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A geometrical method for consensus building in GDM with incomplete heterogeneous preference information

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ABSTRACT

In real-life group decision-making (GDM) problems, the preferences given by decision-makers(DMs) are often incomplete, because the complexity of decision-making problems and the limitation of knowledge of DM make it difficult for DMs to take a determined evaluation of alternatives. In addition, preference relations provided by DMs are often heterogeneous because they always have different decision habits and hobbies. However, the consensus method for GDM under incomplete heterogeneous preference relations is rarely studied. For four common preference relations: utility values, preference orderings, and (incomplete) multiplicative preference relations and (incomplete) fuzzy preference relations, this paper proposes a geometrical method for consensus building in GDM. Specifically, we integrate incomplete heterogeneous preference structures using a similarity-based optimization model and set a corresponding geometrical consensus measurement. Then, preference modification and weighting processes are proposed to improve consensus degree. Finally, we conduct a comparison analysis based on a qualitative analysis and algorithm complexity analysis of existing consensus reaching methods. Numerical analyses and convergence tests show that our method can promote the improvement of the consensus degree in GDM, and has less time complexity than the previous methods. The proposed geometrical method is a more explainable model due to operability and simplicity.

1. Introduction

The group decision-making (GDM) is a process of aggregating the preference information of each decision expert in a group into a collective opinion under certain decision criteria. Many major and complex decision-making problems need to be completed by relying on the experience and wisdom of the group and promoting democracy. Recent years, group decision-making (GDM) problems have long been identified as a hot topic in the field of decision science [1–7].

Consensus in GDM typically means reach a consent, not necessarily the agreement of all group participants [8]. Many factors influence the consensus reaching process, involving ambiguity of DMs' expertise, the incompleteness of decision-making information, the multiplicity of decision-making goals, the uncertainty of decision-making environments. Consequently, many DMs may provide uncertain evaluations and even incomplete preference representations in the decision-making process. On the other hand, DMs are always come from different areas and therefore have their varying experience, knowledge, personalities, and educational backgrounds, and may tend to provide their preference using different structures of preference relations according to their own will. As a result, consensus reaching in GDM with incomplete heterogeneous preference information is a common real-life management issues and a difficult problem needs to be urgently solved [2,8–22].

In the past decade, consensus reaching with heterogeneous preference forms has received more and more attention. Researchers have developed many methods to handle the problem of GDM with heterogeneous preference structures. For example, Herrera-Viedma et al. [23] proposed a consensus process for dealing with GDM with heterogeneous preference structures. Their framework performs the iterative feedback adjustment until an acceptable consensus degree is reached. In the consensus process, the feedback adjustment suggestions are provided to DMs to revise individual judgments. Pérez et al. [24] developed a

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new feedback mechanism considering the relevance or importance level of different DMs and heterogeneity criterion. Hereafter many feedback adjustment rules are also studied in the existing literature [9,17,25]. Another important recent research direction is to acquire collective opinion simultaneously associated with individual preference consistency and group consensus degree. Zhang and Guo [17] introduced a bi-objective optimization model which includes two targets maximizing both the group consensus and the individual consistency of each decision maker. The solution can be regarded as a priority vector possessing the consensus degree in GDM. Li et al. [12] also proposed a consensus model for the heterogeneous preference information which can assure that the consistency of an individual preference relation cannot be decreased in consensus reaching process (CRP). Recently, more situations have also been brought into heterogeneous GDM, such as self-confidence [26] and individual concerns and satisfactions [27,28].

Incomplete preference information is also a popular topic in GDM questions [16,29–33]. Vetschera [34] proposed a complete ranking method for incomplete preferences based on probabilistic information, which is the relative size of regions in parameter space. Zhang and Guo [17] proposed a group preference derivation method and used to consensus reaching with an uncertain situation. To fill in missing values is another approach to deal with the incomplete preference issue and the management of ignorance situations too [35]. Cheng et al. [26] developed a new four-way iteration step to estimate the missing preference values, but it is difficult to assert whether the filling values can reflect the true judgment of the experts.

From the above analysis, CRP with incomplete preference information and heterogeneous preference relations has significantly advanced in the domain of GDM as a result of previous studies. However, many studies for incomplete heterogeneous GDM need to derive the priority vector from individual preference relations or transform various preference relations into a unified structure, or fill in missing values of incomplete individual preference relations. The disadvantage of these methods is the loss or disturbance of preference information or the increasing complexity of the decision-making process. There is very little literature which directly addresses consensus building in terms of incomplete preference information without deriving the individual priority vector. Thus, the collective opinion directly aggregated from incomplete heterogeneous preference information still a problem which need to be resolved. On the one hand, more operational and explainable methods avoiding more steps and complex optimization need to be developed in order to reduce the complexity of CRP. On the other hand, the existing consensus measurement cannot always be used directly in GDM due to incomplete and heterogeneous preference information, so a suitable measurement for this problem must be proposed.

Motivated to deal with the above questions, we propose a geometrical consensus building a model for GDM with incomplete heterogeneous preference structures (utility values, preference orderings, multiplicative preference relations, and fuzzy preference relations). In this model, integrating different preference relations, modifying individual preference and weighting processes will be systemically developed based on a geometric insight. A geometrical consensus degree is also introduced and compared to existing indexes using a qualitative analysis. Finally, we simulate a convergence test for our proposed model and show the effectiveness. The main contribution of this paper is to propose a geometrical insight to build consensus in GDM with incomplete heterogeneous preference information. The collective opinion can be obtained directly through the similarity maximization optimization model of heterogeneous preference relations avoiding to filling in missing values or transforming different preference formats into unified formats. In addition, we propose a mathematically proven individual preference adjustment mechanism that has convergence of consensus degree, rather than a simulation experiment method used in existing literatures. Compared to previous methods, it is operability and simplicity because of its less time complexity and fewer steps.

The remainder of this paper is organized as follows. In Section 2, we introduce related preliminaries about different preference relations. Section 3 we propose an optimization model for aggregation of the heterogeneous preference information, and in Section 4, we construct a consensus reaching model for four incomplete different preference structures. Section 5 provides a numerical example of the proposed model, and Section 6 tests the convergence of the proposed model. We conclude the paper in Section 7.

2. Preliminaries

In this section, we firstly introduce the logical structure of this article, then introduce four different preference relations and main properties corresponding to various preference formats.

In order to facilitate the expression of the subsequent content of this article, the method flow of the CRP in our methods is arranged as follows (Fig. 1). The detailed procedure of this method is described in Section 2 to Section 4.

2.1. Different preference relation and basic notation

First, we explain some notations and review related definitions used in our model. We assume that $A = \{A_1, A_2, \ldots, A_n\}$ is a set of feasible alternatives selected by DMs, and that $w = (w_1, w_2, \ldots, w_n)^T$ is a priority vector that comprises the collective opinions of DMs.

(1) Utility Values. Assume $U = \{u_1, u_2, \ldots, u_n\}$ is the set of utility values provided by one of the DMs. $u_i, i = 1, 2, \ldots, n$ represents the utility values corresponding to alternative A_i. Generally, u_i range from 0 to 1, and the higher the utility value, the more important the alternative possesses [16,23,36,37].

(2) Preference orderings. Let $O = \{o_1, o_2, \ldots, o_n\}$ be a preference ordering set. This set is the permutation function over the set $\{1, 2, \ldots, n\}$. For example, if an individual decision maker provides the ordering $\{3, 1, 4, 2\}$ for four alternatives $\{A_1, A_2, A_3, A_4\}$, then the preference ordering is priority sequence of alternatives, that means the alternative A_3 is the best selection among the candidate alternatives, and A_2 is the least preferable of the alternatives [16,23,36,38].

(3) Multiplicative preference relation [25,39–41]. In contrast to the utility values and preference orderings, the multiplicative preference relation is always represented by a pair-wise comparison matrix (PCM). Assume the matrix $A = \left(a_{ij}\right)_{n\times n}, i, j = 1, 2, \ldots, n$ is the PCM provided by DMs. The entry a_{ij} of the PCM is the degree of preference for alternative A_i over A_j. The PCM satisfies $a_{ij}a_{ji} = 1$ and $a_{ij} > 0$.

(4) Fuzzy preference relation [38,42–45]. A fuzzy preference relation is determined by a comparison matrix whose entries represent preference degree for two alternatives. This differs to multiplicative preference relation in that a fuzzy preference relation uses real numbers from 0 to 1 rather than crisp values. The values of a fuzzy PCM range from zero to one and satisfy $b_{ij}+b_{ji} = 1$. If b_{ij} is 0.5, then the alternatives are equally important. If the value of b_{ij} (resp.b_{ji},) is one (resp., zero), then the alternative A_i is the most (resp., least) important than A_j.

To facilitate searching and understanding, the symbols used in this study are summarize at follows (Table 1):

