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# Modularly equidistant numerical semigroups

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**Abstract:** If $S$ is a numerical semigroup and $s \in S$, we denote by $\text{next}_S(s) = \min\{x \in S \mid s < x\}$. Let $a$ be an integer greater than or equal to two. A numerical semigroup is equidistant modulo $a$ if $\text{next}_S(s) - s - 1$ is a multiple of $a$ for every $s \in S$. In this note, we give algorithms for computing the whole set of equidistant numerical semigroups modulo $a$ with fixed multiplicity, genus, and Frobenius number. Moreover, we will study this kind of semigroups with maximal embedding dimension.

**Key words:** Embedding dimension, Frobenius number, genus, multiplicity, modularly equidistant numerical semigroups, MED semigroups, numerical semigroup

## 1. Introduction

Let $\mathbb{Z}$ be the set of integers an let $\mathbb{N} = n \in \mathbb{Z} \mid n \geqslant 0$ the set of nonnegative integers. A submonoid of $(\mathbb{N}, +)$ is a subset of $\mathbb{N}$ closed addition and containing $0$. A submonoid of $(\mathbb{N}, +)$ with finite complement in $\mathbb{N}$ is a numerical semigroup.

Given a numerical semigroup $S$ and $s \in S$, we denote by $\text{next}_S(s) = \min\{x \in S \mid s < x\}$. For $a \in \mathbb{N} \setminus \{0, 1\}$, we say that $S$ is an equidistant numerical semigroup modulo $a$ if $\text{next}_S(s) - s - 1$ is a multiple of $a$ for every $s \in S$.

We denote by

$$\mathsf{E}(a) = \{S \mid S \text{ is equidistant numerical semigroup modulo } a\}.$$

Our aim in this note is to study of this kind of numerical semigroups.

This work is a generalization of the study of the parity numerical semigroups [3]. Indeed a numerical semigroup $S$ is parity if $s + \text{next}_S(s)$ is odd for every $s \in S$. Clearly $s + \text{next}_S(s)$ is odd if and only if $\text{next}_S(s) - s - 1$ is a multiple of $2$. Therefore, the parity numerical semigroups are equidistant numerical semigroups modulo $2$.

A numerical semigroup $S$ is prefect if $\{x-1, x+1\} \subseteq S$ then $x \in S$ (see for instance [4, 5]). Observe that every equidistant modularly numerical semigroup is a perfect numerical semigroup. In fact, if $S$ is equidistant numerical semigroup modulo $a$ and $\{x-1, x+1\} \subseteq S$ then $x \in S$, because otherwise $\text{next}_S(x-1) = x+1$ and then $\text{next}_S(x-1) - (x-1) - 1 = x + 1 - x + 1 - 1 = 1$ that is not a multiple of $a$.

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We briefly outline the structure of this note. In Section 2, we will order the elements of $\mathsf{E}(a)$ in a tree rooted in $\mathbb{N}$. We will characterize the sons of a vertex and this will allow us to build recursively the elements of $\mathsf{E}(a)$.

If $S$ is a numerical semigroup, then $\mathrm{m}(S) = \min(S\backslash\{0\})$, $\mathrm{F}(S) = \max\{z \in \mathbb{Z} \mid z \notin S\}$ and $\mathrm{g}(S) = \mathrm{card}(\mathbb{N}\backslash S)$ (cardinality of $\mathbb{N}\backslash S$) are three important invariants of $S$ called multiplicity, Frobenius number, and genus of $S$, respectively. See for instance [6] and [2] to understand the importance of the study of these invariants.

As a consequence of the results of Section 2, we present an algorithmic procedure to compute all elements in $\mathsf{E}(a)$ with a given multiplicity and genus in Section 3. In Section 4, we get an algorithm for computing the set $\mathsf{E}(a)$ with given Frobenius number and we also characterize the maximal elements of this set.

An interesting class of numerical semigroups theory are the numerical semigroups with maximal embedding dimension (see [2]). In Section 5, we characterize the elements in $\mathsf{E}(a)$ with maximal embedding dimension.

## 2. The tree associated to $\mathsf{E}(a)$

In this section, we characterize the set $\mathsf{E}(a)$, for given $a$ integer greater than or equal to 2. We see how this set can bee arranged in a tree.

**Lemma 2.1** *If* $S \in \mathsf{E}(a)$ *then* $\mathrm{m}(S) \equiv 1 (mod \;\; a)$.

**Proof** Clearly, $\mathrm{next}_S(0) = \mathrm{m}(S)$ and thus $\mathrm{m}(S) - 0 - 1$ is a multiple of $a$. Therefore, we have that $\mathrm{m}(S) \equiv 1(\mathrm{mod} \;\; a)$. □

It is clear that $\mathbb{N}$ is a numerical semigroup that belongs to $\mathsf{E}(a)$. If $S \in \mathsf{E}(a)$ and $S \neq \mathbb{N}$ then $\mathrm{m}(S) \geq 2$, and so there exists $k \in \mathbb{N}\backslash\{0\}$ such that $\mathrm{m}(S) = 1 + ka$. Therefore, $a < \mathrm{m}(S)$.

**Lemma 2.2** *If* $S \in \mathsf{E}(a)\backslash\{\mathbb{N}\}$*, then* $\{\mathrm{F}(S), \mathrm{F}(S) - 1, \ldots, \mathrm{F}(S) - (a-1)\} \subseteq \mathbb{N}\backslash S$.

