Theory of Symplectic Twist Maps applied to Point-vortex Dynamics

TESIS DOCTORAL

por

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Programa de Doctorado *FisyMat*

Marzo, 2021

Editor: Universidad de Granada. Tesis Doctorales
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ISBN: 978-84-1306-901-2
URI: http://hdl.handle.net/10481/69078

Agradecimientos

Después de tanto tiempo y tantas idas y vueltas, ya me dispongo a escribir estas líneas de agradecimientos. Intento echar la vista atrás y no han sido pocas cosas las que han ocurrido a lo largo de estos años en esta querida ciudad.

En primer lugar, agradezco a mis directores Rafa y Pedro. Por darme la oportunidad de hacer el doctorado con vosotros y por la paciencia, especialmente en este último proceso de escritura de la memoria. Las reuniones con Rafa siempre han sido esclarecedoras. Esa visión sobre las matemáticas que, con aparente sencillez, sabes transmitir es muy valiosa, sobretodo para alguien que viene de la Física. Agradezco también la disponibilidad y amabilidad de Pedro para cualquier consulta.
A Stefano, gracias por acogerme tan amablemente en Madrid y en Pisa. Desde el primer momento me sentí muy cómodo trabajando contigo. Gracias por la paciencia y por todas las facilidades que me has brindado. Me acuerdo también de la buena acogida por parte de Ángela.
Anche ai miei coinquilini a Pisa: Dami, Samu e Matteo. Apprezzo la buona accoglienza e la bella atmosfera che c'era nella casa. Un ringraziamento speciale a Dami, per avermi permesso di suonare la sua chitarra. Tre mesi senza suonare la chitarra sarebbe stati molto lunghi.
Por supuesto, a todos los compañeros de despacho que he tenido a lo largo de este tiempo: Luis, Ricarda, David, Claudia, P.G., Sevilli y Tawer. El buen ambiente que se ha respirado han hecho más llevaderos los momentos difíciles a lo largo de la investigación. Por no mencionar lo que he aprendido sobre citonemas, Kuramotos, etc. No me olvido de los compañeros del Máster: Carlos, Paco... Ni tampoco del resto del Departamento de Matemática Aplicada y compañeros de docencia, especialmente a Juan Calvo, Antonio Ureña, Juan Soler y Juan Campos. También me acuerdo de otros colegas: Björn Gebhard, David Rojas, Rafa López, Javi Aguilera y Óscar Rodríguez.

A todas las amistades cosechadas en estos años y lo vívido juntos: Salva, P.G. gracias por esa *little help*, Tawer, Sevilli, Cris, por todo lo que hemos compartido, Gimmy, Ana, los Lemons: Simo y Toni, Lore, Guille, Julián, Jesús, Pablo y esas rutas del pedal, especialmente el episodio INFOCA, ¿cuántos?¿más de 5 pinchazos en cada rueda?. No me olvido, Rocío, gracias por estos buenos momentos.
A los *Mulhaceners*, especialmente a Loren y a Esther (este año hacemos la Integral!!...y nos aprendemos la Jácara). La imagen del cometa NEOWISE entre el Mulhacén y la Alcazaba dificilmente se me va a olvidar.
A mi gente de Toledo: Javi, Luai, Patri, Raquel, Giuseppe, Volti, Segovia,

Ester y Carlos. Este trabajo también tiene un poco de vosotros.

Especialmente, y con mucho cariño, a mi familia: mis padres, Pepe y Nines, a Miguel y Celia, Marina y Vane, a mis sobrinos muniqueses, Elena y Gabriel. Gracias a vuestro apoyo he podido llegar hasta este punto en mi carrera, aunque no haya sido fácil. También al resto de mi familia, tíos y primos. Algunos que tanto se han preocupado por esta etapa.

Para terminar quiero acordarme también del Cineclub Universitario y de la amplia vida cultural de esta ciudad, especialmente de la vida musical. Esos momentos de belleza (sin desmerecer a las rutas en bici) han hecho el complemento perfecto a la vida predoctoral.

Resumen

Esta tesis se centra en el estudio de la dinámica de un modelo idealizado en el campo de la dinámica de fluidos, conocido como el problema del vórtice puntual. En un fluido ideal, un vórtice puntual es esencialmente una singularidad de la vorticidad, es decir una delta de Dirac. Los vórtices surgen naturalmente en la atmósfera o en el océano. Su influencia en el transporte pasivo de partículas juega un papel importante en una variedad de disciplinas relacionadas con la hidrodinámica y la geofísica. Este tipo de dinámica aparece también en la física de los condensados de Bose-Einstein.
Matemáticamente, nuestro interés también ha sido el estudio de la dinámica alrededor de una singularidad del campo de velocidades en un sistema dinámico de baja dimensión. En concreto, hemos estudiado perturbaciones periódicas a un sistema hamiltoniano integrable en el plano que tiene una singularidad en una o más variables canónicas.

El modelo estudiado ha sido el siguiente. Consideremos el hamiltoniano perturbado

$$\Psi(t,x,y) = \frac{1}{2}\ln(x^2+y^2) + p(t,x,y), \tag{0.0.1}$$

y el sistema asociado

$$\begin{cases} \dot{x} = \partial_y \Psi(t,x,y) = \frac{y}{x^2+y^2} + \partial_y p(t,x,y) \\ \\ \dot{y} = -\partial_x \Psi(t,x,y) = -\frac{x}{x^2+y^2} - \partial_x p(t,x,y) \end{cases} \tag{0.0.2}$$

definido en un entorno $\mathcal{U} \setminus \{0\}$ del origen.

Físicamente, el sistema puede interpretarse como un modelo de la advección o transporte de una partícula bajo la acción de un vórtice colocado en el origen con un flujo periódico de fondo. Sin perturbación, $p = 0$, la dinámica del sistema es integrable y las soluciones se corresponden con circulos concéntricos alrededor del origen. La frecuencia de rotación es inversamente proporcional al radio del círculo, de tal manera que tiende a infinito cuando nos acercamos a la singularidad en el origen. A lo largo de la investigación hemos estudiado cómo esta dinámica integrable se ve afectada por la superposición de una función dependiente del tiempo periódica externa $p(t,x,y)$. Los resultados se han plasmado en los siguientes artículos:

- R. Ortega, V. Ortega, P.J. Torres, *Point-vortex stability under the influence of an external periodic flow*, Nonlinearity, **31** (2018) pp. 1849-1867.

- S. Maro, V. Ortega, *Twist dynamics and Aubry-Mather sets around a periodically perturbed point-vortex*. J. Differential Equations **269** (2020) pp. 3624-3651.

Metodología y resultados

En el primer artículo hemos estudiado la estabilidad (en sentido Lyapunov) del origen en el sistema perturbado. Sin perturbación presente, el sistema resulta integrable y el origen es estable en sentido Lyapunov. Mediante un teorema KAM, el teorema Twist de Moser o también llamado el teorema de la curva invariante, hemos obtenido condiciones suficientes sobre la perturbación para garantizar la estabilidad de la singularidad. La aplicación del teorema twist nos permite encontrar una familia de curvas invariantes por la aplicación de Poincaré asociada al sistema. Esto resulta importante ya que, debido a la baja dimensionalidad del sistema, éstas actuarán como barreras para las soluciones y por tanto, quedará garantizada la estabilidad del origen.

En el segundo artículo hemos aplicado la teoría de Aubry-Mather a nuestro problema. Esta teoría nos ha dado la existencia de conjuntos invariantes para perturbaciones más generales que las consideradas en el artículo anterior. Estos conjuntos invariantes se corresponden con la existencia de soluciones con un número de rotación definido: periódicas para un número de rotacion racional y cuasi-periódicas para uno irracional. Para aplicar la teoría necesitamos que la aplicación de Poincaré asociada al sistema sea una aplicación twist exacto simpléctica definida en el cílindro. Habitualmente, el teorema de Mather exige que la función generatriz (definida a partir de la aplicación de Poincaré) esté definida en todo el plano $\mathbb{R}^2$, sin embargo, en nuestra aplicación la función generatriz queda definida en un subconjunto del plano. Por lo tanto, necesitamos un teorema de extensión para extender el dominio de la función generatriz a todo el plano y así poder aplicar el Teorema de Mather.

Contenido de la tesis

Esbozemos el contenido de los capítulos que se incluyen en esta tesis.
En el primer Capítulo daremos una panorámica general al contenido de la memoria.

En el Capítulo 2, presentaremos el modelo físico: *vórtice puntual con perturbación*. Motivaremos las ecuaciones de movimiento de nuestro modelo y analizaremos algunas propiedades de interés para el estudio.

El tercer capítulo está dedicado a revisar las principales herramientas matemáticas utilizadas a lo largo de la investigación. En la Sección 3.1 introduciremos algunos hechos básicos de aplicaciones twist exacto simplécticas definidas en el cilindro y sus propiedades en el correspondiente *levantamiento*, es decir, la aplicación definida en el recubrimiento universal del cilindro. Definiremos las propiedades exacto simpléctica y su interpretación geométrica. Continuaremos con la definición de la propiedad twist y la propiedad de intersección. La siguiente sección, 3.2 está dedicada a estudiar algunas propiedades

de la aplicación de Poincaré.
En las siguientes secciones, 3.3 y 3.4, presentaremos las teorías aplicadas en nuestros resultados, la teoría de KAM y la teoría de Aubry-Mather. En la Sección 3.3 daremos una breve introducción histórica a la teoría KAM con un significado intuitivo y enunciaremos el teorema de la curva invariante de Moser en la versión analítica. La sección 3.4 presenta un teorema de Aubry-Mather generalizado adaptado al estudio del mapa de Poincaré correspondiente al problema del vórtice puntual perturbado (0.0.2).

Los siguientes capítulos tratan de los resultados de nuestra investigación. El capítulo 4 presentará el resultado relativo a la estabilidad alrededor de la singularidad a través de la existencia de curvas invariantes alrededor del origen.
En primer lugar, desde un punto de vista más analítico, parece necesario enunciar una noción matemática precisa de estabilidad en torno a una singularidad. La Sección 4.1 está dedicada a establecer una definición rigurosa de estabilidad de una singularidad en este contexto. Consideraremos la estabilidad perpetua en el sentido de Lyapunov, lo que significa que una solución de (0.0.2) con una condición inicial pequeña permanecerá cerca de la singularidad para siempre.
Las secciones restantes de este capítulo están dedicadas a enunciar el teorema de estabilidad para el problema del vórtice puntual perturbado y la demostración correspondiente.

El capítulo 5 contiene el resultado relativo a la aplicación de la teoría de Aubry-Mather y la existencia de soluciones periódicas y cuasi-periódicas cercanas a la singularidad.

Finalmente, en el capítulo 6 presentaremos un resumen de las conclusiones de ambos resultados. También mostraremos algunos posibles trabajos futuros en la dirección de esta tesis.

Contents

Chapter 1

Introduction

In this monograph we present an example of the application of the qualitative theory of ordinary differential equations (ODE's) to a problem with a clear physical inspiration. Physically, we want to study the dynamics of a toy model in ideal fluid dynamics known as the *point-vortex problem*. Vortices arise naturally in the atmosphere or in the ocean, also they can be created artificially in stirred fluids. Its influence in the passive transport of particles plays an important role in a variety of disciplines related to Hydrodynamics and Geophysics. Furthermore this kind of dynamics appears in the physics of Bose-Einstein condensates (see [14, 25, 71, 83, 84] for additional information). Mathematically, our interest is also the study of the dynamics near the singularities in the velocity field of a low dimensional dynamical system. Specifically, we treat perturbations of an integrable Hamiltonian system which have a singularity in one or more canonical variables. By singularity we mean that there exists a jump discontinuity or an essential discontinuity in the velocity field.

We are concerned with the description of the dynamics of the perturbed point-vortex problem. Later, we will give the details of this model, but it is important to note that we only consider classes of time-periodic perturbations. In our work we have studied a stability question in the periodically perturbed model. Furthermore, we have looked for a description and a classification of orbits in the same model. So, it is important to fix some ideas about the mathematical methods that we have used.

Dynamics through symplectic maps. Poincaré map. The dynamics of our model with a periodic field can be described through a *symplectic map*. As we can see in [9], [13] and [47], this kind of maps appears in the qualitative analysis of Hamiltonian systems, convex billiards, and many other instances. The relation between the continuous motion of a Hamiltonian system of 2 degrees of freedom with the dynamics of symplectic maps was shown by Poincaré in the study of periodic orbits in Celestial Mechanics. This resulted in the so-called *Poincaré map*, *Poincaré return mapping* or *first return map* and has become an essential tool in the qualitative theory of ordinary differential equations. More generally, to study the flow in an n-dimensional phase space near a periodic orbit one can associate a mapping of an $(n-1)$-

dimensional neighborhood of a fixed point of this mapping. So, a Poincaré map can be interpreted as a discrete dynamical system with a phase space that is one dimension smaller than the original continuous dynamical system. There is a correspondence between many qualitative properties of this map with the qualitative properties of the original system.

For example, a *periodic orbit* of a periodic vector field in the plane corresponds to a fixed point of the associated Poincaré map. *Generalized periodic orbits* or solutions with rational frequencies, are associated with cycles of the Poincaré map. Furthermore, solutions with incommensurable frequencies, that is, *quasi-periodic orbits* are associated to certain invariant sets of the Poincaré map. These invariant sets are the so-called *invariant curves* or *invariant tori*. In the course notes [66] and [65] we can find more information about this relation.

A Poincaré map of a Hamiltonian system is an example of a symplectic map. Hence the importance of the study of these maps in an abstract setting.

Exact symplectic twist maps. Generating function. In this monograph we restrict our study to maps with specific properties. These properties have relation with the usual maps coming from a large class of low dimensional Hamiltonian systems. Specifically, we consider maps that are diffeomorphisms defined on the cylinder. The reason is that we can transform our system to the corresponding action-angle variables, and the phase space becomes a strip of the cylinder. Also, this is reflected assuming certain hypotheses of regularity and periodicity in our model. Furthermore we assume that the map is *exact symplectic*. This condition implies the preservation of the area, and it is related with the Hamiltonian structure of our model. In addition, it is common for maps coming from a Hamiltonian context to have the so-called *twist property*. In fact, it is related with a non-degeneracy condition for the frequency map of the Hamiltonian.

We denote by $\mathfrak{C} = \mathbb{R} \times \mathbb{T}$ with $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ the cylinder and consider the strip $\Sigma :=]a, b[\times\mathbb{T}$ with $-\infty \leq a < b \leq \infty$. So we consider a $\mathcal{C}^k$ embedding

$$\begin{array}{rccl} \Phi : & \Sigma & \longrightarrow & \mathfrak{C} \\ & (r,\theta) & \longmapsto & (r_1,\theta_1) = (\mathcal{F}(r,\theta), \mathcal{G}(r,\theta)) \,. \end{array}$$

We suppose that Φ is exact symplectic with respect to the 1-form $r\mathrm{d}\theta$, and that Φ has the twist property, that is

$$\partial_r \mathcal{G}(r,\theta) > 0 \qquad \forall (r,\theta) \in \Sigma. \tag{1.0.1}$$

To understand (1.0.1) suppose that θ is an angle and r a radial variable. The twist condition means that the evolution of the angular variable θ by the map Φ depends on the radial variable r. If the initial radial position r increases then the angular displacement $\theta_1 - \theta$ increases. Geometrically, the twist condition means that Φ maps each vertical segment in the cylinder into a curve that always twists around the cylinder to the right. For example, the Poincaré maps associated to the equations of motion of the pendulum or the integrable point-vortex system have the twist property. In the last case it is important to note that the twist property has the equivalent definition $\partial_r \mathcal{G}(r,\theta) < 0$.

On the other hand, the exact symplectic condition is related with the preservation of the area and it is a stronger property than being symplectic with respect to the symplectic form $\lambda = \mathrm{d}r \wedge \mathrm{d}\theta$. Let us recall that being symplectic means that

$$\mathrm{d}r_1 \wedge \mathrm{d}\theta_1 = \mathrm{d}r \wedge \mathrm{d}\theta, \qquad \forall (r,\theta) \in \Sigma.$$

Or equivalently, the Jacobian $\det \Phi' = 1$ on Σ, which is the classical definition of an area preserving map. However, a map is said exact symplectic if the differential form

$$\beta = r_1\,\mathrm{d}\theta_1 - r\,\mathrm{d}\theta, \quad (r,\theta) \in \Sigma.$$

is exact in the cylinder, that means that there exists a $\mathcal{C}^2$ function

$$\begin{array}{rccc} \mathcal{S}: & \Sigma & \longrightarrow & \mathbb{R} \\ & (r,\theta) & \longmapsto & \mathcal{S}(r,\theta) \end{array}$$

such that

$$\mathrm{d}\mathcal{S}(r,\theta) = r_1\,\mathrm{d}\theta_1 - r\,\mathrm{d}\theta, \quad \forall (r,\theta) \in \Sigma$$

and a 2π-periodicity in θ.

Exact symplectic twist maps in the cylinder have become an essential tool in the modern theory of low dimensional dynamical systems and there is an extensive published literature. See for instance [9, 13, 23, 31, 46, 50, 81]. As said before, it is common for these maps to appear in the study of autonomous Hamiltonian systems with 2 degrees of freedom, or in the case of 1 degree of freedom and time periodic. In the last case it is also known as a Hamiltonian system of 1.5 degrees of freeedom.

Consider the lift map of Φ, that is, the map defined in a subset of the universal cover of the cylinder ($\tilde{\Sigma} :=]a,b[\times \mathbb{R} \subset \mathbb{R}^2$) with a periodicity condition and an additional hypothesis for the lifted angular variable x,

$$\begin{array}{rccc} \tilde{\Phi}: & \tilde{\Sigma} & \longrightarrow & \mathbb{R}^2 \\ & (r,x) & \longmapsto & (r_1,x_1) = (\mathcal{F}(r,x), \mathcal{G}(r,x)) \end{array} \tag{1.0.2}$$

where

$$\begin{aligned} \mathcal{F}(r,x+2\pi) &= \mathcal{F}(r,x), \\ \mathcal{G}(r,x+2\pi) &= \mathcal{G}(r,x) + 2\pi. \end{aligned}$$

The previous properties are translated to this lift, nevertheless the exact symplectic condition deserves an additional comment. The Poincaré lemma in the context of differential forms says that on a contractible domain, for example $\mathbb{R}^2$, every closed form is exact. But this is not true on non-contractible domains or not simply connected domains, as a cylinder: not all closed forms are exact in these domains. As the condition of area preserving maps or symplectic maps implies that the form β is closed, a symplectic map defined on the plane implies the exact symplectic condition. Therefore, we can expect that there are symplectic maps defined on the cylinder which are not exact. Later, we will illustrate this difference with some examples.

An important fact is that this map $\tilde{\Phi}$, can be expressed implicitly through a function $h = h(x, x_1)$:

$$\begin{cases} \partial_2 h(x, x_1) = r_1, \\ \partial_1 h(x, x_1) = -r \end{cases}$$

where ∂_i represents the derivative with respect to the i-th argument. In Lemma 3.4.1 we will see that $h(x, x_1)$ and $\mathcal{S}(r, \theta)$ are related by the twist condition on the lift map and the implicit function problem $x_1 = \mathcal{G}(r, x)$. So, we recover the original map $\tilde{\Phi}$ from a single function h. This function h is called *generating function* and its role is analogue to the Hamiltonian function in Hamiltonian systems. More accurately, h can be seen as the discrete analogous of the Lagrangian function in the continuous system, see [59]. Following this analogy, a variational principle is proposed in this discrete setting, constructing an action functional and studying the critical points:

$$\sum_n h(x_n, x_{n+1})$$

These critical points are in correspondence to some orbits generated by the symplectic twist map Φ. As a consequence h appears in the *discrete Euler-Lagrange equation*

$$\partial_1 h(\overline{x}_n, \overline{x}_{n+1}) + \partial_2 h(\overline{x}_{n-1}, \overline{x}_n) = 0, \qquad \forall n \in \mathbb{Z}$$

where $(\overline{r}_n, \overline{x}_n)_{n\in\mathbb{Z}}$ is a sequence of iterates of $\tilde{\Phi}$. Solutions of this equation are the orbits that minimize the action functional. We will see that the generating function is a key tool in the usual formulation of the Aubry-Mather theory for exact symplectic twist maps.

KAM theory and Aubry-Mather theory. In this thesis we want to apply a couple of theorems coming from two important theories in the context of twist maps of the cylinder: The KAM theory and the Aubry-Mather theory. Let us motivate the two theorems in an informal way. Consider the twist integrable map

$$\Phi_0 : \ r_1 = r, \qquad\qquad \theta_1 = \theta + \eta(r). \tag{1.0.3}$$

Suppose that the map is defined in the strip Σ and the function η is smooth in $[a, b]$ with $\eta' > 0$. This map is exact symplectic with $\mathrm{d}\mathcal{S}(r) = r\eta' \, \mathrm{d}r$ and has the twist property. The dynamics is simply the following. Every circle around the origin $r = r_*$ is an invariant set for the map Φ_0. That is, if $\Gamma = \{(r, \theta) \in \Sigma : r = r_*\}$, the invariance condition means $\Phi_0(\Gamma) = \Gamma$. In each invariant circle the angle is rotated according to the expression of $\eta(r_*)$. So, the dynamics on every such circle is described by the so-called rotation number α, defined as

$$\alpha := \frac{1}{2\pi} \lim_{n\to\infty} \frac{\theta_n}{n}.$$

where $(r_n, \theta_n) = \Phi_0^n(r, \theta)$. In this example we simply have $\alpha = \eta(r_*)/2\pi$ since $\theta_n = \theta + n\eta(r_*)$.

If $\alpha = s/q \in \mathbb{Q}$, then the solutions are (s,q)-*periodic* and satisfy $\theta_{n+q} = \theta_n + 2\pi s$, $r_{n+q} = r_n$. These solutions make s revolutions around the circle $r = r_*$ in time q. On the contrary, if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ every orbit is quasi-periodic with frequencies $(1, \alpha)$.

Now consider a small perturbation of Φ_0 in the class of exact symplectic maps,

$$\Phi_p : \; r_1 = r + f(r,\theta), \qquad\qquad \theta_1 = \theta + \eta(r) + g(r,\theta). \tag{1.0.4}$$

Here we assume that f and g are 2π-periodic in θ. So the map is defined in the strip Σ of the cylinder and is exact symplectic. It occurs that many invariant sets of Φ_0 are preserved. This fact depends on the arithmetic properties of the rotation number, α.

Firstly, a KAM theorem shows that if the perturbed twist integrable map Φ_p is sufficiently $\mathcal{C}^k$-close to Φ_0 with $k \geq 3+\varepsilon$ (we can say that Φ_p is a small perturbation in the $\mathcal{C}^k$-topology of the integrable case). Then a family of invariant curves are preserved and the restriction of Φ_p to each invariant curve is conjugate to a rotation by α. Furthermore, for each invariant curve, α satisfies a Diophantine condition

$$\left| \frac{\alpha}{2\pi} - \frac{p}{q} \right| \geq \frac{C}{q^{\mu+1}}, \qquad \forall \frac{p}{q} \in \mathbb{Q}, \tag{1.0.5}$$

where C is a positive constant and $\mu > 1$.

By an invariant curve Γ for the map Φ_p we mean a non-contractible simple Jordan curve contained in Σ such that $\Phi_p(\Gamma) = \Gamma$. Also it could exist curves Γ such that $\Phi_p(\Gamma) = \Gamma$ that are contractible, also called *librational* invariant curves: we are not considering this kind of curves and in this monograph by an invariant curve we will refer to the so-called *rotational* invariant curve. See for instance the reviews [47, 48] for more details in the definitions of librational and rotational curves.

The previous result is known as the *Invariant Curve Theorem* or *Moser's Twist Theorem* due to the work of Jürgen Moser in the early sixties [55]. Rüssmann, Herman and Moser himself among others gave different versions of this theorem. For a detailed formulation of different versions of this theorem see [58, 77] (for $k \geq 5$), [21] (for $k \geq 3+\varepsilon$) or [32, 76, 81] (for the analytic case). On the other hand, Takens [82] gave a first counterexample to Moser's Twist Theorem in class $\mathcal{C}^1$, by showing that one can construct a sequence of iterates which come close to $r = a$ as well as $r = b$, that is, the boundaries of the strip Σ. It is known that an invariant curve separates the cylinder into two disjoint regions, so this result implies the non-existence of invariant curves in the strip Σ. Later, Herman [21] improved it giving another counterexample in class $\mathcal{C}^{3-\varepsilon}$ where ε is a small positive constant. Thus it seems that k has to be chosen greater than 3. See also [39] for another counterexample.

Usually, the hypotheses of this theorem are that the map Φ has to be a "small" perturbation of the integrable twist map Φ_p defined in the annulus, with the *intersection property* in Σ and the corresponding regularity assumption. The intersection property means that $\Phi(\Gamma) \cap \Gamma \neq \emptyset$ for every non-contractible Jordan curve $\Gamma \subset \Sigma$. It occurs that being exact symplectic implies the intersection property, so we can apply the theorem to the map (1.0.3). Additionally, Diophantine condition (3.3.4) expresses the difficulty to approximate an irrational number by rational numbers whose denominators are arbitrarily large, in fact, $\left|\frac{\alpha}{2\pi} - \frac{p}{q}\right|$ is the distance between the number $\alpha/2\pi$ and the rational p/q. The theorem implies that the more differentiable a system is, the more invariant curves can be produced by the theorem, since μ can be larger and the inequality (3.3.4) is less restrictive.

In this thesis we are interested in the analytic version of the Moser's Twist Theorem applied to the perturbed point-vortex problem. The hypotheses of this theorem implies some bounds of the perturbations in complexified domains contained in $\mathbb{C}^2$.

So, after the application of the Invariant Curve Theorem we have the following picture: for a twist map in the cylinder with the intersection property we can find a family of invariant curves. These invariant curves divide the phase space into disjoint regions: any motion starting from an initial condition within a region cannot escape through the invariant curve. This fact is crucial when one deals with questions of stability or boundedness of orbits. The result is not entirely obvious: since we are dealing with discrete iterations it might seem possible that the orbit could "hop across" the invariant curve. However, we know that the invariant curve divides the cylinder in two connected components and that the Poincaré map is a homeomorphism isotopic to the identity (the map conserves the $+\infty$ and the $-\infty$ in the cylinder). As a consequence the image of a connected component is in the same connected component. Since there is a continuous deformation $H(\lambda, p)$ from the identity map to the Poincaré map, there cannot exist orbits connecting the top and the bottom of the cylinder.

The KAM theory gives sufficient conditions for finding invariant curves of area-preserving maps of the cylinder but we need certain regularity assumptions and it follows a perturbative approach, so we need the perturbation to be sufficiently small. The theory does not give any information on the invariant curves that are not preserved after the perturbation. Nevertheless the Aubry-Mather theory does not need this perturbative approach, it uses variational techniques to require weaker regularity hypotheses and to drop the condition on nearly integrability.

Going back to our example (1.0.4), the usual Mather's Theorem gives sufficient conditions on the map Φ_p or on the associated generating function $h(x, x_1)$ in order to get orbits with rotation number α that minimize the action. In a version of the theorem by Mather [46], these conditions are that the exact symplectic map Φ_p has to be a twist diffeomorphism of the infinite cylinder $\mathfrak{C}$ and the technical hypothesis of an infinite twist as $r \to \infty$. In fact, in this work he proved it not only for a single exact symplectic twist diffeomorphism but Mather considered a finite composition of them. This result is not trivial, since the composition of twist maps, in general is not a twist map. See also [40] for related results in the case of composition of maps.

The orbits obtained are contained in closed minimal invariant sets, the *Aubry-Mather sets* $\mathcal{M}_\alpha$. The arithmetic properties of α determine the structure of $\mathcal{M}_\alpha$ and the dynamics on it.

- If α is rational then $\mathcal{M}_\alpha$ contains periodic orbits. These periodic orbits correspond to not elliptic fixed points of the Poincaré map. Also $\mathcal{M}_\alpha$ may contain heteroclinic or homoclinic orbits.

- If α is irrational, then Mather's Theorem gives us an alternative: either $\mathcal{M}_\alpha$ is an invariant curve of Φ and every orbit on it is quasi-periodic with frequencies $(1, \alpha)$, or the minimal invariant set become a Cantor set. In the last case the motion is not quasi-periodic in the classical sense but the dynamics on a Cantor set is of Denjoy type with rotation number α.

Another important property of $\mathcal{M}_\alpha$ is that the "ordering" of an orbit is the same as for the rotation by α of a circle. This is consequence of the topological semi-conjugacy to a rotation. In Section 3.4 we explain what it means a "dynamics of Denjoy type."

So, with weaker assumptions, Mather's Theorem gives an elegant explanation about the "missing" invariant curves from KAM theory and shows how those curves disappear. In fact one finds invariant sets but it is not possible to guarantee that they are curves as in KAM theory.

Therefore Aubry-Mather theory gives the existence of orbits classified and ordered by their rotation number α. See [9, 46, 47, 59] for a detailed exposition of the Aubry-Mather theory. This theory has appeared in the eighties and has an almost simultaneous origin in the works of Aubry-LeDaeron on solid state physics [7] and Mather on twist mappings [45]. The Aubry-Mather theory was developed by Aubry and LeDaeron, studying the stationary states in the Frenkel-Kontorova model. The Frenkel-Kontorova model consists of a periodic chain of atoms, coupled in pairs by an elastic potential. On the other hand, Mather developed independently the theory from a Percival's related variational principle for area preserving twist homeomorphisms. Percival introduced the variational principle in a numerical work for the standard mapping [69]:

$$\begin{cases} r_1 = r - \frac{\lambda}{2\pi}\sin(2\pi\theta) \\ \theta_1 = \theta + r_1. \end{cases}$$

An additional related field of applications of the theories of Aubry and Mather is the study of geodesic flow on 2-dimensional surfaces. Minimal geodesics on the torus were investigated already by Morse [53] in 1924 and Hedlund [19] in 1932. In [9], Bangert has extended this results by using the Aubry-Mather theory.

Therefore, in both theories, we have a point of view in which the orbits of a symplectic twist map are classified by their rotation number or frequency. In a sense, these two theorems do not have as a purpose to know what happens to an orbit with a particular initial condition, but to consider the properties of all orbit with the same frequency. In Chapter 3, we will give an exposition of the two theories.

Stability. The concept of stability around a point-vortex in our model is mathematically equivalent to the stability of a singular fixed point. We deal with perpetual stability in the Lyapunov sense and the definition is analogous to the one known when studying the stability of solutions near an equilibrium point. The main difference with the theory of the stability of the equilibrium is a sharp contrast between equilibria and singularities. In the second case continuous dependence may be lost. Therefore, a stable singularity in the continuous framework will produce a stable fixed point of the Poincaré map. However, the stability of the fixed point of the Poincaré map is not sufficient to guarantee the stability of the origin as a singularity of the differential equation. This fact makes necessary to define a kind of *continuous dependence around the singularity*. In Section 4.1 of Chapter 4 we will make a more precise definition of this concept.

Point-vortex model. After taking a glimpse of the mathematical methods used in this research, now we give an outline of our specific work in point-vortex dynamics and the passive transport of particles. Even if we lose information in the formulation of the *point-vortex model* since we are not using the Euler equation, our point of view has an important advantage: as we are working with ODE's, we want to obtain a deep qualitative description of the physical phenomena.

