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Proinov-Type Fixed-Point Results in Non-Archimedean Fuzzy Metric Spaces

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Abstract: Very recently, Proinov introduced a great family of contractions in the setting of complete metric spaces that has attracted the attention of many researchers because of the very weak conditions that are assumed on the involved functions. Inspired by Proinov's results, in this paper, we introduce a new class of contractions in the setting of fuzzy metric spaces (in the sense of George and Veeramani) that are able to translate to this framework the best advantages of the abovementioned auxiliary functions. Accordingly, we present some results about the existence and uniqueness of fixed points for this class of fuzzy contractions in the setting of non-Archimedean fuzzy metric spaces.

Keywords: fuzzy metric; fuzzy metric space; fixed point; Proinov-type theorem; non-Archimedean space

MSC: 47H10; 47H09; 54H25; 46T99

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1. Introduction

Fixed-point theory is currently one of the most active fields in the area of nonlinear analysis and even in mathematics in general. Its results can be applied to an extensive set of distinct types of equations (integral, differential, matricial [1], etc.) in order to prove the existence and uniqueness of several classes of nonlinear problems. Given an arbitrary self-operator, two main factors must be considered in order to achieve a novel result in the field of fixed-point theory: an appropriate inequality that can serve as a contractivity condition and a reasonable, abstract metric structure on the underlying set in which the operator is defined.

On the one hand, the contractivity condition usually establishes that the distance between the images through the operator of two distinct points of the space is lower than (or even, in some cases, equal to) the distance of such initial points. However, this condition can be stated by using hundreds of analytical or algebraic tools, among which the real functions of real variables stand out. Very recently, Proinov introduced in [2] a great family of contractions in the setting of complete metric spaces that has attracted the attention of many researchers. The main advantage of this new class of contractions is the wide family of auxiliary functions that they are able to handle. Very weak constraints are imposed on the pairs of functions that Proinov handled in the abovementioned manuscript.

Consequently, the variety of contractivity conditions that can be considered starting from a general Proinov property, and even the generality that such functions contribute, has encouraged many mathematicians to deepen the study of this class of contractions (see, for instance, [3–5]).

On the other hand, the second key ingredient for success in fixed-point theory is to handle an appropriate abstract metric structure. In general, a "*metric*" is a mapping that associates a real number with each pair of points of the space and that satisfies four properties: non-negativity, the identity of indiscernibles, symmetry, and the triangle inequality. Many generalizations of such a concept have been introduced in the last seventy years (see [6,7]). A class of significant extensions of the family of metric spaces is formed by all *fuzzy metric spaces*. In a fuzzy space, the distance between two objects is not given by a precise real number, but as a distance distribution function, that is a distribution function that modelizes the probability of the event in which two arbitrary points are placed at a distance less than a certain real parameter. This contributes to maintaining the imprecision/vagueness that is inherent to this class of spaces.

There are several ways to introduce the notion of the metric in the fuzzy setting. After the approaches due to Menger [8] (statistical metric spaces), Kaleva and Seikkala [9], Schweizer and Sklar [10] (probabilistic metric spaces), Kramosil and Michálek [11] (fuzzy metric spaces), and Roldán López de Hierro et al. [12,13] (fuzzy spaces), among others, taking into account its potential applications in several fields of study, George and Veeramani introduced in [14] a wide class of fuzzy metric spaces that have enjoyed great success because they are particularly easy to use and interpret. Furthermore, this category of fuzzy metrics has been demonstrated to be special according to the needs of the theory of fixed points (see, for instance, [15–23] in several contexts). Some interrelationships among these fuzzy metric structures can be found in [24]. Fuzzy metrics have been demonstrated to be a very consistent notion, leading to significant improvements in many fields (for instance, in fuzzy regression theory; see [25–30]).

The triangular inequality that a fuzzy metric space satisfies provides a certain control on how the distances between two points of a triplet are related. However, sometimes, it is not strong enough to complete the proofs of certain results in the field of fixed-point theory. In such a case, an additional assumption is often useful: the non-Archimedean property. This condition establishes that the same real parameter can relate the fuzzy distances between any three points of the underlying space. Such a hypothesis is very useful in practice because the main examples of fuzzy metric spaces that are handled in applications usually satisfy such a constraint.

Inspired by Proinov's results, in this paper, we introduce a new class of contractions in the setting of fuzzy metric spaces (in the sense of George and Veeramani) that are able to translate to this framework the best advantages of the auxiliary functions due to the the abovementioned researcher. In this context, we prove some fixed-point results that improve some previous theorems by using a very general class of restrictions on the involved functions. These results help researchers to better understand what topological, analytical, and geometric properties must be satisfied when the main aim is to develop the fixed-point theory in the setting of fuzzy metric spaces.

Before that, we describe some necessary background to develop the main contents.

2. Preliminaries

We include here the appropriate preliminaries to understand the contents of this manuscript (see also [21,22,31]). Let $\mathbb{R}$ be the family of all real numbers, and let $\mathbb{N} = \{1, 2, 3, \ldots\}$ denote the set of all positive integers. Let $\mathsf{T} : \mathsf{X} \to \mathsf{X}$ be a map from X into itself. If a point $\mathsf{u} \in \mathsf{X}$ verifies $\mathsf{Tu} = \mathsf{u}$, then u is a *fixed point of* T. We denote by $\mathrm{Fix}(\mathsf{T})$ the set of all fixed points of T.

Following [32], a sequence $\{\mathsf{u}_\ell\}$ in X is *infinite* if $\mathsf{u}_\ell \neq \mathsf{u}_j$ for all $\ell \neq j$. A sequence $\{\mathsf{u}_\ell\}_{\ell \in \mathbb{N}} \subseteq \mathsf{X}$ is called a *Picard sequence of* T *based on* $\mathsf{u}_1 \in \mathsf{X}$ if $\mathsf{u}_{\ell+1} = \mathsf{Tu}_\ell$ for all $\ell \in \mathbb{N}$. Notice that, in such a case, $\mathsf{u}_{\ell+1} = \mathsf{T}^\ell \mathsf{u}_1$ for each $\ell \in \mathbb{N}$, where $\{\mathsf{T}^\ell : \mathsf{X} \to \mathsf{X}\}_{\ell \in \mathbb{N}}$ are the *iterates of T* defined by T^0 asthe identity mapping on X, T^1 asT, and $T^{\ell+1} = T \circ T^{\ell}$ for all $\ell \geq 2$.

2.1. Proinov Contractions

Very recently, Proinov announced some results that unify many known results.

