$\begin{cases} -\Delta_{\tilde{g}} w - w = -w^{\frac{n+2}{n-2}} & \text{in } \mathbb{R}^n_+, \\ \frac{\partial w}{\partial \eta} + \sqrt{\frac{n-2}{n}}\mathfrak{D}_n(p) w = \sqrt{\frac{n-2}{n}}\mathfrak{D}_n(p) w^{\frac{n}{n-2}} & \text{on } \partial\mathbb{R}^n_+, \end{cases} \tag{5.54}$$

whose associated quadratic form is given by the expression

$$P_w(\phi) = \int_{\mathbb{R}^n_+} |\nabla_{\tilde{g}}\phi|^2 \, dV_{\tilde{g}} + \frac{n+2}{n-2}\int_{\mathbb{R}^n_+} w^{\frac{4}{n-2}}\phi^2 dV_{\tilde{g}} - \int_{\mathbb{R}^n_+} \phi^2 dV_{\tilde{g}}$$
$$- \sqrt{\frac{n}{n-2}}\mathfrak{D}_n(p)\int_{\partial\mathbb{R}^n_+} w^{\frac{2}{n-2}}\phi^2 ds_{\tilde{g}} + \sqrt{\frac{n-2}{n}}\mathfrak{D}_n(p)\int_{\partial\mathbb{R}^n_+} \phi^2 ds_{\tilde{g}}.$$

Since $w = 1$ is a trivial solution of (5.54), we can consider

$$P_1(\phi) = \int_{\mathbb{R}^n_+} |\nabla_{\tilde{g}}\phi|^2 \, dV_{\tilde{g}} + \frac{4}{n-2}\int_{\mathbb{R}^n_+} \phi^2 dV_{\tilde{g}} - \frac{2\mathfrak{D}_n(p)}{\sqrt{n(n-2)}}\int_{\partial\mathbb{R}^n_+} \phi^2 ds_{\tilde{g}}. \tag{5.55}$$

In our next step, we use Proposition 5.21 to show the similarities between (5.52) and (5.55). Indeed, the following relation holds:

$$\int_{\mathbb{R}^n_+} |\nabla\varphi|^2 \, dx = \int_{\mathbb{R}^n_+} |\nabla_{\tilde{g}}\hat{\varphi}|^2 \, dV_{\tilde{g}} - \int_{\mathbb{R}^n_+} \hat{\varphi}^2 dV_{\tilde{g}} + \sqrt{\frac{n-2}{n}}\mathfrak{D}_n(p)\int_{\partial\mathbb{R}^n_+} \hat{\varphi}^2 ds_{\tilde{g}}, \tag{5.56}$$

where $\hat{\varphi} = \varphi v^{-1}$. Inserting (5.56) into (5.52), we get the relation

$$Q_v(\varphi) = P_1(\hat{\varphi}),$$

for every $\varphi \in C_0^\infty(\mathbb{R}^n_+)$. Thus, it is enough to work with the quadratic form P_1, and the problem becomes finding ϕ_0 such that $P_1(\phi_0) < 0$.

In what follows, we will use the Poincaré Disk description for the hyperbolic space $(\mathbb{R}^n_+, \tilde{g})$, consisting on the ball

$$B_{\mathfrak{R}} = \left\{x \in \mathbb{R}^n_+ : \, |x|^2 < \mathfrak{R}^2\right\}, \text{ with } \mathfrak{R} = \frac{1}{\sqrt{-S_{\tilde{g}}}} = \frac{1}{2}\sqrt{\frac{n-2}{n-1}},$$

and the conformal metric

$$\mathfrak{g} = \frac{4\,|dx|^2}{\left(1 + S_{\tilde{g}}\,|x|^2\right)^2}.$$

Observe that

$$P_1(1) = \frac{4}{n-2}\,|B_{\mathfrak{R}}| - \frac{n-2}{2}H(B_{\mathfrak{R}})\,|\partial B_{\mathfrak{R}}|\,, \tag{5.57}$$

where $H(B_{\mathfrak{R}})$ denotes the mean curvature of $\partial B_{\mathfrak{R}}$. The first variation of the area gives us a relation between the $(n-1)-$dimensional volume of $\partial B_{\mathfrak{R}}$ and its mean curvature, namely,

$$\left.\frac{d\,|\partial B_s|}{ds}\right|_{s=\mathfrak{R}} = \int_{\partial B_{\mathfrak{R}}} H(\partial B_{\mathfrak{R}}) ds_{\mathfrak{g}}.$$

Since $H(\partial B_{\mathfrak{R}})$ is constant, (5.57) becomes

$$P_1(1) = \frac{4}{n-2}\,|B_{\mathfrak{R}}| - \frac{n-2}{2}\,\frac{d\,|\partial B_s|}{ds}\bigg|_{s=\mathfrak{R}}. \tag{5.58}$$

We recall that a hyperbolic ball of radius $s>0$ is isometric to an Euclidean ball of radius $\mathfrak{R}\sinh\frac{s}{\mathfrak{R}}$. Therefore, $|\partial B_s| = \omega_{n-1}\mathfrak{R}^{n-1}\left(\sinh\frac{s}{\mathfrak{R}}\right)^{n-1}$ and we can compute the different quantities that appear in (5.58) as follows:

$$\frac{\partial\,|\partial B_s|}{ds}\bigg|_{s=\mathfrak{R}} = (n-1)\omega_{n-1}\mathfrak{R}^{n-2}\,(\sinh 1)^{n-2}\cosh 1. \tag{5.59}$$

On the other hand, integrating by parts:

$$\begin{aligned} |B_{\mathfrak{R}}| &= \int_0^{\mathfrak{R}} \omega_{n-1}\mathfrak{R}^{n-1}\left(\sinh\frac{s}{\mathfrak{R}}\right)^{n-1} ds \\ &= \omega_{n-1}\left(\frac{\mathfrak{R}^n}{n-1}\,(\sinh 1)^{n-2}\cosh 1 - \mathfrak{R}^{n-1}\frac{n-2}{n-1}\int_0^{\mathfrak{R}}\left(\sinh\frac{s}{\mathfrak{R}}\right)^{n-3} ds\right). \end{aligned} \tag{5.60}$$

Pluggin (5.59) and (5.60) into (5.58), we get

$$\begin{aligned} P_1(1) = \omega_{n-1}\mathfrak{R}^{n-2}&\left(\frac{1}{(n-1)^2} - \frac{(n-2)(n-1)}{2}\right)(\sinh 1)^{n-2}\cosh 1 \\ &- \frac{4}{n-1}\omega_{n-1}\mathfrak{R}^{n-1}\int_0^{\mathfrak{R}}\left(\sinh\frac{s}{\mathfrak{R}}\right)^{n-3} ds < 0, \end{aligned}$$

finishing the proof. ∎

Conjecture 5.23. *Let v be a solution of* (1.26) *with $\mathfrak{D}_n(p) > 1$, then* $\operatorname{ind}(v) = 1$.

As it happens in the 2−dimensional case (see [74]), we are strongly convinced that *bubbles* have Morse index equal to 1. In order to simplify the notation, let us call

$$\rho(s) = \frac{4}{\left(1 - \frac{4(n-1)}{n-2}s^2\right)^2},$$

and consider the quadratic form (5.55) on $B_{\mathfrak{R}}$, given by

$$P_1(\phi) = \int_{B_{\mathfrak{R}}} |\nabla\phi|^2\,\rho(|x|)^{\frac{n-2}{2}} + \frac{4}{n-2}\int_{B_{\mathfrak{R}}}\phi^2\rho(|x|)^{\frac{n}{2}} - \frac{2\mathfrak{D}_n(p)}{\sqrt{n(n-2)}}\int_{\partial B_{\mathfrak{R}}}\phi^2\rho(\mathfrak{R})^{\frac{n-1}{2}}. \tag{5.61}$$

The linear operator associated to (5.61) has the following expression:

$$\begin{cases} -\rho\Delta\phi - \frac{n-2}{2}\nabla\phi\cdot\nabla\rho = -\frac{4}{n-2}\rho^2\phi & \text{in } B_{\mathfrak{R}}, \\ \frac{\partial\phi}{\partial\eta} = \frac{2\mathfrak{D}_n(p)}{\sqrt{n(n-2)}}\sqrt{\rho(\mathfrak{R})}\phi & \text{on } \partial B_{\mathfrak{R}}. \end{cases}$$

Compared to the 2−dimensional case, here the presence of the term $-\frac{n-2}{2}\nabla\phi\cdot\nabla\rho$ difficults the search for explicit solutions of the first equation. Inspired by the results obtained in [74], we study existence of solutions of the form $\phi=\phi(r,x_i)$, where $r=|x|$ is the radial variable and x_i is a fixed coordinate function. We are led to solve:

$$\begin{aligned} &-\rho(r)\left(\frac{\partial^2\phi}{\partial {x_i}^2}+\frac{\partial^2\phi}{\partial r^2}+2\frac{\partial^2\phi}{\partial r\partial x_i}\frac{x_i}{r}+\frac{n-1}{r}\frac{\partial\phi}{\partial r}\right)\\ &-\frac{n-2}{2}\rho'(r)\left(\frac{\partial\phi}{\partial x_i}\frac{x_i}{r}+\frac{\partial\phi}{\partial r}\right)=\frac{-4}{n-2}\rho(r)^2\phi, \quad \text{in } B_{\mathfrak{R}}. \end{aligned} \tag{5.62}$$

Different versions of (5.62) have been considered, but we are yet to find explicit solutions for any of them. If $\phi(r,x_i)=\phi(r)$, then

$$-\rho\left(\phi''+\frac{n-1}{r}\phi'\right)-\frac{n-2}{2}\rho'\phi'=\frac{-4}{n-2}\rho^2\phi, \quad \text{in } B_{\mathfrak{R}}. \tag{5.63}$$

Alternatively, we can set $\phi(r,x_i)=\psi(r)x_i$. In that case, we can use (5.62) to see that the radial function $\psi(r)$ solves

$$-\rho\left(\psi''+\frac{n+1}{r}\psi'\right)-\frac{n-2}{2}\rho'\left(\frac{\psi}{r}+\psi'\right)=\frac{-4}{n-2}\rho^2\psi, \quad \text{in } B_{\mathfrak{R}}. \tag{5.64}$$

In the next step, we see that the Morse index of a blowing-up sequence of solutions controlls the sum of the Morse index of its limiting profiles at different points. Consequently, if we assume a bound on the Morse index of a blowing-up sequence, the number of blow-up points whose associated blow-up profiles have positive Morse index is finite.

Lemma 5.24. *Let (u_i) be a blowing-up sequence of solutions of* (1.24) *such that* $\operatorname{ind}(u_i)$ *is uniformly bounded. Then $\mathscr{S}_1$ is a finite collection of points.*

Proof. Without loss of generality, for large values of i we set $\operatorname{ind}(u_i)=m\in\mathbb{N}_0$. We recall that this number stands for the maximum of the dimensions of the subspaces of $C_0^\infty(M)$ on wich the following quadratic form is negative definite:

$$\begin{aligned} Q_i(\varphi)=&\frac{4(n-1)}{n-2}\int_M|\nabla\varphi|^2+\int_M S\varphi^2-p_i\int_M K_iu_i\,|u_i|^{p_i-2}\varphi^2\\ &-(n-1)(p_i+1)\int_{\partial M}H_iu_i\,|u_i|^{\frac{p_i-3}{2}}\varphi^2. \end{aligned}$$

Now we rescale (u_i) as in (5.13): let (z_i) be the sequence of points given by Ekeland's variational principle, and define $\delta_i=u_i(z_i)^{-\frac{2}{p_i-1}}$ and

$$v_i(x)=\delta_i^{\frac{2}{p_i-1}}u_i(\delta_ix+z_i).$$

The proof of Proposition 5.6 gives us the existence of v solving (1.26) such that $u_i \to v$ in $C^2_{loc}(\mathbb{R}^n_+)$. We claim that $\mathrm{ind}(v) \le m$.

Assuming otherwise, there exists a set of $m+1$ linearly independent functions $\{\varphi_1, \dots, \varphi_{m+1}\}$, with compact and disjoint supports, such that

$$\begin{aligned} Q_v(\varphi_j) = \frac{4(n-1)}{n-2} \int_{\mathbb{R}^n_+} |\nabla \varphi_j|^2 - \frac{n+2}{n-2} \int_{\mathbb{R}^n_+} K(0) v^{\frac{4}{n-2}} {\varphi_j}^2 \\ - \frac{2n(n-1)}{n-2} \int_{\partial \mathbb{R}^n_+} H(0) v^{\frac{2}{n-2}} {\varphi_j}^2 < 0, \ \ \forall j = 1 \dots, m+1. \end{aligned}$$

Now, let us define

$$\overline{\varphi}_j = \delta_i^{-\frac{2}{p_i - 1}} \varphi_j \left(\frac{x - z_i}{\delta_i} \right). \tag{5.65}$$

A rescaling argument, together with the fact that $u_i \to v$ in $C^2_{loc}(\mathbb{R}^n_+)$, give us:

$$Q_i(\overline{\varphi}_j) = Q_v(\varphi_j) + o_i(1).$$

For large values of i, this implies the existence of a family of linearly independent and compactly supported functions $\{\overline{\varphi}_1, \dots, \overline{\varphi}_{m+1}\}$, with disjoint supports, such that

$$Q_i(\overline{\varphi}_j) < 0, \ \ \forall j = 1, \dots, m+1.$$

Hence, $\mathrm{ind}(u_i) \ge m+1$, finishing the proof of the claim.

Finally, take $p_1, p_2 \in \mathscr{S}_1$, $p_1 \ne p_2$, with associated blow-up sequences u_i^1, u_i^2, and limiting profiles v^1, v^2, respectively. We write $\mathrm{ind}(v^j) = m_j \in \mathbb{N}_0$, for $j = 1, 2$. Thus, for each j, there exists a collection of m_j linearly independent functions $\{\phi_1^j, \dots, \phi_{m_j}^j\}$, such that

$$Q_{v^j}(\phi_k^j) < 0, \ \ \forall k = 1, \dots, m_j. \tag{5.66}$$

Rescaling each family of functions as in (5.65), we obtain a collection of linearly independent functions with compact and disjoint supports (because $p_1 \ne p_2$),

$$\mathcal{C} = \{\overline{\phi_1^1}, \dots, \overline{\phi_{m_1}^1}, \overline{\phi_1^2}, \dots, \overline{\phi_{m_2}^2}\},$$

which makes true the following inequality for large enough values of i:

$$Q_i(\phi) < 0, \ \text{ for every } \phi \in \mathrm{span}(\mathcal{C}).$$

The consequence is $\mathrm{ind}(u_i) \ge \mathrm{ind}(v^1) + \mathrm{ind}(v^2)$. An inductive reasoning, together with Proposition 5.22 (2), shows that if $\mathrm{ind}(u_i)$ is uniformly bounded, the number of blow-up profiles of the form (3.34) respects the same bound. ∎

Proposition 5.25. *Let* (u_i) *be a blowing-up sequence of solutions of* (1.24) *with* $\mathrm{ind}(u_i) \le \mathfrak{b}$*, where* $\mathfrak{b}$ *is as in Proposition 5.22. Then* $\mathscr{S}_1$ *is a single and isolated blow-up point.*

Proof. By Lemma 5.24, we know that $\mathscr{S}_1 = \{p\}$, with $p \in \partial M$ and $\mathfrak{D}_n(p) > 1$. Let $x_i \to p$, $\overline{R}_i \to +\infty$ and $r_i \to 0$ the quantities given by Lemma 5.8.

In order to check Definition (5.9) for u_i, we take a **fixed** radius $\varepsilon > 0$ such that $\mathfrak{D}_n > 1$ on $\partial M \cap B(p, \varepsilon)$, whose existence is granted by the continuity of K and H, and we will prove that

$$\sup_{x \in B(p,\varepsilon) \setminus B(p,r_i)} u_i(x) \operatorname{dist}(x, x_i)^{\frac{2}{p_i - 1}} \leq C, \tag{5.67}$$

for some uniform constant $C > 0$. Suppose by contradiction that there exists a sequence y_i, with $r_i \leq |y_i - x_i| < \varepsilon$, such that

$$u_i(y_i) \operatorname{dist}(y_i, x_i)^{\frac{2}{p_i - 1}} \to +\infty. \tag{5.68}$$

The above equation shows that $u_i(y_i) \to +\infty$, so there exists a singular point $y \in \mathscr{S}_1$ such that, up to a subsequence, $y_i \to y$. If $y \neq p$, then u_i developes a second bubble, and the proof of Lemma 5.24 shows that in that case $\operatorname{ind}(u_i) \geq 2\mathfrak{b}$, in contradiction with our hypothesis.

Hence, $y = p$. We now take geodesic normal coordinates on x_i and define σ_i as $\sigma_i = \operatorname{dist}(y_i, x_i) = |y_i| \to 0$, and the sequence of rescaled functions

$$v_i(y) = \sigma_i{}^{\frac{2}{p_i - 1}} u_i(\sigma_i y).$$

By rescaling variables, one can show that $\operatorname{ind}(v_i) \leq \operatorname{ind}(u_i) = \mathfrak{b}$, but we will prove that v_i produces two bubbles, getting a contradiction.

Let $z_i = \sigma_i{}^{-1} y_i$. Clearly $\operatorname{dist}(z_i, x_i) = 1$, so up to a subsequence we can assume $z_i \to z$, with $\operatorname{dist}(z, x_i) = |z| = 1$. On one hand, by (5.68) we have:

$$v_i(z_i) = u_i(y_i) \sigma_i^{\frac{2}{p_i - 1}} = u_i(y_i) \operatorname{dist}(y_i, x_i)^{\frac{2}{p_i - 1}} \to +\infty.$$

On the other hand, since $\sigma_i \geq r_i$, we have:

$$v_i(x_i) = \sigma_i{}^{\frac{2}{p_i - 1}} u_i(x_i) \geq \overline{R}_i \to +\infty.$$

Therefore, $\{0, z\} \subset \mathscr{S}_1(v_i)$, giving $\operatorname{ind}(v_i) \geq 2\mathfrak{b}$ again by the proof of Lemma 5.24. ∎

5.3 Conclusion of the proof of Theorem 1.9

In this section we conclude the proof of Theorem 1.9. Let us recall that Proposition 5.3 and Theorem 1.9 imply Theorem 1.8.

5.3.1 Proof of Theorem 1.9, (2.2)

In this subsection we prove that if $n = 3$ and the scalar curvature S satisfies $S \leq 0$, then $\mathscr{S}_1 = \emptyset$. For this purpose, we present a capacity argument that exploits fundamental differences between sequences of solutions of (5.10) and sequences of bubbles. Since we proved in the previous section that solutions resemble bubbles at scales of order 1 near points in $\mathscr{S}_1$, we are able to dismiss this kind of blow-up in three dimensions. We have first two propositions, which hold true for all dimensions $n \geq 3$.

