Tesis doctoral de **Enrique Martín Fernández**

Meanings shown by students and teachers in training on the sine and cosine of an angle

DOCTORAL THESIS

Meanings shown by students and teachers in training on the sine and cosine of an angle

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Abstract

Meaning and understanding are notions used in didactic to work on concepts, comprehension, curricular design, and knowledge assessment. This document aims to delve into the meaning of school mathematical concepts through their semantic analysis. This analysis is used to identify and establish the basic meaning of mathematical concepts and to value their understanding. We gathered the study data through a semantic questionnaire, and analised the responses using an established framework, included in the didactic analysis, that is developed along the dissertation. To illustrate the study, we have chosen the trigonometry relational system. Understanding the trigonometry relational system is one of high school mathematics most demanding topic. The angle, the unit circle, and the trigonometric functions are its foundational notions. Trigonometric contents meaning and their understanding involve these three concepts and their relationships. The study involves two stages.

The aim of the first stage is to analyse the representations, concepts, notions, and the senses handled by secondary school students when describing the sine and cosine of an angle. This part of the report exemplifies some findings of an exploratory study carried out with high school students between 16 and 17 years of age on the several ways of expressing and interpreting the trigonometric notions aforementioned; it collects the variety of emergent notions and elements related to the trigonometric concepts involved when answering on the categories of meaning which have been asked for. From the analysis on the answers of this group of students emerges a categorization, whose relations are discussed and interpreted. The results show several types of representations and senses, some of which have already been recognized in previous studies, while some others are new. The subjects provide a diversity of meanings, interpreted and structured by semantic categories. These meanings underline different understandings of the sine and

cosine, according to the inferred themes, such as length, ratio, angle and the calculation of a magnitude.

Secondly, this research also aims to provide evidence on how pre-service mathematics teachers use trigonometric concepts, and to give examples and arguments that explain how they move between partial goniometry and partial analytic geometry systems; which structures and strategies use, and how they convert notions between representation systems. We characterize responses, as well as organized and interpreted the data obtained. The results indicate that participants' meanings of the angle concept mediate their understanding of the conversions between the trigonometric representation systems involved. The scarcity of research related with school meaning of trigonometric contents provides an extra interest to the study.

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1. INTRODUCTION, STATEMENT OF THE PROBLEM, AND AIMS OF THE RESEARCH

1.1. Introduction

"Everything must have a beginning, […] and that beginning must be linked to something that went before" (Mary Shelley, Frankenstein)

The planning, design, realization, and implementation in the classroom of a didactic unit on trigonometry for 4th ESO (Enseñanza Secundaria Obligatoria), within an initial training programme for secondary school teachers, was the origin of this report, and the first phase of the approach to the research problem in the year 2010-2011. The research is structured around the theoretical perspective of didactic analysis (Rico, Lupiañez & Molina, 2013).

In this first phase of the study, from a secondary teacher's point of view, different analysis of the didactic contents involved were carried out: conceptual analysis, meaning analysis, cognitive analysis, instructional analysis, and performance or evaluative analysis.

During the Master's Degree in Didactics of Mathematics at the University of Granada in the 2012-2013 academic year, a study was designed and carried out with students of Grade 11th. Our focus was on the three semantic categories of the meanings analysis (conceptual structure, systems of representation, and sense). These categories form a triad, giving rise to the semantic triangle that establishes the meaning of a school content. To identify the diversity of meanings for the sine and cosine of an angle, a semantic questionnaire was designed, based on the aforementioned semantic triad. In addition, a study of the historical evolution of this topic was carried out, which made it possible to distinguish different types of instruction. Through data analysis, we came to define themes, subthemes, and finally identified and characterized the meanings attributed by high school students to the sine and cosine of an angle. A synthesis of part of the work carried out and its results was presented at the XVIII SEIEM (Sociedad Española de Investigación en Educación Matemática) symposium as communication, and at the 38th PME (Psychology of Mathematics Education) as short communication, both in 2014. This work is presented in more depth in two papers published in the journals: *Enseñanza de las ciencias and Eurasia Journal of Mathematics, Science and Technology Education*.

Using the results obtained, we designed a second semantic questionnaire to analyse the meanings and content evoked by teachers in training on the sine and cosine of an angle. We characterized in detail different concepts of angle in the plane according to the modalities of their semantic components. We singled out three main types: absolute angles, oriented angles, and analytical angles, each characterized by its corresponding representation system. We established their measurements, and we studied the conversion processes between them, delimiting a trigonometry relational system made up of those types of angles, their conversion processes, the construction of their trigonometric lines and the relationships between them.

Next, the statement of the problem will be described, and the research question and the aims of this thesis will be formulated.

1.2. Statement of the problem

Trigonometry is an interesting, unifying topic for high school mathematics. "It is conceptually rich and contains connections with other mathematical ideas and structures" (Fi, 2003, p. 13). Trigonometry uses notions from various parts of mathematics that allow students to develop skills, reasoning, and strategies to solve problems (Sarac & Aslan-Tutak, 2017, p. 70). In fact, Tuna (2013) stated that "trigonometry is important in terms of improving students' reasoning skills" (p. 1). What is more, "the teaching of trigonometry theorems and concepts is important to develop students' creative, logical and analytical thinking skills" (Dündar, 2015, p. 1380). Furthermore, according to the Mathematical Association of England, "trigonometry fuses arithmetic, algebra, geometry and mechanics" (MA, 1950, p. 3), which gives us an idea of the relevance of this topic in secondary school education. Finally, trigonometry has many applications in different disciplines (Army, 1991).

Despite its importance, trigonometry is a part of mathematics that is difficult for students to understand (De Kee, et al., 1996; Maldonado, 2005; Tuna, 2011). There are many factors related to this fact such as its conceptual complexity, its various relational subsystems, the connection between them, the various approaches to them, the ways of approaching and representing their basic notions, the great diversity of contexts, modes of use and phenomena in which it participates, etc.

Some parts related to trigonometry have been studied in depth, such as the history of its teaching in some countries. However, at the beginning, we found little research on what make trigonometry difficult, and on the intuitive ideas that students have about trigonometric concepts (Brown, 2005, p. 10). As Weber (2005) said, the research that tells us how to overcome students' difficulties in trigonometry is scarce and dispersed. Despite its wealth of concepts, representations, problem-solving opportunities, which implies reasoning and proof abilities, and connections with other mathematical ideas, Koyunkaya (2016, p. 2) points out that there is little interest in the study of the concepts of the trigonometry relational system because it is not an extensive part of the mathematics curriculum. Furthermore, it suggests that the understanding of trigonometric ratios should be investigated. Similarly, Chin (2013) states that: "it would be fruitful to investigate the transition from circular trigonometry to analytical trigonometry in future research" (p. 253).

Additionally, in order to build a solid and coherent framework of trigonometric content, it is important to investigate the links between the different trigonometric notions (Brown, 2005). Demir (2012) states that the graphs of the sine and cosine functions continue to be full of mystery, highlighting a deficit related to different trigonometric notions, and therefore a research gap. Weber (2005, p. 103) emphasizes the importance of the role that geometric figures play in understanding trigonometric functions. In fact, "it is necessary for students to construct the geometric objects of trigonometry as tools for reasoning" (Demir, 2012, p. 104). However, Flores et al. (2015) indicate that it is important to consider "not only how students can be supported, but also how their teachers can be better prepared to support and help their students" (p. 278). In fact, Dündar (2015, p. 1382) highlights that teachers in training must have the ability to make use of a variety of representations, to establish links between them, and to be unable to make the desired transitions.

Finally, due to the necessary knowledge of the trigonometry relational system for other mathematical topics, and in light of the opportunities for reasoning between trigonometric representations involved, more attention must be paid to understand trigonometric notions. In addition, to improve the initial training on trigonometry relational system of secondary school teachers, the connections between different partial trigonometry systems will be investigated.

1.3. Structure of the Thesis

In order to facilitate the reading and to comply with the requirements established for writing a thesis, this report is structured in 7 chapters. Below we describe the content of each of them.

After this section, the introduction chapter ends with the definition of the research questions and with the proposal of the objectives of this report.

The second chapter presents a synthesis of previous research related to the trigonometry relational system. Specifically, we focus on those that refer to each of the categories of the meaning of a school mathematical content, the instructional strategies used in high school students, the conversions between partial trigonometry systems and the existing research on teachers in training.

In the third chapter, the theoretical framework of this study is presented in detail. We firstly frame our research in the didactic analysis and emphasize the importance of meaning. Then, we mention some semantic triads previously used. After that, we present the Rico's (2018) characterization for the meaning of school mathematics concepts and the Freudenthal's (1973) about school trigonometry relational system. Finally, we highlight the trigonometry relational system.

The fourth chapter provides details of the research design, research instruments, and methods of data analysis used in this dissertation. The characteristics of the samples, their selection and the implementation of the research instruments will also be explained.

The fifth and sixth chapter include the data analysis and results collected from the questionnaires. The data collected from the questionnaires is analysed by means of themes, subthemes…etc. These are related to the semantic categories which are based on the theoretical framework presented in chapter three.