Theorem 1 (Proinov [2], Theorem 3.6)**.** *Let (X,d) be a complete metric space and $T : X \to X$ be a mapping such that:*

$$\psi(d(Tu, Tv)) \leq \phi(d(u, v)) \quad \text{for all } u, v \in X \text{ with } d(Tu, Tv) > 0, \tag{1}$$

where the functions $\psi, \phi : (0, \infty) \to \mathbb{R}$ satisfy the following conditions:

(a_1) *ψ is nondecreasing;*
(a_2) *$\phi(s) < \psi(s)$ for any $s > 0$;*
(a_3) $\limsup_{s \to e^+} \phi(s) < \lim_{s \to e^+} \psi(s)$ *for any $e > 0$.*

Then, T has a unique fixed point $v_0 \in X$, and the iterative sequence $\{T^k u\}_{k \in \mathbb{N}}$ converges to v_0 for every $u \in X$.

2.2. Fuzzy Metric Spaces

A *triangular norm* [10,33] (*t-norm*) is a function $* : [0,1] \times [0,1] \to [0,1]$ satisfying the following properties: associativity, commutativity, non-decreasing on each argument, and has one as unity (that is, $t * 1 = t$ for all $t \in [0,1]$). Usually, authors only consider continuous t-norms. Examples of classical continuous t-norms are the following ones:

$$t *_m s = \min\{t, s\}, \qquad t *_P s = t\,s, \qquad t *_L s = \max\{0, t + s - 1\}.$$

Definition 1 (cf. George and Veeramani [14])**.** *A triplet $(X, M, *)$ is called a* fuzzy metric space in the sense of George and Veeramani *(* fuzzy metric space*) if X is an arbitrary nonempty set, $*$ is a continuous t-norm, and $M : X \times X \times (0, \infty) \to [0,1]$ is a fuzzy set satisfying the following conditions, for each $u, v, w \in X$, and $t, s > 0$:*

(GV-1) $M(u, v, t) > 0$;

(GV-2) $M(u, v, t) = 1$ *for all $t > 0$ if, and only if,* $u = v$;

(GV-3) $M(u, v, t) = M(v, u, t)$;

(GV-4) $M(u, w, t + s) \geq M(u, v, t) * M(v, w, s)$;

(GV-5) $M(u, v, \cdot) : (0, \infty) \to [0,1]$ *is a continuous function.*

The following is the canonical way in which a metric space can be seen as a fuzzy metric space.

Example 1. *Each metric space (X, d) can be seen as a fuzzy metric space $(X, M^d, *)$, where $*$ is any t-norm, by defining $M^d : X \times X \times (0, \infty) \to [0,1]$ as:*

$$M^d(u, v, t) = \frac{t}{t + d(u, v)} \qquad \text{for all } u, v \in X \text{ and all } t > 0.$$

Notice that $0 < M^d(u, v, t) < 1$ for all $t > 0$ and all $u, v \in X$ such that $u \neq v$. Furthermore, $\lim_{t \to \infty} M^d(u, v, t) = 1$ for all $u, v \in X$.

Lemma 1 (cf. Grabiec [34])**.** *If $(X, M, *)$ is a fuzzy metric space and $u, v \in X$, then each function $M(u, v, \cdot)$ is nondecreasing on $(0, \infty)$.*

Definition 2. *In a fuzzy metric space $(X, M, *)$, we say that a sequence $\{u_\ell\} \subseteq X$ is:*

- M-Cauchy *if for all* $\mathsf{L} \in (0,1)$ *and all* $\mathsf{t} > 0$*, there is* $\ell_0 \in \mathbb{N}$ *such that* $\mathsf{M}(\mathsf{u}_\ell, \mathsf{u}_j, \mathsf{t}) > 1 - \mathsf{L}$ *for all* $\ell, j \geq \ell_0$*;*
- M-convergent to $\mathsf{u} \in \mathsf{X}$ *if for all* $\mathsf{L} \in (0,1)$ *and all* $\mathsf{t} > 0$*, there is* $\ell_0 \in \mathbb{N}$ *such that* $\mathsf{M}(\mathsf{u}_\ell, \mathsf{u}, \mathsf{t}) > 1 - \mathsf{L}$ *for all* $\ell \geq \ell_0$ *(in such a case, we write* $\{\mathsf{u}_\ell\} \to \mathsf{u}$*).*

We say that the fuzzy metric space $(\mathsf{X}, \mathsf{M}, *)$ *is* M-complete *if each* M*-Cauchy sequence in* X *is* M*-convergent to a point of* X*.*

Definition 3 (Istrăţescu [35])**.** *A fuzzy metric space* $(\mathsf{X}, \mathsf{M}, *)$ *is said to be* non-Archimedean *if:*

$$\mathsf{M}(\mathsf{u}, \mathsf{w}, \mathsf{t}) \geq \mathsf{M}(\mathsf{u}, \mathsf{v}, \mathsf{t}) * \mathsf{M}(\mathsf{v}, \mathsf{w}, \mathsf{t}) \quad \textit{for all } \mathsf{u}, \mathsf{v}, \mathsf{w} \in \mathsf{X} \textit{ and all } \mathsf{t} > 0. \tag{2}$$

This property can also be implemented in the following way:

$$\mathsf{M}(\mathsf{u}, \mathsf{w}, \max\{\mathsf{t}, \mathsf{s}\}) \geq \mathsf{M}(\mathsf{u}, \mathsf{v}, \mathsf{t}) * \mathsf{M}(\mathsf{v}, \mathsf{w}, \mathsf{s}) \quad \textit{for all } \mathsf{u}, \mathsf{v}, \mathsf{w} \in \mathsf{X} \textit{ and all } \mathsf{t}, \mathsf{s} > 0.$$

Example 2 (Altun and Miheţ [16], Example 1.3)**.** *Let* (X, d) *be a metric space, and let* ϑ *be a nondecreasing and continuous function from* $(0, \infty)$ *into* $(0,1)$ *such that* $\lim_{\mathsf{t} \to \infty} \vartheta(\mathsf{t}) = 1$*. Let* $*$ *be a t-norm such that* $* \leq *_P$*. For each* $\mathsf{u}, \mathsf{v} \in \mathsf{X}$ *and all* $\mathsf{t} \in (0, \infty)$*, define:*

$$\mathsf{M}(\mathsf{u}, \mathsf{v}, \mathsf{t}) = [\vartheta(\mathsf{t})]^{\mathsf{d}(\mathsf{u}, \mathsf{v})}.$$