In Fluid Mechanics, a point-vortex is by definition a singularity of the vorticity. That is, given by a Dirac delta in the vorticity. In the Lagrangian description of the dynamics of an ideal 2D fluid, a point-vortex moves subject to the interaction with other point-vortices present in the fluid. This gives a formal analogy to the N-body problem in Celestial Mechanics, but with totally different properties. An important difference is that, in one case we have first order differential equations, and in the Celestial mechanics case, we have second order differential equations. That is, in the case of the N point-vortices, the dynamics does not come from the interaction with a Newtonian potential.

Hence, each point particle of the fluid is subject to the influence of a point-vortices configuration present in the flow or to the influence of other background flows. We call this the passive transport of the particle, or the advection of the particle. The dynamics follows a Hamiltonian formulation with the streamfunction Ψ playing the role of the Hamiltonian. These ideas were introduced in the seminal works of Helmholtz and Kirchoff in the XIXth century. See [20] and Lecture 20 of [26].

Here a word of caution is necessary, because the notion of point-vortex as a singularity of the vorticity of a 2D fluid is a highly idealized version of real-life vortices. From a physical point of view, a vortex is just a concentrated region of high vorticity, but at the moment the Fluids Dynamics community works with several (non-equivalent) mathematical definitions of a vortex, see [17] for a summary of different interpretations available in the literature. Many definitions require a compact set of finite vorticity (vortex patch) rather than a singularity, in other cases the effect of a third spatial dimension is considered (filament or tubular vortices). We refer to [1, 17, 29, 37, 61] for more information about the existing models and a summary on the various definitions.

Given the vorticity ω of an incompressible fluid, the streamfunction is defined as a solution of the Poisson equation $-\Delta\Psi = \omega$. As it said previously, we are concerned with a point-vortex, defined as a Dirac delta of the vorticity. Under this definition, the streamfunction is the fundamental solution of the 2-dimensional Laplacian. So, a point-vortex in the plane induces a streamfunction $\Psi_0 = \frac{\Gamma}{4\pi}\ln(x^2+y^2)$ where Γ is the circulation (or strength) of the vortex and, up to a re-scaling of units, it can be set $\Gamma = 2\pi$. As said before, the solution of the corresponding Hamiltonian system describes the trajectories of a passive particle under the influence of the vortex.
It is easy to see that, without the perturbation, the Hamiltonian system is integrable with periodic solutions and the origin is stable in a Lyapunov sense. The particle rotates around the vortex on circular paths. The frequency of rotation is inversely proportional to the radius of the path and tends to infinity as the radius tends to zero.
We study how this integrable dynamics is affected by the superposition of an external periodic time dependent streamfunction $p(t,x,y)$. More precisely, we consider the Hamiltonian

$$\Psi(t,x,y) = \frac{1}{2}\ln(x^2+y^2) + p(t,x,y), \tag{1.0.6}$$

and the associated Hamiltonian system

$$\begin{cases} \dot{x} = \partial_y \Psi(t,x,y) \\ \\ \dot{y} = -\partial_x \Psi(t,x,y) \end{cases} \qquad (x,y) \in \mathcal{U} \setminus \{0\}, \tag{1.0.7}$$

defined in a neighborhood $\mathcal{U}$ of the origin.

Physically, system (1.0.7) can be interpreted to model the advection of a particle under the action of a steady vortex placed at the origin and a periodic time dependent background flow. This point of view of looking at the motion of single particles is known as the Lagrangian description for Fluid Dynamics, in contrast to the Eulerian perspective, which looks at the global properties of the flow.

Different aspects of the dynamics of advected particles in non-stationary flows have been studied in the literature from different perspectives, see for instance [8, 10, 18, 30, 70, 75, 85]. Numerical studies in related models [8, 10] strongly suggest the presence of stability (regular) islands around a point vortex and other complex dynamics. So this physical phenomenon is well known on a numerical and experimental level, but there are not many results from a purely analytical perspective.

Chapter 2 will be dedicated to an explanation of this model.

Results and methodology. In this report we include the two results obtained during my research period ([44] and [68]). Considering the perturbed system (1.0.7), we have obtained the following results:

1) In [68] we get some sufficient conditions over a class of analytical perturbations $p(t, x, y)$ to guarantee the stability of the origin using a KAM theorem. Specifically, after some transformations and estimates in subdomains of $\mathbb{C}^2$, the analytical version of Moser's invariant curve theorem (or twist theorem) applies. Therefore it gives the existence of invariant curves of the Poincaré map close to the singularity. Due to a low dimensionality of the phase space, these invariants curves acts like barriers and give the desired stability of the origin.

   The requirement of a low dimensionality in these systems deserves an additional comment. If we have 2 or 1 and $1/2$ degrees of freedom, after the perturbation the remaining invariant tori act as barriers for the flux. That is, we have a family of compact level sets in the reduced phase space of the system that prevents the escape of the trajectories. Arnold diffusion cannot occur in this scheme.

   To obtain this result we have adopted the following methodology: First, we need to transform the variables (x, y) to action-angle variables by using symplectic polar coordinates (r, θ). Also the Kelvin transform reverses the role of the origin and infinite and introduces a weighted symplectic form, $\omega = \frac{1}{4r^2}\mathrm{d}r \wedge \mathrm{d}\theta$. This change of variable makes it easier for us to treat the singularity. After this, we pass to complex formulation by considering the system as complex-valued. This formalism is necessary to apply the invariant curve theorem in the analytical version. In this framework, we make some estimates to assure that the Poincaré map is well-defined in a suitable open set of $\mathbb{C}^2$ and the perturbation satisfies the hypothesis of the invariant curve theorem. Finally, we show the intersection property for the Poincaré map, therefore the invariant curve theorem, provides the existence of a sequence of invariant Jordan curves near infinite. After inversion of the Kelvin transform, such invariant curves surround the vortex, acting as flux barriers. So, the stability is guaranteed. In the last step it is important to check the continuous dependence around the singularity over a period.

   As a byproduct of this first result, there exist quasi-periodic solutions of the Hamiltonian systems with (sufficiently large) Diophantine frequencies. These can be seen as a reminiscent of the trajectories of the unperturbed Hamiltonian with Diophantine frequency.

2) In [44] we prove that, close to the singularity, quasi-periodic solutions exist for all frequencies sufficiently large. Actually, our solutions are

a generalization of standard quasi-periodic solutions. In case of commensurable frequencies, we get periodic solutions. These solutions exist also when KAM theory cannot be applied. Indeed, we require a very low regularity of $p(t,x,y)$ that prevents standard KAM theory from being applied. This result comes from the application of a suitable version of Aubry-Mather theory [6, 45] to the Poincaré map of the perturbed system. A similar scheme have been used to describe the dynamics of different systems [42, 38, 64, 79, 86].

For each sufficiently large real number α, we give a proof for the existence of an invariant set $\mathcal{M}_\alpha$ (called *Aubry-Mather set*) with very interesting dynamical properties, among them, each orbit in $\mathcal{M}_\alpha$ has rotation number α. For irrational rotation numbers, the corresponding Aubry-Mather sets are either curves or Cantor sets. Solutions of system (1.0.7) with initial conditions in this set are our generalized quasi-periodic solutions. In the rational case, the Aubry-Mather sets contain periodic and heteroclinic orbits.

In suitable variables, the Poincaré map of the perturbed system (1.0.7) is an exact symplectic twist map of the cylinder. However, it is not defined on the whole cylinder. Hence we cannot apply directly the result of Mather and we need an adapted version to this situation. Using an extension theorem, we can formulate a generalized Aubry-Mather theorem.

To apply this generalized theorem, we only need to show that the Poincaré map is exact symplectic and twist. The first property comes from the Hamiltonian character of the system. The twist condition is more delicate and relies on the behavior of the variational equation. We give a proof following a perturbative approach. Here, we ask that the perturbation has the origin as a zero of order 4.

From the point of view of dynamics of symplectic diffeomorphisms, we describe some aspects of the dynamics around a singularity. In the integrable case, the flow can be continuously extended to the singularity, defining it as a fixed point. However, this extension is not $\mathcal{C}^1$. In the perturbed case, in general it is not even possible to guarantee the continuity of this extension. Since the flow is not regular, all the results coming from the theory of elliptic fixed points and transformation to Birkhoff normal form cannot be applied directly. As in the first result, we overcome the problem of the singularity performing a change of variable that sends the singularity at infinity and makes it easier to treat the singularity. At this stage, the assumption of having the zero of order 4 in the perturbation play a fundamental role.

To sum up, with this result we obtain periodic points, invariant curves and Cantor sets that give us an important dynamical information of the orbits in the perturbed system. These sets have an associated rotation number, α. As we saw in the previous example concerning the twist integrable map. If $\alpha = s/q$ is rational, the invariant set contains periodic points of period q that correspond to s-periodic motions in q iterations. Additionally, this invariant set also contains the heteroclinic orbits. If

α is irrational, then Mather's theorem give us an alternative: either an invariant curve corresponding to quasi-periodic motions in the classical sense with frequencies $(1;\alpha)$ or the invariant set is a Cantor with dynamics of Denjoy type.

Structure of this thesis. The structure of this thesis is the following.

In Chapter 2, we will present the physical model of our thesis, *perturbed point-vortex dynamics*. We will motivate the equations of motion of our model, that is, the system of equations (1.0.7); and we will analyze some properties of interest for our study.

In Chapter 3 we will review the main mathematical tools used throughout the research. In Section 3.1 we will introduce some basic facts of exact symplectic twist maps of the cylinder and its properties in the corresponding *lift map*, that is, the map defined on the universal cover of the cylinder with the inheriting periodic condition in the angle variable. We will start with a definition of the simplectic and exact symplectic properties and its geometrical interpretation. We will continue with the definition of the twist property and the intersection property. The next Section, 3.2 is devoted to study some properties of the Poincaré map.
In the following sections, 3.3 and 3.4, we will present the theories applied in our results, KAM theory and Aubry-Mather theory. In Section 3.3 we will give a brief historical introduction to KAM theory with an intuituve meaning and enunciate Moser's invariant curve theorem in the analytic version. Section 3.4 presents a generalized Aubry-Mather theorem adapted to the study of the Poincaré map corresponding to the perturbed point-vortex problem (1.0.7). The usual Mather's theorem asks for a generating function defined in the whole plane $\mathbb{R}^2$, however, in our application to the perturbed point-vortex problem, we will see that the generating function is defined on a subset of the plane. Therefore, we need an extension theorem to extend the domain of the generating function to the whole plane and so we can apply the Mather's Theorem. In this Section we will give the details.

The following chapters deal with our research results. Chapter 4 will present the result concerning the stability around the singularity via the existence of invariant curves surrounding the origin.
First of all, from a more analytical point of view, it seems necessary to state a precise mathematical notion of stability around a singularity. To fill this gap, Section 4.1 is devoted to settle a rigorous definition of stability of a singularity in this context. We will consider perpetual stability in the Lyapunov sense, meaning that a solution of (1.0.7) with a small initial condition will remain close to the singularity forever.
The remaining sections in this Chapter are devoted to enunciate the stability Theorem for the perturbed point-vortex problem and the correspondent proof.

Chapter 5 contains the result concerning the application of Aubry-Mather theory and the existence of quasi-periodic solutions close to the singularity.

Finally, in Chapter 6 we will present a summary of conclusions of both results. We also show some possible future works in the direction of this thesis.

Chapter 2

Physical model: Perturbed point-vortex dynamics

2.1 Introduction: Point-vortex dynamics

In this chapter we will introduce the physical model used in this monograph: dynamics of a point-vortex under a periodic perturbation. Point-vortex dynamics has been historically used to describe the dynamics of a passive particle (or test particle) in a particular velocity field given by a configuration of N point-vortices. Or, in a clear analogy with celestial mechanics, the dynamics of a configuration of N point-vortices, the so-called wet N-body problem. This is only a formal analogy since there is an important difference. In one case we have first order differential equations, and in the other case, we have second order differential equations. That is, in the case of the N point-vortices, the dynamics does not come from the interaction with a Newtonian potential. Also, the integrability properties are different. The 3-vortex problem is integrable and is well-known the non-integrability of the 3-body problem.

In this monograph we will follow the point of view of H. Aref and others authors [1], in which we want to know the evolution of a passive particle under the action of a prescribed velocity field generated for a configuration of N point-vortices or an external flow. It is assumed that each point-vortex does not have a self-interaction. This description follows the movement of every fluid particle, in analogy with particle mechanics. It is called a *material description* or *Lagrangian description*. A different point of view, which we will not treat, is the *Eulerian perspective* or the *field description*. In this framework, instead of following the motion of each particle, we may concentrate on the spatial distribution of physical quantities and their temporal variation at every point x, for example, the velocity of the fluid. Both points of view give us different and complementary information of the phenomena.

The history of vortex dynamics starts from the works of Helmholtz in 1858, [20]. Specifically, in section V of this paper, Helmholtz introduces the point-vortex model. We can describe this model as follows. Consider a set of N infinitely thin, straight and parallel vortex filaments in $3D$, with an invariant amount of circulation $\Gamma := \oint_C v \cdot \mathrm{d}l$ where v is the velocity field and C is a closed path. If one takes perpendicular sections to this family of vortex filaments, one obtains a set of N points of intersection, that is, a configuration of N point-vortices in the plane. Later, Kirchhoff [26] has developed a Hamiltonian formulation, identifying the streamfunction of the fluid as the Hamiltonian of the system. He also derived some integrals of motion in the N-vortices configuration. Other authors have contributed in the historical development of this area. Just to cite a few, Kelvin with his well-known circulation theorem and the vortical atomic model [24]; Gröbli [16] with the work for the three point-vortex problem and the four point-vortex problem with axial symmetry; even Poincaré in 1893 [74] with his treatise on vortices; also Love [34] with the problem of four point vortex with axial symmetry and the phenomenon of "leapfrogging" of point-vortex pairs. For instance, in [2] the authors give an interesting historical context in this field.

Nowadays, point-vortex dynamics are studied as a branch of Fluid Mechanics with deep connections with Celestial Mechanics and Hamiltonian systems. Also it is an exciting field of research related with the physics of Bose-Einstein condensates (see [14, 25, 71, 83, 84]), hydrodynamics, geophysics, astrophysics, etc.

2.1.1 Equations of motion for a point-vortex

Point-vortex equations of motion can be viewed as a discrete version of the Euler field equations for a 2D inviscid flow. In [35] the authors give a rigorous derivation of the model.

Is important to say that the notion of point-vortex as a singularity of the vorticity of a 2D fluid is a highly idealized version of real life vortices. From a physical point of view, a vortex is a zone of high vorticity. Mathematically, a point-vortex in the plane can be defined in different ways. It can be defined as a singularity of the vorticity or through a compact set of finite vorticity (vortex patch). See [1, 17, 29, 37, 61] for a summary on the various definitions and more information about the existing models. We will define a point-vortex as a Dirac delta of the vorticity ω of an incompressible two-dimensional fluid. With this definition, the associated streamfunction is the fundamental solution of the 2-dimensional Laplacian, that is, as a solution of the Poisson equation $-\Delta\Psi_0 = \omega = \delta$. In consequence a point-vortex situated in the origin induces a streamfunction $\Psi_0 = \frac{\Gamma}{4\pi}\ln(x^2+y^2)$ where Γ is the circulation (or strength) of the vortex. This streamfunction $\Psi_0(x, y)$ will be our Hamiltonian function. The solution of the corresponding Hamiltonian system describes the trajectories of a passive particle under the influence of the vortex.

The Hamiltonian associated to a point-vortex with circulation $\Gamma = 2\pi$ on the *xy* plane

$$\Psi(x, y) = \frac{1}{2}\ln(x^2 + y^2), \tag{2.1.1}$$

and the correspondent Hamiltonian system is

$$\begin{cases} \dot{x} = \frac{y}{x^2+y^2} \\ \\ \dot{y} = -\frac{x}{x^2+y^2} \end{cases} \qquad (x,y) \in \mathcal{U} \setminus \{0\}\,, \tag{2.1.2}$$

We have an autonomous Hamiltonian system with one degree of freedom and the canonical variables (x, y) represents the position of the passive particle on the plane. As it is known, this system is integrable with the Hamiltonian function as an integral of motion with radial symmetry. The passive particle rotates around the vortex on circular paths and the frequency of rotation is inverse proportional to the square of the radius of the path and tends to infinity as the radius tends to zero (see Figure 2.1). It is easy to obtain an expression for the period of this paths. Consider the following transformation from (x, y) to polar symplectic variables

$$x = \sqrt{2\rho}\cos\theta, \qquad v = \sqrt{2\rho}\sin\theta.$$

Hamiltonian (2.1.1) transforms into

$$\Psi(x,y) = H(\rho) = \frac{1}{2}\ln(2\,\rho)$$

and the equations of motion are

$$\begin{cases} \dot{\rho} = 0 \\ \dot{\theta} = -\frac{1}{2\rho} \end{cases} \tag{2.1.3}$$

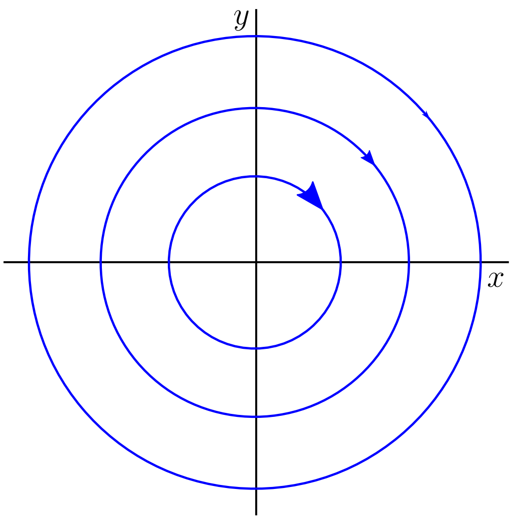

Figure 2.1: Orbits of system (2.1.2). A larger size of the arrows corresponds to a higher velocity of rotation.

Integrating this equations we can see explicitly that the solutions have constant ρ, that is, the orbits are circular paths. Additionally, the angular variable changes linearly with the time

$$\theta(t) = \theta_0 - \frac{1}{2\rho}t$$

with a frequency $\omega := \frac{\mathrm{d}\theta}{\mathrm{d}t} = -\frac{1}{2\rho}$. The period of the orbits is $T = \frac{2\pi}{|\omega|} = 4\,\pi\,\rho$.

Remark 1. Sometimes is useful to write the system (2.1.2) in complex notation. If we take the complex variable as usual: $z = x + iy$, we have:

$$\dot{z} = \frac{1}{i\bar{z}} \tag{2.1.4}$$

2.1.2 Superposition principle

Our equation for a point-vortex gives a velocity field on the plane so its clear that we can consider a superposition principle to add different velocity fields, coming from other configuration of point-vortex or from an external perturbation. In the following examples it is illustrated this superposition principle.

Example 1. Equations of motion for a configuration of N point-vortices.

Consider a configuration of N point-vortices on the plane, being the position and circulation of the j-th point-vortex (x_j, y_j) and Γ_j, respectively. The Hamiltonian is

$$\Psi_N(x_j, y_j) = \frac{1}{4\pi}\sum_{j<k}^{N}\Gamma_j\Gamma_k \ln\left[(x_j - x_k)^2 + (y_j - y_k)^2\right], \tag{2.1.5}$$

and the equations of motion for the j-th point-vortex are

$$\begin{cases} \dot{x}_j = \frac{1}{2\pi}\sum\limits_{k\neq j}^{N}\Gamma_j\Gamma_k\frac{y_j - y_k}{[(x_j-x_k)^2+(y_j-y_k)^2]} \\ \\ \dot{y}_j = -\frac{1}{2\pi}\sum\limits_{k\neq j}^{N}\Gamma_j\Gamma_k\frac{x_j - x_k}{[(x_j-x_k)^2+(y_j-y_k)^2]} \end{cases} \tag{2.1.6}$$

The equations (2.1.6) represent the dynamical influence of the configuration of the $N-1$ vortices on the j-th point-vortex. That is, the j-th point-vortex is advected by the superposed velocity field produced by all the other vortices. Recall that each point-vortex sets up about itself a circular velocity field of magnitude $\frac{\Gamma_j}{2\pi\rho_j}$, where ρ_j is the radius of the circle centered in that vortex.

Remark 2. In complex notation the equations (2.1.6) can be written as

$$\dot{z}_j = \frac{1}{2\pi i}\sum_{k\neq j}^{N}\Gamma_j\Gamma_k\frac{z_j - z_k}{|(z_j - z_k)|^2} = \frac{1}{2\pi i}\sum_{k\neq j}^{N}\frac{\Gamma_j\Gamma_k}{\overline{(z_j - z_k)}}. \tag{2.1.7}$$

Remark 3. *Integrals of motion and integrability.* In contrast with the 3-body problem in Celestial Mechanics, the 3-vortex problem is integrable. In fact, there are 4 integrals of motion coming from the following symmetries:

- time-translational invariance, $I_1 := H$;
- translational invariance, $I_2 + iI_3 := \sum\limits_{j=1}^{N}\Gamma_j z_j$;

- rotational invariance, $I_4 := \sum_{j=1}^{N} \Gamma_j |z_j|^2$.

However, these integrals are not in involution. One can find the three independent integrals in involution that assure the integrability of the N-vortex problem for $N \leq 3$ and any values of the vortex circulations. The three integrals are I_1, I_4 and $I_2^2 + I_3^2$.

Example 2. Equations of motion for a point-vortex with an external time-dependent perturbation.

Consider an external flow $p(t,x,y)$ and a point-vortex situated in the origin. The Hamiltonian of this system is

$$\Psi(t,x,y) = \frac{\Gamma}{4\pi} \ln(x^2+y^2) + p(t,x,y), \tag{2.1.8}$$

and the associated system

$$\begin{cases} \dot{x} = \partial_y \Psi(t,x,y) = \frac{\Gamma}{2\pi}\frac{y}{x^2+y^2} + \partial_y p(t,x,y) \\ \\ \dot{y} = -\partial_x \Psi(t,x,y) = -\frac{\Gamma}{2\pi}\frac{x}{x^2+y^2} - \partial_x p(t,x,y) \end{cases} \qquad (x,y) \in \mathcal{U} \setminus \{0\}\,. \tag{2.1.9}$$

Physically, system (2.1.9) models the passive advection of particles in a fluid subjected to the action of a steady point-vortex placed at the origin and a time-dependent background flow.

2.2 Equations of motion of a periodically perturbed point-vortex

Let us take a 1-periodic perturbation in the Hamiltonian (2.1.8). This system will be our model for the problem in this monograph. Also for simplicity we will consider a point-vortex, with circulation $\Gamma = 2\pi$, centered at the origin of the plane. The Hamiltonian is

$$\Psi(t,x,y) = \frac{1}{2} \ln(x^2+y^2) + p(t,x,y), \tag{2.2.1}$$

and the associated system

$$\begin{cases} \dot{x} = \partial_y \Psi(t,x,y) = \frac{y}{x^2+y^2} + \partial_y p(t,x,y) \\ \\ \dot{y} = -\partial_x \Psi(t,x,y) = -\frac{x}{x^2+y^2} - \partial_x p(t,x,y) \end{cases} \qquad (x,y) \in \mathcal{U} \setminus \{0\}\,, \tag{2.2.2}$$

defined in a neighborhood $\mathcal{U}$ of the origin.

2.3 Objectives, motivation and results

As we have seen, without the perturbation the system is integrable and the passive particle rotates around the vortex with a frequency that tends to ∞ as we approach to the origin. This solutions are trivially stable (in a Lyapunov sense).

The idea is to see what happens to the integrable system if we introduce a periodic external flow $p(t, x, y)$. A first question could be: If we want to preserve the stability in the origin, which are the conditions on $p(t, x, y)$? We give the answer: If $p(t, x, y)$ is analytic and the origin is a zero of order 4, then by means of KAM theory (Invariant Curve Theorem) we obtain a family of invariant curves of the Poincaré map. Then, these invariant curves act as barriers and give the stability of the singularity. This result is contained in the paper [68].

For an analytic perturbation, we have shown the existence of quasi-periodic solutions, that is, the family of invariant curves. So, a second question motivated by this first result is: Which are the sufficient conditions on $p(t, x, y)$ for the existence of periodic and quasi periodic orbits? What can we say about the periodicity of the orbits after the perturbation? Our answer is: If $p(t, x, y)$ is $\mathcal{C}^3$ in a domain around the origin and satisfies a condition zero of order 4, we obtain the existence via Aubry-Mather theory of periodic or generalized periodic orbits for rational rotation number α, and quasi-periodic orbits for an irrational α. Observe that with this regularity KAM results can not be applied. Nevertheless, Aubry-Mather theory does not distinguish if we obtain invariant curves or Cantor sets for an irrational rotation number α. In the paper [44] we give a proof of this result.

So, our results would help us to understand the dynamics of this model as well as the dynamics around a singular point.

In the next Chapter, we give the necessary mathematical tools to prove this results.

Chapter 3

Mathematical tools

In this chapter we introduce some mathematical background necessary to obtain our results. We are studying a dynamical system with a singularity in the velocity field located at the position of the point-vortex. Our unperturbed model is an integrable Hamiltonian system in the plane. Under suitable conditions, we can make transformations of the canonical variables to get the action-angle variables. So the phase map of the system becomes a cylinder. In Section 3.1 we introduce some properties of maps of the cylinder: symplectic, exact symplectic, twist condition and intersection property.

The connection between continuous and discrete dynamics is given by the so-called Poincaré map, in Section 3.2 we give some properties of this map.

Furthermore, we consider perturbations on integrable Hamiltonian systems. A main result in this field is the theory developed by Kolmogorov, Arnold, Moser and others authors. This theory is usually known using the acronym KAM. After an introduction with some historical remarks, in Section 3.3 we enunciate the analytical version of the Invariant Curve Theorem or Moser's Twist Theorem.

Finally, in Section 3.4 we present the Aubry-Mather theory and we give a generalized theorem. Usually, Aubry-Mather theorem requires the generating function to be defined in the whole plane $\mathbb{R}^2$. But in the perturbed point-vortex problem the generating function is defined in a subset of $\mathbb{R}^2$, so we give an extension Theorem that allows to apply the Aubry-Mather theory in our problem.

3.1 Maps in the cylinder

In this section we follow the lines of [31] and [40]. Let us start denoting the cylinder by $\mathfrak{C} = \mathbb{R} \times \mathbb{T}$ with $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ and consider the strip $\Sigma :=]a, b[\times \mathbb{T}$ with $-\infty \leq a < b \leq \infty$. The notation related to the quotient set $\mathbb{T}$ will be: the real variable will be denoted by x and the angle variable by θ. So x will be the lift variable and $\theta = x + 2\pi\mathbb{Z}$ the variable in the covering map.

Consider a $\mathcal{C}^k(\Sigma)$-embedding with $k \geq 0$:

$$\begin{array}{rccl} \Phi: & \Sigma & \longrightarrow & \mathfrak{C} \\ & (r,\theta) & \longmapsto & (r_1,\theta_1) = (\mathcal{F}(r,\theta), \mathcal{G}(r,\theta)) \,. \end{array}$$

We recall than an embedding is a homeomorphism onto its image and a $\mathcal{C}^k(\Sigma)$-embedding is a $\mathcal{C}^k$-diffeomorphism onto the image.

Also we work on the plane $\mathbb{R}^2$. Consider the lift map of Φ, that is, the map defined in a subset of the universal cover of the cylinder ($\tilde{\Sigma} :=]a,b[\times \mathbb{R} \subset \mathbb{R}^2$)

$$\begin{array}{rccl} \tilde{\Phi}: & \tilde{\Sigma} & \longrightarrow & \mathbb{R}^2 \\ & (r,x) & \longmapsto & (r_1,x_1) = (\mathcal{F}(r,x), \mathcal{G}(r,x)) \end{array} \tag{3.1.1}$$

with a periodicity condition

$$\mathcal{F}(r, x + 2\pi) = \mathcal{F}(r,x),$$

and the additional hypothesis for the lifted angular variable x,

$$\mathcal{G}(r, x + 2\pi) = \mathcal{G}(r,x) + 2\pi.$$

We will also suppose that $\tilde{\Phi}$ (or Φ) is isotopic to the identity. This property means that if u is an element of a topological space Y, there exists a continuous map

$$\begin{array}{rcl} H : [0,1] \times Y & \longrightarrow & Y \\ (\lambda, u) & \longmapsto & H(\lambda,u) = H_\lambda(u) \end{array} \tag{3.1.2}$$

such that:

- $H_0(u) = \mathrm{Id}(u) = u$ and $H_1(u) = \Phi(u)$.
- $H_\lambda(u)$ is a homeomorphism for each $\lambda \in [0,1]$.

To clarify the notation, (r,x) are Cartesian coordinates and (r,θ) coordinates in the cylinder. We denote both diffeomorphisms by Φ. Only in cases of possible misunderstandings we use the notations $\tilde{\Phi}$ (defined on the plane) and Φ (defined in the cylinder).

The class of these maps in the plane will be denoted by $\mathcal{E}^k(\tilde{\Sigma})$, equivalently for maps defined in a strip Σ of the cylinder. The case $k = 0$ is interpreted as a homeomorphism.

Now, we introduce some definitions and properties for these maps.

3.1.1 Symplectic and exact symplectic properties

Consider the standard symplectic form on the plane

$$\tilde{\lambda} = \mathrm{d}r \wedge \mathrm{d}x. \tag{3.1.3}$$

Definition 1. A map in $\mathcal{E}^1(\tilde{\Sigma})$ is said *symplectic* if it preserves the symplectic form (3.1.3), that is,

$$\mathrm{d}r_1 \wedge \mathrm{d}x_1 = \mathrm{d}r \wedge \mathrm{d}x \qquad \forall (r,x) \in \tilde{\Sigma}.$$

Equivalently, this condition says that the Jacobian $\det \Phi' = 1$ on $\tilde{\Sigma}$, that is the classical definition of an area and orientation preserving map.

Remark 3.1.1. (*Interpretation in measure theory*) A map $\Phi \in \mathcal{E}^1(\tilde{\Sigma})$ is symplectic if and only if satisfies:

- Φ is orientation-preserving.
- for each (Lebesgue) measurable set $\Omega \subset \tilde{\Sigma}$ the image $\Omega_1 = \Phi(\Omega)$ is also measurable and $\mu(\Omega) = \mu(\Omega_1)$.

Here μ is the usual Lebesgue measure in the plane:

$$\mu(\mathcal{A}) = \int\int_{\mathcal{A}} \mathrm{d}r\, \mathrm{d}x$$

for each measurable set $\mathcal{A} \subset \mathbb{R}^2$.

There is a related stronger property than being symplectic, the *exact symplectic* property.

