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A COMPLEX MULTI-STATE *k*-OUT-OF-*n*: *G* SYSTEM WITH PREVENTIVE MAINTENANCE AND LOSS OF UNITS

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ABSTRACT

In this study, a multi-state *k*-out-of-*n*: *G* system subject to multiple events is modelled through a Markovian Arrival Process with marked arrivals. The system is composed initially of *n* units and is active when at least *k* units are operational. Each unit is multi-state, each of which is classified as minor or major according to the level of degradation presented. Each operational unit may undergo internal repairable or non-repairable failures, external shocks and/or random inspections. An external shock can provoke extreme failure, while cumulative external damage can deteriorate internal performance. This situation can produce repairable and non-repairable failures. When a repairable failure occurs the unit is sent to a repair facility for corrective repair. If the failure is non-repairable, the unit is removed. When the system has insufficient units with which to operate, it is restarted. Preventive maintenance is employed in response to random inspection. The system is modelled in an algorithmic and computational form. Several interesting measures of performance are considered. Costs and rewards are included in the system. All measures are obtained for transient and stationary regimes. A numerical example is analysed to determine whether preventive maintenance is profitable, financially and in terms of performance.

1. Introduction

Redundant systems present considerable research interest as a means of improving reliability and of avoiding possible catastrophic failure provoking severe damage and major financial losses. In the reliability literature, various types of redundant systems have been proposed, including cold, warm and hot, as well as the *k*-out-of-*n*: *F and G* system. In a cold standby system, the redundant units are not subject to failure before being put into full operation; in a warm one, the units can fail at any time but at different rates; and in hot standby, all units can fail at any time, with an identical pattern of behaviour. These redundant systems can be applied in many different fields. For instance, in industrial engineering, any facility that requires a reliable electrical supply must have generating sets available, capable of supplying electricity in case of need. The *k*-out-of-*n*: *G* is an *n*-system which is active if at least *k* units are operational. This system was introduced by Birnbaum et al. (1961) and it is applied in various fields, such as electronic, industrial and military systems to model telecommunications, oil pipelines, vacuum systems in accelerators, computer networks and space relay stations. A generalised *k*-out-of-*n* with parallel modules was developed by Cui and Xie (2005). Li and Zuo (2008) extended these systems for modelling complex engineering systems with weighted *k*-out-of-*n* systems. Recently, Balakrishnan et al. (2018) extends known results in the literature concerning comparisons of *k*-out-of-*n* systems in the exponential model to the Weibull model. Kumar and Ram (2017) have investigated the simultaneous effect of *k*-out-of-*n*: G/F and parallel redundancy in a complex industrial system by using Markov processes from a classical point of view.

Redundant systems were introduced as a means of extending the lifetime of reliability systems and of avoiding the considerable losses of system failure. Reliability systems can also be improved by the application of appropriate maintenance policies, such as preventive maintenance, which is widely recommended as an effective way to minimise system downtime, prevent system failures and increase the benefit derived from the system. A maintained system where each component is repaired independently of the others according to an exponential distribution is developed by Papastavridis and Koutras (1992). Standard and advanced problems of maintenance policies for reliability models have been discussed by Nakagawa (2005). A reliability model and a maintenance policy for a *k*-out-of-*n* system, in which all components have the same properties and experience two dependent/correlated failure processes, is developed in Song et al. (2012). Coit et al. (2015) describe a new *k*-out-of-*n* system reliability model, appropriate when the minimum number of required components changes dynamically in response to failures, to maximize the utility of the available collection of functioning components. In this field, Chalabi et al. (2016) presented an optimised grouping strategy of preventive maintenance actions for multi-unit series production systems. This proposal had two main aims: to maximise the availability of resources and to minimise the impact of their cost on the total cost of the product. Recently, Julanto-Endharta and Young-Yun (2017) develop a preventive maintenance policy, based on the system critical condition which is related to the number of working components, for a circular consecutive-*k*- ot-of-*n*: *F* system.

The literature also contains several interesting studies based on dynamic grouping strategies. With a view to improving performance, replacement policies are also of interest in the field of reliability studies. Thus, Ahmadi (2014) described an inspection and replacement strategy for a deteriorating complex multi-component manufacturing system undergoing damage processes and imperfect repair. On the other hand, Park and Yoo (2004) considered three different replacement policies for a group of identical units in which minimal repair was always considered. Among other outcomes, the expected cost rate under each policy was derived. A paper by Yoo (2011) proposed a maintenance policy based on deriving the failure count, the expected cost rate function and the optimum value. In another approach, Barron and Yechiali (2017) used a Markovian model to analyse a redundant deteriorating repairable system. This study considered state-dependent operating costs, repair costs dependent on the extent of the repair, and failure penalty costs. Dynamic programming showed that a generalised control-limit policy is optimal for the expected total discounted criterion for both cold and warm standby systems.

A significant research advance for *k*-out-of-*n* systems is their generalisation to multi-state *k*-out-of-*n* system reliability when components have different levels of performance or degradation. Multi-state systems (MSS) are of particular importance in ensuring reliability. Under traditional reliability theory, systems are considered in terms of binary models composed of only two states: up (performing) or down (failure). However, many real-life systems contain multiple components with different levels of performance. Lisnianski et al. (2010) provides a comprehensive presentation of MSS reliability theory. Modern mathematical methods for MSS reliability analysis and applications are given by Lisnianski et al. (2018). Levintin (2013) introduces a new general model, named the multi-state vector-*k*-out-of-*n* system. Lisnianski and Frenkel (2012) considered Markov processes in the analysis of multi-state systems, highlighting the benefits of their application. A *k*-out-of-*n*: *G* system with multi-state components was modelled by using matrix analytic methods in Ruiz-Castro and Li (2011).

With respect to applicability, serious calculation difficulties can arise when complex multi-state systems must be modelled. In order to obtain expressions in a well structured way, algorithms must be developed and the results implemented computationally and then applied. One way of analysing multi-state systems is to consider a Markovian environment, incorporating a phase-type (PH) distribution. Phase-type distributions were introduced and described in detail by Neuts (1975, 1981). A probability distribution is a phase-type distribution if and only if it is the distribution of the time until absorption in an irreducible finite Markov process with one state being absorbent and the other transient. Many well-known continuous probability distributions are PH distributed, including the exponential, hyperexponential, Erlang, mixture of Erlang and Coxian among others. One of the main properties of PH distributions is that they comprise a dense class within the set of non-negative probability distributions. Neuts (1975) pointed out that all discrete distributions with finite support are PH distributed. These types of distributions have also been used to model multi-state systems. Barron et al. (2004) developed an *R*-out-of-*N* system where each component was subject to repairable failure and the embedded distributions were phase-type. Matrix algorithmic methods were used and several reliability measures (availability) were built, using Markov renewal and semi-regenerative processes. This approach was later extended by Barron et al. (2006) to the situation in which multiple repairpersons were available. In addition, phase-type distributions were also introduced in this context by Barron (2015), who analysed a multi-component repairable cold standby system with multiple costs, such as downtime cost when failed components are not repaired or replaced, and fixed and replacement costs associated with the maintenance facility. Three classes of group replacement policies (m-failure, T-age and (m, T, τ)) were considered. Barron (2018) also extended this approach to include an R-out-of-N repairable system in which the lifetimes of the units followed a phase-type (PH) distribution, and derived the expected discounted costs under these three classes of group maintenance policies. For each one, a replacement and downtime cost was determined, using matrix-geometric methods.

In this study, in addition to PH distributions, an important role is played by Markovian Arrival Processes (MAPs). In a MAP, the number of events in an underlying Markov chain are counted. MAPs were introduced by Neuts (1979) and a description in detail is given by He (2014). A special case of this is the Marked Markovian Arrival Process (MMAP), in which several types of arrivals are counted. This counting process can be seen in He and Neuts (1998). In all cases, the arrival rates of events can be customised for different situations, which highlights the inherent versatility of this class of process. MMAPs can be applied to model multi-state devices subject to multiple events with a dependence relationship. With PH processes and MAPs, the main results of multi-state complex systems can be expressed in a matrix-algebraic form. Multiple redundant systems have been modelled by considering discrete phase-type distributions and MAPs (Ruiz-Castro, 2016a; Ruiz-Castro, 2016b; Ruiz-Castro and Dawabsha, 2019).

A common assumption in reliability studies is that when a non-repairable failure occurs the unit affected will be replaced with negligible delay. However, this is not always the case and it may not even be the optimal option. Another possibility that has been considered is that in redundant systems whenever a unit undergoes a non-repairable failure it is removed and no further action is taken, providing the system still has enough units to continue functioning. Various such redundant systems have been modelled, assuming a tolerable loss of units. Ruiz-Castro and Fernández-Villodre (2012) studied a warm standby system with loss of units. A general cold standby system with loss of units is modelled by Ruiz-Castro (2015). Recently, this modelling has been extended and optimized by including several types of failures and multiple repairpersons by using Markovian Arrival Processes by Ruiz-Castro et al. (2018).

In the present study, we consider a k-out-of-n: G system with multi-state components and loss of units, evolving in discrete time. This system is developed in a matrix algorithmic form. The embedded lifetimes in the system are PH distributed, and a Markovian arrival process with marked arrivals is assumed. The following potential events are considered: internal failures (repairable or otherwise), inspections and external shocks.

An external shock can produce extreme failure (non-repairable), cumulative damage and/or degraded performance, whether or not a failure occurs. Such a shock may affect one or more operational units. When a unit undergoes a repairable failure, it is sent to the repair facility for corrective repair. Internal and cumulative external damage is composed of various phases, and classed as minor or major according to the degree of damage produced. When a random inspection is performed, all operational units in the system are inspected and if major damage (internal or cumulative external damage) is observed the unit is sent to the repair facility for preventive maintenance. A system presenting the above-described features is described as complex due to the dependence between events and the complexity of the model, and because it cannot be broken down into a group of series and parallel systems. The general complex system is developed in a matrix, algorithmic and computational form and several interesting measures are incorporated in the transient and stationary regimes. A comparative analysis was performed between a system with and without preventive maintenance in order to determine the profitability of the latter.

The model presented can be applied to real-life systems in fields such as civil, industrial and computer engineering, computer and communication systems, and power transmission and distribution systems. Thus, a car with a V8 engine that works if only four cylinders are firing can be modelled by a 4-out-of-8: *G* system. Another case would be that of a communications system with *n* transmitters, which may pass through multiple performance stages; here, the average message load may be such that at least *k* transmitters must be operational at all times or critical messages may be lost. Thus, the transmission subsystem would function as a *k*-out-of-*n*: *G* system.

The paper is organised as follows. The system and the assumptions made are introduced in Section 2. The system is modelled in detail in in Section 3, from the state space up to the Markovian Arrival Process. The transient distribution is built in Section 4. Section 5 is focused on the stationary distribution and on interesting measures associated with the system. Costs and rewards are addressed in Section 6. A numerical example, comparing two similar systems, with and without preventive maintenance, is shown in Section 7. The main conclusions drawn are presented in Section 8.

2. The system

A complex multi-state *k*-out-of-*n*: *G* system that evolves in discrete time is considered. Initially, this system is composed of *n* units and it is active when at least *k* units are operational. Each unit is multi-state and its internal behaviour is partitioned into minor and major degradation stages. The units are subject to internal failure, external shocks and random inspections. Internal failure is provoked by wear and may be either repairable or non-repairable. When an internal failure occurs, the unit is sent to the repair facility for corrective repair. If the failure is non-repairable, the unit is removed provided the number of units remaining is enough for the system to function. The system is reinitialised when the number of units falls to less than k, i.e. when the number of units is insufficient for the system to work. When the system has fewer than k operational units, it does not work and the units outside the repair facility are stopped while the non-operational units are repaired. In this case, the operational units are not subject to internal failure but they are exposed to external shocks.

External shock can produce three different types of events; aggravated internal degradation, cumulative external damage and extreme failure. The first type can produce an internal failure, repairable or otherwise, the second provokes cumulative damage (multiple external damage stages) and the unit undergoes a non-repairable failure if a given threshold is exceeded. Finally, if a unit undergoes extreme failure it must be removed.

Preventive maintenance is introduced in conjunction with random inspections. Periodical inspection is a particular case of the latter. When an inspection takes place, all operational units are observed and if the level of internal degradation and/or external cumulative damage warrants it, the unit is sent to the repair facility for preventive maintenance. The system verifies the following assumptions.

Assumptions

1. The lifetime of each unit follows a discrete-time phase-type distribution with representation $\left(\boldsymbol{\alpha}, \mathbf{T}\right)$ of order m (number of internal operational stages).

2. Each unit can undergo a repairable or non-repairable failure due to internal wear out. We assume two absorbing states, one for each kind of failure. The probability of failure depends on the internal operational stage. Thus, the probability of repairable failure or non-repairable is given by the column vectors $\mathbf{T}_r^0$ and $\mathbf{T}_{nr}^0$, repectively. Clearly, the total absorbing vector produced by any transient state is given by $\mathbf{T}^0 = \mathbf{e} - \mathbf{T}\mathbf{e} = \mathbf{T}_r^0 + \mathbf{T}_{nr}^0$ [1].

3. External shocks that produce events on the operational units occur according to a phase type renewal process. The time between two consecutive events is *PH* distributed with representation ($\boldsymbol{\gamma}$, $\mathbf{L}$) of order t.

4. One external shock can produce, in each operational unit, external cumulative damage, aggravation of the internal degradation where a repairable or non-repairable failure can occur, or an extreme failure (non-repairable failure).

4.1. External damage of each operational unit can pass through an indeterminate number of external degradation stages. The number of external degradation states is equal to d, and these are partitioned in minors (the first d_1 states) and major stages (states d_1+1,...,d). If the external degradation state is i, then the external shock changes

[1] Throughout the paper **e** denotes a column vector of ones with appropriate order

this one to state j with probability d_{ij}. These probabilities are contained in the matrix $\mathbf{Q}$. A cumulative external damage threshold is reached from the external damage states after an external shock through the probability column vector $\mathbf{Q}^0$. If it occurs then the unit undergoes a non-repairable failure. Initially, previously to an external shock, the unit is in external degradation state 1 (no damage due to external shock). The initial distribution for external damage when one unit occupies the online place initially is $\boldsymbol{\omega} = (1,0,...,0)_{1\mathrm{x}d}$.

4.2. One external shock can also produce modification in the internal degradation state. If the internal degradation state is i, then the external shock changes this one to state j with probability w_{ij}. These probabilities are given in the matrix $\mathbf{W}$. An internal failure, repairable or not, can occur due to this fact from any performance state with a probability column vector $\mathbf{W}_r^0$ and $\mathbf{W}_{nr}^0$ respectively. In this case $\mathbf{W}^0 = \mathbf{e} - \mathbf{We} = \mathbf{W}_r^0 + \mathbf{W}_{nr}^0$.

4.3. One external shock can produce an extreme failure (non-repairable failure) in each unit. Each one occurs with a probability equal to ω^0.

5. When a non-repairable failure occurs, the unit is removed. The number of units in the system is always greater than or equal to k. If the number of units after non-repairable failures is less than k, then the system is reinitialized.

6. While there are operational units, random inspections can occur. The time between two consecutive inspections is *PH* distributed with representation ($\boldsymbol{\eta}$, $\mathbf{M}$) of order ε.

7. The corrective repair time for any unit is *PH* distributed with representation ($\boldsymbol{\beta}^1$, $\mathbf{S}_1$) of order z_1.

8. The preventive maintenance time is *PH* distributed with representation ($\boldsymbol{\beta}^2$, $\mathbf{S}_2$) of order z_2.

9. Each operational unit and the unit under repair can undergo events at the same time. The priority of event over the same unit is the following; non-repairable failure, repairable failure and major inspection.

10. If the number of units in the system is greater than or equal to k, then the system is operational only if at least k units are operational. Otherwise, the system is stopped, in which case the repairperson continue operating and external shocks and inspection can happen.

11. Unit quality after a repair is as good as new.

12. The times involved in the model are independent.

Table 1 shows the main parameters and variables addressed in modelling the system.