Then, $(\mathsf{X}, \mathsf{M}, *)$ *is a non-Archimedean fuzzy metric space.*

Proposition 1 (cf. [36], Proposition 2)**.** *Let* $\{\mathsf{u}_\ell\}$ *be a Picard sequence (of an operator* $\mathsf{T} : \mathsf{X} \to \mathsf{X}$*) in a fuzzy metric space* $(\mathsf{X}, \mathsf{M}, *)$ *such that* $\{\mathsf{M}(\mathsf{u}_\ell, \mathsf{u}_{\ell+1}, \mathsf{t})\}_{\ell \in \mathbb{N}} \to 1$ *for all* $\mathsf{t} > 0$*. If there are* $i_0, j_0 \in \mathbb{N}$ *such that* $i_0 < j_0$ *and* $\mathsf{u}_{i_0} = \mathsf{u}_{j_0}$*, then there is* $\ell_0 \in \mathbb{N}$ *and* $\mathsf{w} \in \mathsf{X}$ *such that* $\mathsf{u}_\ell = \mathsf{w}$ *for all* $\ell \geq \ell_0$ *(that is,* $\{\mathsf{u}_\ell\}$ *is constant from a term onwards). In such a case,* w *is a fixed point of the self-mapping for which* $\{\mathsf{u}_\ell\}$ *is a Picard sequence.*

Lemma 2 (cf. [37])**.** *If a sequence* $\{\mathsf{u}_\ell\}_{\ell \in \mathbb{N}}$ *in a non-Archimedean fuzzy metric space* $(\mathsf{X}, \mathsf{M}, *)$ *such that:*

$$\lim_{\ell \to \infty} \mathsf{M}(\mathsf{u}_\ell, \mathsf{u}_{\ell+1}, \mathsf{t}) = 1 \quad \textit{for all } \mathsf{t} > 0$$

is not M*-Cauchy, then there are* $\mathsf{L}_0 \in (0,1)$*,* $\mathsf{t}_0 > 0$*, and two partial subsequences* $\{\mathsf{u}_{\mathsf{p}(\ell)}\}_{\ell \in \mathbb{N}}$ *and* $\{\mathsf{u}_{\mathsf{q}(\ell)}\}_{\ell \in \mathbb{N}}$ *of* $\{\mathsf{u}_\ell\}_{\ell \in \mathbb{N}}$ *such that, for all* $\ell \in \mathbb{N}$*,*

$$\begin{aligned} &\ell < \mathsf{p}(\ell) < \mathsf{q}(\ell) < \mathsf{p}(\ell+1) \qquad \textit{and} \\ &\mathsf{M}(\mathsf{u}_{\mathsf{p}(\ell)}, \mathsf{u}_{\mathsf{q}(\ell)-1}, \mathsf{t}_0) > \mathsf{L}_0 \geq \mathsf{M}(\mathsf{u}_{\mathsf{p}(\ell)}, \mathsf{u}_{\mathsf{q}(\ell)}, \mathsf{t}_0), \end{aligned}$$

and also:

$$\lim_{\ell \to \infty} \mathsf{M}(\mathsf{u}_{\mathsf{p}(\ell)}, \mathsf{u}_{\mathsf{q}(\ell)}, \mathsf{t}_0) = \lim_{\ell \to \infty} \mathsf{M}(\mathsf{u}_{\mathsf{p}(\ell)-1}, \mathsf{u}_{\mathsf{q}(\ell)-1}, \mathsf{t}_0) = \mathsf{L}_0.$$

3. Proinov-Type Fixed-Point Theory in Non-Archimedean Fuzzy Metric Spaces

Inspired by the main theorems of Proinov included in [2], in this section, we introduce some distinct ways to extend such results to the setting of fuzzy metric spaces. To carry out this aim, the non-Archimedean property (2) is important because it provides us certain control on the third parameter of the fuzzy metric.

As in Theorem 1, two functions φ and η play a key role in a Proinov-type contractivity condition as the following one:

$$\varphi(\mathsf{M}(\mathsf{Tu}, \mathsf{Tv}, \mathsf{s})) \geq \eta(\mathsf{M}(\mathsf{u}, \mathsf{v}, \mathsf{s})) \quad \text{for all } \mathsf{u}, \mathsf{v} \in \mathsf{X} \text{ and all } \mathsf{s} > 0. \tag{3}$$

As the fuzzy metric M only takes values in the semi-closed interval $(0,1]$, the domain of the functions φ and η is not necessarily the nonbounded interval $(0, \infty)$, but the bounded interval $(0,1]$. Hence, the functions φ and η are defined as follows:

$$\varphi, \eta : (0,1] \to \mathbb{R}.$$

Notice that the open interval $(0,1)$ would not be wide enough because, when $u = v$, then $M(u,v,s) = 1$ for all $s > 0$, so the real number $t = 1 \in (0,1]$ will play an important role.

Next, we introduce here the family of auxiliary functions that we employ.

Definition 4. *We denote by $\mathcal{L}$ the family of pairs (φ, η) of functions $\varphi, \eta : (0,1] \to \mathbb{R}$ verifying the following properties:*

(Þ_1) *φ is nondecreasing;*

(Þ_2) *$\eta(s) > \varphi(s)$ for any $s \in (0,1)$;*

(Þ_3) *$\liminf_{s \to L^-} \eta(s) > \lim_{s \to L^-} \varphi(s)$ for any $L \in (0,1)$;*

(Þ_4) *if $t \in (0,1]$ is such that $\varphi(t) \geq \eta(1)$, then $t = 1$.*

The reader can easily check that this family is nonempty because the following ones are examples of pairs of functions in $\mathcal{L}$:

- $\varphi(s) = s$ and $\eta(s) = \sqrt{s}$ for all $s \in (0,1]$;
- $\varphi(s) = s^2$ and $\eta(s) = s$ for all $s \in (0,1]$.

The following result describes sufficient conditions in order to ensure that a self-mapping enjoys a fixed point, and it is one of our main statements.

Theorem 2. *Let $(X, M, *)$ be an M-complete non-Archimedean fuzzy metric space, and let $T : X \to X$ be a mapping for which there exists $(\varphi, \eta) \in \mathcal{L}$ such that:*

$$\varphi(M(Tu, Tv, s)) \geq \eta(M(u,v,s)) \quad \textit{for all } u, v \in X \textit{ with } Tu \neq Tv \textit{ and all } s > 0. \tag{4}$$

Then, each iterative Picard sequence $\{T^\ell u\}_{\ell \in \mathbb{N}}$ converges to the unique fixed point of T for every initial condition $u \in X$.