Proposition 5.26. *Consider a sequence of positive solutions (u_i) of (5.10) with $K_i \to K < 0$ in $C^1(\overline{M})$, $S \leq 0$, and $H_i \to H$ in $C^2(\partial M)$. Then, given a small $\delta > 0$, for large enough values of i, there exists a positive constant $C = C(\delta)$ such that*

$$\int_{\{u_i \leq 1\}} \frac{|\nabla u_i|^2}{u_i^{\frac{p_i+3}{2}}} + \int_{\{u_i > 1\}} \frac{|\nabla u_i|^2}{{u_i}^{p_i+1+\delta}} < C. \tag{5.69}$$

Proof. Consider the continuous function

$$f(u) = \begin{cases} u^{-\left(\frac{p_i+1}{2}\right)} & \text{if } 0 < u \leq 1, \\ u^{-(p_i+\delta)} & \text{if } u > 1. \end{cases}$$

Multiplying (5.10) by $f(u_i)$ and integrating by parts, we obtain:

$$\begin{aligned} &\frac{4(n-1)}{n-2} \int_M f'(u_i) \, |\nabla u_i|^2 = -\int_M S u_i f(u_i) + \int_{\{u_i \leq 1\}} K_i u_i^{\frac{2}{n-2}+o_i(1)} \\ &+ \int_{\{u \geq 1\}} K_i u_i^{-\delta+o_i(1)} - 2(n-1) \int_{\{u_1 \leq 1\} \cap \partial M} H_i \\ &+ 2(n-1) \int_{\{u_i > 1\} \cap \partial M} H_i u_i^{-\frac{2}{n-2}-\delta+o_i(1)}. \end{aligned}$$

Notice that all terms in the right-hand side are uniformly bounded, with the possible exception of the first one: however, that term is non-negative by our assumptions. Hence, we get an upper bound for the left-hand side:

$$\frac{4(n-1)}{n-2} \left\{ \frac{p_i+1}{2} \int_{\{u \leq 1\}} \frac{|\nabla u|^2}{u^{\frac{p_i+3}{2}}} + (p_i - \delta) \int_{\{u > 1\}} \frac{|\nabla u|^2}{u^{p_i+1+\delta}} \right\} < C,$$

concluding the proof. ■

Next, we show that the above property is not satisfied for the *bubbles*, i.e. the solutions to (1.26) when $\mathfrak{D}_n(p) > 1$, described in (3.34). We recall that this one-parameter family of solutions can be written as follows:

$$b_\beta(x) = \frac{C_n \beta^{\frac{n-2}{2}}}{\left(|x - x_0(\beta)|^2 - \beta^2\right)^{\frac{n-2}{2}}} \quad \forall x \in \mathbb{R}^n_+, \tag{5.70}$$

with $C_n = (n(n-2))^{\frac{n-2}{4}}$ and $x_0(\beta) = (0, \ldots, 0, -\mathfrak{D}_n(p)\beta)$. Straightforward computations show that

$$\frac{|\nabla b_\beta|^2}{{b_\beta}^\mu} = \tilde{C}_n \beta^{(2-\mu)\frac{n-2}{2}} \frac{|x - x_0(\beta)|^2}{\left(|x - x_0(\beta)|^2 - \beta^2\right)^{n-\mu\frac{n-2}{2}}},$$

where $\tilde{C}_n = (n-2)^2 {C_n}^{2-\mu}$ and $\mu > 0$. Moreover, the domain $\{b_\beta \leq 1\}$ is the complement of a ball centered in $x_0(\beta)$ with radius tending to zero. Notice that $b_\beta(x) \leq 1$ whenever $|x - x_0(\beta)|^2 > \beta^2 + \sqrt{n(n-2)}\beta =: {r_\beta}^2$, thus

$$\{b_\beta \leq 1\} = \mathbb{R}^n_+ \backslash B^n(x_0(\beta), r_\beta).$$

For small enough values of β, we can take $0 < r_\beta < r < R$ so that

$$A_\beta(r, R) := A(x_0(\beta), r, R) \cap \mathbb{R}^n_+ \subset \{b_\beta \leq 1\}.$$

The aim is now to prove that

$$\lim_{\beta \to 0} \int_{A_\beta(r,R)} \frac{|\nabla b_\beta|^2}{{b_\beta}^\mu} = +\infty \tag{5.71}$$

for some $0 < \mu < 2^\sharp$, obtaining an opposite conclusion to that of Proposition 5.26. For a fixed $\beta > 0$,

$$\int_{A_\beta(r,R)} \frac{|\nabla b_\beta|^2}{{b_\beta}^\mu} = \tilde{C}_n \int_r^R \int_{\mathtt{SC}^{n-1}(x_0(\beta), s, s-\beta\mathfrak{D}_n(p))} \frac{\beta^{(2-\mu)\frac{n-2}{2}} s^2}{(s^2 - \beta^2)^{n-\mu\frac{n-2}{2}}} dx ds,$$

where $\mathtt{SC}^{n-1}(x_0, r, h)$ denotes the $(n-1)-$dimensional spherical cap centered at $x_0 \in \mathbb{R}^n$, with radius $r > 0$ and height $0 \leq h < r$. We know that

$$\left|\mathtt{SC}^{n-1}(x, r, h)\right| = w_n r^{n-1} J(r, h), \tag{5.72}$$

for some uniformly bounded function J. Thus, for some dimensional constant $\hat{C}_n$,

$$\int_{A_\beta(r,R)} \frac{|\nabla b_\beta|^2}{{b_\beta}^\mu} \geq \hat{C}_n \beta^{(2-\mu)\frac{n-2}{2}} \int_r^R \frac{s^{n+1}}{(s^2 - \beta^2)^{n-\mu\frac{n-2}{2}}} ds.$$

Since $r > r_\beta > \beta$, the integral is uniformly bounded when $\beta \to 0$, so it is enough to take $\mu > 2$ to have (5.71). We then proved the following result.

Proposition 5.27. *Let b_β be the family of functions in* (5.70). *Then,*

$$\lim_{\beta \to 0} \int_{A_\beta(r,R)} \frac{|\nabla b_\beta|^2}{{b_\beta}^\mu} = +\infty,$$

for all $A_\beta(r, R) \subset \{b_\beta \leq 1\}$ and $\mu > 2$.

To prove that $\mathscr{S}_1$ is empty, it is then sufficient to combine Proposition 5.26, Proposition 5.27, Proposition 5.20 and (5.23) in Lemma 5.8.

5.3.2 Proof of Theorem 1.9, (2.3)

In this subsection we consider a sequence of solutions (u_i) to (5.10) and we assume that $\mathscr{S}_0 \neq \emptyset$. We recall that the limit profile is one-dimensional, as given in (3.33). In particular, we have that $\rho_i := \int_{\partial M} H_i {u_i}^{\frac{p_i+3}{2}} \to +\infty$.

By assumption, $I_i(u_i)$ is uniformly bounded:

$$4\int_M |\nabla u_i|^2 + \frac{1}{2}\int_M S{u_i}^2 - \frac{1}{p_i+1}\int_M |K_i|\, {u_i}^{p_i+1} - \frac{2}{p_i+3}\int_{\partial M} H_i {u_i}^{\frac{p_i+3}{2}} = O(1). \quad (5.73)$$

A second relation between the integrals is given by the fact that $I'(u_i)[u_i] = 0$, namely

$$8\int_M |\nabla u_i|^2 + \int_M S{u_i}^2 + \int_M |K_i|\, {u_i}^{p_i+1} - 4\int_{\partial M} H_i {u_i}^{\frac{p_i+3}{2}} = 0. \quad (5.74)$$

Using the previous two relations, we will try to estimate the integrals in terms of ρ_i. First notice that, by the positivity of the first and third terms in (5.74) and Hölder's inequality,

$$\int_M S u_i^2 = O\left(\rho_i^{\frac{2}{p_i+1}}\right).$$

Therefore, by (5.73) and (5.74)

$$8\int_M |\nabla u_i|^2 = \frac{1}{3}\int_M |K_i|\, {u_i}^{p_i+1} + o(\rho_i) = \int_{\partial M} H_i {u_i}^{\frac{p_i+3}{2}} + o_i(\rho_i). \quad (5.75)$$

Now, take an arbitrary $\varphi \in C^2(M)$, multiply (5.10) by $u_i\varphi$ and integrate by parts. Then, the following identity holds:

$$\int_M \left(8\,|\nabla u_i|^2 + |K_i|\, {u_i}^{p_i+1}\right)\varphi - 2\int_{\partial M} H_i {u_i}^{\frac{p_i+3}{2}}\varphi$$
$$= -\int_M S{u_i}^2\varphi - 8\int_M u_i \nabla u_i \cdot \nabla\varphi. \quad (5.76)$$

By Hölder inequality

$$\int_M |S|\, {u_i}^2\varphi = O\left(\rho_i^{\frac{2}{2^*}}\right), \qquad \int_M u_i\nabla u_i \nabla\varphi = O\left(\rho_i^{\frac{n-1}{n}}\right). \quad (5.77)$$

The combination of (5.75), (5.76) and (5.77) yields that there exists a positive measure σ defined in $\bar{M}$ such that

(i) $4\rho_i^{-1} H_i {u_i}^{\frac{p_i+3}{2}} \rightharpoonup \sigma|_{\partial M}$, and

(ii) $\rho_i^{-1}\left(8\,|\nabla u_i|^2 + |K_i|\, {u_i}^{p_i+1}\right) \rightharpoonup \sigma$ weakly in the sense of measures.

Observe that $\operatorname{supp}\sigma \subset \mathscr{S} \subset \{p \in \partial M : \mathfrak{D}_n(p) \geq 1\}$ by Theorem 1.9.
In dimension $n = 3$, each blow-up point in $\mathscr{S}_1$ is isolated and simple by Theorem 5.17, and around such points we can control u_i with good precision by Proposition 5.12. We combine both results to conclude the following:

Lemma 5.28. *If* $n = 3$, *then* $\operatorname{supp}\sigma \subset \mathscr{S}_0$.

Proof. Supose by contradiction that there exists $p \in \mathscr{S}_1$ and take $\varphi \in C^2(M)$, with compact support contained in $B_\varepsilon(p)$, such that

$$\int_{\partial M} \varphi \, d\sigma > 0. \tag{5.78}$$

Then, for large enough i, there exists a constant $C > 0$ such that

$${\rho_i}^{-1} \int_{\partial M} H_i {u_i}^{\frac{p_i+3}{2}} \varphi ds_g = {\rho_i}^{-1} \int_{B_\varepsilon \cap \partial M} H_i {u_i}^{\frac{p_i+3}{2}} \varphi ds_g > C^{-1}.$$

However, taking normal coordinates at p and using Proposition 5.12 jointly with Lemma 5.15, we find

$$\begin{aligned} {\rho_i}^{-1} \int_{\partial M \cap B_\varepsilon} H_i {u_i}^{\frac{p_i+3}{2}} \varphi ds_g &\simeq {\rho_i}^{-1} H(p) \int_{\mathbb{R}^2} \frac{1}{(1+|\tilde{x}|^2)^2} d\tilde{x} \\ &\leq {\rho_i}^{-1} O(1) \to 0, \end{aligned}$$

where '$\simeq$' represents equality up to a factor $(1+O(\varepsilon^2))(1+o_i(1))$ and $\tilde{x} = (x_1, \ldots, x_{n-1}, 0)$. This is a contradiction to (5.78). ∎

We can extract more information from the finite-energy hypothesis on (u_i). In fact, following the steps of the proof for (5.1), we obtain the following identities for the limiting measure σ.

Proposition 5.29. *For* $n = 3$, *let* (u_i) *be a sequence of solutions of* (5.10) *with* $I_i(u_i)$ *uniformly bounded and* $\mathscr{S}_0 \neq \emptyset$. *Then,*

(i) $4{\rho_i}^{-1} H_i {u_i}^{\frac{p_i+3}{2}} \rightharpoonup \sigma$,

(ii) ${\rho_i}^{-1} |K_i| {u_i}^{p_i+1} \rightharpoonup \frac{3}{4}\sigma$, *and*

(iii) $8{\rho_i}^{-1} |\nabla u_i|^2 \rightharpoonup \frac{1}{4}\sigma$.

Proof. Take any test function $\phi \in C^2_c(M)$ and a vector field N on M with $|N| \leq 1$ and such that $N = \eta$ on ∂M. Straightforward computations show that, calling $X = H_i \phi u^{\frac{p_i+3}{2}} N$,

$$\operatorname{div}(X) = u^{\frac{p_i+3}{2}} \left(H_i \phi \operatorname{div} N + \phi N \cdot \nabla H_i + H_i \nabla\phi \cdot N + \frac{p_i+3}{2} H_i \phi \frac{\nabla u}{u} \cdot N \right). \tag{5.79}$$

Thus, by the Divergence Theorem, there exists a constant C, which depends on $\|\phi\|_{C^1}$, $\|N\|_{C^1}$ and $\|H\|_\infty$, such that

$$\int_{\partial M} H_i \phi u^{\frac{p_i+3}{2}} \leq C \int_M u^{\frac{p_i+3}{2}} + \frac{p_i+3}{2} \int_M H_i \phi \left|\nabla u\right| u^{\frac{p_i+1}{2}}.$$

By Cauchy-Schwartz's inequality,

$$\int_{\partial M} H_i \phi u^{\frac{p_i+3}{2}} \leq C \int_M u^{\frac{p_i+3}{2}} + \frac{p_i+3}{2} \left(\frac{\delta^2}{2} \int_M \left|\nabla u\right|^2 \phi + \frac{1}{2\delta^2} \int_M H_i^2 u^{p_i+1} \phi \right), \quad (5.80)$$

for every $\delta > 0$. Now, let us define the measures σ_1, σ_2 by

(i) $8\rho_i{}^{-1} \left|\nabla u_i\right|^2 \rightharpoonup \sigma_1$, and

(ii) $\rho_i{}^{-1} \left|K\right| u_i{}^{p_i+1} \rightharpoonup \sigma_2$.

Notice that by (5.75) and the definition of ρ_i these measures are finite. By Lemma 5.28, we also know that $\sigma_1 + \sigma_2 = \sigma$, and that $\operatorname{supp} \sigma_j \subset \{\mathfrak{D}_n = 1\}$. Multiplying by $\rho_i{}^{-1}$ and taking limits, using the fact that $H^2 = \frac{1}{6}\left|K\right|$ on $\{\mathfrak{D}_n = 1\}$, we obtain

$$\frac{1}{4} \int_{\partial M} \phi \, d\sigma \leq \frac{\delta^2}{4} \int_{\partial M} \phi \, d\sigma_1 + \frac{1 + o_\varepsilon(1)}{3\delta^2} \int_{\partial M} \phi \, d\sigma_2.$$

Therefore,

$$\left(\frac{1}{4} - \frac{\delta^2}{4} \right) \int_{\partial M} \phi \, d\sigma_1 \leq \left(\frac{1}{3\delta^2} - \frac{1}{4} \right) \int_{\partial M} \phi \, d\sigma_2.$$

Choosing $\delta^2 = \frac{2}{3}$, we get the inequality

$$\int_{\partial M} \phi \, d\sigma_1 \leq 3 \int_{\partial M} \phi \, d\sigma_2.$$

Equality follows from (5.75). ■

Our final step will be to show that the support of the measure σ is formed by critical points of $\mathfrak{D}_n$. In this way, the assertion (2.3) of Theorem 1.9 would be proved.

Proposition 5.30. *For $n = 3$, let (u_i) be a sequence of solutions of* (5.10) *with $I_i(u_i)$ uniformly bounded and $\mathscr{S}_0 \neq \emptyset$. Then, $\nabla^T \mathfrak{D}_n = 0$ on* $\operatorname{supp} \sigma$.

Proof. Integrating by parts in (3.38), we obtain the following identity:

$$\begin{aligned} \frac{1}{p_i+1} \int_M u_i^{p_i+1} F \cdot \nabla K_i + \frac{8}{p_i+3} \int_{\partial M} u_i^{\frac{p_i+3}{2}} F \cdot \nabla^T H_i + \frac{1}{p_i+1} \int_M K_i u_i^{p_i+1} \operatorname{div} F \\ + \frac{8}{p_i+3} \int_{\partial M} H_i u_i^{\frac{p_i+3}{2}} \operatorname{div}^T F - 4 \int_M \left|\nabla u_i\right|^2 \operatorname{div} F + \frac{1}{2} \int_M S F \cdot \nabla(u_i^2) = \\ \frac{1}{p_i+1} \int_{\partial M} K_i u_i^{p_i+1} F \cdot \eta - 8 \int_M DF(\nabla u_i, \nabla u_i) - 4 \int_{\partial M} \left|\nabla u_i\right|^2 F \cdot \eta. \end{aligned} \quad (5.81)$$

The general idea is to choose suitable vector fields F in this identity, divide by ρ_i and take the limit. Using Hölder's inequality, one can see that

$$\rho_i^{-1} \int_M S\ F \cdot \nabla(u_i^2) \to 0.$$

As a first step we consider the distance function $\mathtt{d}$ from the boundary, and with positive sign inside M. Given a small number $\delta > 0$, we consider smooth non-increasing cut-off function $\chi_\delta(t)$ such that

$$\begin{cases} \chi(t) = 1 & t \leq \delta; \\ \chi(t) = 0 & t \geq 2\delta, \end{cases}$$

and the vector field $F = \mathtt{d}\chi(\mathtt{d})\nabla\mathtt{d}$. It is well known that the divergence of $\nabla\mathtt{d}$ equals the opposite of the mean curvature of the level sets of $\mathtt{d}$, see e.g. [**?**, Chapter 6: Exercise 11.d], which are smooth near ∂M. Therefore we obtain that

$$\operatorname{div} F = 1 + O(\mathtt{d}); \quad F|_{\partial M} = 0.$$

By (5.81), we obtain

$$\frac{1 + o_i(1)}{2^*} \int_M |K_i|\, u^{p_i+1} + (4 + o_i(1)) \int_M |\nabla u_i|^2 = (8 + o_i(1)) \int_M (\nabla u_i \cdot \nabla \mathtt{d})^2.$$

Using Proposition 5.29 we then deduce

$$\int_M |\nabla u_i|^2 = (1 + o_i(1)) \int_M (\nabla u_i \cdot \nabla \mathtt{d})^2, \tag{5.82}$$

which means that the gradient of u_i is mostly normal to the boundary.