	Distribution/ probability vector	Order
Lifetime each unit	$(\boldsymbol{\alpha}, \mathbf{T})$	m
Transition probability of internal repairable failure	$\mathbf{T}_r^0$	m x 1
Transition probability of internal non-repairable failure	$\mathbf{T}_{nr}^0$	m x 1
Time between external shocks	$(\boldsymbol{\gamma}, \mathbf{L})$	t
Transition probability of external damage	$\mathbf{Q}$	d
Transition probability to the cumulative external threshold damage	$\mathbf{Q}^0$	d x 1
Initial distribution for the external damage	$\boldsymbol{\omega}$	1 x d
Transition probability of internal modification after external shock	$\mathbf{W}$	m
Transition probability of internal repairable failure after external shock	$\mathbf{W}_r^0$	m x 1
Transition probability of internal non-repairable failure after external shock	$\mathbf{W}_{nr}^0$	m x 1
Transition probability of internal failure after external shock	$\mathbf{W}^0$	m x 1
Extreme failure due to external shock	ω^0	scalar
Time between consecutive inspections	$(\boldsymbol{\eta}, \mathbf{M})$	ε
Corrective repair time	$(\boldsymbol{\beta}^1, \mathbf{S}_1)$	z_1
Preventive maintenance time	$(\boldsymbol{\beta}^2, \mathbf{S}_2)$	z_2

Table 1. Main parameters and variables in the model

Figures 1 and 2 illustrate the behaviour of the system. Figure 1 shows the behaviour of a unit which is subject to internal failure, external shock and inspection, and the time phases until each of these events occur. The consequences of these events – corrective repair, degradation or extreme failure, and preventive maintenance, respectively – are also shown. The last unit to enter the repair facility determines the type of task the repairperson must perform. If a unit undergoes a non-repairable failure, it is removed. Figure 2 shows the system when there are *l* operational units (distinguishing between minor and major stages of degradation) with *a* units in the repair facility.

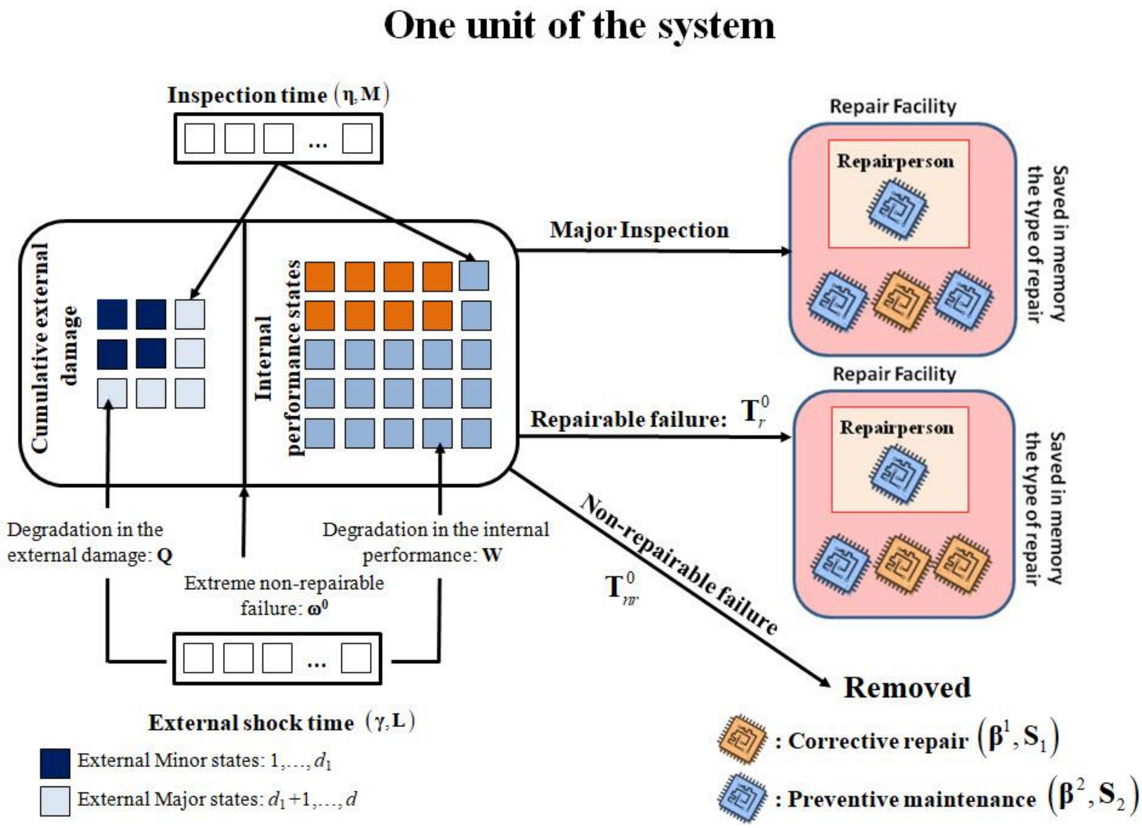

Figure 1. Behaviour of one unit of the complex system

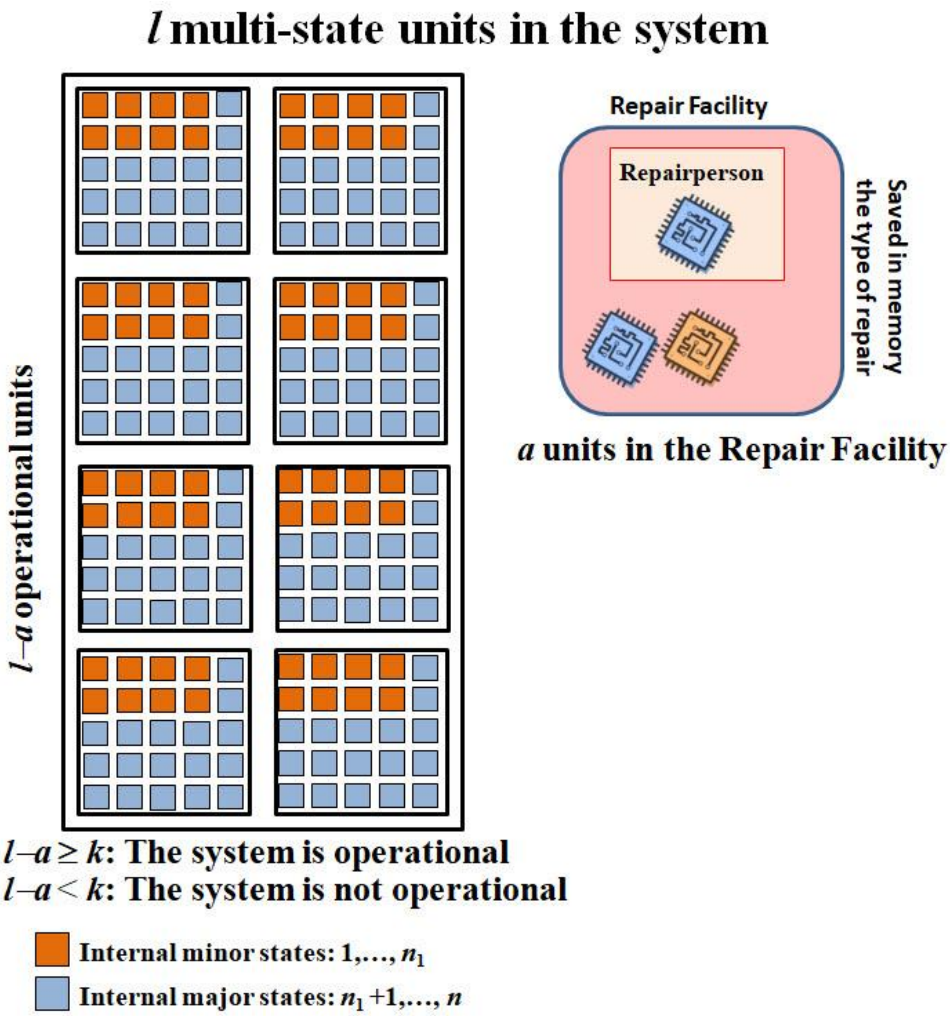

Figure 2. Behaviour of units and the repair facility

3. The model

The modelling of the complex system described above is not easy. In this section the state-space is given, several auxiliary matrices are built and several matrix functions are developed for the operational units. Examples are given.

3.1. The state-space

The system described in the section above is governed by a vector Markov process with a state space $\mathbf{U}$ that composed of three levels of macro-states. It is given by $\mathbf{U}=\left\{\mathbf{U}^n,\mathbf{U}^{n-1},\ldots,\mathbf{U}^k\right\}$. The first level of macro-states, $\mathbf{U}^l$, describes the situation when there are l units in the system. Each macro-state $\mathbf{U}^l$ contains the macro-states $\mathbf{U}_a^l$, l units in the system and a of them in the repair facility (second level). Finally, the third level of macro-states contain the specific order of the units in the repair facility. Therefore,

$\mathbf{U} = \left\{ \mathbf{U}^n, \mathbf{U}^{n-1}, \ldots, \mathbf{U}^k \right\}$ where $\mathbf{U}^l = \left\{ \mathbf{U}_0^l, \mathbf{U}_1^l, \ldots, \mathbf{U}_l^l \right\}$; l = k,…, n (first level). The second level is given by

$$\mathbf{U}_0^l = \left\{ \left(l, 0; \mathbf{v}^1, \mathbf{v}^2, \ldots, \mathbf{v}^l, j, \upsilon \right); j = 1, \ldots, t; \upsilon = 1, \ldots, \varepsilon \right\} \text{ with}$$

$$\mathbf{v}^h = \left\{ \left(i^h, u^h \right); i^h = 1, \ldots, m; u^h = 1, \ldots, d \right\},$$

$$\mathbf{U}_a^l = \left\{ \mathbf{U}_{x_1, \ldots, x_a}^l ; x_y = 1, 2 \text{ and } y = 1, \ldots, a \right\} \text{ for } a = 1, \ldots, l$$

where $\mathbf{v}^h$ contains the phases of the internal degradation level (i^h) and the cumulative external damage (u^h) associated to the h-th operational unit. The elements $x_1, x_2, \ldots, x_a$ indicates the type of failure ordered in queue in the repair facility (x_y =1, the y-th unit in queue is for corrective repair; x_y =2, the y-th unit in queue is for preventive maintenance). The third level are given by

$$\mathbf{U}_{x_1, \ldots, x_a}^l = \left\{ \left(l, a; \mathbf{v}^1, \mathbf{v}^2, \ldots, \mathbf{v}^{l-a}, j, \upsilon, r \right); j = 1, \ldots, t; \upsilon = 1, \ldots, \varepsilon; r = 1, \ldots, z_{x_1} \right\}, \ a = 1, \ldots, l-1,$$

$$\mathbf{U}_{x_1, \ldots, x_a}^l = \left\{ \left(l, a; \mathbf{v}^1, \mathbf{v}^2, \ldots, \mathbf{v}^{l-a}, j, r \right); j = 1, \ldots, t; r = 1, \ldots, z_{x_1} \right\}, \ a = l.$$

The parameter j indicates the phase of the external shock time and υ the phase of the inspection time. Finally, the corrective repair or the preventive maintenance phase of the unit that is being repaired is given by r.

Throughout the paper, the order of these macro-states will be used in the modelling of the system. The order of the macro-state $\mathbf{U}_a^l$ is given by o_a^l and it is equal to

$$o_a^l = \begin{cases} \left(m \cdot d \right)^l \cdot t \cdot \varepsilon & ; \quad a = 0 \\ \left(m \cdot d \right)^{l-a} \cdot t \cdot \varepsilon \cdot \left(z_1 + z_2 \right) \cdot 2^{a-1} & ; \quad a \neq l \\ t \cdot \left(z_1 + z_2 \right) \cdot 2^{a-1} & ; \quad a = l \end{cases} \qquad \text{for } a = 1, \ldots, l.$$

Clearly, the order of the macro-state $\mathbf{U}^l$ is given by $o^l = \sum_{s=0}^{l} o_s^l$.

Therefore, the phases of the macro-state $\mathbf{U}_a^l$ are the ones which occupy in the state space $\mathbf{U}$ the order $I_a^l \equiv I_{\{l \leq n-1\}} \sum_{u=l+1}^{n} o^u + I_{\{a \geq 1\}} \sum_{s=0}^{a-1} o_s^l + 1 : I_{\{l \leq n-1\}} \sum_{u=l+1}^{n} o^u + I_{\{a \geq 0\}} \sum_{s=0}^{a} o_s^l$.

3.2. Types of events and operational conditions

Each operational unit can undergo different types of events. These ones are partitioned as

- O: No events
- b_r, b_{nr}, b_{mr} : b_r repairable failures, b_{nr} non-repairable failures and b_{mr} major inspections occur. The system is not restarted after this number of non-repairable failures.
- NS: Non-repairable failures occur such that the system is reinitialized.

Four important conditions are given according to the situation of the system. For this, we assume that the system is composed of l units with a of them in the repair facility at a certain time.

Condition OP: The system is operational

The system is operational when the number of operational units is greater or equal to k ($l-a \geq k$).

Condition NOP: The system is not operational.

The system is non-operational when the number of operational units is less to k ($l-a \leq k-1$)

Condition R: The system is reinitialized

The system is reinitialized when the number of units that undergo a non-repairable failure (b_{nr}) is greater than l–k ($b_{nr} \geq l-k+1$) at a certain time. It is clear that the number of non-repairable failures, the number of repairable failures (b_r) and the number of major inspections (b_{mr}) always verify $b_r + b_{nr} + b_{mr} \leq l-a$.

Condition NR: The system is not reinitialized

When the number of non-repairable failures is less or equal to l–k then the system is not reinitialized ($b_{nr} \leq l-k$ and $b_r + b_{nr} + b_{mr} \leq l-a$).

rep indicates if one repair is produced (=1) or not (=0)

3.3. Auxiliary matrices

Several auxiliary matrices are defined to ease the modelling of this complex system. This section is focused on the modelling of the operational units.

3.3.1. Inspection

The following matrices will be considered when one inspection occurs. The matrix $\mathbf{U}_l$ and $\mathbf{V}_l$, for l =1,2, are square matrices of order n and d respectively, whose element (s, t) is given by,

$$U_1(s,t)=\begin{cases}1 & ; \quad 1\le s=t\le n_1\\ 0 & ; \quad \text{otherwise}\end{cases},\ U_2(s,t)=\begin{cases}1 & ; \quad s=t>n_1\\ 0 & ; \quad \text{otherwise}\end{cases},$$

$$V_1(s,t)=\begin{cases}1 & ; \quad 1\le s=t\le d_1\\ 0 & ; \quad \text{otherwise}\end{cases},\ V_2(s,t)=\begin{cases}1 & ; \quad s=t>d_1\\ 0 & ; \quad \text{otherwise}\end{cases}.$$

The matrices **U** and **V** will be taken into account when one inspection occurs and the internal degradation level and cumulative external damage are observed respectively. The subscripts 1 and 2 are associated with minor or major damages, respectively.

3.3.2. Operational units

Different functions are defined to model the behaviour of the operational units. The modelling of the shock and inspection is given in first place and later functions for the operational units are given.

Shocks and inspection

The matrix behaviour of external shocks and random inspections will depend on whether the system after events is reinitialized or not. External events occur independently to the performance of the system. If the system is reinitialized then the inspection time is also reinitialized although an inspection does not occur. The matrix function $f_r(shock, insp)$ give the matrix transition probability for external shocks and inspections. The variable *shock* and *insp* are equal to 1 and 0 if they occurs or not respectively. Under condition **R** it is

$$f_r(shock, insp)=\begin{cases}\mathbf{L}\otimes\mathbf{Me\eta} & ; \quad shock=0, insp=0\\ \mathbf{L}\otimes\mathbf{M}^0\boldsymbol{\eta} & ; \quad shock=0, insp=1\\ \mathbf{L}^0\boldsymbol{\gamma}\otimes\mathbf{Me\eta} & ; \quad shock=1, insp=0\\ \mathbf{L}^0\boldsymbol{\gamma}\otimes\mathbf{M}^0\boldsymbol{\eta} & ; \quad shock=1, insp=1.\end{cases}$$

The case when the system is not reinitialized after events (**NR**) is given in Appendix A.