**Proof** Since $\mathrm{F}(S) \geq \mathrm{m}(S) - 1 \geq a$ we have that $\{\mathrm{F}(S), \mathrm{F}(S) - 1, \ldots, \mathrm{F}(S) - (a-1)\} \subseteq \mathbb{N}\backslash\{0\}$. Assume that there exists $i \in \{1, \ldots, a-1\}$ such that $\mathrm{F}(S) - i \in S$ and let $t = \min\{i \in \{1, \ldots, a-1\} \mid \mathrm{F}(S) - i \in S\}$. Then $\mathrm{next}_S(\mathrm{F}(S) - t) = \mathrm{F}(S) + 1$. As $S \in \mathsf{E}(a)$ then $\mathrm{F}(S) + 1 - (\mathrm{F}(S) - t) - 1$ is a multiple of $a$. Hence, $t = la$ for some $l \in \mathbb{N}\backslash\{0\}$ and consequently $t \geq a$, which is impossible. □

**Lemma 2.3** *If* $S \in \mathsf{E}(a)\backslash\{\mathbb{N}\}$*, then* $S \cup \{\mathrm{F}(S), \mathrm{F}(S) - 1, \ldots, \mathrm{F}(S) - (a-1)\} \in \mathsf{E}(a)$.

**Proof** Since $a < \mathrm{m}(S)$, by applying Lemma 2.2, we get that $T = S \cup \{\mathrm{F}(S), \mathrm{F}(S) - 1, \ldots, \mathrm{F}(S) - (a-1)\}$ is a numerical semigroup. Let $s \in S$ such that $\mathrm{next}_S(s) = \mathrm{F}(S) + 1$. In order to conclude the proof, it suffices to show that $\mathrm{next}_T(s) - s - 1$ is a multiple of $a$. As $\mathrm{next}_T(s) = \mathrm{F}(S) - (a-1)$ then $\mathrm{next}_T(s) - s - 1$ is multiple of $a$ if and only if $\mathrm{F}(S) - (a-1) - s - 1 = \mathrm{F}(S) - s - a$ is multiple of $a$. However, this is true, because if $S \in \mathsf{E}(a)$ then $\mathrm{F}(S) + 1 - s - 1 = \mathrm{next}_S(s) - s - 1$ is multiple of $a$. □

The previous result can be viewed as a procedure to construct a sequence $\{S_n \mid n \in \mathbb{N}\}$ of elements in $\mathsf{E}(a)$. Given a numerical semigroup $S$, we define recursively

- $S_0 = S$,

- $S_{n+1} = \begin{cases} S_n \cup \{\mathrm{F}(S_n), \mathrm{F}(S_n)-1, \dots, \mathrm{F}(S_n)-(a-1)\} & \text{if } S_n \neq \mathbb{N} \\ \mathbb{N} & \text{otherwise.} \end{cases}$

The next result can be easily proved.

**Proposition 2.4** *If* $S \in \mathsf{E}(a)$ *and* $\{S_n \mid n \in \mathbb{N}\}$ *is the previous sequence, then there exists* $k \in \mathbb{N}$ *such that* $S_k = \mathbb{N}$. *Furthermore, for every* $i \in \{0, 1, \dots, k-1\}$ *then* $\mathrm{card}(S_{i+1} \backslash S_i) = a$ *and* $k = \frac{\mathrm{g}(S)}{a}$.

We illustrate the previous result with an example.

**Example 2.5** *In the following sequence of elements of* $\mathsf{E}(2)$, *we have* $k = 3$.

$$S_0 = < 5, 6, 7 > \subsetneq S_1 = < 5, 6, 7, 8, 9 > \subsetneq S_2 = < 3, 4, 5 > \subsetneq S_3 = \mathbb{N}.$$

A graph $G$ is a pair $(V, E)$, where $V$ is a nonempty set and $E$ is a subset of $\{(v, w) \in V \times V \mid v \neq w\}$. The elements of $V$ and $E$ are called vertices and edges of $G$, respectively. A path of length $n$ connecting the vertices $u$ and $v$ of $G$ is a sequence of distinct edges of the form $(v_0, v_1)$, $(v_1, v_2), \dots,$ $(v_{n-1}, v_n)$ with $v_0 = u$ and $v_n = v$.

A graph $G$ is a tree if there exists a vertex $r$ (known as the root of $G$) such that for every other vertex $v$ of $G$, there exists a path connecting $v$ and $r$. If $(u, v)$ is a edge of the tree then we say that $u$ is a son of $v$.

We define the graph $G(\mathsf{E}(a))$ as graph whose vertices are the elements of $\mathsf{E}(a)$ and $(S, T) \in \mathsf{E}(a) \times \mathsf{E}(a)$ is an edge if $T = S \cup \{\mathrm{F}(S), \mathrm{F}(S)-1, \dots, \mathrm{F}(S)-(a-1)\}$. From Proposition 2.4, we deduce the following.

**Theorem 2.6** *The graph* $G(\mathsf{E}(a))$ *is a tree with root equal to* $\mathbb{N}$.

Note that the tree $G(\mathsf{E}(a))$ can be constructed recursively, from the root $\mathbb{N}$ in each step we are joining each of the vertices with its sons. Our next goal is to characterize the sons of an arbitrary vertex in the tree $G(\mathsf{E}(a))$. We distinguish two cases depending on whether or not the vertex is $\mathbb{N}$.

**Lemma 2.7** *The vertex* $\mathbb{N}$ *has a unique son in the tree* $G(\mathsf{E}(a))$ *that is* $\{0, a+1, \rightarrow\}$.

**Proof** If $S$ is a son of $\mathbb{N}$, then $S \cup \{\mathrm{F}(S), \mathrm{F}(S)-1, \dots, \mathrm{F}(S)-(a-1)\} = \mathbb{N}$ and thus $S = \{0, a+1, \rightarrow\}$. □

If $\mathcal{X}$ is a nonempty subset of $\mathbb{N}$, then we will write $\langle \mathcal{X} \rangle$ for the submonoid of $(\mathbb{N}, +)$ generated by $\mathcal{X}$, that is,

$$\langle \mathcal{X} \rangle := \left\{ \sum_{i=1}^{n} \lambda_i x_i \mid n \in \mathbb{N} \backslash \{0\},\ x_1, \dots, x_n \in \mathcal{X}, \text{and } \lambda_1, \dots, \lambda_n \in \mathbb{N} \right\}.$$

It is well known (see for example [8]) that $\langle \mathcal{X} \rangle$ is a numerical semigroup if and only if $\gcd(\mathcal{X}) = 1$.

