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Counting the Ideals with a Given Genus of a Numerical Semigroup with Multiplicity Two

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Abstract: Let S and T be two numerical semigroups. We say that T is an $I(S)$ -semigroup if $T \setminus \{0\}$ is an ideal of S . Given k a positive integer, we denote by $\Delta(k)$ the symmetric numerical semigroup generated by $\{2, 2k + 1\}$. In this paper we present a formula which calculates the number of $I(S)$ -semigroups with genus $g(\Delta(k)) + h$ for some nonnegative integer h and which we will denote by $i(\Delta(k), h)$. As a consequence, we obtain that the sequence $\{i(\Delta(k), h)\}_{h \in \mathbb{N}}$ is never decreasing. Besides, it becomes stationary from a certain term.

Keywords: numerical semigroup; symmetric numerical semigroup; ideal; $I(S)$ -semigroup; genus

MSC: 20M14 (Primary); 05A15 (Secondary)



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1. Introduction

Let \mathbb{Z} be the set of integer numbers and let $\mathbb{N} = \{0, 1, 2, \dots\}$ the set of nonnegative integers. A nonempty subset S of \mathbb{N} is a *numerical semigroup*, if it verifies the following conditions: it is closed under addition, $0 \in S$ and $\mathbb{N} \setminus S = \{x \in \mathbb{N} \mid x \notin S\}$ is finite.

If S is a numerical semigroup, then there exist three important invariants to the study of a numerical semigroups. They are the *multiplicity*, the *Frobenius number* and the *genus* of S , which are defined, respectively, as $m(S) = \min(S \setminus \{0\})$, $F(S) = \max\{z \in \mathbb{Z} \mid z \notin S\}$ and $g(S) = \sharp(\mathbb{N} \setminus S)$, (where $\sharp X$ denotes the cardinality of X).

Given A a nonempty subset of \mathbb{N} , we consider the submonoid of $(\mathbb{N}, +)$ generated by A , that is, $\langle A \rangle = \{\lambda_1 a_1 + \dots + \lambda_n a_n \mid n \in \mathbb{N} \setminus \{0\}, \{a_1, \dots, a_n\} \subseteq A \text{ and } \{\lambda_1, \dots, \lambda_n\} \subseteq \mathbb{N}\}$.

In ([1], Lemma 2.1) it is proven that $\langle A \rangle$ is a numerical semigroup if and only if $\gcd(A) = 1$.

If S is a numerical semigroup where $S = \langle A \rangle$, then A is called a *system of generators* of S . Moreover, if $S \neq \langle B \rangle$ for $B \subsetneq A$, then A will be called a *minimal system of generators* of S . In [1], it is proven that every numerical semigroup admits a unique minimal system of generators. Besides, this minimal system of generators is finite. The minimal system of generators of S will denoted by $\text{msg}(S)$ and its cardinality is called the *embedding dimension* of S and it is denoted by $e(S)$.

A numerical semigroup S is *symmetric* if it verifies the following condition: if $x \in \mathbb{N}$, then $x \notin S$ if and only if $F(S) - x \in S$. In ([1], Corollary 4.7) it is proven that every numerical semigroup of embedding dimension two is symmetric. The introduction of these semigroups is due mainly to Kunz, who proves in [2] that a one-dimensional analytically irreducible Noetherian local ring is Gorenstein if and only if its value semigroup is symmetric. This has motivated that the symmetric numerical semigroups have been the numerical semigroup most studied in the literature.

If $k \in \mathbb{N} \setminus \{0\}$, it is clear that $\Delta(k) = \langle \{2, 2k + 1\} \rangle$ is a numerical semigroup with multiplicity 2 and it is known that every numerical semigroup with multiplicity 2 has this form.

If S is a numerical semigroup, then by Proposition 2.10 from [1], we know that $e(S) \leq m(S)$. When $e(S) = m(S)$, we say that S is a numerical semigroup with *maximal*

embedding dimension (MED-semigroup, see [3]). By Remark 4.21 from [1], we know that $\{\Delta(k) \mid k \in \mathbb{N}\}$ is the set formed by all the symmetric numerical semigroups with maximal embedding dimension.

If A and B are nonempty subsets of \mathbb{Z} , then we will consider the following set $A + B = \{a + b \mid a \in A, b \in B\}$. Let S be a numerical semigroup. An *ideal* of S is a subset nonempty P of S such that $P + S \subseteq P$. The ideals of a numerical semigroup have been extensively studied, see for example [4–7].

If S is a numerical semigroup and P is an ideal of S , then $P \cup \{0\}$ is a numerical semigroup. This fact induces us to give the following definition. A numerical semigroup T is an $I(S)$ -semigroup if $T \setminus \{0\}$ is an ideal of S .