We now choose an arbitrary vector field V tangent to ∂M, and extend it as a vector field F in the interior of M such that $\operatorname{div} F = 0$ in a neighbourhood of ∂M. Observe that now all terms in (5.81) involving $F \cdot \nu$ on ∂M or $\operatorname{div} F$ will cancel.

If one splits on ∂M a vector field $\tilde{F}$ into its tangential component $\tilde{F}^T$ and it normal one $\tilde{F}_\eta$, for a tangent vector v to ∂M there holds $\nabla_v^M \tilde{F}^T = \nabla_v^{\partial M} \tilde{F}^T + A(v, \tilde{F}^T)\eta$, where A is the second fundamental form of ∂M. Taking the trace of this relation and adding the covariant derivative of the normal component, one finds for $\tilde{F} = F$ as above

$$0 = (\operatorname{div}_M F)|_{\partial M} = \operatorname{div}^T F + h\langle \eta, F\rangle + D_\eta F_\eta,$$

where h is the mean curvature of the boundary. By the fact that $\langle \eta, F\rangle = 0$ and $F = V$ on ∂M, this implies in particular

$$D_\eta F_\eta = -\operatorname{div}^T V \text{ on } \partial M.$$

By (5.82) we then find

$$-8\rho_i^{-1}\int_M DF(\nabla u_i, \nabla u_i) \longrightarrow \frac{1}{4}\int_{\partial M} \mathrm{div}^T V\, d\sigma.$$

Observe also that, by Proposition 5.29,

$$\rho_i^{-1}\frac{8}{p_i+3}\int_{\partial M} H_i u_i^{\frac{p_i+3}{2}}\,\mathrm{div}^T F \to \frac{1}{4}\int_{\partial M} \mathrm{div}^T V\, d\sigma.$$

Hence, when we divide (5.81) by ρ_i and pass to the limit, the above two terms cancel and we obtain:

$$\frac{1}{4}\int_{\partial M}\frac{1}{H}\left(V\cdot\nabla^T H\right)d\sigma - \frac{1}{8}\int_{\partial M}\frac{1}{K}\left(V\cdot\,\nabla^T K\right)d\sigma = 0.$$

Notice however that, by Lemma 5.28, σ is supported in $\{\mathfrak{D}_n = 1\}$, and hence on its support we have that $\frac{1}{H}\nabla^T H - \frac{1}{2K}\nabla^T K = \frac{1}{\mathfrak{D}_n}\nabla^T\mathfrak{D}_n = \nabla^T\mathfrak{D}_n$. This observation and the last formula then imply

$$\int_{\partial M}\left(V\cdot\,\nabla^T\mathfrak{D}_n\right)d\sigma = 0$$

for all vector fields V on ∂M. This means that $\nabla^T\mathfrak{D}_n = 0$ on the support of σ, as desired. ∎

The proposition implies (2.3) in Theorem 1.9, since we were assuming that 1 is a regular value of $\mathfrak{D}_n$.

Remark 5.31. *The proof of the latter proposition would work with minor changes in general dimension if we had anyway the conclusion of Lemma 5.28 about the support of the measure σ.*

5.4 Appendix: proof of Lemma 5.5

Proof. Since M is a smooth manifold with boundary, we can find an extension $(\hat{M}, \hat{g})$ of the Riemannian structure including an exterior neighbourhood of ∂M. For $p \in \partial M$, let $\eta(p)$ be the the exterior unit normal vector: given two two positive parameters β and D, with β small and $D > \frac{1}{\sqrt{n(n-1)}}$, define the point

$$P_{\beta,D} = Exp_p^{\hat{g}}(\sqrt{n(n-1)}D\beta\eta(p)),$$

where $Exp_p^{\hat{g}}$ stands for the exponential map of $\hat{g}$ at p. Inspired by the solutions to problem (1.26) classified in [30] , we consider the family of functions defined on M

$$\varphi_{\beta,D}(x) = \frac{\beta^{\frac{n-2}{2}}}{\left(\mathrm{dist}_{\hat{g}}\left(x, P_{\beta,D}\right)^2 - \beta^2\right)^{\frac{n-2}{2}}}, \tag{5.83}$$

and the modified family

$$\overline{\varphi}_{\beta,D}(x) = \mu^{\frac{n-2}{2}}\varphi_{\beta,D}(x), \tag{5.84}$$

where μ is a positive constant, yet to be set. The strategy will be to estimate the energy I on such functions, choosing an adequate value for μ so that $I(\overline{\varphi}_{\beta,D}) \to -\infty$ as $D \to \frac{1}{\sqrt{n(n-1)}}$. In order to simplify the notation, let us also define

$$\varepsilon_D{}^2 = n(n-1)D^2 - 1,$$

and notice that $\varepsilon_D \to 0$ as $D \to \frac{1}{\sqrt{n(n-1)}}$. It is easy to see that

$$\begin{aligned} I(\overline{\varphi}_{\beta,D}) = {} & \frac{2(n-1)}{n-2}\mu^{n-2}\int_M |\nabla\varphi_{\beta,D}|^2 + \frac{\mu^{n-2}}{2}\int_M S\varphi_{\beta,D}{}^2 + \\ & + \frac{n-2}{2n}\mu^n\int_M |K|\,\varphi_{\beta,D}{}^{2^*} - (n-2)\mu^{n-1}\int_{\partial M} H\varphi_{\beta,D}{}^{2^\sharp}. \end{aligned} \tag{5.85}$$

Take $r > 0$ small but fixed: it is easy to show that all the above integrals, restricted to the regions (both at the interior and at the boundary) outside $B(p,r)$ tend to zero as $\beta \to 0$, uniformly for D close to $\frac{1}{\sqrt{n(n-1)}}$.
Having observed this, we proceed by estimating the *boundary term* as

$$\begin{aligned} \text{(B)} := \int_{\partial M} H\varphi_{\beta,D}{}^{2^\sharp} &= \int_{\partial M\cap B^{n-1}(p,r)} H\varphi_{\beta,D}{}^{2^\sharp} + o_\beta(1) \\ &\geq \left(\min_{\partial M\cap B^{n-1}(p,r)} H\right)\int_{\partial M\cap B^{n-1}(p,r)} \varphi_{\beta,D}{}^{2^\sharp} + o_\beta(1), \end{aligned}$$

where $o_\beta(1)$ tends to zero as β does. Take normal coordinates $\{x_1,\dots,x_n\}$ around $P_{\beta,D}$ in such a way that the geodesic joining p to $P_{\beta,D}$ is mapped into the x_n-axis, and set $x' = (x_1,\dots,x_{n-1})$. We can rewrite the previous integral in the following way, up to multiplicative errors of the form $(1+o_r(1))$, with $o_r(1)$ tending to zero as $r \to 0$, due to the presence of integral volume elements:

$$\begin{aligned} \text{(B)} &\geq \left(\min_{\partial M\cap B^{n-1}(p,r)} H\right)\int_{B^{n-1}(p,r)} \frac{\beta^{n-1}dx_1\cdots dx_{n-1}}{(|x'|^2 + n(n-1)D^2\beta^2 - \beta^2)^{n-1}} + o_\beta(1) \\ &= \left(\min_{\partial M\cap B^{n-1}(p,r)} H\right)\int_0^r\int_{\mathbb{S}^{n-2}(0,s)} \frac{\beta^{n-1}dx_1\cdots dx_{n-1}ds}{(s^2+\beta^2\varepsilon_D{}^2)^{n-1}} + o_\beta(1) \\ &= \left(\min_{\partial M\cap B^{n-1}(p,r)} H\right)|\mathbb{S}^{n-2}|\beta^{n-1}\int_0^r \frac{s^{n-2}ds}{(s^2+\beta^2\varepsilon_D{}^2)^{n-1}} + o_\beta(1). \end{aligned}$$

The latter integral can be addressed via the change of variable $t = \frac{s}{\varepsilon_D\beta}$:

$$\beta^{n-1}\int_0^r \frac{s^{n-2}ds}{(s^2+\beta^2\varepsilon_D{}^2)^{n-1}} = \frac{1}{\varepsilon_D{}^{n-1}}\int_0^{\frac{r}{\varepsilon_D\beta}} \frac{t^{n-2}dt}{(1+t^2)^{n-1}}. \tag{5.86}$$

We can always take r, β and D in such a way that $\frac{r}{\varepsilon_D \beta} \to +\infty$ as ε_D tends to zero. Calling

$$\gamma_n = \int_0^{+\infty} \frac{t^{n-2} dt}{(1+t^2)^{n-1}} = \frac{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n}{2}\right)},$$

we finally have:

$$\int_{\partial M} H {\varphi_{\beta,D}}^{2^\sharp} \geq H(p)|\mathbb{S}^{n-2}|\gamma_n \frac{1+o_r(1)}{{\varepsilon_D}^{n-1}} + o_\beta(1). \tag{5.87}$$

Now, in (5.85) we bound the *critical term in the interior* of M as:

$$\begin{aligned}
\text{(I)} :&= \int_M |K| \, {\varphi_{\beta,D}}^{2^*} = \int_{M\cap B^n(p,r)} |K| \, {\varphi_{\beta,D}}^{2^*} + o_\beta(1) \\
&\leq \left(\max_{M\cap B^n(p,r)} |K| \right) \int_{M\cap B^n(p,r)} {\varphi_{\beta,D}}^{2^*} + o_\beta(1).
\end{aligned}$$

Taking normal coordinates as before we obtain, up to multiplicative constants of order $(1 + o_r(1))$:

$$\text{(I)} \leq \left(\max_{M\cap B^n(p,r)} |K| \right) \int_{B^n_+(0,r)} \frac{\beta^n dx_1 \cdots dx_n}{\left(|x'|^2 + (x_n + \sqrt{n(n-1)}D\beta)^2 - \beta^2 \right)^n} + o_\beta(1).$$

Notice that

$$\begin{aligned}
\text{(I')} &:= \int_{B^n_+(0,r)} \frac{\beta^n dx_1 \cdots dx_n}{\left(|x'|^2 + (x_n + \sqrt{n(n-1)}D\beta)^2 - \beta^2 \right)^n} \\
&= \int_0^r \int_0^{\sqrt{r^2 - x_n^2}} \int_{\mathbb{S}^{n-2}} \frac{\beta^n dx_1 \cdots dx_n ds}{\left(s^2 + (x_n + \sqrt{n(n-1)}D\beta)^2 - \beta^2 \right)^n} \\
&\leq \left|\mathbb{S}^{n-2}\right| \beta^n \int_0^r \int_0^r \frac{s^{n-2} dx_n ds}{\left(s^2 + x_n^2 + {\varepsilon_D}^2\beta^2 + 2x_n\sqrt{n(n-1)}D\beta \right)^n}.
\end{aligned}$$

Since we are taking $r \to 0$, x_n^2 is negligible compared to x_n. This fact, together with Fubini's Theorem, gives:

$$\begin{aligned}
\text{(I')} &\leq \left|\mathbb{S}^{n-2}\right| \beta^n \int_0^r \left(\int_0^r \frac{s^{n-2} dx_n}{\left(s^2 + {\varepsilon_D}^2\beta^2 + 2x_n\sqrt{n(n-1)}D\beta \right)^n} \right) ds = \\
&= \frac{|\mathbb{S}^{n-2}|\,\beta^{n-1}}{2(n-1)D\sqrt{n(n-1)}} \left\{ \int_0^r \frac{-s^{n-2} ds}{\left(s^2 + 2r\sqrt{n(n-1)}D\beta + {\varepsilon_D}^2\beta^2 \right)^{n-1}} + \right. \\
&\left. + \int_0^r \frac{s^{n-2} ds}{(s^2 + {\varepsilon_D}^2\beta^2)^{n-1}} \right\}.
\end{aligned}$$

Since β and r are small but fixed parameters, the first term of the sum is non-singular and can be uniformly bounded. On the other hand, the second term has been computed in (5.86). If we put everthing together, we find:

$$\int_M |K|\,{\varphi_{\beta,D}}^{2^*} \leq |K(p)|\,\gamma_n \frac{|\mathbb{S}^{n-2}|}{2(n-1)} \frac{1+o_r(1)}{{\varepsilon_D}^{n-1}} + o_\beta(1). \tag{5.88}$$

We can repeat these computations for the *quadratic term* in (5.85), and eventually obtain:

$$\int_M S{\varphi_{\beta,D}}^2 \leq S(p) \frac{|\mathbb{S}^{n-2}|\,\Gamma\left(\frac{n-5}{2}\right)\Gamma\left(\frac{n-1}{2}\right)}{4(n-3)\beta^2\Gamma\left(n-3\right)} \frac{1+o_r(1)}{{\varepsilon_D}^{n-5}} + o_\beta(1) = o\left(\frac{1}{{\varepsilon_D}^{n-2}}\right). \tag{5.89}$$

Finally, we address the *Dirichlet energy.* A direct computation shows that

$$|\nabla\varphi_{\beta,D}(x)| = \frac{n-2}{2}\beta^{\frac{n-2}{2}} \frac{|\nabla \operatorname{dist}(x,P_{\beta,D})^2|}{(\operatorname{dist}(x,P_{\beta,D})^2-\beta^2)^{\frac{n}{2}}}.$$

Using the inequality $|\nabla \operatorname{dist}(x,p)^2| \leq 2\operatorname{dist}(x,p)$, we have

$$|\nabla\varphi_{\beta,D}(x)| \leq (n-2)\beta^{\frac{n-2}{2}} \frac{\operatorname{dist}(x,P_{\beta,D})}{(\operatorname{dist}(x,P_{\beta,D})^2-\beta^2)^{\frac{n}{2}}}. \tag{5.90}$$

Reasoning as before, we find

$$\int_M |\nabla\varphi_{\beta,D}|^2 \leq \int_{M\cap B^n(p,r)} |\nabla\varphi_{\beta,D}|^2 + o_\beta(1). \tag{5.91}$$

Again, taking normal coordinates as above we have, up to multiplicative errors of the form $(1+o_r(1))$

$$\begin{aligned}
&(\mathrm{D'}) := \int_{M\cap B^n(p,r)} |\nabla\varphi_{\beta,D}|^2 = \\
&= (n-2)^2\beta^{n-2}\int_{B^n_+(0,r)} \frac{\left(|x'|^2+\left(x_n+\sqrt{n(n-1)}D\beta\right)^2\right)dx}{\left(|x'|^2+\left(x_n+\sqrt{n(n-1)}D\beta\right)^2-\beta^2\right)^n} \\
&= (n-2)^2\left\{\int_{B^n_+(0,r)} \frac{\beta^{n-2}dx_1\cdots dx_n}{\left(|x'|^2+\left(x_n+\sqrt{n(n-1)}D\beta\right)^2-\beta^2\right)^{n-1}}\right. \\
&\left.+\int_{B^n_+(0,r)} \frac{\beta^{n}dx_1\cdots dx_n}{\left(|x'|^2+\left(x_n+\sqrt{n(n-1)}D\beta\right)^2-\beta^2\right)^{n}}\right\}.
\end{aligned} \tag{5.92}$$

The first integral term is of lower order $o\left(\frac{1}{\varepsilon_D{}^{n-2}}\right)$, and the second one is exactly (I'), which is already calculated. Finally, combining (5.91) and (5.92), we conclude:

$$\int_M |\nabla \varphi_{\beta,D}|^2 \leq \gamma_n \frac{(n-2)^2\,|\mathbb{S}^{n-2}|}{2(n-1)} \frac{1+o_r(1)}{\varepsilon_D{}^{n-1}} + o\left(\frac{1}{\varepsilon_D{}^{n-2}}\right). \tag{5.93}$$

Recalling that $\overline{\varphi}_{\beta,D}(x) = \mu^{\frac{n-2}{2}} \varphi_{\beta,D}(x)$, and substituting (5.87), (5.88), (5.89) and (5.93) into (5.85) we finally have, up to multiplicative contants of the form $(1+o_r(1))$ in each term:

$$\begin{aligned} I(\overline{\varphi}_{\beta,D}) \leq \gamma_n \mu^{n-2} \left|\mathbb{S}^{n-2}\right| (n-2) \left(\frac{|K(p)|}{4n(n-1)}\mu^2 - H(p)\mu + 1\right) \frac{1}{\varepsilon_D{}^{n-1}} + \\ + o\left(\frac{1}{\varepsilon_D{}^{n-2}}\right). \end{aligned} \tag{5.94}$$

Denoting by P the polynomial in μ

$$P(\mu) = \frac{|K(p)|}{4n(n-1)}\mu^2 - H(p)\mu + 1, \tag{5.95}$$

we know that there exist values for which $P(\mu) < 0$ if, and only if, its discriminant is positive, that is,

$$H(p)^2 - \frac{|K(p)|}{n(n-1)} > 0, \tag{5.96}$$

which is equivalent to our hypothesis $\mathfrak{D}_n(p) > 1$. In view of (5.94), we can assert that there exist $\beta, r > 0$ 9small) and $\mu > 0$ such that $I(\overline{\varphi}_{\beta,D}) \to -\infty$ as $D \to \frac{1}{\sqrt{n(n-1)}}$. ■

Remark 5.32. *Minimizing in* (5.95) *we can compute the optimal value for the constant, namely,*

$$\mu = \frac{2n(n-1)H(p)}{|K(p)|} = 2\frac{\mathfrak{D}_n(p)}{\sqrt{|K(p)|}}.$$

Chapter 6

Conclusions and future perspectives

In this thesis we have dealt with two boundary cases of the classical Kazdan-Warner problem in Riemannian Geometry. In two dimensions, we have considered the problem of prescribing Gaussian and geodesic curvatures on topological disks via conformal transformation of the metric, while we have addressed the existence of conformal metrics with prescribed scalar and mean curvatures on Riemannian manifolds with boundary of dimensions greater or equal to three. In particular, our contribution is focused on the prescription of **nonconstant** curvatures.

In the case of a disk in $\mathbb{R}^2$, we obtain existence of solutions when the prescribed curvatures are nonnegative and not simultaneously zero and satisfy a symmetry condition by setting the problem in a variational framework which seems to be completely new in the literature (see [31]). Thanks to compactness of solutions, we were also able to prescribe sign changing curvatures whose negative part is small.

Concerning the problem in higher dimensions, we have treated the case of negative scalar curvature and boundary mean curvature of arbitrary sign, which has not been dealt with in the literature. Using a variational approach, we prove new existence results, especially on dimension three (see [32]).
The following conclusions can be drawn from our work:

i. *There exist substantial differences with respect to the analogous problems on closed manifolds.* It is false, in general, that the theory and results available for this kind of problems in manifolds without boundary can be extended to the boundary cases, and our work perfectly exemplifies it.