Proof. We start by analyzing the existence of fixed points. Let $\{u_\ell\}_{\ell \in \mathbb{N}}$ be the Picard sequence of T starting from an arbitrary initial point $u_1 \in X$. If there is $\ell_0 \in \mathbb{N}$ such that $u_{\ell_0} = u_{\ell_0 + 1}$, then u_{ℓ_0} is a fixed point of T. In this case, the first part of the proof is finished. Next, suppose that $u_\ell \neq u_{\ell+1}$ for all $\ell \in \mathbb{N}$, which also means that:

$$Tu_\ell \neq Tu_{\ell+1} \quad \text{for all } \ell \in \mathbb{N}. \tag{5}$$

As $(X, M, *)$ is a fuzzy metric space in the sense of George and Veeramani,

$$M(u_\ell, u_{\ell+1}, s) > 0 \quad \text{for all } \ell \in \mathbb{N} \text{ and all } s > 0.$$

Taking into account (5) and applying the contractivity condition (4), we deduce that, for all $\ell \in \mathbb{N}$ and all $s > 0$,

$$\varphi(M(u_{\ell+1}, u_{\ell+2}, s)) = \varphi(M(Tu_\ell, Tu_{\ell+1}, s)) \geq \eta(M(u_\ell, u_{\ell+1}, s)). \tag{6}$$

For a better understanding of the rest of the proof, we use six steps to prove the statement.

Step 1. For all $s > 0$, the sequence $\{ M(u_\ell, u_{\ell+1}, s) \}_{\ell \in \mathbb{N}} \subset (0,1]$ is nondecreasing.

Let $s > 0$ be arbitrary. We consider two cases depending on $M(u_\ell, u_{\ell+1}, s) = 1$ or $M(u_\ell, u_{\ell+1}, s) < 1$.

- If $\mathsf{M}(\mathsf{u}_\ell, \mathsf{u}_{\ell+1}, \mathsf{s}) = 1$, then:

$$\varphi(\mathsf{M}(\mathsf{u}_{\ell+1}, \mathsf{u}_{\ell+2}, \mathsf{s})) \geq \eta(\mathsf{M}(\mathsf{u}_\ell, \mathsf{u}_{\ell+1}, \mathsf{s})) = \eta(1).$$

In such a case, Property (Þ_4) leads to:

$$\mathsf{M}(\mathsf{u}_{\ell+1}, \mathsf{u}_{\ell+2}, \mathsf{s}) = \mathsf{M}(\mathsf{u}_\ell, \mathsf{u}_{\ell+1}, \mathsf{s}) = 1.$$

In particular,

$$\mathsf{M}(\mathsf{u}_{\ell+1}, \mathsf{u}_{\ell+2}, \mathsf{s}) \geq \mathsf{M}(\mathsf{u}_\ell, \mathsf{u}_{\ell+1}, \mathsf{s}).$$

- If $\mathsf{M}(\mathsf{u}_\ell, \mathsf{u}_{\ell+1}, \mathsf{s}) \in (0,1)$, then (6) and Property (Þ_2) ensure that:

$$\varphi(\mathsf{M}(\mathsf{u}_{\ell+1}, \mathsf{u}_{\ell+2}, \mathsf{s})) \geq \eta(\mathsf{M}(\mathsf{u}_\ell, \mathsf{u}_{\ell+1}, \mathsf{s})) > \varphi(\mathsf{M}(\mathsf{u}_\ell, \mathsf{u}_{\ell+1}, \mathsf{s})).$$

As φ is nondecreasing by (Þ_1), then:

$$\mathsf{M}(\mathsf{u}_{\ell+1}, \mathsf{u}_{\ell+2}, \mathsf{s}) > \mathsf{M}(\mathsf{u}_\ell, \mathsf{u}_{\ell+1}, \mathsf{s}).$$

In any of the previous cases, we proved that the sequence $\{ \mathsf{M}(\mathsf{u}_\ell, \mathsf{u}_{\ell+1}, \mathsf{s}) \}_{\ell \in \mathbb{N}} \subset (0,1]$ is nondecreasing. This property permits us to define the function $\kappa : (0,\infty) \to (0,1]$ as:

$$\kappa(\mathsf{s}) = \lim_{\ell \to \infty} \mathsf{M}(\mathsf{u}_\ell, \mathsf{u}_{\ell+1}, \mathsf{s}) \quad \text{for all } \mathsf{s} > 0.$$

Step 2. $\kappa(\mathsf{s}) = 1$ *for all* $\mathsf{s} > 0$.

Let $\mathsf{s} > 0$ be arbitrary. If there is some $\ell_0 \in \mathbb{N}$ such that $\mathsf{M}(\mathsf{u}_{\ell_0}, \mathsf{u}_{\ell_0+1}, \mathsf{s}) = 1$, then $\mathsf{M}(\mathsf{u}_{\ell_0+1}, \mathsf{u}_{\ell_0+2}, \mathsf{s}) \geq \mathsf{M}(\mathsf{u}_{\ell_0}, \mathsf{u}_{\ell_0+1}, \mathsf{s}) = 1$, so $\mathsf{M}(\mathsf{u}_{\ell_0+1}, \mathsf{u}_{\ell_0+2}, \mathsf{s}) = 1$. In this case, by induction, we can check that $\mathsf{M}(\mathsf{u}_\ell, \mathsf{u}_{\ell+1}, \mathsf{s}) = 1$ for all $\ell \geq \ell_0$, which implies that $\kappa(\mathsf{s}) = \lim_{\ell\to\infty} \mathsf{M}(\mathsf{u}_\ell, \mathsf{u}_{\ell+1}, \mathsf{s}) = 1$. Next, suppose that:

$$\mathsf{M}(\mathsf{u}_\ell, \mathsf{u}_{\ell+1}, \mathsf{s}) < 1 \quad \text{for all } \ell \in \mathbb{N}.$$

In this case, (6) and Property (Þ_2) ensure that:

$$\varphi(\mathsf{M}(\mathsf{u}_{\ell+1}, \mathsf{u}_{\ell+2}, \mathsf{s})) \geq \eta(\mathsf{M}(\mathsf{u}_\ell, \mathsf{u}_{\ell+1}, \mathsf{s})) > \varphi(\mathsf{M}(\mathsf{u}_\ell, \mathsf{u}_{\ell+1}, \mathsf{s})). \tag{7}$$

As φ is nondecreasing, then:

$$\mathsf{M}(\mathsf{u}_{\ell+1}, \mathsf{u}_{\ell+2}, \mathsf{s}) > \mathsf{M}(\mathsf{u}_\ell, \mathsf{u}_{\ell+1}, \mathsf{s}) \quad \text{for all } \ell \in \mathbb{N}.$$

In order to prove that $\kappa(\mathsf{s}) = 1$, suppose, by contradiction, that $\kappa(\mathsf{s}) < 1$. In such a case,

$$0 < \mathsf{M}(\mathsf{u}_\ell, \mathsf{u}_{\ell+1}, \mathsf{s}) < \mathsf{M}(\mathsf{u}_{\ell+1}, \mathsf{u}_{\ell+2}, \mathsf{s}) < \kappa(\mathsf{s}) < 1 \quad \text{for all } \ell \in \mathbb{N}.$$