This thesis ends with a conclusion and discussion chapter. We present the conclusions that are derived from the development of this research. We express an overview and discussion of the results and their link to the research aims. Then, limitations of the study are presented because of issues of methodology. As a consequence of this study, suggestions for future research are proposed.

1.4. Research questions

This section presents the research questions explored in this dissertation. The questions emerge from the review of the literature, and from my personal interest which stems from training programme for secondary school teachers. In fact, a review of the literature on trigonometry relational system allows us to appreciate that the meaning of trigonometric contents is a little explored area. Its identification and its importance for the understanding of that system has not been deepened. Therefore, the following questions arise:

- How do secondary school students express the sine and cosine through its conceptual structure, its representation systems and its sense?
- What typologies of meaning regarding the sine and cosine of an angle do secondary school students provide?
- How do the pre-service teachers represent an angle, its cosine and how do they give meaning to them in the partial goniometry system?
- What contents do pre-service teachers utilize to represent a point P as an angle, and how do they convert this related angle from the partial goniometry system to the partial analytic geometry system, and how do they give meaning to the cosine of that angle in the partial analytic geometry system?
- How does the meaning that pre-service teachers attribute to the angle concept in the goniometry system determine the moving from this partial system to the analytic geometry partial one?
- Is the used framework useful for examining and explaining the understanding of concepts and procedures of the trigonometry relational system?

These questions guide the general objectives of our research.

1.5. The purpose of the research

The general aim of the research is to explore and describe the meanings revealed by secondary school students and pre-service school teachers about the sine and cosine of an angle, when evoking previously studied knowledge.

In addition, the following are set as specific aims:

Aim 1. To build a valid and reliable instrument to identify and gather the meanings revealed by secondary school students, following established methodological criteria.

Aim 2. To build a valid and reliable instrument to identify and gather the meanings revealed by secondary school teachers in training, following established methodological criteria.

Aim 3. To identify, describe and interpret the meanings about the sine and cosine of an angle that schoolchildren show when high school students respond to tasks strongly connected with each of the categories of meaning according to the perspective of Rico (2013).

Aim 4. To identify the meanings about the angle concept and its cosine shown by secondary school teachers in training, and to describe the conceptual and procedural content when moving between the partial goniometry system and the partial analytic geometry system.

Aim 5. To investigate the understanding of secondary school students and teachers in training on these contents by means of the characterization of their meanings and components.

Aim 6. To examine the influence of the interpreted meanings for the angle concept in teachers in training.

2.LITERATURE REVIEW

2.1. Introduction

> *"Before you begin to rock the boat, be sure you are in it"* (Wolcott, 2001, p. 71).

In the last decade, research has been developed on the learning and teaching of trigometry. Therefore, we are going to review the studies carried out on this topic, and the problems that researchers have tried to answer through the trigonometry relational system. The literature related to the trigonometry relational system that is presented in this chapter is divided into seven sections. Because studies about meanings of the trigonometry relational system in secondary school students are limited, we underline the current results about the categories of the semantic triad, highlighting the existing literature about this issue. In fact, although there have been multiple studies that present useful results associated with students´ difficulties with this topic, more research is needed on students´ meaning of trigonometric concepts (Moore & Laforest, 2014; Koyunkaya, 2016). This work is located within the research on the meanings used in the teaching and learning of the trigonometry relational system, to which little attention has been paid (Byers, 2010, p. 1).

Thus, we will first take into account the investigations with students on each of the categories of the semantic triad: research on the knowledge and understanding of the contents -conceptual structure and procedures- of the trigonometry relational system, research on representations of the sine and cosine of an angle, and finally studies that include the senses of the sine and cosine of an angle. After that, given that the meanings shown by the students are linked to various factors, among which the teaching methodology stands out, we will review different instructional strategies.

Finally, we review the research focused on pre-service teachers, and the literature on the understanding of conversions between trigonometric notions through the trigonometry relational system. We highlight in the summary section that this literature has the purpose of providing a literature base for this research.

2.2. Research on mathematical structure of the trigonometry relational system

Research on the mathematical concepts structure by secondary school students has generated some information related to trigonometry relational system. Firstly, the origin of the problems seems to come from a poor knowledge of main concepts such as angles, measuring angles, and the unit circle (Brown, 2005; Challenger, 2009; De Kee, et al., 1996; De Villiers & Jugmohan, 2012; Gur, 2009; Orhun, 2004; Thompson, 2007).Indeed, De Villiers and Jugmohan (2012) express that "learners appear to have little understanding of the underlying trigonometric principles and thus resort to memorizing and applying procedures and rules, while their procedural success masks underlying conceptual gaps or difficulties" (p. 9). Moreover, they are unable to use their own words in order to define trigonometric concepts or formulate their own knowledge (Challenger, 2009). Similarly, Gur (2009) also claimed that the majority of students only memorized definitions for trigonometry relational system, and had no understanding of what those definitions actually meant and their knowledge was not retained long term. Thus, research reveals students have no strong enough mathematics background with understanding on trigonometric concepts (Challenger, 2009).

As for the ideas involved in learning about trigonometry relational system, whereas Brown (2005) created a content framework to describe the trigonometric mathematical content, and the forms in which secondary students understand its main ideas, ground on the triangle and the unit circle, Demir (2012) presented in his thesis a model of trigonometric understanding based on developing coherent connections among three different contexts of the trigonometry relational system: right triangle trigonometry, unit circle trigonometry, and trigonometric function graphs. In addition, he said that "it is difficult to find a firm framework for trigonometry understanding in the literature." (p.16)

Finally, findings illustrate that trigonometry relational system is more strongly related to the triangle and to a lesser extent with the unit circle.

2.3. Research on representations of the sine and cosine of an angle

We provide some insight into the used representations by students when they dealt with trigonometry relational system. Challenger (2009) said that the ideas of trigonometry relational system are mediated by algebraic representations like ratios and formulae, and by graphical representations such as triangles, circumferences or graphs. Marchi (2012) argued that the right triangle, the unit circle, the graphs, and the equations are closely related to the trigonometric representations. Finally, Byers (2010) stated that when trigonometric notions are represented a variety of themes appears: the right triangle, the trigonometric ratios, the trigonometric function, the unit circle, the sinusoidal waveform, and vectors.

In previous works, several ways in which students can represent sine and cosine are connected with a variety of meanings (Weber, 2005). With regard to these representations, they are unique to the trigonometry relational system

(Sickle, 2011). In this way, Brown (2005) claimed that the most usual way to represent them are as coordinates of the terminal point on the unit circle, as directed distances from the horizontal and vertical axes on the unit circle, and as ratios of sides of a right-angle triangle. Otherwise, Weber (2008) stated that the sine and cosine are mainly represented by means of ratios and functions.

2.4. Research on sense of the sine and cosine of an angle

Principally, research on textbooks and on answers of secondary school maths teachers have generated information that contributes to the study on sense (Allen, 1977; Sickle, 2011; Hertel, 2013; Tavera & Villa-Ochoa, 2016). The sense is the semantic category related to make trigonometry more 'real' to the students, through the use of the trigonometric concepts in real world scenarios.

Sickle (2011, p. 189) mentioned that at the beginning of the twentieth century, the most frequently cited topics in trigonometry relational system textbooks were calculus, surveying, and navigation. Similarly, Allen (1977) stated that students from USA and Canada, from 1890 to 1970, dealt with surveying, carpentry, and ballistics using mainly triangles in trigonometry courses. Tavera and Villa-Ochoa (2016) reported the misuse of the term function and ratio in textbooks. Besides, they said that the trigonometry relational system is mostly utilized for finding out missing sides in a right-angle triangle. This last aspect is also mentioned in Challenger's work (2009). Indeed, Thompson (2007) pointed out that "students understanding is relegated only to exist in the mathematics classroom and concepts lack any connection to their life in the world" (p.14).

The wider study regarding the sense is responsibility of Hertel (2013). He stated that the triangle dominates the real-world activities in high school. Similarly, Dogân (2001) argued that in spite of being the trigonometry relational system a relevant topic of secondary school, as it is utilised in daily life, trigonometry is not positioned accordingly. In line with Dogän (2001), Kamber and Takaci (2018) in their respective research express that students ignore in which practical situations trigonometry relational system is used. However, nowadays, modern sciences (biology, physical, and social sciences) utilize frequently trigonometry relational system to performance model periodic phenomena. It is clearly important to know the role of the trigonometry relational system in the mathematics curriculum, and in the modern sciences to offer a wide variety of tasks to the students. Indeed, students should recognize phenomena with periodic features (NCTM, 2000). In addition to this, given that different ways of experiencing phenomena or concepts can be representative of different capabilities of dealing with those concepts, it is obvious that there is a gap on real-world experiences in trigonometry which should be studied to improve the teaching and learning of the trigonometry relational system (Thompson, 2007; Hertel, 2013).