If $M$ is a submonoid of $(\mathbb{N}, +)$ and $M = \langle \mathcal{X} \rangle$ then we say that $\mathcal{X}$ is a system of generators of $M$. Moreover, if $M \neq \langle \mathcal{Y} \rangle$ for all $\mathcal{Y} \subsetneq \mathcal{X}$, then we say that $\mathcal{X}$ is a minimal system of generators of $S$. The following result can be deduced from [8, Corollary 2.8].

**Corollary 2.8** *If* $M$ *is a submonoid of* $(\mathbb{N}, +)$, *then* $M$ *has a unique minimal system of generators, which in addition is finite.*

This minimal system of generators of $M$ will be denoted by $\mathrm{msg}(M)$.

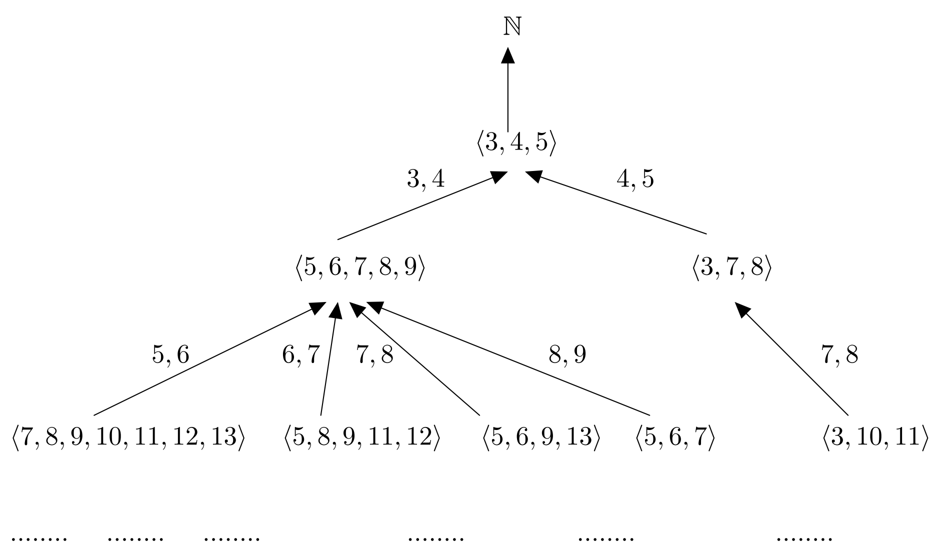
**Lemma 2.9** *[7, Lemma 1.7] Let $S$ be a numerical semigroup and $x \in S$. Then $S \backslash \{x\}$ is a numerical semigroup if and only if $x \in \mathrm{msg}(S)$.*

**Proposition 2.10** *Let $T \in \mathsf{E}(a) \backslash \{\mathbb{N}\}$. Then the set of sons of $T$, in the tree $G(\mathsf{E}(a))$, is equal to*
$\big\{T \backslash \{x, x+1, \ldots, x+(a-1)\} \mid \{x, x+1, \ldots, x+(a-1)\} \subseteq \mathrm{msg}(T)$ *and* $x > \mathrm{F}(T)\big\}$.

**Proof** If $T \in \mathsf{E}(a) \backslash \{\mathbb{N}\}$, then by Lemma 2.1, we have that $\mathrm{m}(T) > a$. Since $S$ is a son of $T$ then $T = S \cup \{\mathrm{F}(S), \mathrm{F}(S)-1, \ldots, \mathrm{F}(S)-(a-1)\}$. Hence, $\{\mathrm{F}(S), \mathrm{F}(S)-1, \ldots, \mathrm{F}(S)-(a-1)\} \subseteq \mathrm{msg}(T)$ and $\mathrm{F}(S)-(a-1) > \mathrm{F}(T)$.

Conversely, if $\{x, x+1, \ldots, x+(a-1)\} \subseteq \mathrm{msg}(T)$ and $x > \mathrm{F}(T)$ then, using repeatedly Lemma 2.9, we obtain that $S = T \backslash \{x, x+1, \ldots, x+(a-1)\}$ is a numerical semigroup with $\mathrm{F}(S) = x+a-1$. Therefore, $T = S \cup \{\mathrm{F}(S), \mathrm{F}(S)-1, \ldots, \mathrm{F}(S)-(a-1)\}$ and thus $S$ is a son of $T$. □

**Example 2.11** *Let us construct recursively the tree $G(\mathsf{E}(2))$.*




The numbers that appear on either side of the edges is the elements that we remove from the semigroup to obtain its son. For example $\langle 3,4,5\rangle \backslash \{3,4\} = \langle 5,6,7,8,9\rangle$.

Observe that $\mathsf{E}(a)$ has infinite cardinality, because $\{0, ka+1, \rightarrow)\} \in \mathsf{E}(a)$ for all $k \in \mathbb{N}$.

## 3. The set $\mathsf{E}(a)$ with a given multiplicity and genus

We will denote by $\mathsf{E}(a,m) = \{S \in \mathsf{E}(a) \mid \mathrm{m}(S) = m\}$. By Lemma 2.1, we obtain that $\mathsf{E}(a,m) \neq \emptyset$ if and only if $m = ka+1$ for some $k \in \mathbb{N}$. It is clear that $\mathsf{E}(a,1) = \{\mathbb{N}\}$.