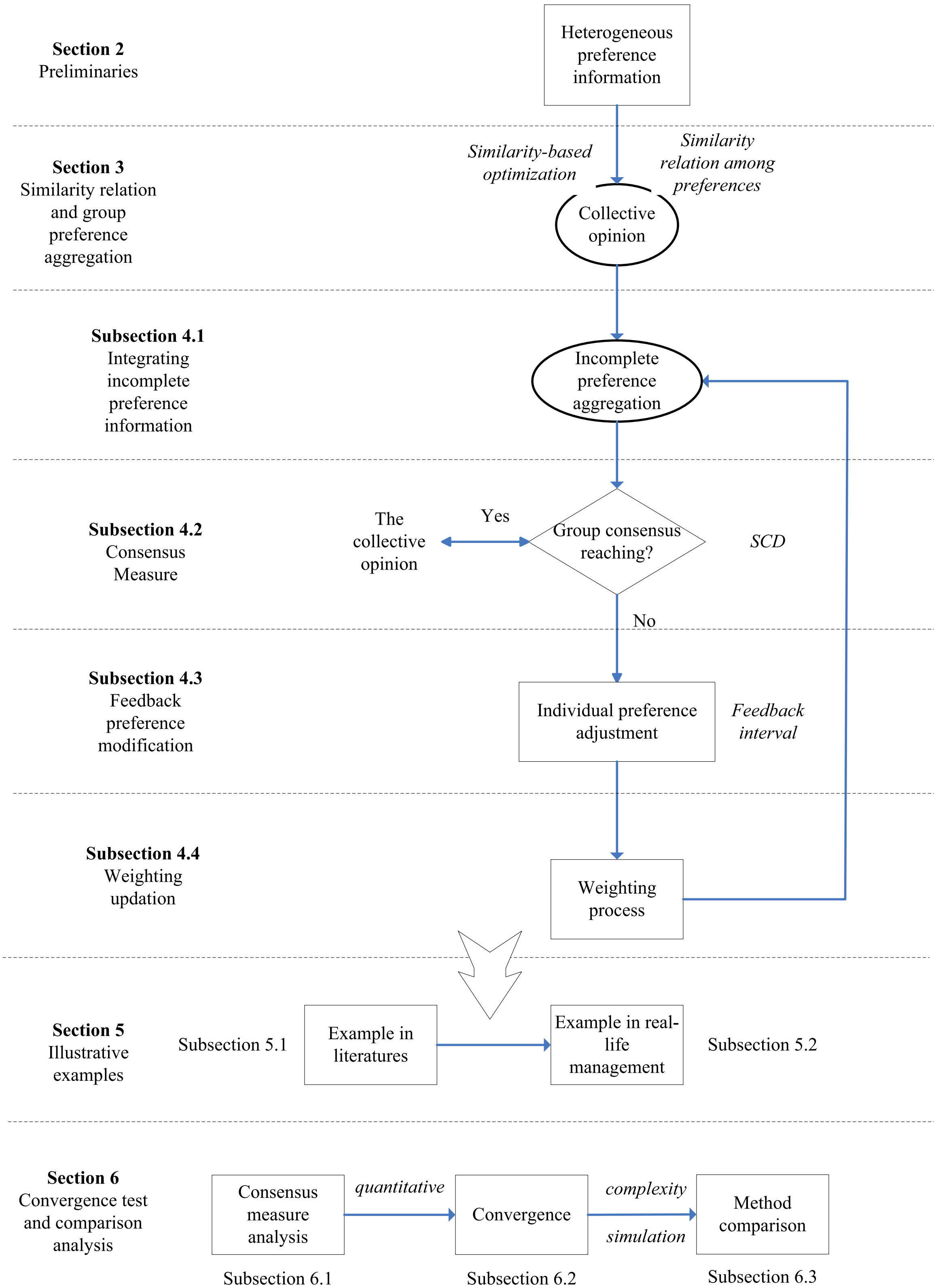

Fig. 1. The flow of CRP.

3. Similarity relation and group preference aggregation

In this section, we illustrate the cosine similarity relation between heterogeneous preference relations and their priority vector, and then introduce an optimization model to obtain the collective opinion from individual preference information.

3.1. Cosine similarity relation

The cosine similarity measure is one of the most widely used similarity measurement indices for two vectors. The cosine similarity value of vectors, $\vec{v}_1 = (a_1, a_2, \ldots, a_n)$ and $\vec{v}_2 =$

Table 1
Summary of the symbols used in this study.

Symbols	Meaning
$E = \{e_1, \ldots, e_i, \ldots, e_m\}$	Set of DMs, e_i is the i-th DM of m DMs.
$A = \{A_1, A_2, \ldots, A_n\}$	Set of alternatives, A_j is the j-th alternative.
$\overrightarrow{v}_i = (a_1, a_2, \ldots, a_n)$	n-vectors.
$w = \{w_1, \ldots, w_i, \ldots, w_n\}$	Priority vector obtained from the preference relation.
$\sigma = \{\sigma_1, \ldots, \sigma_i, \ldots, \sigma_m\}$	The corresponding weights of DMs.
$u = \{u_1, u_2, \ldots, u_n\}$	Set of utility values provided by one of the DMs.
$u_i, i = 1, 2, \ldots, n$	Utility values corresponding to alternativex_i.
$U = (u_{ij})_{n\times n}$	$u_{ij} = u_i/u_j$in utility value matrix
$o = \{o_1, o_2, \ldots, o_n\}$	Preference ordering set.
o_i	Order of alternative x_i.
$O = (o_{ij})_{n\times n}$	$o_{ij} = (n - o_i)/(n - o_j)$in preference ordering matrix
$A = \left(a_{ij}\right)_{n\times n}, i, j = 1, 2, \ldots, n$	A PCM provided by DMs.
a_{ij}	The degree of preference for alternative x_i over x_j.
$B = (b_{ij})_{n\times n}, i, j = 1, 2, \ldots, n$	An additive PCM.
b_{ij}	The fuzzy degree of preference for alternative x_i over x_j.
p_{ij}	A transformed value of b_{ij} using $b_{ij}/(1 + b_{ij})$
$\Omega = \{\Omega_u, \Omega_o, \Omega_A, \Omega_B\}$	A set of DMs with utility values, preference ordering, and multiplicative and additive preference relations, respectively.
$\overrightarrow{u}^{(k)}{}_j, \overrightarrow{o}^{(k)}{}_j, \overrightarrow{a}^{(k)}_j, \overrightarrow{p}^{(k)}_j$	Column vectors of matrixes $U, O, A, B$of the kth DMs.
$\overline{u}_{ij}{}^{(k)}, \overline{o}^{(k)}{}_{ij}, \overline{a}^{(k)}_{ij}, \overline{p}^{(k)}_{ij}$	Normalized entries of matrixes $U, O, A, B$of the k-th DMs.
Θ	The set of positions in the preference relations that contains preference values.
$u_i{}^{(k,t)}, o_i{}^{(k,t)}, a_{ij}{}^{(k,t)}, b_{ij}{}^{(k,t)}$	The preference of k-th DMs at t-th preference iteration.
$\sigma_k{}^{(t)}$	Weight of k-th DMs at the t-th preference iteration.
$P_C{}^{(t)}$	The collective opinion at the t-th preference iteration.
$\cdot^{(G,t)}$	Feedback information from the collective opinion after the t-th preference iteration.

$(b_1, b_2, \ldots, b_n)$, is defined as follows:

$$\left\langle \overrightarrow{v}_1, \overrightarrow{v}_2 \right\rangle = \frac{\sum_{i=1}^{n} a_i b_i}{\sqrt{\sum_{i=1}^{n}(a_i)^2}\sqrt{\sum_{i=1}^{n}(b_i)^2}} \tag{1}$$

The application of the cosine similarity measure to decision making was introduced by Kou & Lin [46]. They discovered that there is a similarity relation between a priority vector and a pairwise comparison matrix (PCM) in the analytic hierarchy process. They found that the cosine value is 1 if the PCM is perfectly consistent, and that the more consistent the PCM, the higher the cosine value of the priority vector and each column of the PCM.

For utility value, if a priority vector is consistent with utility values, then $w_i = u_i/\sum_{i=1}^{n} u_i, i = 1, 2, \ldots, n$, and $\frac{w_i}{w_j} = \frac{u_i}{u_j}$, and so the cosine similarity measure in this case is

$$\left\langle \overrightarrow{u}_j, w \right\rangle = \frac{\sum_{i=1}^{n} \frac{u_i w_i}{u_j}}{\sqrt{\sum_{i=1}^{n}\left(\frac{u_i}{u_j}\right)^2}\sqrt{\sum_{i=1}^{n} w_i^2}} = \frac{\sum_{i=1}^{n} \frac{w_i}{w_j} w_i}{\sqrt{\sum_{i=1}^{n}\left(\frac{w_i}{w_j}\right)^2}\sqrt{\sum_{i=1}^{n} w_i^2}} = 1 \tag{2}$$

where the vector $\overrightarrow{u}_j = (\frac{u_1}{u_j}, \frac{u_2}{u_j}, \ldots, \frac{u_n}{u_j})^T, j = 1, 2, \ldots, n.$

It is clear that the collective opinion in GDM should be close to the utility value, that is, the cosine value $\left\langle \overrightarrow{u}_j, w \right\rangle$ must approach one.

For preference ordering, Herrera et al. [47] proposed a non-decreasing function f that assigns ordering values to utilities as follows:

$$u_i = f(n - o_i) = \frac{n - o_i}{n - 1}, i = 1, 2, \ldots, n \tag{3}$$

Therefore, the priority vector w_i is equal to $u_i/\sum_{i=1}^{n} u_i, i = 1, 2, \ldots, n$ in preference ordering. That is,

$$w_i = \frac{n - o_i}{n - 1} / \sum_{i=1}^{n} \frac{n - o_i}{n - 1}, i = 1, 2, \ldots, n \tag{4}$$

It is obvious that the following formula (5) must be satisfied based on (4), if a priority vector is consistent with preference ordering.

$$\frac{w_i}{w_j} = \frac{n - o_i}{n - o_j} \tag{5}$$

and the cosine similarity measure is one in this case when preference ordering is consistent, i.e.,

$$\left\langle \overrightarrow{o}_j, w \right\rangle = \frac{\sum_{i=1}^{n} \frac{n - o_i}{n - o_j} w_i}{\sqrt{\sum_{i=1}^{n}\left(\frac{n - o_i}{n - o_j}\right)^2}\sqrt{\sum_{i=1}^{n} w_i^2}} = \frac{\sum_{i=1}^{n} \frac{w_i}{w_j} w_i}{\sqrt{\sum_{i=1}^{n}\left(\frac{w_i}{w_j}\right)^2}\sqrt{\sum_{i=1}^{n} w_i^2}} = 1 \tag{6}$$

where the vector $\overrightarrow{o}_j = (\frac{n-o_1}{n-o_j}, \frac{n-o_2}{n-o_j}, \ldots, \frac{n-o_n}{n-o_j})^T, j = 1, 2, \ldots, n.$

Generally, the collective opinion is not perfectly consistent to the individual preference relation, but it should be close to the preference vector, that is $\left\langle \overrightarrow{o}_j, w \right\rangle$ will be close to 1.

Specially, let $B = (b_{ij})_{n\times n}, i, j = 1, 2, \ldots, n$ be a fuzzy PCM. We can obtain the cosine similarity relation under the following transformation for a fuzzy preference relation be means of (7):

$$p_{ij} = \frac{b_{ij}}{1 - b_{ij}} \tag{7}$$

The multiplicative preference relation and fuzzy preference relation also have a similarity relation between the column vector of PCM and priority vector [3,22,46]. That is $\langle \overrightarrow{a}_{ij}, w \rangle$ and $\langle \overrightarrow{p}_{ij}, w \rangle$ is equal to 1 when two preference relations have perfect consistency in which $\overrightarrow{a}_{ij}$ and $\overrightarrow{p}_{ij}$ is the column vector of two preference relations.