2.5. Research on instructional strategies about trigonometry relational system

Numerous studies have investigated instructional strategies to help students and teachers connect the different trigonometric representations (Kendal & Stacey, 1997; Weber, 2005; Demir, 2012; Moore, 2014; Fanning, 2016). Moore (2012) argues that trigonometry relational system is a part of mathematics that lacks coherence in its teaching due to the difficulties that students and teachers present in its use in multiple contexts. These problems suggest that the approach to this topic of mathematics does not facilitate that students and teachers establish connections between its different components of meaning. Thus, Moore (2014) proposes that trigonometry should be taught firstly using a circle approach, then, connecting with the trigonometric function and the triangle trigonometry so that students will acquire and develop more comprehensive knowledge. He suggests that it is more productive to present the unit circle rather than providing definitions of the triangle trigonometry (static definition) given that this approaching allows students to achieve a solid covariational, and quantitative thinking. He argues that using the unit circle enables students to have a better preparation for learning not only for the rest of representations related trigonometry relational system, but also for more advanced mathematical contents. On the other hand, following the suggestion of *The Common Core State Standards for Mathematics* (2010), Kendal and Stacey (1997) recommend that teachers to use the ratio method to approach trigonometry relational system, defining trigonometric ratios as ratios of lengths of sides in right angle triangles (ratio system). This is due to the higher results obtained by students who were taught using this method in contrast to students who were taught using the unit circle method. Weber (2005) also proposes using the unit circle method or "line system" (in which the cosine and sine are defined as the x and y coordinates of a point on the goniometric circle) to approach the trigonometry relational system given that it is more useful for its learning, particularly trigonometric functions. Regarding the last aspect, Kendal and Stacey (1997) indicate that although a long term benefit of being teaching using the unit circle is provided, the benefit for students with low-ability is another advantage so as to introduce trigonometry relational system using the ratio method. For these reasons, they both suggest that both methods are necessary to build meanings in students of the trigonometry relational system so that pupils can construct a solid and coherent foundation for advanced mathematical thinking. Finally, Fanning (2016) and Demir (2012) report on mathematical alternatives of the trigonometry´s relational system instruction. Concretely, the rational trigonometry, and a designed lesson sequence based on a new theoretical approach respectively. On the one hand, rational trigonometry implies replacing the units of distance and angle with the unit of quadrance and spread. However, these units are less intuitive and more research is needed about this new hypothetical instruction. On the other hand, Demir (2012, p. 124) proposes the use of arcs of a unit circle, and of a metaphor of travelling along the rim of a geometric object to develop coherent meanings based on arcs of a unit circle. However, there is considerable need for further research given that one of his difficulties was about assessing students′ understanding, which was based on an untested framework of understanding.

Other studies that have used real situations, graphing calculator and dynamic geometry software to approach the topic, have concluded that they help to establish numerical and geometric relationships (Army, 1991; Blackett & Tall, 1991; Thompson, 2007; Zengin, et al., 2012), and thus facilitating the mastery of concepts and the construction of meanings.

2.6. Research literature on trigonometry conversions

The ability of high school students to convert trigonometric notions and to move between partial trigonometry systems has been investigated by recent research, although this topic has been neglected for a long period of time (Brown, 2005, p. 26; Byers, 2010, p. 1). Firstly, Brown (2005, p. 225) created two models, one to describe the content, and the another to describe how the participants understood trigonometric content based primarily on the movement between the partial elementary geometry system and the partial goniometry system. According to Brown (2005, p. 233), the understanding of the participants was very weak and disjointed because they were unable to interpret the cosine as ratios of sides lengths of right-angle triangles, as neither distances nor directed coordinates. Participants do could not process and convert notions. Furthermore, she highlighted the inability of the students to connect the partial goniometry system, and the partial analytic system. Secondly, Marchi (2012, p. 43) also examined students' understanding, the representations of the sine and what connections students have among their representations. He posited that although students possess a great deal of knowledge for each individual representation, the connections among them are not consistent enough to allow students to develop a deep understanding of the sine. Additionally, Demir (2012, p.121) explored the effect of a new learning trajectory for trigonometry relational system, based on a new theoretical approach grounded on a conceptual analysis of trigonometry relational system. Although he points out that students appear to understand the links between three partial trigonometry systems based on the right-angled triangle, the unit circle and the trigonometric functions, his results seem to show that students are only making short ranged connections between them. These results are aligned with Challenger (2009, p. 126, p. 137, p. 184) in whose research participants seem to hold a scarcely developed functional reasoning, due to the tendency of students to use the triangle and operational aspects of trigonometry relational system. He mainly drew on the theoretical frameworks proposed by Sfard (1991) and Dubinksky (1991) for studying the development of students' understanding. Most relevant to our study, in his doctoral thesis, Chin (2013, p. 68) explored how student teachers cope with the changes of meaning in trigonometric concepts to reflect how this may impact on their future teaching. This research is based on the theoretical framework of Tall (2004, 2013) that intends to make sense of mathematics (perception, operation and reason); on the changing of meaning in mathematics that allows looking at trigonometry as three partial systems, and on the notion of extensional blend. He concludes that subjects do not build coherent links across partial goniometry system, and the partial analytic geometry system. Finally, Martinez-Planell and Cruz Delgado (2016) analyse the mental constructions when developing an approach to the sine and cosine and their inverse trigonometric functions restricted to the unit circle.

2.7. Research literature on pre-service teachers in trigonometry relational system

Since studies about pre-service secondary school teachers´ knowledge in trigonometry relational system are sparse, our view now focuses on that existing literature. Firstly, Fi (2003) sheds light on the pedagogical content knowledge of the trigonometry relational system, and the knowledge of this relational system of pre-service secondary school teachers. Fi's work about pre-service secondary school teachers is the widest existing study. He investigates different issues such as: definitions and terminology; angles of rotations, co-terminal angles and reference angles; trigonometric functions and their graphs; domain and range; trigonometric identities; the use of trigonometry relational system in solving, and modelling mathematical and real world situations. Other researchers have focused explicitly on trigonometric concepts. Akkoc (2008, p. 859) examined the possible sources of the understanding of a trigonometric concept, the radian, under the theoretical framework of concept image of Tall and Vinner (1981). The research revealed a lack of knowledge of the radian content, which caused problems for understanding the trigonometric functions in the field of the real numbers. However, the main finding was that the image of the radian concept was dominated by the image of the concept of degree. Similarly, Chaar (2015) also investigated the knowledge of the radian, and the unit circle content in secondary in-service, pre-service, and student teachers. She highlighted the significance of the subject matter, pedagogical content knowledge, and the interpretative language when building on students' work and thinking. Additionally, a deeper understanding of trigonometric concepts was found in in-service teachers compared to their pre-service counterparts. Using a dynamic geometry environment, Hertel and Cullen´s (2011) developed an instructional sequence in order to investigate its influence on pre-service secondary teachers' understanding of trigonometric functions. This sequence was designed to promote a directed length interpretation of the six trigonometric functions. They concluded that "the instructional sequence aided students in developing ways to reason about the trigonometric functions" (p. 1406). Grounding on the meaning of Thompson

(2016), Paoletti et al. (2015) studied the strategies used to make sense of a variety of tasks to shed light into participants' meanings for inverse functions. They stated that the majority of subjects' meanings were the result of generalizations tied to the representations, situations or product of that activity. Moore, et al. (2016, p. 240) gained insight into the meanings for the unit circle through a perspective of the goniometric circumference based on the quantitative reasoning when participants are exposed to an approach, which develops the previous perspective. Their study described the connections between the students´ reasoning about measurement and their unit circle meanings. They found that the capacity to coordinate changes in unit magnitudes with changes in a quantity´s measure was critical for students in order to understand the unit circle. However, Moore (2016) remarks that he is uncertain about how such meanings might develop when working with secondary students, given that his work is mostly situated in undergraduate settings. Finally, using the theory of Harel and Tall (1991), Çekmez (2020) investigates the students' generalizations from the unit circle to the Cartesian coordinate system of trigonometric functions. In his study, participants represent values of trigonometric functions, which are given with different types of input on the unit circle, and they determine relationships between trigonometric values listed. Additionally, he implements an instructional sequence to define trigonometric functions for real numbers.

2.8. Summary

On balance, previous researchers look into the meaning of the sine and cosine of secondary school students, but they mainly concentrate on one of the semantic categories of our framework. Whereas some of these studies focus on examining the trigonometric representations and how secondary school students link them (Marchi, 2012), others examine how secondary school students understand the concepts involved (Brown, 2005; Demir, 2012). Finally, others investigate the presence of trigonometry in current sciences and their relation to educational practices and modern curricula (Hertel, 2013). We have not found previous research on the connections between these three semantic categories in secondary school students. A goal of this study is to contribute to overcome that lack.

Additionally, although there are some studies on instructional strategies, there is no consensus about the best strategy to teach the trigonometry relational system. Thus, more research is needed in order to shed light on how to overcome the difficulties of people involved in education on trigonometry relational system.