   On one hand, for the problem in the disk and its boundary, the paremeter ρ is not quantized, and therefore we cannot rewrite the problem in the form of a mean-field equation. Therefore, ρ should be treated as another variable of the problem and the variational approach changes significantly. Furthermore, the presence of the boundary produces a nonnegligible interaction between the integral terms associated to the curvatures, especially when both of them are non zero. This phenomena does not find a counterpart in the closed version of the problem.

On the other hand, under the assumption $K < 0$ the problem of prescribing scalar curvature on a compact and closed manifold of negative Yamabe constant always admits a solution, as proved in [61], while the boundary version exhibits a large variety of phenomena:

Firstly, *there exists a strong interaction between the prescribed curvatures.* Evidenced by the trace inequality (5.1), which is also available for the two-dimensional case in [74], the relation between the prescribed curvatures in the boundary of the manifold plays a major role in the behaviour of our energy functional. Being the critical term in the interior negative, if the prescribed scalar curvature *shadows* the mean curvature, the boundary term gets absorbed, putting us in a nice coercive regime. On one hand, the use of min-max theory and compactness of solutions is needed if the prescribed mean curvature stands out in a single point in the boundary. In the closed case with $K < 0$ and negative Yamabe class, the absence of the boundary term makes it always coercive, simplifying its study. The analogous scenario on a manifold with boundary would be that of prescribing $K < 0$ and $h < 0$.

Moreover, *the presence of the boundary leads crucially to new non compactness phenomena.* Contrarily to what happens in the closed case, at least in low dimensions, there exist blow-up profiles with infinite mass. In fact, an explicit example of a sequence of solutions whose singular set contains the whole boundary is given in the present work. Therefore, we need to develop a blow-up analysis in a situation in which the number of blow-up points or the volume of the blow-up profiles can be infinite. Our main tool for this was the Ekeland variational principle.

ii. *The particularities of the problem in dimension two, and the added difficulty of the case of the disk.* The variational study of the problem on a surface requires the use of specific techniques, such as the Moser-Trudinger inequalities. In the case of the disk, the constant that appears in the inequalities is sharp, and refined versions of them, both for interior and boundary terms, are necessary to achieve coercivity. This phenomenon is one of the consequences of the non-compact group of Möbius transformations of the disk.

   In that regard, we also remark *the utility of our variational formulation.* The energy functional used in this thesis stands out both for its novelty, as it seems to be completely new in the literature of this kind of problems, and for its geometry, since it depends on two variables of very different natures. The variational study of the functional permits us to recover some classical results for the limiting cases where one of the curvatures is zero, and to give the first known result when both of them are nonconstant.

iii. In Chapter 5 we followed the ideas of [74], but *the adaptation to the Scalar-mean curvature prescription problem was nontrivial.*

In what concerns the analysis of the singular set of blowing-up solutions, while in higher dimensions we have more rigidity in the classification of blow-up profiles, we lack the complex-analytical tools that are exclusive to the two-dimensional case. Moreover, the domain-variation techniques applied in the case of blow-up with unbounded mass required arbitrary deformations of the tangent space at the boundary.

Because of the lack of compactness that the high-dimensional equation exhibits, a subcritical approach becomes necessary to guarantee that bounded Palais-Smale sequences admit a convergent subsequence.

iv. *The importance of Struwe's monotony trick and the subcritical approach in the search for min-max solutions of the Scalar-mean prescription problem.* As we mentioned, two problems arise when we try to check the compactness condition for the Palais-Smale sequences associated to our functional. The first one is that the sequences may not be bounded because of the triple homogeneity of the energy functional. Struwe's monotonicity trick allows us to consider bounded Palais-Smale sequences for approximated problems, but even then it is not clear that we can pass to the limit because of the presence of the critical exponents for the Sobolev embeddings. Only the combination of the two strategies permits us to bypass these obstacles and get a min-max solution.

The results obtained in this thesis represent a first step in the study of the nonconstant versions of these prescription problems. At the same time, though, we have shown that there are fundamental issues yet to be understood. This opens a research door, behind which the most immediate objective is the extension of the results of this thesis to more general settings. With this in mind, some interesting research goals could be the following:

with respect to the Gaussian-geodesic prescription problem on the disk,

i. *Remove the symmetry assumptions.* In that case, the coercivity of the functional is lost, and the search for critical points of minimum type is hampered. One possible direction is to look for additional conditions ensuring coercivity under less restrictive symmetry assumptions, like the group of symmetries having a finite number of fixed points in $\mathbb{S}^1$.

   In addition, we could look for solutions of min-max type, taking advantage of the recent development of the blow-up analysis in [59] and [11].

ii. *Prescribe curvatures with different sign.* If we consider curvatures with opposite signs, from the Gauss-Bonnet equation we cannot deduce the boundedness of any of the integral terms because of a possible compensation. Hence, the study of the interaction between both terms will require techniques different from the usual ones. In addition, the presence of bubbling solutions, confirmed in [11], shows the loss of compactness of this model. The blow-up analysis carried out in [59] is a good starting point in the search for conditions to rule out this behaviour.

The prescription of sign-changing curvatures is also an interesting question, but the wide variety of cases that it covers makes it a challenging problem.

Moreover, the research in the Scalar-mean prescription problem can continue in the following ways:

iii. *Extend the compactness theorem, namely Theorem 1.9 (2.2) and (2.3), to dimensions $n > 3$*, and consequently the min-max existence result, Theorem 1.8. To this aim, we will work under the additional hypothesis that our blowing-up sequences of solutions have bounded Morse index. This is not restrictive when considering sequences of mountain-pass type solutions, for instance. We conjecture that, in such circumstances, the singular set $\mathscr{S}_1$ is a finite collection of isolated and simple blow-up points, and therefore we could apply our energy estimates, valid for all dimensions greater or equal to three, to dismiss the bubbling solutions.

Some steps are already given in that direction, and the proof of that result may involve a classification of the singular solutions of the problem

$$\begin{cases} \frac{-4(n-1)}{n-2}\Delta v = K(0)v^{\frac{n+2}{n-2}} & \text{on } \mathbb{R}^n_+, \\ \frac{2}{n-2}\frac{\partial v}{\partial \eta} = H(0)v^{\frac{n}{n-2}} & \text{on } \partial\mathbb{R}^n_+\setminus\{0\}, \end{cases}$$

or at least an estimate over the Morse index of such solutions.

iv. *Prescribe $K < 0$ on manifolds with $S > 0$.* Partial positive results are known for low dimensions, but the problem in dimensions higher than three has hardly been considered. In this scenario, a priori, the above two types of blow-up could coexist, and the compactness of solutions arises as a challenging question, as well as the interaction between the critical terms. To start with, one could ask for conditions under which only one type of blow-up is possible, and derive new existence results.

In the case of $K > 0$ on a manifold with $S > 0$, the classification of the limiting profiles in [30] discards the presence of one-dimensional blow-up. Then, in order to gain compactness, a deep study around blow-up points of $\mathscr{S}_1$ becomes essential.

v. *Search for obstructions to the existence of solutions.* These are expected to exist, but the known results for the two-dimensional cases rely on rigidity results that are not available in higher dimensions.

Finally, we mention another curvature prescription problem that is related to the problems studied here and that can become a subject of study in the future.

The Q-T prescription problem.

In the case of a compact and closed Riemannian manifold (M, g) of dimension $n = 4$, there exists another differential operator P_g which is invariant under conformal transformations, called the *Paneitz operator*, which is associated to a natural concept of curvature. This operator, discovered by Paneitz in [84], and the corresponding Q curvature introduced by Branson ([14]), are defined in terms of the Ricci tensor Ric_g and the scalar curvature S_g as

$$\begin{aligned} P_g f &= -\Delta_g^2 f + \operatorname{div}\left(\frac{2}{3} S_g g - 2\mathrm{Ric}_g\right) df, \\ Q_g &= \frac{-1}{12}\left(\Delta_g S_g - R_g^2 + 3|\mathrm{Ric}_g|^2\right) \end{aligned}$$

for every differentiable function f on M. Similarly to how the Laplace-Beltrami operator governs the transformation rule of the Gaussian curvature, the same does the Paneitz operator with the Q curvature. Indeed, under a conformal transformation of the metric of the form $\tilde{g} = e^{2v} g$, we have:

$$P_g v + 2Q_g = 2Q_{\tilde{g}} e^{4v} \text{ on } M. \tag{6.1}$$

As before, one could ask if given a function Q defined on M, there exists a conformal metric $\tilde{g}$ such that $Q_{\tilde{g}} = Q$. In the literature we can find some partial results, see [16, 34, 22]. The analogue of the Nirenberg's problem on $\mathbb{S}^4$ is again intrinsically complex. In [92] positive results were obtained using Degree Theory, which are the counterpart of those of [24] for the equation (1.2) on $\mathbb{S}^2$ (see also the flow approach in [76]). On the other hand, a Kazdan-Warner type condition was obtained in [22].

If M has a boundary, Chang and Qing ([20]) discovered a differential operator P_g^3 defined on ∂M, and introduced a third order curvature related to it called the T curvature, which will be denoted by T_g. Therefore, on a compact Riemannian 4-manifold with boundary, the pair (P_g, P_g^3) controls the transformation equations of (Q, T) under conformal changes of the metric. Indeed, if $\tilde{g} = e^{2v} g$, then

$$\begin{cases} P_g v + 2Q_g = 2Q_{\tilde{g}} e^{4v} & \text{in } M \\ P_g^3 + T_g = T_{\tilde{g}} e^{3v} & \text{on } \partial M. \end{cases} \tag{6.2}$$

Consequently, the pair (Q_g, T_g) can be understood as a natural extension of (K_g, h_g) of a Riemannian surface in the context of the Conformal Geometry. Besides this analogy with (1.8), we also have an extension of the Gauss-Bonnet theorem, namely the Gauss-Bonnet-Chern theorem:

$$\int_M \left(Q_g + \frac{1}{8}|W_g|^2\right) dV_g + \int_{\partial M} (T_g + Z_g) dS_g = 4\pi^2 \chi(M),$$

where W_g denotes the Weyl tensor of (M, g), and $Z_g = 0$ if ∂M is totally geodesic. In particular, in a locally conformally flat manifold with totally geodesic boundary, we have the exact analogue of (1.9).

Prescribing Q and T curvatures on M and ∂M, respectively, consists in solving (6.2) with $Q_{\tilde{g}} = Q$ and $T_{\tilde{g}} = T$. Since it is a fourth order problem with a third order boundary condition, from the point of view of the PDEs it is natural to impose a second condition of first order on the boundary. In this spirit of solvability, one can impose that the boundary is minimal, that is,

$$\frac{\partial v}{\partial \nu} = 0 \text{ on } \partial M.$$

This condition is not restrictive since it can always be obtained via a conformal transformation of the metric. As far as we know, the $Q - T$ prescription problem has not been much considered, and the only available results at the present day are [80], in which the case $Q = 0$ and constant T is treated, and [81], for the problem of prescribing Q under $T = 0$. We would like to start our research on the problem with the case of the half-space $\mathbb{R}^4_+$, and prescribed curvatures Q and T equal to constants.

Chapter 7

Summary

7.1 Spanish

Esta tesis comprende el estudio de dos problemas elípticos semilineales que aparecen en el ámbito de la Geometría Riemanniana. En concreto, estamos interesados en prescribir determinadas cantidades geométricas en variedades Riemannianas con borde mediante transformaciones conformes de la métrica, a saber, las curvaturas Gaussiana y geodésica en una superficie compacta y su borde, y las curvaturas escalar y media en una variedad de dimensión superior.

La mayor parte de los resultados disponibles se centran en el estudio de estas ecuaciones en variedades cerradas, mientras que el caso con borde ha sido mucho menos tratado. En este sentido, destacamos que la presencia de borde da lugar a una mayor cantidad de fenómenos, muchos de los cuales no encuentran análogo en las versiones cerradas de estos problemas. En particular, la formulación variacional del capítulo 4, y los argumentos de compacidad y existencia del capítulo 5 están íntimamente relacionados con la presencia de borde.

Además, el foco de nuestra investigación está puesto en el caso en que *las curvaturas prescritas son no constantes*, para los cuales hay solo unos pocos resultados conocidos.

Este tipo de problemas admite una estructura variacional, de modo que discutiremos la existencia de soluciones desde el punto de vista del Cálculo de Variaciones. A veces los funcionales de energía considerados estarán minorados y será posible encontrar un mínimo global; en otros casos, sin embargo, esto no es posible y el uso de la teoría mín-máx se hace necesario. En esta última situación, esto nos conduce al análisis de soluciones de blow-up de problemas aproximados.

El trabajo desarrollado en esta tesis ha dado lugar a dos artículos de investigación, [31] y [32].

Motivación.

El principal objetivo de este trabajo es contribuir a la profundización en el conocimiento de las propiedades de las clases de métricas conformes en variedades Riemannianas. Recomendamos el trabajo [6] para conocer las nociones básicas de esta área de investigación.

El estudio de este tipo de problemas empezó con el clásico *Teorema de Uniformización*. Éste fue conjeturado por Klein y Poincaré ([62, 85]), y afirma que toda superficie Riemanniana simplemente conexa es conformemente equivalente a uno de los tres espacios modelo: $\mathbb{R}^2$, $\mathbb{S}^2$ o $\mathbb{H}^2$. Este resultado fue demostrado por Koebe y Poincaré ([63, 64, 65]) y como consecuencia toda superficie compacta y orientable admite una métrica conforme con curvatura de Gauss constante.

Llegados a este punto, uno puede preguntarse: *dada una superficie compacta* (Σ, g) *y una función* $K(x)$ *definida en* Σ*, ¿puede encontrarse una métrica conforme* $\tilde{g} \in [g]$*, tal que su curvatura Gaussiana sea igual a* K*?* Este problema se conoce como el problema de la curvatura Gaussiana prescrita, y fue propuesto por Kazdan y Warner en [60]. Si denotamos por $\tilde{g}$ a la métrica conforme $\tilde{g} = e^u g$, el problema equivale a resolver la ecuación

$$-\Delta_g u + 2K_g = 2Ke^u \quad \text{en } \Sigma. \tag{7.1}$$

Integrando (7.1) en Σ y aplicando el teorema de Gauss-Bonnet, nos damos cuenta de que existe una restricción de carácter topológico: el signo de K está condicionado por el de la característica de Euler de la superficie, $\chi(\Sigma)$.

$$\int_\Sigma Ke^u = \int_\Sigma K_g = 2\pi\chi(\Sigma). \tag{7.2}$$

Hasta el momento, solo los casos $\chi(\Sigma) = 0$ y $\chi(\Sigma) = 1$ han sido completamente resueltos, véase [60, 78, 79]. Un caso especialmente delicado es el llamado *problema de Nirenberg*, $\Sigma = \mathbb{S}^2$, debido al efecto del grupo no compacto de transformaciones conformes de la esfera. En este caso, además, se conocen otras obstrucciones para la existencia de soluciones aparte de (7.2), como las dadas en [60, 18].

La literatura sobre el problema de Nirenberg es extensa, y hay disponibles muchas condiciones suficientes para la existencia de soluciones. Por ejemplo, en [79], Moser demostró que es posible prescribir curvaturas de Gauss con simetría antipodal, propiciando el estudio de (7.1) bajo condiciones de simetría. Otros resultados sin simetrías fueron obtenidos en [23, 24, 89].

En dimensiones superiores, un resultado equivalente al Teorema de Uniformización no es esperable debido a la naturaleza tensorial de la curvatura. Sin embargo, es natural considerar contracciones de esta que aún aporten información. Por ejemplo, en una variedad Riemanniana (M, g) compacta y cerrada de dimensión $n \geq 3$, si consideramos una métrica conforme de la forma $\tilde{g} = u^{\frac{4}{n-2}} g$, con $u > 0$, las curvaturas escalares S_g y $S_{\tilde{g}}$ verifican la siguiente igualdad:

$$-\frac{4(n-1)}{n-2}\Delta_g u + S_g u = S_{\tilde{g}} u^{\frac{n+2}{n-2}} \text{ en } M. \tag{7.3}$$

La cuestión de encontrar métricas conformes con curvatura escalar constante fue propuesta por primera vez por Yamabe en [94], y resuelta completamente en [91, 4, 86]. Cuando la curvatura scalar a prescribir es una función arbitraria $K(x)$, este problema recibe el nombre de *problema de la curvatura escalar prescrita.*

En este caso también tenemos una restricción sobre el signo de $S_{\tilde{g}}$ dependiendo de la clase conforme de M, aunque no se trata de una condición topológica como en el caso dos-dimensional.

Cuando consideramos (7.3) con $S_{\tilde{g}} = K$ igual a cero o negativa, siempre existen soluciones (véase [61]). Sin embargo, aparecen obstrucciones para la existencia de estas en el caso positivo, y es necesario imponer hipótesis adicionales. Inspirados por el trabajo pionero [79], en los artículos [43, 54, 56] los autores dan teoremas de existencia con K positiva verificando una condición de simetría. Teoremas para funciones más generales aparecieron en [8, 9, 88].

Como observación final, señalamos que las ecuaciones (7.1) y (7.3) son de tipo crítico desde el punto de vista de las Ecuaciones en Derivadas Parciales; el exponente $\frac{n+2}{n-2}$ es el exponente crítico de Sobolev en (7.3), proveniente del significado geométrico de esta, mientras que la no linealidad $u \to e^u$ en (7.1) es, de alguna manera, el análogo al crecimiento crítico en dimensión $n = 2$.

El problema de las curvaturas Gaussiana y geodésica prescritas

El primer objeto de estudio de esta tesis ha sido la ecuación (7.1) en una superficie con borde, por lo que es necesario imponer condiciones de contorno. Las condiciones Dirichlet y Neumann homogéneas en el borde han sido estudiadas en la literatura, sin embargo, motivados por su significado geométrico consideramos una condición de contorno no lineal.

En efecto, nuestro objetivo es prescribir no solo la curvatura Gaussiana en Σ, sino también la curvatura geodésica en $\partial\Sigma$. Más concretamente, dada una métrica conforme $\tilde{g} = e^u g$, si K_g y $K_{\tilde{g}} = K$ son las curvaturas Gaussianas y $h_g, h_{\tilde{g}} = h$ son las curvaturas geodésicas de $\partial\Sigma$ relativas a esas métricas, entonces el logaritmo del factor conforme, u, verifica el siguiente problema de contorno:

$$\begin{cases} -\Delta_g u + 2K_g = 2Ke^u & \text{en } \Sigma, \\ \frac{\partial u}{\partial \eta} + 2h_g = 2he^{u/2} & \text{en } \partial\Sigma, \end{cases} \tag{7.4}$$

donde η denota el normal exterior unitario a $\partial\Sigma$. Integrando (7.4) en Σ, por el teorema de Gauss-Bonnet se tiene:

$$\int_\Sigma K_{\tilde{g}} e^v + \int_{\partial\Sigma} h_{\tilde{g}} e^{v/2} = 2\pi\chi(\Sigma). \tag{7.5}$$

Algunas versiones de este problema han sido ya estudiadas. El caso $h = 0$ ha sido tratado en [21]. Además, si $K = 0$ se conocen los resultados [19, 70, 72] (véase también [33] para un desarrollo más reciente del problema bajo la perspectiva de operadores no locales). El caso K y h constantes también ha sido trabajado en [15, 53, 58], al igual que el problema en el semi-plano ([71, 46, 95]).