Definition 2. A map $\Phi \in \mathcal{E}^1(\tilde{\Sigma})$ is *exact symplectic* with respect to the form $r\mathrm{d}\theta$ if the differential form

$$\beta = r_1\, \mathrm{d}\theta_1 - r\, \mathrm{d}\theta, \quad (r,\theta) \in \Sigma.$$

is exact in the cylinder, that means that there exists a $\mathcal{C}^2$ function

$$\begin{array}{rccc} \mathcal{S}: & \Sigma & \longrightarrow & \mathbb{R} \\ & (r,\theta) & \longmapsto & \mathcal{S}(r,\theta) \end{array}$$

such that

$$\mathrm{d}\mathcal{S}(r,\theta) = r_1\, \mathrm{d}\theta_1 - r\, \mathrm{d}\theta, \quad \forall (r,\theta) \in \Sigma.$$

Note that the function $\mathcal{S}(r,\theta)$ is defined in the cylinder (is 2π-periodic in θ), hence the lift $\mathcal{S}(r,x)$ must be a 2π-periodic function in the variable x such that

$$\mathcal{S}_r(r,x) = \mathcal{F}(r,x)\mathcal{G}_r(r,x), \qquad \mathcal{S}_x(r,x) = \mathcal{F}(r,x)\mathcal{G}_x(r,x) - r. \tag{3.1.4}$$

Remark 3.1.2. (*Interpretation in measure theory*) The notion of exact symplectic map can also be characterized in terms of measure theory. Informally, the area under a non-contractible Jordan curve Γ does not change under the map. Given a non-contractible regular Jordan curve $\Gamma \subset \Sigma$ which is $\mathcal{C}^1$, the image $\Gamma_1 := \Phi(\Gamma) \subset \mathfrak{C}$ is another Jordan curve with the same properties. This is a consequence of assuming that Φ is an embedding. Let us fix some $r_0 > a$

such that $\Gamma \cup \Gamma_1 \subset]r_0, +\infty[\times\mathbb{T}$ and let $\mathcal{R}_i(\Gamma)$ and $\mathcal{R}_i(\Gamma_1)$ denote the bounded components of $\{]r_0, +\infty[\times\mathbb{T}\} \setminus \Gamma$ and $\{]r_0, +\infty[\times\mathbb{T}\} \setminus \Gamma_1$ respectively.

Then, if Φ is exact symplectic, $\mu(\mathcal{R}_i(\Gamma)) = \mu(\mathcal{R}_i(\Gamma_1))$.

To prove this, recall that by the theory of differential forms, we have that Φ is exact symplectic on Σ so its integral over any closed path in Σ must vanish, then for every closed path c in Σ we have

$$\int_c r\mathrm{d}\theta = \int_{\Phi(c)} r\mathrm{d}\theta. \tag{3.1.5}$$

Now, Stoke's Theorem gives ($\gamma = r\mathrm{d}\theta$)

$$\int_{\mathcal{R}_e(\Gamma)} \mathrm{d}\gamma = \int_{\partial\mathcal{R}_e(\Gamma)} \gamma = \int_{r=r_0} \gamma + \int_{\Gamma} \gamma = \int_{r=r_0} \gamma + \int_{\Gamma_1} \gamma.$$

Last equality follows from the previous characterization (3.1.5). Now, again Stoke's theorem gives:

$$\int_{r=r_0} \gamma + \int_{\Gamma_1} \gamma = \int_{\partial\mathcal{R}_e(\Gamma_1)} \gamma = \int_{\mathcal{R}_e(\Gamma_1)} \mathrm{d}\gamma.$$

This means that $\mu(\mathcal{R}_e(\Gamma)) = \mu(\mathcal{R}_e(\Gamma_1))$.

Let us compare these properties according to the domain of the map: plane or cylinder.

A map $\Phi \in \mathcal{E}^1(\tilde{\Sigma})$ is symplectic if and only if the 1-form $\tilde{\beta} = r_1\mathrm{d}x_1 - r\mathrm{d}x$ is closed. That is, $\mathrm{d}\tilde{\beta} = \mathrm{d}r_1 \wedge \mathrm{d}x_1 - \mathrm{d}r \wedge \mathrm{d}x = 0$. We have the equivalent result for a map in the cylinder.
By Poincaré lemma on a contractible domain, for example $\mathbb{R}^2$, every closed form is exact. But this is not true on non-contractible domains or not simply connected domains, as a cylinder: not all closed forms are exact in this domains. Therefore, we can expect that there are symplectic maps which are not exact.
On the other hand, exact symplectic maps are always symplectic. Indeed we have that, in the cylinder

$$\beta = \mathrm{d}\mathcal{S}(r,\theta) = r_1\,\mathrm{d}\theta_1 - r\,\mathrm{d}\theta, \quad (r,\theta) \in \Sigma,$$

and β is a closed form: $\mathrm{d}\beta = \mathrm{d}^2\mathcal{S}(r,\theta) = 0$.

Example 3. Consider the following maps:

- $\Phi(r,\theta) = (r, \theta + \alpha)$ for $\alpha \in]0, 2\pi[$.
  In the plane this map is a translation in the horizontal direction. It can be seen as the lift of a rotation. From $r_1\mathrm{d}\theta_1 - r\mathrm{d}\theta = r\mathrm{d}(\theta+\alpha) - r\mathrm{d}\theta = 0$. We deduce that rotations are exact symplectic maps with $\mathcal{S} \equiv 0$.

- $\Phi(r,\theta) = (r+\varepsilon,\theta)$ for $\varepsilon \in \mathbb{R}\setminus\{0\}$.
  This map can be interpreted as a vertical translation. It is clearly symplectic but it cannot satisfy the measure characterization for exact symplectic maps that we gave in the previous Remark 3.1.2.

  Alternatively, $r_1\mathrm{d}\theta_1 - r\mathrm{d}\theta = (r+\varepsilon)\mathrm{d}\theta - r\mathrm{d}\theta = \varepsilon\mathrm{d}\theta$. In the plane is an exact symplectic map with $\tilde{\mathcal{S}}(r,x) = \varepsilon x$. On the contrary the differential form $\varepsilon\mathrm{d}\theta$ is not exact in the cylinder (the periodicity condition fails) and so Φ is symplectic but not exact.

Now we are going to give another criterion to decide whether the diffeomorphism Φ is exact symplectic or not.

Let us consider the strip immersed in the cylinder Σ. Since $\tilde{\Sigma}$ is its universal covering, all 1-forms on Σ of class $\mathcal{C}^1$ can be expressed as

$$\beta = A(r,\theta)\mathrm{d}\theta + B(r,\theta)\mathrm{d}r$$

with $A, B \in \mathcal{C}^1(\Sigma)$ and 2π-periodic in θ. When β is closed it is possible to find a function $\mathcal{S} = \mathcal{S}(r,\theta)$ with $\mathrm{d}\mathcal{S} = \beta$. The problem is that sometimes $\mathcal{S}$ is not periodic in θ and so it becomes a multi-valued function when regarded in the cylinder, as in the previous example.

To decide if it is exact on the cylinder or not, we consider the following integral over the closed form β:

$$\int_c \beta$$

on the closed path $c = \{r_*\}\times\mathbb{T}$ for some $r_* \in [a,b]$ and we show that it vanishes. This translates in the condition

$$\int_0^{2\pi} A(r_*,\theta)\mathrm{d}\theta = 0,$$

since $r = r_*$ along the path c.
In our case, the differential form is $r_1\,\mathrm{d}\theta_1 - r\,\mathrm{d}\theta$. Identifying

$$A(r,\theta) = \mathcal{F}(r,\theta)\mathcal{G}_\theta(r,\theta) - r, \qquad B(r,\theta) = \mathcal{F}(r,\theta)\mathcal{G}_r(r,\theta).$$

So, we have that the diffeomorphism Φ is exact symplectic if and only if there exists $r_* \in [a,b]$ such that

$$\int_0^{2\pi} \mathcal{F}(r_*,\theta)\mathcal{G}_\theta(r_*,\theta)\mathrm{d}\theta = 2\pi r_*. \tag{3.1.6}$$

3.1.2 The twist condition and the intersection property

We introduce two key properties in the formulation of the theories presented in this monograph and also in the framework of symplectic maps.

Twist condition

The twist condition is essential since it is fundamental for the application of the Aubry-Mather theory and Moser's Twist Theorem. We have the following definition:

Definition 3. A map in the cylinder $\Phi \in \Sigma$ has the *twist condition* if

$$\partial_r \mathcal{G}(r,\theta) > 0 \qquad \forall (r,\theta) \in \Sigma. \tag{3.1.7}$$

To understand (3.1.7) let us recall that θ is an angle and r a radial variable. The twist condition means that the evolution of the angular variable θ by the map Φ depends on the radial variable r. If the initial radial position r increases then the angular displacement $\theta_1 - \theta$ increases. Geometrically, the twist condition means that Φ maps each vertical segment in the cylinder into a curve that always twists around the cylinder to the right (see Figure 3.1).

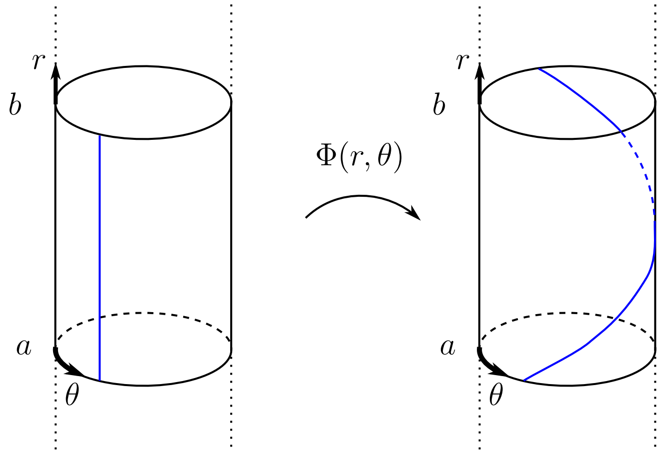

Figure 3.1: Twist condition $\partial_r \mathcal{G}(r,\theta) > 0$.

An equivalent definition of the twist condition is $\partial_r \mathcal{G}(r,\theta) < 0$. In this case, each vertical segment in the cylinder is mapped into a curve that twists to the left.

Considering the lift map $\tilde{\Phi} \in \tilde{\Sigma}$, we have a similar definition of the twist condition, $\partial_r \mathcal{G}(r,x) > 0$. The equivalent geometrical interpretation is that vertical segments in the universal cover of the cylinder, that is $\mathbb{R}^2$, are twisted to the right. This twist property of the lift map is employed to solve the implicit function problem

$$x_1 = \mathcal{G}(r,x)$$

to obtain a unique function $\mathcal{R}(x,x_1)$ in a domain of the plane $(x,x_1) \subset \mathbb{R}^2$. With the exact symplectic condition, we can define the so-called *generating function*

$$h(x,x_1) := \mathcal{S}(\mathcal{R}(x,x_1),x).$$

An important fact is that the map $\tilde{\Phi}$, can be expressed implicitly through a function $h = h(x,x_1)$:

$$\begin{cases} \partial_2 h(x,x_1) = r_1, \\ \partial_1 h(x,x_1) = -r \end{cases}$$

where ∂_i represents the derivative with respect to the i-th argument. In Lemma 3.4.1 we will present more details of this program to define the generating function and its properties.

Intersection property

There is another property of interest in the formulation of the Moser's Twist Theorem, the intersection property. We have the following definition:

Definition 4. A map $\Phi \in \mathcal{E}^0(\Sigma)$ has the *intersection property* if for every non-contractible Jordan curve $\Gamma \subset \Sigma$,

$$\Phi(\Gamma) \cap \Gamma \neq \emptyset.$$

We give an equivalent definition for maps in the plane. Consider the punctured domain $\mathcal{D} \subset \mathbb{R}^2 \setminus \{0\}$ and the homeomorphism $\mathcal{H} \in \mathcal{E}^0(\mathcal{D})$. The equivalent concept of non-contractible Jordan curve Γ in the cylinder is that the curve surrounds the origin.

Definition 5. The map $\mathcal{H} \in \mathcal{E}^0(\mathcal{D})$ has the *intersection property* if for every Jordan curve $\Gamma \subset \mathcal{D}$ that surrounds the origin,

$$\mathcal{H}(\Gamma) \cap \Gamma \neq \emptyset.$$

Remark 3.1.3. The intersection property is topological in the following sense: Consider the homeomorphism Ψ, if Φ has the intersection property then $\Psi \circ \Phi \circ \Psi^{-1}$ also has the intersection property.

An additional comment is that from the *Jordan Curve Theorem* we have that any non-contractible Jordan curve Γ divides the strip of the cylinder in two connected components: $\mathcal{R}_i(\Gamma)$ the lower one and $\mathcal{R}_e(\Gamma)$ the other one. Similarly the curve image $\Gamma_1 := \Phi(\Gamma)$ divides the strip in other two connected components, say, $\mathcal{R}_i(\Gamma_1)$ for the lower one and $\mathcal{R}_e(\Gamma_1)$ for the other. Observe that if $a = 0$ and $b = \infty$, the lower components of the cylinder $\mathcal{R}_i(\Gamma)$ and $\mathcal{R}_i(\Gamma_1)$ are bounded, and the others $\mathcal{R}_e(\Gamma)$ and $\mathcal{R}_e(\Gamma_1)$ are unbounded. Finally, using the characterization in the Remark 3.1.2, is easy to demonstrate that if $\Phi \in \mathcal{E}^1(\Sigma)$ is an exact symplectic map then it has the intersection property.

In Section 4.3.5 we will check if a map related with the system (2.2.2), the Poincaré map associated to this system, satisfies the intersection property.

Let us give some examples illustrating the two previous properties of maps in the cylinder:

Example 4. Consider the following maps:

- $\Phi(r, \theta) = ((1 + \varepsilon)r, \theta + \alpha + \beta(r))$ for $\beta'(r) > 0$ and $\varepsilon > 0$.

  This map is a twisted dilation. It expands the radial variable and in the angular variable we have a rotation given by $\alpha + \beta(r)$. Since $\beta'(r) > 0$, this map has the twist property but there exist some non-contractible Jordan curves such that the intersection property fails. For instance, curves with $r = r_c$ where r_c is a constant.

- $\Phi(r,\theta) = (r + \sin^2(\theta), \theta)$.

  This map has the intersection property since $r_1 = r$ only if $\theta = k\pi$ for $k \in \mathbb{Z}$ and is impossible to construct a non-contractible Jordan curve such that the image do not assume some of these values for θ. Additionally, it is clear that is not a twist map.

- From [77], $\Phi(r,\theta) = (r + f(\theta + r), \theta + r)$ with $r \in [a,b]$, $f \in \mathcal{C}^1$, 2π-periodic and
$$\int_0^{2\pi} f(\theta)\mathrm{d}\,\theta = 0.$$
  Using (3.1.6) we can check that this map is exact symplectic, for $r_* \in]a,b[$
$$\int_0^{2\pi} (r_* + f(\theta + r_*))\mathrm{d}\,\theta - 2\pi r_* = 0,$$
  so it has the intersection property. Also it has the twist property since $\partial_r \mathcal{G}(r,\theta) = 1$.

3.2 The Poincaré map

In this section we establish the link between the continuous dynamics of a periodic Hamiltonian system (1.5 degree of freedom) and the discrete dynamics of an area-preserving map in the cylinder. Poincaré shown this relation in the study of periodic orbits in Celestial Mechanics. This map is called *Poincaré map*, *Poincaré return mapping* or *first return map*. The Poincaré map of a T-periodic equation is the mapping $\mathcal{P}$ that assign to every initial point $(0,\xi_0)$ the value of the solution at $t = T$ with this initial condition, $\xi(T,\xi_0)$. Since there is a correspondence between many qualitative properties of this map with the qualitative properties of the original system, the Poincaré map has become an essential tool in the qualitative theory of ordinary differential equations.

Obviously this process can be generalized, so to study the flow in an n-dimensional phase space near a periodic orbit one can associate a mapping of an $(n-1)$-dimensional neighborhood of a fixed point of this mapping. In this text, we concentrate in the definition of this map for a periodic Hamiltonian system of 1 degree of freedom or equivalently, a Hamiltonian system of 2 degrees of freedom.

Let us assume that Ω is an open subset of $\mathbb{R}^2$. Consider a T-periodic field $X(t,\xi)$, continuous in t and differentiable in ξ:
$$\begin{aligned} X : \mathbb{R}\times\Omega &\longrightarrow \mathbb{R}^2 \\ (t,\xi) &\longmapsto X(t,\xi). \end{aligned}$$

We denote this field as $X \in \mathcal{C}^{0,1}(\mathbb{R} \times \Omega, \mathbb{R}^2)$. We suppose also that $\operatorname{div}_x X = 0$. This condition comes from the Hamiltonian character of the field. Due the regularity of the field there is uniqueness for the associated initial value problem, for simplicity let us assume $t_0 = 0$:

$$\begin{cases} \dot{\xi} = X(t, \xi), \\ \xi(0) = \xi_{_0}, \qquad \xi_{_0} \in \Omega. \end{cases} \tag{3.2.1}$$

The corresponding solution will be denoted by $\xi(t, \xi_{_0})$ and we suppose that is defined on a maximal interval I_{ξ_0}. For each $t \in I_{\xi_0}$ we can define the *time-t map* as

$$\begin{aligned} \phi_t : \mathcal{D}_t \subset \Omega &\longrightarrow \mathbb{R}^2 \\ \xi_0 &\longmapsto \phi_t(\xi_0) = \xi(t, \xi_{_0}), \end{aligned}$$

with $\mathcal{D}_t = \{\xi_0 \in \Omega \ : \ t \in I_{\xi_0}\}$. This map gives the evolution to time t of the solution of the initial value problem (3.2.1).

It is well known that, by the theorem on continuous dependence, $\mathcal{D}_t$ is open and ϕ_t is a homeomorphism from $\mathcal{D}_t$ onto $\phi_t(\mathcal{D}_t)$. In general, as the field $X(t, \xi)$ is not constant, this map does not define a flow, that is

$$\phi_t \circ \phi_s \neq \phi_{t+s} \quad \forall t, s \in \mathbb{R} \quad s.t. \quad t + s \in I_{\xi_0}.$$

Nevertheless, the map ϕ_t satisfies the property

$$\phi_t \circ \phi_T = \phi_{t+T} \quad \forall t \in I_{\xi_0}.$$

Remark 3.2.1. [67] There exist periodic equations such that ϕ_t and ϕ_T do not commute for some $t \in I_{\xi_0}$. For instance, consider the initial value problem:

$$\begin{cases} \dot{\xi} = \xi + \sin t \\ \xi(t_0) = \xi_{_0}. \end{cases}$$

The solution is:

$$\phi_t(\xi_0) = \xi(t, \xi_0) = \left(\xi_0 + \frac{1}{2}\right) \mathrm{e}^t - \frac{\sin(t) + \cos(t)}{2}$$

and

$$\phi_{t+2\pi}(\xi_0) = \left(\xi_0 + \frac{1}{2}\right) \mathrm{e}^{t+2\pi} - \frac{\sin(t) + \cos(t)}{2} = \phi_t(\phi_{2\pi}(\xi_0)).$$

Nevertheless,

$$\phi_{2\pi}(\phi_t(\xi_0)) = \left(\xi_0 + \frac{1}{2}\right) \mathrm{e}^{t+2\pi} - \frac{\sin(t) + \cos(t) - \frac{1}{2}}{2} \mathrm{e}^{2\pi} \neq \phi_{t+2\pi}(\xi_0).$$

for any $t \in I_{\xi_0}$.

From the theorem of differentiability with respect to the initial conditions this map is a diffeomorphism onto its image. Additionally, the Liouville theorem implies that ϕ_t is an orientation and area preserving map, that is, $\det \phi_t'(\xi_0) = 1$. Therefore, the *time-t map* ϕ_t is a symplectic map in the plane.

Let us define the map $\tilde{\mathcal{P}} := \phi_T$ as the Poincaré map of (3.2.1). It is clear that satisfies $\tilde{\mathcal{P}}^n := \phi_{nT}$ with $n \in \mathbb{Z}$. We recall that an important property of periodic equations is that if $\xi(t, \xi_0)$ is a solution then $\xi(t + T, \xi_0)$ is also a solution. From this fact and the uniqueness we deduce that

$$\xi(t + T, \xi_0) = \xi(t, \tilde{\mathcal{P}}(\xi_0))$$

whenever it is defined.
Additionally, the Poincaré map is a homeomorphism isotopic to the identity. The isotopy H_λ would be the time-t map with $\phi_0 = \mathrm{Id}$ and $\phi_T = \tilde{\mathcal{P}}$. Starting at the plane $t = 0$ and then move in the direction of time until reaching $t = T$, the flow will deforms continuously the identity into $\tilde{\mathcal{P}}$.

As we said, the dynamics of (3.2.1) can be studied through the Poincaré map $\tilde{\mathcal{P}}$ since many properties of the periodic Hamiltonian system are translated to the framework of discrete dynamics via $\tilde{\mathcal{P}}$. For example, a fixed point of the Poincaré map corresponds to the initial condition of a *T-periodic solution*. Cycles or periodic points of the Poincaré map are associated with *Generalized periodic solutions* or solutions with rational frequencies. Furthermore, solutions with incommensurable frequencies, that is, *quasi-periodic orbits* are associated to certain invariant sets of the Poincaré map. These invariant sets are the so-called *invariant curves* or *invariant tori* and they are related with questions of stability or boundedness of orbits. The property of isotopy of the Poincaré map is crucial to prove the stability when one has an invariant curve. In the course notes [66] and in the works [40, 65, 67] we can find more information about this relation.

So the Poincaré map $\tilde{\mathcal{P}}$ is a symplectic map in the plane. As we work with the 1-periodic Hamiltonian system (2.2.2). After taking action-angle variables (r, θ), we can interpret the Poincaré map as a system in a strip of the cylinder (in Chapters 4 and 5 we have an explicit expression of this system, equations (4.3.5) or (5.2.2)). Let us introduce the Poincaré map $\mathcal{P}$ in this context as

$$\begin{array}{rccl} \mathcal{P}: & \Sigma & \longrightarrow & \Sigma' \subset \mathbb{R} \times \mathbb{T} \\ & (r_0, \theta_0) & \longmapsto & (r_1, \theta_1) = (r(1; r_0, \theta_0), \theta(1; r_0, \theta_0)), \end{array}$$

where $(r(t; r_0, \theta_0), \theta(t; r_0, \theta_0))$ is the solution defined on $t \in [0, 1]$ with initial condition $(r(0), \theta(0)) = (r_0, \theta_0)$.

Therefore, we have a connection between a Hamiltonian system and the discrete dynamics of a map defined in a strip Σ of the cylinder, concretely is a diffeomorphism of a section of the cylinder and $\mathcal{P} \in \mathcal{E}^1(\Sigma)$. Due to the orientation and area preserving property of the original field, the Poincaré map $\mathcal{P}$ is a symplectic map in the cylinder.

In the following Chapters we should assure that the Poincaré map is well defined in each case and has the appropriate properties. The remains sections of this Chapter are devoted to present the main theorems used in this research.

3.3 A KAM theorem

The main purpose of the "KAM theory" or the "KAM technique" (Moser points out that he liked this name better than "KAM theory", see [57]) in the context of Hamiltonian Dynamical Systems is to study what is preserved when an integrable system is perturbed. In other words, what happens to the invariant tori that foliate the phase space in an integrable system when it is perturbed. That is, to study the so-called *nearly integrable systems*.

According to Poincaré, the investigation of these systems is the "fundamental problem of Dynamics" [73]. This problem was studied by Poincaré motivated by the problem of the stability of the planetary system, but the appearance of "small denominators" in the formal series expansion (Lindstedt series) originated the difficulties to prove the convergence. Over a long period Poincare, Birkhoff, Siegel and others have not been able to solve the problems in this perturbative approach. Then, at the International Mathematical Congress of 1954, Kolmogorov presented an important result in the framework of Hamiltonian systems [28]: a small analytic perturbation of an integrable analytic Hamiltonian systems preserve most of the invariant tori. This result also appeared in a classic paper [27], where the proof of the theorem is discussed. He gave the sketch of how to control small divisors through the Newton's method of convergence. It is in this point where a diophantine condition appears. We quote that also Siegel worked with the onset of small divisors and diophantine conditions in the context of holomorphic functions [80]. A few years later, Arnold gave a detailed proof in [3]. See also [4] where Arnold proved an important result in the N-body problem of Celestial Mechanics: If the masses of $N-1$ "planets" are sufficiently small in comparison with the mass of the central body, there exists a set of initial conditions having a positive Lebesgue measure such that, if the initial positions and velocities of the bodies belong to this set, the distances of the bodies from each other will remain perpetually bounded and without collisions, like their Keplerian approximation.

Informally, the theorem says that if we have a small analytic perturbation to an integrable analytic Hamiltonian system with n degrees of freedom, $H(I,\varphi) = H_0(I) + \varepsilon H_1(I,\varphi) \in \mathbb{R}^n \times \mathbb{T}^n$, and a non-degeneracy condition for the frequency map $\eta(I) := -\partial_I H_0$, that is, the map $\eta(I)$ is locally a diffeomorphism and we can classify the tori of the integrable system by their frequencies η. Then many of the invariant tori from the integrable system do not break but are only slightly deformed. The preserved tori still carry quasi-periodic motions.

"Many of the invariant tori" means those tori whose frequency satisfies the following Diophantine condition: For constants $\mu > n-1$ and $\gamma > 0$, it is required that $\forall k \in \mathbb{Z}^n \setminus \{0\}$,

$$|\langle \eta, k\rangle| \geq \frac{\gamma}{|k|^{\mu}}, \quad \text{with} \quad \langle \eta, k\rangle = \sum_{j=1}^{n} \eta_j k_j. \tag{3.3.1}$$

The relation of γ with the size of the perturbation is

$$|\varepsilon| < \gamma^2\delta$$

with $\delta > 0$. So if ε increases, the Diophantine condition becomes more restrictive because γ also has to increase. In the case of a perturbed Hamiltonian of finite class, say $\mathcal{C}^r$, the condition for μ is $2n < 2\mu + 2 < r$. This implies that the more differentiable a system is, the more invariant tori are preserved, since μ can be larger and the Diophantine condition is less restrictive.

The preserved tori contain a nowhere dense closed set whose complement has measure that tends to zero if the size of the perturbation also tends to zero. That is, invariant tori are not isolated objects, on the contrary, they cover a substantial part of the phase space: their union has positive Lebesgue measure, although it is usually nowhere dense. In the same way that the Diophantine numbers are distributed on the real line $\mathbb{R}$.

In the case of a system with $n = 2$ degrees of freedom the phase space is 4-dimensional. The 3-dimensional invariant manifold where the autonomous Hamiltonian H is constant, is divided by 2-dimensional deformed invariant tori of the perturbed system. The complementary domain has the form of gaps through which the trajectories cannot leave, since they cannot intersect the invariant tori. In this low dimensional case, we can imagine the "effective phase space" as a book with an infinite number of pages ordered according to their frequency $\eta(I)$. The perturbation can tear off the pages corresponding to a rational frequency and the irrational ones that do not fulfill a Diophantine condition. If we increase the size of the perturbation, the Diophantine condition becomes more restrictive and the book will be losing its pages.

For $n > 2$ the invariant n-dimensional tori do not separate the $(2n-1)$-dimensional energy levels where H is constant, and trajectories from the gaps can a priori go to infinity, these phenomena is known as *Arnold diffusion*.

On the other hand, Moser focused on the discrete version of this theorem. In the sixties, he proved the existence of invariant curves for small perturbations of area preserving twist mappings of the annulus [55]. This result changes the hypothesis of analyticity by a finite regularity for this mappings, say $\mathcal{C}^k$. The proof of this result combines Newton's method in abstract linear spaces with an implicit function theorem due to Nash ([60]) to overcome the "loss of derivatives" in the convergence problem of the iterative process. In Moser's own words [54]:

"In many applications this method (*Newton's method*) fails, however, because of a phenomenon which may be described as "loss of derivatives". To be more precise, we consider u, v, u_n as elements of a function space, i.e., as r-times differentiable functions of the variables $x = (x_1, x_2, ..., x_d)$ ranging in an open set S. As norm we introduce

$$|v|_r = \max_{0\leq\rho\leq r} \sup_{x\in\mathcal{S}} |\, \mathrm{D}^\rho\, v(x)|,$$

where D denotes any derivative of order ρ. The difficulty frequently encountered is the following: In the attempt to construct successive approximations it may occur that one can estimate $|u_{n+1} - u_n|_r$ only in terms of $|u_n - u_{n-1}|_{r+s}$ where the positive integer s designates the loss of derivatives. It is clear that such a process can not lead to an approximation of the solution, since after finitely many steps all derivatives are exhausted unless one works with infinitely often differentiable or analytic functions. In the latter case, however, the convergence of the procedure is doubtful and in most cases divergence prevails."

In fact, in [55] the regularity hypothesis is that $k = 333$. Later, Rüssmann, Herman and Moser himself among others gave different versions of this theorem with weaker regularity assumptions. For a detailed formulation of different versions of this theorem see for instance [58, 77] (for $k \geq 5$) or [21, 22] (for $k \geq 3 + \varepsilon$). Some counterexamples due especially by Herman [21] suggest that k has to be chosen greater than 3.

So, as we said, this result is related with the previous theorem for Hamiltonian systems of $n = 2$ degrees of freedom. The associated Poincaré map is an area preserving map, and it has the twist property. Clearly, a small perturbation of an integrable system results in a small perturbation of the Poincaré map. Also, the existence of a preserved torus for the perturbed system near an invariant torus for the integrable system is equivalent to the existence of an invariant curve for the perturbed Poincaré map near a circle $r = r_0$. On the other hand, the non-degeneracy condition for the frequency map turns into the twist condition for the Poincaré map, and the Diophantine condition (3.3.1) becomes a Diophantine condition for the rotation number $\alpha(r)$. Recall that the rotation number α of a map $\Phi(r_n, \theta_n) = (r_{n+1}, \theta_{n+1})$ for $n \in \mathbb{Z}$, is defined as

$$\alpha := \frac{1}{2\pi} \lim_{n \to \infty} \frac{\theta_n}{n}.$$

The Poincaré maps for Hamiltonian systems are area preserving and exact. This, in particular, implies that it has the intersection property. Moser used only this weaker property to prove the preservation of invariant curves with Diophantine rotation number. Thus, his result holds for a more general class of systems and not only for the associated Poincaré maps of Hamiltonian systems with 2 degrees of freedom.

Let us illustrate the previous paragraphs. Consider the twist map defined on a strip of the cylinder Σ:

$$\Phi : \begin{cases} r_1 = r \\ \theta_1 = \theta + \beta(r) \end{cases} \tag{3.3.2}$$

with $\beta' > 0$. This map leaves invariant each curve Γ of the form $r = r_*$, rotating it an angle $\beta(r_*)$. We can say that the map Φ restricted to Γ is conjugated to a rotation $\mathcal{R}_{\beta(r_*)}$, in the sense that there exists an homeomorphism ψ such that $\psi \circ \Phi = \mathcal{R}_{\beta(r_*)} \circ \psi$. That is, the diagram in Figure 3.2 commutes. In this case the homeomorphism is $\psi(\theta, r) = \theta$.

$$\begin{array}{ccc} \Gamma & \xrightarrow{\Phi} & \Gamma \\ \psi \downarrow & & \downarrow \psi \\ \mathbb{T} & \xrightarrow[\mathcal{R}_{\beta(r_*)}]{} & \mathbb{T} \end{array}$$

Figure 3.2: Commutative diagram $\psi \circ \Phi = \mathcal{R}_{\beta(r_*)} \circ \psi$.