Finally, we would also like to contrast and deepen how the links across partial goniometry system and the partial analytic geometry system are. Indeed, according to Martinez-Planell and Cruz-Delgado (2016), one of the unanswered question is the relation between the construction of the cosine function in the partial goniometry system and its graphical representation in the partial analytic geometry system, on which we would like to concentrate.

3.THEORETICAL FRAMEWORK

3.1. Introduction

> *"What does this mean? […] What is the point for this to me?"* (Kilpatrick et al., 2005, p. 2-3).

This research is grounded on a theoretical framework, which is described in this chapter. As has been shown in the statement of the problem and in the purpose of the research, this thesis has its origin in the meaning of a notion, a concept or a school mathematical content. In this way, firstly, the curricular approach on which this thesis is based is addressed. In addition, didactic analysis is introduced as the instrument for the analysis of the content which is utilised in this report. Next, the importance of the meaning for the learning and teaching of mathematical content is highlighted. The fourth section refers to different authors who have used semantic triads in mathematics education as instruments to establish the meaning of a school mathematical content. Afterward, we define the meaning of a school mathematical content that frames the papers published regarding this thesis. Finally, we explain what we consider as mathematics systems and relational system.

3.2. Curricular framework and dimensions of the didactic analysis

Our study is based on the general framework called Didactic Analysis, proposed by Rico, Lupiañez and Molina (2013), Rico and Moreno (2016) and Rico and Ruiz-Hidalgo (2018). In its beginnings, the didactic analysis was conceived as a method to approach the organization, design and realization of didactic units in school mathematics. It provided criteria so as to classify the contents, the knowledge about cognition, the instruction and evaluation of a school topic. Over time didactic analysis has extended its usefulness. Currently, the didactic analysis of a mathematical content is considered "a method to deepen, structure and clarify the curricular content with a view to its programming and implementation" (Rico & Ruiz-Hidalgo, 2018, p. 7).

The didactic analysis aims to be a pillar to guide, and facilitate the implementation of teaching, and to improve the learning of school mathematical contents, based on how they have been ordered and set by the educational community. That is why school curricula and mathematics teacher training plans become its focus of study (Rico, 2013, pp. 19-20).

In this work, curriculum is understood as:

> "any training plan whose determination is given by some subjects who must be trained; the training purposes that are intended, and the needs to which one wants to attend; the institution, staff, and resources with which the training is carried out; the type of training to be provided: standards and codes, values, knowledge and capabilities, abilities and techniques, attitudes, and skills; finally, the evaluation system of the training plan, determined by some criteria and instruments."(Rico, 2013, p. 20)

As the previous definition indicates, a given curriculum is marked by a series of aims that can be classified into: conceptual, cognitive, formative and social. They allow establishing dimensions: cultural/conceptual, cognitive, ethical/formative and social that delimit the curriculum as we understand it. These dimensions use similar methodological tools (content analysis, and conceptual analysis).

Concretely, our study focuses on the cultural/conceptual dimension that has as its main aim the meanings of school mathematical contents. It is necessary to mention that this dimension also is related to the history of the notions involved. In fact, knowledge about the history of mathematics has implications for the teaching of mathematics (Cajori, 1985, pp. 1-3). More specifically, a better understanding of the history of a mathematical topic, and of the history of its education will be a source of vocation, motivation, orientation, inspiration and self-training for mathematics education researchers, teachers and students. Furthermore, teachers with adequate knowledge of mathematical content and of its history are more flexible in their teaching, and tend to encourage their students to establish mathematical connections. We recognize that history is useful to determine the origin of some notions, to compare different systems of representation, to locate basic problems (Furingueti, 2007), to find the meanings of different notions, and to create conceptual frameworks that allow us to reach a greater understanding of these notions.

To characterize the meanings of school contents, this theoretical framework uses three categories: structural, representational, and the sense that come from the notion of curriculum organizer (Rico, 1997). These categories are described in the section called meaning of a school mathematical content.

3.3. Importance of the meaning in Mathematics Education

Recent studies on the meaning of school mathematical concepts prove that the semantic approach is a solid path to research the teaching, and learning of mathematics. One strong argument in favour of this approach sustains that the meaning of mathematical content provides an essential framework to explain the fundaments of pupils' mathematics knowledge, to describe their understanding, and to clarify the foundation of the decisions for students' orientation and instruction (Thompson, 2016, p. 438; Rico, 2019). In fact, "Mathematical knowledge that matters most for teachers resides in the mathematical meaning" (Thompson, 2016, p. 437). Furthermore, the meaning plays a fundamental role in organizing content (Kumar, 2017, p. 559). Consequently, meaning should be the pillar for the future learning of the mathematical contents by the students (Castro-Rodriguez et al., 2016; Thompson, 2016, p. 461). Given that, "meaning may change by new applications, by new conceptual relationships or by new representations" (Bielher, 2005, p. 69), semantic categories also play crucial role. Finally, a semantic focus on the mathematical contents should provide guidance and aid pre-service mathematics teachers and in-service in planning instruction so that it allows students to improve their understanding and develop meanings (Thompson, 2016, p. 438, Castro-Rodriguez et al., 2015).

If we want students to develop a mathematical understanding that helps them to be creative and spontaneous thinkers outside of class, the study of meanings is a must (Thompson, 2013, p. 61). In fact, each mathematical notion is determined by its different meanings, and uses, and therefore, by its various representations (Wittgenstein, 1988). Representation makes sense within a system of meanings and relationships (Rico, 2009). Furthermore, it is relevant that meanings become a way of seeing mathematical notions (Thompson, 2008, p. 35). As a consequence, we consider that the breadth and depth of the meanings that schoolchildren construct depend on the different modes of expression and use with which concepts are handled, the ability to connect various structures and to use different procedures, the rich of connections that are established for a certain mathematical notion or notions (Gómez, 2007), and the conceptual elements that must be taken into consideration.

3.4. Semantic triangles in Mathematics Education

There are several proposals, in which a three semantic categories frame is utilised, in order to investigate how to endow a mathematic content with meaning (Vergnaud, 1990; Radford, 2003; Sáenz-Ludlow, 2003; Biehler, 2005; Steinbring, 2006). On the one hand, for Saénz-Ludlow, the sign has a triadic nature and dyadic relations among its three elements: object, representamen, and interpretant. A chain of the signification arise by means of processes of representation, and determination in which, the triad is in perpetual motion (Saénz-Ludlow, 2003).

> "Peirce calls the object of a sign that mental idea or that physical object the sign stands for; the interpretant of a sign that idea produced in the mind of the interpreter; and the representamen of a sign that material or mental vehicle fit to stand for the object" (Sáenz-Ludlow, 2003, p. 182).

On the other hand, our approaching is similar to the one utilized by Steinbring (2006). His approaching is based on the determination of an epistemological triangle, which has three reference points "*concept*", "*sign/symbol*" and "*object/reference context*", which form a balanced, and supported system. He considers that mathematics requires certain signs or symbols system in order to keep a record of and code the mathematical knowledge and mathematical concepts. The mathematical sign means the specific way of writing the "concept". The meaning has also to be produced by the learner by means of establishing mediation to suitable reference contexts (Steinbring, 2006, p. 135). "Therefore, it is really important to distinguish between the aspect of the application and the aspect of the representation" (Steinbring, 1989, p. 29). This triad is a way to characterize aspects of mathematical knowledge, and at the same time, it can be utilized as a methodical instrument to analyse the meaning of mathematical contents and their related understanding (Steinbring, 1998, p. 172).

3.5. Meaning of a mathematic school content

We define mathematical school content as a set of concepts, procedures, structures and attitudes that teachers communicate and teach so that students can learn and use them.

Following Frege (1962) in "Über Sinn und Bedeutung", we consider the meaning of a mathematical concept from a wider point. It is composed by its definitions, representations and senses.

> "Sign (name, combinations of words, letter), besides that to which the sign refers, which may be called the reference of the sign, also what I should like to call the sense of the sign, wherein the mode of presentation is contained" (Frege, 1962, p. 36).

From our point of view, we understand by meaning the distinction between sense and denotation, based in Frege's semiotic triangle, in the same way that been used in the PME by Bazzini et al., (2001), which was interpreted by Rico and Gómez (Gómez, 2007). Thus, we take the notion of the meaning of a mathematical school concept developed by Rico (2012), based on reference, sign and sense. By means of these categories, a mathematical concept is identified, expressed and utilised. These three categories are on the basis of the meaning of a school mathematical concept: the conceptual structure, the systems of representations, and the sense.

In this paper we use the three categories aforementioned to analyse the mathematical knowledge and the meaning shown by participants of our study. Meaning categories allow us to interpret the value and adequacy of students' knowledge when they try to define, represent or use the mathematical concepts considered in the high school. It is clear that this system is organized to achieve the meaning of a school mathematical content. In addition, the system of categories, fields, and components is uncomplicated for its interpretation, and allows to conjecture a theoretical explanation about the local knowledge of a singular student or of a group of them, and help to detect how it has been understood at a determined moment. To achieve this, the elected framework identifies, organize, synthesize, analyse, and interpret elements of content, their relationships, and rules of processing and conversion, all of which play a significant role within the categories used to carry out the content analysis of school mathematical contents (Rico & Ruiz-Hidalgo, 2018, p. 1-6). We characterize these categories below.