Thus, the collective opinion in GDM should be close to the column vector of multiplicative preference relations and fuzzy preference relations.

In sum, it is concluded that the similarity relation is 1 between the column vector (or project vector of them) of the individual preference relation and collective opinion if each individual preference relation satisfies the consistency.

3.2. Group preference aggregation

Integrating different preference forms has become a hot topic in GDM [13,16,17,23,36,38,48,49]. The natural idea and most applied method is to unify heterogeneous preference structures into unified formats and implement selection operators to rank alternatives [50]. Other transformation objects include converting utility values and preference orderings [50], utility and multiplicative preference relations [43,47,51], fuzzy and multiplicative preference relations [22,42,46], interval Fuzzy preference relations [25,52] and linguistic preference relations [53,54]. Optimization-based methods were also used to derive collective opinions, instead of transforming various preference structures into unified formats. Existing optimization models include linear goal programming [55], nonlinear optimization [17,42–44], and chi-square optimization [51]. Other preference formats integrated by optimization methods include interval preference relations, intuitionistic fuzzy sets, and trapezoidal fuzzy numbers [43,53]. In this subsection, a preference aggregation method will be proposed based on a vector operation.

We simplify the matrix through normalization. We set $u_{ij} = \frac{u_i}{u_j}$ and obtain the matrix

$$\overline{U} = \left(\overline{u}_{ij}\right)_{n\times n} = \begin{pmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ u_{21} & u_{22} & \dots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n1} & u_{n2} & \dots & u_{nn} \end{pmatrix} \tag{8}$$

Then, we set $o_{ij} = \frac{n-o_i}{n-o_j}$ and normalize the proposed matrix as follows:

$$\overline{O} = (\overline{o}_{ij})_{n\times n} = \begin{pmatrix} o_{11} & o_{12} & \dots & o_{1n} \\ o_{21} & o_{22} & \dots & o_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ o_{n1} & o_{n2} & \dots & o_{nn} \end{pmatrix} \tag{9}$$

We repeat the multiplicative preference relation and fuzzy the preference relation PCM as in (10) and (11):

$$\overline{A} = (\overline{a}_{ij})_{n\times n} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \tag{10}$$

and

$$\overline{B} = (\overline{p}_{ij})_{n\times n} = \begin{pmatrix} p_{11} & p_{12} & \dots & p_{1n} \\ p_{21} & p_{22} & \dots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \dots & p_{nn} \end{pmatrix} \tag{11}$$

Let $\overrightarrow{\cdot}$ indicate a column vector of a matrix in (8)–(11). We assume that $\Lambda = \{\sigma_1, \sigma_2, \dots, \sigma_K\}$ is a finite set of degrees of importance pre-specified by DMs. Let $\Omega = \{\Omega_U, \Omega_O, \Omega_A, \Omega_B\}$ be a set of DMs with utility values, preference ordering, and multiplicative and fuzzy preference relations, respectively. The basic insight of the following optimization is that the collective opinion should be the closest vector to each preference relation.

The optimization model is as follows:

$$\begin{aligned} \textit{Maximize} \quad C = & \sum_{k\in\Omega_U}\sum_{j=1}^{n}\sigma_k\left\langle \overrightarrow{u}^{(k)}{}_j, w\right\rangle + \sum_{k\in\Omega_O}\sum_{j=1}^{n}\sigma_k\left\langle \overrightarrow{o}^{(k)}{}_j, w\right\rangle \\ & + \sum_{k\in\Omega_A}\sum_{j=1}^{n}\sigma_k\left\langle \overrightarrow{a}_j^{(k)}, w\right\rangle + \sum_{k\in\Omega_B}\sum_{j=1}^{n}\sigma_k\left\langle \overrightarrow{p}_j^{(k)}, w\right\rangle \end{aligned} \tag{12}$$

$$\textit{Subject to} \begin{cases} \sum_{i=1}^{n} w_i = 1; \\ 0 \le w_i \le 1. \end{cases}$$

where $\cdot^{(k)}$ indicate the kth decision maker's normalized PCM, and $\overrightarrow{u}_{ij}^{(k)}$, $\overrightarrow{o}_{ij}^{(k)}$, $\overrightarrow{a}_{ij}^{(k)}$ and $\overrightarrow{p}_{ij}^{(k)}$ are the column vectors of matrices (8)–(11), respectively. C is total cosine similarity measure between column vectors of matrices and the collective vector.

The model (12) can be transformed into following (13) :

$$\begin{aligned} \textit{Maximize} \quad & \left\langle \sum_{k\in\Omega_U}\sum_{j=1}^{n}\sigma_k\overrightarrow{u}^{(k)}{}_j, w\right\rangle \\ & + \left\langle \sum_{k\in\Omega_O}\sum_{j=1}^{n}\sigma_k\overrightarrow{o}^{(k)}{}_j, w\right\rangle \\ & + \left\langle \sum_{k\in\Omega_A}\sum_{j=1}^{n}\sigma_k\overrightarrow{a}_j^{(k)}, w\right\rangle + \left\langle \sum_{k\in\Omega_B}\sum_{j=1}^{n}\sigma_k\overrightarrow{p}_j^{(k)}, w\right\rangle \end{aligned} \tag{13}$$

The (13) is also equal to (14):

$$\textit{Maximize} \quad \left\langle \begin{matrix} \sum_{k\in\Omega_U}\sum_{j=1}\sigma_k\overrightarrow{u}^{(k)}{}_j + \sum_{k\in\Omega_O}\sum_{j=1}^{n}\sigma_k\overrightarrow{o}^{(k)}{}_j \\ + \sum_{k\in\Omega_A}\sum_{j=1}^{n}\sigma_k\overrightarrow{a}_j^{(k)} + \sum_{k\in\Omega_B}\sum_{j=1}^{n}\sigma_k\overrightarrow{p}_j^{(k)}, \end{matrix} \quad w\right\rangle \tag{14}$$

Let $v = \sum_{k\in\Omega_U}\sum_{j=1}^{n}\sigma_k\overrightarrow{u}^{(k)}{}_j + \sum_{k\in\Omega_O}\sum_{j=1}^{n}\sigma_k\overrightarrow{o}^{(k)}{}_j,$ $+\sum_{k\in\Omega_A}\sum_{j=1}^{n}\sigma_k\overrightarrow{a}_j^{(k)} + \sum_{k\in\Omega_B}\sum_{j=1}^{n}\sigma_k\overrightarrow{p}_j^{(k)}$, thus the objective function $\langle v, w\rangle$has a unique solution $w = v$ since in a vector space, the maximum cosine similarity is held when the vector is coincident (the angle of two vectors is 0).

Remark. In particular, in practice, we use $p_{ij} = 0.9999$ and $p_{ji} = 0.0001$, instead of $p_{ij} = 1$ and $p_{ji} = 0$, to avoid potential mathematical obstacles. In addition, $n - o_j$ is replaced by 0.0001, when $o_j = n$ in the preference ordering.

4. Consensus reaching model

In this section, we consider the question of reaching consensus in GDM with incomplete heterogeneous preference information, which is a hot topic in information fusion in recent years [16,31,33,42,55,56]. The incomplete preference information in this

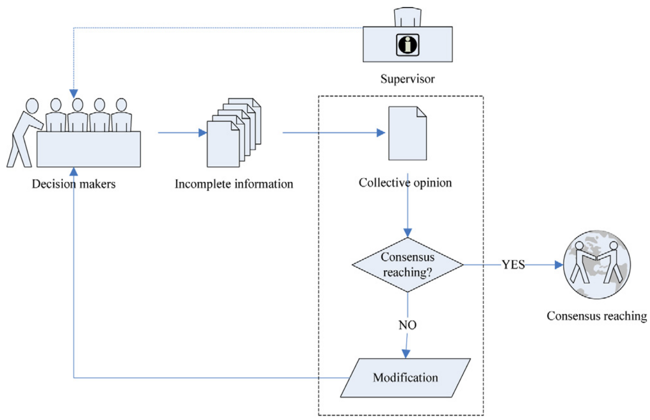

Fig. 2. Consensus process.

paper includes incomplete multiplicative and fuzzy preference relations. The consensus process is implemented as Fig. 2 and the detailed steps will be illustrated as follows:

4.1. Integrating incomplete preference information

We propose an optimization model to integrate varying preference structures with incomplete preference information. In our paper, we still use an optimization model to obtain a priority vector to enable the model to reduce the number of steps and decrease complexity to solve this question.

Let $\Omega' = \{\Omega'_A, \Omega'_B\}$ be a set of DMs with incomplete multiplicative and fuzzy preference relations, and $\Theta = \{\Theta_{A'}, \Theta_{B'}\}$ the subscript set of known values in the PCM, respectively. Without loss of generality, we suppose an entry of DMs with an incomplete PCM for multiplicative and fuzzy preference relations is as follows:

$$a'_{ij} = \begin{cases} a_{ij}, & (i,j) \in \Theta_{A'} \\ x, & (i,j) \notin \Theta_{A'}. \end{cases} \tag{15}$$

and

$$p'_{ij} = \begin{cases} \dfrac{b_{ij}}{1-b_{ij}}, & (i,j) \in \Theta_{B'} \\ x, & (i,j) \notin \Theta_{B'}. \end{cases} \tag{16}$$

As in (10) and (11), we normalize the incomplete PCM, using $\overline{a}'_{ij}$ and $\overline{p}'_{ij}$ as normalized entries of (15) and (16).

$$\overline{a}'_{ij} = \begin{cases} \dfrac{a_{ij}}{\sqrt{\sum_i a_{ij}^2}}, & (i,j) \in \Theta_{A'} \\ x, & (i,j) \notin \Theta_{A'}. \end{cases} \tag{17}$$

$$\overline{p}'_{ij} = \begin{cases} \dfrac{p'_{ij}}{\sqrt{\sum_i {p'_{ij}}^2}}, & (i,j) \in \Theta_{B'} \\ x, & (i,j) \notin \Theta_{B'}. \end{cases} \tag{18}$$

We construct a cosine similarity measure-based optimization model for incomplete preference information. The basic insight is that the column vector with known entries and the corresponding project vector of the collective opinion in the same position have a maximum similarity degree, which is $Max \sum_i \overline{a}'_{ij}\overline{w}_i$, $(i,j) \in \Theta_{A'}$ and $Max \sum_i \overline{p}'_{ij}\overline{w}_i$, $(i,j) \in \Theta_{B'}$ where $\overline{a}'_{ij}$ and $\overline{p}'_{ij}$ is denoted in Eq. (17) and (18). For example, assume one of the column vectors of the PCM is $\overrightarrow{a}_j = (a_{1j}, a_{2j}, \ldots, a_{mj}, x, \ldots, x)^T$, then $\langle \overrightarrow{a}_j, w \rangle = \frac{\sum_{i=1}^m a_{ij}w_i}{\sqrt{\sum_{i=1}^m a_{ij}^2}\sqrt{\sum_{i=1}^m w_i^2}}$. In geometrical insight, the maximum similarity degree should be kept between the column vector with non-missing values and the corresponding project vector of collective preference opinion in the subspace.