If T is an $I(S)$ -semigroup, then T is a numerical semigroup and $T \subseteq S$. Therefore, $g(S) \leq g(T)$.

In [8] Bras-Amorós computed the number of numerical semigroups with genus g for $g \in \{0, \dots, 50\}$ and she proved that the sequence of the number of numerical semigroup with a fixed genus less than or equal to 50 has a behaviour similar to Fibonacci sequence. She also surmised that, in general, for a fixed integer g there are more numerical semigroups with genus $g + 1$ than numerical semigroups with genus g . Despite extensive research into this problem, it remains to be solved. For this reason, determining the number of numerical semigroups with a given genus has become a question of great interest (see for instance [8–14]).

If $h \in \mathbb{N}$, we denote by $\mathcal{J}(S, h) = \{T \mid T \text{ is an } I(S)\text{-semigroup and } g(T) = g(S) + h\}$. Denote by $i(S, h)$ the cardinality of $\mathcal{J}(S, h)$.

In ([15]) we have conjectured the following: if S is a numerical semigroup, then there exists $n_S \in \mathbb{N}$ such that if $0 \leq a < b \leq n_S$, then $i(S, a) \leq i(S, b)$ and $i(S, n_S + n) = i(S, n_S)$ for all $n \in \mathbb{N}$.

A numerical semigroup S is *ordinary* if $S = \{0, m(S), \rightarrow\}$ (the symbol \rightarrow means that every integer greater than $m(S)$ belongs to the set). In [15] we also show that the previous conjecture is true for ordinary numerical semigroups.

The purpose of the present work is to demonstrate that, given k a positive integer and $\Delta(k) = \langle \{2, 2k + 1\} \rangle$, then

$$i(\Delta(k), h) = \begin{cases} \lceil \frac{h+2}{2} \rceil & \text{if } h \leq 2k - 1, \\ k + 1 & \text{if } h \geq 2k - 1, \end{cases}$$

where $\lceil q \rceil = \min\{z \in \mathbb{Z} \mid q \leq z\}$ for all $q \in \mathbb{Q}$.

Therefore, if $0 \leq a < b \leq 2k - 1$, then we have $i(S, a) \leq i(S, b)$ and $i(\Delta(k), 2k - 1 + n) = k + 1$ for all $n \in \mathbb{N}$.

This result proves that the conjecture proposed by M. Bras-Amorós in [8] is true for this family of symmetric numerical semigroups. Moreover, we have taken a step forward towards the resolution of the conjecture, since our result asserts that there exist families of numerical semigroups where the number of semigroups with genus g is equal to the number of semigroups with genus $g + 1$ from some genus.

The contents of this work are organized as follows. In Section 2 we give some concepts and results used during this work. All of them appear in [15]. In Section 3 we show a characterization of $\Delta(k)$ -incomparable set. This is the content of Proposition 4. Afterwards, we obtain the expression of genus of an $I(S)$ -semigroup, according to the $\Delta(k)$ -incomparable set, X . Finally, we prove Theorem 1, which is the main result of the work. We finish the Section and the paper by providing an example illustrating the theorem. Specifically, this example calculates the monotone increasing sequence of the number of $I(\Delta(5))$ -semigroups with a fixed genus. In this example we can see that the sequence becomes stationary in the genus 9.

2. Results and Basic Concepts

In Proposition 1 from [15], appears the following result.

Proposition 1. Let S be a numerical semigroup and let X be a subset nonempty of S , then $X + S$ is an ideal of S . Moreover, every ideal of S has this form.

Given S a numerical semigroup, then we define the following order relation in S : $a \leq_S b$ if $b - a \in S$.

A S -incomparable set is a subset nonempty X of S verifying that $x - x' \notin S$ for all $(x, x') \in X \times X$ such that $x \neq x'$.

The content of the following proposition is Theorem 5 from [15].

Proposition 2. If S is a numerical semigroup, then the set

$$\{X + S \mid X \text{ is a } S\text{-incomparable set}\}$$

is the set composed by all the ideals of S . Moreover, if X and Y are two S -incomparable sets with $X \neq Y$, then $X + S \neq Y + S$.

The following result is Proposition 7 from [15].

Proposition 3. Let S be a numerical semigroup and let X be a S -incomparable set, then X is finite.

3. The Main Result

Given $k \in \mathbb{N} \setminus \{0\}$, the numerical semigroup generated by $\{2, 2k + 1\}$, will be denoted by $\Delta(k)$. Our first purpose in this section will be to show how the $\Delta(k)$ -incomparable sets are.