Taking into account that:

$$\lim_{\ell \to \infty} \mathsf{M}(\mathsf{u}_\ell, \mathsf{u}_{\ell+1}, \mathsf{s}) = \lim_{\ell \to \infty} \mathsf{M}(\mathsf{u}_{\ell+1}, \mathsf{u}_{\ell+2}, \mathsf{s}) = \kappa(\mathsf{s}),$$

it follows that:

$$\lim_{\ell \to \infty} \varphi(\mathsf{M}(\mathsf{u}_\ell, \mathsf{u}_{\ell+1}, \mathsf{s})) = \lim_{\ell \to \infty} \varphi(\mathsf{M}(\mathsf{u}_{\ell+1}, \mathsf{u}_{\ell+2}, \mathsf{s})) = \lim_{\mathsf{r} \to \kappa(\mathsf{s})^-} \varphi(\mathsf{r}).$$

This limit exists and is finite because φ is well defined on $(0,1]$ and is nondecreasing on $(0,1)$. Letting $\ell \to \infty$ in (7), we deduce that the following limit exists and is finite:

$$\lim_{\ell \to \infty} \eta(\mathsf{M}(\mathsf{u}_\ell, \mathsf{u}_{\ell+1}, \mathsf{s})) = \lim_{\mathsf{r} \to \kappa(\mathsf{s})^-} \varphi(\mathsf{r}).$$

However, this fact contradicts Property $(\mathrm{Þ}_3)$ because:

$$\lim_{r\to\kappa(s)^-}\varphi(r)=\lim_{\ell\to\infty}\eta(M(u_\ell,u_{\ell+1},s))\geq\liminf_{r\to\kappa(s)^-}\eta(r)>\lim_{r\to\kappa(s)^-}\varphi(r).$$

This contradiction shows that $\kappa(s)=1$ for all $s>0$, which completes Step 2 and proves that:

$$\lim_{\ell\to\infty}M(u_\ell,u_{\ell+1},s)=1\quad\text{for all } s>0. \tag{8}$$

Step 3. The sequence $\{u_\ell\}_{\ell\in\mathbb{N}}$ *is either almost constant or infinite, and in this last case,*

$$Tu_{\ell_1}\neq Tu_{\ell_2}\quad\text{for any } \ell_1,\ell_2\in\mathbb{N}\text{ such that }\ell_1\neq\ell_2. \tag{9}$$

If we suppose that there are two distinct indices $\ell_1,\ell_2\in\mathbb{N}$ such that $u_{\ell_1}=u_{\ell_2}$, taking into account (8), Proposition 1 guarantees that the sequence $\{u_\ell\}_{\ell\in\mathbb{N}}$ is almost constant, that is there are $\ell_0\in\mathbb{N}$ and $w\in X$ such that $u_\ell=w$ for all $\ell\geq\ell_0$. In this case, w is a fixed point of T, and the part of the proof about the existence of fixed points of T is finished. On the contrary, assume that $u_{\ell_1}\neq u_{\ell_2}$ for any $\ell_1,\ell_2\in\mathbb{N}$ such that $\ell_1\neq\ell_2$ (that is, $\{u_\ell\}_{\ell\in\mathbb{N}}$ is an infinite sequence). We continue the proof in this second case, where we also know that the property (9) holds.

Step 4. We claim that $\{u_\ell\}_{\ell\in\mathbb{N}}$ *is an* M*-Cauchy sequence.*

We reason in the contrary case. Suppose that $\{u_\ell\}_{\ell\in\mathbb{N}}$ is not an M-Cauchy sequence. In such a case, Lemma 2 guarantees that there are $L_0\in(0,1)$, $t_0>0$, and two partial subsequences $\{u_{p(\ell)}\}_{\ell\in\mathbb{N}}$ and $\{u_{q(\ell)}\}_{\ell\in\mathbb{N}}$ of $\{u_\ell\}$ such that, for all $\ell\in\mathbb{N}$,

$$\begin{gathered}\ell<p(\ell)<q(\ell)<p(\ell+1)\qquad\text{and}\\ M(u_{p(\ell)},u_{q(\ell)-1},t_0)>L_0\geq M(u_{p(\ell)},u_{q(\ell)},t_0),\end{gathered} \tag{10}$$

and also:

$$\lim_{\ell\to\infty}M(u_{p(\ell)},u_{q(\ell)},t_0)=\lim_{\ell\to\infty}M(u_{p(\ell)-1},u_{q(\ell)-1},t_0)=L_0. \tag{11}$$

Since $\lim_{\ell\to\infty}M(u_{p(\ell)-1},u_{q(\ell)-1},t_0)=L_0<1$, there is $\ell_0\in\mathbb{N}$ such that:

$$M(u_{p(\ell)-1},u_{q(\ell)-1},t_0)<1\quad\text{for all }\ell\geq\ell_0.$$

In order not to complicate the notation, assume that:

$$M(u_{p(\ell)-1},u_{q(\ell)-1},t_0)<1\quad\text{for all }\ell\in\mathbb{N}. \tag{12}$$

Applying the contractivity condition (4), Property $(\mathrm{Þ}_2)$, and (12), we deduce that, for all $\ell\in\mathbb{N}$,

$$\begin{aligned}\varphi(M(u_{p(\ell)},u_{q(\ell)},t_0))&=\varphi(M(Tu_{p(\ell)-1},Tu_{q(\ell)-1},t_0))\\&\geq\eta(M(u_{p(\ell)-1},u_{q(\ell)-1},t_0))>\varphi(M(u_{p(\ell)-1},u_{q(\ell)-1},t_0)).\end{aligned}$$

In particular,

$$\varphi(M(u_{p(\ell)},u_{q(\ell)},t_0))\geq\eta(M(u_{p(\ell)-1},u_{q(\ell)-1},t_0))>\varphi(M(u_{p(\ell)-1},u_{q(\ell)-1},t_0)). \tag{13}$$

Since φ is nondecreasing, then:

$$M(u_{p(\ell)},u_{q(\ell)},t_0)>M(u_{p(\ell)-1},u_{q(\ell)-1},t_0),$$

which means, by (11), that:

$$M(u_{p(\ell)-1},u_{q(\ell)-1},t_0)<M(u_{p(\ell)},u_{q(\ell)},t_0)\leq L_0<1\quad\text{for all }\ell\in\mathbb{N}. \tag{14}$$

Using (11) and (14), we deduce that:

$$\lim_{\ell\to\infty}\varphi(\mathsf{M}(\mathsf{u}_{\mathsf{p}(\ell)},\mathsf{u}_{\mathsf{q}(\ell)},\mathsf{t}_0)) = \lim_{\ell\to\infty}\varphi(\mathsf{M}(\mathsf{u}_{\mathsf{p}(\ell)-1},\mathsf{u}_{\mathsf{q}(\ell)-1},\mathsf{t}_0)) = \lim_{\mathsf{r}\to\mathsf{L}_0^-}\varphi(\mathsf{r}).$$