Operational Units

The following functions have been defined to model the behaviour of the internal degradation level and the external cumulative damage taking into account the phases of the operational units.

1. Function *C*

   Transition matrix for the operational units when the system is composed of *l* units, of which *a* are in the repair facility, when b_{nr} non-repairable failures, b_r repairable failures and b_{mr} major inspections occur, in a specific failure order ($k_r^1,\ldots,k_r^{b_r}$ is the ordinal of the repairable failures, $k_{mr}^1,\ldots,k_{mr}^{b_{mr}}$ the ordinal of major inspections and

$k_{nr}^{1},\ldots,k_{nr}^{b_{nr}}$ is the ordinal of the non-repairable failures). The parameters *shock* and *insp* indicates if one external shock and one inspection occurs or not. This situation is described as $C\left(l,a,b_r,b_{nr},b_{mr},shock,insp;k_r^1,\ldots,k_r^{b_r},k_{nr}^1,\ldots,k_{nr}^{b_{nr}},k_{mr}^1,\ldots,k_{mr}^{b_{mr}}\right)$. The subscript will indicate the conditions **OP**, **NOP**, **R** and **NR**.

2. Function *d*

   Transition matrix for the operational units when the system is composed of *l* units, of which *a* are in the repair facility, when b_{nr} non-repairable failures, b_r repairable failures and b_{mr} major inspections occur, where the failure order is not established. The parameter *rep* indicates if one repair is finished at this time. This situation is described as $d\left(l,a,rep,b_r,b_{nr},b_{mr}\right)$. The subscript of this function will indicate the conditions **OP**, **NOP**, **R** and **NR**.

The case **OP** and **R** is describe next. The rest functions are further developed in *Appendix A*.

3.3.3. Functions *C* and *d* under OP and NR

In this section, we describe the transition matrix, considering each of the phases associated with the behaviour of the operational units. We then consider the particular case in which the system is operational (OP) and not restarted after transition (NR).
To describe one situation, we assume that one external shock is not produced (**L**) but one inspection takes place ($\mathbf{M}^0$). If all operational units undergo an event and one repairing is not produced, then the inspection time is not reinitialized. Therefore,

$$f_{nr}\left(0,1\right)=\mathbf{L}\otimes\mathbf{M}^0\left(I_{\{b_r+b_{nr}+b_{mr}=l-a \text{ and } rep=0\}}+I_{\{b_r+b_{nr}+b_{mr}<l-a \text{ or } (b_r+b_{nr}+b_{mr}=l-a \text{ and } rep=1)\}}\boldsymbol{\eta}\right).$$

In this situation, each unit of the system can undergo an internal failure, a non-repairable failure, a major inspection and no event. Given that an external shock is not produced in this case, then the failures are only possible from internal degradation.
In the repairable case, it occurs and the external damage stage does not change. The unit goes to the repair facility ($\mathbf{T}_r^0\otimes\mathbf{e}$). Analogously it occurs for the non-repairable case. It happens ($\mathbf{T}_{nr}^0\otimes\mathbf{e}$) and it is removed with reinitializing.
In this case, the inspection is produced. If only the internal degradation level ($\mathbf{U}_2\left(\mathbf{e}-\mathbf{T}^0\right)\otimes\mathbf{V}_1\mathbf{e}$) is major or the cumulative external damage state observed is major ($\left(\mathbf{e}-\mathbf{T}^0\right)\otimes\mathbf{V}_2\mathbf{e}$), then the unit goes to the repair facility.
Finally, if no shock occurs and inspection, and the unit continues working then it is because there is no failure and in both cases minor damage ($\mathbf{U}_1\mathbf{T}\otimes\mathbf{V}_1$).
Then, weather all operational units (*l*–*a*) are considered and the failure order is established then

$$C_{op\_nr}\left(l,a,b_r,b_{nr},b_{mr},shock=0,insp=1;k_r^1,\ldots,k_r^{b_r},k_{nr}^1,\ldots,k_{nr}^{b_{nr}},k_{mr}^1,\ldots,k_{mr}^{b_{mr}}\right)$$

$$=\phi^2\left(1\right)\otimes\cdots\otimes\phi^2\left(l-a\right),$$

where

$$\phi^2(\nu)=\begin{cases}\mathbf{T}_r^0\otimes\mathbf{e} & ;\quad \nu=k_r^1,\ldots,k_r^{b_r}\\ \mathbf{T}_{nr}^0\otimes\mathbf{e} & ;\quad \nu=k_{nr}^1,\ldots,k_{nr}^{b_{nr}}\\ \mathbf{U}_2\left(\mathbf{e}-\mathbf{T}^0\right)\otimes\mathbf{V}_1\mathbf{e}+\left(\mathbf{e}-\mathbf{T}^0\right)\otimes\mathbf{V}_2\mathbf{e} & ;\quad \nu=k_{mr}^1,\ldots,k_{mr}^{b_{mr}}\\ \mathbf{U}_1\mathbf{T}\otimes\mathbf{V}_1 & ;\quad \text{otherwise,}\end{cases}$$

for $\nu = 1,\ldots,l{-}a$.

Example 1. Assume a 3–*out-of*–5 multi-state system with l = 4 units in which three units are operational ($l{-}a$ = 3). Assume then that the first unit undergoes a non-repairable failure, that the second has a major inspection and that the third presents a repairable failure ($k_{nr}^1=1, k_r^1=3, k_{mr}^1=2$). The necessary repair is performed. If the operational phases and external damage are considered for all units, then

$$\begin{aligned}&C_{op\_nr}\left(4,1,1,1,1,shock=0,insp=1;3,1,2\right)\\&=\mathbf{T}_{nr}^0\otimes\mathbf{e}\otimes\left[\mathbf{U}_2\left(\mathbf{e}-\mathbf{T}^0\right)\otimes\mathbf{V}_1\mathbf{e}+\left(\mathbf{e}-\mathbf{T}^0\right)\otimes\mathbf{V}_2\mathbf{e}\right]\otimes\mathbf{T}_r^0\otimes\mathbf{e}.\end{aligned}$$

In this case, the absence of external shock and the performance of the inspection are included as

$$\begin{aligned}&C_{op\_nr}\left(4,1,1,1,1,shock=0,insp=1;3,1,2\right)\otimes f_{nr}\left(0,1\right)\\&=\mathbf{T}_{nr}^0\otimes\mathbf{e}\otimes\left[\mathbf{U}_2\left(\mathbf{e}-\mathbf{T}^0\right)\otimes\mathbf{V}_1\mathbf{e}+\left(\mathbf{e}-\mathbf{T}^0\right)\otimes\mathbf{V}_2\mathbf{e}\right]\otimes\mathbf{T}_r^0\otimes\mathbf{e}\otimes\mathbf{L}\otimes\mathbf{M}^0\boldsymbol{\eta}.\end{aligned}$$

When all operational units ($l{-}a$) are considered and the type of failure is not established, all combinations must be taken into account. In this case,

$$d_{op\_nr}\left(l,a,rep,b_r,b_{nr},b_{mr}\right)$$

$$\begin{aligned}&C_{op\_nr}\left(5,1,1,1,1,shock=0,insp=1;4,1,2\right)\\&=\mathbf{T}_{nr}^0\otimes\mathbf{e}\otimes\left[\mathbf{U}_2\left(\mathbf{e}-\mathbf{T}^0\right)\otimes\mathbf{V}_1\mathbf{e}+\left(\mathbf{e}-\mathbf{T}^0\right)\otimes\mathbf{V}_2\mathbf{e}\right]\otimes\mathbf{U}_1\mathbf{T}\otimes\mathbf{V}_1\otimes\mathbf{T}_r^0\otimes\mathbf{e}.\end{aligned}$$

$$\begin{aligned}&=\sum_{shock=0}^{1}\sum_{insp=0}^{1}\sum_{k_r^1=1}^{l-a-b_r+1}\sum_{k_r^2=k_r^1+1}^{l-a-b_r+2}\cdots\sum_{k_r^{b_r}=k_r^{b_r-1}+1}^{l-a}\sum_{\substack{k_{nr}^1=1\\k_{nr}^1\neq k_r^u\\u=1,\ldots,b_r}}^{l-a-b_{nr}+1}\cdots\sum_{\substack{k_{nr}^{b_{nr}}=k_{nr}^{b_{nr}-1}+1\\k_{nr}^{b_{nr}}\neq k_r^u\\u=1,\ldots,b_r}}^{l-a}\sum_{\substack{k_{mr}^1=1\\k_{mr}^1\neq k_r^u\\k_{mr}^1\neq k_{nr}^v\\u=1,\ldots,b_r\\v=1,\ldots,b_{nr}}}^{l-a-b_{mr}+1}\cdots\sum_{\substack{k_{mr}^{b_{mr}}=k_{mr}^{b_{mr}-1}+1\\k_{mr}^{b_{mr}}\neq k_r^u\\k_{mr}^{b_{mr}}\neq k_{nr}^v\\u=1,\ldots,b_r\\v=1,\ldots,b_{nr}}}^{l-a}\\&C_{op\_nr}\left(l,a,b_r,b_{nr},b_{mr},shock,insp;k_r^1,\ldots,k_r^{b_r},k_{nr}^1,\ldots,k_{nr}^{b_{nr}},k_{mr}^1,\ldots,k_{mr}^{b_{mr}}\right)\\&\otimes\left(I_{\{rep=0\}}+I_{\{rep=1\}}\boldsymbol{\alpha}\otimes\boldsymbol{\omega}\right)\otimes f_{nr}\left(shock,insp\right)\end{aligned}$$

Example 2. We assume the example 1 where the system undergoes a non-repairable failure, one major inspection and one repairable failure, without shock but with inspection. The order of event is not established and we assume that one repairing occurs. If the operational phases and external damage are considered for all units then it is

$$\begin{aligned}&\left[C_{op\_nr}\left(4,1,1,1,1,shock=0,insp=1;1,2,3\right)+C_{op\_nr}\left(4,1,1,1,1,shock=0,insp=1;1,3,2\right)\right.\\&+C_{op\_nr}\left(4,1,1,1,1,shock=0,insp=1;2,1,3\right)+C_{op\_nr}\left(4,1,1,1,1,shock=0,insp=1;2,3,1\right)\\&\left.+C_{op\_nr}\left(4,1,1,1,1,shock=0,insp=1;3,1,2\right)+C_{op\_nr}\left(4,1,1,1,1,shock=0,insp=1;3,2,1\right)\right]\otimes f_{nr}\left(0,1\right)\end{aligned}$$

$$\begin{aligned}&=\left[\mathbf{T}_r^0\otimes\mathbf{e}\otimes\mathbf{T}_{nr}^0\otimes\mathbf{e}\otimes\left[\mathbf{U}_2\left(\mathbf{e}-\mathbf{T}^0\right)\otimes\mathbf{V}_1\mathbf{e}+\left(\mathbf{e}-\mathbf{T}^0\right)\otimes\mathbf{V}_2\mathbf{e}\right]\right.\\&+\mathbf{T}_r^0\otimes\mathbf{e}\otimes\left[\mathbf{U}_2\left(\mathbf{e}-\mathbf{T}^0\right)\otimes\mathbf{V}_1\mathbf{e}+\left(\mathbf{e}-\mathbf{T}^0\right)\otimes\mathbf{V}_2\mathbf{e}\right]\otimes\mathbf{T}_{nr}^0\otimes\mathbf{e}\\&+\mathbf{T}_{nr}^0\otimes\mathbf{e}\otimes\mathbf{T}_r^0\otimes\mathbf{e}\otimes\left[\mathbf{U}_2\left(\mathbf{e}-\mathbf{T}^0\right)\otimes\mathbf{V}_1\mathbf{e}+\left(\mathbf{e}-\mathbf{T}^0\right)\otimes\mathbf{V}_2\mathbf{e}\right]\\&+\left[\mathbf{U}_2\left(\mathbf{e}-\mathbf{T}^0\right)\otimes\mathbf{V}_1\mathbf{e}+\left(\mathbf{e}-\mathbf{T}^0\right)\otimes\mathbf{V}_2\mathbf{e}\right]\otimes\mathbf{T}_r^0\otimes\mathbf{e}\otimes\mathbf{T}_{nr}^0\otimes\mathbf{e}\\&+\mathbf{T}_{nr}^0\otimes\mathbf{e}\otimes\left[\mathbf{U}_2\left(\mathbf{e}-\mathbf{T}^0\right)\otimes\mathbf{V}_1\mathbf{e}+\left(\mathbf{e}-\mathbf{T}^0\right)\otimes\mathbf{V}_2\mathbf{e}\right]\otimes\mathbf{T}_r^0\otimes\mathbf{e}\\&\left.+\left[\mathbf{U}_2\left(\mathbf{e}-\mathbf{T}^0\right)\otimes\mathbf{V}_1\mathbf{e}+\left(\mathbf{e}-\mathbf{T}^0\right)\otimes\mathbf{V}_2\mathbf{e}\right]\otimes\mathbf{T}_{nr}^0\otimes\mathbf{e}\otimes\mathbf{T}_r^0\otimes\mathbf{e}\right]\otimes\mathbf{L}\otimes\mathbf{M}^0\boldsymbol{\eta}\end{aligned}$$

3.4. The Markovian arrival process with marked arrivals

The system is modelled by a MMAP by considering the different types of events described in Section 3.2. The representation is given by

$$\left(\mathbf{D}^O,\left\{\mathbf{D}^{b_{nr},b_r,b_{mr}};b_{nr},b_r,b_{mr}=0,\ldots,n;b_{nr}\le l-k;1\le b_{nr}+b_r+b_{mr}\le n\right\},\mathbf{D}^{NS}\right).$$

The matrix $\mathbf{D}^Y$ contains the transition probabilities when event Y occurs, and is composed of three matrix block levels The third level corresponds to the transitions from the macro-state $\mathbf{U}^l$ to $\mathbf{U}^q$ with $q \le l$ or $q = 0$. This transition is given by the matrix $\mathbf{R}^{Y,l,q}$. These matrix blocks are composed of the matrices $\mathbf{B}_{ij}^{Y,l,q}$ which correspond to the transitions between the macro-states from $\mathbf{U}_i^l$ to either $\mathbf{U}_j^q$ or $\mathbf{E}_h^{k-1}$ (level 2) under Y.

The matrix $\mathbf{D}^Y$ contains the transition probabilities when event Y occurs, and is composed of three matrix block levels. The matrices $\mathbf{B}_{ij}^{Y,l,q}$ are composed of matrix blocks corresponding to the transition from the macro-states $\mathbf{U}_{x_1,\ldots,x_i}^{Y,l,q}$ to $\mathbf{U}_{x_1,\ldots,x_q}^{Y,l,q}$. The matrix block $\mathbf{B}_{ij}^{Y,l,q}\left(s_1,\ldots,s_j;x_1,\ldots,x_i\right)$ contains the transition probabilities described above where the type of repair in the repair facility is ordered for the case before and after transition. These blocks are built by considering the auxiliary matrices defined in section 3.3.1 and developed in Appendix *A* (level 1).

Next, the case b_r, b_{nr}, b_{mr} at least one event occurs and the system is not reinitialized, is described in detail. The rest is given in an algorithmic form in Appendix *B*.

3.4.1. Matrix $\mathbf{D}^{b_{nr},b_r,b_{mr}}$

The matrix block $\mathbf{D}^{b_{nr},b_r,b_{mr}}$ contains the transitions when b_{nr} non-repairable, b_r repairable and b_{mr} major inspections occur. At least, one event must have happened, $1 \leq b_{nr} + b_r + b_{mr} \leq n$. This matrix block has a matrix block order equal to $(n-k+1)$ x $(n-k+1)$, the system is composed with at least k units,

$$\mathbf{D}^{b_{nr},b_r,b_{mr}} = \left(\mathbf{R}^{b_{nr},b_r,b_{mr};l,l-b_{nr}}\right)_{(n-k+1)\ x\ (n-k+1)}.$$

The matrices $\mathbf{R}^{b_{nr},b_r,b_{mr};l,l-b_{nr}}$ are different to matrix zero when the number of non-repairable failures does not provoke a restarting of the system ($b_{nr} \leq l-k$) and the number of operational units are greater or equal to the sum of events ($l \geq b_r + b_{nr} + b_{mr}$).