From now on, assume that $m = ka + 1$ with $k \in \mathbb{N} \backslash \{0\}$. It is clear that maximum element in $\mathsf{E}(a)$ is $\langle m, m+1, \ldots, 2m-1 \rangle = \{0, m, \rightarrow\}$. From Lemma 2.3, we deduce the next result.

**Lemma 3.1** *If* $S \in \mathsf{E}(a,m)$ *and* $S \neq \{0, m, \rightarrow\}$*, then* $S \cup \{\mathrm{F}(S), \mathrm{F}(S)-1, \ldots, \mathrm{F}(S)-(a-1)\} \in \mathsf{E}(a,m)$.

The previous lemma allows us to define recurrently the sequence $\{S_n \mid n \in \mathbb{N}\}$ of elements of $S \in \mathsf{E}(a,m)$. If $S \in \mathsf{E}(a,m)$, then

- $S_0 = S$,
- $S_{n+1} = \begin{cases} S_n \cup \{\mathrm{F}(S_n), \mathrm{F}(S_n)-1, \ldots, \mathrm{F}(S_n)-(a-1)\} & \text{if } S_n \neq \{0, m, \rightarrow\} \\ \{0, m, \rightarrow\} & \text{otherwise.} \end{cases}$

The next result has immediate proof.

**Proposition 3.2** *If* $S \in \mathsf{E}(a,m)$ *and* $\{S_n \mid n \in \mathbb{N}\}$ *is the previous sequence, then there exists* $k \in \mathbb{N}$ *such that* $S_k = \{0, m, \rightarrow\}$*. Moreover, for every* $i \in \{0, 1, \ldots, k-1\}$ *then* $\mathrm{card}(S_{i+1} \backslash S_i) = a$ *and* $k = \frac{\mathrm{g}(S)-(m-1)}{a}$.

As a consequence of Lemma 2.1 and Proposition 3.2, we have the following result.

**Corollary 3.3** *If* $S \in \mathsf{E}(a,m)$ *then* $\mathrm{g}(S)$ *is a multiple of* $a$.

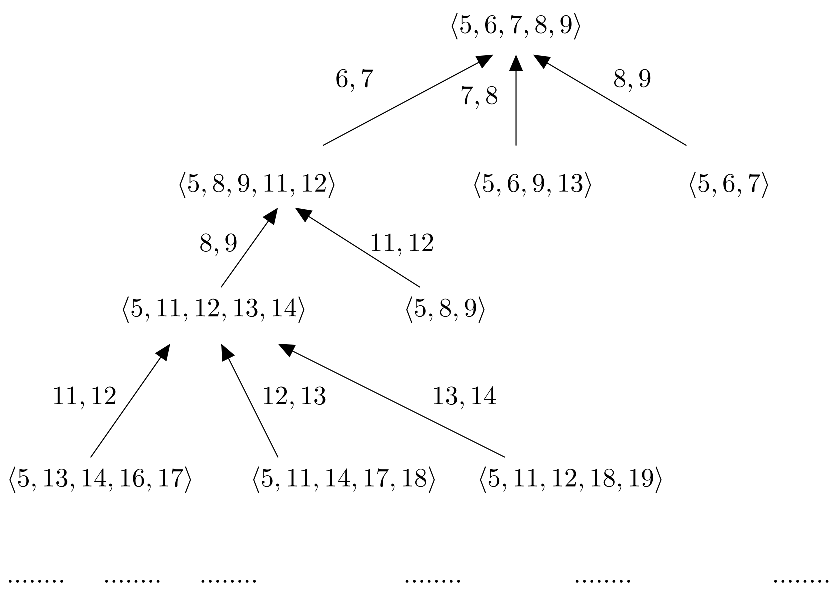
We define the graph $G\big(\mathsf{E}(a,m)\big)$ as follows: $\mathsf{E}(a,m)$ is its set of vertices and $(S,T) \in \mathsf{E}(a) \times \mathsf{E}(a)$ is an edge if $T = S \cup \{\mathrm{F}(S), \mathrm{F}(S)-1, \ldots, \mathrm{F}(S)-(a-1)\}$.

The following result is a consequence of Propositions 2.10 and 3.2.

**Theorem 3.4** *The graph* $G\big(\mathsf{E}(a,m)\big)$ *is a tree rooted in* $\{0, m, \rightarrow\}$*. Moreover, the set of sons of* $T$ *is equal to*

$$\{T \backslash \{x, x+1, \ldots, x+(a-1)\} \mid \{x, x+1, \ldots, x+(a-1)\} \subseteq msg(T),\ x > \mathrm{F}(T)) \text{ and } x \neq m\}.$$

**Example 3.5** *We are going to build the tree* $G\big(\mathsf{E}(2,5)\big)$.




Observe that for $t \in \mathbb{N}\backslash\{0\}$ we have that $\langle m\rangle \cup \{tm, \rightarrow\} \in \mathsf{E}(a,m)$ and so $\mathsf{E}(a,m)$ has infinite cardinality. Our next aim in this section will be to show an algorithm that allows us to compute the set $\mathsf{E}(a,m)$ with given genus.

If $G = (V,E)$ is a tree and $v \in V$, then the depth of $v$, which we will denote by $d(v)$, is the length of a unique path that connects $v$ with the root. Given $k$ a nonnegative integer, denote by $N(G,k) = \{v \in V \mid d(v) = k\}$.

As a consequence of Proposition 3.2 we have the following result.

**Proposition 3.6** *Let* $S \in \mathsf{E}(a,m)$. *Then* $S \in N\big(G\left(\mathsf{E}(a,m)\right), k\big)$ *if and only if* $\mathrm{g}(S) = m-1+ak$.

The next lemma follows immediately from the definitions.