3.5.1. The conceptual structure

We remark that mathematics works with abstract notions, and their interrelationships. Mathematical results are based on, and are inferred from basic concepts. Mathematical concepts and methods are general and abstract; they are recognized by their logical foundations and are derived from each other by deductive and necessary reasoning. Proving a property or a theorem means that its truth or falsity is proven by logical argumentation based on properties of the basic concepts involved, which are based on a reference that allows evaluating its truth value.

The mathematical disciplines have an internal method of work, a conventional regulation, justification, and exposition. In general, the organization of disciplines respects a sequence. This process begins with definitions and notations, is continued by axioms, statements, relations, operations and properties, is completed by theorems and corollaries, and ends with applications (Alexandrov et al., 1981). Mathematics has an internal structure.

The term structure is used in a variety of fields, and it generally refers to an organized whole. A structure is generally considered "as a non-empty set together with relations, operations and distinguished elements" (Demopoulos, 1994, p. 213). In mathematics, Shapiro (1997) defines a structure as "the abstract way of a system, emphasizing the interrelationships among objects, ignoring any feature of them that not affect to the way in which they relate with others objects in the system" (p.74).

In this thesis, the conceptual structure comprises notions, concepts, properties, propositions, procedures and their relationships implied in a mathematical concept. It establishes priorities and links; it shows the trajectories so as to organize the learning expectations; it provides references to establish the truth or false quality of a statement.

Following Bell et al., (1983), Hiebert and Lefevre (1986), and Rico (1997; 2012), in order to characterize the conceptual structure, we consider two fields to classify mathematical contents: conceptual and procedural. Complementarily, they differentiate three cognitive levels of complexity in each of them, with which they structure the different contents in the fields considered (Bell et al., 1983). It is true that experts consider three general fields to characterize this structure: conceptual, procedural, and attitudinal field (Rico, 1997). However, given the scope of our work, we limit ourselves to the first two fields.

Conceptual content is organized in three levels of increasing complexity: basic, middle, and higher, that correspond to three different kinds of components: facts, concepts, and structures respectively. Concepts are the ideas by means of which we think, that link and organize facts as singular pieces of information. Otherwise, concepts (or even concepts and facts) are related, taking part in the structures that organize them (Skemp, 1987, pp. 53-55).

Procedural content corresponds to operations, properties, and mathematical methods, the way of handling them as well as rules, the logical reasoning, and strategies (Hiebert & Lefebvre, 1986). The procedural field is also organized in three corresponding components and increasing levels of complexity: skills, reasoning, and strategies. Skills are procedures to manipulate facts; reasoning consists of logical procedures to infer between concepts or among concepts and facts, and strategies are procedures to work within, and between structures.

These two fields organize a system useful to identify and classify the units of information obtained from any mathematical school content. In particular, the components allow to integrate facts, data, and curricular contents at their corresponding level, and are useful for carrying out a methodological analysis structured by categories or themes.

3.5.2. Systems of representation

Systems of representation generate, express and communicate meaning of a concept. Each representation system has its own grammar codes or laws. They remark or hide a variety of relations. A system of representation is defined by a set of signs, graphics and rules, to make the concept present and to establish relationships with other concepts. Moreover, representation systems help students to think, express and communicate their mathematical ideas, which are involved in the processes of construction of structures (Castro & Castro, 1997).

It is clear that learning mathematics involves working with signs, symbols, and other representations. In educational research, the term representation is open to many interpretations. This word may be used to describe mental structures, ideas' frames with which subjects think about concepts and its relationships. It is also associated with external displays: words, pictures, diagrams, physical objects, and so on. Hiebert and Carpenter (1992) stated that "to think about mathematical ideas we need to represent them internally (…). Communication requires that the representations be external" (p. 65). In that way, this thesis belongs to the tradition that consider a representation as any external thing that is used for stand for any other thing (Golding & Kaput, 1996). Futhermore, Morgan and Kynigos (2014) state that representations create meaning. "Representations are seen as expressions of meaning, the ways in which representations are manipulated also represent meaning" (Morgan & Kynigos, 2014, p. 363). The various representations of the concepts and their connections are fundamental in the understanding of school mathematics to capture all its complexity and the peculiarities of its different notions (NCTM, 2000).

Due to the systemic nature of a mathematical structure, each of its representations is encoded within a system. The different representation systems organize diverse experiences, highlight some properties and relationships and hide others (Golding & Kaput, 1996). Thus, the systems of representation influence the meaning expressed by a content. "The power and utility of the representations depend on their being part of a structured system, and on the degree of flexibility or versatility in what they can be represented" (Golden & Kaput, 1996, p. 400). In this work, by representation system we understand those combinations of signs and rules whose use transmits specific mathematical contents in a structured, operative and coherent way (Rico, 2009).

3.5.3. Sense

Taking into account a functional view of the curriculum, the meaning of a school mathematical concept must include possible uses and its different functions, in other words, what it is apt for, and how is it used; which are the problems that the mathematical concept provides response; the terms, the owns words used by people involved in education to refer mathematical notions, ideas, concept and their use adequate. Indeed, we consider that to deepen in the sense of a concept students need to know a variety of terms that they utilize when they express and work with them. Thus, sense includes those contexts, situations, and terms that provide mathematical ideas with sense (Rico& Moreno, 2016, p. 139).

Mathematical concepts, structures and ideas help organize and give sense of phenomena. These phenomena are the roots of these mathematical ideas and they contribute to the development of the mathematical contents (Freudenthal, 1983). Phenomena are presented in a variety of problem situations, which are the source and may serve as contexts for this development (Van den Heuvel-Panhuizen, 2003; Van den Heuvel-Panhuizen, 2014). "These sites provide the context in which the learning of mathematics takes place so that both the meaningfulness and the utility of the mathematical ideas are ensured" (Lamon, 1995, p. 168). Finally, Biehler (2005, p. 61) stated that different contexts provide differentiated senses for a mathematical concept.

We consider sense when applying mathematical concepts and procedures in different contexts, which are the questions which the mathematical concept give answer –contexts-; when properly choosing the words for a concept -terms-, and so as to mention the settings in which they are involved –situations.

3.6. Understanding a mathematical concept

According to Rico (2018), the meaning of a school mathematical content implies knowing and giving its definitions, representing its relationships, establishing and processing its operations as well as giving them sense in different contexts and situations. Thus, understanding a school mathematical content is to give it meaning, that is, to define it, to represent it, to identify its operations, relationships and properties, its modes of use, its interpretation and application. In others words, our notion of meaning also holds that mastering a mathematical content involves knowing its definition, how to represent it, and how to show its operations, properties, relations, and senses. These considerations follow from Frege's (1998) notion of the meaning of a mathematical concept.

Understanding a mathematical content also consists of interpreting and using it with meaning. Understanding a mathematical content in depth involves providing its concepts and performing its procedures with coherent meaning, "to understand something means to assimilate it into an appropriate schema" (Skemp, 1987, p. 29). Similarly, according to Skovsmose (2005, p. 85), what the person can do by means of a concept is related to the meaning of the concept.

In the understanding of school mathematical concepts, the representations of mathematical notions and their links play an important role. Representations provide sense within a mathematical structure (Rico, 2009). Thus, a way of broadening the understanding of concepts and procedures includes using and blending different systems of representation in solving problems to convert and process one representation into a different one (Skemp, 1987, pp. 55-56; Even, 1990, p. 105; Camacho & Depool, 2003, p. 2; Rico, 2009). In fact, as Kaput (1992) states, "all aspects of a complex idea cannot be adequately represented by a single notation system, and hence require multiple systems for their full expression, meaning that multiple" (p. 530). Similarly, Duval (1993) claims the need for various systems linked to the same mathematical content. This plurality leads to consider the relationships between different systems for the same mathematical content. Janvier (1987) writes about translations between different systems for the same content, while Duval (1993) refers to these relationships with the "*conversion*" term, which we follow in this work. Its significance is such that, according to NCTM (2000), students should "select, apply and translate amongst mathematical representations to solve problems" (p. 64). Thus, converting representations plays a crucial role to improve understanding.

Finally, it must be remarked that we have identified and chosen three different terms: converting, transforming one representation into another one between different representation systems; processing, transforming expressions within a representation system; moving, transforming one representation into another one between partial systems.

3.7. Mathematics systems

A Mathematical System gives a way of working, organizing or doing something in which you follow a fixed plan or set of rules, mainly mathematics, which are used to count, order, measure, estimate or calculate. If a whole of entities participates in a system, it gives rules for its organization and a sense of orderliness (Reinhardt & Soeder, 1984, p. 37; RAC, 1996, p. 929). An abstract mathematics relational system is a kind of structured organization for mathematical contents, which is formed by a set A of mathematical entities and k relationships established between them. Relational systems distinguish between basic and multiple structures (Rico et al., 2020). The basic structures may be of three types: algebraic, order and topological; a multiple structure is combination of basic kind ones, and they are not as rich in content as initial structure. Likewise, a derived structure comes from one or several basic structures appropriately combined. Commonly there are three main derived modalities: Substructure, Cartesian product, and Quotient structure.