The Frobenius problem for numerical semigroups consists in finding formulas for the Frobenius number and genus of a numerical semigroup by using its minimal system of generators. This problem was solved by Sylvester in [16] for numerical semigroups with embedding dimension two. Specifically, Sylvester proves that if a and b are positive integers greater than or equal to two and $\gcd\{a, b\} = 1$, then $F(\langle\langle\{a, b\}\rangle\rangle) = ab - a - b$ and $g(\langle\langle\{a, b\}\rangle\rangle) = \frac{(a-1)(b-1)}{2}$. At the present time, the Frobenius problem is still open for embedding dimension greater than or equal to three (see [17]). The following result has an immediate proof.

Lemma 1. If k is a positive integer, then

$$\Delta(k) = \{2x \mid x \in \mathbb{N}\} \cup \{2x + 2k + 1 \mid x \in \mathbb{N}\}.$$

Moreover, $F(\Delta(k)) = 2k - 1$ and $g(\Delta(k)) = k$.

Proposition 4. Let X be a nonempty subset of \mathbb{N} . Then X is a $\Delta(k)$ -incomparable set if and only if one of the following conditions is verified:

- (1) $X = \{a\}$ for some $a \in \mathbb{N}$ such that a is even.
- (2) $X = \{a\}$ for some $a \in \mathbb{N}$ such that a is odd and $a \geq 2k + 1$.
- (3) $X = \{a, b\}$ with $\{a, b\} \subseteq \mathbb{N}$, a is even, b is odd, $a \geq 2$, $b \geq 2k + 1$ and $|a - b| \leq 2k - 1$.

Proof. (Necessity). Initially, note that if $\{x, y\} \subseteq X$ and $x \neq y$, then x and y have different parity. Therefore, if X has cardinality greater than or equal to two, then $X = \{a, b\}$ where $\{a, b\} \subseteq \mathbb{N}$, a is even, b is odd, $a \geq 2$ and $b \geq 2k + 1$. In addition, $\{a - b, b - a\} \cap \Delta(k) = \emptyset$ and so $|a - b| \leq 2k - 1$.

(Sufficiency). Trivial. \square

The following result is deduced from Proposition 9 in [7].

Lemma 2. Given S a numerical semigroup, $a \in S \setminus \{0, 1\}$ and we consider the set $T = (\{a\} + S) \cup \{0\}$, then $F(T) = F(S) + a$, $g(T) = g(S) + a - 1$ and $m(T) = a$.

Proposition 5. Let $X \neq \{0\}$ be a $\Delta(k)$ -incomparable set and we consider the numerical semigroup $S = (X + \Delta(k)) \cup \{0\}$. Then

- (1) If $X = \{a\}$, then $g(S) = g(\Delta(k)) + a - 1$.
- (2) If $X = \{a, b\}$ where a is even and b is odd, then

$$g(S) = g(\Delta(k)) + \frac{a-2}{2} + \frac{b-2k-1}{2}.$$

Proof. (1) It is a trivial consequence from Lemma 2.

- (2) By Proposition 4, $a \geq 2$, $b \geq 2k + 1$ and $|a - b| \leq 2k - 1$. And by Lemma 1,

$$g(S) = g(\Delta(k)) + \#\{x \in \mathbb{N} \mid 2 \leq x \leq a - 2 \text{ and } x \text{ is even}\} + \#\{x \in \mathbb{N} \mid 2k + 1 \leq x \leq b - 2 \text{ and } x \text{ is odd}\}$$

because $b \leq a + 2k - 1$ and $a \leq b + 2k - 1$. Therefore the genus of S is $g(S) = g(\Delta(k)) + \frac{a-2}{2} + \frac{b-2k-1}{2}$.

□

At this point, after these results, it is possible for us to validate the Theorem announced in the Introduction.

Theorem 1. If $h \in \mathbb{N} \setminus \{0\}$, then

$$i(\Delta(k), h) = \begin{cases} \lceil \frac{h+2}{2} \rceil & \text{if } h \leq 2k - 1, \\ k + 1 & \text{if } h \geq 2k - 1. \end{cases}$$

Proof. (1) In the first place, we focus on the calculation of all the elements of the set $\mathcal{J}(\Delta(k), h)$ which have the form $S = (\{a\} + \Delta(k)) \cup \{0\}$, being $a \in \Delta(k)$. By 1) from Proposition 5, $g(S) = g(\Delta(k)) + a - 1$. Thus, if $S \in \mathcal{J}(\Delta(k), h)$, then $a = h + 1$. As $a \in \Delta(k)$, then by applying Lemma 1, we deduce the following:

- If h is odd, then $\mathcal{J}(\Delta(k), h)$ contains a unique element which has the shape $(\{a\} + \Delta(k)) \cup \{0\}$.
 - If h is even and $h + 1 \leq 2k - 1$, then $\mathcal{J}(\Delta(k), h)$ does not contain any element with the form $(\{a\} + \Delta(k)) \cup \{0\}$.
 - If h is even and $h + 1 \geq 2k + 1$, then $\mathcal{J}(\Delta(k), h)$ contains a unique element with the shape $(\{a\} + \Delta(k)) \cup \{0\}$.
- (2) In the second place, let us see how many elements there are in $\mathcal{J}(\Delta(k), h)$ which have the form $S = (\{a, b\} + \Delta(k)) \cup \{0\}$, where a is even, b is odd, $a \geq 2$, $b \geq 2k + 1$ and $|a - b| \leq 2k - 1$.