Note that we can assign to every invariant curve a rotation number, defined through the diagram since $\beta' > 0$. The rotation number for each invariant curve Γ of the form $r = r_* \in [a, b]$ is $\alpha = \frac{\beta(r_*)}{2\pi}$. Observe that for every $\alpha \in [\beta(a), \beta(b)]$ there exists an invariant curve with rotation number α. Also note that the map Φ restricted to Γ is still isotopic to the identity so that the orientation is preserved on Γ. So, we have that the rotation number is well defined.

In this context, if we consider a small perturbation of the map (3.3.2), some questions are relevant: ¿does some invariant curve is preserved? ¿how many of them are preserved after the perturbation?. Moser's Twist Theorem provides some answers.

Moser's Twist Theorem

In this thesis we will use Moser approach, precisely we study the analytical version of his invariant curve theorem. As we said, Moser considers small perturbations of a twist integrable map and gives sufficient conditions that guarantee the preservation of some rotational or non-contractible invariant curve. Rotational invariant curves are particularly important as they act as barriers in the cylinder. This fact is crucial when one deals with question of stability or boundedness. Therefore the concept of a (rotational) invariant curve for a map $\Phi \in \mathcal{E}(\Sigma)$ is clear:

Definition 6. An *invariant curve* Γ is a non-contractible simple Jordan curve contained in $\mathcal{E}$ such that $\Phi(\Gamma) = \Gamma$.

Also could exist curves Γ such that $\Phi(\Gamma) = \Gamma$ that are contractible, called *librational* invariant curves: our definition implies that we are not considering this kind of curves. See for instance the reviews [47, 48] for more details in the definitions of librational and rotational curves.

The analytic version of *Moser's Twist Theorem* or the *Invariant Curve Theorem* is the following, see [76, 81]. Consider a strip Σ of the cylinder of width 1, i.e $b - a = 1$, and a perturbation of the map (3.3.2) with $\beta(r) = \gamma r$:

$$\Phi_p : \begin{cases} r_1 = r + f(r, \theta) \\ \theta_1 = \theta + \gamma r + g(r, \theta) \end{cases} \qquad \gamma > 0.$$

Assume that f, g are real analytic functions and 2π-periodic in θ, then they can be extended to a complex domain which we may take to be of the form

$$\mathcal{D}_{\mathbb{C}} = \mathcal{K}([a,b]) \times \mathcal{C}_{\Delta}$$

with

$$\mathcal{C}_{\Delta} = \{\theta \in \mathbb{C} \;:\; |\mathrm{Im}\,(\theta)| < \Delta\}$$

and being $\mathcal{K}([a,b])$ a complex neighborhood of the interval $[a,b]$.
Moreover, let us assume that this map has the intersection property restricted for non-contractible Jordan curves, which is the graph of an analytic and 2π-periodic function in θ, φ with $r = \varphi(\theta)$.

Theorem 3.3.1. *With the previous setting and considering the positive constants $C < \frac{1}{2}$ and $\mu \geq 1$. For each ε there exists a positive M_0 (depending on ε, C, μ, Δ and the size of $\mathcal{K}$ but not on γ), such that for*

$$|f(r,\theta)| + |g(r,\theta)| < \gamma M_0, \qquad (r,\theta) \in \mathcal{D}_{\mathbb{C}},$$

the map Φ_p has an invariant real analytic curve in parameter representation:

$$\begin{cases} r = u(\zeta), & u(\zeta + 2\pi) = u(\zeta) \\ \theta_1 = \zeta + v(\zeta) & u(\zeta + 2\pi) = u(\zeta) \end{cases} \qquad |\mathrm{Im}\,\zeta| < \frac{\Delta}{2}, \tag{3.3.3}$$

such that the induced analytic map on this curve is the rotation

$$\zeta_1 = \zeta + \alpha$$

where $\alpha \in [\gamma a, \gamma b]$ and satisfies the Diophantine condition:

$$\left| \frac{\alpha}{2\pi} - \frac{p}{q} \right| \geq \frac{C}{q^{\mu+1}}, \qquad \forall\, \frac{p}{q} \in \mathbb{Q}. \tag{3.3.4}$$

The functions u and v satisfy

$$\left| v - \frac{\alpha}{\gamma} \right| + |u| < \varepsilon.$$

A proof of this theorem can be found in the book by Siegel and Moser [81] or in the works [32, 76]. In [32], in contrast to the Hamiltonian approach, where one looks for the invariant curves in the phase space, the authors use a Lagrangian approach which is based on the search for the solution of a second-order difference equation in the configuration space.

Remark 3.3.1. Let us give some additional comments:

- The intersection property is a key hypothesis for the existence of invariant curves. It is clear that a smallness condition alone on f and g is not suffice, as can be seen by taking f to be a small positive constant. In that case r always increases under application of the map, and no invariant curve can exist. Recall also the example 4 of Subsection 3.1.2.

- The existence of invariant curves for the map Φ_p is strongly related with the arithmetic properties of the rotation number α. In fact, each choice of α in the twist interval $[\gamma a, \gamma b]$ and satisfying the condition (3.3.4) gives rise to an invariant curve (if we adjust the size of the perturbation with M_0).

- In this version of the theorem, the invariant curves are analytic and analytically conjugated to a rotation with rotation number α. In a finite differentiable version, the regularity is reduced, that is, the theorem concludes that there exists an invariant $\mathcal{C}^1$-curve and the map induces a $\mathcal{C}^1$-diffeomorphism with rotation number α. See for instance the books by Moser [58, 59].

- As we said previously, the existence of invariant curves Γ for a perturbed twist map $\Phi \in \mathcal{E}^k(\Sigma)$ with $k > 3$ guarantees the stability or boundedness of the orbits with initial conditions $p := (r, \theta)$ in the bounded component of the cylinder. This is consequence of the isotopy to the identity of the map. After an iteration, the image of a connected component will be in the same connected component. Since there is a continuous deformation $H(\lambda, p)$ from the identity map to the map Φ, there cannot exist orbits connecting the top and the bottom of the cylinder.

The following example illustrates the beautiful relation between the invariant curves and the condition for Diophantine numbers.

Example 5. *Last invariant curve for the standard map:* Consider the standard map

$$\begin{cases} r_1 = r - \frac{\lambda}{2\pi}\sin(2\pi\theta) \\ \theta_1 = \theta + r_1. \end{cases}$$

In [15], Greene discovered numerically that the last invariant curve corresponds to a frequency very close to the golden number, $\alpha = \gamma = \frac{\sqrt{5}-1}{2}$, and the value $\lambda \approx 0.971635$.

3.4 A generalized Aubry-Mather theorem

The usual Mather's Theorem [46, 47] gives sufficient conditions on an exact symplectic twist diffeomorphism of the infinite cylinder $\mathfrak{C}$ to get the existence of Aubry-Mather sets $\mathcal{M}_\alpha$ for every rotation number α. This sets are minimizers of an action functional defined via the generating function h and contain some orbits of the diffeomorphism classified by its rotation number. From a topological point of view, Aubry-Mather sets $\mathcal{M}_\alpha$ sets are limits of monotone (p, q)-periodic orbits, see [9, 47]. Also, the first work of Mather [45] applies to the class of area-preserving twist homeomorphisms of the annulus (or defined in a strip Σ of the cylinder). In both cases, the arithmetic properties of α determine the structure of $\mathcal{M}_\alpha$ and the dynamics on it.

- If α is rational then $\mathcal{M}_\alpha$ contains periodic orbits. These periodic orbits correspond to not elliptic fixed points of the Poincaré map. Also $\mathcal{M}_\alpha$ may contain heteroclinic or homoclinic orbits.

- If α is irrational, then Mather's Theorem gives us an alternative: either $\mathcal{M}_\alpha$ is an invariant curve of Φ and every orbit on it is quasi-periodic with frequencies $(1, \alpha)$, or the minimal invariant set become a Cantor set. In the last case the motion is not quasi-periodic in the classical sense but the dynamics on a Cantor set is of Denjoy type with rotation number α.

All these orbits of $\mathcal{M}_\alpha$ have an order preserving property, that is, the points on the orbit come in the same order as those obtained from a trivial rotation with the same frequency. This is consequence of the topological semi-conjugacy to a rotation of angle α.

Remark 4. [11, 36, 72] The classical results of of Poincaré and Denjoy for topological classification of homeomorphisms or diffeomorphisms f of the circle show these two distinct possibilities. If the rotation number is rational, say $\frac{p}{q}$ then one has the following cases:

- f has a periodic orbit, every periodic orbit has period q, and the order of the points on each such orbit coincides with the order of the points for a rotation by p/q. Moreover, every forward orbit of f converges to a periodic orbit. The same is true for backward orbits, corresponding to iterations of f^{-1}, but the limiting periodic orbits in forward and backward directions may be different.

Nevertheless, given an invariant curve Γ if the rotation number α of $f|_\Gamma$ is irrational then f has no periodic orbit and one has the following:

- if $f|_\Gamma \in \mathcal{C}^2$ then $f|_\Gamma$ is topologically conjugate to a rotation of angle α, $\mathcal{R}_\alpha$ and every orbit is dense in Γ.
- (Denjoy counterexamples) There exist examples of $f|_\Gamma \in \mathcal{C}^{2-\varepsilon}$, $\varepsilon > 0$, such that is not topologically conjugated to a rotation, no orbit is dense in Γ and the limit set of the orbit of every point of Γ is the same Cantor set (Denjoy minimal set). Cantori can be visualized as invariant circles with deleted "gaps."

See also [56] or [59] for a presentation of Denjoy's theory in the context of the Aubry-Mather theory.

Therefore Aubry-Mather theory gives the existence of orbits for twist maps in the cylinder, classified and ordered by their rotation number α. See for instance [9, 46, 47, 59] for a detailed exposition of the Aubry-Mather theory. Let us illustrate with the simplest example what does and does not contain the Aubry-Mather set, $\mathcal{M}_\alpha$.

Example 6. Consider the phase map of the pendulum. Aubry-Mather set $\mathcal{M}_\alpha$ contains orbits with a rotation number α. If $\alpha = 0$, $\mathcal{M}_0$ contains the inestable equilibrioum point, the superior heteroclinic orbit, i.e. the orbit from $(x, x') = (\pi, 0)$ to $(x, x') = (\pi, 0)$; and the inferior heteroclinic, i.e. the orbit from $(x, x') = (\pi, 0)$ to $(x, x') = (\pi, 0)$. Clearly, if α is irrational, all the orbits are associated to an invariant curve.

Nevertheless, $\mathcal{M}_\alpha$ does not contains these orbits with a rotation number not well defined: the orbits in the interior region of the union of the heteroclinics. So, Aubry-Mather set does not contain the stable equilibrium point. This critical point is not a minimizer of the action.

Now, for our purposes let us consider a different symplectic form, a weighted symplectic form. Being the weight a $\mathcal{C}^2$ function with Lipschitz inverse

$$\begin{array}{rcl} f :]a,b[& \longrightarrow & \mathbb{R} \\ r & \longmapsto & f(r), \end{array}$$

such that f' never vanishes. Without loss of generality we fix $f' > 0$.
Consider a $\mathcal{C}^2$ diffeomorphism defined in a strip of the cylinder

$$\begin{array}{rrcl} \Phi : & \Sigma & \longrightarrow & \mathfrak{C} \\ & (r,\theta) & \longmapsto & (r_1,\theta_1) = (\mathcal{F}(r,\theta), \mathcal{G}(r,\theta)) \,. \end{array}$$

and the corresponding lift by

$$\begin{array}{rrcl} \Phi : & \tilde{\Sigma} & \longrightarrow & \mathbb{R}^2 \\ & (r,x) & \longmapsto & (r_1,x_1) = (\mathcal{F}(r,x), \mathcal{G}(r,x)) \end{array} \tag{3.4.1}$$

where

$$\begin{aligned} \mathcal{F}(r,x+2\pi) &= \mathcal{F}(r,x), \\ \mathcal{G}(r,x+2\pi) &= \mathcal{G}(r,x) + 2\pi. \end{aligned}$$

We suppose that Φ is exact symplectic with respect to the weighted form $f(r)\mathrm{d}\theta$. That is, there exists a $\mathcal{C}^2$ function

$$\begin{array}{rrcl} \mathcal{S} : & \Sigma & \longrightarrow & \mathbb{R} \\ & (r,\theta) & \longmapsto & \mathcal{S}(r,\theta) \end{array}$$

such that

$$\mathrm{d}\mathcal{S}(r,\theta) = f(r_1)\,\mathrm{d}\theta_1 - f(r)\,\mathrm{d}\theta, \quad \forall (r,\theta) \in \Sigma.$$

Remark 3.4.1. Note that the function $\mathcal{S}(r,\theta)$ is defined in the cylinder, hence the lift $\mathcal{S}(r,x)$ must be a 2π-periodic function in the variable x such that

$$\mathcal{S}_r(r,x) = f(\mathcal{F}(r,x))\mathcal{G}_r(r,x), \qquad \mathcal{S}_x(r,x) = f(\mathcal{F}(r,x))\mathcal{G}_x(r,x) - f(r). \tag{3.4.2}$$

The weighted symplectic form is:

$$\lambda = \mathrm{d}f(r) \wedge \mathrm{d}\theta = f'(r)\mathrm{d}r \wedge \mathrm{d}\theta$$

We also suppose that Φ is twist, that is

$$\partial_r \mathcal{G}(r,\theta) > 0 \qquad \forall (r,\theta) \in \Sigma. \tag{3.4.3}$$

Suppose additionally that the following uniform limits (w.r.t. x) exist

$$\alpha^+(x) := \frac{1}{2\pi}\left(\lim_{r\to b}\mathcal{G}(r,x) - x\right),$$
$$\alpha^-(x) := \frac{1}{2\pi}\left(\lim_{r\to a}\mathcal{G}(r,x) - x\right).$$

Note that $\alpha^{\pm}(x)$ are 2π-periodic $\mathcal{C}^2$ functions and define

$$W^+ = \min_x \alpha^+(x), \qquad W^- = \max_x \alpha^-(x).$$

The main result of this section deals with the existence of special orbits of the diffeomorphism Φ. To state the Theorem, we recall that a sequence $(x_n)_{n\in\mathbb{Z}}$ of real numbers is increasing if $x_n < x_{n+1}$ for all $n \in \mathbb{Z}$ and we say that any two translates are comparable if for any $(s,q) \in \mathbb{Z}^2$ only one of the following alternatives holds

$$\overline{x}_{n+q} + 2\pi s > \overline{x}_n \quad \forall n, \qquad \overline{x}_{n+q} + 2\pi s = \overline{x}_n \quad \forall n, \qquad \overline{x}_{n+q} + 2\pi s < \overline{x}_n \quad \forall n.$$

We are now ready to state the main result of this section:

Theorem 3.4.1. *With the previous setting, suppose that $W^+ - W^- > 8\pi$ and fix α such that $2\pi\alpha \in (W^- + 4\pi, W^+ - 4\pi)$. Then*

- *if $\alpha = s/q \in \mathbb{Q}$ there exists a (s,q)-periodic orbit $(\overline{r}_n, \overline{x}_n)_{n\in\mathbb{Z}}$ such that*

$$\overline{r}_{n+q} = \overline{r}_n, \qquad \overline{x}_{n+q} = \overline{x}_n + 2\pi s \quad \forall n \in \mathbb{Z};$$

- *if $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ there exists a compact invariant subset $\mathcal{M}_\alpha \subset \Sigma$ (and a corresponding subset $\tilde{\mathcal{M}}_\alpha \subset \tilde{\Sigma}$) with the following properties:*
  - *denoting $\pi : \Sigma \to \mathbb{T}$ the projection, $\pi|_{\mathcal{M}_\alpha}$ is injective and $\mathcal{M}_\alpha = \operatorname{graph} u$ for a Lipschitz function $u : \pi(\mathcal{M}_\alpha) \to \mathbb{R}$,*
  - *each orbit $(\overline{r}_n, \overline{x}_n)_{n\in\mathbb{Z}} \in \tilde{\mathcal{M}}_\alpha$ is such that the sequence $(\overline{x}_n)$ is increasing and any two translates are comparable,*
  - *each orbit $(\overline{r}_n, \overline{x}_n)_{n\in\mathbb{Z}} \in \tilde{\mathcal{M}}_\alpha$ has rotation number α, i.e.*

$$\frac{1}{2\pi}\lim_{n\to\infty}\frac{\overline{x}_n}{n} = \alpha;$$

  - *the set $\mathcal{M}_\alpha$ is either an invariant curve or a Cantor set.*

The following corollary gives an equivalent interpretation of the result and has been proven in Section 6 of [42].

Corollary 3.4.1. *For each α there exists two functions $\phi, \eta : \mathbb{R} \to \mathbb{R}$ such that, for every $\xi \in \mathbb{R}$*

$$\phi(\xi + 2\pi) = \phi(\xi) + 2\pi, \quad \eta(\xi + 2\pi) = \eta(\xi),$$
$$\Phi(\phi(\xi), \eta(\xi)) = (\phi(\xi + 2\pi\alpha), \eta(\xi + 2\pi\alpha))$$

where ϕ is monotone (strictly if $\alpha \in \mathbb{R}\setminus\mathbb{Q}$) and η is of bounded variation.

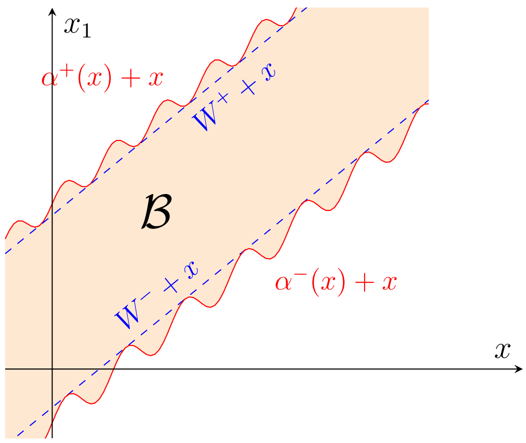

Figure 3.3: Domain $\mathcal{B}$.

The proof of Theorem 3.4.1 will make use of the *generating function*. We introduce it in the following

Lemma 3.4.1. *There exists an open connected set $\mathcal{B} \subset \mathbb{R}^2$ and a function $h : \mathcal{B} \to \mathbb{R}$, called generating function such that*

o) $\mathcal{B}$ is invariant under the translation $(x, x_1) \mapsto (x + 2\pi, x_1 + 2\pi)$;

i) $h \in \mathcal{C}^3(\mathcal{B})$;

ii) $h(x + 2\pi, x_1 + 2\pi) = h(x, x_1)$ for all $(x, x_1) \in \mathcal{B}$;

iii) $\partial^2_{xx_1} h(x, x_1) < 0$ for all $(x, x_1) \in \mathcal{B}$;

iv) a sequence $(\overline{r}_n, \overline{x}_n)_{n\in\mathbb{Z}}$ is an orbit of $\tilde{\Phi}$ iff for all $n \in \mathbb{Z}$

$$\partial_1 h(\overline{x}_n, \overline{x}_{n+1}) + \partial_2 h(\overline{x}_{n-1}, \overline{x}_n) = 0 \quad \textit{and} \quad f(\overline{r}_n) = -\partial_1 h(\overline{x}_n, \overline{x}_{n+1}).$$

Proof. By the twist property, $\alpha^+(x) > \alpha^-(x) \quad \forall x \in \mathbb{R}$, so that we can consider the open connected set (see figure 3.3)

$$\mathcal{B} = \left\{(x, x_1) \in \mathbb{R}^2 : \alpha^-(x) < x_1 - x < \alpha^+(x)\right\}.$$

The periodic property of the functions $\alpha^{\pm}(x)$ implies that this set is invariant under the translation

$$T_{2\pi,2\pi}(x, x_1) = (x + 2\pi, x_1 + 2\pi).$$

By the twist condition we can solve the implicit function problem

$$x_1 = \mathcal{G}(r, x)$$

and obtain a unique $\mathcal{C}^2$ function $\mathcal{R}(x, x_1) : \mathcal{B} \longrightarrow,]a, b[$ such that

$$x_1 = \mathcal{G}(r, x) \Longleftrightarrow r = \mathcal{R}(x, x_1)$$

and, by implicit differentiation,

$$\mathcal{G}_r(\mathcal{R}, x)\mathcal{R}_x + \mathcal{G}_x(\mathcal{R}, x) = 0, \qquad \mathcal{G}_r(\mathcal{R}, x)\mathcal{R}_{x_1} = 1. \tag{3.4.4}$$

Moreover, uniqueness implies that $\mathcal{R}(x+2\pi, x_1+2\pi) = \mathcal{R}(x,x_1)$. Analogously we get

$$r_1 = \mathcal{F}(r,x) \Longleftrightarrow r_1 = \mathcal{F}(\mathcal{R}(x,x_1),x) := \mathcal{R}_1(x,x_1)$$

with $\mathcal{R}_1(x+2\pi, x_1+2\pi) = \mathcal{R}_1(x,x_1)$. Hence, the map (5.5.1) is equivalent to

$$\begin{cases} r_1 = \mathcal{R}_1(x,x_1), \\ r = \mathcal{R}(x,x_1) \end{cases} \quad \text{with } (x,x_1) \in \mathcal{B}.$$

Now, we use the exact symplectic condition and define the generating function

$$h(x,x_1) := \mathcal{S}(\mathcal{R}(x,x_1),x).$$

This map is clearly $\mathcal{C}^2(\mathcal{B})$, and, a posteriori, we will get $\mathcal{C}^3$ regularity. From the periodicity conditions of $\mathcal{S}$ and $\mathcal{R}$, one can prove the periodicity condition *ii)*.
To prove point *iii)*, we use (3.4.2),(3.4.4) to get that for all $(x,x_1) \in \mathcal{B}$,

$$\begin{aligned} \partial_x h(x,x_1) &= \partial_x \mathcal{S}(\mathcal{R}(x,x_1),x) = \mathcal{S}_r(\mathcal{R}(x,x_1),x)\mathcal{R}_x + \mathcal{S}_x(\mathcal{R}(x,x_1),x) \\ &= f(\mathcal{F}(\mathcal{R},x))\mathcal{G}_r(\mathcal{R},x)\mathcal{R}_x + f(\mathcal{F}(\mathcal{R},x))\mathcal{G}_x(\mathcal{R},x) - f(\mathcal{R}) \\ &= -f(\mathcal{R}), \end{aligned} \tag{3.4.5}$$

so that the twist condition and the monotonicity of f imply

$$\partial_{x,x_1} h(x,x_1) = -\partial_{x_1} f(\mathcal{R}(x,x_1)) = -f'(\mathcal{R})\partial_{x_1}\mathcal{R}(x,x_1) = -\frac{f'(\mathcal{R})}{\partial_r \mathcal{G}(x,x_1)} < 0.$$

To prove the last point, a similar computation as (3.4.5) gives for all $(x,x_1) \in \mathcal{B}$,

$$\begin{aligned} \partial_{x_1} h(x,x_1) &= \partial_{x_1} \mathcal{S}(\mathcal{R}(x,x_1),x) = \mathcal{S}_r(\mathcal{R}(x,x_1),x)\mathcal{R}_{x_1} \\ &= f(\mathcal{F}(\mathcal{R},x))\mathcal{G}_r(\mathcal{R},x)\mathcal{R}_{x_1} = f(\mathcal{F}(\mathcal{R},x)) \\ &= f(\mathcal{R}_1). \end{aligned} \tag{3.4.6}$$

Equations (3.4.5)-(3.4.6), together with the regularity of $f, \mathcal{R}, \mathcal{R}_1$ have the consequence that $h \in \mathcal{C}^3(\mathcal{B})$, proving point *i)*.
Less formally (3.4.5)-(3.4.6) also imply that the map Φ can be expressed implicitly:

$$\begin{cases} \partial_{x_1} h(x,x_1) = f(r_1), \\ \partial_x h(x,x_1) = -f(r) \end{cases} \quad \text{with } (x,x_1) \in \mathcal{B}.$$

It means that an orbit $(\overline{r}_n, \overline{x}_n)_{n\in\mathbb{Z}}$ is such that for every $n \in \mathbb{Z}$

$$\begin{cases} f(\overline{r}_{n+1}) = \partial_2 h(\overline{x}_n, \overline{x}_{n+1}), \\ f(\overline{r}_n) = -\partial_1 h(\overline{x}_n, \overline{x}_{n+1}). \end{cases}$$

This implies $f(\overline{r}_n) = -\partial_1 h(\overline{x}_n, \overline{x}_{n+1}) = \partial_2 h(\overline{x}_{n-1}, \overline{x}_n)$ so that

$$\partial_1 h(\overline{x}_n, \overline{x}_{n+1}) + \partial_2 h(\overline{x}_{n-1}, \overline{x}_n) = 0, \qquad \forall n \in \mathbb{Z}.$$

□

Remark 3.4.2. The equation

$$\partial_1 h(x_n, x_{n+1}) + \partial_2 h(x_{n-1}, x_n) = 0, \qquad \forall n \in \mathbb{Z}$$

is known as *discrete Euler-Lagrange equation.*

The usual Mather's theorem (see Theorem 3.4.2), gives sufficient conditions on the generating function in order to get orbits with rotation number. In particular it is required $h \in \mathcal{C}^2(\mathbb{R}^2)$ and properties $ii)$ and $iii)$ of Lemma (3.4.1) should hold in the whole plane. For this reason we need the following extension lemma. A version of this lemma is stated in [47, chapter 8] and for the sake of completeness, we report here a detailed proof (see also [41, 38]).

Lemma 3.4.2. *Let $\mathcal{B}^+, \mathcal{B}^- : \mathbb{R} \longrightarrow \mathbb{R}$ be $\mathcal{C}^r$ diffeomorphisms satisfying*

$$\mathcal{B}^{\pm}(x + 2\pi) = \mathcal{B}^{\pm}(x) + 2\pi$$

for some $r \geq 2$. Suppose that

$$\mathcal{B}^+(x) > \mathcal{B}^-(x) \qquad \forall x \in \mathbb{R}.$$

Define the following set

$$\mathcal{W} = \left\{(x, x_1) \in \mathbb{R}^2 : \mathcal{B}^-(x) \leq x_1 \leq \mathcal{B}^+(x)\right\}$$

and let $h : \mathcal{W} \longrightarrow \mathbb{R}$ be a $\mathcal{C}^{r+1}$ function such that:

- $h(x + 2\pi, x_1 + 2\pi) = h(x, x_1), \qquad (x, x_1) \in \mathcal{W}$;
- $\partial_{x,x_1} h(x, x_1) < 0, \qquad (x, x_1) \in \mathcal{W}$.

Then there exists $\tilde{h} \in \mathcal{C}^r(\mathbb{R}^2)$ such that:

- $\tilde{h}(x + 2\pi, x_1 + 2\pi) = \tilde{h}(x, x_1), \qquad (x, x_1) \in \mathbb{R}^2$ *;*
- $\partial_{x,x_1} \tilde{h}(x, x_1) < -\delta < 0,$ *with* $\delta > 0 \qquad (x, x_1) \in \mathbb{R}^2$ *;*
- $\tilde{h} = h$ *on* $\mathcal{W}$.

Proof. The domain $\mathcal{W}$ is invariant under the translation $T_{2\pi,2\pi}$, in consequence the quotient set $\mathcal{W}/(2\pi\mathbb{Z})^2$ is compact so that

$$\partial_{x,x_1} h(x, x_1) \leq -\delta', \qquad (x, x_1) \in \mathcal{W}$$

for some $\delta' > 0$. Consider the $\mathcal{C}^{r-1}$-extension of $\partial_{x,x_1} h(x, x_1)$ to $\mathbb{R}^2$ satisfying the translation invariance under $T_{2\pi,2\pi}$ and keep denoting it $\partial_{x,x_1} h$. By continuity, there exists $\varepsilon > 0$, $\delta' \geq \delta > 0$ such that $\partial_{x,x_1} h \leq -\delta$ in the domain

$$\mathcal{W}_\varepsilon = \left\{(x, x_1) \in \mathbb{R}^2 : \mathcal{B}^-(x) - \varepsilon \leq x_1 \leq \mathcal{B}^+(x) + \varepsilon\right\}.$$

Consider a $\mathcal{C}^\infty$ real valued function $\chi : \mathbb{R}^2 \to [0, 1]$ such that $\chi(x+2\pi, x_1+2\pi) = \chi(x, x_1)$ and

$$\begin{cases} \chi = 1 & (x, x_1) \in \mathcal{W}, \\ \chi = 0 & (x, x_1) \in \mathbb{R}^2 \setminus \mathcal{W}_\varepsilon. \end{cases}$$

Let's define the function

$$\mathcal{D}(x,x_1) := \chi\partial_{x\,x_1}h - (1-\chi)\delta\ .$$

Then by the definition of χ we have that $\mathcal{D} \in \mathcal{C}^{r-1}(\mathbb{R}^2)$ and $\mathcal{D}(x+2\pi, x_1+2\pi) = \mathcal{D}(x,x_1)$. Moreover,

$$\begin{cases} \mathcal{D} = \partial_{x\,x_1}h(x,x_1) & (x,x_1) \in \mathcal{W}, \\ \mathcal{D} = -\delta & (x,x_1) \in \mathbb{R}^2 \setminus \mathcal{W}_\varepsilon. \end{cases}$$

In particular, with the hypotheses on h we have :

$$\mathcal{D} \le -\delta < 0 \quad (x,x_1) \in \mathbb{R}^2.$$

Now, let us consider the Cauchy problems for the wave equation (with periodic boundary conditions):

$$\begin{cases} \partial_{x\,x_1}u(x,x_1) = \mathcal{D}(x,x_1), \\ u(x,\mathcal{B}^\pm(x)) = h(x,\mathcal{B}^\pm(x)), \\ (\partial_{x_1}u - \frac{1}{(\mathcal{B}^\pm)'(x)}\partial_x u)(x,\mathcal{B}^\pm(x)) = (\partial_{x_1}h - \frac{1}{(\mathcal{B}^\pm)'(x)}\partial_x h)(x,\mathcal{B}^\pm(x)). \end{cases} \tag{3.4.7}$$

The change of variable

$$t = \frac{x_1 - \mathcal{B}^\pm(x)}{2}, \quad y = \frac{x_1 + \mathcal{B}^\pm(x)}{2}$$

conjugates system (3.4.7) to the classical wave equation

$$\begin{cases} v_{tt} - v_{yy} = f(t,y), \\ v(0,y) = \phi(y) \\ v_t(0,y) = \psi(y) \end{cases} \tag{3.4.8}$$

where, denoting $x(t,y) = (\mathcal{B}^\pm)^{-1}(y-t)$, $x_1(t,y) = t+y$,

$$v(t,y) = u\left(x(t,y), x_1(t,y)\right), \quad f(t,y) = -\frac{4}{(\mathcal{B}^\pm)'(x(t,y))}\mathcal{D}\left(x(t,y),x_1(t,y)\right),$$

$$\phi(y) = h(x(0,y),x_1(0,y)), \quad \psi(y) = \left(\partial_{x_1}h - \frac{1}{(\mathcal{B}^\pm)'(x(0,y))}\partial_x h\right)(x(0,y),x_1(0,y)).$$

Note that $f,\psi \in \mathcal{C}^r$, $\phi \in \mathcal{C}^{r+1}$ and $r \ge 2$ so that problem (3.4.8) has a unique solution $v(t,y) \in \mathcal{C}^r$ (see [51]). Moreover, since $f(t,y+2\pi) = f(t,y)$, $\phi(y+2\pi) = \phi(y)$ and $\psi(y+2\pi) = \phi(y)$, the solution satisfies $v(t,y+2\pi) = v(t,y)$. Undoing the change of variable, we get a unique solution $u \in \mathcal{C}^r(\mathbb{R}^2)$ of problem (3.4.7) such that $u(x+2\pi,x_1+2\pi) = u(x,x_1)$. Hence, setting $\tilde{h} = u$ proves the lemma.
□

Using the terminology introduced in Theorem 3.4.1, we recall some of the conclusions of Mather theory.