The study and expansion of some examples of mathematics relational systems such as number systems have been repeatedly carried out by different authors, who have highlighted the problems addressed, the needs met, the laws and the shared structures. Indeed, the extension of the different properties to new numbers sets justify the blend of these different types of numbers and operations (Skemp, 1987; Feferman, 1989). Concretely, the morphism between whole numbers and integer fractions is a sum and product isomorphism in both sets that is expressed by the notation of "fractions or mixed numbers". The main justification of working with combined numbers is that existing isomorphism allows us to deal with the representation systems of both.

3.8. Relational systems of school trigonometry

Symbolically we define the trigonometry relational system as an N-tuple $(A, R_1, R_2, R_3, ..., R_k)$ mainly formed by a set A of trigonometric concepts and k relationships established between them (Rico et al., 2020).

The trigonometry relational system helps to give solution to questions associated with the value of an angle. The trigonometry relational system is built by expanding the Euclidean system that consider angles as part of the plane, to angles view as central angles. Afterwards, the system is again widened, and angles become arguments of the goniometric function. Likewise, angle measures are normally expressed by an arc of a subtending circumference, and they are given in the sexagesimal system in the Elementary geometry; in Goniometry, angle measures are numbers without measure unit, in other words, they are measured in radians, and may be notated by a dot in the goniometric circumference; finally, angle measures are expressed by a dot in the real line, an abscissa, and are measured by means of a real number. Thus, the trigonometry relational system consists of subsystems, each of which has a unique notation. Those notations are the trigonometric representation systems.

Based on various assumptions, in the second half of the 20th century several authors reviewed the definition of the trigonometry relational system to give answer to a variety of theoretical and practical problems derived from angles measure. These works were developed attending to the emerging educational needs of the time. Some of their remarkable authors were Choquet (1964, pp. 117-116), Dieudonnè (1971, pp. 181- 190), Freudenthal (1973, pp. 476- 494), and Lakoff and Nunes (2000, pp. 387- 397). As it has been mentioned above, the trigonometry relational system comprises several subsystems, each with its own notation and rules of representation. Trigonometry relational system combines these rules and notations and share them. Such combined notations are considered and used as a representation system for the trigonometry relational system as a whole. Given a concept, some of the aforementioned studies allow to interpret its conversion between trigonometric representation systems as change in its meaning. The moving between partial trigonometry systems is structured by the organization of the contents, and it can be used as methodological tool to analyse meanings and their related understanding.

On the one hand, Freudenthal (1973, p. 479) describes the instrumental way in which angles have been measured and organized, highlighting their main definitions and changes throughout the history of mathematics. As can be seen in table below, he establishes the following subsystems within the global trigonometry relational system: elementary geometry, goniometry, and analytic geometry.

Table 3.1. *Partial trigonometry systems to the angle concept*

	Elementary geometry	Goniometry	Analytic geometry
Angles sides	Order half-lines	Non-ordered half-lines	An ordered lines
Type of plane	Non-oriented	Oriented	Oriented
Representation model	Right triangle	Oriented unit circle	Analytic function
Module	Between 0º and 180º	Mod 2π	Mod π

On the other hand, through a conceptual analysis, Lakoff and Nunes (2000, p. 388-398) characterize the concepts involved and the relationships among the concepts in trigonometry relational system in order to give an account in cognitive terms of its meaning. Specifically, they look in from a cognitive perspective different domains and their blends which play an important role in understanding. Indeed, according to them, understanding a use of a concept implies understanding a blend. Concretely, the blend of remarkable complexity from the unit circle to the state the trigonometric function is the most important for our study. For this reason, we expressed their considerations about this blend.

Table 3.2. *The trigonometry metaphor between unit circle and trigonometric function*

Domain Unit circle	Domain Trigonometric Functions
The length of the arc subtended by angle ϕ	The number assigned to angle ϕ
The length of a side a in the x-axis	The value of the function cos (ϕ)
The length of side b parallel to the y-axis	The value of the function sin (ϕ)

Following Lakoff and Nunes (2000, pp. 385-387), we also highlight that trigonometry relational system have their foundation on the Cartesian plane, that Descartes described and theorized along with many other mathematicians who developed new concepts and relationship, and advanced the analytic system. Its characteristics enable conceptualize the points, the functions and the angles of the trigonometry relational system as numbers. Firstly, it is evident that trigonometric functions are not numbers. However, when we conceptualize them as ordered pairs of points in the Cartesian plane, the operations of arithmetic can be extended from numbers to functions. As it is also stated, trigonometric functions can be conceptualized as periodic curves in the Cartesian plane, so that they may be added, subtracted, multiplied or divided. Secondly, these authors define the unit circle domain by means of blends of different domains: a circle in the Euclidean plane with centre and radius 1 unit; the Cartesian plane with x-axis, y-axis and origin at (0,0); an angle in the Euclidean plane, and finally a right-angle triangle. When we compare these domains with the trigonometric function, the angles can also be conceptualized as numbers. Finally, several correspondences are not extended in these blends, which increase the complexity of this relational system.

Some questions derived from an approach to trigonometry as a relational system will serve as a common thread to study a semantic approach for conversions amongst trigonometric systems of representation, and the meanings that secondary school teachers in training attribute to trigonometric contents.

Furthermore, the complexity of trigonometry relational system, with its wealth of contents, which may convey several senses, and the variety of notations and rules, make this global system an ideal choice to approach a research about conversions between trigonometric representation systems together with their relationships (Reinhardt & Soeder, 1984, p. 37).

4. METHODOLOGICAL GROUNDING

4.1. Introduction

"If there were only one truth, you couldn´t paint a hundred canvases of the same theme"
*(*Pablo Picasso, 1993)

This chapter firstly presents the methodological foundations and the design study. The next section presents the main characteristics of a semantic questionnaire given that this was our method of data collection in the two stages in which our study is divided. Afterwards, we compare and describe the grounded theory and content analysis given that they are our methods of analysis. This is followed by the description of the two stages of our study in two different sections. Each section describes the data collections tools, the participants and settings, the implementation of the instruments and the data analysis.

4.2. Methodological foundations and design study

This study is part of a research project on school mathematical concepts meaning and their understanding. Former works have examined the meaning of other mathematical concepts (Fernández-Plaza et al., 2013, Castro-Rodriguez et al., 2016; Vargas-González et al., 2020).

This research study is composed of a two-stage main study. The first stage of this study introduces a descriptive study that concentrates on the meanings that a group of Spanish Non-Compulsory Secondary Education students (16-17 year-olds) associate with the notions of sine and cosine of an angle. In other words, the first study was intended to identify and characterize the meanings that a group of students attribute to the aforementioned concepts. The meanings given by the answers come from their basic elements. Several meanings might be identified. The accuracy of the meaning and the mastery of each concept will be firstly established from its richness of elements and breadth of its relationships. The data analysis of the provided answers will allow the investigation and evaluation of the manner in which students understand these contents.

The second stage intends to study and interpret the conversions between two trigonometric representation systems in secondary school teachers in training. In other words, it describes the understanding that an available group of high school pre-service teachers express about the angle concept and its cosine, which arises from the links between two partial trigonometry systems.

Both stage-studies allow to show the usefulness and convenience of the framework described to analyse the understanding of teachers in training and secondary school students about school mathematical contents and their meanings. This framework has already been developed in this report and we exemplify how it can be applied.

This is a qualitative study based on the survey method. This method is used to collect data with the purpose of describing, interpreting and structuring the existing situation and developing possible explanations based on the chosen semantic framework (Fraenkel & Warren, 2006). The instrument is a semantic questionnaire, designed by the researchers, grounded on the triad semantic for two trigonometric concepts: the sine and cosine of an angle. As it has been said, this research is built on a semantic framework that analyses the meaning of a content using three semantic categories: its conceptual structure, its systems of representation, and its sense (Bunge, 2008, pp. 24-25). These three categories and their components are employed as analytical and interpretive tools to describe the meaning of a content (Martín-Fernández, et al., 2016). As a procedure, we use a combination of the content analysis and of the grounded theory.

4.3. Semantic questionnaire

When we use a determined technique to collect primary data, we should wonder whether it suffices to obtain the result that we seek. Besides, we should contemplate if the results depend on the used technique. Among the techniques used and developed, one of the main means of obtaining information is the questionnaire.

A questionnaire is a formalized set of questions employed for getting information from respondents to describe, compare, understand and/ or explain knowledge, attitudes, behaviour, etc. It allows the collection of data which are internally consistent and coherent for the analysis in a standardized manner.

Over the past year, a lot of attention has been paid to the validity of questionnaires, the way of selection, or its consistence (García & Román, 1998, p. 477). Although there are not scientific principles for designing a questionnaire, some guidelines are available to assist researchers not only to develop a questionnaire, but also to avoid mistakes in its design.