Sin embargo, el caso en que ambas curvaturas son no constantes apenas ha sido estudiado. En [29] se obtienen resultados parciales, pero están lastrados por la presencia de un multiplicador de Lagrange fuera de control. Además, obstrucciones para la existencia de soluciones en el caso del disco fueron encontradas en [50]. En el reciente trabajo [74], el caso $K < 0$ en dominios diferentes del disco es tratado, junto a un análisis de soluciones de blow-up.

El problema en el disco

En esta tesis, consideramos el caso $\chi(\Sigma) = 1$. Por el Teorema de Uniformización, podemos pasar mediante una transformación conforme al disco, obteniendo $K_g = 0$, y $h_g = 1$. Teniendo esto en cuenta, consideraremos el problema:

$$\begin{cases} -\Delta u = 2Ke^u & \text{in } \mathbb{D}^2, \\ \frac{\partial u}{\partial \eta} + 2 = 2he^{u/2} & \text{on } \mathbb{S}^1, \end{cases} \tag{7.6}$$

donde K y h son las curvaturas a prescribir.

Cuando K y h son funciones no constantes, hay algunos resultados parciales de existencia disponibles. Por ejemplo, cuando una de las curvaturas es nula: [19, 72, 73] para el caso $K = 0$, y [21] para $h = 0$. Puede decirse que la acción no compacta del grupo de transformaciones conformes del disco es lo que hace desafiante el problema, como pasa para el problema de Nirenberg en $\Sigma = \mathbb{S}^2$. Este fenómeno ha sido tratado en [19] para $K = 0$ (consúltese también [33]). En [59] se realiza un análisis de blow-up con $K < 0$ y h no constantes. Además, la presencia de *bubbles* fue confirmada en [11], mediante técnicas de perturbación singular.

Desde el punto de vista del Cálculo de Variaciones, uno de los principales obstáculos que el problema de prescribir curvaturas Gaussiana y geodésica presenta es que, a priori, no hay una estrategia variacional clara. Encontrar un funcional de energía adecuado y estudiar sus propiedades es uno de los puntos fuertes de este trabajo; no solo porque es nuevo en la literatura, sino también por su inusual geometría.

Integrando (7.6), obtenemos:

$$\int_{\mathbb{D}^2} Ke^u + \int_{\mathbb{S}^1} he^{u/2} = 2\pi,$$

lo cual deja claro que K y h no pueden ser escogidas arbitrariamente: por ejemplo, no pueden ser simultáneamente no positivas. Definiremos el parámetro ρ como $\rho = \int_{\mathbb{D}^2} Ke^u = 2\pi - \int_{\mathbb{S}^1} he^{u/2}$. Solo para fijar las ideas, supongamos que $0 < \rho < 2\pi$ y que K y h son funciones positivas. Nuestra intención es demostrar que (7.6) es equivalente a:

$$\begin{cases} -\Delta u = 2\rho \frac{Ke^u}{\int_{\mathbb{D}^2} Ke^u} & \text{en } \mathbb{D}^2, \\ \frac{\partial u}{\partial \eta} + 2 = 2(2\pi - \rho)\frac{he^{u/2}}{\int_{\mathbb{S}^1} he^{u/2}} & \text{en } \mathbb{S}^1, \\ \frac{(2\pi-\rho)^2}{\rho} = \frac{\left(\int_{\mathbb{S}} he^{u/2}\right)^2}{\int_{\mathbb{D}^2} Ke^u} & \text{para } 0 < \rho < 2\pi. \end{cases} \tag{7.7}$$

Comparado con el problema de prescribir curvatura Gaussiana en una superficie cerrada, aquí la masa ρ no está cuantizada, y por tanto (7.7) no puede ser escrita como una ecuación de campo medio. En su lugar, ρ debe considerarse como una variable más del problema.

Por otra parte, obsérvese que ahora el problema (7.7) es invariante ante la adición de constantes a u. Esta formulación variacional puede parecer artificiosa, pero tiene la ventaja de estar relacionada con los puntos críticos de un funcional de energía con buenas propiedades, el cual definimos a continuación:

Definición 7.1. *Sean $K : \mathbb{D}^2 \to \mathbb{R}$ y $h : \mathbb{S}^1 \to \mathbb{R}$ funciones Hölder-continuas y positivas en algún punto. Se define el espacio de funciones*

$$\mathbb{X} = \left\{ u \in H^1(\mathbb{D}^2) : \int_{\mathbb{D}^2} Ke^u > 0, \int_{\mathbb{S}^1} he^{u/2} > 0 \right\},$$

que es no vacío por las hipótesis sobre K y h, y el Lagrangiano $I : \mathbb{X} \times (0, 2\pi) \to \mathbb{R}$ dado por

$$\begin{aligned} I(u, \rho) = \frac{1}{2} \int_{\mathbb{D}^2} |\nabla u|^2 - 2\rho \log \int_{\mathbb{D}^2} Ke^u + 2 \int_{\mathbb{S}^1} u - 4(2\pi - \rho) \log \int_{\mathbb{S}^1} he^{u/2} \\ + 4(2\pi - \rho) \log(2\pi - \rho) + 2\rho + 2\rho \log \rho. \end{aligned} \tag{7.8}$$

Volvemos a destacar el hecho de que el funcional de energía anterior depende del par (u, ρ), donde $u \in H^1(\mathbb{D}^2)$ y ρ es un número real positivo. Para simplificar la notación, para un $\rho \in (0, 2\pi)$ fijo, denotaremos por I_ρ al funcional $u \to I_\rho(u)$, definido en $\mathbb{X}$.

Los resultados de existencia derivan de procesos de minimización. Si congelamos la variable ρ, la familia de funcionales I_ρ es apta para la aplicación de las desigualdades de tipo Moser-Trudinger (o desigualdades tipo Onofri), que tienen sus análogos para los términos de borde. De hecho, interpolando estas desigualdades junto a variaciones de las mismas podemos demostrar que el funcional I está acotado por debajo, puesto que la cota inferior no depende de ρ, pero no conseguimos coercividad.

Como primer paso en el estudio del problema, imponemos condiciones de simetría en K y h para eliminar este fenómeno, *à la Moser* [79].

En concreto, trabajaremos con un grupo de simetrías de la siguiente forma:

Definición 7.2. *Denotaremos por G a uno de los siguientes grupos de simetría del disco:*

G es el grupo diédrico $\mathbb{D}_k$ con $k \geq 3$, o
G es el grupo de rotaciones generado por el giro de ángulo $2\pi/k$, $k \geq 2$, o
G es el grupo completo de simetrías del disco $O(2)$.

Nótese que ninguno de los grupos enumerados en la definición anterior tiene puntos fijos en $\mathbb{S}^1$, esto es, para cada $x \in \mathbb{S}^1$ existe $\phi \in G$ de modo que $\phi(x) \neq x$. Además, diremos que una función f es $G-$simétrica si $f(x) = f(\phi(x))$ para toda $\phi \in G$ y x en el dominio de f.

Cuando nos restringimos a espacios de funciones $G-$simétricas, versiones locales o mejoradas de las desigualdadesde Moser-Trudinger aplican y nos garantizan coercividad para I_ρ, permitiéndonos encontrar un mínimo global. Nuestro resultado de existencia principal para el caso del disco es el siguiente:

Teorema 7.1. *Sea G como en la definición 7.2, y $K : \mathbb{D}^2 \to \mathbb{R}$, $h : \mathbb{S}^1 \to \mathbb{R}$ funciones $G-$simétricas, Hölder continuas y no negativas, no simultáneamente cero. Entonces el problema* (7.6) *admite una solución.*

Este teorema se obtiene minimizando la función $\rho \to \min_{\mathbb{X}^G} I_\rho$. Para descartar la posibilidad de que el mínimo se alcance en los extremos del intervalo, son necesarias estimas de energía junto a un análisis de los problemas límite.

Gracias a la compacidad de soluciones del problema (7.6), podemos lidiar con el caso de curvaturas K y h cambiando de signo, siempre que su parte negativa sea suficientemente pequeña:

Teorema 7.2. *Sea G como en la definición 7.2, y $K_0 : \mathbb{D}^2 \to \mathbb{R}$, $h_0 : \mathbb{S}^1 \to \mathbb{R}$ funciones $G-$simétricas, Hölder continuas y no negativas, no simultáneamente nulas. Entonces existe $\varepsilon > 0$ tal que el problema* (7.6) *admite solución para cualesquiera funciones Hölder continuas y $G-$simétricas K, h con $\|K - K_0\|_{L^\infty} + \|h - h_0\|_{L^\infty} < \varepsilon$.*

El problema de las curvaturas escalar y media prescritas

El segundo problema que estudiamos en esta tesis hace referencia a la ecuación (7.3) en una variedad Riemanniana con borde, bajo condiciones geométricas de contorno. Para ser precisos, si (M, g) es una variedad Riemanniana compacta de dimensión $n \geq 3$ con borde ∂M, estamos interesados en la transformación de la curvatura escalar S_g y de la curvatura media de ∂M bajo transformaciones conformes de la métrica.

Si $\tilde{g} = u^{\frac{4}{n-2}} g$ es una métrica conforme y escribimos $S_{\tilde{g}} = K$, y $h_{\tilde{g}} = H$, entonces se tiene la siguiente relación:

$$\begin{cases} -c_n \Delta_g u + S_g u = K u^{\frac{n+2}{n-2}} & \text{en } M, \\ \frac{2}{n-2} \frac{\partial u}{\partial \eta} + h_g u = H u^{\frac{n}{n-2}} & \text{en } \partial M. \end{cases} \tag{7.9}$$

En la literatura, podemos encontrar varios problemas asociados a esta ecuación. Han sido menos investigados que el caso cerrado, pero aún así hay resultados que merece la pena comentar.

El primero es el análogo al problema de Yamabe, esto es, estudiar si es posible deformar g de forma conforme de tal manera que las nuevas curvaturas escalar y media sean constantes. Un primer criterio para la existencia de soluciones fue dado en [29], aunque depende de multiplicadores de Lagrange. Escobar trabajó en el caso en que $H = 0$ y K es una constante positiva, ahora conocido como *problema de Escobar*, y dio algunos resultados parciales positivos [40, 41, 42], que fueron posteriormente completados en los trabajos [52, 51]. Consúltese también [77] y sus respectivas referencias.

El caso de curvaturas variables ha sido estudiado en situaciones específicas. El caso $H = 0$ en la semiesfera fue tratado en [68, 12, 13]. En [25] se da un resultado perturbativo, a saber, los autores estudian el problema de prescribir curvatura escalar nula y curvatura media cercana a una constante. Los trabajos [1, 36, 25, 93] también versan sobre el caso $K = 0$.

Cuando ambas curvaturas son variables, destacamos el trabajo [2], el cual contiene resultados perturbativos para curvaturas cercanas a constantes en la esfera unidad de $\mathbb{R}^n$, y [35], en el que se considera el problema con $K > 0$ en la semiesfera $\mathbb{S}^3_+$, además de realizar un estudio minucioso de las soluciones de blow-up. Finalmente, en [28] también se estudia el problema para curvaturas negativas, pero las soluciones arrastran multiplicadores de Lagrange.

Nuestro objetivo aquí es considerar curvaturas variables $K < 0$ y H de signo arbitrario, y dar resultados acerca de la existencia de soluciones y del comportamiento de las soluciones tipo *bubble*. Algunos de nuestros resultados son la contraparte de otros que aparecen en [74], en el que se estudia el caso dos-dimensional en dominios con género positivo. Como veremos, aparecen muchas diferencias cuando la dimensión es mayor que dos.

Para el enunciado de nuestros teoremas, primero reducimos el problema a un caso más sencillo usando un resultado de Escobar ([41]), que afirma que toda variedad Riemanniana compacta de dimensión $n \geq 3$ con borde admite una métrica conforme con curvatura escalar que no cambia de signo y borde minimal. Esto implica que, sin perder generalidad, mediante una transformación conforme de la métrica podemos considerar una métrica de partida con $h_g = 0$ y $S_g = S$ que no cambia de signo. Siendo así, en lo que sigue supondremos que la métrica inicial de nuestra variedad es aquella dada por el resultado de Escobar, así como el hecho de que $n \geq 3$.

En vista de (7.9), nuestro objetivo es encontrar soluciones *positivas* del problema de contorno:

$$\begin{cases} -\frac{4(n-1)}{n-2}\Delta_g u + Su = Ku^{\frac{n+2}{n-2}} & \text{en } M, \\ \frac{2}{n-2}\frac{\partial u}{\partial \eta} = Hu^{\frac{n}{n-2}} & \text{en } \partial M. \end{cases} \tag{7.10}$$

La formulación variacional de (7.10) es clásica; las soluciones débiles del problema se corresponden con los puntos críticos del siguiente funcional de energía, definido en $H^1(M)$:

$$I(u) = \frac{2(n-1)}{n-2}\int_M |\nabla u|^2 + \frac{1}{2}\int_M Su^2 - \frac{1}{2^*}\int_M K|u|^{2^*} - (n-2)\int_{\partial M} H|u|^{2^\sharp}, \tag{7.11}$$

siendo $2^* = \frac{2n}{n-2}$ y $2^\sharp = \frac{2(n-1)}{n-2}$ los exponentes críticos de Sobolev para M y ∂M, respectivamente. Como hemos mencionado antes, asumimos que $K < 0$, de modo que el tercer término a la derecha de la igualdad (7.11) es positivo. La interacción entre este término y el término crítico de borde es crucial para el comportamiento del funcional.

De hecho, a través de una desigualdad de traza demostramos que la naturaleza del funcional está fuertemente condicionada por el cociente de las curvaturas prescritas en el borde, lo cual a su vez nos permite comparar ambos términos críticos. Por comodidad, definimos la función invariante por dilataciones $\mathfrak{D}_n : \partial M \to \mathbb{R}$ como

$$\mathfrak{D}_n(x) = \sqrt{n(n-1)}\frac{H(x)}{\sqrt{|K(x)|}}. \tag{7.12}$$

Dependiendo de si $\mathfrak{D}_n$ es estrictamente menor que 1 o no, nos encontramos en escenarios completamente diferentes. Remarcamos el hecho de que las fronteras de esferas geodésicas en espacios hiperbólicos satisfacen $\mathfrak{D}_n > 1$, mientras que $\mathfrak{D}_n = 1$ en los bordes de horoesferas. Por tanto, si $\mathfrak{D}_n \geq 1$, podrían existir soluciones de blow-up para (7.10) con dichos perfiles.

Suponiendo que $\mathfrak{D}_n(x) < 1$ para todo punto $x \in \partial M$, resulta que K *eclipsa* a H, y el término positivo asociado en I domina sobre el término de borde con H. El resultado es que el funcional es coercivo y admite un mínimo global.

Nuestro primer resultado hace referencia al caso en que la métrica de Escobar satisface $S < 0$, y si lo comparamos con [29, 28], resolvemos el problema geométrico original sin multiplicadores de Lagrange.

Teorema 7.3. *Supongamos que $K < 0$ en M, y que $\mathfrak{D}_n$ dada por* (7.12) *satisface $\mathfrak{D}_n < 1$ en todo punto de ∂M. Entonces, si $S < 0$, el problema* (7.10) *tiene solución.*

Si $S = 0$, necesitamos imponer hipótesis adicionales sobre H para evitar la posibilidad de que el mínimo sea idénticamente cero, y que por tanto la solución obtenida sea geométricamente admisible.

Teorema 7.4. *Supongamos que $K < 0$ en M, y que $\mathfrak{D}_n < 1$ en ∂M. Entonces, si $S = 0$ y $\int_{\partial M} H > 0$, el problema* (7.10) *tiene solución.*

Por otro lado, si existe un punto $p \in \partial M$ con $\mathfrak{D}_n(p) > 1$, podemos construir una sucesión de soluciones u_i, con masas concentradas en torno a p, de modo que la energía $I(u_i)$ tienda a $-\infty$. A pesar de que esto impide la existencia de minimizantes, en dimensión $n = 3$ podemos servirnos del Teorema de Paso de Montaña para obtener una solución de (7.10).

Teorema 7.5. *Sea $n = 3$, y supongamos que $S = 0$, $K < 0$ y que H es tal que*

(1) $\int_{\partial M} H < 0$,
(2) $\mathfrak{D}_n(\overline{p}) > 1$ *para algún* $\overline{p} \in \partial M$, *y*
(3) 1 *es un valor regular de* $\mathfrak{D}_n$.

Entonces, (7.10) *admite una solución positiva.*

Más adelante explicamos por qué aparece la restricción dimensional $n = 3$ en el Teorema 7.5, y damos un esbozo de su demostración. Para demostrar la existencia de soluciones de mín-máx es necesario probar que las sucesiones de Palais-Smale de soluciones aproximadas convergen. Al hacer esto, dos obstáculos aparecen: en primer lugar, necesitamos demostrar que están acotadas en norma, lo cual no está claro en nuestro caso debido a la triple homogeneidad del funcional de Euler-Lagrange. En segundo lugar, debido a la presencia de los exponentes críticos en (7.10), incluso las sucesiones de Palais-Smale acotadas podrían no converger.