Therefore, for $l = \max\{k+b_{nr}, b_r+b_{nr}+b_{mr}\},\ldots,n$

$$\mathbf{R}^{b_{nr},b_r,b_{mr};l,l-b_{nr}} = \left(\mathbf{B}_{ij}^{b_{nr},b_r,b_{mr};l,l-b_{nr}}\right)_{(l+1)x(l-b_{nr}+1)},$$

where the only elements different to matrix zero are when the number of operational units are greater or equal to the sum of events ($i \leq l - b_{nr} - b_r - b_{mr}$). Also, the number of units in the repair facility after transition is the previous number plus the number of repairable failures and major inspections. If there is a unit being repaired then this one must be subtracted if the repair is finished. Then,

$$j = \begin{cases} b_r + b_{mr} + i - 1, b_r + b_{mr} + i & ; \quad i > 0 \\ b_r + b_{mr} & ; \quad i = 0 \end{cases}.$$

The auxiliary matrices described in Section 3.2.2 are used next. If initially there is zero units in the repair facility, then the system is always operational and then

$$\mathbf{B}_{00}^{b_{nr},b_r=0,b_{mr}=0;l.l-b_{nr}} = d_{op\_nr}\left(l,0,0,b_{nr},0,0\right).$$

Under **OP** condition,

- the transition from zero units in the repair facility will happen up to the number of major inspections and repairable failures. The repairable failure (if any) begins the repair (it has priority versus major inspection)

  For b_r >0 or/and b_{mr} >0

$$\mathbf{B}_{0,b_{nr}+b_{mr}}^{b_{nr},b_r,b_{mr};l.l-b_{nr}}\left(s_1,\dots,s_j\right)=d_{op\_nr}\left(l,0,b_{nr},b_r,b_{mr}\right)\otimes\boldsymbol{\beta}^{s_1}\quad;\quad \begin{array}{l} s_y=1;\ y=1,\dots,b_r \\ s_y=2;\ y=b_r+1,\dots,j \end{array}$$

- the transition from one unit in the repair facility to zero is due to the corrective repair or preventive maintenance finishes and only non-repairable failures are possible.

$$\mathbf{B}_{10}^{b_{nr},b_r=0,b_{mr}=0;l.l-b_{nr}}\left(x_1\right)=d_{op\_nr}\left(l,1,1,b_{nr},0,0\right)\otimes\mathbf{S}_{x_1}^0\quad;\quad x_1=1,2\,.$$

- if initially the repair facility is not empty (i >0) and the unit being repaired does not finish it, then

$$\mathbf{B}_{i,i+b_r+b_{mr}}^{b_{nr},b_r,b_{mr};l.l-b_{nr}}\left(s_1,\dots,s_{i+b_r+b_{mr}}\mid x_1,\dots,x_i\right)=d_{op\_nr}\left(l,i,0,b_{nr},b_r,b_{mr}\right)\otimes\mathbf{S}_{x_1}\quad;\quad \begin{array}{l} x_y=s_y,\ y=1,\dots,i \\ s_y=1;\ y=i+1,\dots,i+b_r\ \left(\text{if } b_r>0\right) \\ s_y=2;\ y=i+b_r+1,\dots,i+b_r+b_{mr} \end{array}$$

- if initially the repair facility is not empty (i >0), also after transition there are units (i =1 and (b_r >0 or b_{mr} >0) or i >1) and the unit being repaired finishes, then

  $\mathbf{B}_{i,i+b_r+b_{mr}-1}^{b_{nr},b_r,b_{mr};l.l-b_{nr}}\left(s_1,\dots,s_{i+b_r+b_{mr}-1}\mid x_1,\dots,x_i\right)=d_{op\_nr}\left(l,i,1,b_{nr},b_r,b_{mr}\right)\otimes\mathbf{S}_{x_1}^0\otimes\boldsymbol{\beta}^{x_1}$ for

  $x_y=s_y,\ y=1,\dots,i,$

  $s_y=1;\ y=i+1,\dots,i+b_r\ \left(\text{if } b_r>0\right),$

  $s_y=2;\ y=i+b_r+1,\dots,i+b_r+b_{mr}.$

A similar reasoning can be carried out under option **NOP**.

4. Transition probability matrix and transient distribution

The complex system is modelled through a Markovian Arrival Process with marked arrivals as described in Section 3.4. It is well known that, in this case, the transition probability matrix associated to the Markov chain is given by

$$\mathbf{P}=\mathbf{D}^0+\mathbf{D}^{NS}+\sum_{b_{nr}=0}^{n}\sum_{b_r=0}^{n-b_{nr}}\sum_{b_r=0}^{n-b_{nr}-b_{mr}}\mathbf{D}^{b_{nr},b_r,b_{mr}}\quad.$$

The initial distribution of the system is $\boldsymbol{\theta} = [\boldsymbol{\alpha} \otimes \boldsymbol{\omega} \otimes \overset{n}{\cdots} \otimes \boldsymbol{\alpha} \otimes \boldsymbol{\omega} \otimes \boldsymbol{\gamma}_{st} \otimes \boldsymbol{\eta}, \mathbf{0}]$ where $\boldsymbol{\gamma}_{st} = [1, \mathbf{0}]\left[\mathbf{e} \,|\, \left(\mathbf{L} + \mathbf{L}^0\boldsymbol{\gamma} - \mathbf{I}\right)^*\right]^{-1}$ is the stationary distribution of the Markov chain associated to the external shock with generator $\mathbf{L} + \mathbf{L}^0\boldsymbol{\gamma}$ and $\mathbf{A}^*$ denotes the matrix $\mathbf{A}$ without the first column.

The transient distribution can be expressed by considering the macro-states defined in Section 3.1. This one is given through the vector $\mathbf{p}^\nu$.

$$\mathbf{p}^\nu = \left(\mathbf{p}^\nu_{U_0^n}, \mathbf{p}^\nu_{U_1^n}, \ldots, \mathbf{p}^\nu_{U_n^n}, \mathbf{p}^\nu_{U_0^{n-1}}, \mathbf{p}^\nu_{U_1^{n-1}}, \ldots, \mathbf{p}^\nu_{U_{n-1}^{n-1}}, \ldots, \mathbf{p}^\nu_{U_0^k}, \mathbf{p}^\nu_{U_1^k}, \ldots, \mathbf{p}^\nu_{U_k^k}\right),$$

where $\mathbf{p}^\nu_{U_a^l} = \left(\boldsymbol{\theta}\mathbf{P}^\nu\right)_{I_a^l}$ is the probability of being in the macro-state U_a^l at time ν.

5. Measures in transient and stationary regime

Availability, reliability, mean times, conditional probability of failure and mean number of events are interesting measures of reliability, and are studied in transient and stationary regimes.

The long-run distribution, $\boldsymbol{\pi}$, is obtained by matrix-analytic methods. This distribution has been calculated for the macro-states $\mathbf{U}^l$. The stationary probability of being in this macro-state is denoted by $\boldsymbol{\pi}_{\mathbf{U}^l}$, thus $\boldsymbol{\pi} = \left(\boldsymbol{\pi}_{\mathbf{U}^n}, \boldsymbol{\pi}_{\mathbf{U}^{n-1}}, \ldots, \boldsymbol{\pi}_{\mathbf{U}^k}\right)$. The transition probability matrix can be expressed according to these macro-states. Let $\mathbf{R}^{s,l}$ be the matrix block in $\mathbf{P}$ associated to the transition from $\mathbf{U}^s$ to $\mathbf{U}^l$. The stationary distribution verifies $\boldsymbol{\pi}\mathbf{P} = \boldsymbol{\pi}$ jointly to the normalisation condition. This equation by considering the matrix blocks is

$$\boldsymbol{\pi}_{\mathbf{U}^n} = \sum_{s=k}^{n} \boldsymbol{\pi}_{\mathbf{U}^s}\mathbf{R}^{s,n}, \qquad \boldsymbol{\pi}_{\mathbf{U}^l} = \sum_{s=l}^{n} \boldsymbol{\pi}_{\mathbf{U}^s}\mathbf{R}^{s,l}; \; l = k, \ldots, n-1, \qquad \sum_{s=k}^{n} \boldsymbol{\pi}_{\mathbf{U}^s}\mathbf{e} = 1.$$

The solution is given by $\boldsymbol{\pi}_{\mathbf{U}^l} = \boldsymbol{\pi}_{\mathbf{U}^n}\mathbf{R}^l$; $l = k, \ldots, n-1$ where

$$\mathbf{R}^l = \left(\mathbf{R}^{n,l} + I_{\{l \le n-2\}} \sum_{s=l+1}^{n-1} \mathbf{R}^s\mathbf{R}^{s,l}\right)\left(\mathbf{I} - \mathbf{R}^{l,l}\right)^{-1}; \quad l = k, \ldots, n-1.$$

The vector $\boldsymbol{\pi}_{\mathbf{U}^n}$ is obtained from the first balance equation and the normalization condition. This is equal to

$$\boldsymbol{\pi}_{\mathbf{U}^n} = (1, \mathbf{0})\left[\left(\mathbf{e} + \sum_{s=k}^{n-1} \mathbf{R}^s\mathbf{e}\right) \middle| \left(\mathbf{I} - \sum_{s=k}^{n-1} \mathbf{R}^s\mathbf{R}^{s,n} - \mathbf{R}^{n,n}\right)^*\right]^{-1},$$

where the matrix $\mathbf{A}^*$ is a matrix $\mathbf{A}$ without the first column.

The vector $\boldsymbol{\pi}_{\mathbf{U}^l}$ can be partitioned by considering the macro-state $\mathbf{U}_a^l$. It is expressed as $\boldsymbol{\pi}_{\mathbf{U}^l} = \left(\boldsymbol{\pi}_{\mathbf{U}_0^l}, \ldots, \boldsymbol{\pi}_{\mathbf{U}_l^l}\right)$.

One of the most significant measures in reliability is the availability. The availability is the probability that the system is operational at a certain time ν. This is expressed according to the macro-states as

$$A(\nu)=\sum_{l=k}^{n}\sum_{a=0}^{l-k}\mathbf{p}_{U_a^l}^{\nu}\mathbf{e}.$$

This value in stationary regime is

$$A=\sum_{l=k}^{n}\sum_{a=0}^{l-k}\boldsymbol{\pi}_{U_a^l}\mathbf{e}.$$

The reliability function is the probability of a given (harmful) event not taking place by a certain time. Two different reliability functions are defined: the distribution of the time up to first non-operational time and up to first restarting time.

An event is defined as the first time that the system is non-operational because fewer than k units are operational. The distribution time of this event is PH distributed with representation $(\boldsymbol{\theta}^*,\mathbf{P}^*)$ where $\boldsymbol{\theta}^*,\mathbf{P}^*$ are $\boldsymbol{\theta},\mathbf{P}$ restricted to the macro-states U_a^l under the condition **OP**. The mean time up to first time that the system is not operational is $\boldsymbol{\theta}^*(\mathbf{I}-\mathbf{P}^*)^{-1}\mathbf{e}$.

Given that the units in the system can undergo non-repairable failure (in which case they are removed), it is interesting to analyse the time elapsed until the first time that the system must be restarted. The distribution time of this event is PH distributed with representation $(\boldsymbol{\theta},\mathbf{P}^{'})$ where $\mathbf{P}^{'}$ is

$$\mathbf{P}^{'}=\mathbf{D}^{O}+\sum_{b_{nr}=0}^{n}\sum_{b_r=0}^{n-b_{nr}}\sum_{b_{mr}=0}^{n-b_{nr}-b_{mr}}D^{b_{nr},b_r,b_{mr}}.$$

In analysing system behaviour, it is important to study the mean time in each macro-state associated with the system in question. The mean time in the macro-state $\mathbf{U}_a^l$ up to a certain time ν is given by

$$\psi_{l,a}(\nu)=\sum_{m=0}^{\nu}\mathbf{p}_{U_a^l}^{m}\cdot\mathbf{e}. \tag{1}$$

In the stationary regime it is $\psi_{l,a}=\boldsymbol{\pi}_{U_a^l}\mathbf{e}$. (2)

This value can be interpreted as the proportional time in this macro-state up to this time.

Clearly, from this measure the mean time in the macro-state $\mathbf{U}^l$ is

$$\psi_l(\nu) = \sum_{a=0}^{l} \psi_{l,a}(\nu). \tag{3}$$

The proportional time in the macro-state $\mathbf{U}^l$ is $\psi_l = \sum_{a=0}^{l} \psi_{l,a}$. (4)

The system developed in this work is subject to multiple events. The probability of an event occurring before time ν is given by the conditional probability of the event. The conditional probability of a non-repairable failure occurring before time ν, requiring the system to be restarted, is

$$CONPRO_{NS}(\nu) = \boldsymbol{\theta} \cdot \mathbf{P}^{\nu-1} \cdot \mathbf{D}^{NS} \cdot \mathbf{e}.$$

In the stationary case it id given by $CONPRO_{NS} = \boldsymbol{\pi}\mathbf{D}^{NS} \cdot \mathbf{e}$.

The conditional probability of happening b_{nr} non-repairable failures, b_r repairable failures and b_{mr} major inspection at time ν is given by

$$CONPRO_{b_{nr},b_r,b_{mr}}(\nu) = \boldsymbol{\theta} \cdot \mathbf{P}^{\nu-1} \cdot \mathbf{D}^{b_{nr},b_r,b_{mr}} \cdot \mathbf{e},$$

and the stationary value is $CONPRO_{b_{nr},b_r,b_{mr}} = \boldsymbol{\pi}\mathbf{D}^{b_{nr},b_r,b_{mr}} \cdot \mathbf{e}$.

Over time the system passes through several situations: operational, with the repairperson idle or with the repairperson busy. Accordingly, it is of interest to determine the mean time in which the system will be in each these situations, up to a given time.

The *k*-out-of-*n* system is operational when at least *k* units are operational. The mean time that que system is operational up to a certain time ν is given by

$$\mu_{op}(\nu) = \sum_{l=k}^{n} \sum_{a=0}^{l-k} \psi_{l,a}(\nu). \tag{5}$$

The proportional time that the system is operational (stationary regime) is

$$\mu_{op} = \sum_{l=k}^{n} \sum_{a=0}^{l-k} \psi_{l,a} . \tag{6}$$

Another question of interest is to analyse the mean time elapsed while the repairperson is idle (busy) up to time ν. These mean times are respectively

$$\mu_{idle}(\nu)=\sum_{l=k}^{n}\psi_{l,0}(\nu) \quad , \quad \mu_{busy}(\nu)=\sum_{l=k}^{n}\sum_{a=1}^{l}\psi_{l,a}(\nu). \tag{7}$$

The proportional time that the repairperson is idle and busy (stationary regime) respectively is $\mu_{idle}=\sum_{l=k}^{n}\psi_{l,0}$ and $\mu_{busy}=\sum_{l=k}^{n}\sum_{a=1}^{l}\psi_{l,a}$. (8)

Finally, multiple events can occur up to a certain time. The system modelled in this work is subject to different types of events given in section 3.2. For each one the mean number up to a certain time and in stationary regime is calculated. Table 2 shows these measures.