Theorem 3.4.2 (Mather [9, 47]). *Consider a $\mathcal{C}^2$ function $h : \mathbb{R}^2 \to \mathbb{R}$ such that $h(x+2\pi,x_1+2\pi) = h(x,x_1)$ and $\partial^2_{xx_1}h \le \bar{\delta} < 0$ for all $(x,x_1) \in \mathbb{R}^2$. Fix $\alpha \in \mathbb{R}$. Then* *(i) if $\alpha = s/q \in \mathbb{Q}$ there exists an increasing sequence $(\overline{x}_n)_{n\in\mathbb{Z}}$ and an homeomorphism of the circle g_α such that*

- *$g_\alpha(\overline{x}_n) = \overline{x}_{n+1}$ and $|\overline{x}_n - \overline{x}_0 - 2\pi n\alpha| < 2\pi$ for every $n \in \mathbb{Z}$,*
- *$\partial_1 h(\overline{x}_n, \overline{x}_{n+1}) + \partial_2 h(\overline{x}_{n-1}, \overline{x}_n) = 0$ and $\overline{x}_{n+q} = \overline{x}_n + 2\pi s$ for every $n \in \mathbb{Z}$;*

(ii) If $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ there exists a set $\tilde{M}_\alpha$ of increasing sequences $x = (\overline{x}_n)_{n\in\mathbb{Z}}$ such that

- *if $x \in \tilde{M}_\alpha$ then $\partial_1 h(x_n, x_{n+1}) + \partial_2 h(x_{n-1}, x_n) = 0$ for every $n \in \mathbb{Z}$, any two translates are comparable and $|x_n - x_0 - n2\pi\alpha| < 2\pi$ for all $n \in \mathbb{Z}$,*
- *there exists a Lipschitz homeomorphism of the circle g_α with rotation number α and a closed set $A_\alpha \subset \mathbb{R}$ such that $x \in \tilde{M}_\alpha$ iff $x_0 \in A_\alpha$ and $g^n_\alpha(x_0) = x_n$ for all n,*
- *the set $Rec(g_\alpha) \subset A_\alpha$ of recurrent points of g_α is either the whole $\mathbb{R}$ or a Cantor set.*

Remark 3.4.3. We recall that the homeomorphism g_α satisfies $g_\alpha(x + 2\pi) = g_\alpha(x) + 2\pi$ for all $x \in \mathbb{R}$ and the set $Rec(g_\alpha)$ is defined as the set of accumulation points of $\{g^n_\alpha(x) + 2\pi k \ : \ (n, k) \in \mathbb{Z}^2\}$ and is independent on the choice of the point $x \in \mathbb{R}$. Moreover, the condition $|\overline{x}_n - \overline{x}_0 - n2\pi\alpha| < 2\pi, \ \forall n \in \mathbb{Z}$ implies that
$$\frac{1}{2\pi} \lim_{n\to\infty} \frac{\overline{x}_n}{n} = \alpha,$$
and α is called the rotation number of the orbit.

We are now ready for the proof of our result.

Proof of Theorem 3.4.1. We apply Lemma 3.4.1 and get the generating function h defined on the set

$$\mathcal{B} = \left\{(x, x_1) \in \mathbb{R}^2 : \alpha^-(x) < x_1 - x < \alpha^+(x)\right\},$$

and satisfying the corresponding properties *o)–iv)*.
Consider the case in which both W^+, W^- are finite. Since $W^+ - W^- > 8\pi$, for every α such that
$$W^- + 4\pi < 2\pi\alpha < W^+ - 4\pi,$$
we can choose $\varepsilon > 0$ such that

$$W^- + \varepsilon < 2\pi\alpha - 4\pi < 2\pi\alpha + 4\pi < W^+ - \varepsilon.$$

Now, consider the set

$$\mathcal{W} = \left\{(x, x_1) \in \mathbb{R}^2 : W^- + \varepsilon \le x_1 - x \le W^+ - \varepsilon\right\} \subset \mathcal{B},$$

and apply Lemma 3.4.2 with $\mathcal{B}^\pm(x) := x + W^\pm \mp \varepsilon$ that clearly are diffeomorphisms. We can extend the function h to the whole $\mathbb{R}^2$ getting a function $\tilde{h}$ satisfying the conditions in Theorem 3.4.2 and such that $\tilde{h} = h$ in $\mathcal{W}$.

By applying Theorem 3.4.2 we obtain sequences $(\tilde{x}_n)$, such that

$$\partial_1 \tilde{h}(\tilde{x}_n, \tilde{x}_{n+1}) + \partial_2 \tilde{h}(\tilde{x}_{n-1}, \tilde{x}_n) = 0, \qquad \forall n \in \mathbb{Z}$$

and

$$|\tilde{x}_n - \tilde{x}_0 - n2\pi\alpha| < 2\pi, \qquad \forall n \in \mathbb{Z}.$$

From this inequality we obtain:

$$2\pi\alpha - 4\pi < \tilde{x}_{n+1} - \tilde{x}_n < 2\pi\alpha + 4\pi, \qquad \forall n \in \mathbb{Z},$$

that means that for every $n \in \mathbb{Z}$, $(\tilde{x}_{n+1}, \tilde{x}_n) \in \mathcal{W}$. But since $\tilde{h} = h$ in $\mathcal{W}$,

$$\partial_1 h(\tilde{x}_n, \tilde{x}_{n+1}) + \partial_2 h(\tilde{x}_{n-1}, \tilde{x}_n) = 0, \qquad \forall n \in \mathbb{Z}.$$

Hence, in case of rational α we define

$$\tilde{r}_n = f^{-1}(-\partial_1 h(\tilde{x}_n, \tilde{x}_{n+1})) = f^{-1}(-\partial_1 h(\tilde{x}_n, g_\alpha(\tilde{x}_n)))$$

such that $(\tilde{x}_n, \tilde{r}_n) \subset \tilde{\Sigma}$ is the (s, q)-periodic orbit of Φ. In the irrational case, the set $\tilde{\mathcal{M}}_\alpha$ is given by

$$\tilde{\mathcal{M}}_\alpha = \{(\xi, \eta) \in \mathbb{R}^2 \ : \ \xi \in Rec(g_\alpha), \quad \eta = f^{-1}(-\partial_1 h(\xi, g_\alpha(\xi)))\} \subset \tilde{\Sigma}.$$

Note that the Lipschitz regularity of f^{-1} plays a role at this stage.

In case $-\infty < W^- < W^+ = \infty$ is enough to choose α such that $2\pi\alpha > W^- + 4\pi$ and fix $M > 2\pi\alpha + 8\pi$. Fix ε such that

$$W^- + \varepsilon \leq 2\pi\alpha - 4\pi$$

and apply extension lemma with $\mathcal{B}^-(x) = x + W^- + \varepsilon$ and $\mathcal{B}^+(x) = x + M$, in order to get the same result.

The other cases are similar.

□

The next two chapters are devoted to show the results obtained. The first one use the KAM theorem introduced in Section 3.3. The second one use the generalized Aubry-Mather theorem presented in Section 3.4.

Chapter 4

Stability theorem

Consider the Hamiltonian (2.2.2) in Chapter 2:

$$\Psi(t,x,y) = \frac{1}{2}\ln(x^2+y^2) + p(t,x,y), \tag{4.0.1}$$

and the associated system

$$\begin{cases} \dot{x} = \partial_y \Psi(t,x,y) = \frac{y}{x^2+y^2} + \partial_y p(t,x,y) \\ \\ \dot{y} = -\partial_x \Psi(t,x,y) = -\frac{x}{x^2+y^2} - \partial_x p(t,x,y) \end{cases} \qquad (x,y) \in \mathcal{U} \setminus \{0\}\,, \tag{4.0.2}$$

defined in a neighborhood $\mathcal{U}$ of the origin.

The main aim of this chapter is to provide sufficient conditions for the stability of the particle advection around a fixed point-vortex in a two-dimensional ideal fluid under the action of a periodic background flow. This sufficient conditions are imposed to the perturbative flow $p(t,x,y)$. The proof relies on the identification of a family of closed invariant curves around the origin by means of Moser's *Invariant Curve Theorem*. This family will converge to the origin. Due the low dimensionality of the problem, this family will be a barrier for the evolution of the solutions with initial conditions near the singularity. This fact guarantees the stability of the singularity.

From a more analytical point of view, it seems necessary to state a precise mathematical notion of stability. To fill this gap, section 4.1 is devoted to settle a rigorous definition of stability of a singularity in this context. We consider perpetual stability in the Lyapunov sense, meaning that a solution of (4.0.2) with a small initial condition will remain close to the singularity forever. Once such definition is precisely stated, the main aim of this paper is to provide a simple sufficient condition for stability.

The following Sections are organized as follows. In Section 4.2, the main result is stated, where we identify a class of functions $p(t,x,y)$ for which the vortex singularity is stable. The model example would be a polynomial

$$p(t,x,y) = \sum_{4\le h+k\le N} \alpha_{h,k}(t) x^h y^k$$

with 1-periodic coefficients $\alpha_{h,k} \in \mathcal{C}(\mathbb{R}/\mathbb{Z})$ and $\alpha_{h,4-h} \in \mathcal{C}^1(\mathbb{R}/\mathbb{Z})$. The proof is presented in Section 4.3. The first step, performed in Subsection 4.3.1 is to invert the role of the origin and infinite by the Kelvin transform. Also, we have to consider symplectic polar coordinates in order to preserve the symplectic structure of the system. In Subsection 4.3.2, we pass to complex formulation by considering the system as complex-valued. This formalism is necessary to apply the *Invariant Curve Theorem* in the version presented in Section 32 of [81]. Subsections 4.3.3 and 4.3.4 are devoted to prove that the Poincaré map is well-defined for a suitable domain. Finally, Subsection 4.3.5 proves the *intersection property* for the Poincaré map, therefore the *Invariant Curve Theorem* provides the existence of a sequence of invariant Jordan curves near infinite. After inversion of the Kelvin transform, such invariant curves surround the vortex, acting as flux barriers and giving rise to the desired stability.

4.1 The stability of a singularity

Assume that Ω is an open subset of $\mathbb{R}^d$ with $d \geq 2$ and let $\xi_* \in \Omega$ be a fixed point. We consider the punctured region $\Omega_* = \Omega \setminus \{\xi_*\}$ and a continuous vector field

$$\begin{array}{rcl} X : \mathbb{R} \times \Omega_* & \longrightarrow & \mathbb{R}^d \\ (t, \xi) & \longmapsto & X(t, \xi). \end{array}$$

From now on we assume that there is global uniqueness for each initial value problem of the type

$$\begin{cases} \dot{\xi} = X(t, \xi), \\ \xi(0) = \xi_0, \qquad \xi_0 \in \Omega_*. \end{cases} \tag{4.1.1}$$

The corresponding solution will be denoted by $\xi(t, \xi_0)$.

The point $\xi = \xi_*$ will be interpreted as a singularity of the equation.

Definition 7. The singularity ξ_* will be called *stable* (in a perpetual sense) if given $\varepsilon > 0$, there exists $\delta > 0$ such that if $\xi_0 \in \Omega_*$ with

$$|\xi_0 - \xi_*| < \delta$$

then $\xi(t, \xi_0)$ is well defined everywhere and

$$|\xi(t, \xi_0) - \xi_*| < \varepsilon \quad \text{for each} \quad t \in \mathbb{R}.$$

Example 7. The model example is the unperturbed system (2.2.2) with $p \equiv 0$. Taking $z = \begin{pmatrix} x \\ y \end{pmatrix}$, it can be written as

$$\dot{z} = \frac{1}{|z|^2} J\, z \quad \text{with} \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The explicit solution is

$$z(t, z_0) = \mathcal{R}\left[\frac{t}{|z_0|^2}\right] z_0, \quad \text{where} \quad \mathcal{R}[\theta] = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

The stability of the point $z_* = 0$ is clear because $|z(t, z_0)| = |z_0|$. In contrast, the derivative satisfies

$$|\dot{z}(t, z_0)| = \frac{1}{|z_0|} \to \infty \quad \text{as} \quad z_0 \to z_*.$$

The previous example was autonomous but our definition also works in time-dependent situations. We will restrict to periodic vector fields satisfying

$$X(t+T, \xi) = X(t, \xi) \quad \text{for some} \quad T > 0.$$

In this case, we can define the Poincaré map

$$\begin{array}{rcl} \mathcal{P} : \mathcal{E} \subset \Omega_* & \longrightarrow & \Omega_* \\ \xi_0 & \longmapsto & \mathcal{P}(\xi_0) = \xi(T, \xi_0) \end{array}$$

where $\mathcal{E} = \{\xi_0 \in \Omega_* : \ \xi(t, \xi_0) \text{ is well defined on } [0, T]\}$.

The standard theorem on continuous dependence implies that $\mathcal{E}$ is open. When ξ_* is a stable singularity there exists a neighborhood $\mathcal{O} \subset \mathbb{R}^d$ of ξ_* such that $\mathcal{O} \setminus \{\xi_*\}$ is contained in $\mathcal{E}$. Then it is natural to define $\mathcal{P}(\xi_*) = \xi_*$ so that $\mathcal{P}$ is extended to a one-to-one and continuous map from $\mathcal{E} \cup \{\xi_*\}$ into Ω. The standard notion of stable fixed point of a map is now applicable. A stable singularity $\xi = \xi_*$ will produce a stable fixed point of the Poincaré map. This means that for each neighborhood $\mathcal{V}$ of ξ_* there exists another neighborhood $\mathcal{W} \subset \mathcal{E} \cup \{\xi_*\}$ such that all the iterates $\mathcal{P}^n(\mathcal{W})$ are well defined and satisfy $\mathcal{P}^n(\mathcal{W}) \subset \mathcal{V}$, $n \in \mathbb{Z}$.

The next example will show that the stability of ξ_* as a fixed point of $\mathcal{P}$ is not sufficient to guarantee the stability of $\xi = \xi_*$ as a singularity of the differential equation.

Example 8. The equation for $z \in \mathbb{R}^2 \setminus \{0\}$,

$$\dot{z} = \sin(2t) \frac{z}{|z|}$$

is T-periodic with $T = \pi$ and

$$z(t, z_0) = \left(|z_0| + \sin^2 t\right) \frac{z_0}{|z_0|}, \qquad z_0 \in \mathbb{R}^2 \setminus \{0\}.$$

We observe that $\mathcal{P}$ is the identity and so $z_* = 0$ is stable as a fixed point of $\mathcal{P}$. On the other hand,

$$\left| z\left(\frac{T}{2}, z_0\right) - z_* \right| = \left| z\left(\frac{T}{2}, z_0\right) \right| = |z_0| + 1 > 1$$

and so $z_* = 0$ is not stable as a singularity (see Fig.4.1).

The reader who is familiar with the theory of the stability of the equilibrium will notice the sharp contrast between equilibria and singularities. In the second case continuous dependence may be lost. For this reason we make the following definition.

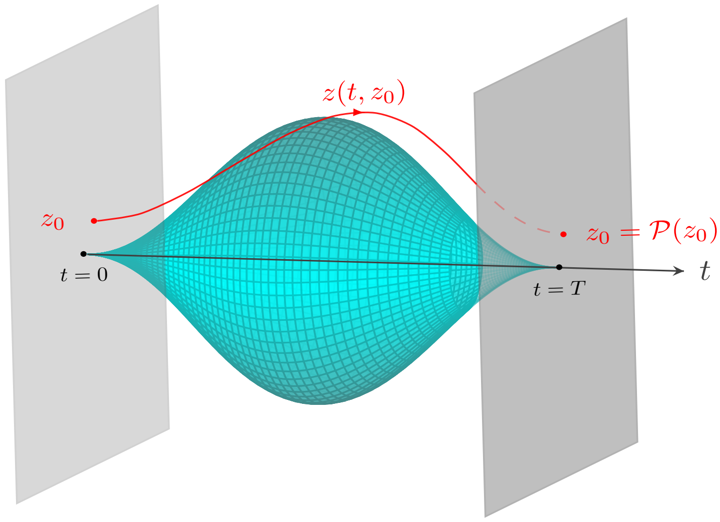

Figure 4.1: The singularity evolution over a period forces the solution to get away from $z_* = 0$.

Definition 8. We say that there is *continuous dependence around the singularity* $\xi = \xi_*$ if given $\varepsilon > 0$, there exists $\delta > 0$ such that if

$$0 < |\xi_0 - \xi_*| < \delta$$

then $\xi(t, \xi_0)$ is well defined on $[0, T]$ and

$$|\xi(t, \xi_0) - \xi_*| < \varepsilon \quad \text{for each} \quad t \in [0, T]\,.$$

Remark 4.1.1. Note that the choice of $t_0 = 0$ in the initial value problem 4.1.1 is completely arbitrary. A change in t_0 does not affect the continuous dependence around the singularity since the new Poincaré map is conjugated and by periodicity the continuous dependence is for every t.

The following result is a direct consequence of the previous definitions.

Proposition 1. *In the previous setting assume that there is continuous dependence around the singularity $\xi = \xi_*$ and moreover ξ_* is stable as a fixed point of $\mathcal{P}$. Then, $\xi = \xi_*$ is a stable singularity.*

4.2 Main result

For our purpose we will consider the complexification of the variables x and y. Given $\varepsilon > 0$, consider the disk $\mathbb{D}_\varepsilon = \{x \in \mathbb{C} : |x| < \varepsilon\}$. A function

$$\begin{array}{rcl} p : \mathbb{R} \times \mathbb{D}_\varepsilon^2 & \longrightarrow & \mathbb{C} \\ (t, x, y) & \longmapsto & p(t, x, y) \end{array}$$

belongs to the class $\mathcal{H}_\varepsilon$ if it satisfies the conditions below:

i) p is continuous.

ii) $p(t+1,x,y) = p(t,x,y)$.

iii) p extends a real-valued function; that is,

$$p(t,x,y) \in \mathbb{R} \quad \text{if} \quad x,y \in \mathbb{D}_\varepsilon \cap \mathbb{R}.$$

iv) For each $t \in \mathbb{R}$,

$$(x,y) \in \mathbb{D}_\varepsilon^2 \longmapsto p(t,x,y) \in \mathbb{C}$$

is a holomorphic function in two variables.

v) $\|p\|_\infty := \sup \{|p(t,x,y)| \ : \ (t,x,y) \in \mathbb{R} \times \mathbb{D}_\varepsilon^2\} < \infty$.

Example 9. A simple example of $p(t,x,y)$ is a polynomial

$$p(t,x,y) = \sum_{h+k \geq N} \alpha_{h,k}(t) x^h y^k$$

with $\alpha_{h,k} \in \mathcal{C}(\mathbb{R}/\mathbb{Z})$.

Definition 9. Given a function p in $\mathcal{H}_\varepsilon$ we say that the origin is a *zero of order* N if for each $t \in \mathbb{R}$,

$$p(t,0,0) = 0 \quad \text{and} \quad \frac{\partial^{h+k}}{\partial x^h \partial y^k} p(t,0,0) = 0 \quad \text{when} \quad h+k < N.$$

Now, we are ready to state the main result of the Chapter.

Theorem 1. *Assume that* $p \in \mathcal{H}_\varepsilon$ *and the origin is a zero of order* 4*. In addition, the functions*

$$\alpha_h(t) = \frac{\partial^4}{\partial x^h \partial y^{4-h}} p(t,0,0) \tag{4.2.1}$$

are of class $\mathcal{C}^1$ *for each* h, $0 \leq h \leq 4$. *Then, the origin of system* (4.0.2) *is stable in the sense of Definition 7.*

4.3 Proof of the main result

4.3.1 Some transformations: from zero to infinity

The system (4.0.2) is defined on some neighborhood $\mathcal{U}$ of the origin. It will be convenient to transform it into another system defined in a neighborhood of infinity. To do this we employ the Kelvin transform:

$$\begin{array}{rcl} \kappa : \mathbb{R}^2 \setminus \{0\} & \longrightarrow & \mathbb{R}^2 \setminus \{0\} \\ (u,v) & \longmapsto & \kappa(u,v) = (x,y) \end{array}$$

where

$$x = \frac{u}{u^2+v^2}\,, \qquad y = -\frac{v}{u^2+v^2}\,.$$

This map is an analytic involution ($\kappa^2 = \mathrm{Id}$) satisfying

$$\mathrm{d}x \wedge \mathrm{d}y = \frac{1}{(u^2+v^2)^2}\,\mathrm{d}u \wedge \mathrm{d}v. \tag{4.3.1}$$

Hence it is not a canonical map. To remain in a Hamiltonian framework we will introduce a new symplectic structure in the phase space. First we recall some well known facts, see [62] for more details. Let Ω be our phase space, an open subset of $\mathbb{R}^2$. We consider the symplectic form

$$\hat{\omega}_z = \Phi(z)\,\mathrm{d}q \wedge \mathrm{d}p$$

where $z = (q,p) \in \Omega$ and $\Phi : \Omega \longrightarrow \mathbb{R}$ is a smooth and positive function. Given $H \in \mathcal{C}^\infty(\Omega)$, the Hamiltonian system associated to the triplet $(\Omega, \hat{\omega}, H)$ is

$$\dot{z} = \frac{1}{\Phi(z)}\,J\,\nabla H(z) \quad \text{with} \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4.3.2}$$

Let $\omega = \mathrm{d}x \wedge \mathrm{d}y$ be the standard symplectic form in $\mathbb{R}^2$, the identity (4.3.1) implies that κ is a symplectic diffeomorphism between $(\mathbb{R}^2 \setminus \{0\}, \hat{\omega})$ and $(\mathbb{R}^2 \setminus \{0\}, \omega)$. The previous remark can be applied with

$$\Phi(u,v) = \frac{1}{(u^2+v^2)^2}$$

and so the system (4.0.2) is transformed into

$$\left\{ \begin{aligned} \dot{u} &= (u^2+v^2)^2\,\partial_v \hat{\Psi}(t,u,v) \\ \\ \dot{v} &= -\,(u^2+v^2)^2\,\partial_u \hat{\Psi}(t,u,v) \end{aligned} \right. \quad \text{with} \quad \hat{\Psi} = \Psi \circ \kappa. \tag{4.3.3}$$

This system is defined in $\kappa(\mathcal{U})$, a neighborhood of infinity. Let us remark that the effect of the Kelvin transform is a change by the previous factor $\Phi(u,v)$ of the usual simplectic form.

Next we pass to symplectic polar coordinates, defined by the diffeomorphism

$$\begin{array}{rcl} \varphi : \,]0,\infty[\, \times \mathbb{T} & \longrightarrow & \mathbb{R}^2 \setminus \{0\} \\ (r,\theta) & \longmapsto & \varphi(r,\theta) = (u,v) \end{array}$$

where

$$u = \sqrt{2r}\cos\theta, \qquad v = \sqrt{2r}\sin\theta.$$

We observe that

$$\mathrm{d}x \wedge \mathrm{d}y = \frac{1}{(u^2+v^2)^2}\,\mathrm{d}u \wedge \mathrm{d}v = \frac{1}{4r^2}\mathrm{d}r \wedge \mathrm{d}\theta,$$

and so $\kappa \circ \varphi$ is a symplectic diffeomorphism from $\Omega = \,]0,\infty[\, \times \mathbb{T}$ with the form $\tilde{\omega} = \frac{1}{4r^2}\,\mathrm{d}r \wedge \mathrm{d}\theta$ onto $(\mathbb{R}^2 \setminus \{0\}, \omega)$. Now the original system is transformed into

$$\left\{ \begin{aligned} \dot{r} &= 4r^2\,\partial_\theta H(t,r,\theta) \\ \\ \dot{\theta} &= -4r^2\,\partial_r H(t,r,\theta) \end{aligned} \right. \quad \text{with} \quad H = \Psi \circ \kappa \circ \varphi. \tag{4.3.4}$$

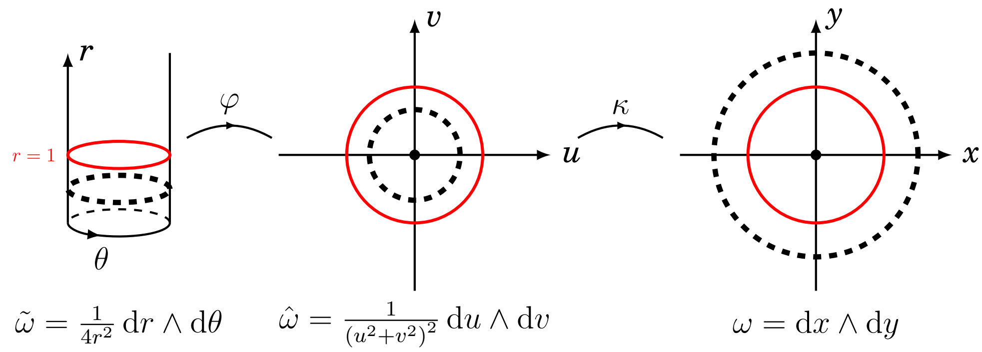

Figure 4.2: The transformations κ, φ and related symplectic forms.

Thus $H(t,r,\theta) = -\frac{1}{2}\ln(2r) + h(t,r,\theta)$ with $h(t,r,\theta) = p\left(t, \frac{\cos\theta}{\sqrt{2r}}, -\frac{\sin\theta}{\sqrt{2r}}\right)$.

The previous transformations and the different symplectic forms are illustrated in Fig. 4.2.

4.3.2 The complexified system

System (4.3.4) can be expressed in the form

$$\left\{\begin{array}{l} \dot{r} = F(t,r,\theta) \\ \dot{\theta} = 2r + G(t,r,\theta) \end{array}\right. \tag{4.3.5}$$

where

$$F(t,r,\theta) = -(2\,r)^{3/2}\,(\mathrm{p}_1 \sin\theta + \mathrm{p}_2 \cos\theta)\,,$$
$$G(t,r,\theta) = \sqrt{2\,r}\,(\mathrm{p}_1 \cos\theta - \mathrm{p}_2 \sin\theta)$$

with

$$\mathrm{p}_1 = \partial_x p\left(t, \frac{\cos\theta}{\sqrt{2r}}, -\frac{\sin\theta}{\sqrt{2r}}\right), \qquad \mathrm{p}_2 = \partial_y p\left(t, \frac{\cos\theta}{\sqrt{2r}}, -\frac{\sin\theta}{\sqrt{2r}}\right),$$

and

$$\partial_\theta p\left(t, \frac{\cos\theta}{\sqrt{2r}}, -\frac{\sin\theta}{\sqrt{2r}}\right) = -\frac{1}{\sqrt{2\,r}}\,(\mathrm{p}_1 \sin\theta + \mathrm{p}_2 \cos\theta)\,,$$
$$\partial_r p\left(t, \frac{\cos\theta}{\sqrt{2r}}, -\frac{\sin\theta}{\sqrt{2r}}\right) = .\frac{1}{(2\,r)^{3/2}}\,(\mathrm{p}_1 \cos\theta - \mathrm{p}_2 \sin\theta)\,.$$

This is a periodic planar system defined with coordinates $r > 0$ and $\theta \in \mathbb{R}$ or on a cylinder $]0,\infty[\times\mathbb{T}$. As we have announced in Section 4.2, for convenience we embed it into the complex system. This is understood in the following sense: the equations are the same but the unknowns $r = r(t)$ and $\theta = \theta(t)$ will be complex valued, the independent variable t will remain in $\mathbb{R}$.

Remark. To avoid multi-valued functions, $w = \sqrt{z}$ will be interpreted as the holomorphic function defined on the half plane $\mathrm{Re}(z) > 0$ which extends the positive square root of positive real numbers. Also the function $z^{3/2} = z\sqrt{z}$ is holomorphic in $\mathrm{Re}\,(z) > 0$.

From now on, $r_* > 0$ and $\Delta_* > 0$ are two fixed numbers satisfying

$$\frac{\cosh(\Delta_*)}{\sqrt{2r_*}} < \frac{\varepsilon}{2}. \tag{4.3.6}$$

If we define the domain

$$\mathcal{D} = \left\{(r,\theta) \in \mathbb{C}^2 : \mathrm{Re}\,(r) > r_*, \quad |\mathrm{Im}\,(\theta)| < \Delta_*\right\} \tag{4.3.7}$$

then the inequalities

$$\left|\frac{\cos\theta}{\sqrt{2r}}\right| < \frac{\varepsilon}{2}, \qquad \left|\frac{\sin\theta}{\sqrt{2r}}\right| < \frac{\varepsilon}{2}$$

holds for any point lying on $\mathcal{D}$. In consequence $F, G : \mathbb{R} \times \mathcal{D} \longrightarrow \mathbb{C}$ are continuous functions which are 1-periodic in t and such that $F(t,\cdot,\cdot), G(t,\cdot,\cdot)$ are holomorphic for each $t \in \mathbb{R}$.

4.3.3 Some preliminary estimates

The main aim of Subsections 4.3.3 and 4.3.4 is to prove that, under the assumptions of Theorem 1, the Poincaré map is well-defined in a suitable open set. In this subsection we replace the main assumption of Theorem 1 by a stronger condition. More concretely, let us assume that the origin is a zero of order 5 of the function $p \in \mathcal{H}_\varepsilon$. In particular, the functions α_h vanish and so the assumptions of Theorem 1 are satisfied. The proof under the general assumptions will be presented later in Subsection 4.3.4.

The following Lemma states some bounds for functions $p \in \mathcal{H}_\varepsilon$ with a zero of order $N+1$.

Lemma 4.3.1. *Assume that $p \in \mathcal{H}_\varepsilon$ and the origin is a zero of order $N+1$. Then there exists a constant C, depending on $\|p\|_\infty$ and ε, such that*

I) $|p(t,x,y)| \le C\left(|x|^{N+1} + |y|^{N+1}\right)$.

II) $|\partial_x\, p(t,x,y)| + |\partial_y\, p(t,x,y)| \le C\left(|x|^N + |y|^N\right)$ if $|x| < \frac{\varepsilon}{2}$ and $|y| < \frac{\varepsilon}{2}$.

Proof. From Cauchy estimates for functions of several complex variables it is easy to find an estimate

$$\left|\frac{\partial^{h+k}}{\partial\, x^h \partial\, y^k} p(t,x,y)\right| \le 2\frac{h!k!\,\|p\|_\infty}{(\varepsilon/2)^{h+k}} =: \widetilde{C}$$

valid if $h+k \le N+1$ and $|x| < \frac{\varepsilon}{2}$ and $|y| < \frac{\varepsilon}{2}$.

The Taylor formula with integral remainder is valid for holomorphic functions (see Section 14 of Chapter VIII in [12]). Then, since the origin is a zero of order $N+1$, we have

$$p(t,x,y) = \left(\int_0^1 \frac{(1-\xi)^N}{N!} p^{(N+1)}(t,\xi x,\xi y)\,\mathrm{d}\xi\right)(x,y)^{(N+1)}$$

where $p^{(N+1)}(t,x,y)$ is the multilinear form associated to the derivative of order $N+1$ and $(x,y)^{(N+1)}$ stands for the repetition of (x,y) during $N+1$ times.