Para sortear el primer obstáculo usaremos el *truco de monotonía de Struwe*, véase [57, 90], que consiste en perturbar el problema mediante un parámetro de forma que la energía sea monótona. Además, utilizamos una aproximación subcrítica que nos garantizará la compacidad para las sucesiones de Palais-Smale. Por tanto, consideraremos la siguiente situación:

Sea $(K_i)_i$ una sucesión de funciones regulares definida en M tal que $K_i \to K$ en $C^2(\overline{M})$, y sea $(H_i)_i$ una sucesión de funciones diferenciables en ∂M tal que $H_i \to H$ en $C^2(\partial M)$. Suponiendo que $K < 0$, consideramos soluciones positivas $(u_i)_i$ del problema perturbado

$$\begin{cases} -4\frac{n-1}{n-2}\Delta_g u_i + S u_i = K_i {u_i}^{p_i} & \text{en } M, \\ \frac{2}{n-2}\frac{\partial u_i}{\partial \eta} = H_i {u_i}^{\frac{p_i+1}{2}} & \text{en } \partial M, \end{cases} \tag{7.13}$$

es decir, puntos críticos del funcional de energía:

$$I_i(u) = \frac{2(n-1)}{n-2}\int_M |\nabla u|^2 + \frac{1}{2}\int_M Su^2 - \frac{1}{p_i+1}\int_M K_i|u|^{p_i+1} - 4\frac{n-1}{p_i+3}\int_{\partial M} H_i|u|^{\frac{p_i+3}{2}}, \tag{7.14}$$

con $p_i \nearrow \frac{n+2}{n-2}$. Nos preguntamos si esta sucesión de soluciones está uniformemente acotada, en cuyo caso convergería a una solución del problema original (7.10). Razonando por contradicción, tomamos (u_i) de la forma detallada anteriormente, y definimos su *conjunto singular* como

$$\mathscr{S} = \{p \in \overline{M} : \exists x_i \to p \text{ tal que } u_i(x_i) \text{ es no acotada}\}.$$

A este respecto, obtenemos el siguiente resultado de compacidad:

Teorema 7.6. *Sea* (u_i) *una sucesión de soluciones de* (7.13)*, y* $\mathscr{S}$ *el conjunto sigular asociado. Entonces*

(1) $\mathscr{S} \subset \{p \in \partial M : \mathfrak{D}_n(p) \geq 1\}$.

Por tanto, podemos escribir $\mathscr{S} = \mathscr{S}_0 \sqcup \mathscr{S}_1$*, con* $\mathscr{S}_1 = \mathscr{S} \cap \{\mathfrak{D}_n > 1\}$ *y* $\mathscr{S}_0 = \mathscr{S} \cap \{\mathfrak{D}_n = 1\}$*. En dimensión* $n = 3$*, tenemos además:*

(2.1) $\mathscr{S}_1$ *es una colección finita de puntos.*

(2.2) Si $S \leq 0$*, entonces* $\mathscr{S}_1 = \emptyset$.

(2.3) Si $I_i(u_i)$ *está uniformemente acotada y* 1 *es un valor regular de* $\mathfrak{D}_n$*, entonces* $\mathscr{S}_0 = \emptyset$.

El resultado anterior describe dos tipos de puntos de blow-up, agrupados en los subconjuntos $\mathscr{S}_0$ y $\mathscr{S}_1$. Estos perfiles están en correspondencia con las diferentes soluciones del siguiente problema en el semiespacio

$$\begin{cases} \frac{-4(n-1)}{n-2}\Delta v = K(p)v^{\frac{n+2}{n-2}} & \text{en } \mathbb{R}^n_+, \\ \frac{2}{n-2}\frac{\partial v}{\partial \eta} = H(p)v^{\frac{n}{n-2}} & \text{en } \partial\mathbb{R}^n_+, \end{cases} \tag{7.15}$$

donde $p \in \mathscr{S}$. Las soluciones de (7.15) fueron clasificadas en [30] (véase también [71]) de la siguiente forma:

- ⋆ Si $\mathfrak{D}_n(p) < 1$, entonces (7.15) no tiene soluciones.
- ⋆ Si $\mathfrak{D}_n(p) = 1$, las únicas soluciones son $1-$*dimensionales* y vienen dadas por:

$$v(x) = v_\alpha(x) := \left(\frac{2}{\sqrt{n(n-2)}}x_n + \alpha\right)^{-\frac{n-2}{2}}, \tag{7.16}$$

para todo $\alpha > 0$.

- ⋆ Si $\mathfrak{D}_n(p) > 1$, las soluciones reciben el nombre de *bubbles* y están dadas por la expresión:

$$v(x) = b_\beta(x) := \frac{(n(n-2))^{\frac{n-2}{4}} \beta^{\frac{n-2}{2}}}{\left(|x - x_0(\beta)|^2 - \beta^2\right)^{\frac{n-2}{2}}}, \tag{7.17}$$

con $x_0(\beta) = -\mathfrak{D}_n(p)\beta, e_n \in \mathbb{R}^n$, para $\beta > 0$ arbitrario.

Nos gustaría enfatizar que el perfil de blow-up puede tener masa infinita, contrariamente a lo que sucede en el caso sin borde, al menos en dimensión baja. El desarrollo de un análisis de blow-up en una situación en la que podrían aparecer perfiles con masa infinita, o un número infinito de puntos de blow-up es uno de los principales objetivos de este trabajo. Además, ambos tipos de blow-up podrían coexistir; el blow-up en puntos de $\mathscr{S}_1$ puede entenderse por la invarianza del problema frente a transformaciones conformes del disco, en analogía a lo que sucede en el caso cerrado. Sin embargo, en este contexto podemos tener blow-up alrededor de un conjunto infinito $\mathscr{S}_0$. De hecho, damos un ejemplo explícito de este fenómeno.

Comparado con el caso dos-dimensional estudiado en [74], disfrutamos de una mayor rigidez en la clasificación de los perfiles límite, puesto que para el semiplano existen otras soluciones generadas por funciones meromorfas, véase [46]. Por otro lado, en dimensión $n = 2$ podemos hacer uso de herramientas del Análisis Complejo, que no están presentes en dimensiones superiores.

Para tratar la pérdida de compacidad en puntos con $\mathfrak{D}_n > 1$, realizamos un estudio minucioso del comportamiento de las soluciones de blow-up alrededor de ellos, demostrando que en dimensión $n = 3$ son aislados y simples, y por tanto forman un conjunto finito de puntos (véase también [35]). Una vez que esto está demostrado, es posible controlar el comportamiento de estas soluciones también lejos de estos puntos, excluyendo la presencia de bubbles mediante estimas integrales válidas para $S \leq 0$.

Por otra parte, cerca de puntos de blow-up con $\mathfrak{D}_n = 1$, los términos $\int_M |\nabla u_i|^2$, $\int_M |K_i| \, {u_i}^{p_i+1}$ y $\int_{\partial M} H_i {u_i}^{\frac{p_i+3}{2}}$ divergen. Asumiendo la acotación de las energías $I_i(u_i)$, (que es una condición natural para sucesiones de soluciones de mín-máx) demostramos que convergen débilmente a una misma medida en el borde después de una normalización apropiada. Entonces, mediante una técnica de variación de dominio demostramos que en tales puntos el gradiente de $\mathfrak{D}_n$ a lo largo de ∂M en $\{\mathfrak{D}_n = 1\}$ se anula, contradiciendo las hipótesis de regularidad impuestas en el nivel $\{\mathfrak{D}_n = 1\}$. Comparado con un paso similar de [74], en este caso necesitamos considerar deformaciones arbitrarias tangentes a ∂M.

7.2 Italian

Questa tesi riguarda lo studio di due problemi ellittici semilineari che appaiono nel campo della Geometria Riemanniana. In particolare, siamo interessati a prescrivere certe quantità geometriche su varietà Riemanniane con bordo per mezzo di trasformazioni conformi della metrica, cioè le curvature Gaussiana e geodetica su una superficie compatta e il suo bordo, e le curvature scalare e media su una varietà di dimensione superiore.

La maggior parte dei risultati disponibili si concentra sullo studio di queste equazioni in varietà chiuse, mentre il caso con bordo è stato trattato molto meno. In relazione a ciò, evidenziamo che la presenza del bordo produce una più ampia varietà di fenomeni, molti dei quali non trovano una controparte sulle versioni chiuse di questi problemi. In particolare, la formulazione variazionale del capitolo 4, e gli argomenti di compattezza ed esistenza del capitolo 5 sono intimamente legati alla presenza del bordo.

Inoltre, la nostra ricerca è focalizzata sul caso in cui *le curvature prescritte sono non costanti*, per il quale ci sono solo pochi risultati noti.

Questo tipo di problemi ammette una struttura variazionale, quindi discuteremo l'esistenza di soluzioni dal punto di vista del Calcolo delle Variazioni. A volte i funzionali di energia considerati saranno limitati dal basso e sarà possibile trovare un minimo globale; in altri casi, tuttavia, questo non è possibile e l'uso della teoria min-max diventa necessario. In quest'ultima situazione, questo ci porta all'analisi di blow-up delle soluzioni dei problemi approssimati.

Il lavoro sviluppato in questa tesi ha portato a due articoli di ricerca, [31] e [32].

Motivazione.

L'obiettivo principale di questo lavoro è quello di contribuire all'approfondimento della conoscenza delle proprietà delle classi conformi delle varietà Riemanniane. Consigliamo la referenza [6] per le nozioni di base di questa area di ricerca.

Lo studio di questi problemi è iniziato con il classico *Teorema di uniformizzazione*. Questo è stato ipotizzato da Klein e Poincaré ([62, 85]), e afferma che ogni superficie Riemanniana semplicemente connessa è conformemente equivalente a uno dei tre spazi modello: $\mathbb{R}^2$, $\mathbb{S}^2$ o $\mathbb{H}^2$. Questo risultato fu dimostrato da Koebe e Poincaré ([63, 64, 65]) e come conseguenza ogni superficie compatta e orientabile ammette una metrica conforme con curvatura di Gauss costante.

A questo punto, ci si può chiedere: *data una superficie compatta* (Σ, g) *e una funzione* $K(x)$ *definita su* Σ*, si può trovare una metrica conforme* $\tilde{g} \in [g]$ *tale che la sua curvatura gaussiana sia uguale a* K*?* Questo problema è noto come problema della curvatura Gaussiana prescritta, ed è stato proposto da Kazdan e Warner in [60]. Se denotiamo con $\tilde{g}$ la metrica conforme $\tilde{g} = e^u g$, il problema è equivalente a risolvere l'equazione:

$$-\Delta_g u + 2K_g = 2Ke^u \quad \text{in } \Sigma. \tag{7.18}$$

Integrando (7.18) in Σ e applicando il teorema di Gauss-Bonnet, ci rendiamo conto che esiste una restrizione topologica: il segno di K è condizionato da quello della caratteristica di Eulero della superficie, $\chi(\Sigma)$.

$$\int_\Sigma Ke^u = \int_\Sigma K_g = 2\pi\chi(\Sigma). \tag{7.19}$$

Finora, solo i casi $\chi(\Sigma) = 0$ e $\chi(\Sigma) = 1$ sono stati completamente risolti, vedi [60, 78, 79]. Un caso particolarmente difficile è il cosiddetto *problema di Nirenberg*, $\Sigma = \mathbb{S}^2$, dovuto all'effetto del gruppo non compatto delle trasformazioni conformi della sfera. In questo caso, inoltre, sono noti altri ostacoli all'esistenza di soluzioni oltre a (7.19), come quelli dati in [60, 18].

La letteratura sul problema di Nirenberg è ampia e sono disponibili molte condizioni sufficienti per l'esistenza di soluzioni. Per esempio, in [79], Moser ha dimostrato che è possibile prescrivere curvature gaussiane con simmetria antipodale, portando allo studio di (7.18) sotto condizioni di simmetria. Altri risultati senza simmetrie sono stati ottenuti in [23, 24, 89].

In dimensioni superiori, un risultato equivalente al Teorema di uniformizzazione non è da aspettarsi a causa della natura tensoriale della curvatura. Pertanto, è naturale considerare contrazioni della metrica che forniscono ancora informazioni. Per esempio, in una varietà Riemanniana compatta e chiusa (M, g) di dimensione $n \geq 3$, se consideriamo una metrica conforme della forma $u^{\tilde{g}} = u^{\frac{4}{n-2}} g$, con $u > 0$, le curvature scalare S_g e $S_{\tilde{g}}$ verificano la seguente uguaglianza:

$$-\frac{4(n-1)}{n-2}\Delta_g u + S_g u = S_{\tilde{g}} u^{\frac{n+2}{n-2}} \quad \text{in } M. \tag{7.20}$$

La questione di trovare metriche conformi con curvatura scalare costante è stata proposta per la prima volta da Yamabe in [94], e completamente risolta in [91, 4, 86]. Quando la curvatura scalare da prescrivere è una funzione arbitraria $K(x)$, questo problema si chiama *problema della curvatura scalare prescritta.*

In questo caso abbiamo anche una restrizione sul segno di $S_{\tilde{g}}$ che dipende dalla classe conforme di M, sebbene questa non sia una condizione topologica come nel caso bidimensionale.

Quando consideriamo (7.18) con $S_{\tilde{g}} = K$ uguale a zero o negativa, ci sono sempre soluzioni (vedi [61]). Tuttavia, nel caso positivo appaiono degli ostacoli all'esistenza di questi, ed è necessario imporre delle ipotesi supplementari. Ispirati dal lavoro pioneristico [79], negli articoli [43, 54, 56] gli autori danno teoremi di esistenza con K positivo verificando una condizione di simmetria. Teoremi per funzioni più generali sono apparsi in [8, 9, 88].

Come osservazione finale, sottolineiamo che dal punto di vista delle equazioni differenziali parziali, le equazioni (7.18) e (7.20) sono di tipo critico; l'esponente $\frac{n+2}{n-2}$ è l'esponente di Sobolev critico in (7.20), derivante dal suo significato geometrico, mentre la non linearità $u \to e^u$ in (7.18) è, in un certo senso, l'analogo della crescita critica in dimensione $n = 2$.

Il problema delle curvature Gaussiana e geodetica prescritte

Il primo oggetto di studio di questa tesi è stata l'equazione (7.18) su una superficie con bordo, quindi è necessario imporre condizioni al bordo. Le condizioni omogenee di Dirichlet e Neumann sul bordo sono state studiate nella letteratura, tuttavia, motivate dal suo significato geometrico, consideriamo una condizione al contorno non lineare.

Infatti, il nostro obiettivo è di prescrivere non solo la curvatura gaussiana in Σ, ma anche la curvatura geodetica in $\partial\Sigma$. Più concretamente, data una metrica conforme $\tilde{g} = e^u g$, se K_g e $K_{\tilde{g}} = K$ sono le curvature gaussiane e $h_g, h_{\tilde{g}} = h$ sono le curvature geodetiche di Σ relative a quelle metriche, allora il logaritmo del fattore conforme, u, sodisfa il seguente problema al contorno:

$$\begin{cases} -\Delta_g u + 2K_g = 2Ke^u & \text{in } \Sigma, \\ \frac{\partial u}{\partial \eta} + 2h_g = 2he^{u/2} & \text{in } \partial\Sigma, \end{cases} \tag{7.21}$$

dove η denota la normale esterna unitaria a Σ. Integrando (7.21) in Σ, dal teorema di Gauss-Bonnet abbiamo:

$$\int_\Sigma K_{\tilde{g}} e^v + \int_{\partial\Sigma} h_{\tilde{g}} e^{v/2} = 2\pi\chi(\Sigma). \tag{7.22}$$

Alcune versioni di questo problema sono già state studiate. Il caso $h = 0$ è stato trattato in [21], mentre che il caso $K = 0$ in [19, 70, 72] (vedi anche [33] per uno sviluppo più recente del problema nella prospettiva degli operatori non locali). Inoltre, il caso con K e h costanti è stato trattato in [15, 53, 58], così come il problema nel semipiano ([71, 46, 95]).

Tuttavia, il caso in cui entrambe curvature sono non costanti non è stato appena studiato. Risultati parziali sono ottenuti in [29], ma sono appesantiti dalla presenza di un moltiplicatore di Lagrange fuori controllo. Inoltre, ostacoli all'esistenza di soluzioni nel caso del disco sono stati trovati in [50]. Nel recente articolo [74], viene trattato il caso $K < 0$ in domini diversi dal disco, insieme ad un'analisi delle soluzioni blow-up.

Il problema sul disco

In questa tesi, consideriamo il caso $\chi(\Sigma) = 1$. Usando il Teorema di uniformizzazione, possiamo passare attraverso una trasformazione conforme al disco, ottenendo $K_g = 0$, e $h_g = 1$. Fatta questa osservazione, consideriamo il problema:

$$\begin{cases} -\Delta u = 2Ke^u & \text{in } \mathbb{D}^2, \\ \frac{\partial u}{\partial \eta} + 2 = 2he^{u/2} & \text{in } \mathbb{S}^1, \end{cases} \tag{7.23}$$

dove K e h sono le curvature da prescrivere.

Quando K e h sono funzioni non costanti, sono disponibili alcuni risultati parziali di esistenza. Per esempio, quando una delle curvature è zero: [19, 72, 73] per il caso $K = 0$, e [21] per $h = 0$. Si può dire che l'azione non compatta del gruppo di trasformazioni conformi del disco è ciò che rende il problema impegnativo, come accade per il problema di Nirenberg in $\Sigma = \mathbb{S}^2$. Questo fenomeno è stato trattato in [19] per $K = 0$ (consultare anche [33]). In [59], viene fatto un'analisi di blow-up con $K < 0$ e h non costante. Inoltre, la presenza di *bolle* è stata confermata in [11], utilizzando tecniche di perturbazione singolare.

Dal punto di vista del Calcolo delle Variazioni, uno dei principali ostacoli che presenta il problema delle curvature Gaussiana e geodetica prescritte è che, a priori, non esiste una chiara strategia variazionale. Trovare un funzionale di energia adatto e studiare le sue proprietà è uno dei punti salienti di questo lavoro; non solo perché è nuovo nella letteratura, ma anche per la sua insolita geometria.

Integrando (7.23), abbiamo:

$$\int_{\mathbb{D}^2} Ke^u + \int_{\mathbb{S}^1} he^{u/2} = 2\pi,$$

da dove è chiaro che K e h non possono essere scelti arbitrariamente: per esempio, non possono essere contemporaneamente non positive. Definiamo il parametro ρ come $\rho = \int_{\mathbb{D}^2} Ke^u = 2\pi - \int_{\mathbb{S}^1} he^{u/2}$. Solo per fissare le idee, supponiamo che $0 < \rho < 2\pi$ e che K e h siano funzioni positive. La nostra intenzione è mostrare che (7.23) è equivalente a:

$$\begin{cases} -\Delta u = 2\rho \frac{Ke^u}{\int_{\mathbb{D}^2} Ke^u} & \text{in } \mathbb{D}^2, \\ \frac{\partial u}{\partial \eta} + 2 = 2(2\pi - \rho) \frac{he^{u/2}}{\int_{\mathbb{S}^1} he^{u/2}} & \text{in } \mathbb{S}^1, \\ \frac{(2\pi-\rho)^2}{\rho} = \frac{\left(\int_{\mathbb{S}} he^{u/2}\right)^2}{\int_{\mathbb{D}^2} Ke^u} & \text{per } 0 < \rho < 2\pi. \end{cases} \tag{7.24}$$

Se confrontiamo questo con il problema di prescrivere la curvatura gaussiana su una superficie chiusa, qui la massa ρ non è quantizzata, e quindi (7.24) non può essere scritta come un'equazione di campo medio. Invece, ρ deve essere considerata come un'altra incognita del problema.