**Lemma 3.7** *If* $k \in \mathbb{N}$, *then* $N\big(G\left(\mathsf{E}(a,m)\right), k+1\big) = \big\{S \mid S \text{ is a son of an element in } N\big(G\left(\mathsf{E}(a,m)\right), k\big)\big\}$.

We are already able to give an algorithm that allows us to compute the set $\mathsf{E}(a,m)$ with a given genus. Note that $S \in \mathsf{E}(a,m)$ then $\{1,\ldots,m-1\} \subseteq \mathbb{N}\backslash S$ and thus $\mathrm{g}(S) \geq m-1$. Moreover, by Corollary 3.3, we have that $\mathrm{g}(S)$ is a multiple of $a$.

**Algorithm 1**

**INPUT:** $a$, $m$ and $g$ nonnegative integers such that $2 \leq a \leq m-1 \leq g$,
$\mathrm{m} \equiv 1 (\mathrm{mod} \;\; a)$ and $g \equiv 0 (\mathrm{mod} \;\; a)$.

**OUTPUT:** The set $\{S \in \mathsf{E}(a,m) \mid \mathrm{g}(S) = g\}$.

1: Set $i = m-1$ and $A = \{\langle m, m+1, \ldots, 2m-1\rangle\}$
2: **while** True **do**
3:   **if** $i = g$ **then**
4:     **return** $A$
5:   **for** $S \in A$ **do**
6:     $B_S = \{\{x, x+1, \ldots, x+(a-1)\} \subseteq \mathrm{msg}(S), x > F(S)$ and $x \neq m\}$
7: $A := \bigcup_{S \in A} \{S \backslash \{x, x+1, \ldots, x+(a-1)\} \mid \{x, x+1, \ldots, x+(a-1)\} \in B_S\}$,
$i = i + a$ and go to step 2.

**Example 3.8** *Let us compute the set* $\{S \in \mathsf{E}(2,5) \mid \mathrm{g}(S) = 8\}$ *using Algorithm* 1.

1. *Set* $i = 4$ *and* $A = \{\langle 5,6,7,8,9\rangle\}$.
2. *The first loop constructs* $B_{\langle 5,6,7,8,9\rangle} = \{\{6,7\},\{7,8\},\{8,9\}\}$ *then*
   $A = \{\langle 5,6,7,8,9\rangle \backslash \{6,7\}, \langle 5,6,7,8,9\rangle \backslash \{7,8\}, \langle 5,6,7,8,9\rangle \backslash \{8,9\}\}$, $i = 6$,
3. *the second loop constructs* $B_{\langle 5,8,9,11,12\rangle} = \{\{8,9\},\{11,12\}\}$, $B_{\langle 5,6,9,13\rangle} = \emptyset$ *and* $B_{\langle 5,6,7\rangle} = \emptyset$ *then*
   $A = \{\langle 5,8,9,11,12\rangle \backslash \{8,9\}, \langle 5,8,9,11,12\rangle \backslash \{11,12\}\}$, $i = 8$.

   *Hence,* $\{S \in \mathsf{E}(2,5) \mid \mathrm{g}(S) = 8\} = \{\langle 5,11,12,13,14\rangle, \langle 5,8,9\rangle\}$.

## 4. The set $\mathsf{E}(a)$ with a given Frobenius number

Our aim in this section will be to show an algorithm that allows us to compute the set $\mathsf{E}(a)$ with a given Frobenius number. If $S$ is a numerical semigroup such that $S \neq \mathbb{N}$, then we have that $\mathrm{F}(S) \geq \mathrm{m}(S) - 1$. Besides, by Lemma 2.1, if $S \in \mathsf{E}(a) \backslash \{\mathbb{N}\}$ then $\mathrm{m}(S) = 1 + ka$ for some $k \in \mathbb{N} \backslash \{0\}$. Since $\mathrm{m}(S) \leq \mathrm{F}(S) + 1$ and $\mathrm{m}(S) = 1 + ka$ then $k \leq \frac{\mathrm{F}(S)}{a}$.

Given a rational number $q$ we denote by $\lfloor q \rfloor$ its integer part, that is, $\lfloor q \rfloor = \max\{z \in \mathbb{Z} \mid z \leq q\}$. We can announce the following result.

**Lemma 4.1** *Let* $\mathrm{F}(S)$ *be an integer greater than or equal to two. Then*
$\{S \in \mathsf{E}(a) \mid \mathrm{F}(S) = F\} = \bigcup_{k=1}^{\lfloor \frac{F}{a} \rfloor} \{S \in \mathsf{E}(a, ka+1) \mid \mathrm{F}(S) = F\}$.

Clearly, $\{S \in \mathsf{E}(a, ka+1) \mid \mathrm{F}(S) = ka\} = \{0, ka+1 \rightarrow\}$.

**Lemma 4.2** *Let* $k \in \mathbb{N} \backslash \{0\}$ *such that* $ka + 1 < F$. *Then* $\{S \in \mathsf{E}(a, ka+1) \mid \mathrm{F}(S) = F\} \neq \emptyset$ *if and only if* $F$ *mod* $(ka+1) \notin \{0, 1, \ldots, a-1\}$.

**Proof** *Necessity.* If $S \in \mathsf{E}(a, ka+1)$ and $\mathrm{F}(S) = F$ then, applying Lemma 2.2, we have that $\{\mathrm{F}(S), \mathrm{F}(S)-1, \ldots, \mathrm{F}(S)-(a-1)\} \subseteq \mathbb{N} \backslash S$. Therefore, $\{\mathrm{F}(S), \mathrm{F}(S)-1, \ldots, \mathrm{F}(S)-(a-1)\} \cap \langle ka+1\rangle = \emptyset$ and so $F$ mod $(ka+1) \notin \{0, 1, \ldots, a-1\}$.