There are different formats of questionnaires, which represent potentially invaluable tools for determining a wide range of factual information, subjective views and perceptions from a representative sample of a particular population.

The three main types are: structured, semi-structures and unstructured (Cohen & Manion, 2007, p. 320).

Human beings use mathematics to seek and interpret the world. Their method is based on abstraction data and facts and on the accuracy of its terms, concepts, sentences and reasoning. Many scientists state that a school mathematical content has a meaning for the student, when they are able to answer, at least partially, some of the following questions: What is? What does the school mathematical content consist of? How is it expressed? What is it used for? These questions are related to concepts, properties, and relations; signs and rules representing them and the situations, contexts and ways of using (Rico et al., 2015). This knowledge is showed by means of the categories of the meaning of a school mathematical content: the conceptual structure, the representation systems and the sense used by students.

Finally, the semantic questionnaires enable us to collect traces to investigate the meaning of the school mathematical concepts, content and notions showed by students. The main goal of this kind of questionnaires is to examine what properties of the concepts are identified by students, how the concepts are represented and defined, what values are given and what kind of words and reasoning are used so as to make sense of them. That is to say, the traces collected by the semantic questionnaires are words, terms, signs, graphs, sentences, relations, reasonings, descriptions that show how the student has acquired the concept. When a subject keeps in contact with and/ or receives training of mathematical notions, these are totally or partially internalized and used by students. There is a huge variability of these traces, which can be identified and classified by means of the categories of the meaning of a mathematical concept (Rico, 2016; Martín-Fernández et al., 2019).

4.4. Grounded theory and Content analysis

The analysis of this study is based on two methodological perspectives: grounded theory and content analysis.

"Grounded theory that is a general methodology for developing theory that is grounded in data systematically gathered and analysed" (Strauss & Corbin, 1994, p. 273). Strauss and Corbin (1994) consider that theories "consist of plausible relationships proposed among concepts and set of concepts", […] its plausibility is to be strengthened through continued research" (p. 278). Furthermore, "theory evolves during actual research, and it does this through continuous interplay between analysis and data collection" (Strauss & Corbin, 1994, p. 273).

The strategies used in grounded theory are few and flexible, allowing researchers to choose and create new methods (Charmaz, 2008), and also to adapt these strategies to the demands of their study. Similarly, Corbin and Strauss (2008) affirm that techniques and procedures are tools. "The analytic process, like any thinking process, should be relaxed, flexible, and driven by insight gained through interaction with data rather than being overly structured and based only on procedures" (Corbin & Strauss, 2008, p. 12)

Consequently, grounded theory is an emerging, inductive and endless method, formed by a set of systematic but flexible strategies that guide data collection, coding, synthesis, categorization and integration of concepts for the generation of a theory, and that includes the verification of categories that emerge from successive levels of analysis through hypothetical-deductive reasoning. The researcher makes hypotheses on the observed data and tests them until reaching the most feasible interpretation. This method mainly uses constant comparison methods for data analysis and for the improvement of the iterative and simultaneous process of analysis and data collection (Bryant & Charmaz, 2010).

Following a constructivist sense of grounded theory, the emerging theory will be situated in a particular time and context. Furthermore, we assume that the researcher does not enter the investigation in a clean slate. Researchers have knowledge and experience before starting the research that they will use in collecting and analysing data (Bryant & Charmaz, 2010). Furthermore, what researchers perceive depends on the context of the research, the mode of generation and collection of data (Charmaz & Mitchel, 2001), focusing on the way in which subjects expose meanings (Bryant & Charmaz, 2010), invoking the creativity and originality, and reaching the unanticipated. All of this helps minimize the preconceptions of the research problem.

In the past, grounded theory used to be considered disjoint from other research methods (Bryant & Charmaz, 2010). However, the use of grounded theory along with other types of analysis is useful, since grounded theory offers tools for the explanation and construction of theories. Thus, grounded theory makes ethnography more analytical, and it enables more in-depth interview data analysis methods and more focused content analysis (Bryant & Charmaz, 2010). Although the general philosophy of grounded theory is shared, another analysis technique suitable for this study has been developed and used, the qualitative content analysis.

Nowadays, content analysis is considered as one of the main educational research methods (Sándorová, 2014). “Content analysis is a research for making replicable and valid inferences from data to their context” (Krippendorff, 1989, p. 403). Its purpose is to discover the internal structure of communication by studying its semantic content (Rico, 2013). "It is a method for systematically describing the meaning of qualitative material" (Schreier, 2012, p.1). Content analysis is a “research method that uses a set of procedures to make valid inferences from a text” (Weber, 1990, p.117). Cohen et al. (2011, p. 563) indicate that content analysis defines a set of strict and systematic procedures for rigorous analysis, and also the examination and verification of written data content, understood as any written communication material, which can be read, interpreted, and understood by people other than the one who analyses them. This method uses categorization as an essential trait.

In this research we consider content analysis as a method that is carried out through a systematic process of classification, coding, and identification of themes and patterns (Hsieh & Shannon, 2005, p. 1278), allowing us to infer meanings, to discover patterns, to make conceptual maps, to check previous hypotheses, etc. (Rico, 2013).

Also, in this thesis we have used these two emerging theories, closely related to the appearance of concepts and relationships during the analysis, which demand a great openness on the part of the researcher (Strauss & Corbin, 1998). These two methods originate from data and allow for greater knowledge and understanding of productions (Strauss & Corbin, 1998), providing significant insights for carrying out actions aimed at improving both teaching and learning.

However, while content analysis focuses on describing the meaning of qualitative material and listing categories or themes, grounded theory goes further (Cho & Lee, 2014), looking beyond data and placing data in interpretative and explanatory structures, generating a theory.

4.5. Stage 1. Methodological aspects

4.5.1. Data collection tools

In the first stage, we designed two semantic questionnaires (Blok, 2014), with eight tasks each, presented as two different options, questionnaires A and B. The items were designed taking into account other researches (Fi, 2003; Weber, 2005; Brown, 2005; Dominic, 2012) and several schoolbooks (Ibañes, Ortega & Piñeiro, 1998; Arias & Maza, 2008; Bescós & Pena, 2010; Vizmanos et al., 2008). The choice of the items that made up the questionnaires covered the three categories of the semantic triangle of a school mathematical concept (Rico, 2012). All the questions of questionnaire A were the same as in questionnaire B, with the exception that questionnaire A was referred to the sine of an angle less than 90º and questionnaire B was related to the cosine of an angle less than 90º. The sixth item was the same in both questionnaires. The differentiation of these two options was motivated by the possibility that there were differences between them in the mode of presentation and in the meanings communicated by the secondary school students.

We evaluated these instruments by means of a pilot study, in order to ensure the comprehensibility of the items and the questionnaires' structural validity (Martín-Fernández et al., 2013).

Previous to the implementation of the final questionnaire and in order to ensure their validity, we took into account the considerations provided by a field specialist who work in mathematics education. The specialists evaluate the goals of the items, the order of its presentation, and the accuracy of the questions, always taking into account our theoretical perspective.

The items were clearly understood on account of the low number of unanswered, irrelevant or illegible answers in the productions of the students. Indeed, no questions were posed by the students during the implementation of the questionnaire.

In this stage, we study the productions of four items of both questionnaires. These items show a set of tasks useful to explore the meaning ascribed to the students through the triad semantic.

Task 1-A. "Draw a picture in which it is showed sin (30º)"

Task 1-B. "Draw a picture in which it is showed cos (30º)"

Task 6-A and 6-B: "Draw a picture showing a difference between cos (30º) and sin (30º)".

Task 2-A. "Explain in your own words sin (30º)"

Task 2-B. "Explain in your own words cos (30º)"

Task 8-A. "Write a problem in which you use the sine of 60º"

Task 8-B. "Write a problem in which you use the cosine of 60º"

Figure 4.1. Items of the questionnaire

The aim of the item 2, about two concepts from trigonometry structure, is to evoke the basic elements of the sine and cosine of an angle. We identify terms, notions and definitions that students show when they explain in their own words the aforementioned concepts. Furthermore, we sought to explore if students distinguish the elements of a right triangle using the most specific vocabulary possible and whether they know how to express the studied concepts.

The first and sixth questions are connected with the representation systems. The main goal of these items is to keep a record of the variety of representations used by students and its complexity. The sixth question is also related to the discrimination of two similar concepts. This task allows students to utilize several systems of representation including the graphic and symbolic ones, as it indicates them to do a drawing that shows some difference between the sine and cosine of the same angle.

The eighth question is related to the category sense. We intended to explore how the trigonometric concepts under study are used. We described the contexts in which they appear, how pupils use the mathematical language, the terms associated with the studied concepts, and what situations they present. We sought to see if students understood a functional meaning for these trigonometric concepts and if they spontaneously related the use of the analysed concepts with their everyday life. Additionally, the diversity of created tasks, their way of expression, and the made-up picture helped us gain insight into the students′ meanings.