	Up to time ν	Stationary case
Mean number of repairable failures	$\Lambda^{rep}(\nu)=\sum_{u=1}^{\nu}\mathbf{p}^{u-1}\sum_{b_{nr}=0}^{n}\sum_{b_r=0}^{n-b_{nr}}\sum_{b_{mr}=0}^{n-b_{nr}-b_r} b_r\cdot\mathbf{D}^{b_{nr},b_r,b_{mr}}\cdot\mathbf{e}$ (9)	$\Lambda^{rep}=\boldsymbol{\pi}\sum_{b_{nr}=0}^{n}\sum_{b_r=0}^{n-b_{nr}}\sum_{b_{mr}=0}^{n-b_{nr}-b_r} b_r\cdot\mathbf{D}^{b_{nr},b_r,b_{mr}}\cdot\mathbf{e}$ (10)
Mean number of major inspections	$\Lambda^{major}(\nu)=\sum_{u=1}^{\nu}\mathbf{p}^{u-1}\sum_{b_{nr}=0}^{n}\sum_{b_r=0}^{n-b_{nr}}\sum_{b_{mr}=0}^{n-b_{nr}-b_r} b_{mr}\cdot\mathbf{D}^{b_{nr},b_r,b_{mr}}\cdot\mathbf{e}$ (11)	$\Lambda^{major}=\boldsymbol{\pi}\sum_{b_{nr}=0}^{n}\sum_{b_r=0}^{n-b_{nr}}\sum_{b_{mr}=0}^{n-b_{nr}-b_r} b_{mr}\cdot\mathbf{D}^{b_{nr},b_r,b_{mr}}\cdot$ (12)
Mean number of new systems	$\Lambda^{newsystems}(\nu)=\sum_{u=1}^{\nu}\mathbf{p}^{u-1}\cdot\mathbf{D}^{NS}\cdot\mathbf{e}$ (13)	$\Lambda^{newsystems}=\boldsymbol{\pi}\cdot\mathbf{D}^{NS}\cdot\mathbf{e}$ (14)

Table 2. Mean number of events up to a given time, in the stationary case

6. Costs and rewards

Costs and rewards are associated with the system according to its operational phase. For each unit of time elapsed during which the system is operational, a reward equal to *B* is produced. However, a cost equal to *R* per unit of time for the repairperson must also be taken into account. When the system is not operational, a loss equal to *C* per unit of down time is provoked. The model incorporates a cost vector reflecting the state space of the system.

While the system is operational, each unit has a cost per unit of time depending on the operational phase. This is expressed by the column vector $\mathbf{c}_0$. Any unit in the repair facility can come from a repairable failure or from a major inspection. Different costs for both cases depending on the repair phase are included. The corrective repair and the preventive maintenance cost vector, depending on the repair phase, are given by **cr** and **cmp** respectively.

A cost vector associated to the state space has been built according to the macro-states $\mathbf{U}_a^l$, this column vector is denoted as $\mathbf{c}_a^l$ and it is equal to

$$\mathbf{c}_a^l = \begin{cases} (B-R)\cdot\mathbf{e}_{(md)^l t\varepsilon} - \overbrace{\mathbf{c}_0\otimes\mathbf{e}_d \odot\cdots\odot \mathbf{c}_0\otimes\mathbf{e}_d}^{l} \otimes \mathbf{e}_{t\varepsilon} & ; \quad a=0 \\ (B-R)\cdot\mathbf{e}_{(md)^{l-a} t\varepsilon 2^{a-1}(z_1+z_2)} - \overbrace{\mathbf{c}_0\otimes\mathbf{e}_d \odot\cdots\odot \mathbf{c}_0\otimes\mathbf{e}_d}^{l-a} \otimes \mathbf{e}_{t\varepsilon 2^{a-1}(z_1+z_2)} & \\ -\mathbf{e}_{(md)^{l-a} t\varepsilon} \otimes \begin{pmatrix} \mathbf{e}_{2^{a-1}} \otimes \mathbf{cr} \\ \mathbf{e}_{2^{a-1}} \otimes \mathbf{cpm} \end{pmatrix} & ; \quad a\neq 0; l-a\geq k \\ -(C+R)\cdot\mathbf{e}_{(md)^{l-a} t\varepsilon 2^{a-1}(z_1+z_2)} - \mathbf{e}_{(md)^{l-a} t\varepsilon} \otimes \begin{pmatrix} \mathbf{e}_{2^{a-1}} \otimes \mathbf{cr} \\ \mathbf{e}_{2^{a-1}} \otimes \mathbf{cpm} \end{pmatrix} & ; \quad a\neq 0; a\neq 0; l-a< k \\ -(C+R)\cdot\mathbf{e}_{t 2^{a-1}(z_1+z_2)} - \mathbf{e}_t \otimes \begin{pmatrix} \mathbf{e}_{2^{a-1}} \otimes \mathbf{cr} \\ \mathbf{e}_{2^{a-1}} \otimes \mathbf{cpm} \end{pmatrix} & ; \quad a=l. \end{cases}$$

The operator $\odot$ is defined as $\mathbf{a}\odot\mathbf{b} = \mathbf{a}\otimes\mathbf{e}_m + \mathbf{e}_n\otimes\mathbf{a}$ being **a** and **b** column vectors with order *n* and *m* respectively.

The cost vector for the state-space from the macro-states is

$$\mathbf{c} = \left(\mathbf{c}_0^n, \mathbf{c}_1^n, \ldots, \mathbf{c}_n^n, \mathbf{c}_0^{n-1}, \mathbf{c}_1^{n-1}, \ldots, \mathbf{c}_{n-1}^{n-1}, \ldots, \mathbf{c}_0^k, \mathbf{c}_1^k, \ldots, \mathbf{c}_k^k\right)'.$$

Besides considering the cost vector to add to the net reward obtained over time as the system operates, three fixed costs have been considered, according to the events taking place. Each time that a repairable failure or major inspection occurs, a fix cost equal to *fcr* or *fpm* takes place respectively. Each unit has a cost equal to *fnu*, therefore a full system costs $n\cdot fnu$.

In section 5. the mean number of events up to a certain time is developed. From these measures the fix cost due to this measures are easily worked out, $fcr\cdot\Lambda^{rep}(\nu)$, $fpm\cdot\Lambda^{major}(\nu)$ and $n\cdot fnu\cdot\left(1+\Lambda^{newsystems}(\nu)\right)$. In the last case the initial system is included.

Thus, the mean net profit up to time ν is given by

$$\Phi(\nu) = \sum_{m=0}^{\nu} \mathbf{p}^m \mathbf{c} - fcr\cdot\Lambda^{rep}(\nu) - fpm\cdot\Lambda^{major}(\nu) - n\cdot fnu\cdot\left(1+\Lambda^{newunits}(\nu)\right). \qquad (15)$$

The mean net profit ratio up to time ν is $\Phi(\nu)/(\nu+1)$ and in the stationary case the mean net profit per unit of time is

$$\Phi = \lim_{\nu \to \infty} \frac{\Phi(\nu)}{\nu + 1} = \lim_{\nu \to \infty} \left(\boldsymbol{\pi}\mathbf{c} - fcr \cdot \Lambda^{rep} - fpm \cdot \Lambda^{major} - n \cdot fnu \cdot \Lambda^{newunits} \right).$$

(16)

7. Numerical Example

In this section, we consider a 2-out-of-3: *G* system. This kind of system is motivated by situations such as the following. In a communications system with three transmitters, the average message load may be such that at least two must be operational at all times or critical messages may be lost. Therefore, a 2-out-of-3 system subject to multiple events is assumed. This numerical example illustrates the versatility of the model, modelling and analysing, in an algorithmic form, the effect of preventive maintenance applied to a complex reliability system such as that described in this paper. When preventive maintenance is introduced, new costs are incurred and the performance of the device usually improves. The following question then arises: does the performance improvement justify the extra cost? Is it profitable from an economic standpoint? A system comparison is performed, with and without preventive maintenance.

Each operational unit passes through three levels of degradation. The first two are minor, but the last is major. The system is subject to external shocks, any one of which can produce an extreme failure in each unit with a probability equal to ω^0=0.05, deteriorated internal behaviour, as shown in matrix **W**, and cumulative damage according to the transitions shown in matrix **Q**. In response to an external shock, the internal degradation can change between any two levels. This matrix is

$$\mathbf{W} = \begin{pmatrix} 0.6 & 0.2 & 0.1 \\ 0 & 0.6 & 0.2 \\ 0 & 0 & 0.6 \end{pmatrix}.$$

Also, an external shock can provoke an internal repairable or non-repairable failure. The probability vectors depending on the internal degradation level are given by

$$\mathbf{W}_r^0 = \begin{pmatrix} 0.1 \\ 0.2 \\ 0 \end{pmatrix}, \quad \mathbf{W}_{nr}^0 = \begin{pmatrix} 0 \\ 0 \\ 0.4 \end{pmatrix}.$$

Cumulative damage to the system is composed of three stages, rising to a threshold beyond which a non-repairable failure occurs. The two first states are minor external damage (in fact, the state 1 is no external damage) and the last one is major external damage. If one external shock occurs and one unit is in state 1 then it pass to state 2 or 3 with a probability equal to 0.3 or 0.7 respectively. If the cumulative external damage is in state 2 or 3 the unit undergoes a non-repairable failure with a probability 0.4 and 1 respectively at next shock. The matrix is given by

$$\mathbf{Q} = \begin{pmatrix} 0 & 0.3 & 0.7 \\ 0 & 0 & 0.6 \\ 0 & 0 & 0 \end{pmatrix}.$$

Random inspections over the system takes place. When one inspection occurs the internal degradation level and the cumulative external damage of all operational units are inspected. The distribution times are given in Table 3.

Internal time distribution	**External shock time distribution**	**Inspection time**
$\boldsymbol{\alpha} = (1,0,0)$	$\boldsymbol{\gamma} = (1,0)$	$\boldsymbol{\eta} = (1,0)$
$\mathbf{T} = \begin{pmatrix} 0.99 & 0.002 & 0 \\ 0 & 0.9 & 0.001 \\ 0 & 0 & 0.9 \end{pmatrix}$	$\mathbf{L} = \begin{pmatrix} 0.89 & 0.1 \\ 0.1 & 0.8 \end{pmatrix}$	$\mathbf{M} = \begin{pmatrix} 0.75 & 0.1 \\ 0.25 & 0.4 \end{pmatrix}$
Mean time: 102.02	Mean time: 25	Mean time: 5.6

Table 3. Phase-type distributions of the internal behavior, external shock time and inspection time

The units in the repair facility can be there for corrective repair or for preventive maintenance. These time distributions are given in Table 4.

Corrective repair time distribution	**Preventive maintenance time distribution**
$\boldsymbol{\beta}^1 = (1,0)$	$\boldsymbol{\beta}^2 = (1,0)$
$\mathbf{S}_1 = \begin{pmatrix} 0.91 & 0.01 \\ 0 & 0.8 \end{pmatrix}$	$\mathbf{S}_2 = \begin{pmatrix} 0.006 & 0.002 \\ 0 & 0.008 \end{pmatrix}$
Mean time: 11.67	Mean time: 1.0081

Table 4. Phase-type distributions in the repair facility

As it has been mentioned above a comparison between the system with and without preventive maintenance is carried out. Firstly the proportional time in each macro-state up to a certain time has been calculated for both systems (with and without preventive maintenance) in transient and stationary case from (1-4). It is shown in Table 5.

	$\nu = 50$	$\nu = 100$	$\nu = 200$	$\nu = 500$	***Stationary***
$\psi_{3,0}(\nu)/(\nu+1)$	0.5806 (0.6082)	0.4904 (0.5343)	0.4186 (0.4911)	0.3638 (0.4649)	0.3258 (0.4473)
$\psi_{3,1}(\nu)/(\nu+1)$	0.1583 (0.1976)	0.1475 (0.1897)	0.1313 (0.1812)	0.1167 (0.1760)	0.1065 (0.1726)
$\psi_{3,2}(\nu)/(\nu+1)$	0.0749 (0.0590)	0.0773 (0.0607)	0.0713 (0.0595)	0.0645 (0.0588)	0.0596 (0.0583)
$\psi_{3,3}(\nu)/(\nu+1)$	0.0528 (0.0020)	0.0559 (0.0022)	0.0520 (0.0022)	0.0471 (0.0022)	0.0436 (0.0023)
$\psi_{3}(\nu)/(\nu+1)$	0.8666 (0.8668)	0.7711 (0.7869)	0.6731 (0.7341)	0.5921 (0.7019)	0.5354 (0.6804)
$\psi_{2,0}(\nu)/(\nu+1)$	0.0798 (0.0689)	0.1449 (0.1187)	0.2143 (0.1545)	0.2722 (0.1766)	0.3127 (0.1913)
$\psi_{2,1}(\nu)/(\nu+1)$	0.0305 (0.0502)	0.0498 (0.0764)	0.0692 (0.0920)	0.0851 (0.1015)	0.0963 (0.1078)
$\psi_{2,2}(\nu)/(\nu+1)$	0.0231 (0.0140)	0.0342 (0.0181)	0.0434 (0.0193)	0.0506 (0.0201)	0.0556 (0.0205)
$\psi_{2}(\nu)/(\nu+1)$	0.1334 (0.1332)	0.2289 (0.2131)	0.3269 (0.2659)	0.4079 (0.2981)	0.4646 (0.3196)

Table 5. Proportional time in each macro-state for the systems with and without preventive maintenance (in parenthesis)

This table shows that the operational time when there are three units in the system is greater when no preventive maintenance is performed, but not when there are only two.

In this study, we calculated the mean operational time and the mean time during which the repairperson is busy (idle), both up to a certain time and in the stationary case from (5-8). We can observe that the mean proportional operational time is 0.8112 for the system without preventive maintenance, a 8.89% higher than the case with preventive maintenance. The mean proportional time that the repairperson is busy is similar for both cases. It is given in Table 6.

	$\nu = 50$	$\nu = 100$	$\nu = 200$	$\nu = 500$	***Stationary***
$\mu_{op}(\nu)/(\nu+1)$	0.8186 (0.8748)	0.7829 (0.8426)	0.7641 (0.8268)	0.7527 (0.8174)	0.7450 (0.8112)
$\mu_{busy}(\nu)/(\nu+1)$	0.3396 (0.3228)	0.3647 (0.3471)	0.3671 (0.3544)	0.3641 (0.3586)	0.3615 (0.3614)
$\mu_{idle}(\nu)/(\nu+1)$	0.6603 (0.6772)	0.6353 (0.6529)	0.6329 (0.6456)	0.6359 (0.6414)	0.6385 (0.6386)

Table 6. Mean proportional operational time and mean proportional time that the repairperson is busy and idle for the systems with and without preventive maintenance (in parenthesis)

Our model shows that the system performs better when there is no preventive maintenance. Nevertheless, it is interesting to examine the performance cost of this situation. For example, how many repairable failures, major inspections and system restarts would occur? These questions are illustrated in Table 7 (9-14). In the stationary case, the mean number of repairable failures per unit of time is 0.0344 when there is no preventive maintenance, which is 17.01% higher than when preventive maintenance is performed. Even more important is the number of system restarts, which is 137.5% higher in the absence of preventive maintenance.

	$\nu = 50$	$\nu = 100$	$\nu = 200$	$\nu = 500$	***Stationary***
$\Lambda^{rep}(\nu)/(\nu+1)$	0.0350 (0.0390)	0.0330 (0.0368)	0.0314 (0.0356)	0.0302 (0.0349)	0.0294 (0.0344)
$\Lambda^{major}(\nu)/(\nu+1)$	0.0363	0.0352	0.0340	0.0330	0.0323
$\Lambda^{newsystems}(\nu)/(\nu+1)$	0.0035 (0.0094)	0.0041 (0.0112)	0.0047 (0.0122)	0.0052 (0.0129)	0.0056 (0.0133)

Table 7. Mean proportional number of repairable, major inspections and new systems for the systems with and without preventive maintenance (in parenthesis)

The reliability function for both systems up to first restarting has been plotted by considering the section 6.2.2. It is shown in Figure 3. The mean time up to first restarting for the model with preventive maintenance is 175.5678 and for the system without preventive maintenance is 72.2522.

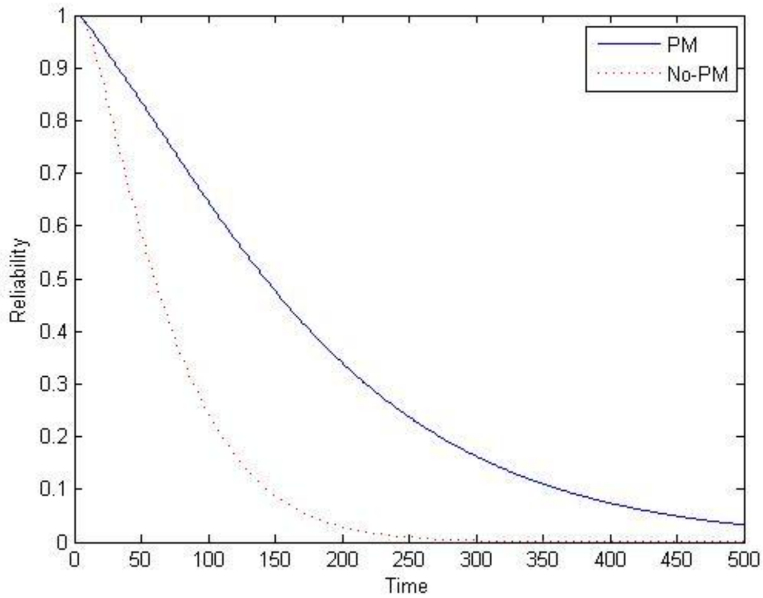

Figure 3. Reliability function between two consecutive restarting for both systems.