Then we can obtain the first estimate using the norm of a multilinear form. Note that C only depends on $\widetilde{C}$ and N. The second estimate can be obtained in the same way after applying Taylor formula to the functions $\partial_x p$ and $\partial_y p$. □

Using the previous Lemma, we have the estimate

$$|\partial p\,(t,x,y)| \leq C(|x|^4 + |y|^4) \quad \text{if} \quad |x| < \frac{\varepsilon}{2},\ |y| < \frac{\varepsilon}{2}. \tag{4.3.8}$$

Here $\partial = \partial_x$ or ∂_y and C only depends upon $\|p\|_\infty$ and ε. Then, from the definition of F and G, we have

$$|F(t,r,\theta)| + |2r|\,|G(t,r,\theta)| \leq C_1 \frac{e^{5|\mathrm{Im}\,(\theta)|}}{|r|^{1/2}} \tag{4.3.9}$$

for any $t \in \mathbb{R}$ and $(r,\theta) \in \mathcal{D}$. Here C_1 is a constant depending upon $r_*, \Delta_*, \varepsilon$ and C.

Next, we introduce a family of domains in $\mathbb{C}^2$ whose geometry is well adapted to our differential equation.

Given real numbers a, b, R, Δ such that

$$b - a = 1 \quad \text{and} \quad R > 0,\ \Delta > 0,$$

we define

$$\mathcal{C}_\Delta = \{\theta \in \mathbb{C} \ : \ |\mathrm{Im}\,(\theta)| < \Delta\}$$

and

$$[a,b]_R = \{r \in \mathbb{C} \ : \ \mathrm{dist}\,(r,[a,b]) < R\}$$

where $\mathrm{dist}\,(z,\mathcal{K})$ is the distance from a point z to a compact set $\mathcal{K} \subset \mathbb{C}$. The set $\mathcal{C}_\Delta$ is a horizontal strip and $[a,b]_R$ has the shape of a stadium. We consider the domain in $\mathbb{C}^2$

$$\Omega = [a,b]_R \times \mathcal{C}_\Delta.$$

Sometimes, to emphasize the dependence on R and Δ, we will write $\Omega(R,\Delta)$. Obviously this set also depends of the interval $[a,b]$ but this dependence will not be made explicit.

The domain $\Omega(R,\Delta)$ is contained in $\mathcal{D}$ defined by (4.3.7) as soon as

$$\Delta < \Delta_* \qquad \text{and} \qquad a - R > r_*. \tag{4.3.10}$$

This is important to be sure that the complexified system (4.3.5) is well defined on $\Omega(R,\Delta)$.

For $0 < \rho < R,\ 0 < \delta < \Delta$ it is clear that $\Omega(\rho,\delta)$ is contained in $\Omega(R,\Delta)$. In the next result we will prove that the solutions starting at $\Omega(\rho,\delta)$ do not exit $\Omega(R,\Delta)$ in one period (see Fig.4.3). This will require some smallness on the parameters ρ and δ, and a large enough.

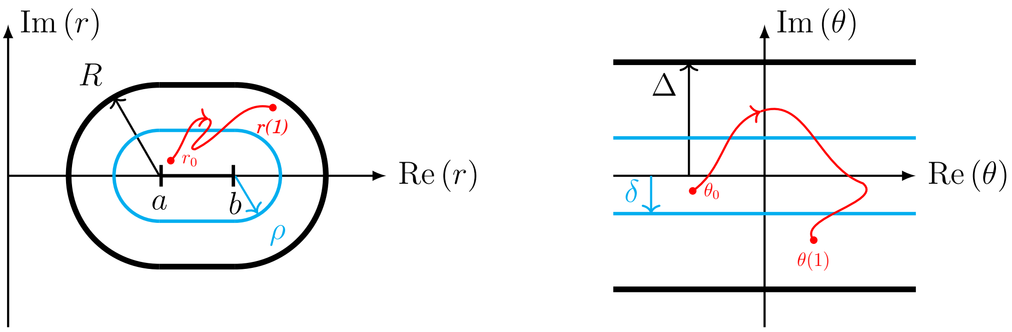

Figure 4.3: Domains $[a,b]_R$, $\mathcal{C}_\Delta$, subdomains $[a,b]_\rho$, $\mathcal{C}_\delta$ and solution $(r(t),\theta(t))$ with $t \in [0,1]$.

Lemma 4.3.2. *Let us assume that the origin is a zero of order* 5 *of the function* $p \in \mathcal{H}_\varepsilon$. *Assume that* $R > \rho > 0$, $\Delta > \delta > 0$ *are given numbers satisfying* (4.3.10). *In addition,*

$$\delta + 2\rho < \Delta. \tag{4.3.11}$$

Then there exists $a_* > R$ *such that if* $a > a_*$ *the solution of* (4.3.5) *with initial condition* $(r_0, \theta_0) \in \Omega(\rho, \delta)$ *is well defined on* $t \in [0,1]$. *Moreover,*

$$(r(t), \theta(t)) \in \Omega(R, \Delta)$$

and

$$|r(t) - r_0| + |\theta(t) - \theta_0 - 2r_0 t| \le \frac{K e^{5\Delta}}{|r_0|^{1/2}} \quad \text{if} \quad t \in [0,1]. \tag{4.3.12}$$

Proof. Let $[0,\tau]$ be a compact sub-interval of $[0,1]$ where the solution $(r(t),\theta(t))$ is well defined and remains in $\Omega(R,\Delta)$. The geometry of $[a,b]_R$ implies that

$$|r(t)| \ge a - R \quad \text{if} \quad t \in [0,\tau].$$

From the first equation in (4.3.5) and the estimate (4.3.9)

$$|r(t) - r_0| \le \int_0^t |\dot r(s)|\, \mathrm{d}s \le \frac{C_1 e^{5\Delta}}{(a-R)^{1/2}}, \qquad t \in [0,\tau].$$

Note that for a large enough we can assume

$$\frac{C_1 e^{5\Delta}}{(a-R)^{1/2}} < R - \rho.$$

This inequality guarantees that $r(t)$ does not touch the boundary of $[a,b]_R$. Also,

$$|r(t)| \ge |r_0| - |r(t) - r_0| \ge |r_0| - R, \qquad t \in [0,\tau].$$

After increasing a we can assume that $a > \rho + R$ and $|r_0| - R$ is positive.

With this information we go back to the first equation of (4.3.5) to obtain the new estimate

$$|r(t) - r_0| \le \frac{C_1 e^{5\Delta}}{(|r_0| - R)^{1/2}}, \qquad t \in [0,\tau]. \tag{4.3.13}$$

Now, let us consider the second equation of (4.3.5). After integrating from 0 to t, some simple manipulations lead to

$$\theta(t) = \theta_0 + 2r_0 t + 2\int_0^t [r(s) - r_0]\,\mathrm{d}s + \int_0^t G(s, r(s), \theta(s))\,\mathrm{d}s.$$

Using the inequalities (4.3.9) and (4.3.13),

$$|\theta(t) - \theta_0 - 2r_0 t| \le \frac{2C_1 e^{5\Delta}}{(|r_0| - R)^{1/2}} + \frac{C_1 e^{5\Delta}}{2\,(|r_0| - R)^{3/2}}, \qquad t \in [0, \tau].$$

It is the time to employ the condition (4.3.11). It allows to find a_* large enough so that

$$\delta + 2\rho + \frac{2C_1 e^{5\Delta}}{(a - \rho - R)^{1/2}} + \frac{C_1 e^{5\Delta}}{2\,(a - \rho - R)^{3/2}} < \Delta \quad \text{if} \quad a > a_*.$$

Then,

$$|\mathrm{Im}\,(\theta(t))| \le |\mathrm{Im}\,(\theta_0)| + 2\,|\mathrm{Im}\,(r_0)| + |\theta(t) - \theta_0 - 2r_0 t| < \Delta, \qquad t \in [0, \tau].$$

We conclude that $\theta(t)$ cannot touch the boundary of $\mathcal{C}_\Delta$.

The previous discussions imply that the solution cannot touch the boundary of $\Omega(R, \Delta)$. In particular $(r(t), \theta(t))$ is well defined on $[0, 1]$ and we can take $\tau = 1$. The estimate (4.3.12) is a consequence of the above estimates because $(|r_0| - R)^{1/2}$ and $|r_0|^{1/2}$ are of the same order as $|r_0| \to \infty$.

□

4.3.4 Estimates for the general case

As commented in the previous subsection, our intention now is to prove Lemma 4.3.2 under the general assumptions of our main result. Then, let us assume that the function p is in the conditions of Theorem 1 and go back to the setting for the complexified system (4.3.5) proposed in Subsection 4.3.2. The constants r_* and Δ_* satisfying (4.3.6) are determined in the same way. The first difference appears with the estimate (4.3.9) that can be replaced by

$$|F(t, r, \theta)| + |2r|\,|G(t, r, \theta)| \le C_2 e^{4|\mathrm{Im}\,(\theta)|} \tag{4.3.14}$$

for any $t \in \mathbb{R}$ and $(r, \theta) \in \mathcal{D}$. Note that the origin is now a zero of order 4. The estimate (4.3.14) does not seem to provide enough information in order to obtain a result in the line of Lemma 4.3.2.

The new idea will be to split the function F as

$$F(t, r, \theta) = F_*(t, \theta) + \tilde{F}(t, r, \theta) \tag{4.3.15}$$

where $\tilde{F}$ satisfies an estimate of the type (4.3.9) and F_* can be averaged. To describe this splitting in precise terms we write

$$p(t, x, y) = T_4(t, x, y) + \tilde{p}(t, x, y)$$

where T_4 is the Taylor polynomial of degree 4. In consequence the origin is a zero of order 5 for $\tilde{p}$ and we can assume that the first derivatives $\partial\tilde{p}(t,x,y)$ satisfy (4.3.8).

Going back to the equations in (4.3.5) we observe that

$$F(t,r,\theta) = 4r^2\partial_\theta\left[p\left(t,\frac{\cos\theta}{\sqrt{2r}},\frac{-\sin\theta}{\sqrt{2r}}\right)\right].$$

The homogeneity of T_4 allows to write F in the form (4.3.15) with

$$F_*(t,\theta) = \partial_\theta\left[T_4\left(t,\cos\theta,-\sin\theta\right)\right],$$

$$\tilde{F}(t,r,\theta) = 4r^2\partial_\theta\left[\tilde{p}\left(t,\frac{\cos\theta}{\sqrt{2r}},\frac{-\sin\theta}{\sqrt{2r}}\right)\right].$$

To average F_*, we will apply the following Lemma of Riemann-Lebesgue type for the polynomial of degree $N=4$

$$q(t,x,y) = -y\frac{\partial T_4}{\partial x}(t,x,-y) - x\frac{\partial T_4}{\partial y}(t,x,-y).$$

Lemma 4.3.3. *Let* $q(t,x,y)$ *be a polynomial of degree* N*,*

$$q(t,x,y) = \sum_{j+h\le N}\alpha_{j,h}(t)x^jy^h$$

with $\alpha_{j,h}(t)\in\mathcal{C}^1(\mathbb{R}/\mathbb{Z})$*. Assume in addition that for each* t

$$\int_0^{2\pi} q(t,\cos\,\theta,\sin\,\theta)\,\mathrm{d}\theta = 0\,. \tag{4.3.16}$$

Let $\beta:[0,\tau]\longrightarrow\mathbb{C}$ *be a* $\mathcal{C}^1$ *function defined on* $[0,\tau]\subset[0,1]$ *and* $\Lambda>0$*. Then there exists* $C_{RL}>0$ *such that*

$$\left|\int_0^t q(s,\cos\left(\lambda s+\beta(s)\right),\sin\left(\lambda s+\beta(s)\right))\,\mathrm{d}s\right| \le \frac{C_{RL}e^{N\Lambda}}{|\lambda|}$$

if $t\in[0,\tau]$ *and* $\lambda\in\mathbb{C}\setminus\{0\}$ *with* $|\mathrm{Im}\ \lambda|<\Lambda$*. Moreover, the constant* C_{RL} *only depends upon* N*,* $\max_{j,h}\left[\|\alpha_{j,h}\|_\infty+\|\dot\alpha_{j,h}\|_\infty\right]$*,* $\|\beta\|_\infty$ *and* $\|\dot\beta\|_\infty$*.*

Proof. The function $q(t,\cos\,\theta,\sin\,\theta)$ has a finite Fourier expansion with respect to θ, say

$$q(t,\cos\,\theta,\sin\,\theta) = \sum_{|k|\le N} q_k(t)e^{ik\theta}.$$

The coefficients q_k can be expressed in terms of the functions $\alpha_{j,h}$ and belong to $\mathcal{C}^1(\mathbb{R}/\mathbb{Z})$,

$$q_k(t) = \frac{1}{2\pi}\int_0^{2\pi} q(t,\cos\,\theta,\sin\,\theta)e^{-ik\theta}\,\mathrm{d}\theta.$$

The condition (4.3.16) implies that $q_0(t)$ vanishes everywhere and so the integral $I(t)$ we want to estimate can be expressed as the sum

$$I(t) = \sum_{0<|k|\le N} I_k(t) \quad \text{with} \quad I_k(t) = \int_0^t q_k(s) e^{ik\beta(s)} e^{ik\lambda s}\, \mathrm{d}s.$$

where we have used $\theta(s) = \lambda s + \beta(s)$.
Since we have excluded $k = 0$ these integrals can be estimated by a standard procedure in the theory of oscillatory integrals, see for instance [5]. After integrating by parts

$$\begin{aligned} I_k(t) = \frac{1}{ik\lambda} &\Big[q_k(t) e^{ik\beta(t)} e^{ik\lambda t} - q_k(0) e^{ik\beta(0)} \\ &- \int_0^t \left(q_k(s) e^{ik\beta(s)} \right)' e^{ik\lambda s}\, \mathrm{d}s \Big] . \end{aligned}$$

Therefore, using that $1 \le e^{|k||\mathrm{Im}\ \lambda|\tau}$,

$$|I_k(t)| \le \frac{C_k}{|k|\,|\lambda|} e^{|k|\Lambda}$$

with

$$C_k = \left[2 \left\| q_k \right\|_\infty + \left\| q_k' \right\|_\infty + |k| \left\| q_k \right\|_\infty \left\| \beta' \right\|_\infty \right] e^{|k| \|\beta\|_\infty} .$$

□

Remark. The condition (4.3.16) is essential. Consider the polynomial

$$q(t,x,y) = \sum_{j=1}^{N} \alpha_j(t) (x^2 + y^2)^j .$$

Then

$$I(t) = \sum_{j=1}^{N} \int_0^t \alpha_j(s)\, \mathrm{d}s$$

does not depend on λ and the conclusion of the lemma cannot hold.

We observe that $T_4(t, \cos\theta, -\sin\theta)$ is a periodic primitive in the variable θ of the function $q(t, \cos\theta, \sin\theta)$ and so the condition (4.3.16) holds. The assumption (4.2.1) also plays a role since we need to know that q has coefficients that are $\mathcal{C}^1$ in the variable t.

We are ready to prove that Lemma 4.3.2 is also valid in the assumptions of Theorem 1.

Lemma 1. *Under the hypotheses of Theorem 1, the conclusion of Lemma 4.3.2 holds.*

Proof. Let $[0,\tau]$ be a compact sub-interval of $[0,1]$ where the solution $(r(t), \theta(t))$ is well defined and remains in $\Omega(R, \Delta)$. Then $r(t)$ and r_0 belong to $[a,b]_R$ and so

$$|r(t) - r_0| \le d \quad \text{if} \quad t \in [0,\tau],$$

where $d = 1 + 2R$ is the diameter of $[a, b]_R$. In particular, $|r(t)| \geq |r_0| - d$ and we will impose $a - R > d$ to guarantee that $|r_0| - d$ is positive.

Integrating on the second equation of (4.3.5), we deduce that

$$|\theta(t) - \theta_0 - 2r_0 t| \leq 2d + \frac{C_2 e^{4\Delta}}{2(|r_0| - d)}.$$

Here we have employed (4.3.14) and the above estimates on $r(t)$.

Defining $\beta(t) = \theta(t) - \hat{\theta}_0 - 2r_0 t$, where $\hat{\theta}_0 - \theta_0 \in 2\pi\mathbb{Z}$ and $\hat{\theta}_0 \in [0, 2\pi[$. The previous estimate can be interpreted as a bound of $\|\beta\|_\infty$. To get a bound of $\|\dot{\beta}\|_\infty$ we observe that

$$\dot{\beta}(t) = \dot{\theta}(t) - 2r_0 = 2\left(r(t) - r_0\right) + G(t, r(t), \theta(t)).$$

Then

$$\|\dot{\beta}\|_\infty \leq 2d + \frac{C_2 e^{4\Delta}}{2(|r_0| - d)}.$$

and Lemma 4.3.3 can be applied to deduce that

$$\left| \int_0^t F_*(s, \theta(s)) \mathrm{d}s \right| \leq \frac{C_{RL} e^{4\Delta}}{2\,|r_0|}.$$

Going back to the first equation in (4.3.5) and taking into account the splitting (4.3.15) we obtain

$$|r(t) - r_0| \leq \frac{C_{RL} e^{4\Delta}}{2\,|r_0|} + \frac{C_1 e^{5\Delta}}{(|r_0| - d)^{1/2}}, \qquad t \in [0, \tau].$$

At this point we have applied that from (4.3.9), $\tilde{F}$ satisfies

$$\left|\tilde{F}(t, r, \theta)\right| \leq \frac{C_1 e^{5|\mathrm{Im}\,(\theta)|}}{|r|^{1/2}} \quad \text{if} \quad t \in [0, \tau] \quad \text{and} \quad (r, \theta) \in \mathcal{D}.$$

Going back to (4.3.5) with the improved estimate for $|r(t) - r_0|$, one obtains

$$|\theta(t) - \theta_0 - 2r_0 t| \leq \frac{C_{RL} e^{4\Delta}}{2\,|r_0|} + \frac{C_1 e^{5\Delta}}{(|r_0| - d)^{1/2}} + \frac{C_2 e^{4\Delta}}{2(|r_0| - d)}. \tag{4.3.17}$$

The rest of the proof follows along the lines of Lemma 4.3.2. □

4.3.5 Existence of invariant curves

Let us fix numbers $R > \rho > 0$ and $\Delta > \delta > 0$ in the conditions of Lemma 4.3.2. The corresponding numbers a_* and K will be also fixed.

The Poincaré map associated to the system (4.3.5) is given by the formula

$$\mathcal{P}(r(0), \theta(0)) = (r(1), \theta(1))$$

where $(r(t), \theta(t))$ is a solution defined on $t \in [0, 1]$. From the previous section we know that $\mathcal{P}$ is well defined on the open set

$$\mathcal{E}_{\rho,\delta} = \left\{(r_0, \theta_0) \in \mathbb{C}^2 \;:\; \mathrm{dist}\,(r_0, [a_*, +\infty[) < \rho, \qquad |\mathrm{Im}\,(\theta)| < \delta\right\}.$$

Moreover $\mathcal{P}$ maps $\mathcal{E}_{\rho,\delta}$ into a subset of $\mathcal{E}_{R,\Delta}$. The standard theory for the Cauchy problem implies that

$$\mathcal{P} : \mathcal{E}_{\rho,\delta} \subset \mathbb{C}^2 \longrightarrow \mathbb{C}^2$$

is a holomorphic diffeomorphism from $\mathcal{E}_{\rho,\delta}$ onto $\mathcal{P}\left(\mathcal{E}_{\rho,\delta}\right)$. System (4.3.5) is 2π-periodic in θ and this property is inherited by $\mathcal{P}$,

$$\mathcal{P}(r_0, \theta_0 + 2\pi) = \mathcal{P}(r_0, \theta_0) + (0, 2\pi).$$

Moreover, $\mathcal{P}$ can be expressed as

$$\begin{cases} r_1 = r_0 + f(r_0, \theta_0) \\ \theta_1 = \theta_0 + 2r_0 + g(r_0, \theta_0) \end{cases}$$

where $f, g : \mathcal{E}_{\rho,\delta} \longrightarrow \mathbb{C}$ are holomorphic functions, 2π-periodic in θ and satisfying

$$|f(r_0, \theta_0)| + |g(r_0, \theta_0)| \leq \frac{Ke^{5\Delta}}{(a-\rho)^{1/2}} \tag{4.3.18}$$

when $a > a_*$ is such that $\operatorname{dist}(r_0, [a, +\infty[) < \rho$. Note that the bound $M(a) := \frac{Ke^{5\Delta}}{(a-\rho)^{1/2}} \to 0 \quad \text{as} \quad a \to +\infty.$

Let us consider the restriction of the Poincaré map to the real domain

$$\mathcal{P} : E_{\rho,\delta} \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

where

$$E_{\rho,\delta} = \left\{ (r_0, \theta_0) \in \mathbb{R}^2 \ : \ r_0 > a_* - \rho \right\}.$$

We claim that this map has the *intersection property*. In the book by Siegel and Moser [81], this means that

$$\mathcal{P}(\Gamma_\phi) \cap \Gamma_\phi \neq \emptyset$$

when $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ is any analytic and 2π-periodic function with $\phi(\theta) > a_* - \rho$ and Γ_ϕ is its corresponding graph,

$$\Gamma_\phi = \{ (\phi(\theta), \theta) \ : \ \theta \in \mathbb{R} \}.$$

In fact, we will prove that the map $\mathcal{P}$ has a stronger intersection property, with more topological flavour. Keep in mind the Subsection 3.1.2 for the definitions. Let us interpret system (4.3.5) as a system in the cylinder

$$\mathfrak{C} = \left\{ (r, \overline{\theta}) \ : \ r > a_* - \rho, \quad \overline{\theta} \in \mathbb{T} \right\}.$$

Similarly, $\mathcal{P}$ will be understood as an embedding of the cylinder

$$\overline{\mathcal{P}} : \mathfrak{C} \subset \mathbb{R} \times \mathbb{T} \longrightarrow \mathbb{R} \times \mathbb{T}.$$

This means that $\overline{\mathcal{P}}$ is a homeomorphism from $\mathfrak{C}$ onto the open set $\overline{\mathcal{P}}(\mathfrak{C})$. We will prove that

$$\overline{\mathcal{P}}(\overline{\Gamma}) \cap \overline{\Gamma} \neq \emptyset$$

for any Jordan curve $\overline{\Gamma} \subset \mathfrak{C}$ that is not contractible. The key idea to prove this claim is to observe that $\overline{\mathcal{P}}$ preserves a finite measure on the cylinder, namely

$$\mu(\mathcal{A}) = \int\int_{\mathcal{A}} \frac{1}{4r^2}\, \mathrm{d}\overline{\theta}\, \mathrm{d}r$$

for each measurable set $\mathcal{A} \subset \mathbb{R} \times \mathbb{T}$. This is a consequence of the Hamiltonian structure of (4.3.5) described in Subsection 4.3.1. Once we know that $\mu(\overline{\mathcal{P}}(\mathcal{A})) = \mu(\mathcal{A})$ for each $\mathcal{A}$, we can prove that $\overline{\mathcal{P}}$ has the intersection property.

Let $\overline{\Gamma} \subset \mathfrak{C}$ be a non-contractible Jordan curve. Then $(\mathbb{R} \times \mathbb{T}) \setminus \overline{\Gamma}$ splits into two connected components $\mathcal{R}_e(\overline{\Gamma})$ and $\mathcal{R}_i(\overline{\Gamma})$ (see Fig.4.4). Since $\overline{\mathcal{P}}$ is an embedding, the image $\overline{\Gamma}_1 = \overline{\mathcal{P}}(\overline{\Gamma})$ is also a non-contractible Jordan curve with $(\mathbb{R} \times \mathbb{T}) \setminus \overline{\Gamma}_1 = \mathcal{R}_e(\overline{\Gamma}_1) \cup \mathcal{R}_i(\overline{\Gamma}_1)$. Assume by contradiction that $\overline{\Gamma} \cap \overline{\Gamma}_1 = \emptyset$, then either $\mathcal{R}_i(\overline{\Gamma}) \subset \mathcal{R}_i(\overline{\Gamma}_1)$ or $\mathcal{R}_i(\overline{\Gamma}_1) \subset \mathcal{R}_i(\overline{\Gamma})$ and the inclusion is strict. This is impossible because $\mathcal{R}_i(\overline{\Gamma})$ and $\mathcal{R}_i(\overline{\Gamma}_1)$ are open subsets of the cylinder with $\mu(\mathcal{R}_i(\overline{\Gamma})) = \mu(\mathcal{R}_i(\overline{\Gamma}_1)) < \infty$. Note that, except the last step, the argument also is true for the connected component $\mathcal{R}_e$. Nevertheless we cannot assure that the measure is finite.

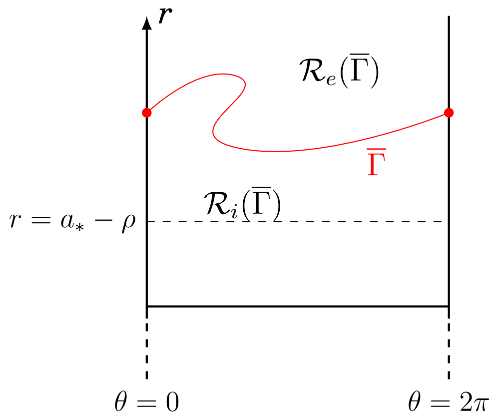

Figure 4.4: Splitting $(\mathbb{R} \times \mathbb{T}) \setminus \overline{\Gamma}$ in connected components $\mathcal{R}_e(\overline{\Gamma})$, $\mathcal{R}_i(\overline{\Gamma})$.

Once we know that the intersection property holds we can go back to the estimate (4.3.18) in order to apply the *Invariant Curve Theorem* as stated in Section 3.3 or in Section 32 of [81]. By taking a large enough we can assume that the map is in the conditions of the theorem. Then there exists an analytic and 2π-periodic function $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ with $a \leq \psi(\theta) \leq a+1$ for each $\theta \in \mathbb{R}$ and such that Γ_ψ is invariant under $\mathcal{P}$; that is $\mathcal{P}(\Gamma_\psi) = \Gamma_\psi$. In this way, we deduce that the map $\overline{\mathcal{P}}$ has a family of invariant Jordan curves that are non-contractible and converge to $r = +\infty$.

4.3.6 Conclusion of the proof

We are ready to prove the stability of the singularity $x = y = 0$ for the original system (4.0.2). The Kelvin transform is a diffeomorphism of $\mathbb{R}^2 \setminus \{0\}$ sending neighborhoods of infinity into neighborhoods of the origin. Then we can transport solutions from (4.3.5) to (4.0.2) and so we know that the solutions of (4.0.2) are well defined in $[0, 1]$ if the initial condition (x_0, y_0) belongs to some small punctured neighborhood of the origin. Let $\mathcal{P}_I$ be the Poincaré map associated to (4.0.2). In the previous section we found a family of invariant curves under $\mathcal{P}_{IV}$, the Poincaré map for (4.3.5).

The corresponding invariant curves under $\mathcal{P}_I$ are Jordan curves which are non-contractible in $\mathbb{R}^2 \setminus \{0\}$ and converge to the origin. The stability of the origin as a fixed point of $\mathcal{P}_I$ is now standard. According to Proposition 1 it remains to check that there is continuous dependence around the singularity $x = y = 0$. The successive changes of variables and the estimate in Lemma 4.3.2 and Lemma 1 lead to

$$\begin{aligned} x(t; x_0, y_0)^2 + y(t; x_0, y_0)^2 &= \frac{1}{u(t)^2 + v(t)^2} = \frac{1}{2r(t)} \\ &\leq \frac{1}{2\left(r_0 - \frac{K_1}{r_0^{1/2}}\right)} = \frac{x_0^2 + y_0^2}{1 - K_1 \sqrt[3]{2(x_0^2 + y_0^2)}} . \end{aligned}$$

This inequality is valid if $(x_0, y_0) \neq (0, 0)$ is small enough and $t \in [0, 1]$. The continuous dependence of the origin follows.

Chapter 5

Existence of periodic and quasi-periodic solutions

This result explores the dynamics for a more general perturbation using Aubry-Mather theory. Let us consider the perturbed Hamiltonian system given by (2.2.1)-(2.2.2):

$$\Psi(t, x, y) = \frac{1}{2}\ln(x^2 + y^2) + p(t, x, y), \tag{5.0.1}$$

and the associated Hamiltonian system

$$\begin{cases} \dot{x} = \partial_y \Psi(t, x, y), \\ \\ \dot{y} = -\partial_x \Psi(t, x, y) \end{cases} \qquad (x, y) \in \mathcal{U} \setminus \{0\}\,, \tag{5.0.2}$$

defined in a neighborhood $\mathcal{U}$ of the origin.

In this Chapter, we will prove that, close to the singularity, quasi-periodic solutions exist for all frequency sufficiently large. Actually, our solutions will be a generalization of standard quasi-periodic solutions and in case of commensurable frequencies, we will get periodic solutions. These solutions exist also when KAM theory cannot be applied. Indeed, we will require very low regularity that prevents standard KAM theory from being applied.

To prove our result, we will apply a suitable version of Aubry-Mather theory (Theorem 3.4.1) to the Poincaré map of system (5.0.2). A similar scheme has been used to describe the dynamics of different systems [38, 42, 64, 79, 86].

For each sufficiently large real number α, we will prove the existence of an invariant set $\mathcal{M}_\alpha$ (called Aubry-Mather set) with very interesting dynamical properties, among them each orbit in $\mathcal{M}_\alpha$ has rotation number α. For irrational rotation numbers, the corresponding Aubry-Mather sets are either curves or Cantor sets. Solutions of system (5.0.2) with initial conditions in this set will be our generalized quasi-periodic solutions. In the rational case, the Aubry-Mather sets contain a periodic orbit.

In suitable variables, the Poincaré map will be an exact symplectic twist map of the cylinder. However, it will not be defined on the whole cylinder. Hence we cannot apply directly the result of Mather and we will use an adapted version to this situation, Theorem 3.4.1.

To apply our theorem, we will need to prove that the Poincaré map is exact symplectic and twist. The first property comes from the Hamiltonian character of the system. The twist condition is more delicate and relies on the behavior of the variational equation. We will give a proof following a perturbative approach. Here, we will ask that the perturbation has the origin as a zero of order 4.

From the point of view of dynamics of symplectic diffeomorphisms, we will describe some aspects of the dynamics around a singularity. In the integrable case, the flow can be continuously extended to the singularity, defining it as a fixed point. However, this extension is not $\mathcal{C}^1$. In the perturbed case, in general is not even possible to guarantee continuity of this extension. Since the flow is not regular, all the results coming from the theory of elliptic fixed points and transformation to Birkhoff normal form cannot be applied directly. We will overcome the problem of the singularity performing a change of variable that sends the singularity at infinity and has a regularizing effect. At this stage, the assumption of having the zero of order 4 in the perturbation play a fundamental role.

This Chapter is organized as follows. In Section 5.1 we state the problem and the main result. The definition of generalized quasi-periodic solution will be given in this section. In Section 5.2 we introduce the regularizing variables and the Poincaré map together with some preliminary estimates. In Section 5.3 it is proved the property of exact symplectic and Section 5.4 is dedicated to the proof of the twist property. Previously, in Section 3.4 of Chapter 3 we have stated and proved a suitable version of the Aubry-Mather theorem. So, in Section 5.5 we just recall the statement of this Theorem and a proof of the main result will be given.