By 2) from Proposition 5, we know that $\frac{a-2}{2} + \frac{b-2k-1}{2} = h$ and, in consequence we have $a + b = 2h + 2k + 3$. Therefore, $(\{a, b\} + \Delta(k)) \cup \{0\}$ belongs to $\mathcal{J}(\Delta(k), h)$ if and only if $a + b = 2h + 2k + 3$, $a \geq 2$, $b \geq 2k + 1$, a is even, b is odd and $|a - b| \leq 2k - 1$. We distinguish the following cases:

- As $b \geq 2k + 1$, then $a \leq 2h + 2$.
- As $b = 2h + 2k + 3 - a$ and $b - a \leq 2k - 1$, then $h + 2 \leq a$.
- As $b = 2h + 2k + 3 - a$ and $a - b \leq 2k - 1$, then $a \leq h + 2k + 1$.

As a consequence of these three previous cases, we can say that the number of elements of $\mathcal{J}(\Delta(k), h)$ which have the form $(\{a, b\} + \Delta(k)) \cup \{0\}$ coincides with the cardinality of the set

$$\{a \in \mathbb{N} \mid a \text{ is even and } h + 2 \leq a \leq \min\{2h + 2, h + 2k + 1\}\}.$$

Note that $\min\{2h + 2, h + 2k + 1\} = \begin{cases} 2h + 2 & \text{if } h \leq 2k - 1, \\ h + 2k + 1 & \text{if } h \geq 2k - 1. \end{cases}$

Now, we are going to study these two cases:

- If $h \leq 2k - 1$, then $\#\{a \in \mathbb{N} \mid a \text{ is even and } h + 2 \leq a \leq 2h + 2\} = \lfloor \frac{h+2}{2} \rfloor$, where $\lfloor q \rfloor = \max\{z \in \mathbb{Z} \mid z \leq q\}$ for all $q \in \mathbb{Q}$.
 - If $h \geq 2k - 1$, then $\#\{a \in \mathbb{N} \mid a \text{ is even and } h + 2 \leq a \leq h + 2k + 1\} = k$.
- (3) Finally, we are able to affirm the following:
- If $h \leq 2k - 1$ then, according to previous points 1) and 2), we deduce that

$$i(\Delta(k), h) = \begin{cases} \lfloor \frac{h+2}{2} \rfloor + 1 & \text{if } h \text{ is odd,} \\ \lfloor \frac{h+2}{2} \rfloor & \text{if } h \text{ is even.} \end{cases}$$

So $i(\Delta(k), h) = \lceil \frac{h+2}{2} \rceil$.

- If $h \geq 2k - 1$ then, according again to the previous points 1) and 2), we assert that $i(\Delta(k), h) = k + 1$.

□

Example 1. The increasing sequence of the number of $I(\Delta(5))$ -semigroups with a fixed genus is the following:

- $i(\Delta(5), 0) = 1,$
- $i(\Delta(5), 1) = i(\Delta(5), 2) = 2,$
- $i(\Delta(5), 3) = i(\Delta(5), 4) = 3,$
- $i(\Delta(5), 5) = i(\Delta(5), 6) = 4,$
- $i(\Delta(5), 7) = i(\Delta(5), 8) = 5,$
- $i(\Delta(5), h) = 6$ for all $h \geq 9$.

4. Conclusions

This paper is devoted to study the number of $I(S)$ -semigroups with a fixed genus for a particular case of symmetric numerical semigroup S . That is, numerical semigroups which have the form $\Delta(k) = \langle \{2, 2k + 1\} \rangle$, being $k \in \mathbb{N}$. We have proved that, in this case, the sequence of the number of $I(\Delta(k))$ -semigroups with a fixed genus, follows a nondecreasing sequence.

We have performed many calculations for this problem and all of them show that this sequence is a nondecreasing sequence for every numerical semigroup considered. For this reason, in future works, we are going to study the behaviour of this sequence in other families of numerical semigroups. Also, we want to obtain the formula of the bound where the sequence becomes stationary.

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