In this case, the cosine similarity optimization model can be formulated as follows:

$$\begin{aligned} \text{Maximize} \quad & \left\langle \sum_{k\in\Omega_U}\sum_{j=1}^n \sigma_k \overrightarrow{u}_j^{(k)}, w \right\rangle + \left\langle \sum_{k\in\Omega_O}\sum_{j=1}^n \sigma_k \overrightarrow{o}_j^{(k)}, w \right\rangle \\ & + \left\langle \sum_{k\in\Omega_A}\sum_{j=1}^n \sigma_k \overrightarrow{a}^{(k)}_i, w \right\rangle + \left\langle \sum_{k\in\Omega_{A'}}\sum_{(i,j)\in\Theta_{A'}} \sigma_k \overrightarrow{a}'^{(k)}_j, w \right\rangle \\ & + \left\langle \sum_{k\in\Omega_B}\sum_{j=1}^n \sigma_k \overrightarrow{p}^{(k)}_j, w \right\rangle + \left\langle \sum_{k\in\Omega'_B}\sum_{(i,j)\in\Theta_{B'}} \sigma_k \overrightarrow{p}'^{(k)}_j, w \right\rangle \end{aligned} \tag{19}$$

$$\text{Subject to} \begin{cases} \sum_{i=1}^n w_i = 1; \\ 0 \le \overline{w}_i \le 1. \end{cases}$$

The solution can also be solved using the same method as (13). The result is as follows (20):

$$\begin{aligned} w = & \sum_{k\in\Omega_U}\sum_{j=1}^n \sigma_k \overrightarrow{u}_j^{(k)} + \sum_{k\in\Omega_O}\sum_{j=1}^n \sigma_k \overrightarrow{o}_j^{(k)} + \sum_{k\in\Omega_A}\sum_{i,j=1}^n \sigma_k \overrightarrow{a}_j^{(k)} \\ & + \sum_{k\in\Omega_{A'}}\sum_{(i,j)\in\Theta_{A'}} \sigma_k \overrightarrow{a}'^{(k)}_j + \sum_{k\in\Omega_B}\sum_{i,j=1}^n \sigma_k \overrightarrow{p}_j^{(k)} + \sum_{k\in\Omega'_B}\sum_{(i,j)\in\Theta_{B'}} \sigma_k \overrightarrow{p}'^{(k)}_j \end{aligned} \tag{20}$$

4.2. Consensus measurement

Consensus measure and control is used to judge whether the consensus has been reached in GDM. The "soft" index is most widely used method to evaluate the consensus degree in GDM. It can be divided into two classes. The one is measured by a total "distance" between collective opinion and individual preference relations, for instances, the position deviation of individual priority vector and group opinion [23], ordinal consensus degree and cardinal consensus degree [9]. The other one is the total deviation among individual preference relations [13,24]. Other common indexes include Geometric Consistency Index (Escobar et al. 2004), and extend Geometric Consistency Index [57] and intuitionistic measurement for consensus reaching [8]. The mathematical properties of consensus index were also discussed in [1, 58]. However, the existing methods are not very applicable to decision surroundings with incomplete heterogeneous preference information (the detailed analysis will be shown in Section 6.1).

In our model, a similarity consensus measure (*SCD*) is established to judge consensus degree (see the equation in Box I). where w is the collective opinion in GDM, m is the number of DMs and n is the size of alternatives.

It is clear that the more *SCD* values obtained, the higher the consensus degree will be. The *SCD* is also a weighted average result considering impacts from different DMs.

4.3. Feedback modification

In fact, consensus reaching cannot come to an end at once time as it may be difficult to get all DMs to accept the collective opinion. (In other words, some individual preference relations do not have higher similarity to the collective opinion). Thus, feedback information about collective opinion needs to turn back to DMs in order to persuade DMs to change part of their judgments based on referred information in GDM [59,60].

The modification vector will move toward to collective opinion compared to the original vector in this step so that the consensus

$$SCD = \frac{1}{mn}\left\langle \begin{array}{l} \sum\limits_{k\in\Omega_U}\sum\limits_{j=1}^{n}\sigma_k \overrightarrow{u}_j^{(k)} + \sum\limits_{k\in\Omega_O}\sum\limits_{j=1}^{n}\sigma_k \overrightarrow{o}_j^{(k)} + \sum\limits_{k\in\Omega_A}\sum\limits_{i,j=1}^{n}\sigma_k \overrightarrow{a}^{(k)}_j \\ + \sum\limits_{k\in\Omega_{A'}}\sum\limits_{(i,j)\in\Theta_{A'}}\sigma_k \overrightarrow{a}'^{(k)}_j + \sum\limits_{k\in\Omega_B}\sum\limits_{i,j=1}^{n}\sigma_k \overrightarrow{p}^{(k)}_j + \sum\limits_{k\in\Omega'_B}\sum\limits_{(i,j)\in\Theta_{B'}}\sigma_k \overrightarrow{p}'^{(k)}_j \end{array}, w \right\rangle \tag{21}$$

Box I.

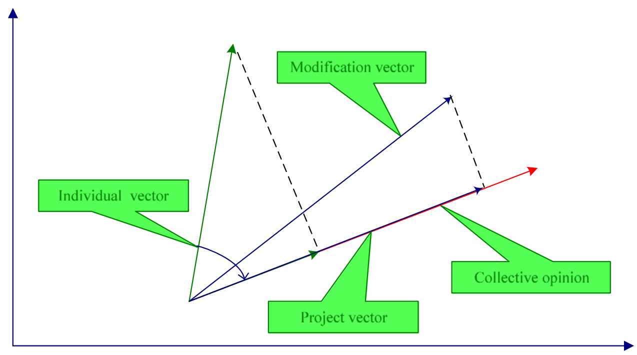

Fig. 3. Modification principle of feedback information.

degree will increase along with the vector transformation. That is, after modification, the column vector should be located in "middle" of individual preference and collective opinion (Fig. 3).

Without loss of generality, let $\alpha = \left(a_1, a_{2,}, \ldots, a_n\right)^T$ and $\beta = \left(b_1, b_{2,}, \ldots, b_n\right)^T$ be two given unit vectors. Assume α rotates to β, and $\gamma = (r_1, r_2, \ldots, r_n)^T$ transformed vector from α. The project vector α toward to β is $c\beta$ and the following dot product holds:

$$(\alpha - c\beta)\cdot\beta = 0 \tag{22}$$

Thus, $c = \frac{\alpha\cdot\beta}{\beta\cdot\beta}$. Then the project vector α toward to β (note $\Pr_\beta oj(\alpha)$) is as follows:

$$\Pr_\beta oj(\alpha) = \left(\frac{\alpha\cdot\beta}{\beta\cdot\beta}\right)\beta \tag{23}$$

In a similar way, the project vector of transformed vector γ is the following:

$$\Pr_\beta oj(\gamma) = \left(\frac{\gamma\cdot\beta}{\beta\cdot\beta}\right)\beta \tag{24}$$

Then,

$$\begin{aligned}\left\langle \Pr_\beta oj(\alpha) - \Pr_\beta oj(\gamma), \Pr_\beta oj(\beta)\right\rangle &= \left\langle\left(\frac{(\alpha-\gamma)\cdot\beta}{\beta\cdot\beta}\right)\beta, \beta\right\rangle \\ &= \left(\frac{(\alpha-\gamma)\cdot\beta}{\beta\cdot\beta}\right)\langle\beta,\beta\rangle \\ &= \left\|\Pr_\beta oj(\alpha) - \Pr_\beta oj(\gamma)\right\|\end{aligned} \tag{25}$$

And

$$(\alpha-\gamma)\cdot\beta = \left\|\Pr_\beta oj(\alpha) - \Pr_\beta oj(\gamma)\right\|. \tag{26}$$

Therefore, the vector equation $(\alpha-\gamma)/\left\|\Pr_\beta oj(\alpha) - \Pr_\beta oj(\gamma)\right\| = \beta$ is one of the necessary conditions in the transformation process, and the similarity can be corrected by the vector scale invariant as follows:

$$0 \le |a_i - r_i| \le |a_i - b_i| \tag{27}$$

Thus, $Min\{a_i, b_i\} \le r_i \le Max\{a_i, b_i\}$ can be deduced as a necessary condition of (27).

The following relation also holds from (27):

$$Min\{a_i, b_i\}/Max\{a_j, b_j\} \le r_i/r_j \le Max\{a_i, b_i\}/Min\{a_j, b_j\} \tag{28}$$

Based on above result, we provide corresponding forms for different preference relations. Clearly, utility value and preference ordering can be directly modified by means of group preference [9].