D'altra parte, ora il problema (7.24) è invariante per la somma di costanti a u. Questa formulazione variazionale può sembrare artificiosa, ma ha il vantaggio di essere legata ai punti critici di un funzionale di energia con buone proprietà, che definiamo qui di seguito:

Definizione 7.3. *Siano $K : \mathbb{S}^1 \to \mathbb{R}$ e $h : \mathbb{S}^1 \to \mathbb{R}$ funzioni Hölder-continue e positive in qualche punto. Definiamo lo spazio di funzioni*

$$\mathbb{X} = \left\{ u \in H^1(\mathbb{D}^2) : \int_{\mathbb{D}^2} Ke^u > 0, \int_{\mathbb{S}^1} he^{u/2} > 0 \right\},$$

che è non vuoto per le condizioni su K e h, e la Lagrangiana $I : \mathbb{X} \times (0, 2\pi) \to \mathbb{R}$ data da

$$\begin{aligned} I(u,\rho) = \frac{1}{2}\int_{\mathbb{D}^2} |\nabla u|^2 - 2\rho \log \int_{\mathbb{D}^2} Ke^u + 2\int_{\mathbb{S}^1} u - 4(2\pi - \rho)\log \int_{\mathbb{S}^1} he^{u/2} \\ + 4(2\pi - \rho)\log(2\pi - \rho) + 2\rho + 2\rho \log \rho. \end{aligned} \tag{7.25}$$

Sottolineiamo il fatto che il funzionale I dipende dalla coppia (u, ρ), dove $u \in H^1(\mathbb{D}^2)$ e ρ è un numero reale positivo. Per semplificare la notazione, per un $\rho \in (0, 2\pi)$ fissato, denotiamo con I_ρ il funzionale $u \to I_\rho(u)$, definito su $\mathbb{X}$.

I risultati di esistenza derivano dai processi di minimizzazione. Se congeliamo la variabile ρ, la famiglia di funzionali I_ρ è adatta per l'applicazione delle disuguaglianze di tipo Moser-Trudinger (o disuguaglianze di tipo Onofri), che hanno i loro analoghi per i termini al bordo. Infatti, interpolando queste disuguaglianze insieme a variazioni di esse possiamo mostrare che il funzionale I è limitato dal basso, poiché il limite inferiore non dipende da ρ, ma non abbiamo coercitività. Come primo passo nello studio del problema, imponiamo condizioni di simmetria su K e h per eliminare questo fenomeno, *à la Moser* [79].

In particolare, lavoreremo con un gruppo di simmetrie come segue:

Definizione 7.4. *Denotiamo con G uno dei seguenti gruppi di simmetria del disco:*

G è il gruppo diedrale $\mathbb{D}_k$ con $k \geq 3$, oppure

G è il gruppo ciclico generato dalla rotazione di angolo $2\pi/k$, $k \geq 2$, oppure

G è il gruppo completo delle simmetrie del disco $O(2)$.

Si noti che nessuno dei gruppi elencati nella definizione precedente ha punti fissi in $\mathbb{S}^1$, cioè, per ogni $x \in \mathbb{S}^1$ esiste $\phi \in G$ in modo che $\phi(x) \neq x$. Inoltre, diremo che una funzione f è $G-$simmetrica se $f(x) = f(\phi(x))$ per tutti $\phi \in G$ e x nel dominio di f.

Quando ci limitiamo a spazi di funzioni $G-$simmetriche, si applicano versioni locali o migliorate delle disuguaglianze di Moser-Trudinger e ci garantiscono la coercitività per I_ρ, permettendoci di trovare un minimo globale. Il nostro principale risultato di esistenza per il caso del disco è il seguente:

Teorema 7.7. *Sia G come nella Definizione 7.4, e $K : \mathbb{D}^2 \to \mathbb{R}$, $h : \mathbb{S}^1 \to \mathbb{R}$, funzioni $G-$simmetriche, Hölder continue e non negative, non simultaneamente nulle. Allora il problema* (7.23) *ammette una soluzione.*

Questo teorema si ottiene minimizzando la funzione $\rho \to \min_{\mathbb{X}^G} I_\rho$. Per escludere la possibilità che il minimo sia raggiunto agli estremi dell'intervallo, sono necessarie stime dell'energia e un'analisi dei problemi limite.

Grazie alla compattezza delle soluzioni del problema (7.23), possiamo affrontare il caso di curvature K e h che cambiano segno, a condizione che la loro parte negativa sia abbastanza piccola:

Teorema 7.8. *Sia G come nella definizione 7.4, e $K_0 : \mathbb{D}^2 \to \mathbb{R}$, $h_0 : \mathbb{S}^1 \to \mathbb{R}$ funzioni $G-$simmetriche, Hölder-continue e non negative, non contemporaneamente nulle. Allora esiste $\varepsilon > 0$ tale che il problema* (7.23) *ammette soluzione per qualsiasi funzioni Hölder-continue e $G-$simmetriche K, h con $\|K - K_0\|_{L^\infty} + \|h - h_0\|_{L^\infty} < \varepsilon$.*

Il problema delle curvature scalare e media prescritte

Il secondo problema che abbiamo studiato in questa tesi si riferisce all'equazione (7.20) su una varietà Riemanniana con bordo, sotto condizioni geometriche al contorno. Per essere precisi, se (M, g) è una varietà riemanniana compatta di dimensione $n \geq 3$ con bordo ∂M, siamo interessati alla trasformazione della curvatura scalare S_g e la curvatura media di ∂M sotto trasformazioni conformi della metrica.

Se $\tilde{g} = u^{\frac{4}{n-2}} g$ è una metrica conforme e scriviamo $S_{\tilde{g}} = K$, e $h_{\tilde{g}} = H$, allora abbiamo la seguente legge di evoluzione:

$$\begin{cases} -c_n \Delta_g u + S_g u = K u^{\frac{n+2}{n-2}} & \text{in } M, \\ \frac{2}{n-2} \frac{\partial u}{\partial \eta} + h_g u = H u^{\frac{n}{n-2}} & \text{in } \partial M. \end{cases} \tag{7.26}$$

Nella letteratura, possiamo trovare diversi problemi associati a questa equazione. Sono stati meno indagati del caso chiuso, ma ci sono ancora risultati che vale la pena commentare.

Il primo è l'analogo del problema di Yamabe, cioè studiare se è possibile deformare g conformemente in modo tale che le nuove curvature scalare e media siano costanti. Un primo criterio per l'esistenza di soluzioni è stato dato in [29], anche se dipende da moltiplicatori di Lagrange. Escobar ha lavorato sul caso in cui $h = 0$ e K è una costante positiva, ora noto come *problema di Escobar*, e ha dato alcuni risultati parziali positivi, che sono stati poi completati negli articoli [52, 51]. Vedi anche [77] e i loro rispettivi riferimenti.

Il caso in cui le curvature sono variabili è stato studiato in situazioni specifiche. Il caso $h = 0$ nella semisfera è stato trattato in [68, 12, 13]. In [25] viene dato un risultato perturbativo, cioè gli autori studiano il problema di prescrivere curvatura scalare zero e curvatura media vicina ad una costante. Gli articoli [1, 36, 25, 93] considerano pure il caso $K = 0$.

Quando entrambe curvature sono variabili, evidenziamo il lavoro [2], che contiene risultati perturbativi per curvature prossime a costanti sulla sfera unitaria di R^n, e [35], dove considerano il problema con $K > 0$ sulla semisfera $\mathbb{S}^3_+$, oltre ad uno studio approfondito delle soluzioni di blow-up. Infine, in [28] è studiato anche il problema per curvature negative, ma le soluzioni includono moltiplicatori di Lagrange.

Il nostro obiettivo qui è quello di considerare curvature variabili $K < 0$ e H di segno arbitrario, e di dare risultati sull'esistenza di soluzioni e sul comportamento delle soluzioni di tipo *bolle*. Alcuni dei nostri risultati sono la controparte di altri che appaiono in [74], in cui il caso bidimensionale è studiato in domini con genere positivo. Come vedremo, appaiono molte differenze quando la dimensione è maggiore di due.

Per enunciare i nostri teoremi, riduciamo prima il problema a un caso più semplice usando un risultato di Escobar ([41]), che afferma che ogni varietà Riemanniana compatta di dimensione $n \geq 3$ con bordo ammette una metrica conforme con curvatura scalare che non cambia segno e bordo minimale. Questo implica che, senza perdita di generalità, per mezzo di una trasformazione conforme della metrica possiamo considerare una metrica di partenza con $h_g = 0$ e $S_g = S$ che non cambia segno. Alla luce di questo fatto, in ciò che segue assumeremo che la metrica iniziale della nostra varietà sia quella data dal risultato di Escobar, e anche il fatto che $n \geq 3$.

In virtù della (7.26), il nostro obiettivo è trovare soluzioni positive del problema al contorno:

$$\begin{cases} -\frac{4(n-1)}{n-2}\Delta_g u + Su = Ku^{\frac{n+2}{n-2}} & \text{in } M, \\ \frac{2}{n-2}\frac{\partial u}{\partial \eta} = Hu^{\frac{n}{n-2}} & \text{in } \partial M. \end{cases} \tag{7.27}$$

La formulazione variazionale di (7.27) è classica; le soluzioni deboli del problema corrispondono ai punti critici del seguente funzionale di energia, definito su $H^1(M)$:

$$I(u) = \frac{2(n-1)}{n-2}\int_M |\nabla u|^2 + \frac{1}{2}\int_M Su^2 - \frac{1}{2^*}\int_M K|u|^{2^*} - (n-2)\int_{\partial M} H|u|^{2^\sharp}, \tag{7.28}$$

essendo $2^* = \frac{2n}{n-2}$ y $2^\sharp = \frac{2(n-1)}{n-2}$ gli esponenti critici di Sobolev per M e ∂M, rispettivamente. Come abbiamo detto prima, assumiamo che $K < 0$, quindi il terzo termine a destra dell'uguaglianza (7.28) è positivo. L'interazione tra questo termine e il termine critico al bordo è cruciale per il comportamento del funzionale.

Infatti, attraverso una disuguaglianza di traccia mostriamo che la natura del funzionale è fortemente condizionata dal quoziente delle curvature prescritte sul bordo, che a sua volta ci permette di confrontare entrambi i termini critici. Per comodità, definiamo la funzione invariante per riscalamento $\mathfrak{D}_n : \partial M \to \mathbb{R}$ come

$$\mathfrak{D}_n(x) = \sqrt{n(n-1)}\frac{H(x)}{\sqrt{|K(x)|}}. \tag{7.29}$$

A seconda che $\mathfrak{D}_n$ sia strettamente inferiore a 1 o meno, ci troviamo in scenari completamente diversi. Sottolineiamo il fatto che i bordi delle sfere geodetiche negli spazi iperbolici soddisfano $\mathfrak{D}_n > 1$, mentre $\mathfrak{D}_n = 1$ ai bordi delle orosfere. Quindi, se $\mathfrak{D}_n \geq 1$, potrebbero esistere soluzioni di blow-up per (7.27) con tali profili.

Assumendo che $\mathfrak{D}_n(x) < 1$ per ogni punto $x \in \partial M$, risulta che K eclissa H, e il termine positivo associato in I domina sul termine di bordo con H. Il risultato è che il funzionale è coercitivo e ammette un minimo globale.

Il nostro primo risultato è per il caso in cui la metrica di Escobar soddisfa $S < 0$, e confrontato con [29, 28], risolviamo il problema geometrico originale senza moltiplicatori di Lagrange.

Teorema 7.9. *Supponiamo che $K < 0$ in M, e che $\mathfrak{D}_n$ dato da* (7.29) *soddisfa $\mathfrak{D}_n < 1$ in ogni punto di ∂M. Allora, se $S < 0$, il problema* (7.27) *ha una soluzione.*

Se $S = 0$, dobbiamo imporre ipotesi su H per evitare la possibilità che il minimo sia identicamente zero, e quindi che la soluzione ottenuta sia geometricamente ammissibile.

Teorema 7.10. *Supponiamo che $K < 0$ in M, e che $\mathfrak{D}_n < 1$ in ∂M. Allora, se $S = 0$ e $\int_{\partial M} H > 0$, il problema* (7.27) *ha una soluzione.*

D'altra parte, se esiste un punto $p \in M$ con $\mathfrak{D}_n(p) > 1$, possiamo costruire una successione di soluzioni u_i, con masse concentrate intorno a p, in modo che l'energia $I(u_i)$ tenda a $-\infty$. Anche se questo impedisce l'esistenza di minimizzatori, in dimensione $n = 3$ possiamo usare il Teorema del passo montano per ottenere una soluzione di (7.27).

Teorema 7.11. *Sia $n = 3$, e supponiamo che $S = 0$, $K < 0$ e che H sia tale che*

(1) $\int_{\partial M} H < 0$,
(2) $\mathfrak{D}_n(\overline{p}) > 1$ *per qualche* $\overline{p} \in \partial M$, *e*
(3) 1 *è un valore regolare di* $\mathfrak{D}_n$.

Allora, (7.27) *ammette una soluzione positiva.*

Più avanti spiegheremo perché la restrizione $n = 3$ appare nel Teorema 7.11, e daremo un'idea della sua dimostrazione. Per provare l'esistenza di soluzioni min-max è necessario provare che le successioni di Palais-Smale di soluzioni approssimate convergono. Nel fare questo, appaiono due ostacoli: in primo luogo, abbiamo bisogno di dimostrare che sono limitate in norma, e questo non è chiaro nel nostro caso a causa della triplice omogeneità del funzionale di energia. In secondo luogo, a causa della presenza degli esponenti critici in (7.27), le successioni di Palais-Smale potrebbero non convergere.

Per aggirare il primo ostacolo useremo il *trucco della monotonia di Struwe*, vedi [57, 90], che consiste in perturbare il problema inserendo un parametro in modo che l'energia sia monotona. Inoltre, usiamo un'approssimazione sottocritica che garantisce la compattezza per le successioni di Palais-Smale. Pertanto, considereremo la seguente situazione:

Sia $(K_i)_i$ una successione di funzioni regolari definite su M tale che $K_i \to K$ in $C^2(\overline{M})$, e siaa $(H_i)_i$ una successione di funzioni differenziabili su ∂M tale che $H_i \to H$ su $C^2(\partial M)$. Assumendo $K < 0$, consideriamo soluzioni positive $(u_i)_i$ del problema perturbato

$$\begin{cases} -4\frac{n-1}{n-2}\Delta_g u_i + S u_i = K_i {u_i}^{p_i} & \text{in } M, \\ \frac{2}{n-2}\frac{\partial u_i}{\partial \eta} = H_i {u_i}^{\frac{p_i+1}{2}} & \text{in } \partial M, \end{cases} \tag{7.30}$$

cioè, i punti critici del funzionale di energia:

$$I_i(u) = \frac{2(n-1)}{n-2}\int_M |\nabla u|^2 + \frac{1}{2}\int_M S u^2 - \frac{1}{p_i+1}\int_M K_i |u|^{p_i+1} - 4\frac{n-1}{p_i+3}\int_{\partial M} H_i |u|^{\frac{p_i+3}{2}}, \tag{7.31}$$

con $p_i \nearrow \frac{n+2}{n-2}$. Ci chiediamo se questa successione di soluzioni è uniformemente limitata, nel qual caso convergerebbe ad una soluzione del problema originale (7.27). Ragionando per contraddizione, prendiamo (u_i) come sopra, e definiamo il suo *insieme singolare* come

$$\mathscr{S} = \{p \in \overline{M} : \exists x_i \to p \text{ tale che } u_i(x_i) \text{ non è limitata}\}.$$

A questo proposito, otteniamo il seguente risultato di compattezza:

Teorema 7.12. *Sia (u_i) una successione di soluzioni di (7.30), e $\mathscr{S}$ l'insieme sigolare associato. Allora*

(1) $\mathscr{S} \subset \{p \in \partial M : \mathfrak{D}_n(p) \geq 1\}$.

Quindi, possiamo scrivere $\mathscr{S} = \mathscr{S}_0 \sqcup \mathscr{S}_1$, con $\mathscr{S}_1 = \mathscr{S} \cap \{\mathfrak{D}_n > 1\}$ y $\mathscr{S}_0 = \mathscr{S} \cap \{\mathfrak{D}_n = 1\}$. In dimensione $n = 3$, abbiamo inoltre

(2.1) $\mathscr{S}_1$ *è una collezione finita di punti.*

(2.2) Se $S \leq 0$*, allora* $\mathscr{S}_1 = \emptyset$.

(2.3) Se $I_i(u_i)$ *è uniformemente limitata e* 1 *è un valore regolare di* $\mathfrak{D}_n$*, poi* $\mathscr{S}_0 = \emptyset$.

Il risultato precedente descrive due tipi di punti di blow-up, raggruppati nei sottoinsiemi $\mathscr{S}_0$ e $\mathscr{S}_1$. Questi profili corrispondono alle diverse soluzioni del seguente problema nel semispazio.

$$\begin{cases} \frac{-4(n-1)}{n-2}\Delta v = K(p)v^{\frac{n+2}{n-2}} & \text{in } \mathbb{R}^n_+, \\ \frac{2}{n-2}\frac{\partial v}{\partial \eta} = H(p)v^{\frac{n}{n-2}} & \text{in } \partial\mathbb{R}^n_+, \end{cases} \tag{7.32}$$

dove $p \in \mathscr{S}$. Le soluzioni di (7.32) sono state classificate in [30] (vedi anche [71]) come segue:

- ⋆ Se $\mathfrak{D}_n(p) < 1$, allora (7.32) non ha soluzioni.
- ⋆ Se $\mathfrak{D}_n(p) = 1$, le uniche soluzioni sono *1-dimensionali* e vengono date da:

$$v(x) = v_\alpha(x) := \left(\frac{2}{\sqrt{n(n-2)}}x_n + \alpha\right)^{-\frac{n-2}{2}}, \tag{7.33}$$

  per tutti $\alpha > 0$.

- ⋆ Se $\mathfrak{D}_n(p) > 1$, le soluzioni sono chiamate *bolle* e hanno l'espressione:

$$v(x) = b_\beta(x) := \frac{\left(n(n-2)\right)^{\frac{n-2}{4}}\beta^{\frac{n-2}{2}}}{\left(|x - x_0(\beta)|^2 - \beta^2\right)^{\frac{n-2}{2}}}, \tag{7.34}$$

  con $x_0(\beta) = -\mathfrak{D}_n(p)\beta, e_n \in \mathbb{R}^n$, per $\beta > 0$ arbitrario.