Rewards and costs

The performance and the number of failures have been studied previously. Costs and rewards are included next. We assume that while the system is operational a reward per unit of time equal to *B*=10 is produced, the same amount than the lost when the system is not operational (*C*=10). Also, each unit has a cost while is operational by considering the degradation level state given by the column vector $\boldsymbol{c_0}$ = (0.5, 0.7, 0.8)'.

In corrective repair and preventive maintenance states, the cost varies according to whether a unit is being repaired. The latter case is only considered when preventive maintenance is scheduled. These costs are given by the column vectors **cr**= (2, 3)' and **cpm =** (0.4, 0.6)' respectively.

Fixed costs have also introduced. Each time that a repairable failure and a major inspection occur a fix cost equal to *fcr* = 2 and *fpm* = 0.15 is produced respectively. Finally, a new unit cost *fnu* = 100.

The mean net profit per unit of time has been studied for both systems by using (15) and (16). It is plotted in Figure 4 and some values are given in Table 8. We can observe that, for the case without preventive maintenance, the system is always deficit. The situation is different for the system with preventive maintenance. In this last case, the system is profitable from the time 273. The stationary value is 0.6850 monetary units per unit of time.

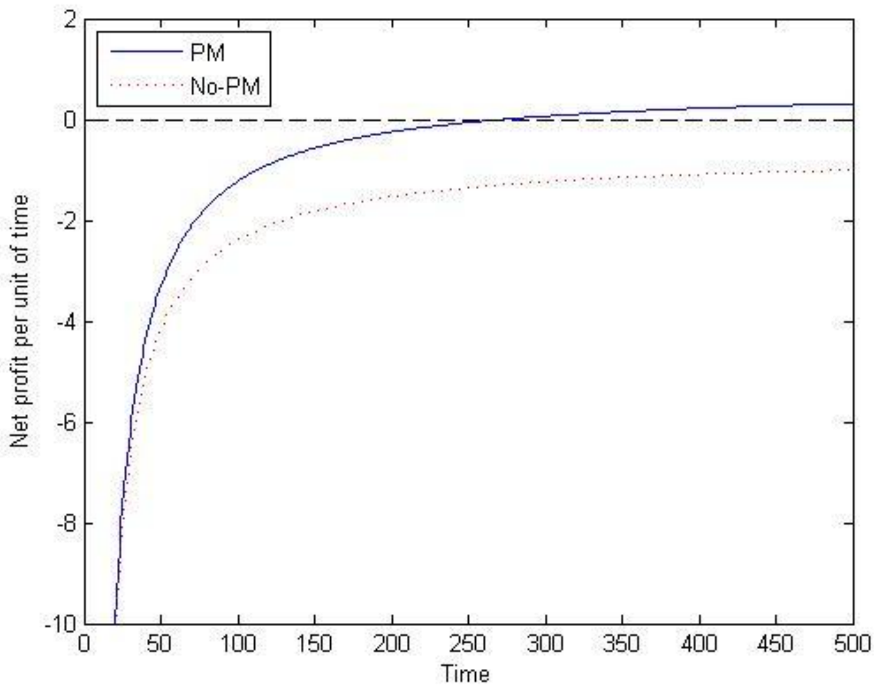

Figure 4. Mean net profit per unit of time for both systems

	$\nu = 50$	$\nu = 100$	$\nu = 200$	$\nu = 500$	***Stationary***
$\Phi(\nu)/(\nu+1)$	−3.2957 (−4.1574)	−1.2275 (−2.3966)	−0.2494 (−1.5294)	0.3121 (−1.0040)	0.6850 (−0.6520)

Table 8. Mean net profit per unit of time for the systems with and without preventive maintenance (in parenthesis)

The example shows that preventive maintenance may not improve system performance as much as would be desirable, but nevertheless it produces economic benefits.

Computational times

One of the main objectives of this study is to model a complex system using a well-structured algorithmic methodology. The results obtained were determined by this standpoint, which enabled computational implementation. The order of the state space depends on the number of units reaching a high dimension when many units are considered. This order for multiple units presenting similar features, as described in the example, is given in Table 9.

	n=1	*n*=2	*n*=3	*n*=4	*n*=5	*n*=6	*n*=7	*n*=8
k=1	44 (22)	528 (224)	5060 (2046)	46200 (18448)	417164 (166070)	3757248 (1494672)	33820820 (13452094)	304398600 (121068896)
k=2		484 (202)	5016 (2024)	46156 (18426)	417120 (166048)	3757204 (1494650)	33820776 (13452072)	304398556 (121068874)
k=3			4532 (1822)	45672 (18224)	416636 (165846)	3756720 (1494448)	33820292 (13451870)	304398072 (121068672)
k=4				41140 (16402)	412104 (164024)	3752188 (1492626)	33815760 (13450048)	304393540 (121066850)
k=5					370964 (147622)	3711048 (1476224)	33774620 (13433646)	304352400 (121050448)
k=6						3340084 (1328602)	33403656 (13286024)	303981436 (120902826)
k=7							30063572 (11957422)	300641352 (119574224)
k=8								270577780 (107616802)

Table 9. Order of the state space for systems with multiple units and similar characteristics, with and without (in parenthesis) preventive maintenance

Nevertheless, in real life, the availability and reliability of redundant systems are improved considerably when only a limited number of units are included. To analyse the utility and importance of the algorithms applied to the system, we calculated the computational times needed to build the transition probability matrix, the stationary distribution and the different measures described in this paper and applied in the numerical example. The computational time depends on the processor used and also on whether multiple computers are connected in parallel. In the present case, the study was performed with a computer with the following characteristics: Intel (R) Core (TM) i5-5257U CPU @ 2.70GHz 2.70 GHz; 8,00 GB of RAM; MATLAB. Version 7.12.0.635 (R2011a).

Given that the computational time required to calculate any measure is a random variable, the process was repeated 100 times. The mean and standard deviation values of these times (expressed in seconds) are shown in Table 10.

	Mean time (seconds)	**Standard deviation**
D	28.2068 (3.1011)	2.2907 (0.0406)
$\boldsymbol{\pi}$	15.5850 (1.1016)	1.0931 (0.0411)
A	0.000069 (0.000045)	0.000082 (0.000052)
μ_{op}	0.000136 (0.000105)	0.000195 (0.000096)
μ_{busy}	0.000084 (0.000067)	0.000164 (0.000079)
μ_{idle}	0.000102 (0.000067)	0.000614 (0.000235)
Λ^{rep}	0.4290 (0.0531)	0.0336 (0.0076)
Λ^{major}	0.4251	0.0456
$\Lambda^{newsystems}$	0.0135 (0.0024)	0.0019 (0.000955)
$\boldsymbol{\Phi}$	0.0011 (0.0011)	0.000356 (0.000304)

Table 10. Computational times for calculating multiples measures associated to the system (case without preventive maintenance in parenthesis)

As can be seen, the greatest computational times are for the transition probability matrix and the stationary distribution. The remaining times are all of less than half a second and in many cases are negligible.

8. Conclusions

In this paper a multi-state complex k-out-of-n: G system is described and modeled in a matrix and algorithmic-computational form. The system is operational when at least k units are operational. Several types of events are considered in the modeling; failures, external shocks, inspections, corrective repair and preventive maintenance. Both repairable and non-repairable failures are included in the system, the first one as a consequence of internal degradation or after one external shock. If a non-repairable failure occurs, the corresponding unit is removed without replacing. When the number of units in the system is less than or equal to $n - k - 1$, the system is replaced by a new one, otherwise it would never restart.

The complex system has been modeled and several interesting reliability measures have been built in transient and stationary regime. Costs and rewards are introduced to analyze the profitability of the system. A numerical example shows this fact. Phase-type distributions and Markovian arrival processes with marked arrivals have been considered in the development.

One of the main problems to be addressed in considering a complex multi-state system is that of the number of the phases associated with the model. As the number of units in the system increases, the order of the macro-state may become unmanageable. Nevertheless, in real problems with redundant systems it is often sufficient to consider a small number of units in order to improve the reliability of the system. With the method applied in this study, the results obtained from this complex modelling can be expressed in a matrix-algorithmic and computational form, making it possible to implement the results computationally, thus facilitating applicability. The system we propose is highly generalisable and is applicable to a large number of simultaneous events. Alternatively, smaller cases could be addressed using the same approach.

In this study, a computational time analysis was performed for the numerical example proposed with a normal computer processor. The order of the macro-state in this example is equal to 5016 and the computational time can be considered reasonable. Of course, in other circumstances the computational time will depend on the processor used and on whether multiple computers in parallel are taken into consideration.

In future research, the ideas presented in this paper can be extended in several ways. For example, different types of replacement policies, when minimal repair is considered, might be included. Another possibility could be to consider an indeterminate and variable number of repairpersons in the repair facility in a *k*-out-of-*n*: *G* system, a situation that has previously been applied in redundant systems such as warm and cold standby systems. Each time that a non-repairable failure occurs and the system can continue working, the number of repairpersons can be modified, thus enabling us to optimise the model.

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APPENDIX A

Functions f (·, ·)

- Under condition **R**

$$f_r\left(shock, insp\right)=\begin{cases}\mathbf{L}\otimes\mathbf{Me\eta} & ;\quad shock=0, insp=0\\ \mathbf{L}\otimes\mathbf{M}^0\boldsymbol{\eta} & ;\quad shock=0, insp=1\\ \mathbf{L}^0\boldsymbol{\gamma}\otimes\mathbf{Me\eta} & ;\quad shock=1, insp=0\\ \mathbf{L}^0\boldsymbol{\gamma}\otimes\mathbf{M}^0\boldsymbol{\eta} & ;\quad shock=1, insp=1\end{cases}$$

- Under condition **NR**

$$f_{nr}\left(shock, insp\right)=\begin{cases}\mathbf{L}\otimes\mathbf{M}\left(I_{\{b_r+b_{nr}+b_{mr}=l-a\text{ and }rep=0\}}\mathbf{e}+I_{\{b_r+b_{nr}+b_{mr}<l-a\text{ or }(b_r+b_{nr}+b_{mr}=l-a\text{ and }rep=1)\}}\right) & ;\quad shock=0, insp=0\\ \mathbf{L}\otimes\mathbf{M}^0\left(I_{\{b_r+b_{nr}+b_{mr}=l-a\text{ and }rep=0\}}+I_{\{b_r+b_{nr}+b_{mr}<l-a\text{ or }(b_r+b_{nr}+b_{mr}=l-a\text{ and }rep=1)\}}\boldsymbol{\eta}\right) & ;\quad shock=0, insp=1\\ \mathbf{L}^0\boldsymbol{\gamma}\otimes\mathbf{M}\left(I_{\{b_r+b_{nr}+b_{mr}=l-a\text{ and }rep=0\}}\mathbf{e}+I_{\{b_r+b_{nr}+b_{mr}<l-a\text{ or }(b_r+b_{nr}+b_{mr}=l-a\text{ and }rep=1)\}}\right) & ;\quad shock=1, insp=0\\ \mathbf{L}^0\boldsymbol{\gamma}\otimes\mathbf{M}^0\left(I_{\{b_r+b_{nr}+b_{mr}=l-a\text{ and }rep=0\}}+I_{\{b_r+b_{nr}+b_{mr}<l-a\text{ or }(b_r+b_{nr}+b_{mr}=l-a\text{ and }rep=1)\}}\boldsymbol{\eta}\right) & ;\quad shock=1, insp=1\end{cases}$$

Functions C and *d*

- Under condition **OP** and **NR**

$$C_{op\_nr}\left(l,a,b_r,b_{nr},b_{mr},shock,insp;k_r^1,\ldots,k_r^{b_r},k_{nr}^1,\ldots,k_{nr}^{b_{nr}},k_{mr}^1,\ldots,k_{mr}^{b_{mr}}\right)$$

$$=\begin{cases}\phi^1(1)\otimes\cdots\otimes\phi^1(l-a) & ; \quad shock=0, insp=0\\ \phi^2(1)\otimes\cdots\otimes\phi^2(l-a) & ; \quad shock=0, insp=1\\ \phi^3(1)\otimes\cdots\otimes\phi^3(l-a) & ; \quad shock=1, insp=0\\ \phi^4(1)\otimes\cdots\otimes\phi^4(l-a) & ; \quad shock=1, insp=1\end{cases}$$

where for $\nu = 1,\ldots,l\text{–}a$

$$\phi^1(\nu)=\begin{cases}\mathbf{T}_r^0\otimes\mathbf{e} & ; \quad \nu=k_r^1,\ldots,k_r^{b_r}\\ \mathbf{T}_{nr}^0\otimes\mathbf{e} & ; \quad \nu=k_{nr}^1,\ldots,k_{nr}^{b_{nr}}\\ \mathbf{T}\otimes\mathbf{I} & ; \quad \text{otherwise}\end{cases}$$

$$\phi^2(\nu)=\begin{cases}\mathbf{T}_r^0\otimes\mathbf{e} & ; \quad \nu=k_r^1,\ldots,k_r^{b_r}\\ \mathbf{T}_{nr}^0\otimes\mathbf{e} & ; \quad \nu=k_{nr}^1,\ldots,k_{nr}^{b_{nr}}\\ \mathbf{U}_2\left(\mathbf{e}-\mathbf{T}^0\right)\otimes\mathbf{V}_1\mathbf{e}+\left(\mathbf{e}-\mathbf{T}^0\right)\otimes\mathbf{V}_2\mathbf{e} & ; \quad \nu=k_{mr}^1,\ldots,k_{mr}^{b_{mr}}\\ \mathbf{U}_1\mathbf{T}\otimes\mathbf{V}_1 & ; \quad \text{otherwise}\end{cases}$$

$$\phi^3(\nu)=\begin{cases}\left(\mathbf{T}_r^0+\mathbf{T}\mathbf{W}_r^0\right)\otimes\mathbf{Q}\mathbf{e}\left(1-\omega^0\right) & ; \quad \nu=k_r^1,\ldots,k_r^{b_r}\\ \left(\mathbf{T}_{nr}^0+\mathbf{T}\mathbf{W}_{nr}^0\right)\otimes\mathbf{Q}\mathbf{e}\left(1-\omega^0\right)+\mathbf{e}\otimes\left(\mathbf{e}\omega^0+\mathbf{Q}^0\left(1-\omega^0\right)\right) & ; \quad \nu=k_{nr}^1,\ldots,k_{nr}^{b_{nr}}\\ \mathbf{T}\mathbf{W}\otimes\mathbf{Q}\left(1-\omega^0\right) & ; \quad \text{otherwise}\end{cases}$$

$$\phi^4(\nu)=\begin{cases}\left(\mathbf{T}_r^0+\mathbf{T}\mathbf{W}_r^0\right)\otimes\mathbf{Q}\mathbf{e}\left(1-\omega^0\right) & ; \quad \nu=k_r^1,\ldots,k_r^{b_r}\\ \left(\mathbf{T}_{nr}^0+\mathbf{T}\mathbf{W}_{nr}^0\right)\otimes\mathbf{Q}\mathbf{e}\left(1-\omega^0\right)+\mathbf{e}\otimes\left(\mathbf{e}\omega^0+\mathbf{Q}^0\left(1-\omega^0\right)\right) & ; \quad \nu=k_{nr}^1,\ldots,k_{nr}^{b_{nr}}\\ \mathbf{U}_2\mathbf{T}\mathbf{W}\mathbf{e}\otimes\mathbf{V}_1\mathbf{Q}\mathbf{e}\left(1-\omega^0\right)+\mathbf{T}\mathbf{W}\mathbf{e}\otimes\mathbf{V}_2\mathbf{Q}\mathbf{e}\left(1-\omega^0\right) & ; \quad \nu=k_{mr}^1,\ldots,k_{mr}^{b_{mr}}\\ \mathbf{U}_1\mathbf{T}\mathbf{W}\otimes\mathbf{V}_1\mathbf{Q}\left(1-\omega^0\right) & ; \quad \text{otherwise}\end{cases}$$