If $\ell\to\infty$ in (13), it follows that:

$$\lim_{\ell\to\infty}\eta(\mathsf{M}(\mathsf{u}_{\mathsf{p}(\ell)-1},\mathsf{u}_{\mathsf{q}(\ell)-1},\mathsf{t}_0)) = \lim_{\mathsf{r}\to\mathsf{L}_0^-}\varphi(\mathsf{r}).$$

However, this is a contradiction to Property (Þ_3) because:

$$\lim_{\mathsf{r}\to\mathsf{L}_0^-}\varphi(\mathsf{r}) = \lim_{\ell\to\infty}\eta(\mathsf{M}(\mathsf{u}_{\mathsf{p}(\ell)-1},\mathsf{u}_{\mathsf{q}(\ell)-1},\mathsf{t}_0)) \geq \liminf_{\mathsf{s}\to\mathsf{L}_0^-}\eta(\mathsf{s}) > \lim_{\mathsf{r}\to\mathsf{L}_0^-}\varphi(\mathsf{r}).$$

This contradiction proves that $\{\mathsf{u}_\ell\}_{\ell\in\mathbb{N}}$ is an M-Cauchy sequence.
As $(X,\mathsf{M},*)$ is M-complete, there is $\mathsf{v}\in\mathsf{X}$ such that $\{\mathsf{u}_\ell\}_{\ell\in\mathbb{N}}$ M-converges to v, that is,

$$\lim_{\ell\to\infty}\mathsf{M}(\mathsf{u}_\ell,\mathsf{v},\mathsf{s}) = 1 \quad \text{for all } \mathsf{s}>0. \tag{15}$$

Step 5. The point $\mathsf{v}\in\mathsf{X}$ *is a fixed point of* T.
To prove it, assume, by contradiction, that $\mathsf{v}\in X\setminus\mathrm{Fix}(\mathsf{T})$, that is $\mathsf{v}\neq\mathsf{Tv}$. As the sequence $\{\mathsf{u}_\ell\}_{\ell\in\mathbb{N}}$ is infinite, then there is $\ell_0\in\mathbb{N}$ such that $\mathsf{u}_\ell\neq\mathsf{v}$ and $\mathsf{u}_\ell\neq\mathsf{Tv}$ for all $\ell\geq\ell_0$. Without loss of generality, assume that:

$$\mathsf{u}_\ell\neq\mathsf{Tv} \quad \text{and} \quad \mathsf{Tu}_\ell\neq\mathsf{Tv} \quad \text{for all } \ell\in\mathbb{N}.$$

The hypothesis (4) leads to:

$$\varphi(\mathsf{M}(\mathsf{u}_{\ell+1},\mathsf{Tv},\mathsf{t})) = \varphi(\mathsf{M}(\mathsf{Tu}_\ell,\mathsf{Tv},\mathsf{t})) \geq \eta(\mathsf{M}(\mathsf{u}_\ell,\mathsf{v},\mathsf{t}))$$

for all $\ell\in\mathbb{N}$ and all $\mathsf{t}>0$. Let us prove that $\mathsf{M}(\mathsf{u}_{\ell+1},\mathsf{Tv},\mathsf{t})\geq\mathsf{M}(\mathsf{u}_\ell,\mathsf{v},\mathsf{t})$ by discussing two cases.

- If $\mathsf{M}(\mathsf{u}_\ell,\mathsf{v},\mathsf{t})=1$, then

  $$\varphi(\mathsf{M}(\mathsf{u}_{\ell+1},\mathsf{Tv},\mathsf{t})) \geq \eta(\mathsf{M}(\mathsf{u}_\ell,\mathsf{v},\mathsf{t})) = \eta(1).$$

  In such a case, the assumption (Þ_4) guarantees that $\mathsf{M}(\mathsf{u}_{\ell+1},\mathsf{Tv},\mathsf{t}) = \mathsf{M}(\mathsf{u}_\ell,\mathsf{v},\mathsf{t}) = 1$. In particular, $\mathsf{M}(\mathsf{u}_{\ell+1},\mathsf{Tv},\mathsf{t})\geq\mathsf{M}(\mathsf{u}_\ell,\mathsf{v},\mathsf{t})$;
- If $\mathsf{M}(\mathsf{u}_\ell,\mathsf{v},\mathsf{t})<1$, then:

  $$\varphi(\mathsf{M}(\mathsf{u}_{\ell+1},\mathsf{Tv},\mathsf{t})) \geq \eta(\mathsf{M}(\mathsf{u}_\ell,\mathsf{v},\mathsf{t})) > \varphi(\mathsf{M}(\mathsf{u}_\ell,\mathsf{v},\mathsf{t})),$$

  and the nondecreasing character of φ lets us deduce that $\mathsf{M}(\mathsf{u}_{\ell+1},\mathsf{Tv},\mathsf{t})>\mathsf{M}(\mathsf{u}_\ell,\mathsf{v},\mathsf{t})$.

In both cases, we checked that:

$$\mathsf{M}(\mathsf{u}_\ell,\mathsf{v},\mathsf{t}) \leq \mathsf{M}(\mathsf{u}_{\ell+1},\mathsf{Tv},\mathsf{t}) \leq 1 \quad \text{for all } \ell\in\mathbb{N} \text{ and all } \mathsf{t}>0.$$

Using (15) and the previous inequalities, we conclude that:

$$\lim_{\ell\to\infty}\mathsf{M}(\mathsf{u}_{\ell+1},\mathsf{Tv},\mathsf{t}) = 1 \quad \text{for all } \mathsf{t}>0,$$

which means that the sequence $\{\mathsf{u}_\ell\}_{\ell\in\mathbb{N}}$ also M-converges to Tv. The uniqueness of the limit of a convergent sequence in a fuzzy metric space demonstrates that $\mathsf{Tv}=\mathsf{v}$.
Step 6. The operator T *has a unique fixed point in* X.