*Sufficiency*. Assume that $r = F \mod (ka+1)$. Then $F = q(ka+1)+r$ for some $q \in \mathbb{N}\backslash\{0\}$ and $r \in \{a, \ldots, ka\}$. Hence we obtain that

$$S = \{0, ka+1, 2(ka+1), \ldots, q(ka+1), q(ka+1)+1, \ldots, q(ka+1)+r-a, F+1, \rightarrow\} \in \mathsf{E}(a)$$

and $\mathrm{F}(S) = F$. □

Now by using Lemmas 4.1 and 4.2, in order to compute the set $\mathsf{E}(a)$ with a given Frobenius number $F$, it is enough to give an algorithm that computes this set with $m$ a positive integer and verifies that $2 \le a \le m-1$, $m < F$, $m \equiv 1 (\mathrm{mod}\ a)$, and $(F \mod m) \notin \{0, 1, \ldots, a-1\}$.

**Algorithm 2**

**INPUT:** $a$, $m$ and $F$ nonnegative integers such that $2 \le a \le m-1$, $m < F$, $m \equiv 1(\mathrm{mod}\ a)$ and $(F \mod m) \notin \{0, 1, \ldots, a-1\}$.

**OUTPUT:** The set $\{S \in \mathsf{E}(a, m) \mid \mathrm{F}(S) = F\}$.

$B = \emptyset$ and $A = \{\langle m, m+1, \ldots, 2m-1\rangle\}$
**while** True **do**
  **for** $S \in A$ **do**
    Compute $B_S = \{\{x, x+1, \ldots, x+(a-1)\} \subseteq \mathrm{msg}(S) \mid x \neq m, x > F(S)$ and $x+(a-1) \le F\}$
    $B := B \cup \{S \backslash \{x, x+1, \ldots, x+(a-1)\} \mid S \in A, \{x, x+1, \ldots, x+(a-1)\} \in B_S$ and $x+(a-1) = F\}$
    $A := \bigcup_{S \in A} \{S \backslash \{x, x+1, \ldots, x+(a-1)\} \mid \{x, x+1, \ldots, x+(a-1)\} \in B_S$ and $x+(a-1) < F\}$
  **if** $A = \emptyset$ **then**
    **return** $B$

Next we illustrate this method with an example.

**Example 4.3** *Let us compute the set* $\{S \in \mathsf{E}(2) \mid \mathrm{F}(S) = 12\}$*. First, by using Lemma 4.1, we have that* $\{S \in \mathsf{E}(2) \mid \mathrm{F}(S) = 12\} = \bigcup_{m \in \{3,5,7,9,11,13\}} \{S \in \mathsf{E}(2, m) \mid \mathrm{F}(S) = 12\}$*. From Lemma 4.2, we obtain that* $\{S \in \mathsf{E}(2,3) \mid \mathrm{F}(S) = 12\} = \emptyset$ *and* $\{S \in \mathsf{E}(2,11) \mid \mathrm{F}(S) = 12\} = \emptyset$*. Moreover, by the observation made after the Lemma 4.1, we get that* $\{S \in \mathsf{E}(2,13) \mid \mathrm{F}(S) = 12\} = \{0, 13, \rightarrow\}$*. Therefore, by using Algorithm 2, we have to compute the set* $\{S \in \mathsf{E}(2, m) \mid \mathrm{F}(S) = 12\}$ *with* $m \in \{5, 7, 9\}$*.*

*For example, we will calculate the set* $\{S \in \mathsf{E}(2,5) \mid \mathrm{F}(S) = 12\}$*.*

1. *Start* $B = \emptyset$ *and* $A = \{\langle 5, 6, 7, 8, 9\rangle\}$*.*
2. *The first loop constructs* $B_{\langle 5,6,7,8,9\rangle} = \{\{6,7\}, \{7,8\}, \{8,9\}\}$ *then* $B = \emptyset$*,*
3. *next constructs* $A = \{\langle 5,6,7,8,9\rangle \backslash \{6,7\}, \langle 5,6,7,8,9\rangle \backslash \{7,8\}, \langle 5,6,7,8,9\rangle \backslash \{8,9\}\}$*,*
4. *the second loop constructs* $B_{\langle 5,8,9,11,12\rangle} = \{\{8,9\}, \{11,12\}\}$*,* $B_{\langle 5,6,9,13\rangle} = \emptyset$ *and* $B_{\langle 5,6,7\rangle} = \emptyset$ *then* $B = \{\langle 5,8,9,11,12\rangle \backslash \{11,12\}\}$*,*
5. *next constructs* $A = \{\langle 5,8,9,11,12\rangle \backslash \{8,9\}\}$*,*

6. *the third loop constructs* $B_{\langle 5,11,12,13,14\rangle} = \{\{11,12\}\}$
*then* $B = \{\langle 5,8,9\rangle, \langle 5,11,12,13,14\rangle \backslash \{11,12\}\}$,

7. *next constructs* $A = \emptyset$,

8. $\{S \in \mathsf{E}(2,5) \mid \mathrm{F}(S) = 12\} = \{\langle 5,8,9\rangle, \langle 5,13,14,16,17\rangle\}$.

Next we are interested in characterizing of the maximal elements in the set $\{S \in \mathsf{E}(a) \mid \mathrm{F}(S) = F\}$. The next result is well known.