$$d_{op\_nr}\left(l,a,rep,b_r,b_{nr},b_{mr}\right)$$

$$= \sum_{shock=0}^{1} \sum_{insp=0}^{1} \sum_{k_r^1=1}^{l-a-b_r+1} \sum_{k_r^2=k_r^1+1}^{l-a-b_r+2} \cdots \sum_{k_r^{b_r}=k_r^{b_r-1}+1}^{l-a} \sum_{\substack{k_{nr}^1=1 \\ k_{nr}^1 \neq k_r^u \\ u=1,\ldots,b_r}}^{l-a-b_{nr}+1} \cdots \sum_{\substack{k_{nr}^{b_{nr}}=k_{nr}^{b_{nr}-1}+1 \\ k_{nr}^{b_{nr}} \neq k_r^u \\ u=1,\ldots,b_r}}^{l-a} \sum_{\substack{k_{mr}^1=1 \\ k_{mr}^1 \neq k_r^u \\ k_{mr}^1 \neq k_{nr}^v \\ u=1,\ldots,b_r \\ v=1,\ldots,b_{nr}}}^{l-a-b_{mr}+1} \cdots \sum_{\substack{k_{mr}^{b_{mr}}=k_{mr}^{b_{mr}-1}+1 \\ k_{mr}^{b_{mr}} \neq k_r^u \\ k_{mr}^{b_{mr}} \neq k_{nr}^v \\ u=1,\ldots,b_r \\ v=1,\ldots,b_{nr}}}^{l-a}$$

$$C_{op\_nr}\left(l,a,b_r,b_{nr},b_{mr},shock,insp;k_r^1,\ldots,k_r^{b_r},k_{nr}^1,\ldots,k_{nr}^{b_{nr}},k_{mr}^1,\ldots,k_{mr}^{b_{mr}}\right)$$

$$\otimes\left(I_{\{rep=0\}}+I_{\{rep=1\}}\boldsymbol{\alpha}\otimes\boldsymbol{\omega}\right)\otimes f_{nr}\left(shock,insp\right)$$

- Under condition **OP** and **R**

$$C_{op\_r}\left(l,a,b_r,b_{nr},b_{mr},shock,insp;k_r^1,\ldots,k_r^{b_r},k_{nr}^1,\ldots,k_{nr}^{b_{nr}},k_{mr}^1,\ldots,k_{mr}^{b_{mr}}\right)$$

$$=\begin{cases} \phi^1(1)\otimes\cdots\otimes\phi^1(l-a) & ; \quad shock=0, insp=0 \\ \phi^2(1)\otimes\cdots\otimes\phi^2(l-a) & ; \quad shock=0, insp=1 \\ \phi^3(1)\otimes\cdots\otimes\phi^3(l-a) & ; \quad shock=1, insp=0 \\ \phi^4(1)\otimes\cdots\otimes\phi^4(l-a) & ; \quad shock=1, insp=1 \end{cases}$$

where for $\nu = 1,\ldots,l-a$

$$\phi^1(\nu)=\begin{cases} \mathbf{T}_r^0\boldsymbol{\alpha}\otimes\mathbf{e}\boldsymbol{\omega} & ; \quad \nu=k_r^1,\ldots,k_r^{b_r} \\ \mathbf{T}_{nr}^0\boldsymbol{\alpha}\otimes\mathbf{e}\boldsymbol{\omega} & ; \quad \nu=k_{nr}^1,\ldots,k_{nr}^{b_{nr}} \\ \mathbf{T}\mathbf{e}\boldsymbol{\alpha}\otimes\mathbf{e}\boldsymbol{\omega} & ; \quad \text{otherwise} \end{cases}$$

$$\phi^2(\nu)=\begin{cases} \mathbf{T}_r^0\boldsymbol{\alpha}\otimes\mathbf{e}\boldsymbol{\omega} & ; \quad \nu=k_r^1,\ldots,k_r^{b_r} \\ \mathbf{T}_{nr}^0\boldsymbol{\alpha}\otimes\mathbf{e}\boldsymbol{\omega} & ; \quad \nu=k_{nr}^1,\ldots,k_{nr}^{b_{nr}} \\ \mathbf{U}_2\left(\mathbf{e}-\mathbf{T}^0\right)\boldsymbol{\alpha}\otimes\mathbf{V}_1\mathbf{e}\boldsymbol{\omega}+\left(\mathbf{e}-\mathbf{T}^0\right)\boldsymbol{\alpha}\otimes\mathbf{V}_2\mathbf{e}\boldsymbol{\omega} & ; \quad \nu=k_{mr}^1,\ldots,k_{mr}^{b_{mr}} \\ \mathbf{U}_1\mathbf{T}\boldsymbol{\alpha}\otimes\mathbf{V}_1\boldsymbol{\omega} & ; \quad \text{otherwise} \end{cases}$$

$$\phi^3(\nu)=\begin{cases} \left(\mathbf{T}_r^0+\mathbf{T}\mathbf{W}_r^0\right)\boldsymbol{\alpha}\otimes\mathbf{Q}\mathbf{e}\left(1-\omega^0\right)\boldsymbol{\omega} & ; \quad \nu=k_r^1,\ldots,k_r^{b_r} \\ \left(\mathbf{T}_{nr}^0+\mathbf{T}\mathbf{W}_{nr}^0\right)\boldsymbol{\alpha}\otimes\mathbf{Q}\mathbf{e}\left(1-\omega^0\right)\boldsymbol{\omega}+\mathbf{e}\boldsymbol{\alpha}\otimes\left(\mathbf{e}\boldsymbol{\omega}\omega^0+\mathbf{Q}^0\boldsymbol{\omega}\left(1-\omega^0\right)\right) & ; \quad \nu=k_{nr}^1,\ldots,k_{nr}^{b_{nr}} \\ \mathbf{T}\mathbf{W}\boldsymbol{\alpha}\otimes\mathbf{Q}\left(1-\omega^0\right)\boldsymbol{\omega} & ; \quad \text{otherwise} \end{cases}$$

$$\phi^4(\nu)=\begin{cases} \left(\mathbf{T}_r^0+\mathbf{T}\mathbf{W}_r^0\right)\boldsymbol{\alpha}\otimes\mathbf{Q}\mathbf{e}\left(1-\omega^0\right)\boldsymbol{\omega} & ; \quad \nu=k_r^1,\ldots,k_r^{b_r} \\ \left(\mathbf{T}_{nr}^0+\mathbf{T}\mathbf{W}_{nr}^0\right)\boldsymbol{\alpha}\otimes\mathbf{Q}\mathbf{e}\boldsymbol{\omega}\left(1-\omega^0\right)+\mathbf{e}\boldsymbol{\alpha}\otimes\left(\mathbf{e}\boldsymbol{\omega}\omega^0+\mathbf{Q}^0\boldsymbol{\omega}\left(1-\omega^0\right)\right) & ; \quad \nu=k_{nr}^1,\ldots,k_{nr}^{b_{nr}} \\ \mathbf{U}_2\mathbf{T}\mathbf{W}\mathbf{e}\boldsymbol{\alpha}\otimes\mathbf{V}_1\mathbf{Q}\mathbf{e}\left(1-\omega^0\right)\boldsymbol{\omega}+\mathbf{T}\mathbf{W}\mathbf{e}\boldsymbol{\alpha}\otimes\mathbf{V}_2\mathbf{Q}\mathbf{e}\left(1-\omega^0\right)\boldsymbol{\omega} & ; \quad \nu=k_{mr}^1,\ldots,k_{mr}^{b_{mr}} \\ \mathbf{U}_1\mathbf{T}\mathbf{W}\boldsymbol{\alpha}\otimes\mathbf{V}_1\mathbf{Q}\left(1-\omega^0\right)\boldsymbol{\omega} & ; \quad \text{otherwise} \end{cases}$$

$$d_{op\_r}\left(l,a,b_r,b_{nr},b_{mr}\right)$$

$$= \sum_{shock=0}^{1} \sum_{insp=0}^{1} \sum_{k_r^1=1}^{l-a-b_r+1} \sum_{k_r^2=k_r^1+1}^{l-a-b_r+2} \cdots \sum_{k_r^{b_r}=k_r^{b_r-1}+1}^{l-a} \sum_{\substack{k_{nr}^1=1 \\ k_{nr}^1 \neq k_r^u \\ u=1,\ldots,b_r}}^{l-a-b_{nr}+1} \cdots \sum_{\substack{k_{nr}^{b_{nr}}=k_{nr}^{b_{nr}-1}+1 \\ k_{nr}^{b_{nr}} \neq k_r^u \\ u=1,\ldots,b_r}}^{l-a} \sum_{\substack{k_{mr}^1=1 \\ k_{mr}^1 \neq k_r^u \\ k_{mr}^1 \neq k_{nr}^v \\ u=1,\ldots,b_r \\ v=1,\ldots,b_{nr}}}^{l-a-b_{mr}+1} \cdots \sum_{\substack{k_{mr}^{b_{mr}}=k_{mr}^{b_{mr}-1}+1 \\ k_{mr}^{b_{mr}} \neq k_r^u \\ k_{mr}^{b_{mr}} \neq k_{nr}^v \\ u=1,\ldots,b_r \\ v=1,\ldots,b_{nr}}}^{l-a}$$

$$C_{op\_r}\left(l,a,b_r,b_{nr},b_{mr},shock,insp;k_r^1,\ldots,k_r^{b_r},k_{nr}^1,\ldots,k_{nr}^{b_{nr}},k_{mr}^1,\ldots,k_{mr}^{b_{mr}}\right)$$

$$\otimes\left(\boldsymbol{\alpha}\otimes\boldsymbol{\omega}\otimes\overset{n-l+a}{\cdots}\otimes\boldsymbol{\alpha}\otimes\boldsymbol{\omega}\right)\otimes f_r\left(shock,insp\right)$$

- Under condition **NOP** and **NR**

$$C_{nop\_nr}\left(l,a,b_r,b_{nr},b_{mr},shock,insp;k_r^1,\ldots,k_r^{b_r},k_{nr}^1,\ldots,k_{nr}^{b_{nr}},k_{mr}^1,\ldots,k_{mr}^{b_{mr}}\right)$$

$$=\begin{cases} \phi^1(1)\otimes\cdots\otimes\phi^1(l-a) & ; \quad shock=0, insp=0 \\ \phi^2(1)\otimes\cdots\otimes\phi^2(l-a) & ; \quad shock=0, insp=1 \\ \phi^3(1)\otimes\cdots\otimes\phi^3(l-a) & ; \quad shock=1, insp=0 \\ \phi^4(1)\otimes\cdots\otimes\phi^4(l-a) & ; \quad shock=1, insp=1 \end{cases}$$

$$\phi^1(\nu)=\mathbf{I}_m\otimes\mathbf{I}_d$$

$$\phi^2(\nu)=\begin{cases} \mathbf{U}_2\mathbf{e}\otimes\mathbf{V}_1\mathbf{e}+\mathbf{e}\otimes\mathbf{V}_2\mathbf{e} & ; \quad b_r=b_{nr}=0, \nu=k_{mr}^1,\ldots,k_{mr}^{b_{mr}} \\ \mathbf{U}_1\otimes\mathbf{V}_1 & ; \quad \text{otherwise} \end{cases}$$

$$\phi^3(\nu)=\begin{cases} \mathbf{W}_r^0\otimes\mathbf{Qe}\left(1-\omega^0\right) & ; \quad \nu=k_r^1,\ldots,k_r^{b_r} \\ \mathbf{W}_{nr}^0\otimes\mathbf{Qe}\left(1-\omega^0\right)+\mathbf{e}\otimes\left(\mathbf{e}\omega^0+\mathbf{Q}^0\left(1-\omega^0\right)\right) & ; \quad \nu=k_{nr}^1,\ldots,k_{nr}^{b_{nr}} \\ \mathbf{W}\otimes\mathbf{Q}\left(1-\omega^0\right) & ; \quad \text{otherwise} \end{cases}$$

$$\phi^4(\nu)=\begin{cases} \mathbf{W}_r^0\otimes\mathbf{Qe}\left(1-\omega^0\right) & ; \quad \nu=k_r^1,\ldots,k_r^{b_r} \\ \mathbf{W}_{nr}^0\otimes\mathbf{Qe}\left(1-\omega^0\right)+\mathbf{e}\otimes\left(\mathbf{e}\omega^0+\mathbf{Q}^0\left(1-\omega^0\right)\right) & ; \quad \nu=k_{nr}^1,\ldots,k_{nr}^{b_{nr}} \\ \mathbf{U}_2\mathbf{We}\otimes\mathbf{V}_1\mathbf{Qe}\left(1-\omega^0\right)+\mathbf{We}\otimes\mathbf{V}_2\mathbf{Qe}\left(1-\omega^0\right) & ; \quad \nu=k_{mr}^1,\ldots,k_{mr}^{b_{mr}} \\ \mathbf{U}_1\mathbf{W}\otimes\mathbf{V}_1\mathbf{Q}\left(1-\omega^0\right) & ; \quad \text{otherwise} \end{cases}$$

$$d_{nop\_nr}\left(l,a,rep,b_r,b_{nr},b_{mr}\right)$$

$$= \sum_{shock=0}^{1} \sum_{insp=0}^{1} \sum_{k_r^1=1}^{l-a-b_r+1} \sum_{k_r^2=k_r^1+1}^{l-a-b_r+2} \cdots \sum_{k_r^{b_r}=k_r^{b_r-1}+1}^{l-a} \sum_{\substack{k_{nr}^1=1\\ k_{nr}^1\neq k_r^u\\ u=1,\dots,b_r}}^{l-a-b_{nr}+1} \cdots \sum_{\substack{k_{nr}^{b_{nr}}=k_{nr}^{b_{nr}-1}+1\\ k_{nr}^{b_{nr}}\neq k_r^u\\ u=1,\dots,b_r}}^{l-a} \sum_{\substack{k_{mr}^1=1\\ k_{mr}^1\neq k_r^u\\ k_{mr}^1\neq k_{nr}^v\\ u=1,\dots,b_r\\ v=1,\dots,b_{nr}}}^{l-a-b_{mr}+1} \cdots \sum_{\substack{k_{mr}^{b_{mr}}=k_{mr}^{b_{mr}-1}+1\\ k_{mr}^{b_{mr}}\neq k_r^u\\ k_{mr}^{b_{mr}}\neq k_{nr}^v\\ u=1,\dots,b_r\\ v=1,\dots,b_{nr}}}^{l-a}$$

$$C_{nop\_nr}\left(l,a,b_r,b_{nr},b_{mr},shock,insp;k_r^1,\dots,k_r^{b_r},k_{nr}^1,\dots,k_{nr}^{b_{nr}},k_{mr}^1,\dots,k_{mr}^{b_{mr}}\right)$$

$$\otimes\left(I_{\{rep=0\}}+I_{\{rep=1\}}\boldsymbol{\alpha}\otimes\boldsymbol{\omega}\right)\otimes f_{nr}\left(shock,insp\right)$$