5.1 Statement of the problem and main result

Let us consider the perturbed Hamiltonian system given by (5.0.1)-(5.0.2). We suppose that the perturbation $p(t,x,y)$ belongs to the following class.

Definition 5.1.1. Given $\varepsilon > 0$, consider the open disk around the origin $\mathbb{D}_\varepsilon = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < \varepsilon^2\}$. We say that a continuous function $p : \mathbb{R} \times \mathbb{D}_\varepsilon \longrightarrow \mathbb{R}$ belongs to the class $\mathcal{R}^k_\varepsilon$ if

i) $p(t+1,x,y) = p(t,x,y)$.

ii) $p \in \mathcal{C}^{0,k}(\mathbb{R} \times \mathbb{D}_\varepsilon)$ i.e. p is $\mathcal{C}^k$ w.r.t. the spatial variables (x,y) and all the partial derivatives are continuous w.r.t. (t,x,y).

Now, given any $N \in \mathbb{N}$ we give the notion of *zero of order* N of a function $p \in \mathcal{R}^k_\varepsilon$. Note that it differs from the definition 9 in Section 4.2. The following definition will be of particular interest in the case $N > k$.

Definition 5.1.2. Given a function $p \in \mathcal{R}^k_\varepsilon$ we say that the origin is a *zero of order* N if there exist $T_N, \tilde{p} \in \mathcal{C}^{0,k}(\mathbb{R} \times \mathbb{D}_\varepsilon)$ such that,

$$p(t,x,y) = T_N(t,x,y) + \tilde{p}(t,x,y)$$

and satisfying the following properties.

- $$T_N(t,x,y) = \sum_{i+j=N} \alpha_{i,j}(t) x^i y^j$$

  is a homogeneous polynomial of degree N with $\mathcal{C}^1$ coefficients,

- there exists a constant C such that, for all $(t,x,y) \in \mathbb{R} \times \mathbb{D}_\varepsilon$,

  $$\begin{aligned}&|\tilde{p}(t,x,y)| \leq C(|x|^{N+1} + |y|^{N+1}),\\ &|\partial^{(m)}\tilde{p}(t,x,y)| \leq C(|x|^{N-m+1} + |y|^{N-m+1}) \quad \text{for} \quad 1 \leq m \leq k.\end{aligned}$$

Our result gives the existence of particular families of solutions: periodic and quasi-periodic solutions in a generalized sense. To define them, given a solution $(x(t), y(t))$ of (5.0.2), consider the functions

$$r(t) = \frac{1}{2(x(t)^2 + y(t)^2)}, \quad \theta(t) = -\operatorname{Arg}[x(t) + iy(t)], \tag{5.1.1}$$

having a relation with the standard polar coordinates. Actually $\theta(t)$ represents the angle in the clockwise sense, while $r(t)$ is, up to a scaling constant, the inverse of the square of the radius.

Definition 5.1.3. We say that the solution $(x(t), y(t))$, defined for $t \in \mathbb{R}$

- is *non-singular* if
  $$\sup_{t \in \mathbb{R}} r(t) < \infty;$$
- is *bounded* if there exists $A > 0$ such that
  $$\inf_{t \in \mathbb{R}} r(t) > A;$$
- has *monotone argument* if $\theta(t)$ is monotone;
- has *rotation number* α if
  $$\frac{1}{2\pi} \lim_{t \to \infty} \frac{\theta(t)}{t} = \alpha.$$

Remark 5.1.1. A non-singular bounded solution with monotone argument rotates clockwise in a closed annulus around the origin. Moreover, the rotation number represents the average angular velocity.

We are ready to state the main result.

Theorem 5.1.1. *Suppose that $p \in \mathcal{R}^3_\varepsilon$ is such that the origin is a zero of order 4. Then there exists $\bar{\alpha}$ sufficiently large such that for every $\alpha > \bar{\alpha}$ there exist a family of non-singular, bounded solutions*

$$\{(x(t), y(t))_\xi\}_{\xi \in \mathbb{R}}$$

with monotone argument and rotation number α. These solutions are such that the related functions $r(t), \theta(t)$ defined by (5.1.1) *satisfy, for every $t, \xi \in \mathbb{R}$,*

$$(r(t), \theta(t))_{\xi+2\pi} = (r(t), \theta(t))_\xi + (0, 2\pi), \tag{5.1.2}$$

$$(r(t+1), \theta(t+1))_\xi = (r(t), \theta(t))_{\xi+2\pi\alpha}. \tag{5.1.3}$$

Remark 5.1.2. If $\alpha = s/q \in \mathbb{Q}$, then the solutions satisfy

$$(r(t+q), \theta(t+q))_\xi = (r(t), \theta(t))_\xi + (0, 2\pi s)$$

and are said (s, q)-*periodic*. These solutions make s revolutions around the singularity in time q. If $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, solutions satisfying (5.1.2)-(5.1.3) can be seen as *generalized quasi-periodic*. Actually, consider the function

$$\Phi_\xi(a, b) = (r(a), \theta(a))_{b-2\pi\alpha a+\xi}.$$

This function is doubly-periodic in the sense that

$$\Phi_\xi(a+1, b) = (r(a+1), \theta(a+1))_{b-2\pi\alpha a+\xi-2\pi\alpha} = \Phi_\xi(a, b),$$
$$\Phi_\xi(a, b+2\pi) = (r(a), \theta(a))_{b-2\pi\alpha a+\xi+2\pi} = \Phi_\xi(a, b) + (0, 2\pi).$$

and $\Phi_\xi(t, 2\pi\alpha t) = (r(t), \theta(t))_\xi$. If the function $\xi \mapsto \Phi_\xi$ is continuous, then these solutions are classical quasi-periodic solutions with frequencies $(1, \alpha)$ in the sense of [81] (see also [66]). We will not guarantee the continuity, however, the function $\xi \mapsto \Phi_\xi$ will have at most jump discontinuities and if ξ is a point of continuity then so are $\xi + 2\pi\alpha, \xi + 2\pi$. Finally, the set $Cl\{(x(0), y(0))_\xi : \xi \in \mathbb{R}\}$ is either a curve or a Cantor set. In the first case, the invariant curve is conjugated to a rotation. If we have a Cantor set, it follows a dynamics of Denjoy type. Recall the remark 4 in Section 3.4.

5.2 Some estimates on the solutions and the Poincaré map

Let us consider system (5.0.2) and, following Section 4.3.1 of Chapter 4 or Section 4.1 of [68], consider the change of variables $(x, y) = \varphi(\theta, r)$ defined by

$$x = \frac{\cos\theta}{\sqrt{2r}}, \qquad y = -\frac{\sin\theta}{\sqrt{2r}}.$$

These variables comes from applying first the Kelvin transform and subsequently the change to symplectic polar coordinates. System (5.0.2) transforms into

$$\begin{cases} \dot{r} = 4r^2\, \partial_\theta H(t, r, \theta), \\ \dot{\theta} = -4r^2\, \partial_r H(t, r, \theta) \end{cases} \tag{5.2.1}$$

where $H(t,r,\theta) = -\frac{1}{2}\ln(2r) + h(t,r,\theta)$ and $h(t,r,\theta) = p\left(t, \frac{\cos\theta}{\sqrt{2r}}, -\frac{\sin\theta}{\sqrt{2r}}\right)$. System (5.2.1) is still a periodic planar Hamiltonian system with symplectic form $\tilde{\lambda} = \frac{1}{4r^2}\,\mathrm{d}r \wedge \mathrm{d}\theta$. Moreover, the change of variables φ transforms the domain $\mathbb{R} \times \mathbb{D}_\varepsilon$ into the domain $\mathbb{R} \times \mathcal{D}$ with

$$\mathcal{D} = \left\{ (r,\theta) \in]r_*, \infty[\times \mathbb{T} \;:\; r_* = \frac{1}{2\varepsilon^2} \right\}.$$

Let us write the Cauchy problem associated to system (5.2.1), in the following form:

$$\begin{cases} \dot{r} = F(t,r,\theta), \\ \dot{\theta} = 2r + G(t,r,\theta), \\ (r(0),\theta(0)) = (r_0,\theta_0) \end{cases} \tag{5.2.2}$$

where,

$$F(t,r,\theta) = 4r^2 \partial_\theta \left[p\left(t, \frac{\cos\theta}{\sqrt{2r}}, \frac{-\sin\theta}{\sqrt{2r}}\right)\right],$$
$$G(t,r,\theta) = -4r^2 \partial_r \left[p\left(t, \frac{\cos\theta}{\sqrt{2r}}, \frac{-\sin\theta}{\sqrt{2r}}\right)\right]. \tag{5.2.3}$$

Since $p \in \mathcal{R}^3_\varepsilon$, the vector field in (5.2.2) is continuous and $\mathcal{C}^2$ in the spatial variables. This guarantees existence and uniqueness of the solution.

Remark 5.2.1. The change of variables φ has the effect to transform the phase space from the plane to the cylinder. The singularity is moved from the origin to $r \to \infty$. In this sense, the change of variables has a regularizing effect since the functions F, G in (5.2.2) are bounded for $r \to \infty$. The fact that the origin is a zero of order 4 plays a fundamental role in this discussion. See estimate (5.2.5) in the following Lemma 5.2.1 for more details.

Since the domain $\mathcal{D}$ is not invariant, we need to control the growth of the solutions. For this purpose, given $a > r_*$ we introduce the set

$$\Sigma(a) =]a, \infty[\times \mathbb{T} \subset \mathcal{D}$$

and prove the following lemma, whose meaning is illustrated in Fig. 5.1.

Lemma 5.2.1. *Let us assume that the origin is a zero of order* 4 *of the function* $p \in \mathcal{R}^3_\varepsilon$. *Then there exists* $a_* > r_*$ *such that if* $(r_0,\theta_0) \in \Sigma(a_*)$*, the corresponding solution of* (5.2.2) *is well defined on* $t \in [0,1]$ *and* $(r(t),\theta(t)) \in \mathcal{D}$ *for all* $t \in [0,1]$. *Moreover, the following estimate holds*

$$|r(t) - r_0| + |\theta(t) - \theta_0 - 2r_0 t| \le K \quad \text{if} \quad t \in [0,1] \tag{5.2.4}$$

for some $K > 0$.

Proof. Since the origin is a zero of order 4 of p, there exists a constant $C > 0$ such that

$$\left|\partial^{(1)} p\,(t,x,y)\right| \le C(|x|^3 + |y|^3) \quad \text{in } \mathbb{R} \times \mathbb{D}_\varepsilon.$$

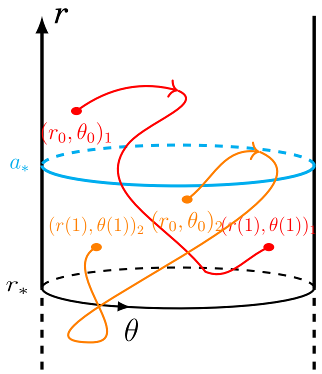

Figure 5.1: Domains and evolution of two solutions over a period. $(r(t), \theta(t))_1$ represents the solution with $(r_0, \theta_0)_1 \in \Sigma(a_*)$ and $(r(t), \theta(t))_2$ represents the solution with $(r_0, \theta_0)_2 \notin \Sigma(a_*)$. Note that the solution $(r(t), \theta(t))_1$ remains in the domain $\mathcal{D}$.

Then, from the definition of $F$and G, we have

$$|F(t, r, \theta)| + |2r|\,|G(t, r, \theta)| \leq C_1 \tag{5.2.5}$$

for any $t \in \mathbb{R}$ and $(r, \theta) \in \mathcal{D}$.

We shall prove that $a_* = r_* + C_1$ satisfies the lemma. Fix $(\theta_0, r_0) \in \Sigma(a_*)$ and consider the corresponding solution $(\theta(t), r(t))$. By continuity there exists τ such that $r(t)$ is well defined and $r(t) > r_*$ for $t \in [0, \tau]$. Suppose that $\tau < 1$, otherwise we are done. Integrating the first equation of (5.2.2) and using (5.2.5) we have

$$|r(t) - r_0| \leq C_1 t \quad \text{if} \quad t \in [0, \tau].$$

In particular, $r(\tau) \geq r_0 - C_1\tau > r_*$. Hence, we can continue the solution until time $\tau + \tau_1$. Suppose that $\tau + \tau_1 < 1$, otherwise we are done. Hence, as before $r(\tau + \tau_1) \geq r_0 - C_1(\tau + \tau_1) > r_*$. Repeating this procedure we can reach $\tau = 1$.

Finally, integrating the second equation of (5.2.2), we deduce that

$$|\theta(t) - \theta_0 - 2r_0 t| \leq 2C_1 + \frac{C_1}{2(r_0 - C_1)}.$$

Here we have employed (5.2.5) and the above estimates on $r(t)$.

□

Now, let us introduce the Poincaré map $\mathcal{P}$ as

$$\begin{array}{rrcl} \mathcal{P}: & \Sigma(a_*) =]a_*, \infty[\times \mathbb{T} & \longrightarrow & \mathcal{D} \subset \mathbb{R} \times \mathbb{T} \\ & (r_0, \theta_0) & \longmapsto & (r_1, \theta_1) = (r(1; r_0, \theta_0), \theta(1; r_0, \theta_0)) \end{array}$$

where $(r(t; r_0, \theta_0), \theta(t; r_0, \theta_0))$ is the solution with initial condition $(r(0), \theta(0)) = (r_0, \theta_0)$. Lemma 5.2.1 together with existence and uniqueness of the solutions of problem (5.2.2) guarantee that the Poincaré map is well defined.
Due to the regularity of the vector field of (5.2.1), $\mathcal{P} \in \mathcal{C}^2(\Sigma(a_*))$, concretely is a diffeomorphism of a section of the cylinder.

The proof of the theorem will be a consequence of a suitable version of the so called Aubry-Mather theory applied to the previous Poincaré map. The following sections is dedicated to prove that the Poincarè map satisfies the hypotheses of Theorem 3.4.1, namely, that is an exact symplectic twist diffeomorphism.

5.3 Exact symplectic properties of the Poincaré map

Fix $r \geq 2$ and a $\mathcal{C}^{r+1}$ function $f :]a, b[\longrightarrow \mathbb{R}$ such that $f'(r)$ never vanishes and consider the associated differential form $\tilde{\lambda} = \mathrm{d}f(r) \wedge \mathrm{d}\theta = f'(r)\mathrm{d}r \wedge \mathrm{d}\theta$ on $]a, b[\times\mathbb{R}$. In local coordinates, the corresponding time dependent Hamiltonian system takes the form

$$\begin{cases} \dot{r} = \frac{1}{f'(r)}\, \partial_\theta H(t, r, \theta) \\ \dot{\theta} = -\frac{1}{f'(r)}\, \partial_r H(t, r, \theta), \\ r(0) = r_0 \\ \theta(0) = \theta_0. \end{cases} \tag{5.3.1}$$

Suppose that $H : \mathbb{R}\times]a, b[\times\mathbb{R} \to \mathbb{R}$ is continuous in t and $\mathcal{C}^{r+1}$ in the phase variables (r, θ) and the following periodicity hold

$$H(t + 1, r, \theta) = H(t, r, \theta) \quad \text{and} \quad H(t, r, \theta + 2\pi) = H(t, r, \theta).$$

By the periodicity in θ we have the phase space is the cylinder $]a, b[\times\mathbb{T}$.

Remark 5.3.1. In our problem $]a, b[=]r_*, \infty[$, $f(r) = -\frac{1}{4r}$ and

$$H(t, r, \theta) = -\frac{1}{2}\ln(2r) + \; p\left(t, \frac{\cos\theta}{\sqrt{2r}}, -\frac{\sin\theta}{\sqrt{2r}}\right).$$

Let us consider the Poincaré map $\mathcal{P}(r_0, \theta_0) = (r_1, \theta_1)$ associated to the Cauchy problem (5.3.1). By the hypothesis on H and f, the map $\mathcal{P}$ belongs to $\mathcal{C}^r(]a, b[\times\mathbb{R})$ and satisfies:

$$\mathcal{P}(r_0, \theta_0 + 2\pi) = \mathcal{P}(r_0, \theta_0) + (0, 2\pi).$$

Lemma 5.3.1. *The Poincaré map $\mathcal{P}$ is exact symplectic with respect to the form $\tilde{\lambda} = f'(r)\mathrm{d}r \wedge \mathrm{d}\theta$.*

Proof. To simplify the notation, let us denote $(r(t), \theta(t))$ the solution $(r(t; r_0, \theta_0), \theta(t; r_0, \theta_0))$ of (5.3.1). Consider the $\mathcal{C}^r$ function

$$\mathcal{S}(r_0, \theta_0) = -\int_0^1 \left[\frac{f(r(t))}{f'(r(t))}\partial_r H(t, r(t), \theta(t)) - H(t, r(t), \theta(t))\right] \mathrm{d}t,$$

and note that by the periodicity assumptions in θ, and the uniqueness, we have

$$\mathcal{S}(r_0, \theta_0 + 2\pi) = \mathcal{S}(r_0, \theta_0).$$

Let us now prove that

$$\mathrm{d}\mathcal{S}(r_0, \theta_0) = f(r_1)\, \mathrm{d}\theta_1 - f(r_0)\, \mathrm{d}\theta_0.$$

We start with

$$\begin{aligned} \partial_{r_0}\mathcal{S}(r_0, \theta_0) = -\int_0^1 \Bigg[&-\frac{f(r(t))f''(r(t))}{[f'(r(t))]^2}[\partial_{r_0} r(t)]\partial_r H(t, r(t), \theta(t)) \\ &+\frac{f(r(t))}{f'(r(t))}\partial_{r_0}\left[\partial_r H(t, r(t), \theta(t))\right] - \partial_\theta H(t, r(t), \theta(t))\left[\partial_{r_0}\theta(t)\right]\Bigg] \mathrm{d}t. \end{aligned} \tag{5.3.2}$$

Note that the term $\partial_r H(t,r(t),\theta(t))[\partial_{r_0} r(t)]$ is canceled. Now, using the first equation in (5.3.1), and integrating by parts the last term, we obtain

$$\begin{aligned}\int_0^1 \partial_\theta H(t,r(t),\theta(t))\,[\partial_{r_0}\theta(t)]\;\mathrm{d}t &= \int_0^1 \dot f(r(t))\,[\partial_{r_0}\theta(t)]\;\mathrm{d}t \\ &= [f(r(t))\partial_{r_0}\theta(t)]_{t=0}^{t=1} - \int_0^1 f(r(t))\left[\partial_{r_0}\dot\theta(t)\right]\;\mathrm{d}t.\end{aligned}$$

Replacing in (5.3.2) and using the second equation in (5.3.1), we get

$$\partial_{r_0}\mathcal{S}(r_0,\theta_0) = [f(r(t))\partial_{r_0}\theta(t)]_{t=0}^{t=1}.$$

Analogously,

$$\partial_{\theta_0}\mathcal{S}(r_0,\theta_0) = [f(r(t))\partial_{\theta_0}\theta(t)]_{t=0}^{t=1}.$$

Hence,

$$\begin{aligned}\mathrm{d}\mathcal{S}(r_0,\theta_0) = \partial_{r_0}\mathcal{S}\,\mathrm{d}r_0 + \partial_{\theta_0}\mathcal{S}\,\mathrm{d}\theta_0 &= [f(r_1)\partial_{r_0}\theta_1 - f(r_0)\partial_{r_0}\theta_0]\,\mathrm{d}r_0 + \\ &+ [f(r_1)\partial_{\theta_0}\theta_1 - f(r_0)\partial_{\theta_0}\theta_0]\,\mathrm{d}\theta_0 = f(r_1)\,\mathrm{d}\theta_1 - f(r_0)\,\mathrm{d}\theta_0.\end{aligned}$$

□

5.4 The twist property for the vortex problem

In this section we consider the Poincaré map $\mathcal{P}$ associated to system (5.2.2). We recall the notation

$$\begin{aligned}\mathcal{P}:\ \Sigma(a_*) =]a_*,+\infty[\times\mathbb{T} &\longrightarrow \mathbb{R}^2 \\ (r_0,\theta_0) &\longmapsto (r_1,\theta_1) = (\mathcal{F}(r_0,\theta_0),\mathcal{G}(r_0,\theta_0)).\end{aligned}$$

The following theorem will clearly imply the twist condition (5.5.2).

Theorem 5.4.1. *Suppose that* $p \in \mathcal{R}^2_\varepsilon$ *and the origin is a zero of order* 4*. Then*

$$\frac{\partial\mathcal{G}}{\partial r_0} \underset{r_0\to\infty}{\longrightarrow} 2\,,\ \textit{uniformly in } \theta_0. \tag{5.4.1}$$

To prove the theorem, let us fix a solution $(r(t;\theta_0,r_0),\theta(t;\theta_0,r_0))$ of problem (5.2.2). Note that, since $p\in\mathcal{R}^2_\varepsilon$, the vector field of system (5.2.2) is $\mathcal{C}^1$ in the variables (r,θ) so that the solution is unique and we can consider the associated variational equation

$$\begin{cases} \dot Y = \mathcal{M}(t,r(t;r_0,\theta_0),\theta(t;r_0,\theta_0))\,Y, \\ Y(0) = \mathbb{I}_2. \end{cases} \tag{5.4.2}$$

Here

$$\mathcal{M}(t;r,\theta) = \frac{\partial\,(F(t,r,\theta),2r+G(t,r,\theta))}{\partial\,(r,\theta)}$$

is the Jacobian of the vector field in (5.2.2). We denote the matrix solution

$$\mathcal{Y}(t;r_0,\theta_0) = \begin{pmatrix}\partial_{r_0}r(t;r_0,\theta_0) & \partial_{\theta_0}r(t;r_0,\theta_0)\\ \partial_{r_0}\theta(t;r_0,\theta_0) & \partial_{\theta_0}\theta(t;r_0,\theta_0)\end{pmatrix}$$

and by the definition of the Poincaré map,

$$\frac{\partial \mathcal{G}}{\partial r_0}(r_0,\theta_0) = \partial_{r_0}\theta(1;r_0,\theta_0).$$

In the integrable case $p = 0$, the Jacobian matrix is $A = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$ and the solution of the corresponding variational equation is

$$\mathcal{Y}_{int}(t;r_0,\theta_0) = \begin{pmatrix} 1 & 0 \\ 2t & 1 \end{pmatrix}, \tag{5.4.3}$$

that shows that the Poincaré map of the unperturbed problem is twist.

To prove the result in the non integrable case, we will follow a perturbative approach. More precisely, we will prove that the solution remains close to that of the integrable case over a period $t \in [0,1]$. For this purpose we begin considering the following splitting. To simplify the notation we will denote a solution of (5.2.2) by $(r(t),\theta(t))$ where we have dropped the dependence on the initial conditions.

Lemma 5.4.1. *Under the hypothesis of Theorem* (5.4.1)*, we have the following splitting:*

$$\mathcal{M}(t;r(t),\theta(t)) = A + B(t,r_0,\theta_0) + C(t,r_0,\theta_0)$$

where $B(t,r_0,\theta_0)$ *is bounded, the entries satisfy* $b_{11} = b_{21} = b_{22} = 0$ *and* $\forall \varphi \in \mathcal{C}^\infty([0,1])$

$$\left| \int_0^t b_{12}(s,r_0,\theta_0)\,\varphi(s)\,\mathrm{d}s \right| \underset{r_0\to\infty}{\longrightarrow} 0 \quad \textit{uniformly in } t\in[0,1], \theta_0 \in \mathbb{T}. \tag{5.4.4}$$

Moreover,

$$\|C(t,r_0,\theta_0)\| \underset{r_0\to\infty}{\longrightarrow} 0 \quad \textit{uniformly in } t\in[0,1], \theta_0 \in \mathbb{T}.$$

Proof. Since the origin is a zero of order 4 for p, we can split the perturbation as

$$p(t,x,y) = T_4(t,x,y) + \tilde{p}(t,x,y),$$

where T_4 is a homogeneous polynomial of degree 4. From system (5.2.2)

$$F(t,r,\theta) = 4r^2\partial_\theta p\left[\left(t, \frac{\cos\theta}{\sqrt{2r}}, \frac{-\sin\theta}{\sqrt{2r}}\right)\right],$$

so that p induce the following splitting on F

$$F(t,r,\theta) = F_*(t,\theta) + \tilde{F}(t,r,\theta),$$

where, using the homogeneity of T_4 w.r.t the variable r,

$$F_*(t,\theta) = \partial_\theta[T_4\,(t,\cos\theta,-\sin\theta)], \quad \tilde{F}(t,r,\theta) = 4r^2\partial_\theta\left[\tilde{p}\left(t,\frac{\cos\theta}{\sqrt{2r}},\frac{-\sin\theta}{\sqrt{2r}}\right)\right].$$

Therefore we write

$$\mathcal{M}(t, r(t), \theta(t)) = \begin{pmatrix} \partial_r F(t,r,\theta) & \partial_\theta F(t,r,\theta) \\ 2 + \partial_r G(t,r,\theta) & \partial_\theta G(t,r,\theta) \end{pmatrix}\Bigg|_{(r(t),\theta(t))}$$
$$= \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} + \left[\begin{pmatrix} 0 & b_{12} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \right]_{(r(t),\theta(t))}$$

where $b_{12} = b_{12}(t,r,\theta)$ and $c_{ij} = c_{ij}(t,r,\theta)$ defined as

$$b_{12} := \partial_\theta F_*(t,\theta) = \partial_{\theta\theta}\left[T_4\left(t, \cos\theta, -\sin\theta\right)\right]$$

and

$$c_{11} := \partial_r \tilde{F}(t,r,\theta) \qquad\qquad c_{12} := \partial_\theta \tilde{F}(t,r,\theta)$$
$$c_{21} := \partial_r G(t,r,\theta) \qquad\qquad c_{22} := \partial_\theta G(t,r,\theta)$$

Concerning the matrix $C = (c_{ij})$, one can first explicitly write the expression of the entries c_{ij} recalling that $\tilde{F}$ has just been introduced and G is defined in (5.2.3). Note that they depend on the derivatives up to second order of the functions $p\left(t, \frac{\cos\theta}{\sqrt{2r}}, \frac{-\sin\theta}{\sqrt{2r}}\right)$, $\tilde{p}\left(t, \frac{\cos\theta}{\sqrt{2r}}, \frac{-\sin\theta}{\sqrt{2r}}\right)$ w.r.t the variables (r,θ).

The following lemma states some bounds for the derivatives of functions $p \in \mathcal{R}^2_\varepsilon$ with a zero of order N.

Lemma 5.4.2. *Suppose the $p \in \mathcal{R}^2_\varepsilon$ and that the origin is a zero of order N. Consider the decomposition given in Definition 5.1.2*

$$p(t,x,y) = T_N(t,x,y) + \tilde{p}(t,x,y)$$

and consider the functions

$$x = x(r,\theta) = \frac{\cos\theta}{\sqrt{2r}}, \quad y = y(r,\theta) = \frac{-\sin\theta}{\sqrt{2r}}$$

defined for $r > \frac{1}{2\varepsilon^2}$ and $\theta \in \mathbb{T}$. Then, there exists a constant $C > 0$ such that

1) $r^{(N+1)/2}\left(|\partial_\theta\,\tilde{p}(t,x,y)| + |\partial_{\theta\theta}\,\tilde{p}(t,x,y)|\right) \le C$,

2) $r^{(N+3)/2}\,|\partial_{r\theta}\,\tilde{p}(t,x,y)| \le C$,

3) $r^{(N+2)/2}\left(|\partial_r\,p(t,x,y)| + |\partial_{r\theta}\,p(t,x,y)|\right) \le C$,

4) $r^{(N+4)/2}\,|\partial_{rr}\,p(t,x,y)| \le C$.

Proof. To get the estimate **1)**, let us calculate explicitly the derivatives with respect to θ:

$$\partial_\theta \tilde{p}(t, x(r,\theta), y(r,\theta)) = -\frac{1}{(2r)^{1/2}}\left[\sin\theta\,\partial_x\tilde{p}(t,x,y) + \cos\theta\,\partial_y\tilde{p}(t,x,y)\right]$$

and

$$\partial_{\theta\theta}\tilde{p}(t, x(r,\theta), y(r,\theta)) = \frac{1}{(2r)^{1/2}}\left[\sin\theta\,\partial_y\tilde{p}(t,x,y) - \cos\theta\,\partial_x\tilde{p}(t,x,y)\right]$$
$$+ \frac{1}{2r}\left[\sin^2\theta\,\partial_{xx}\tilde{p}(t,x,y) + 2\cos\theta\sin\theta\,\partial_{xy}\tilde{p}(t,x,y) + \cos^2\theta\,\partial_{yy}\tilde{p}(t,x,y)\right].$$

From the definition of zero of order N we have:

$$\begin{aligned}|\partial_\theta \tilde{p}(t,x,y)| + |\partial_{\theta\theta} \tilde{p}(t,x,y)| &\le \frac{C_1}{r^{1/2}} \left(|\partial_x \tilde{p}(t,x,y)| + |\partial_y \tilde{p}(t,x,y)|\right)\\ &+ \frac{C_2}{r} \left(|\partial_{xx} \tilde{p}(t,x,y)| + |\partial_{xy} \tilde{p}(t,x,y)| + |\partial_{yy} \tilde{p}(t,x,y)|\right)\\ &\le \frac{C_1}{r^{1/2}}(|x|^N + |y|^N) + \frac{C_1}{r}(|x|^{N-1} + |y|^{N-1})\\ &\le \frac{C_1}{r^{(N+1)/2}} + \frac{C_2}{r^{1+(N-1)/2}} \le \frac{C}{r^{(N+1)/2}}\end{aligned}$$

To obtain **2)**, **3)** and **4)**, the computations are similar.

□

Using the previous Lemma we get the following estimate

$$r^{3/2}\,|c_{11}| + r^{1/2}\,|c_{12}| + r^2\,|c_{21}| + r\,|c_{22}| \le K$$

with K independent on r, θ. We evaluate the entries c_{ij} on a solution $(r(t), \theta(t))$ and we remember that from Lemma 5.2.1 we have that for $r_0 > a_*$,

$$|r(t) - r_0| \le K_1 \quad \forall\, \theta_0 \in \mathbb{T} \text{ and } t \in [0,1]. \tag{5.4.5}$$

This proves that $\|C(t, r_0, \theta_0)\| \longrightarrow 0$ as $r_0 \to \infty$ uniformly in $\theta_0 \in \mathbb{T}, t \in [0,1]$.

Let us study the matrix $B = (b_{ij})$. Since $T_4\,(t, \cos\theta, -\sin\theta)$ is a trigonometric polynomial, $|b_{12}| \le K$ and B is bounded. To obtain (5.4.4) we note that being $T_4\,(t, \cos\theta, -\sin\theta)$ a trigonometric polynomial of degree 4, $\partial_\theta\,[T_4\,(t, \cos\theta, -\sin\theta)]$ will be another trigonometric polynomial (of the same degree) that we denote by $\tilde{T}_4\,(t, \cos\theta, -\sin\theta)$.

To get the estimates of the element b_{12} of matrix B we need the followimg result. It is another lemma of Riemann-Lebesgue type but is more general than Lemma (4.3.3).