**Lemma 4.4** *[9, Lemma 10] Let $S$ and $T$ be two numerical semigroups such that $S \subsetneq T$ and $x = \max(T\backslash S)$. Then $S \cup \{x\}$ is a numerical semigroup.*

**Proposition 4.5** *Let $\{S,T\} \subseteq \mathsf{E}(a)$ such that $S \subsetneq T$, $x = \max(T\backslash S)$ and let $s \in S$ such that $s < x < \mathit{next}_S(s)$. If $\{x_1 < x_2 < \cdots < x_r\} = \{t \in T \mid s < t < \mathit{next}_S(s)\}$ then $S \cup \{x_1, x_2, \ldots, x_r\} \in \mathsf{E}(a)$, $r$ is a multiple of $a$ and $S \cup \left\{x_r, x_{r-1}, \ldots, x_{r-(a-1)}\right\} \in \mathsf{E}(a)$.*

**Proof** By repeatedly applying Lemma 4.4, we get that $S \cup \{x_1, x_2, \ldots, x_r\}$ is a numerical semigroup. Moreover, as $\{S,T\} \subseteq \mathsf{E}(a)$ such that $S \subseteq T$, we deduce that $S \cup \{x_1, x_2, \ldots, x_r\} \in \mathsf{E}(a)$.

Since $s < x_1 < \cdots < x_r < \mathrm{next}_S(s)$ are consecutive elements of $T$ and $T \in \mathsf{E}(a)$, then there exist $\{k_1, \ldots k_{r+1}\} \subseteq \mathbb{N}$ such that $x_1 = s + k_1 a + 1, x_2 = s + k_1 a + 1 + k_2 a + 1, \ldots, x_r = s + k_1 a + 1 + \cdots + k_r a + 1$ and thus $\mathrm{next}_S(s) = s + k_1 a + 1 + \cdots + k_r a + 1 + k_{r+1} a + 1$. Therefore, $\mathrm{next}_S(s) - s = k_1 a + 1 + \cdots + k_r a + 1 + k_{r+1} a + 1$. As $S \in \mathsf{E}(a)$ then $\mathrm{next}_S(s) - s - 1 = ta$ for some $t \in \mathbb{N}$. Consequently, $k_1 a + 1 + \cdots + k_r a + 1 + k_{r+1} a + 1 = ta + 1$. Then $(k_1 + \cdots + k_{r+1})a + r + 1 = ta + 1$ and thus $r$ is a multiple of $a$. Assume that $r = la$ for some $l \in \mathbb{N}\backslash\{0\}$

To conclude the proof, we check that $S \cup \left\{x_{(l-1)a+1}, x_{(l-1)a+2}, \ldots, x_{(l-1)a+a}\right\} \in \mathsf{E}(a)$. In order to see this, it is enough to see that $x_{(l-1)a+1} - s - 1$ is a multiple of $a$. This is true because $x_{(l-1)a+1} - s - 1 = s + k_1 a + 1 + k_2 a + 1 + \cdots + k_{(l-1)a+1} a + 1 - s - 1 = (k_1 + \cdots + k_{(l-1)+1})a + (l-1)a + 1 - 1$ is a multiple of $a$. □

Given a sequence of nonnegative integers $n_1 < n_2 < \cdots < n_p$, we say that it is equidistant modulo $a$ if $n_{i+1} - n_i - 1$ is a multiple of $a$ for all $i \in \{1, \ldots, p-1\}$.

Let $S$ be a numerical semigroup. An element of $s \in S$ is called *a-refinable* if there exists $\{x_1 < x_2 < \cdots < x_a\} \subseteq \{x \in \mathbb{N} \mid s < x < \mathrm{next}_S(s) \text{ and } x_a < \mathrm{F}(S)\}$ such that $S \cup \{x_1, x_2, \ldots, x_a\}$ is a numerical semigroup and the sequence $s, x_1, x_2, \ldots, x_a, \mathrm{next}_S(s)$ is equidistant modulo $a$. We denote by $\mathcal{R}(S) = \{s \in S \mid S \textit{ is a-refinable}\}$.

**Theorem 4.6** *Let $S \in \mathsf{E}(a)$ with $\mathrm{F}(S) = F$. Then $S$ is a maximal element in the set $\{T \in \mathsf{E}(a) \mid \mathrm{F}(T) = F\}$ if and only if $\mathcal{R}(S) = \emptyset$.*

**Proof** *Necessity*. If $\mathcal{R}(S) \neq \emptyset$, then there exists $s \in \mathcal{R}(S)$. Hence, there exist $\{x_1 < x_2 < \cdots < x_a\} \subseteq \mathbb{N}\backslash\{0\}$ such that the sequence $s < x_1 < x_2 < \cdots < x_a < \mathrm{next}_S(s)$ is equidistant modulo $a$ with $x_a < F$ and $S \cup \{x_1, x_2, \ldots, x_a\}$ is a numerical semigroup. We deduce that $S \cup \{x_1, x_2, \ldots, x_a\} \in \mathsf{E}(a)$ with $\mathrm{F}(S \cup \{x_1, x_2, \ldots, x_a\}) = F$ contradicting the maximality of $S$.

*Sufficiency*. If we suppose that $S$ is not maximal, then there exists $T \in \mathsf{E}(a)$ with $\mathrm{F}(T) = F$ and $S \subsetneq T$. Let $x = \max(T \backslash S)$ and $s \in S$ such that $s < x < \mathrm{next}_S(s)$. By applying Proposition 4.5, we obtain that $s \in \mathcal{R}(S)$ and thus $\mathcal{R}(S) \neq \emptyset$. □

## 5. The elements of $\mathsf{E}(a)$ with maximal embedding dimension

Let $S$ be a numerical semigroup. The cardinality of $\mathrm{msg}(S)$ is known as the embedding dimension of $S$ and it is denoted here by $\mathrm{e}(S)$. In [8, Proposition 2.10] shows that $\mathrm{e}(S) \leq \mathrm{m}(S)$. We say that $S$ has maximal embedding dimension (MED-semigroup) if $\mathrm{e}(S) = m(S)$. The following result can be deduced from [2, Proposition $I$.2.9].