Vorremmo sottolineare che il profilo di blow-up può avere una massa infinita, contrariamente a quanto accade nel caso senza bordo, almeno in dimensione bassa. Lo sviluppo di un'analisi di blow-up in una situazione in cui potrebbero apparire profili con massa infinita o un numero infinito di punti di blow-up è uno degli obiettivi principali di questa tesi. Inoltre, entrambi i tipi di blow-up potrebbero coesistere; il blow-up nei punti $\mathscr{S}_1$ può essere capito dall'invarianza del problema per le trasformazioni conformi del disco, in analogia con il caso chiuso. Tuttavia, in questo contesto possiamo avere blow-up intorno ad un insieme infinito $\mathscr{S}_0$. In effetti, diamo anche un esempio esplicito di questo fenomeno.

In confronto al caso bidimensionale studiato in [74], abbiamo una maggiore rigidità nella classificazione dei profili limite, poiché per il semipiano ci sono altre soluzioni generate da funzioni meromorfe, vedi [46]. D'altra parte, in dimensione $n = 2$ possiamo fare uso di strumenti di Analisi Complessa, che non sono presenti in dimensioni superiori.

Per affrontare la perdita di compattezza nei punti con $\mathfrak{D}_n > 1$, eseguiamo uno studio approfondito del comportamento delle soluzioni di blow-up intorno a quei punti, mostrando che in dimensione $n = 3$ sono isolati e semplici, e quindi formano un insieme finito di punti (vedi anche [35]). Una volta dimostrato questo, è possibile controllare il comportamento di queste soluzioni anche lontano da questi punti, escludendo la presenza di bolle per mezzo di stime integrali valide per $S \leq 0$.

Invece, vicino ai punti di blow-up con $\mathfrak{D}_n = 1$, i termini $\int_M |\nabla u_i|^2$, $\int_M |K_i|\, u_i^{p_i+1}$ e $\int_{\partial M} H_i u_i^{\frac{p_i+3}{2}}$ divergono. Assumendo la limitatezza delle energie $I_i(u_i)$, (che è una condizione naturale per le successioni di soluzioni min-max) mostriamo che convergono debolmente alla stessa misura sul bordo dopo un'appropriata normalizzazione. Poi, con una tecnica di variazione del dominio mostriamo che in tali punti il gradiente di $\mathfrak{D}_n$ lungo ∂M in $\{\mathfrak{D}_n = 1\}$ si annulla, contraddicendo le ipotesi di regolarità imposte sul livello $\{\mathfrak{D}_n = 1\}$. Paragonato ad un argomento simile in [74], in questo caso dobbiamo considerare deformazioni arbitrarie tangenti a ∂M.

Bibliography

[1] ABDELHEDI, W., CHTIOUI, H., AHMEDOU, M.O., *A Morse theoretical approach for the boundary mean curvature problem on B^4*. J. Funct. Anal. 254 (2008), no. 5, 1307-1341.

[2] AMBROSETTI, A., LI, Y., MALCHIODI, A., *On the Yamabe problem and the scalar curvature problems under boundary conditions.* Math. Ann. 322 (2002), no. 4, 667-699.

[3] AMBROSETTI, A., RABINOWITZ, P.H., *Dual variational methods in critical point theory and applications.* J. Funct. Anal. 14 (1973) 349–381

[4] AUBIN, T., *Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire*, J. Math. Pures Appl., (9) 55: 269-296, 1976.

[5] AUBIN, T., *Meilleures constantes dans le théorème d'inclusion de Sobolev et un théorèm de Fredholm non linéaire pour la transformation conforme de la courbure scalaire*, J. Funct. Anal. 32 (1979), no. 2, 148-174.

[6] AUBIN T., *Some Nonlinear Problems in Differential Geometry*, Springer-Verlag, Berlin, 1998.

[7] BAO, J., WANG, L., ZHOU C., *Blow-up analysis for solutions to Neumann boundary value problem*, Journal of Math. Analysis and Appl. 418 (2014), 142-162.

[8] BAHRI, A., *Critical points at infinity in some variational problems*, Research Notes in Mathematics, Longman-Pitman, London, 182(1989)

[9] BAHRI, A., CORON J.M., *The Scalar-Curvature problem on the standard three-dimensional sphere,* Journal of Functional Analysis, 95(1991), 106-172.

[10] BARTSCH, T., *Critical Point Theory on Partially Ordered Hilbert Spaces,* Journal of Functional Analysis 186 (2001), 117-152.

[11] Battaglia, L., Medina, M., Pistoia, A., *Large conformal metrics with prescribed Gaussian and geodesic curvatures.* Calc. Var. Partial Differential Equations 60 (2021), no. 1, Paper No. 39, pp. 47.

[12] Ben Ayed, M., El Mehdi, K., Ahmedou, M.O., *Prescribing the scalar curvature under minimal boundary conditions on the half sphere.* Adv. Nonlinear Stud. 2 (2002), no. 2, 93-116.

[13] Ben Ayed, M., El Mehdi, K., Ahmedou, M.O., *The scalar curvature problem on the four dimensional half sphere.* Calc. Var. Partial Differential Equations 22 (2005), no. 4, 465-482.

[14] Branson, T.P., *Differential operators canonically associated to a conformal structure,* Mathematica Scandinavica, 57-2 (1985), 293-345.

[15] Brendle, S., *A family of curvature flows on surfaces with boundary,* Simon Math. Z. 241 (2002), no. 4, 829-869.

[16] Brendle, S., *Global existence and convergence for a higher order ow in conformal geometry,* Annals of Math. 158 (2003), 323-343.

[17] Brezis, H., Lieb, E., *A Relation Between Pointwise Convergence of Functions and Convergence of Functionals* Proc. Amer. Math. Soc., Vol. 88 (3), 1983.

[18] Bourguignon, J.P., Ezin J.P., *Scalar curvature functions in a conformal class of metrics and conformal transformations.* Trans. Amer. Math. Soc., 301(1987), no. 2, 723-736.

[19] Chang, K.C., Liu, J.Q.,, *A prescribing geodesic curvature problem*, Math. Z. 223 (1996), 343-365.

[20] Chang, S.Y.A., Qing, J., *The zeta functional determinants on manifolds with boundary* 1. Formula J. Funct. Anal. 147, 327–362 (1997)

[21] Chang, S.Y.A., Yang, P.C., *Conformal deformation of metrics on* $\mathbb{S}^2$, J. Di. Geom. 27: 259-296, 1988.

[22] Chang, S.Y.A., Yang, P.C., *Extremal metrics of zeta function determinants on 4-manifolds,* Annals of Math. 142 (1995), 171-212.

[23] Chang, S.Y.A., Yang, P.C., *Prescribing Gaussian curvature on* $\mathbb{S}^2$, Acta Math. 159, no. 3-4: 215-259, 1987.

[24] Chang, S.Y.A, Gursky, M., Yang, P.C., *The scalar curvature equation on* 2 *and* 3*-spheres*, Calc. Var. PDE, 1: 205-229, 1993.

[25] Chang, S.A., Xu X., Yang P.,, *A perturbation result for prescribing mean curvature*, Math. Ann. 310-3 (1998), 473-496.

[26] Chanpling, R., *Automorphisms of the Unit Disc, University of Cambridge*, 2015.

[27] Chen, W.X., Li, C., *Prescribing Gaussian curvatures on surfaces with conical singularities,* J. Geom. Anal. 1-4 (1991) pp. 359-372.

[28] Chen, X., Ho, T. T., Sun, L., *Prescribed scalar curvature plus mean curvature flows in compact manifolds with boundary of negative conformal invariant.* Ann. Global Anal. Geom. 53 (2018), no. 1, 121-150.

[29] Cherrier, P., *Problemes de Neumann non lineaires sur les varietes riemanniennes*, Journal of Functional Analysis 57, 154-206 (1984).

[30] Chipot, M., Fila, M., Shafrir, I., *On the Solutions to some Elliptic Equations with Nonlinear Neumann Boundary Conditions.* Advances in Diferential Equations, 1, Vol. 1, 1996, pp. 91-110.

[31] Cruz-Blázquez, S., Ruiz, D., *Prescribing Gaussian and geodesic curvatures on disks*, Adv. Nonlinear Stud. 18 (2018), no. 3, 453–468.

[32] Cruz-Blázquez, S., Malchiodi, A., Ruiz, D., *Conformal metrics with prescribed scalar and mean curvature*, preprint. arxiv.org/pdf/2105.04185.

[33] Da Lio, F., Martinazzi, L., Rivière, T., *Blow-Up Analysis of a Nonlocal Liouville-Type Equation*, Analysis and PDE Vol. 8, No. 7, 2015.

[34] Z. Djadli, A. Malchiodi, *Existence of conformal metrics with constant Q-curvature*, Ann. of Math. (2) 168 (2008), no. 3, 813–858.

[35] Djadli, Z., Malchiodi, A. and Ahmedou, M.O. *Prescribing Scalar and Boundary Mean Curvature on the Three Dimensional Half Sphere.* The Journal of Geometric Analysis 13, no. 2, 2003.

[36] Djadli, Z., Malchiodi, A., Ahmedou, M.O., *The prescribed boundary mean curvature problem on B^4*. J. Differential Equations 206 (2004), no. 2, 373-398.

[37] Dupaigne, L., *Stable solutions to elliptic partial differential equations.* Chapman and Hall, 143 (2011).

[38] Druet, O.,*Compactness for Yamabe metrics in low dimensions.* Int. Math. Res. Not. 2004, no. 23, 1143–1191.

[39] Druet, O., Hebey, E., Robert, F., *Blow-up theory for elliptic PDEs in Riemannian geometry.* Mathematical Notes, 45. Princeton University Press, Princeton, NJ, 2004. pp. viii+218.

[40] Escobar, J., *Conformal deformation of a Riemannian metric to a scalar at metric with constant mean curvature on the boundary*, Ann. Math. 136 (1992), 1-50.

[41] Escobar, J. *The Yamabe problem on manifolds with boundary.* J. Differ. Geom., 35 (1992), pp. 21-84.

[42] Escobar, J., *Conformal metrics with prescribed mean curvature on the boundary*, Calc. Var. 4 (1996), 559-592.

[43] Escobar, J., Schoen R.M., *Conformal metrics with prescribed scalar curvature*, Invent. Math., 86(1986), 243-254.

[44] Felli, V., Ahmedou, M.O., *Compactness results in conformal deformations of Riemannian metrics on manifolds with boundaries*, Math. Z. 244 (2003), 175-210.

[45] Fontana, L., *Sharp borderline Sobolev inequalities on compact Riemannian manifolds*, Comm. Math. Helv. 68, 1993, pp. 415-454.

[46] Gálvez, J.A., Mira, P., *The Liouville equation in a half-plane*, J. Differential Equations 246 (2009), no. 11, 4173-4187.

[47] Gidas, B., Spruck, J. *A Priori Bounds for Positive Solutions of Nonlinear Elliptic Equations.* Communications in Partial Differential Equations, no. 6(8), 1981, pp. 883-901

[48] Gilbarg, D. and Trudinger, N. S. *Elliptic Partial Differential Equations of Second Order.* Springer, 2001.

[49] Ghoussoub, N., *Duality and Perturbation Methods in Critical Point Theory,* Cambridge University Press 1993.

[50] Hamza, H., *Sur les transformations conformes des varietes Riemanniennes a bord*, Journal of Functional Analysis 92 (1990), 403-447.

[51] Han, Z.C., Li Y.Y., *The Yamabe problem on manifolds with boundaries: existence and compactness results*, Duke Math. J. 99 (1999), 489-542.

[52] Han, Z.C., Li Y.Y., *The existence of conformal metrics with constant scalar curvature and constant boundary mean curvature*, Comm. Anal. Geom. 8 (2000), 809–869.

[53] Hang, F., Wang, X., *A new approach to some nonlinear geometric equations in dimension two*, Calc. Var. Partial Differential Equations 26 (2006), 119-135.

[54] Hebey, E., *Changements de métriques conformes sur la sphère - Le problème de Nirenberg,* Bull. Sci. Math., 114(1990), 215-242.

[55] Hebey, E., *Sobolev Spaces in Riemannian Manifolds*, volume 1635 of Lecture notes in mathematics. Springer, Berlin (1996).

[56] Hebey E., Vaugon M., *Le probleme de Yamabe equivariant*, Bull. Sci. Math., 117(1993), no. 2, 241-286.

[57] Jeanjean, L., *On the existence of bounded Palais-Smale sequences and application to a Landesman-Lazer-type problem set on* $\mathbb{R}^n$, Proceedings of the Royal Society of Edinburgh, 129A, 787-809, 1999.

[58] Jiménez, A., *The Liouville equation in an annulus*, Nonlinear Analysis 75 (2012), 2090-2097.

[59] Jevnikar, A., López-Soriano, R., Medina, M., Ruiz, D., *Blow-up analysis of conformal metrics of the disk with prescribed Gaussian and geodesic curvatures*, to appear in Analysis and PDEs, arXiv.org/pdf/2004.14680.

[60] Kazdan, J.L., Warner, F.W., *Curvature functions for compact 2-manifolds*, Ann. of Math. 99: 14-47 (1974).

[61] Kazdan, J.L., Warner F.W., *Scalar curvature and conformal deformation of Riemannian structure*, J. Differential Geometry, 10(1975), 113-134.

[62] Klein, F., *Neue Beiträge zur Riemann'schen Functionentheorie*, Mathematische Annalen, 21 (2): 141–218, 1883.

[63] Koebe, P., *Über die Uniformisierung beliebiger analytischer Kurven*, Göttinger Nachrichten: 191–210, 1907.

[64] Koebe, P., *Über die Uniformisierung beliebiger analytischer Kurven (Zweite Mitteilung)*, Göttinger Nachrichten: 633–66, 1907.

[65] Koebe, P., *Über die Uniformisierung reeller analytischer Kurven*, Göttinger Nachrichten: 177–190, 1907.

[66] Khuri, M.A., Marques F.C., Schoen, R.M. *A Compactness Theorem for the Yamabe* Problem. J. Differential Geometry, no. 81 2009, pp. 143-196.

[67] Li, Y.Y., *The Nirenberg problem in a domain with boundary*, Top. Meth. Nonlin. Anal. 6 (1995), 309-329.

[68] Li, Y.Y., *Prescribing scalar curvature on $\mathbb{S}^n$ and related problems. I,* J. Differential Equations 120 (1995), no. 2, 319-410.

[69] LI, Y., LIU, P., *A Moser-Trudinger inequality on the boundary of a compact Riemann surface*, Math. Z. 250, 2005, pp. 363-386.

[70] LI, P., LIU, J., *Nirenberg's problem on the 2-dimensional hemisphere*, Int. J. Math. 4 (1993), 927-939.

[71] LI, Y.Y., ZHU, M., *Uniqueness theorems through the method of moving spheres*, Duke Math. J. 80 (1995), no. 2, 383-417.

[72] LIU, P., HUANG, W., *On prescribing geodesic curvature on D^2*, Nonlinear Analysis 60 (2005) 465-473.

[73] LIU, P. Y XU, L., *A small note on symmetric geodesic curvature on D^2*, J. Math. Anal. Appl. 322, 2006, pp. 489-494.

[74] LÓPEZ-SORIANO, R., MALCHIODI, A., RUIZ, D., *Conformal metrics with pescribed Gaussian and geodesic curvatures*, Annales Scient. E.N.S., to appear. arxiv.org/1806.11533.

[75] LÓPEZ-SORIANO, R., RUIZ, D.,*Prescribing the Gaussian curvature on a subdomain of S2 with Neumann boundary conditions*, J. Geom. Anal. 26 (2016), no. 1, 630-644.

[76] MALCHIODI, A., STRUWE, M. *Q-curvature flow on S4*. J. Differential Geom. 73 (2006), no. 1, 1–44.

[77] MAYER, M., NDIAYE, C.B., *Barycenter technique and the Riemann mapping problem of Cherrier-Escobar.* J. Differential Geom. 107 (2017), no. 3, 519-560.

[78] MOSER, J., *A sharp form of an inequality by N. Trudinger*, Indiana Univ. Math. J. 20: 1077-1091 1971.

[79] MOSER, J., *On a non-linear Problem in Differential Geometry and Dynamical Systems*, Academic Press, N.Y. (ed M. Peixoto) 1973.

[80] NDIAYE, C.B., *Constant T-Curvature conformal metrics on 4-manifolds with boundary*, Pacific J. Math. 240 (2009), 151-184.

[81] NDIAYE, C.B., *Curvature flows on four manifolds with boundary*, Math. Z. 269 (2011), 83-114.

[82] NECAS, J., *Direct Methods in the Theory of Elliptic Equations*, Springer (2012).

[83] OSGOOD, B., PHILLIPS, R., SARNAK, P., *Extremals of determinants of Laplacians,* J. Functional Analysis 80 (1988), 148-211.

[84] Paneitz, S., *A quartic conformally covariant differential operator for arbitrary pseudo-Riemannian manifolds*, preprint, 1983.

[85] Poincaré, H., *Mémoire sur les fonctions fuchsiennes*, Acta Mathematica, 1: 193–294, 1882.

[86] Schoen, R., *Conformal deformation of a Riemannian metric to constant scalar curvature*, J. Differential Geom. 20: 479-495, 1984.

[87] Schoen, R. *Courses at Stanford University*. 1989.

[88] Schoen, R., Zhang D., *Prescribed scalar curvature on the n-sphere*, Calculus of Variations and Partial Differential Equations, 4(1996), 1-25.

[89] Struwe, M., *A flow approach to Nirenberg's problem*, Duke Mathematical Journal Vol. 128 (2005), No. 1.

[90] Struwe, M., *The existence of surfaces of constant mean curvature with free boundaries*. Acta Math. 160 (1988), no. 1-2, 19-64.

[91] Trudinger, N., *Remarks concerning the conformal deformation of Riemannian structures on compact manifolds*, Ann. Scuola Norm. Sup. 22, 265-274, 1968.

[92] Wei, J., Xu, X., *On conformal deformations of metrics on S^n*, J. Funct. Anal. 157-1 (1998), 292-325.

[93] Xu, X., Zhang, H., *Conformal metrics on the unit ball with prescribed mean curvature*. Math. Ann. 365 (2016), no. 1-2, 497–557.

[94] Yamabe, H., *On a deformation of Riemannian structures on compact manifolds*, Osaka Math. J. 12, 21-37, 1960.

[95] Zhang, L., *Classification of conformal metrics on $\mathbb{R}^2_+$ with constant Gauss curvature and geodesic curvature on the boundary under various integral finiteness assumptions*, Calc. Var. 16 (2003), 405-430.