Lemma 5.4.3. *Let $q(t,\eta,\xi)$ be a polynomial of degree N,*

$$q(t,\eta,\xi) = \sum_{j+h\le N} \alpha_{j,h}(t)\eta^j\xi^h$$

with $\alpha_{j,h}(t) \in \mathcal{C}^1(\mathbb{R}/\mathbb{Z})$. Assume in addition that for each t

$$\int_0^{2\pi} q(t, \cos\,\theta, \sin\,\theta)\,\mathrm{d}\theta = 0\,. \tag{5.4.6}$$

Let $\beta \in \mathcal{C}^1([0,\tau])$ with $[0,\tau] \subset [0,1]$ and $\varphi \in \mathcal{C}^\infty([0,1])$. Then there exists $C_{RL} > 0$ such that

$$\left|\int_0^t q(s, \cos\,(\lambda s + \beta(s)), \sin\,(\lambda s + \beta(s)))\,\varphi(s)\,\mathrm{d}s\right| \le \frac{C_{RL}}{|\lambda|}$$

if $t \in [0,\tau]$ and $\lambda \in \mathbb{R}\setminus\{0\}$. Moreover, the constant C_{RL} depends upon N, $\max_{j,h}\left[\|\alpha_{j,h}\|_\infty + \|\dot{\alpha}_{j,h}\|_\infty\right]$, $\|\beta\|_\infty$, $\|\dot{\beta}\|_\infty$ $\|\varphi\|_\infty$ and $\|\dot{\varphi}\|_\infty$.

Proof. The function $q(t, \cos\,\theta, \sin\,\theta)$ has a finite Fourier expansion with respect to θ, say

$$q(t, \cos\,\theta, \sin\,\theta) = \sum_{|k|\le N} q_k(t) e^{ik\theta}.$$

The coefficients q_k can be expressed in terms of the functions $\alpha_{j,h}$ and belong to $\mathcal{C}^1(\mathbb{R}/\mathbb{Z})$,

$$q_k(t) = \frac{1}{2\pi}\int_0^{2\pi} q(t, \cos\,\theta, \sin\,\theta) e^{-ik\theta}\,\mathrm{d}\theta.$$

The condition (5.4.6) implies that $q_0(t)$ vanishes everywhere and so the integral $I(t)$ we want to estimate can be expressed as the sum

$$I(t) = \sum_{0<|k|\le N} I_k(t) \quad \text{with} \quad I_k(t) = \int_0^t q_k(s) e^{ik\beta(s)} e^{ik\lambda s}\,\varphi(s)\,\mathrm{d}s.$$

Since we have excluded $k = 0$ these integrals can be estimated by a standard procedure in the theory of oscillatory integrals, see for instance [5]. After integrating by parts

$$\begin{aligned} I_k(t) = \frac{1}{ik\lambda} \Big[& q_k(t) e^{ik\beta(t)} e^{ik\lambda t}\,\varphi(t) - q_k(0) e^{ik\beta(0)}\,\varphi(0) \\ & - \int_0^t \left(q_k(s) e^{ik\beta(s)}\,\varphi(s)\right)' e^{ik\lambda s}\,\mathrm{d}s \Big]. \end{aligned}$$

Therefore,

$$|I_k(t)| \le \frac{C_k}{|k|\,|\lambda|}$$

with

$$C_k = \|q_k\|_\infty \left[2\,\|\varphi\|_\infty + |k| \left\|\dot\beta\right\|_\infty \|\varphi\|_\infty + \|\dot\varphi\|_\infty\right] + \|\dot q_k\|_\infty\,\|\varphi\|_\infty\,.$$

□

Let us define

$$P_4(t, \eta, \xi) := \tilde T_4(t, \eta, \xi).$$

We show that we can apply Lemma 5.4.3 choosing the polynomial of degree $N = 4$,

$$q(t, \eta, \xi) = -\xi \frac{\partial P_4}{\partial \eta}(t, \eta, -\xi) - \eta \frac{\partial P_4}{\partial \xi}(t, \eta, -\xi).$$

Since $P_4(t, \cos\theta, -\sin\theta)$ is a periodic primitive in the variable θ of the function $q(t, \cos\theta, \sin\theta)$ the condition (5.4.6) holds. Moreover, the regularity assumption in Definition 5.1.2 guarantees that q is $\mathcal{C}^1$ in the variable t.
Let us define the function $\beta(t) := \theta(t) - 2r_0 t$. The estimate (5.2.4) on the angular evolution gives a bound of $\|\beta\|_\infty$. To get a bound of $\|\dot\beta\|_\infty$ we observe that

$$\dot\beta(t) = \dot\theta(t) - 2r_0 = 2\,(r(t) - r_0) + G(t, r(t), \theta(t)).$$

And again from (5.2.4) and (5.2.5), we have

$$\|\dot\beta\|_\infty \le 2K + \frac{C_1}{(r_0 - K)}.$$

Finally, Lemma 5.4.3 can be applied to deduce that for all $\varphi \in \mathcal{C}^\infty([0,1])$ and $t \in [0,1]$,

$$\begin{aligned}
\left|\int_0^t b_{12}(s,r_0,\theta_0)\varphi(s)\mathrm{d}s\right| &= \left|\int_0^t b_{12}(s,\theta(s))\varphi(s)\mathrm{d}s\right| \\
&= \left|\int_0^t \partial_{\theta\theta}T_4\left(s,\cos\theta(s),-\sin\theta(s)\right)\varphi(s)\mathrm{d}s\right| \\
&= \left|\int_0^t \partial_\theta P_4(s,\cos\theta(s),-\sin\theta(s))\varphi(s)\mathrm{d}s\right| \\
&= \left|\int_0^t q(s,\cos\theta(s),\sin\theta(s))\varphi(s)\mathrm{d}s\right| \\
&= \left|\int_0^t q(s,\cos(2r_0 s+\beta(s)),\sin(2r_0 s+\beta(s)))\varphi(s)\mathrm{d}s\right| \\
&\leq \frac{C_{RL}}{2r_0}.
\end{aligned}$$

□

By the previous discussion, we can write the variational equation (5.4.2) as

$$\begin{cases} \dot{Y} = \left(A + B(t,r_0,\theta_0) + C(t,r_0,\theta_0)\right)Y, \\ Y(0) = \mathbb{I}_2. \end{cases}$$

We claim that the solution $\mathcal{Y}(t;r_0,\theta_0)$ converge uniformly, as $r_0 \to \infty$, to the solution $\mathcal{Y}_{int}(t;r_0,\theta_0)$ of the integrable case (5.4.3).

To prove it, we need the following Lemma concerning the uniform convergence of the solution of a linear ODE whose time-dependent coefficients are bounded and converging weak* in $\mathcal{L}^\infty$. Similar results can be found in [49] and [63]. For the proof we will follow the lines of the proof of Lemma 2.1 in [63].

Consider the following linear system depending on the parameters $(r,\theta) \in]a,b[\times\mathbb{T}$, $a > b$

$$\begin{cases} \dot{Y} = \left(A + M(t;r,\theta)\right)Y, \\ Y(0) = \mathbb{I}_2, \end{cases} \tag{5.4.7}$$

where A, M are 2×2 matrices, A is constant and $M \in \mathcal{C}^1([0,1]\times]a,b[\times\mathbb{T})$. We denote the matrix solution of this system as $\mathcal{Y}(t;r,\theta)$.

Lemma 5.4.4. *Suppose that the family $\{M(t;r,\theta)\}$ is uniformly bounded in $\mathcal{L}^\infty([0,1])$ and that $M(t;r,\theta)$ converges to $\tilde{M}(t;\theta) \in \mathcal{L}^\infty([0,1])$ in the weak* sense as $r \to b$. Then*

$$\mathcal{Y}(t;r,\theta) \underset{r\to b}{\longrightarrow} \tilde{\mathcal{Y}}(t;\theta) \quad \textit{uniformly,} \qquad t\in[0,1] \quad \forall\theta\in\mathbb{T}$$

where $\tilde{\mathcal{Y}}(t;\eta)$ is the matrix solution of the problem

$$\begin{cases} \dot{Y} = \left(A + \tilde{M}(t;\theta)\right)Y, \\ Y(0) = \mathbb{I}_2\,. \end{cases} \tag{5.4.8}$$

Proof. The solution of (5.4.7) can be written as:

$$\mathcal{Y}(t;r,\theta) = \mathbb{I}_2 + \int_0^t (A + M(s;r,\theta))\mathcal{Y}(s;r,\theta)\,\mathrm{d}s, \quad t \in \mathbb{R}, \tag{5.4.9}$$

from which we get the following estimate on the matrix norm

$$\|\mathcal{Y}(t;r,\theta)\| \le 1 + \int_0^t \|A + M(s;r,\theta)\|\,\|\mathcal{Y}(s;r,\theta)\|\,\mathrm{d}s. \tag{5.4.10}$$

Gronwall lemma applied on the interval [0,1] gives us

$$\|\mathcal{Y}(t;r,\theta)\| \le 1 + \int_0^t \|A + M(s;r,\theta)\|\,\mathrm{e}^{\int_s^t \|A+M(\xi;r,\theta)\|\,\mathrm{d}\xi}\,\mathrm{d}s, \quad t \in [0,1].$$

Since $\{M(t;r,\theta)\}$ is uniformly bounded, $\|\mathcal{Y}(t;r,\theta)\|$ is uniformly bounded. Moreover, from (5.4.9) also $\|\dot{\mathcal{Y}}(t;r,\theta)\|$ is uniformly bounded. Hence, we can apply Ascoli-Arzelà theorem to get a subsequence $r_k \to b$ as $k \to \infty$ and a matrix $\Phi \in \mathcal{C}^1([0,1] \times \mathbb{T})$ such that

$$\mathcal{Y}(t;r_k,\theta) \underset{k\to\infty}{\longrightarrow} \Phi(t;\theta), \quad \text{uniformly in} \qquad t \in [0,1] \quad \theta \in \mathbb{T}.$$

The matrices $\mathcal{Y}(t;r_k,\theta)$ satisfies

$$\begin{aligned}\mathcal{Y}(t;r_k,\theta) &= \mathbb{I}_2 + \int_0^t (A + M(s;r_k,\theta))\mathcal{Y}(s;r_k,\theta)\,\mathrm{d}s \\ &= \mathbb{I}_2 + \int_0^t A\mathcal{Y}(s;r_k,\theta)\,\mathrm{d}s + \int_0^t M(s;r_k,\eta)\Phi(s;\theta)\,\mathrm{d}s \\ &+ \int_0^t M(s;r_k,\theta)(\mathcal{Y}(s;r_k,\theta) - \Phi(s;\theta))\,\mathrm{d}s.\end{aligned}$$

Using the uniform convergence, and the weak* convergence of $M(t;r,\theta)$ (recall $\Phi \in \mathcal{L}^1([0,1])$), we have the limit:

$$\lim_{k\to\infty} \mathcal{Y}(t;r_k,\theta) = \mathbb{I}_2 + \int_0^t A\Phi(s;\eta)\,\mathrm{d}s + \int_0^t \tilde{M}(s;\theta)\Phi(s;\theta)\,\mathrm{d}s, \quad t \in [0,1] \quad \forall\theta \in \mathbb{T}.$$

Finally, by uniqueness we observe that this limit is the solution of system (5.4.8) and we obtain

$$\lim_{r\to b} \mathcal{Y}(t;r,\theta) = \Phi(t;\theta) = \tilde{\mathcal{Y}}(t;\theta) \qquad t \in [0,1] \quad \forall\theta \in \mathbb{T}.$$

□

Now we can prove that $\mathcal{Y}(t;r_0,\theta_0)$ converge uniformly, as $r_0 \to \infty$, to the solution $\mathcal{Y}_{int}(t;r_0,\theta_0)$ of the integrable case (5.4.3). Applying Lemma 5.4.4 to the family of matrices $M(t,r_0,\theta_0) = B(t,r_0,\theta_0) + C(t,r_0,\theta_0)$ as $r_0 \to +\infty$. From Lemma 5.4.1 we have that $M(t,r_0,\theta_0)$ is uniformly bounded and $C(t,r_0,\theta_0)$ converge uniformly to 0 so that it converge also in the weak* topology. To study the convergence of the matrix $B(t, r_0, \theta_0)$ it is enough to consider the term b_{12}. From (5.4.4) and using the density of $\mathcal{C}^\infty$ in $\mathcal{L}^1$ we have

$$\left| \int_0^t b_{12}(s, r_0, \theta_0)\, \varphi(s)\, \mathrm{d}s \right| \underset{r_0 \to \infty}{\longrightarrow} 0, \qquad \forall \varphi \in \mathcal{L}^1([0,1]), \quad t \in [0,1].$$

Hence, we can apply Lemma 5.4.4 and get

$$\mathcal{Y}(t; r_0, \theta_0) \underset{r_0 \to \infty}{\longrightarrow} \mathcal{Y}_{int}(t; r_0, \theta_0) \quad \text{uniformly in } t \in [0,1], \theta_0 \in \mathbb{T},$$

from which (5.4.1) follows evaluating in $t = 1$.

5.5 Proof of the main Theorem

Firstly, we recall the version of the Aubry-Mather theorem in Section 3.4 of Chapter 3.
Consider a $\mathcal{C}^2$ diffeomorphism

$$\begin{array}{rccl} \Phi: & \Sigma & \longrightarrow & \mathfrak{C} \\ & (r, \theta) & \longmapsto & (r_1, \theta_1) = (\mathcal{F}(r, \theta), \mathcal{G}(r, \theta)) . \end{array}$$

and the corresponding lift map

$$\begin{array}{rccl} \Phi: & \tilde{\Sigma} & \longrightarrow & \mathbb{R}^2 \\ & (r, x) & \longmapsto & (r_1, x_1) = (\mathcal{F}(r, x), \mathcal{G}(r, x)) \end{array} \tag{5.5.1}$$

where

$$\begin{aligned} \mathcal{F}(r, x + 2\pi) &= \mathcal{F}(r, x), \\ \mathcal{G}(r, x + 2\pi) &= \mathcal{G}(r, x) + 2\pi. \end{aligned}$$

Consider a $\mathcal{C}^2$ function with Lipschitz inverse

$$\begin{array}{rcl} f :]a, b[& \longrightarrow & \mathbb{R} \\ r & \longmapsto & f(r), \end{array}$$

such that f' never vanishes. Without loss of generality we fix $f' > 0$.

We suppose that Φ is exact symplectic with respect to the form $f(r)\mathrm{d}\theta$. The weighted symplectic form is:

$$\lambda = \mathrm{d}f(r) \wedge \mathrm{d}\theta = f'(r)\mathrm{d}r \wedge \mathrm{d}\theta$$

and that Φ is twist, that is

$$\partial_r \mathcal{G}(r, \theta) > 0 \qquad \forall (r, \theta) \in \Sigma. \tag{5.5.2}$$

Suppose additionally that the following uniform limits (w.r.t. x) exist

$$\begin{aligned} \alpha^+(x) &:= \frac{1}{2\pi} \left(\lim_{r \to b} \mathcal{G}(r, x) - x \right), \\ \alpha^-(x) &:= \frac{1}{2\pi} \left(\lim_{r \to a} \mathcal{G}(r, x) - x \right). \end{aligned}$$

Note that $\alpha^{\pm}(x)$ are 2π-periodic $\mathcal{C}^2$ functions and define

$$W^+ = \min_x \alpha^+(x), \qquad W^- = \max_x \alpha^-(x).$$

The following Theorem deals with the existence of special orbits of the diffeomorphism Φ. To state the Theorem, we recall that a sequence $(x_n)_{n\in\mathbb{Z}}$ of real numbers is increasing if $x_n < x_{n+1}$ for all $n \in \mathbb{Z}$ and we say that any two translates are comparable if for any $(s,q) \in \mathbb{Z}^2$ only one of the following alternatives holds

$$\overline{x}_{n+q} + 2\pi s > \overline{x}_n \quad \forall n, \qquad \overline{x}_{n+q} + 2\pi s = \overline{x}_n \quad \forall n, \qquad \overline{x}_{n+q} + 2\pi s < \overline{x}_n \quad \forall n.$$

Theorem 5.5.1. *With the previous setting, suppose that $W^+ - W^- > 8\pi$ and fix α such that $2\pi\alpha \in (W^- + 4\pi, W^+ - 4\pi)$. Then*

- *if $\alpha = s/q \in \mathbb{Q}$ there exists a (s,q)-periodic orbit $(\overline{r}_n, \overline{x}_n)_{n\in\mathbb{Z}}$ such that*

$$\overline{r}_{n+q} = \overline{r}_n, \qquad \overline{x}_{n+q} = \overline{x}_n + 2\pi s \quad \forall n \in \mathbb{Z};$$

- *if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ there exists a compact invariant subset $\mathcal{M}_\alpha \subset \Sigma$ (and a corresponding subset $\tilde{\mathcal{M}}_\alpha \subset \tilde{\Sigma}$) with the following properties:*
  - *denoting $\pi : \Sigma \to \mathbb{T}$ the projection, $\pi|_{\mathcal{M}_\alpha}$ is injective and $\mathcal{M}_\alpha =$ $\operatorname{graph} u$ for a Lipschitz function $u : \pi(\mathcal{M}_\alpha) \to \mathbb{R}$,*
  - *each orbit $(\overline{r}_n, \overline{x}_n)_{n\in\mathbb{Z}} \in \tilde{\mathcal{M}}_\alpha$ is such that the sequence $(\overline{x}_n)$ is increasing and any two translates are comparable,*
  - *each orbit $(\overline{r}_n, \overline{x}_n)_{n\in\mathbb{Z}} \in \tilde{\mathcal{M}}_\alpha$ has rotation number α, i.e.*

$$\frac{1}{2\pi} \lim_{n\to\infty} \frac{\overline{x}_n}{n} = \alpha;$$

  - *the set $\mathcal{M}_\alpha$ is either an invariant curve or a Cantor set.*

The following corollary gives an equivalent interpretation of the result.

Corollary 5.5.1. *For each α there exists two functions $\phi, \eta : \mathbb{R} \to \mathbb{R}$ such that, for every $\xi \in \mathbb{R}$*

$$\begin{aligned} &\phi(\xi + 2\pi) = \phi(\xi) + 2\pi, \quad \eta(\xi + 2\pi) = \eta(\xi), \\ &\Phi(\phi(\xi), \eta(\xi)) = (\phi(\xi + 2\pi\alpha), \eta(\xi + 2\pi\alpha)) \end{aligned}$$

where ϕ is monotone (strictly if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$) and η is of bounded variation.

In this section we apply the previous Theorem 5.5.1 and Corollary 5.5.1 to the Poincaré map $\mathcal{P}$ of system (5.2.2) and get the so called Aubry-Mather orbits of rotation number α. These orbits determine the solutions we announced in our main theorem 5.1.1.

By Lemma 5.2.1 the map $\mathcal{P}$ is well defined in $\Sigma(a_*)$ and is a $\mathcal{C}^2$-diffeomorphism since $p \in \mathcal{R}^3_\varepsilon$. For every initial condition in $\Sigma(a_*)$, the corresponding solution satisfies $|r(t) - r_0| \leq K$ for all $t \in [0,1]$. Moreover, from (5.2.5), we can find $a_1 > 0$ such that, in $\Sigma(a_1)$,

$$|\dot{\theta}| = |2r + G(t; r, \theta)| \geq 2r - \frac{C_1}{2r} > 0.$$

By theorem 5.4.1, the map $\mathcal{P}$ is twist in $\Sigma(a_2)$ for some a_2 large enough. Let us consider the strip $\Sigma(\bar{r})$ where $\bar{r} = \max\{a_*, a_1, a_2\} + K$. Theorem 5.4.1 also imply that the following limits hold:

$$W^+ := \min_x \{\lim_{r \to +\infty} \mathcal{G}(r, x) - x\} = +\infty,$$
$$W^- := \max_x \{\lim_{r \to \bar{r}} \mathcal{G}(r, x) - x\} = c < +\infty.$$

so that $W^+ - W^- > 8\pi$.

Since, from Lemma 5.3.1, the map $\mathcal{P}$ is exact symplectic w.r.t the form $\lambda = \frac{1}{4r^2} dr \wedge d\theta = d(-\frac{1}{4r}) \wedge d\theta$ and $1/(4r)$ is Lipschitz for $r > \bar{r}$, we can apply Theorem 5.5.1 and Corollary 5.5.1 to the Poincaré map restricted to the strip $\Sigma(\bar{r})$. For every $\alpha > (c+2)/2\pi$, we get two functions $\phi, \eta : \mathbb{R} \to \mathbb{R}$ such that, for every $\xi \in \mathbb{R}$

$$\phi(\xi + 2\pi) = \phi(\xi) + 2\pi, \quad \eta(\xi + 2\pi) = \eta(\xi), \tag{5.5.3}$$
$$\mathcal{P}(\phi(\xi), \eta(\xi)) = (\phi(\xi + 2\pi\alpha), \eta(\xi + 2\pi\alpha)). \tag{5.5.4}$$

For every $\xi \in \mathbb{R}$, let us consider the solution of the Cauchy problem (5.2.2) with initial condition $(r(0), \theta(0)) = (\eta(\xi), \phi(\xi))$ and denote it $(r(t), \theta(t))_\xi$. By (5.5.3) and uniqueness we have that

$$(r(t), \theta(t))_{\xi + 2\pi} = (r(t), \theta(t))_\xi + (0, 2\pi),$$

and from (5.5.4) and the definition of $\mathcal{P}$,

$$(r(t+1), \theta(t+1))_\xi = (r(t), \theta(t))_{\xi + 2\pi\alpha}$$

so that conditions (5.1.2)-(5.1.3) are satisfied.

The function $\xi \mapsto \Phi_\xi(a, b)$ introduced in Remark 5.1.2 has the same regularity of the functions ϕ, η that can have at most jump discontinuities. Moreover, from properties (5.5.3),(5.5.4), if ξ is a point of continuity, so are $\xi + 2\pi$ and $\xi + 2\pi\alpha$.

Finally, these solutions have rotation number α, actually,

$$\begin{aligned}\lim_{t\to\infty} \frac{\theta_\xi(t)}{t} &= \lim_{k\to\infty} \frac{\theta_\xi(k)}{k} = \lim_{k\to\infty} \frac{\theta_{\xi + 2\pi k\alpha}(0)}{k} = \lim_{k\to\infty} \frac{\phi(\xi + 2\pi k\alpha)}{k} \\ &= \lim_{k\to\infty} \frac{\phi(\xi + 2\pi\{k\alpha\}) + 2\pi[k\alpha]}{k} = 2\pi\alpha.\end{aligned}$$

where $[z]$ denote the integer part of z and $\{z\} = z - [z]$.

Chapter 6

Conclusions

In this thesis we have answer some questions about the dynamical properties of the perturbed point-vortex problem as well as the dynamics around a singular point in a more mathematical context. In Chapter 4 we gave a stability result by means of the KAM theory.
Additionally, in Chapter 5 with other hypothesis for the perturbation and using the Aubry-Mather theory, we gave an existence result. In this last result, KAM results can not be applied due the regularity of the perturbation : $p(t, x, y)$ has to be $\mathcal{C}^3$.

Let us comment both results:

Stability result for the perturbed point-vortex problem

The result concerning the stability of the perturbed point-vortex problem can be seen as a first step on the study of stable singularities in Hamiltonian systems, in analogy to stable equilibria. To this aim, we have used the basic point-vortex model as a canonical example of singularity. Our study of the complex valued version of the system is inspired by Morris approach in [52].

Recently, in [33] the authors prove a similar stability under the action of a periodic background flow induced by a general polynomial field $\sum\limits_{1\leq i+j\leq N} a_{ij}(t)x\hat{\imath}y^j$ with a_{ij} 2π-periodic continuous differentiable functions. The proof is obtained from a finite differentiable version of the *Invariant Curve Theorem* [55]. Additionally, this improved theorem is used in two applications in the dynamics of point-vortices:

- Planar motion of two point-vortices with arbitrary intensities μ_1, μ_2 under the effect of an external linear deformation flow $f(a(t), b(t)$. The components of the external deformation flow are $a(t)$ (shear component) and $b(t)$ (rotational component).
- Advection of fluid particles induced by a prescribed vortex path $z(t)$ inside a circular domain.

Of course, our initial standpoint admits several generalizations. In principle, it should not be too difficult to find similar results with a quasiperiodic perturbation by using the quasiperiodic version of twist theorem by Zharnitsky [87]. Extensions to 3D flows would be more involved because the Hamiltonian structure is lost, being necessary an extra dimension to recover it, or the use of alternative theorems for volume-preserving maps (see for instance [78]).

By using a standard symbolic computation package, we have computed numerically the Poincaré map with different types of perturbations and it can be said that numerical experiments confirm the main result (see Fig. 6.1).

Let us discuss several alternative paths that we have studied to show the results.

To verify the twist coefficient, we cannot use the common procedure in elliptical fixed points, the Birkhoff normal form. This is because the Poincaré map is not differentiable in the singularity. This is an important mathematical aspect of our result.

With respect the complexification. We tried to complexify before the changes of variables (Section 4.3.1), but it generated two singular planes in $\mathbb{C}^2$ whose intersection is reduced to the singularity at the origin in the real variable, making things more complicated. Therefore the Kelvin transform is very useful since it brings the singularity to the point of infinity and there we can do the estimates.

Another technique that we have tried is to rescale the variables to be able to treat the degree of perturbation in the critical case where the origin is a zero of order 4. Finally, the use of the lemma 4.3.3 helped us with that. In addition, in order to make estimates that may decrease this critical degree, we have also worked on proposing other adaptive geometries of the domains in the complex band. Unfortunately, we did not get any results.

Existence of periodic and quasi-periodic orbits

Our result can be seen as an example of the study of twist dynamics around a singularity in Hamiltonian systems. As a paradigmatic example we choose the point-vortex model. In suitable variables we applied a version of Aubry-Mather theory to get similar results as in the case of exact area-preserving maps of the annulus [45].

We suppose that the origin was a zero of order 4 for the perturbation. This condition played a role in the regularizing change of variable φ. For this reason, it seems unclear how to weaken this assumption.

Our result leaves open the distinction between classical and generalized quasi-periodic solutions. This relies on the nature of the corresponding Mather set with irrational rotation number. Actually it can be either a invariant curve or a Cantor set.

A possible future development of the present work could be finding conditions that break invariant curves. An example of the breaking of invariant curves can be found in [43] for the bouncing ball model, where the author considers the vertical motion of a free falling ball bouncing elastically on a racket moving in the vertical direction according to a regular periodic function f. An

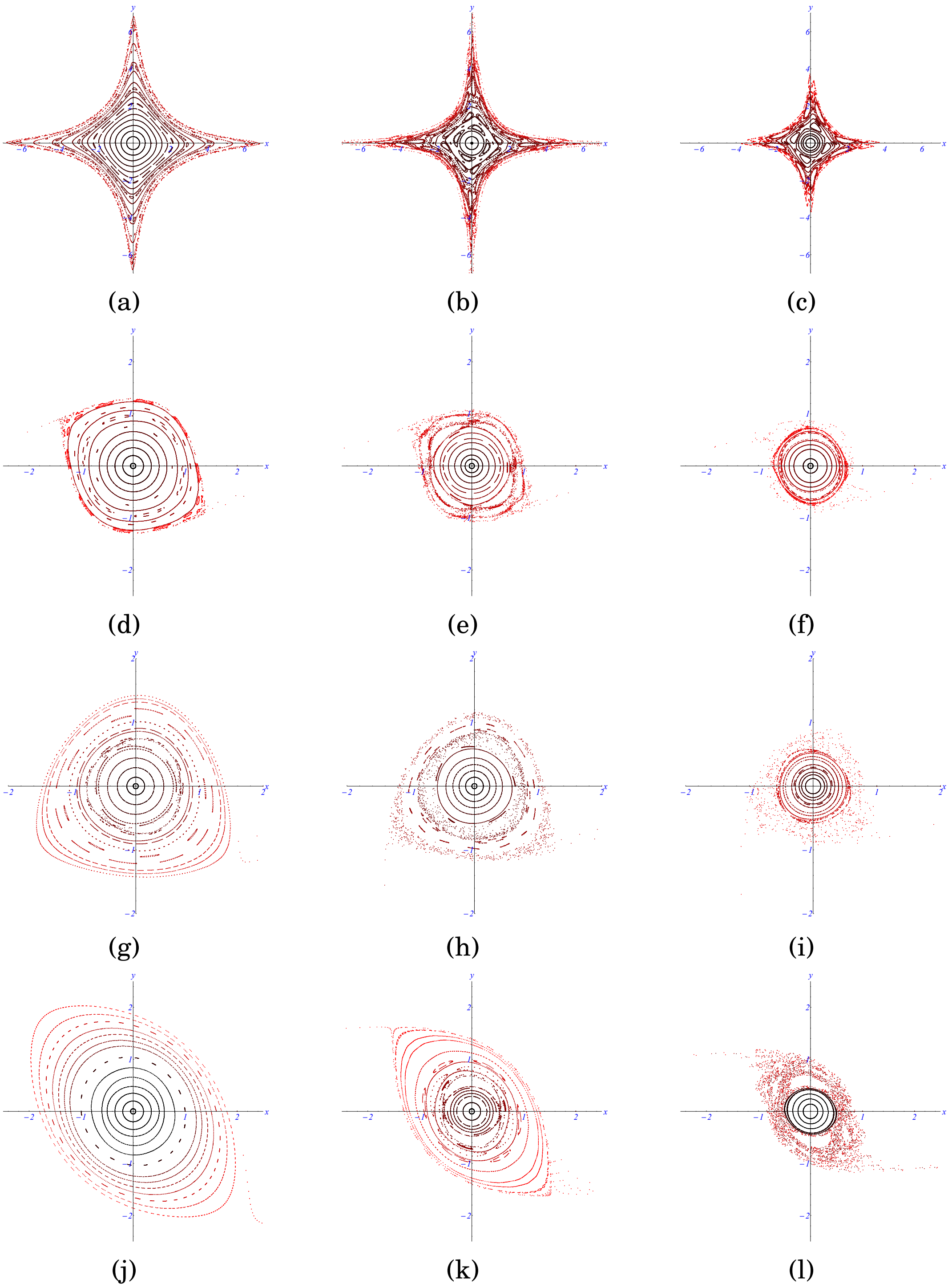

Figure 6.1: Poincaré sections of system (4.0.2) with two different choices of the perturbation $p(t,x,y)$. The first row corresponds to $p(t,x,y) = \epsilon(1+\sin(2\pi t))x^2y^2$ with $\epsilon = 0.1, 0.4, 0.7$, respectively; the second corresponds to $p(t,x,y) = \epsilon(1+\sin(2\pi t))x^3y$ with $\epsilon = 0.1, 0.2, 0.4$; the third one corresponds to $p(t,x,y) = \epsilon(1+\sin(2\pi t))x^2y$ with $\epsilon = 0.1, 0.2, 0.4$ and the fourth row corresponds to $p(t,x,y) = \epsilon(1+\sin(2\pi t))xy$ with $\epsilon = 0.1, 0.2, 0.4$.

advantage of this model is that the generating function can be computed explicitly.

To finish, let us comment that to demonstrate the twist property (Section 5.4) we also tried to apply the Sturm theory on zeros of a solution. Nevertheless, the method with the variational equation gives us a stronger result.

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