**Lemma 5.1** *Let $S$ be a numerical semigroup. Then $S$ is a MED-semigroup if and only if $\{s - \mathrm{m}(S) \mid s \in S \backslash \{0\}\}$ is a numerical semigroup.*

**Proposition 5.2** *Let $S$ be a MED-semigroup. Then $S \in \mathsf{E}(a)$ if and only if $T = \{s - \mathrm{m}(S) \mid s \in S \backslash \{0\}\}$ is an element of $\mathsf{E}(a)$ and $\mathrm{m}(S) \equiv 1 (mod \ \ a)$.*

**Proof** *Necessity*. By Lemmas 5.1 and 2.1, we have that $T$ is a numerical semigroup and $\mathrm{m}(S) \equiv 1(\mathrm{mod} \ \ a)$, respectively. To conclude the proof, it suffices to see that $\mathrm{next}_T(t) - t - 1$ is a multiple of $a$ for every $t \in T$. If $t \in T$, then there exists $s \in S \backslash \{0\}$ such that $t = s - \mathrm{m}(S)$ and so $\mathrm{next}_T(t) = \mathrm{next}_S(t) - \mathrm{m}(S)$. Therefore, $\mathrm{next}_T(t) - t - 1 = \mathrm{next}_S(s) - \mathrm{m}(S) - (s - \mathrm{m}(S)) - 1 = \mathrm{next}_S(s) - s - 1$ is a multiple of $a$, because $S \in \mathsf{E}(a)$.

*Sufficiency*. Let us see that $S \in \mathsf{E}(a)$, that is, if $s \in S$ then $\mathrm{next}_S(s) - s - 1$ is a multiple of $a$. If $s = 0$, then $\mathrm{next}_S(s) - 0 - 1 = \mathrm{m}(S) - 1$ is a multiple of $a$. If $s \neq 0$, then $s - m(s) = t \in T$ and $\mathrm{next}_T(t) = \mathrm{next}_S(s) - \mathrm{m}(S)$. Hence, $\mathrm{next}_S(s) - s - 1 = \mathrm{next}_T(t) - t - 1$ is multiple of $a$, because $T \in \mathsf{E}(a)$. □

From Lemma 5.1, it is easy to deduce the following result.

**Lemma 5.3** *Let $S$ be a numerical semigroup and $x \in S \backslash \{0\}$. Then $S(x) = (\{x\} + S) \cup \{0\}$ is a MED-semigroup with multiplicity $x$. Moreover, every MED-semigroup is of this form.*

**Proposition 5.4** *Let $S \in \mathsf{E}(a)$ and $x \in S \backslash \{0\}$ such that $x \equiv 1 (mod \ \ a)$. Then $S(x) = (\{x\} + S) \cup \{0\}$ is an equidistant MED-semigroup modulo $a$. Moreover, every equidistant MED-semigroup modulo $a$ is of this form.*

**Proof** By Lemma 5.3, we obtain that $S(x)$ is a MED-semmigroup with multiplicity $x$. Clearly $S = \{s - x \mid s \in S(x) \backslash \{0\}\}$ and thus, applying Proposition 5.2, we obtain that $S(x)$ is equidistant modulo $a$.

Let $T$ be an equidistant MED-semigroup modulo $a$. Then by Proposition 5.2, we get that $S = \{T - \mathrm{m}(T) \mid t \in T \backslash \{0\}\} \in \mathsf{E}(a)$ and $\mathrm{m}(T) \equiv 1(\mathrm{mod} \ \ a)$. Therefore, $T = (\{\mathrm{m}(T)\} + S) \cup \{0\}$ with $S \in \mathsf{E}(a)$ and $\mathrm{m}(T) \in S$ such that $\mathrm{m}(T) \equiv 1(\mathrm{mod} \ \ a)$. □

Let $S$ be a numerical semigroup and let $n \in S \backslash \{0\}$. The Apéry set (named so in honour of [1]) of $n$ in $S$ is

$$\mathrm{Ap}(S, n) := \{s \in S \mid s - n \notin S\}.$$

**Lemma 5.5** *[8, Lemma 2.4] Le $S$ be a numerical semigroup and $n \in S \backslash \{0\}$. Then $\mathrm{Ap}(S, n) = \{0 = w(0), w(1), \ldots, w(n-1)\}$, where $w(i)$ is the least element in $S$ congruent with $i$ modulo $n$, for all $i \in \{0, \ldots, n-1\}$.*

Observe that the above lemma in particular implies that the cardinality of $\mathrm{Ap}(S,n)$ is $n$. From [10], we can deduce the next result.

**Proposition 5.6** *Le $S$ be a numerical semigroup, $n \in S\backslash\{0\}$ and $T = (\{n\}+S)\cup\{0\}$ . Then the following conditions hold:*

1. *$T$ is MED-semigroup*
2. $\mathrm{m}(T) = n$.
3. $\mathrm{F}(T) = \mathrm{F}(S) + n$.
4. $\mathrm{g}(T) = \mathrm{g}(S) + n - 1$.
5. $\mathrm{msg}(T) = \mathrm{Ap}(S,n) + \{n\}$.

**Example 5.7** *It is clear that $S = \langle 5,8,9\rangle$ is an an equidistant numerical semigroup modulo $2$. We have that $9$ is an element in $S$ such that $9 \equiv 1 (mod \quad 2)$. Then from Proposition 5.4, we obtain that $T = (\{9\}+S)\cup\{0\}$ is an equidistant MED-semigroup modulo $2$. Since $\mathrm{F}(S) = 12$, $g(S) = 8$ and $\mathrm{Ap}(S,9) = \{0,5,8,10,13,15,16,20,21\}$, by Proposition 5.6 we have that $\mathrm{F}(T) = 12+9 = 21$, $\mathrm{g}(T) = 8+9-1 = 16$ and* $\mathrm{msg}(T) = \{9,14,17,19,22,24,25,29,30\}$.

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