Utility value: Let $u_i^{(k,t+1)}$ be $t+1$ round modification, then $u_i^{(G,t)} = w_i^{(t)}\sum_{k=1}^{n} u_i^{(k,t)}$, where $u_i^{(G,t)}$ is collective at t round. The modification can be selected as follows:

$$u_i^{(k,t+1)} \in [\min\{u_i^{(G,t)}, u_i^{(k,t)}\}, \max\{u_i^{(G,t)}, u_i^{(k,t)}\}] \tag{29}$$

Preference ordering: Let $o_i^{(k,t+1)}$ be $t+1$ round modification, then $o_i^{(G,t)}$ should be t if the $w_i^{(t)}$ is tth largest value in collective opinion. The modification interval is as follows:

$$o_i^{(k,t+1)} \in [\min\{o_i^{(G,t)}, o_i^{(k,t)}\}, \max\{o_i^{(G,t)}, o_i^{(k,t)}\}] \tag{30}$$

However, multiplicative and fuzzy preference relation is PCM and has complex comparison decision-making, the modification need to quickly convergent to collective opinion. For this purpose, the modification interval of each value in individual preference relation can be set up with row average vector of PCM.

Multiplicative preference relation: Let $\overline{a}_i^{(k,t)} = \sum_j a_{ij}^{(k,t)}, i = 1, 2, \ldots, n$ be the average mean of a row vector. The existing preference values can be modified as follows:

$$\begin{aligned} & a_{ij}^{(k,t+1)} \in [Min\{Max\{Min\{\overline{a}_i^{(k,t)}, w_i^{(t)}\} \\ & /Max\{\overline{a}_j^{(k,t)}, w_j^{(t)}\}, \overline{a}_i^{(k,t)}\}, a_{ij}^{(k,t)}\}, \\ & Max\{Min\{Max\{\overline{a}_i^{(k,t)}, w_i^{(t)}\} \\ & /Min\{\overline{a}_j^{(k,t)}, w_j^{(t)}\}, \overline{a}_i^{(k,t)}\}, a_{ij}^{(k,t)}\}], \end{aligned} \quad i > j \tag{31}$$

Fuzzy preference relation: Let $\overline{p}_i^{(k,t)} = \sum_j p_{ij}^{(k,t)}, i = 1, 2, \ldots, n$ be the average mean of a row vector. The fuzzy preference relation can be modified on account of the following interval:

$$\begin{aligned} & p_{ij}^{(k,t+1)} \in [Min\{Max\{Min\{\overline{p}_i^{(k,t)}, w_i^{(t)}\} \\ & /Max\{1+\overline{p}_j^{(k,t)}, 1+w_j^{(t)}\}, \overline{p}_i^{(k,t)}\}, p_{ij}^{(k,t)}\}, \\ & \{Max\{Min\{\overline{p}_i^{(k,t)}, w_i^{(t)}\} \\ & /Min\{1+\overline{p}_j^{(k,t)}, 1+w_j^{(t)}\}, \overline{p}_i^{(k,t)}\}, p_{ij}^{(k,t)}\}], \end{aligned} \quad i > j \tag{32}$$

4.4. Weighting process

To assure a convergence of CRP, a weighting function with respect to the similarity measure is proposed to improve the importance of being "close" to collective opinion while it can decrease the influence of those DMs that are "far away" from collective opinion [61]. The updated weights can be measured by means of relative distance toward to total similarity of the individual column vector and collective opinion at iteration.

$$\sigma_k{}^{(t+1)} = \sigma_k{}^{(t)} \frac{\left\langle P_k{}^{(t)}, P_C{}^{(t)} \right\rangle}{\sum_{j=1}^{m} \left\langle P_j{}^{(t)}, P_C{}^{(t)} \right\rangle}, k = 1, 2, \ldots, m \tag{33}$$

where $\sigma_k{}^{(t)}$ is weight of kth DM at t-th iteration, $P_C{}^{(t)}$ is the collective opinion in tth iteration, and $P_j{}^{(t)}$ is row mean column of jth DM at tth iteration.

It is clear that weights become larger when the individual preference relation moves toward to collective since the cosine similarity increases as the angle of two vectors is reduced.

In sum, the consensus reaching can be implemented in following steps:

Step 1: integrate incomplete heterogeneous preference relations (20);

Step 2: measure consensus degree and judge whether or not reach preset agreement threshold (21). If the consensus cannot be reached, then go to step 3.

Step 3: modify preference value in terms of feedback information corresponding to different forms (29)–(32).

Step 4: update weights and go to step1.

5. Illustrative examples

In this section, we present numeric examples to validate the proposed model and show the detailed consensus building process.

5.1. Examples in literatures

Example 1. Integrating four different preference structures. This example is studied in Chiclana et al. [36], Xu et al. [16], and Fan et al. [31]. We let $\Pi = \{DM_1, DM_2, \ldots, DM_K\}$ be the K^{th} DM and assume that the importance degrees are equal for all the DMs. The varying preference formats are as follows.

The former two DMs provide utility values with the formats DM_1 and DM_2, where

$$DM_1 = \left\{u_i \middle| i = 1, 2, 3, 4\right\} = \{0.5, 0.7, 1.0, 0.1\}$$
$$DM_2 = \left\{u_i \middle| i = 1, 2, 3, 4\right\} = \{0.7, 0.9, 0.6, 0.3\}$$

and with the ranking of alternatives $A_3 \succ A_2 \succ A_1 \succ A_4$ and $A_2 \succ A_1 \succ A_3 \succ A_4$, respectively.

The third and fourth DMs provide preference orderings, with the preference structures DM_3 and DM_4, where

$$DM_3 = \left\{o_i \middle| i = 1, 2, 3, 4\right\} = \{3, 1, 4, 2\}$$
$$DM_4 = \left\{o_i \middle| i = 1, 2, 3, 4\right\} = \{2, 3, 1, 4\}$$

The fifth and sixth DMs express their preference information in terms of a multiplicative preference relation as DM_5 and DM_6, where

$$DM_5 = \begin{pmatrix} 1 & 1/7 & 1/3 & 1/5 \\ 7 & 1 & 3 & 2 \\ 3 & 1/3 & 1 & 1/2 \\ 5 & 1/2 & 2 & 1 \end{pmatrix}$$

$$DM_6 = \begin{pmatrix} 1 & 3 & 1/4 & 5 \\ 1/3 & 1 & 2 & 1/3 \\ 4 & 1/2 & 1 & 2 \\ 1/5 & 3 & 1/2 & 1 \end{pmatrix}$$

The last two DMs provide their preference formats by fuzzy preference relation. The fuzzy PCM is DM_7 and DM_8, respectively.

$$DM_7 = \begin{pmatrix} 0.5 & 0.1 & 0.6 & 0.7 \\ 0.9 & 0.5 & 0.8 & 0.4 \\ 0.4 & 0.2 & 0.5 & 0.9 \\ 0.3 & 0.6 & 0.1 & 0.5 \end{pmatrix}$$

$$DM_8 = \begin{pmatrix} 0.5 & 0.5 & 0.7 & 1 \\ 0.5 & 0.5 & 0.8 & 0.6 \\ 0.3 & 0.2 & 0.5 & 0.8 \\ 0 & 0.4 & 0.2 & 0.5 \end{pmatrix}$$

The ranking of alternatives is $A_2 \succ A_3 \succ A_1 \succ A_4$ using our methods, the same result as that obtained by using existing methods [16,31,36,62]. However, our result has a higher cosine similarity value than other methods. This indicates that our method has a higher consistent for each DM than the other models (in Table 2). The *SCD* measure also shows that our method also has the highest consensus degree compared to other models.

Example 2. This example includes incomplete multiplicative and fuzzy preference relations, which are investigated in Xu et al. [16]. The example has six DMs from different areas. They evaluate six suppliers based on overall cost criteria. The different DMs have the same importance degree. The preference formats are as follows:

$$DM_1 = \{0.15, 0.10, 0.30, 0.20, 0.35, 0.40\};$$
$$DM_2 = \{4, 6, 5, 3, 2, 1\};$$

$$DM_3 = \begin{pmatrix} 1 & 3 & 2 & 1/3 & 1/5 & 1/8 \\ 1/3 & 1 & 1/3 & 1/4 & 1/7 & 1/9 \\ 1/2 & 3 & 1 & 1/2 & 1/5 & 1/6 \\ 3 & 4 & 2 & 1 & 1/2 & 1/4 \\ 5 & 7 & 5 & 2 & 1 & 1/3 \\ 1/8 & 9 & 6 & 4 & 3 & 1 \end{pmatrix}$$

$$DM_4 = \begin{pmatrix} 1 & 4 & x & 1/4 & 1/2 & 1/6 \\ 1/4 & 1 & 1/5 & x & 1/4 & 1/6 \\ x & 5 & 1 & x & 1/3 & x \\ 4 & x & x & 1 & 1/4 & 1/2 \\ 2 & 4 & 3 & 4 & 1 & x \\ 6 & 6 & x & 2 & x & 1 \end{pmatrix}$$

$$DM_5 = \begin{pmatrix} 0.5 & 0.7 & 0.6 & 0.4 & 0.8 & 0.9 \\ 0.3 & 0.5 & 0.4 & 0.3 & 0.2 & 1 \\ 0.4 & 0.6 & 0.5 & 0.4 & 0.3 & 0.2 \\ 0.6 & 0.7 & 0.6 & 0.5 & 0.4 & 0.3 \\ 0.2 & 0.8 & 0.7 & 0.6 & 0.5 & 0.2 \\ 0.1 & 0 & 0.8 & 0.7 & 0.8 & 0.5 \end{pmatrix}$$

Table 2
Comparative result with other methods.

Approaches	Priority vector and ranking of alternatives	C	SCD
Chiclana et al. [36]	0.5651, 0.7826, 0.6619, 0.4973 $A_2 \succ A_3 \succ A_1 \succ A_4$	26.8227	0.8243
Ma et al. [62]	0.2210, 0.3426, 0.2755, 0.1159 $A_2 \succ A_3 \succ A_1 \succ A_4$	26.9149	0.8274
Xu et al. [16]	0.2210, 0.3426, 0.2827, 0.1537 $A_2 \succ A_3 \succ A_1 \succ A_4$	26.8878	0.8289
Our model	0.2303, 0.3588, 0.2563, 0.1547 $A_2 \succ A_3 \succ A_1 \succ A_4$	***27.0200***	***0.8306***

Remark:the ranking of alternatives of Chiclana et al. [36] was computed using a selection operator. We normalized the vector and then calculated the cosine similarity measure.

$$DM_6 = \begin{pmatrix} 0.5 & 0.8 & x & 0.3 & 0.7 & 1 \\ 0.2 & 0.5 & 0.4 & x & 0.1 & x \\ x & 0.6 & 0.5 & 0.2 & 0.4 & 0.2 \\ 0.7 & x & 0.8 & 0.5 & x & 0.3 \\ 0.3 & 0.9 & 0.6 & x & 0.5 & x \\ 0 & x & 0.8 & 0.7 & x & 0.5 \end{pmatrix}$$

where the first decision maker DM_1 provides their preference with utility values. The second decision maker DM_2 provides their preference with a preference ordering. The third and fourth DMs, DM_3 and DM_4, provide their preference information in terms of multiplicative and incomplete multiplicative preference relations. The last two DMs, DM_3 and DM_4, provide their preference information in terms of fuzzy and incomplete fuzzy preference relations. The initial impacts of DMs are regarded as equally weighted coefficients, i.e. $\sigma_i^{(0)} = 1, i = 1, 2, \ldots, 6$.