Finally, suppose that $v_1, v_2 \in X$ are two distinct fixed points of T. Since $Tv_1 \neq Tv_2$, then, for all $s > 0$,

$$\varphi(M(v_1, v_2, s)) = \varphi(M(Tv_1, Tv_2, s)) \geq \eta(M(v_1, v_2, s)).$$

If we suppose that $M(v_1, v_2, s) < 1$ for some $s > 0$, then:

$$\varphi(M(v_1, v_2, s)) \geq \eta(M(v_1, v_2, s)) > \varphi(M(v_1, v_2, s)),$$

which is a contradiction. Then, necessarily, $M(v_1, v_2, s) = 1$ for all $s > 0$, but this fact contradicts that v_1 and v_2 are distinct (recall Axiom GV-2). As a result, the mapping T has a unique fixed point. □

The condition $Tu \neq Tv$ in the contractivity condition (4) is useful in practice to avoid cases such that $u = v$. However, it is possible that two points $u, v \in X$ satisfy $M(u, v, s_0) = 1$ for some $s_0 \in (0, \infty)$ (and, automatically, $M(u, v, s) = 1$ for any $s \in [s_0, \infty)$ because the fuzzy set $M(u, v, \cdot) : (0, \infty) \to [0, 1]$ is nondecreasing). This condition leads to the inequality $\varphi(1) \geq \eta(1)$, which is inevitable in this context. Then, the following result is a direct consequence of Theorem 2 by removing this condition on the contractivity constraint.

Corollary 1. *Let $(X, M, *)$ be an M-complete non-Archimedean fuzzy metric space, and let $T : X \to X$ be a mapping for which there exists $(\varphi, \eta) \in \mathcal{L}$ such that:*

$$\varphi(M(Tu, Tv, s)) \geq \eta(M(u, v, s)) \quad \textit{for all } u, v \in X \textit{ and all } s > 0.$$

Then, each iterative Picard sequence $\{T^\ell u\}_{\ell \in \mathbb{N}}$ converges to the unique fixed point of T for every initial condition $u \in X$.

We next show a case in which Axiom $(Þ_4)$ can be replaced by a stronger condition.

Corollary 2. *Let $(X, M, *)$ be an M-complete non-Archimedean fuzzy metric space, and let $T : X \to X$ be a mapping for which there exist two functions $\varphi, \eta : (0, 1] \to \mathbb{R}$ such that:*

$$\varphi(M(Tu, Tv, s)) \geq \eta(M(u, v, s)) \quad \textit{for all } u, v \in X \textit{ with } Tu \neq Tv \textit{ and all } s > 0.$$

Suppose that the functions φ and η verify the following assumptions:

$(\mathbf{Þ}_1)$ *φ is nondecreasing;*

$(\mathbf{Þ}_2)$ $\eta(s) > \varphi(s)$ *for any* $s \in (0, 1)$;

$(\mathbf{Þ}_3)$ $\liminf_{s \to L^-} \eta(s) > \lim_{s \to L^-} \varphi(s)$ *for any* $L \in (0, 1)$;

$(\mathbf{Þ}'_4)$ $\eta(1) \geq \sup(\{\eta(s) : s \in (0, 1)\})$.

Then, each iterative Picard sequence $\{T^\ell u\}_{\ell \in \mathbb{N}}$ converges to the unique fixed point of T for every initial condition $u \in X$.

Proof. Indeed. we check that, under $(Þ_2)$, Condition $(Þ'_4)$ implies Property $(Þ_4)$. Let $t \in (0, 1]$ be such that $\varphi(t) \geq \eta(1)$. To prove that $t = 1$, suppose, by contradiction, that $t < 1$. In such a case,

$$\varphi(t) \geq \eta(1) \geq \sup(\{\eta(s) : s \in (0, 1)\}) \geq \eta(t).$$

However, $\varphi(t) \geq \eta(t)$ contradicts Property $(Þ_2)$. Therefore, $t = 1$. Hence, Theorem 2 is applicable. □

Corollary 3. *Let* $(X, \mathsf{M}, *)$ *be an* M*-complete non-Archimedean fuzzy metric space, and let* $\mathsf{T} : \mathsf{X} \to \mathsf{X}$ *be a mapping such that:*

$$\mathsf{M}(\mathsf{Tu}, \mathsf{Tv}, \mathsf{s}) \geq \sqrt{\mathsf{M}(\mathsf{u}, \mathsf{v}, \mathsf{s})} \quad \textit{for all } \mathsf{u}, \mathsf{v} \in \mathsf{X} \textit{ with } \mathsf{Tu} \neq \mathsf{Tv} \textit{ and all } \mathsf{s} > 0.$$

Then, each iterative Picard sequence $\{\mathsf{T}^{\ell}\mathsf{u}\}_{\ell \in \mathbb{N}}$ *converges to the unique fixed point of* T *for every initial condition* $\mathsf{u} \in \mathsf{X}$.

Proof. It follows from Theorem 2 by using the functions φ and η defined by $\varphi(\mathsf{s}) = \mathsf{s}$ and $\eta(\mathsf{s}) = \sqrt{\mathsf{s}}$ for all $\mathsf{s} \in (0, 1]$. □

The following consequence is a version of the Banach theorem in the setting of non-Archimedean fuzzy metric spaces that can be directly deduced form our main result.

Corollary 4. *Let* $(X, \mathsf{M}, *)$ *be an* M*-complete non-Archimedean fuzzy metric space, and let* $\mathsf{T} : \mathsf{X} \to \mathsf{X}$ *be a mapping for which there exists* $\lambda \in (0, 1)$ *such that:*

$$1 - \mathsf{M}(\mathsf{Tu}, \mathsf{Tv}, \mathsf{s}) \leq \lambda(1 - \mathsf{M}(\mathsf{u}, \mathsf{v}, \mathsf{s})) \quad \textit{for all } \mathsf{u}, \mathsf{v} \in \mathsf{X} \textit{ with } \mathsf{Tu} \neq \mathsf{Tv} \textit{ and all } \mathsf{s} > 0. \tag{16}$$

Then, each iterative Picard sequence $\{\mathsf{T}^{\ell}\mathsf{u}\}_{\ell \in \mathbb{N}}$ *converges to the unique fixed point of* T *for every initial condition* $\mathsf{u} \in \mathsf{X}$.

Proof. Define $\varphi, \eta : (0, 1] \to \mathbb{R}$ by $\varphi(\mathsf{t}) = \mathsf{t}$ and $\eta(\mathsf{t}) = 1 - \lambda(1 - \mathsf{t})$ for all $\mathsf{t} \in (0, 1]$. It is easy to check that the pair (φ, η) belongs to $\mathcal{L}$ (all assumptions (Þ_1)–(Þ_4) are apparent). Then, for all $\mathsf{u}, \mathsf{v} \in \mathsf{X}$ with $\mathsf{Tu} \neq \mathsf{Tv}$ and all $\mathsf{s} > 0$:

$$\begin{aligned} 1 - \mathsf{M}(\mathsf{Tu}, \mathsf{Tv}, \mathsf{s}) \leq \lambda(1 - \mathsf{M}(\mathsf{u}, \mathsf{v}, \mathsf{s})) \quad &\Leftrightarrow \quad \mathsf{M}(\mathsf{Tu}, \mathsf{Tv}, \mathsf{s}) - 1 \geq \lambda(\mathsf{M}(\mathsf{u}, \mathsf{v}, \mathsf{s}) - 1) \\ &\Leftrightarrow \quad \mathsf{M}(\mathsf{Tu}, \mathsf{Tv}, \mathsf{s}) \geq 1 + \lambda(\mathsf{M}(\mathsf{u}, \mathsf{v}, \mathsf{s}) - 1) \\ &\Leftrightarrow \quad \varphi(\mathsf{M}(\mathsf{Tu}, \mathsf{Tv}, \mathsf{s})) \geq \eta(\mathsf{M}(\mathsf{u}, \mathsf{v}, \mathsf{s})). \end{aligned}$$