- Under condition **NOP** and **R**

$$C_{nop\_r}\left(l,a,b_r,b_{nr},b_{mr},shock,insp;k_r^1,\dots,k_r^{b_r},k_{nr}^1,\dots,k_{nr}^{b_{nr}},k_{mr}^1,\dots,k_{mr}^{b_{mr}}\right)$$

$$=\begin{cases}\phi^1(1)\otimes\cdots\otimes\phi^1(l-a) & ;\quad shock=1, insp=0\\ \phi^2(1)\otimes\cdots\otimes\phi^2(l-a) & ;\quad shock=1, insp=1\end{cases}$$

$$\phi^1(\nu)=\begin{cases}\mathbf{W}_r^0\boldsymbol{\alpha}\otimes\mathbf{Qe\omega}\left(1-\omega^0\right) & ;\quad \nu=k_r^1,\dots,k_r^{b_r}\\ \mathbf{W}_{nr}^0\boldsymbol{\alpha}\otimes\mathbf{Qe\omega}\left(1-\omega^0\right)+\mathbf{e\alpha}\otimes\left(\mathbf{e\omega}\omega^0+\mathbf{Q}^0\boldsymbol{\omega}\left(1-\omega^0\right)\right) & ;\quad \nu=k_{nr}^1,\dots,k_{nr}^{b_{nr}}\\ \mathbf{We\alpha}\otimes\mathbf{Qe\omega}\left(1-\omega^0\right) & ;\quad \text{otherwise}\end{cases}$$

$$\phi^2(\nu)=\begin{cases}\mathbf{W}_r^0\boldsymbol{\alpha}\otimes\mathbf{Qe\omega}\left(1-\omega^0\right) & ;\quad \nu=k_r^1,\dots,k_r^{b_r}\\ \mathbf{W}_{nr}^0\boldsymbol{\alpha}\otimes\mathbf{Qe\omega}\left(1-\omega^0\right)+\mathbf{e\alpha}\otimes\left(\mathbf{e\omega}\omega^0+\mathbf{Q}^0\boldsymbol{\omega}\left(1-\omega^0\right)\right) & ;\quad \nu=k_{nr}^1,\dots,k_{nr}^{b_{nr}}\\ \mathbf{U}_2\mathbf{We\alpha}\otimes\mathbf{V}_1\mathbf{Qe\omega}\left(1-\omega^0\right)+\mathbf{We\alpha}\otimes\mathbf{V}_2\mathbf{Qe\omega}\left(1-\omega^0\right) & ;\quad \nu=k_{mr}^1,\dots,k_{mr}^{b_{mr}}\\ \mathbf{U}_1\mathbf{We\alpha}\otimes\mathbf{V}_1\mathbf{Qe\omega}\left(1-\omega^0\right) & ;\quad \text{otherwise}\end{cases}$$

$$d_{nop\_r}\left(l,a,b_r,b_{nr},b_{mr}\right)$$

$$= \sum_{insp=0}^{1} \sum_{k_r^1=1}^{l-a-b_r+1} \sum_{k_r^2=k_r^1+1}^{l-a-b_r+2} \cdots \sum_{k_r^{b_r}=k_r^{b_r-1}+1}^{l-a} \sum_{\substack{k_{nr}^1=1\\ k_{nr}^1\neq k_r^u\\ u=1,\dots,b_r}}^{l-a-b_{nr}+1} \cdots \sum_{\substack{k_{nr}^{b_{nr}}=k_{nr}^{b_{nr}-1}+1\\ k_{nr}^{b_{nr}}\neq k_r^u\\ u=1,\dots,b_r}}^{l-a} \sum_{\substack{k_{mr}^1=1\\ k_{mr}^1\neq k_r^u\\ k_{mr}^1\neq k_{nr}^v\\ u=1,\dots,b_r\\ v=1,\dots,b_{nr}}}^{l-a-b_{mr}+1} \cdots \sum_{\substack{k_{mr}^{b_{mr}}=k_{mr}^{b_{mr}-1}+1\\ k_{mr}^{b_{mr}}\neq k_r^u\\ k_{mr}^{b_{mr}}\neq k_{nr}^v\\ u=1,\dots,b_r\\ v=1,\dots,b_{nr}}}^{l-a}$$

$$C_{nop\_r}\left(l,a,b_r,b_{nr},b_{mr},1,insp;k_r^1,\dots,k_r^{b_r},k_{nr}^1,\dots,k_{nr}^{b_{nr}},k_{mr}^1,\dots,k_{mr}^{b_{mr}}\right)$$

$$\otimes\left(\boldsymbol{\alpha}\otimes\boldsymbol{\omega}\otimes\overset{n-l+a}{\cdots}\otimes\boldsymbol{\alpha}\otimes\boldsymbol{\omega}\right)\otimes f_r\left(1,insp\right)$$

Appendix B

Matrix no events: $\mathbf{D}^{O}$

Matrix blocks different to zero,

$$\mathbf{D}^{O} = \left(\mathbf{R}^{O,l,l}\right)_{(n-k+1)x(n-k+1)} \ , k \le l \le n$$

$\mathbf{R}^{O,l,l} = \left(\mathbf{B}_{ij}^{O,l,l}\right)_{(l+1)x(l+1)}$ with $i = 0,\ldots, l$ and $j = \max\{0,i-1\},\ldots, i$.

- Under **OP** and **NR**

$$\mathbf{B}_{00}^{O,l,l} = d_{op\_nr}\left(l,0,0,0,0,0\right),$$

If $i = 0, \ldots, l-k$ and $j = \max\{0,i-1\},\ldots, i$ then

$$\mathbf{B}_{ij}^{O,l,l}\left(s_1,\ldots s_j; x_1,\ldots x_i\right) = \begin{cases} \mathbf{B}_{10}^{O,l,l}\left(x_1\right) = d_{op\_nr}\left(l,1,1,0,0,0\right)\otimes \mathbf{S}_{x_1}^{0} & ; \quad x_1 = 1,2 \\ d_{op\_nr}\left(l,i,1,0,0,0\right)\otimes \mathbf{S}_{x_1}^{0} \otimes \boldsymbol{\beta}^{x_2} & ; \quad \begin{cases} i > 1,\ j = i-1 \\ s_{y-1} = x_y;\ y = 2,\ldots, i \end{cases} \\ d_{op\_nr}\left(l,i,0,0,0,0\right)\otimes \mathbf{S}_{x_1} & ; \quad \begin{cases} i > 0,\ j = i \\ s_y = x_y;\ y = 1,\ldots, i \end{cases} \end{cases}$$

- Under **NOP** and **NR**

If $i = l-k+1, \ldots, l$ and $j = \max\{0,i-1\},\ldots, i$ then

$$\mathbf{B}_{10}^{O,1,1}\left(x_1\right) = \boldsymbol{\alpha}\otimes\boldsymbol{\omega}\otimes\left(\mathbf{L}+\mathbf{L}^{0}\boldsymbol{\gamma}\right)\otimes\boldsymbol{\eta}\otimes\mathbf{S}_{x_1}^{0} \quad ; \quad x_1 = 1,2$$

$$\mathbf{B}_{11}^{O,1,1}\left(s_1; x_1\right) = \left(\mathbf{L}+\mathbf{L}^{0}\boldsymbol{\gamma}\right)\otimes\mathbf{S}_{x_1} \quad ; \quad x_1 = s_1 = 1,2$$

For $l > 1$

$$\mathbf{B}_{10}^{O,l,l}\left(x_1\right) = d_{nop\_nr}\left(l,1,1,0,0,0\right)\otimes\mathbf{S}_{x_1}^{0} \quad ; \quad x_1 = 1,2$$

$$\mathbf{B}_{i,i-1}^{O,l,l}\left(s_1,\ldots,s_{i-1};x_1,\ldots,x_i\right)=d_{nop\_nr}\left(l,i,1,0,0,0\right)\otimes\mathbf{S}_{x_1}^{0}\otimes\left(I_{\{i=1\}}+I_{\{i>1\}}\boldsymbol{\beta}^{x_2}\right)\quad;\quad \begin{array}{l} i<l \\ x_y=s_{y-1}=1,2;\,y=2,\ldots,i \end{array}$$

$$\mathbf{B}_{l,l-1}^{O,l,l}\left(s_1,\ldots,s_{i-1};x_1,\ldots,x_i\right)=\boldsymbol{\alpha}\otimes\boldsymbol{\omega}\otimes\left(\mathbf{L}+\mathbf{L}^0\boldsymbol{\gamma}\right)\otimes\boldsymbol{\eta}\otimes\mathbf{S}_{x_1}^{0}\otimes\left(I_{\{l=1\}}+I_{\{l>1\}}\boldsymbol{\beta}^{x_2}\right)\quad;\quad x_y=s_{y-1}=1,2;\,y=2,\ldots,i$$

$$\mathbf{B}_{i,i}^{O,l,l}\left(s_1,\ldots,s_{i-1};x_1,\ldots,x_i\right)=d_{nop\_nr}\left(l,i,0,0,0,0\right)\otimes\mathbf{S}_{x_1}\quad;\quad \begin{array}{l} i<l \\ x_y=s_y=1,2;\,y=1,\ldots,i \end{array}$$

$$\mathbf{B}_{l,l}^{O,l,l}\left(s_1,\ldots,s_l;x_1,\ldots,x_l\right)=\left(\mathbf{L}+\mathbf{L}^0\boldsymbol{\gamma}\right)\otimes\mathbf{S}_{x_1}\quad;\quad x_y=s_y=1,2;\,y=1,\ldots,i$$

Matrix no events: $\mathbf{D}^{NS}$

Matrix blocks different to zero,

For b_{nr} = 1,…, n

$$\mathbf{D}^{NS}=\left(\mathbf{R}^{NS,l,n}\right)_{(n-k+1)x(n-k+1)}\text{ for } l=\max\{b_{nr},k\},\ldots,\min\{b_{nr}+k-1,n\}$$

$$\mathbf{R}^{NS,l,n}=\left(\mathbf{B}_{i0}^{NS,l,n}\right)_{(l+1)x(l+1)}\text{ with } i=\max\{l-b_{nr}-k+1,0\},\ldots,l-b_{nr}$$

- Under **OP** and **R**

$$\mathbf{B}_{i0}^{NS,l,n}=\mathbf{e}_{2^i}\otimes\left[\sum_{rep=0}^{1}\sum_{b_{nr}=0}^{l-i}\sum_{b_r=0}^{l-i-b_{nr}}\sum_{b_{mr}=0}^{l-i-b_{nr}-b_r}d_{op\_r}\left(l,i,b_{nr},b_r,b_{mr}\right)\right]\otimes\mathbf{e}$$

- Under **NOP** and **R**

$$\mathbf{B}_{i0}^{NS,l,n}=\mathbf{e}_{2^i}\otimes\left[\sum_{rep=0}^{1}\sum_{b_{nr}=0}^{l-i}\sum_{b_r=0}^{l-i-b_{nr}}\sum_{b_{mr}=0}^{l-i-b_{nr}-b_r}d_{nop\_r}\left(l,i,b_{nr},b_r,b_{mr}\right)\right]\otimes\mathbf{e}$$

Matrix no events: $\mathbf{D}^{b_{nr},b_r,b_{mr}}$

Matrix blocks different to zero,

For b_{nr}, b_r, b_{mr} s.a. $1\le b_{nr}+b_r+b_{mr}\le n$,

$$\mathbf{D}^{b_{nr},b_r,b_{mr}}=\left(\mathbf{R}^{b_{nr},b_r,b_{mr};l,l-b_{nr}}\right)_{(n-k+1)\,x\,(n-k+1)}\;;\; l=\max\{k+b_{nr},b_r+b_{nr}+b_{mr}\},\ldots,n\,.$$

For b_{nr}, b_r, b_{mr} s.a. $1\le b_{nr}+b_r+b_{mr}\le l$,

$$\mathbf{R}^{b_{nr},b_r,b_{mr};l,l-b_{nr}} = \left(\mathbf{B}_{ij}^{b_{nr},b_r,b_{mr};l,l-b_{nr}}\right)_{(l+1)x(l-b_{nr}+1)} \;;\; i \le l-\max\{k, b_{nr}+b_r+b_{mr}\},\; b_{nr} \le l-k \text{ and}$$

$$j = \begin{cases} b_r+b_{mr}+i-1, b_r+b_{mr}+i & ; \quad i>0 \\ b_r+b_{mr} & ; \quad i=0 \end{cases}$$

- Under **OP** and **NR**

$$\mathbf{B}_{00}^{b_{nr},b_r=0,b_{mr}=0;l.l-b_{nr}} = d_{op\_nr}\left(l,0,0,b_{nr},0,0\right)$$

For b_r >0 or/and b_{mr} >0

$$\mathbf{B}_{0,b_{nr}+b_{mr}}^{b_{nr},b_r,b_{mr};l.l-b_{nr}}\left(s_1,\ldots,s_j\right) = d_{op\_nr}\left(l,0,b_{nr},b_r,b_{mr}\right)\otimes\boldsymbol{\beta}^{s_1} \quad ; \quad \begin{array}{l} s_y=1;\ y=1,\ldots,b_r \\ s_y=2;\ y=b_r+1,\ldots,j \end{array}$$

$$\mathbf{B}_{10}^{b_{nr},b_r=0,b_{mr}=0;l.l-b_{nr}}\left(x_1\right) = d_{op\_nr}\left(l,1,1,b_{nr},0,0\right)\otimes\mathbf{S}_{x_1}^0 \quad ; \quad x_1=1,2$$

For i > 0

$$\mathbf{B}_{i,i+b_r+b_{mr}}^{b_{nr},b_r,b_{mr};l.l-b_{nr}}\left(s_1,\ldots,s_{i+b_r+b_{mr}} \mid x_1,\ldots,x_i\right) = d_{op\_nr}\left(l,i,0,b_{nr},b_r,b_{mr}\right)\otimes\mathbf{S}_{x_1} \quad ; \quad \begin{array}{l} x_y=s_y,\ y=1,\ldots,i \\ s_y=1;\ y=i+1,\ldots,i+b_r\ \left(\text{if } b_r>0\right) \\ s_y=2;\ y=i+b_r+1,\ldots,i+b_r+b_{mr} \end{array}$$

For (i =1 and (b_r >0 or b_{mr} >0)) or i >1

$$\mathbf{B}_{i,i+b_r+b_{mr}-1}^{b_{nr},b_r,b_{mr};l.l-b_{nr}}\left(s_1,\ldots,s_{i+b_r+b_{mr}-1} \mid x_1,\ldots,x_i\right) = d_{op\_nr}\left(l,i,1,b_{nr},b_r,b_{mr}\right)\otimes\mathbf{S}_{x_1}^0\otimes\boldsymbol{\beta}^{x_1} \quad ; \quad \begin{array}{l} x_y=s_y,\ y=1,\ldots,i \\ s_y=1;\ y=i+1,\ldots,i+b_r\ \left(\text{if } b_r>0\right) \\ s_y=2;\ y=i+b_r+1,\ldots,i+b_r+b_{mr} \end{array}$$

- Under **NOP** and **NR**

$$\mathbf{B}_{10}^{b_{nr},b_r=0,b_{mr}=0;l.l-b_{nr}}\left(x_1\right) = b_{nop\_nr}\left(l,1,1,b_{nr},0,0\right)\otimes\mathbf{S}_{x_1}^0 \quad ; \quad x_1=1,2$$

For (i =1 and (b_r >0 or b_{mr} >0)) or i >1

$$\mathbf{B}_{i,i+b_r+b_{mr}}^{b_{nr},b_r,b_{mr};l.l-b_{nr}}\left(s_1,\ldots,s_{i+b_r+b_{mr}} \mid x_1,\ldots,x_i\right)=d_{nop\_nr}\left(l,i,0,b_{nr},b_r,b_{mr}\right)\otimes\mathbf{S}_{x_1} \quad ; \quad \begin{array}{l} x_y=s_y,\ y=1,\ldots,i \\ s_y=1;\ y=i+1,\ldots,i+b_r\ (\text{if } b_r>0) \\ s_y=2;\ y=i+b_r+1,\ldots,i+b_r+b_{mr} \end{array}$$

$$\mathbf{B}_{i,i+b_r+b_{mr}-1}^{b_{nr},b_r,b_{mr};l.l-b_{nr}}\left(s_1,\ldots,s_{i+b_r+b_{mr}-1} \mid x_1,\ldots,x_i\right)=d_{nop\_nr}\left(l,i,1,b_{nr},b_r,b_{mr}\right)\otimes\mathbf{S}_{x_1}^{0}\otimes\boldsymbol{\beta}^{x_1} \quad ; \quad \begin{array}{l} x_y=s_y,\ y=1,\ldots,i \\ s_y=1;\ y=i+1,\ldots,i+b_r\ (\text{if } b_r>0) \\ s_y=2;\ y=i+b_r+1,\ldots,i+b_r+b_{mr} \end{array}$$