Firstly, we can determine the priority vector to find the normalization matrix. The priority vector is

$$w = (0.1719, 0.0591, 0.1100, 0.1609, 0.2296, 0.2684)$$

Therefore, the ranking of alternatives is $A_6 \succ A_5 \succ A_1 \succ A_4 \succ A_3 \succ A_2$. The result is the same as that obtained using the ranking obtained by a nonlinear programming model with a 3-order and larger norm proposed by [16]. However, our model can obtain a higher cosine value, indicating that our model has a higher consistency degree for each DMs (Table 3). Furthermore, our method is far simpler than the nonlinear model in [16], which must be solved using a genetic algorithm that is difficult to implement in management practice.

Secondly, the consensus similarity measure of our model is $SCD = 0.7895$, and the preset threshold should be improved higher than 0.84.

Thirdly, the feedback information will be return to DMs who will be required to modify preference values on account of collective opinion.

For utility value, the transformed modification from collective is $\{0.258, 0.089, 0.165, 0.241, 0.344, 0.403\}$. Hence, the first iteration of DM_1 will be $u_1^{(1,1)}(DM_1) \in [0.15, 0.26]$, $u_2^{(1,1)}(DM_1) \in [0..09, 0.10]$, $u_3^{(1,1)}(DM_1) \in [0..17, 0.30]$, $u_4^{(1,1)}(DM_1) \in [0..20, 0.24]$, $u_5^{(1,1)}(DM_1) \in [0..34, 0.35]$ and $u_6^{(1,1)}(DM_1) \in [0..40, 0.40]$,respectively.

For preference ordering, the collective opinion can be transformed as ordering $\{3, 6, 5, 4, 2, 1\}$. Hence, the DM_2 only needs to consider whether or not compare the alternatives A_1 and A_3 again.

For multiplicative preference relations, the collective opinion should be turn into PCM as follows:

$$a_{ij}^{(G,0)} = \begin{pmatrix} 1 & 3 & 2 & 1 & 1 & 1/2 \\ 1/3 & 1 & 1/2 & 1/3 & 1/4 & 1/5 \\ 1/2 & 2 & 1 & 1/2 & 1/2 & 1/2 \\ 1 & 3 & 2 & 1 & 1/2 & 1/2 \\ 1 & 4 & 2 & 2 & 1 & 1 \\ 2 & 5 & 2 & 2 & 1 & 1 \end{pmatrix}$$

Then the modification interval is provided for DM_3 and DM_4, and they can modify their preference following the same principle.

$$a_{ij}^{(3,1)} = \begin{pmatrix} 1 & 3 & 2 & [1/3, 1] & [1/5, 1] & [1/8, 1/2] \\ & 1 & 1/2 & [1/3, 1/2] & 1/4 & [1/7, 1/5] \\ & & 1 & 1/2 & [1/5, 1/2] & [1/6, 1/2] \\ & & & 1 & 1/2 & [1/4, 1/2] \\ & & & & 1 & [1/3, 1] \\ & & & & & 1 \end{pmatrix}$$

It is a similar process for incomplete multiplicative relation to modify corresponding position to existing value in original PCM.

For fuzzy preference relations and incomplete fuzzy preference relations, the corresponding fuzzy preference relation is the following matrix:

$$p_{ij}^{(G,0)} = \begin{pmatrix} 0.5 & 0.7441 & 0.6099 & 0.5165 & 0.4281 & 0.3904 \\ 0.2559 & 0.5 & 0.3497 & 0.2687 & 0.2047 & 0.1805 \\ 0.3901 & 0.6503 & 0.5 & 0.4059 & 0.3238 & 0.2906 \\ 0.4835 & 0.7313 & 0.5941 & 0.5 & 0.4120 & 0.3748 \\ 0.5719 & 0.7953 & 0.6762 & 0.5880 & 0.5 & 0.4611 \\ 0.6096 & 0.8195 & 0.7094 & 0.6252 & 0.5389 & 0.5 \end{pmatrix}$$

For example, the decision maker DM_5 should be informed to correct some preference values by reconsidering alternatives in terms of the following feedback information. It is similar to the decision maker DM_6.

Table 3
Comparative Results with Xu et al. [16].

Approaches		Priority vector, ranking of alternatives and cosine values	Total cosine value
Xu et al. [16]	$\rho = 1$	0.1314,0.0431,0.0826,0.1653,0.2477,0.3299 $A_6 \succ A_5 \succ A_4 \succ A_1 \succ A_3 \succ A_2$	27.9275
	$\rho = 2$	0.1525,0.0432,0.1038,0.1666,0.2424,0.2915 $A_6 \succ A_5 \succ A_4 \succ A_1 \succ A_3 \succ A_2$	28.3055
	$\rho = 3$	0.1734,0.0432,0.1151,0.1659,0.2311,0.2712 $A_6 \succ A_5 \succ A_1 \succ A_4 \succ A_3 \succ A_2$	28.3894
	$\rho = 4$	0.1878,0.0483,0.1213,0.1744,0.2294,0.2388 $A_6 \succ A_5 \succ A_1 \succ A_4 \succ A_3 \succ A_2$	28.3616
Our model		0.1719,0.0591,0.1100,0.1609,0.2296,0.2684 $A_6 \succ A_5 \succ A_1 \succ A_4 \succ A_3 \succ A_2$	***28.4229***

Remark: The $\rho = 3$ is the best parameter, since its cosine similarity value is the highest in the ranking of alternatives of Xu et al. [16]. Our method has the same ranking with $\rho = 3$ and other norms and obtains a higher cosine similarity measure than all the models in Xu et al. [16].

Table 4
Consensus reaching process by means of Modification.

Iterations	Priority vector, ranking of alternatives	*SCD*
$t = 0$	0.1719,0.0591,0.1100,0.1609,0.2296,0.2684 $A_6 \succ A_5 \succ A_1 \succ A_4 \succ A_3 \succ A_2$	0.7895
$t = 1$	0.1689 ,0.0549 ,0.1098 ,0.1565 ,0.2267 ,0.2832 $A_6 \succ A_5 \succ A_1 \succ A_4 \succ A_3 \succ A_2$	0.8074
$t = 2$	0.1593 ,0.0557 ,0.1119 ,0.1566 ,0.2330 ,0.2835 $A_6 \succ A_5 \succ A_1 \succ A_4 \succ A_3 \succ A_2$	0.8141
$t = 3$	0.1770 ,0.0522 ,0.1276 ,0.1605 ,0.2284 ,0.2543 $A_6 \succ A_5 \succ A_1 \succ A_4 \succ A_3 \succ A_2$	**0.8408**

Remark: In whole process, we assume the preferences of DM_1 and DM_2 hold and the remaining decision makes a modified preference relation.

DM_5

$$= \begin{pmatrix} 0.5 & 0.7 & 0.6 & [0.4, 0.5165] & [0.4281, 0.8] & [0.3904, 0.9] \\ & 0.5 & [0.3497, 0.4] & 0.3 & 0.2 & [0.1805, 1] \\ & & 0.5 & 0.4 & 0.3 & [0.2, 0.2906] \\ & & & 0.5 & 0.4 & [0.3, 0.3748] \\ & & & & 0.5 & [0.2, 0.4611] \\ & & & & & 0.5 \end{pmatrix}$$

Lastly, weights are updated based on the equal weight in the original GDM until consensus reaches the preset degree. Table 4 shows the consensus reaching process using the proposed model. The *SCD* which measures the agreement of GDM is convergent to the preset threshold.

5.2. Example in real-life management issue

Example 3. Urban resettlement is a key public management issue related to social stability, regional economic development, improvement of residents' living conditions and poverty ending [63]. The successful implementation of demolition projects depends on many complex factors, such as politics, economics, sociology and Law [64]. One of the key issues of urban resettlement is the formulation of resettlement plan and the agreement of resettlement plan among residents. However, households have various knowledge backgrounds and social status, thus the individual preferences often have different forms when a set of alternatives are provided to them. In addition, when a decision maker is unable to make a judgment on the comparison between two alternatives, it will give up for evaluation of the preference value among them, resulting in a phenomenon of the missing preference value. Thus, it is a typical incomplete heterogeneous group decision making question.

This example aims to evaluate plans for an urban resettlement project beginning in July 2017. It aims to reset the "69 mail box" residence area built in the 1950s, which is the largest shantytown located in the center of the northern old town at Chengdu, China. This is a commercial project involving households, real estate developers and government representatives who act as mediators in the global management process. In order to ensure the democracy and satisfaction of all participants, the government stipulated that the resettlement plan must be agreed with a higher degree of consensus. Following a preliminary investigation from February 23, 2017 to March 5, 2017, a simulated demolition conference was held on June 18, 2017. In this urban resettlement project, the goal of the GDM process is to propose a resettlement plan or set of plans that takes into account the opinions of all residents and is accepted by residents and real estate developers.

Our example is the decision-making process from the initial plan determination of residents' representatives before the project was officially implemented. The GDM process for this project has four steps. First, real estate developers proposed alternatives and convened several representatives, including a neighborhood committee, random residents' representatives, and a government official to discuss. This project invites 6 members as representatives, and it is necessary to coordinate with the residents during the entire consultation process. Second, all residents thoroughly reviewed the alternatives developed in the first step and expressed their preferences. Third, decision makers evaluate group opinions and then re-evaluate individual preferences based on group opinions. Finally, form preliminary group opinions and return to decision makers. This opinion will be further discussed in the residents' meeting and the simulated demolition meeting if the consensus in GDM is built.