Therefore, the contractivity condition (16) is equivalent to (4). As a consequence, Theorem 2 is applicable, and it guarantees the announced conclusions. □

Corollary 5. *Let* $(X, \mathsf{M}, *)$ *be an* M*-complete non-Archimedean fuzzy metric space, and let* $\mathsf{T} : \mathsf{X} \to \mathsf{X}$ *be a mapping such that:*

$$2\mathsf{M}(\mathsf{Tu}, \mathsf{Tv}, \mathsf{s}) - \mathsf{M}(\mathsf{u}, \mathsf{v}, \mathsf{s}) \geq 1 \quad \textit{for all } \mathsf{u}, \mathsf{v} \in \mathsf{X} \textit{ with } \mathsf{Tu} \neq \mathsf{Tv} \textit{ and all } \mathsf{s} > 0.$$

Then, each iterative Picard sequence $\{\mathsf{T}^{\ell}\mathsf{u}\}_{\ell \in \mathbb{N}}$ *converges to the unique fixed point of* T *for every initial condition* $\mathsf{u} \in \mathsf{X}$.

Proof. Notice that for all $\mathsf{u}, \mathsf{v} \in \mathsf{X}$ with $\mathsf{Tu} \neq \mathsf{Tv}$ and all $\mathsf{s} > 0$:

$$\begin{aligned} 2\mathsf{M}(\mathsf{Tu}, \mathsf{Tv}, \mathsf{s}) - \mathsf{M}(\mathsf{u}, \mathsf{v}, \mathsf{s}) \geq 1 \quad &\Leftrightarrow \quad \mathsf{M}(\mathsf{Tu}, \mathsf{Tv}, \mathsf{s}) \geq \frac{1 + \mathsf{M}(\mathsf{u}, \mathsf{v}, \mathsf{s})}{2} \\ &\Leftrightarrow \quad 1 - \mathsf{M}(\mathsf{Tu}, \mathsf{Tv}, \mathsf{s}) \leq 1 - \frac{1 + \mathsf{M}(\mathsf{u}, \mathsf{v}, \mathsf{s})}{2} \\ &\Leftrightarrow \quad 1 - \mathsf{M}(\mathsf{Tu}, \mathsf{Tv}, \mathsf{s}) \leq \frac{1 - \mathsf{M}(\mathsf{u}, \mathsf{v}, \mathsf{s})}{2}, \end{aligned}$$

so Corollary 4 is applicable by using $\lambda = 1/2$. □

Finally, we show how to take advantage of the families of non-Archimedean fuzzy metric spaces given in Example 2.

Corollary 6. *Let* (X, d) *be a complete metric space, and let* ϑ *be a nondecreasing and continuous function from* $(0, \infty)$ *into* $(0, 1)$ *such that* $\lim_{\mathsf{t}\to\infty} \vartheta(\mathsf{t}) = 1$*. Let* $\mathsf{T} : \mathsf{X} \to \mathsf{X}$ *be a mapping for which there exists* $(\varphi, \eta) \in \mathcal{L}$ *such that:*

$$\varphi\Big([\vartheta(\mathsf{s})]^{\mathsf{d}(\mathsf{Tu},\mathsf{Tv})}\Big) \geq \eta\Big([\vartheta(\mathsf{s})]^{\mathsf{d}(\mathsf{u},\mathsf{v})}\Big) \quad \textit{for all } \mathsf{u}, \mathsf{v} \in \mathsf{X} \textit{ with } \mathsf{Tu} \neq \mathsf{Tv} \textit{ and all } \mathsf{s} > 0. \tag{17}$$

Then, each iterative Picard sequence $\{\mathsf{T}^{\ell}\mathsf{u}\}_{\ell\in\mathbb{N}}$ *converges to the unique fixed point of* T *for every initial condition* $\mathsf{u} \in \mathsf{X}$.

Proof. As we described in Example 2, if $*$ is a t-norm such that $* \leq *_P$ and we define $\mathsf{M} : \mathsf{X} \times \mathsf{X} \times (0, \infty) \to [0, 1]$ such that:

$$\mathsf{M}(\mathsf{u}, \mathsf{v}, \mathsf{t}) = [\vartheta(\mathsf{t})]^{\mathsf{d}(\mathsf{u},\mathsf{v})}$$

for each $\mathsf{u}, \mathsf{v} \in \mathsf{X}$ and all $\mathsf{t} \in (0, \infty)$, then $(\mathsf{X}, \mathsf{M}, *)$ is a non-Archimedean fuzzy metric space. As (X, d) is complete, then $(\mathsf{X}, \mathsf{M}, *)$ is also. Furthermore, the contractivity condition (17) is equivalent, in the setting, to (4). Therefore, Theorem 2 guarantees that each iterative Picard sequence $\{\mathsf{T}^{\ell}\mathsf{u}\}_{\ell\in\mathbb{N}}$ converges to the unique fixed point of T for every initial condition $\mathsf{u} \in \mathsf{X}$. □

The reader can also particularize Theorem 2 to the non-Archimedean fuzzy metric spaces given in Example 1.

4. Conclusions and Open Problems

In this paper, we introduced a novel family of contractions in the setting of non-Archimedean fuzzy metric spaces. The most important advantage of the cited family of contractions is that it involves very general auxiliary functions that were inspired on Proinov's attractive paper [2]. The obtained results showed that there is a wide field of research that must be explored to better understand the topological, analytical, and algebraic structure of fuzzy metric spaces. In this sense, future research should clarify what new theorems in spaces with an abstract metric structure (*b*-metric spaces, generalized metric spaces, etc.) can be obtained from the contractions that we introduced here, even in the fuzzy and probabilistic setting.

To focus the question, the results presented here led us to consider the following issues, which we propose to the reader as open problems.

Open Problem 1: Do our main results hold in general fuzzy metric spaces in the sense of George and Veeramani, avoiding the non-Archimedean condition?

Open Problem 2: Can the hypotheses of nondecreasingness be removed (or replaced by other weaker assumption) from Theorem 2?

Open Problem 3: Can a function be employed on the parameter t so as to prove some extensions of the previous statements?

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