This urban resettlement project has some common features: (1) The project organizer randomly selects some representatives to form a committee, which represents all participants to discuss the initial alternative and the negotiation process to reach a consensus. (2) Representatives involved in decision-making are selected from residents of different strata and industries, and their preferences have different representation forms. (3) There is an incomplete preference relationship, because for some alternatives, decision makers cannot make very precise comparison judgments.

The alternatives are composed of the following 5 plans:

A1: {Buy new house at original address};

A2: {Displacement of new house from original construction area};

A3: {Reform by self-organization};

A4: {Cash compensation};

A5: {Replacement of new house at offsite address}.

A residence committee is established and invited to evaluate the plans of the project. They are asked to provide their preference relations using four different structures in terms of their decision habits. The preference relations are listed as follows:

$$DM_1 = \left\{ 0.8147 \quad 0.9058 \quad 0.127 \quad 0.9134 \quad 0.6324 \right\};$$

$$DM_2 = \left\{ 1 \quad 4 \quad 3 \quad 5 \quad 2 \right\}$$

$$DM_3 = \left\{ \begin{matrix} 1 & 1/4 & 1/2 & 1/4 & 1 \\ 4 & 1 & x & x & 2 \\ 2 & x & 1 & x & 2 \\ 4 & x & x & 1 & 1 \\ 1 & 1/2 & 1/2 & 1 & 1 \end{matrix} \right\}$$

$$DM_4 = \left\{ \begin{matrix} 1 & x & 1 & x & 2 \\ x & 1 & x & 1 & 1/4 \\ 1 & x & 1 & 2 & 1/3 \\ x & 1 & 1/2 & 1 & 1/2 \\ 1/2 & 4 & 3 & 2 & 1 \end{matrix} \right\}$$

$$DM_5 = \left\{ \begin{matrix} 0.5 & 0.4 & 0.6 & 0.4 & 0.3 \\ 0.6 & 0.5 & 0.8 & x & 0.7 \\ 0.4 & 0.2 & 0.5 & 0.4 & x \\ 0.6 & x & 0.6 & 0.5 & 0.6 \\ 0.7 & 0.3 & x & 0.4 & 0.5 \end{matrix} \right\}$$

$$DM_6 = \left\{ \begin{matrix} 0.5 & 0.7 & x & 0.2 & 0.3 \\ 0.3 & 0.5 & 0.4 & x & 0.4 \\ x & 0.6 & 0.5 & 0.2 & 0.5 \\ 0.8 & x & 0.8 & 0.5 & 0.6 \\ 0.7 & 0.6 & 0.5 & 0.4 & 0.5 \end{matrix} \right\}$$

(1) $t = 0$, the collective priority vector of the above preference relation is $w = (0.2117, 0.1784, 0.1535, 0.2368, 0.2197)^T$, and the total consensus degree SCD is 0.7705, which is less than preset consensus degree 0.80. Therefore, feedback information is provided to the DMs and the corresponding matrixes are as follows: $a_{ij}{}^{(3,1)} = \left\{ \begin{matrix} 1 & [1/4, 1] & [1/2, 1] & [1/4, 1] & 1 \\ & 1 & x & x & [1/2, 2] \\ & & 1 & x & [1/2, 2] \\ & & & 1 & 1 \\ & & & & 1 \end{matrix} \right\}$

$$a_{ij}{}^{(4,1)} = \left\{ \begin{matrix} 1 & x & 1 & x & [1, 2] \\ & 1 & x & 1 & [1/4, 1/2] \\ & & 1 & [1/2, 2] & 1/3 \\ & & & 1 & [1/2, 1] \\ & & & & 1 \end{matrix} \right\}$$

$$p_{ij}{}^{(5,1)} = \left\{ \begin{matrix} 0.5 & [0.4, 0.5390] & 0.6 & 0.4 & [0.3.0.4916] \\ & 0.5 & [0.5341, 0.8] & x & [0.4523, 0.7] \\ & & 0.5 & 0.4 & x \\ & & & 0.5 & [0.4185, 0.6] \\ & & & & 0.5 \end{matrix} \right\}$$

$$p_{ij}{}^{(6,1)} = \begin{pmatrix} 0.5 & [0.5390, 0.7] & x & [0.2, 0.4745] & [0.3, 0.4916] \\ & 0.5 & [0.4, 0..5341] & x & [0.4, 0.4526] \\ & & 0.5 & [0.2, 0.4014] & [0.4186, 0.5] \\ & & & 0.5 & [0.5171, 0.6] \\ & & & & 0.5 \end{pmatrix}$$

(2) $t = 1$. The preference relations after modification are as follows:

$$DM_3{}^{(1)} = \left\{ \begin{matrix} 1 & 1/4 & 1/2 & 1 & 1 \\ 4 & 1 & x & x & 1/2 \\ 2 & x & 1 & x & 2 \\ 1 & x & x & 1 & 1 \\ 1 & 2 & 1/2 & 1 & 1 \end{matrix} \right\}$$

$$DM_4{}^{(1)} = \left\{ \begin{matrix} 1 & x & 1 & x & 1 \\ x & 1 & x & 1 & 1/2 \\ 1 & x & 1 & 2 & 1/3 \\ x & 1 & 1/2 & 1 & 1/2 \\ 1 & 2 & 3 & 2 & 1 \end{matrix} \right\}$$

$$DM_5{}^{(1)} = \left\{ \begin{matrix} 0.5 & 0.4 & 0.6 & 0.4 & 0.3 \\ 0.6 & 0.5 & 0.8 & x & 0.5 \\ 0.4 & 0.2 & 0.5 & 0.4 & x \\ 0.6 & x & 0.6 & 0.5 & 0.6 \\ 0.7 & 0.5 & x & 0.4 & 0.5 \end{matrix} \right\}$$

$$DM_6{}^{(1)} = \left\{ \begin{matrix} 0.5 & 0.6 & x & 0.4 & 0.4 \\ 0.4 & 0.5 & 0.4 & x & 0.4 \\ x & 0.6 & 0.5 & 0.4 & 0.5 \\ 0.6 & x & 0.6 & 0.5 & 0.6 \\ 0.6 & 0.6 & 0.5 & 0.4 & 0.5 \end{matrix} \right\}$$

The collective preference opinion in this iteration is $w = (0.2228, 0.1638, 0.1636, 0.2191, 0.2307)^T$ and the consensus degree is 0.8161. In this step, DM_3 agrees to modify its preference $a_{12}{}^{(3,1)} = 1/4$ to $a_{12}{}^{(3,2)} = 1/2$ in terms of the modification

Table 5
Consensus reaching process.

Iterations	Priority vector, ranking of alternatives	*SCD*
$t=0$	0.2117,0.1784,0.1535,0.2368,0.2197 $A_4 \succ A_5 \succ A_1 \succ A_2 \succ A_3$	0.7705
$t=1$	0.2125,0.1694 ,0.1606 , 0.2186 ,0.2387 $A_4 \succ A_5 \succ A_1 \succ A_2 \succ A_3$	0.7778
$t=2$	0.2228, 0.1638, 0.1636, 0.2191,0.2307 $A_5 \succ A_1 \succ A_4 \succ A_3 \succ A_2$	**0.8161**

Remark: In whole process, we assume the preferences of DM_1 and $DM_2$are unchanged and the remaining decision makes modified preference relations.

interval [1/4, 1], and then the decision making is turned into the second iteration.

(3) $t = 2$. The preference relations after modification are as follows:

$$DM_3{}^{(2)} = \begin{Bmatrix} 1 & 1/2 & 1/2 & 1 & 1 \\ 2 & 1 & x & x & 1/2 \\ 2 & x & 1 & x & 2 \\ 1 & x & x & 1 & 1 \\ 1 & 2 & 1/2 & 1 & 1 \end{Bmatrix}$$

$$DM_4{}^{(2)} = DM_4{}^{(1)} = \begin{Bmatrix} 1 & x & 1 & x & 1 \\ x & 1 & x & 1 & 1/2 \\ 1 & x & 1 & 2 & 1/3 \\ x & 1 & 1/2 & 1 & 1/2 \\ 1 & 2 & 3 & 2 & 1 \end{Bmatrix}$$

$$DM_5{}^{(2)} = DM_5{}^{(1)} = \begin{Bmatrix} 0.5 & 0.4 & 0.6 & 0.4 & 0.3 \\ 0.6 & 0.5 & 0.8 & x & 0.5 \\ 0.4 & 0.2 & 0.5 & 0.4 & x \\ 0.6 & x & 0.6 & 0.5 & 0.6 \\ 0.7 & 0.5 & x & 0.4 & 0.5 \end{Bmatrix}$$

$$DM_6{}^{(2)} = DM_6{}^{(1)} = \begin{Bmatrix} 0.5 & 0.6 & x & 0.4 & 0.4 \\ 0.4 & 0.5 & 0.4 & x & 0.4 \\ x & 0.6 & 0.5 & 0.4 & 0.5 \\ 0.6 & x & 0.6 & 0.5 & 0.6 \\ 0.6 & 0.6 & 0.5 & 0.4 & 0.5 \end{Bmatrix}$$

Following above steps after the preference modification, the consensus degree can be improved using iterative preference modification (Table 5).

Judging from the decision-making process, alternatives 1, 4 and 5 are the three main options that the decision makers focus on. Three types of alternatives represent the main opinions: original site resettlement, off-site resettlement and monetary compensation. Although the original monetary compensation(A4) was the mainstream solution in many urban resettlement projects, after the group discussions of the representatives, the final consensus opinion was off-site resettlement (A5), original address demolition(A1), and then monetary compensation. In short, a replacement of house property right without extra costs (combining A5 and A1) is the optimal choice for the project to achieve consensus.

6. Convergence test and comparison analysis

In this section, we aim to test the convergence of the above consensus reaching model. Moreover, the various consensus measurements and consensus building methods for the GDM with incomplete heterogeneous information is also discussed in following contexts.

6.1. Discussion of consensus measurement

The consensus degree aims to measure the agreement for collective opinion among all the DMs. Generally, it is computed by measuring the distance between individual preference values and collective preference values [8,9,23,59,65,66], in addition, some authors also proposed a measurement by mean of the total deviation among individual preference values [13,18,24,61].

However, these measurements are not effective when dealing with incomplete heterogeneous preference relations. Firstly, it is difficult to directly compute the total deviation among heterogeneous preference relations, such as the deviation between preference ordering and multiplication preference relation. Secondly, the priority vector from the individual preference is not easy to get since the preference relation is not complete.

Thus, a simple and easy method to measure consensus degree for incomplete heterogeneous information needs to be developed. We proposed a *SCD* measure, which is a total similarity degree between column vector and collective preference (or project vector of a preference relation in subspace). Compared to the insights measuring the distance between individual preference values and collective preference values, the *SCD* does not need to compute the priority vector from each individual preference value and only needs to obtain the collective opinion using model (19) and Eq. (20). In addition, the missing entries in incomplete preference relations need not be forecasted or filled to evaluate the consensus degree (Table 6).

6.2. Convergence simulation

To test the convergence of the proposed consensus reaching model, we have simulated this experiment in order to show the effectiveness of our model.

In this experiment, we investigate the trend of CRP with different degrees of missing information. The preference relations are randomly generated and eliminate part of the preference values. In each preference modification, we simulate it 1000 times, and then serve average values as outputs. The simulation algorithm is listed in Table 7.

The results of the experiments are shown in Table 8. It can be concluded that our proposed model can improve consensus building for GDM with incomplete heterogeneous preference information. In addition, the degree of missing information also influences the consensus reaching in GDM.

Generally, if we preset threshold *cr*=0.90, the degree can reach consensus in less than 9 iterations and the more information the preference relations provide, the faster the consensus is built.

It can also be observed from Fig. 4 that GDM with 10% missing information has the fastest convergence speed to reach *cr*. Moreover, the preference relations with half the preference values need the longest iteration times to reach the preset consensus.

6.3. Comparisons analysis of different methods

This subsection compares the main properties of several representative methods by means of a qualitative analysis and the results can be found in Table 9.

Table 6
Comparisons of the representative consensus measurements.

Measurements	Methods	Drawbacks	Derive priority vector of individual preference	Use to heterogeneous preference information	Use to incomplete preference information
ordinal consensus degree [59],	Order deviation between individual preference and collective opinion	Must derive priority vector from incomplete preference matrix	✓	✓	✓
cardinal consensus degree[57]	Weights deviation between individual preference and collective opinion		✓	✓	✓
CR[13][24]	Distance among individual preference relations	only used for homogeneous preference matrix	×	×	×
Geometric Compatibility Index[65]	Logarithm deviation between individual preference value and	only used for multiplicative preference matrix; Non-monotonous	×	×	✓
SCD	Similarity between individual preference and collective opinion		×	✓	✓

Table 7
Algorithm for convergence test.

Algorithm1

Inputs: m, n, consensus threshold cr, degree of missing information IR
Outputs: SCD, maximum iteration times i.

1.	Generate m preference relations	# Satisfy individual consistency.
2.	Generate incomplete preference relations	#Eliminate preference values in terms of $1/2 \times n(n-1) \times IR$
3.	For i=1:1:1 do	
4.	Compute collective opinion	# equation (20)
5.	Compute consensus degree SCD	# equation (21)
6.	if $SCD<cr$ do	
7.	Compute feedback information	
8.	Randomly select values from modification interval	# equations (31) and (32)
9.	Weighting process	# equation (33)
10.	Continue	
11.	end	
12.	break	
13.	end	
14.	Return i and SCD	

Table 8
Convergence reaching process.

$m = 52, n = 5$									
IR(%)	SCD, $cr = 0.90$								
	1	2	3	4	5	6	7	8	9
10	0.8748	0.9002							
20	0.8253	0.8565	0.8783	0.9109					
30	0.8047	0.8341	0.8442	0.8633	0.8973	0.9112			
40	0.7866	0.7996	0.8154	0.8355	0.8656	0.8878	0.8990	0.9201	
50	0.7630	0.7795	0.7976	0.8166	0.8477	0.8649	0.8788	0.8905	0.9098

It is observed that our methods can deal with incomplete and heterogeneous preference relations at the same time. Although the same situation can be handled by [26], our methods can obtain an analytical solution of the optimization model so that the computation process can be quickly solved by optimization (19).

Moreover, our method need not derive the individual priority vector from incomplete heterogeneous preference relations to compute consensus degree and collective opinion, and thus it can efficiently reduce the complexity of CRP. The geometrical insights used to integrate individual preference relations and measure consensus degree can also help organize and clearly understand the CRP, thus it is a more operational and explainable consensus building model.

In addition, we compare the proposed methods using a quantitative analysis. By analyzing the algorithm complexity of different methods, the theoretical comparison of the calculation speed of different methods is implemented. Due to our method aims to deal with incomplete preference relations, we compare two known methods that can handle this situation (Zhang and Guo, [17] and Cheng et al. [26]) with our method in theoretical and numerical experiments.

A consensus reaching procedure includes three flows: Aggregation and feedback adjustments of preference relations, and then

Table 9
Comparisons of the representative consensus building methods.

Measurements	Used to incomplete preference relation	Filling missing values	Used to heterogeneous preference relation	Transformation into unified preference structure	Derive individual priority vector	Analytical solution in optimization	Iterative consensus reaching
Herrera-viedma et al. [23]	×	×	✓	✓	✓	×	✓
Dong and Zhang, [9]	×	×	✓	×	✓	×	✓
Zhang and Guo, [17]	✓	×	✓	×	✓	×	×
Cheng et al. [26]	✓	✓	×	×	✓	×	×
Li et al. [12]	×	×	✓	×	✓	×	✓
Our method	✓	×	✓	×	×	✓	✓

Table 10
Algorithm complexity.

Algorithm	Complexity	Remark
Zhang and Guo, [17]	At least $O(n^3)$	Search algorithm and matrix factorization in optimization model. whether triangular factorization or the Cholesky decomposition, is $O(\frac{2}{3}n^3)$ and $O(\frac{1}{3}n^3)$, and the complexity of $n \times n$ matrix multiplication is $O(n^3)$. The complexity of naive matrix multiplication is $O(mnp)$ for matrix $A_{m\times n}$ and $B_{n\times p}$.
Cheng et al. [26]	$O(3n^2 + n!)$	The time complexity of two-layer loop statement in their algorithms is $3n \times n$. The time complexity of estimation function and iterative assignment is $n!$.
Our methods	$O(n^2 + 2n)$	The core calculations of our method are unitization, summation, and normalization of vectors. The complexity of $\left\Vert \vec{p}_j \right\Vert = (\vec{p}_{n\times 1}{}^T \vec{p}_{1\times n})^{1/2}$ for vector $\vec{p}_j$ is $O(n^2)$.

Remark: Similar to Cheng et al. [26], Li et al. [12] and Dong and Zhang, [9] also used a similar aggregation operator, thus, their time complexity is not much different when dealing with complete preference relation, but it is not discussed and undetermined in their methods if the preference relation is incomplete.

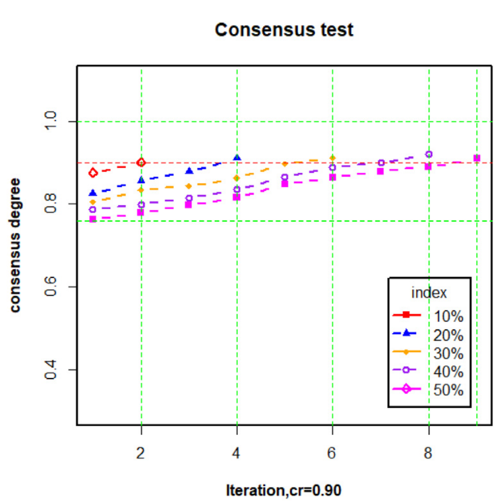

Fig. 4. The trend of convergence reaching process.

aggregation of updated preference relationships. Evaluating missing value is needed in Cheng et al. [26] and thus this procedure is also included in their algorithm analysis. The time complexity is listed at following Table 7:

The Table 10 shows that our methods can obtain the most time-saving complexity because an analytical solution in optimization can be solved. Although Cheng et al. [26] also have a lower time complexity, but the need of their method to evaluate missing preference values complicates the decision-making process. We must highlight that the previous different methods target different decision environments and research objects. For example, our method targets numerical heterogeneous incomplete preference relations, Zhang and Guo [17] aims to handle uncertain (interval and linguistic) heterogeneous incomplete preference relations, and incomplete linguistic preference relations are studied in Cheng et al. [26]. Therefore, it is difficult to compare the performance of the different algorithms on same data scenario through a numerical example or simulation environment. Through qualitative analysis and time complexity analysis in this subsection, we can show the advantages of our proposed method.

7. Conclusions

GDM with heterogeneous and incomplete preference relations has gained an increasing attention in real-life decision questions because DMs always have different decision habits and may not give their opinions on some more difficult alternative comparisons to evaluate.

For utility values, preference orderings, and (incomplete) multiplicative preference relations and (incomplete) additive preference relations, we proposed a consensus reaching process including preference aggregation, consensus measure, preference modification and weighting process. A similarity-based optimization can be directly used to integrate incomplete heterogeneous preference relation without the need to fill missing information and transform heterogeneous information into a uniform form. We proposed consensus measure which is total similarity degree between individual preference and the collective opinion. Following this, we propose and prove a preference adjustment mechanism that satisfies convergence. We illustrate some examples from the existing literatures and real-life decision question to show the operation process of the proposed method. A qualitative comparison and algorithm analysis show our proposed method is operability and simplicity because it can avoid preference information loss or disturbance in the transformation process and decrease the computing complexity of computation.

Further research may include exploring other potential questions in fields where large-scale participants are involved in GDM. For example, it needs to consider complex decision behavior in decision making, such as non-cooperative behavior and self-confidence. The influences of possible social relationships between DMs on decision consensus building are also a valuable research issue. In addition, the development of large-scale group decision support systems and management applications is also worthy of attention.

CRediT authorship contribution statement

Gang Kou: Methodology, Software, Supervision. **Yi Peng:** Methodology, Writing - review & editing, Project administration, Funding acquisition. **Xiangrui Chao:** Writing - original draft, Investigation, Conceptualization. **Enrique Herrera-Viedma:** Visualization, Validation. **Fawaz E. Alsaadi:** Formal analysis.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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