4.5.2. Participants and settings

The pilot study was conducted to 27 students of Grade 11th, of whom 13 surveyed questionnaire A, and 14 surveyed questionnaire B. The participants were students in IES Mediterráneo, in La Línea de la Concepción, Cádiz (Spain). The final version of the questionnaire was implemented in 74 students (37 males and

37 females) of Grade 11th from a high school in the city of Granada (IES Padre Suárez). The participants in this stage were taking Mathematics as a subject for the Science and Technology "Bachillerato" path. Three teachers, who had a work experience of respectively thirty, seventeen and eleven years, were responsible for their instruction. Through self-report and as observers, teachers asserted that they taught the topic using "traditional methods" and "mainly using the textbook". The textbook used in lessons was "Matemáticas I para Bachillerato en la modalidad de Ciencias y Tecnología" (Vizmanos et al., 2008). Among teachers, only one of them occasionally utilized web applications. With regard to their instruction, it can be considered as "standard instruction". Most of the lesson time was devoted to the explanations of the topic and afterwards students were given enough time in order to do some exercises. As stated by their teachers, students had been taught trigonometry relational system in accordance with current regulation (MEC, 2007). Additionally, firstly, trigonometry relational system was introduced by means of "ratio system", and after that, the "line system" was presented to them. Thus, students in the second and last level of curriculum related to trigonometry relational system were selected for the research.

Our decision to work with secondary students was partly strategic and partly functional. Strategically, we hoped to gain insights into their meanings in order to know how to improve the understanding of the students, and to use their results for later studies. From a functional standpoint, this was the last grade in which students study trigonometry relational system in high school and it would be the beginning of an ulterior study in pre-service secondary mathematics teachers.

4.5.3. Implementation of the instruments

The questionnaires were provided on a booklet. The questionnaires were administered in the middle of the academic year 2012/2013. The pilot survey and the final survey were conducted in an ordinary lesson and all students completed the items in up to 60 minutes. Students responded to the items by means of pictures, explanations, or generalizations -in short, showing their own ideas about different questions related to the sine and cosine of an angle.

After a consultation held between the research team and an expert in Didactics of Mathematics, the participants were divided into two equivalent subgroups based on academic performance in the subject of mathematics during that year. A stratification of the sample following the same criteria was carried out in these subgroups as well. The purpose was to plan a similar distribution for the two questionnaires that would allow to obtain results as representative as possible.

4.5.4. Data analysis

There are two types of analysis. Firstly, we analysed the first and the sixth task. In order to examine the productions, we identified the central ideas or themes of the answers. Then, we studied closely all the units of information of the responses. After comparing them, subcategories emerged. Afterwards, the subcategories gave rise to categories. Finally, we identified the preferential students' interpretation of the sine and cosine of an angle.

Another kind of analysis was carried out with the responses of the students to the tasks 1, 2, 6 and 8. Firstly, the answers provided elements which are the expressed facts working as information units. Researchers recognized relationships between the identified elements, giving rise to empirically founded components and subthemes. The themes emerged inductively from the subthemes.

The precision, features and connections of subthemes and themes enabled to achieve the most accurate meaning categories of the students' productions.

Finally, in order to show the data, we used auxiliary methods as tools to express the students' view, and we interpreted them. Through this process, two researchers examined the results of the previous stages. The definition of categories, subcategories, themes, subthemes, and hypothesis about connections among them were verified.

Following a common characteristic of various qualitative research evaluation criteria -the detailed exposition of the content analysis process (Neuendorf, 2002; Schreier, 2012)-, we are going to present exactly the development of the analysis carried out in this investigation. It is necessary to inform about the decisions made during the research in relation to the coding process, and the methods used to establish validity to the research (Zhang & Wildemuth, 2009), since there are no systematic rules for the analysis of data content (Weber, 1990; Neuendorf, 2002; Cohen et al., 2007; Rico, 2013). Thus, content analysis is very flexible and therefore there is no simple and correct way to carry it out. Researchers must judge which option is most appropriate for their problem (Weber, 1990).

Finally, because of its complexity and singularity, we detail the particularities of the technique used for the first data analysis for questions 1 and 6.

An open coding was started, in which the productions of each individual were analysed as a unit of analysis, examining the productions comparatively, allowing us to identify the concepts that emerged from the data, detecting the central ideas and delimiting the main themes (Strauss & Cobin, 1998), which in our case are the triangle and the circumference. The data analysis consisted in carrying out as many readings and data visualizations as were necessary to achieve the researcher's immersion in the productions and obtain a global sense of them, making the pertinent annotations.

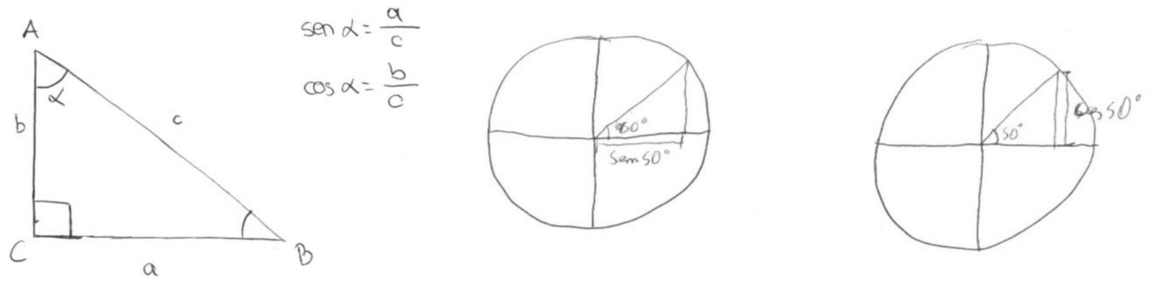


Figure 4.2. Examples of student productions with the themes triangle and circumference

More specifically, if we take into account the productions of the item number six, another issue arises: the trigonometric function; if we add the answers from the pilot study, the angle appears exceptionally.

Figure 4.3. Examples of productions with the trigonometric function

After that, we carried out a more detailed analysis of the productions associated with each of the themes, describing all the aspects contained therein. The concepts that emerged began to accumulate, and were grouped and categorized under more abstract terms, using the constant comparison method (Glaser & Strauss, 1967; Zhang & Wildemuth, 2009) allowing us to reduce the number of categories with which we were working. By doing so, we produced a means to describe the data which increase our understanding and generate knowledge (Cavanagh, 1997). The names of the themes, categories, and subcategories were precise, avoiding ambiguities, and promoting the credibility of the study (Weber, 1990; Zhang and Wildemuth, 2009). When we categorized, we formulated hypotheses and deduced whether the categories fit the data. There was an interaction between induction and deduction (Strauss & Corbin, 1998, p. 137).

As stated above, two main themes were found, the circumference and the triangle. In relation to the circumference theme, three units of information appeared: “axes”, “radius” and “diameter”. After that, two subcategories emerged, “axes” and “segments”, to finally determine the category “elements of division of the circumference”.

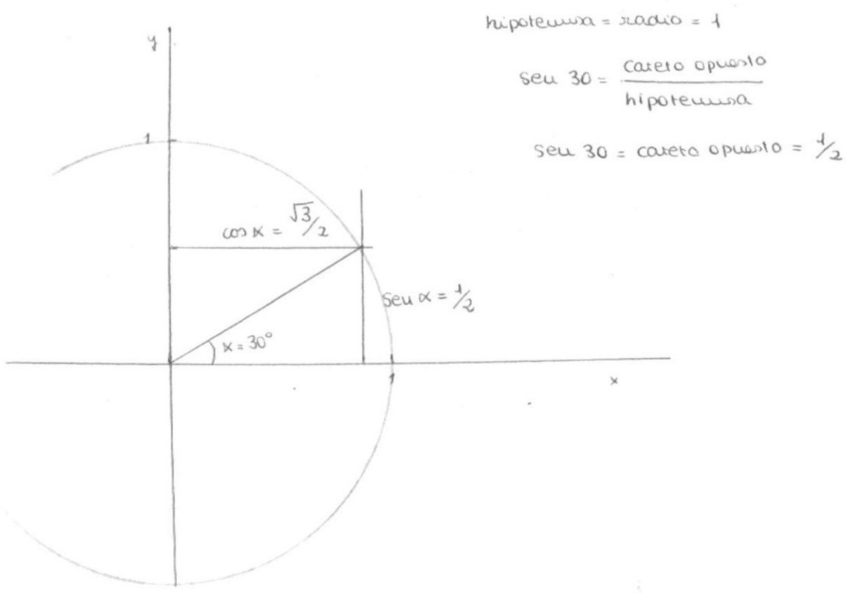


Figure 4.5. Student answer using axes as element of division of the circumference

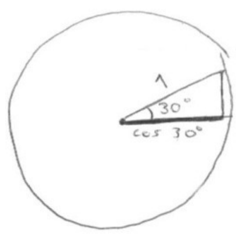


Figure 4.4. Student answer using radii as a division element of the circumference

Next (this step is not disjoint with open coding), another unit of information was identified, the angle. Three subcategories associated with it emerged: "central angle", "interior angle of a triangle", and "point in a circle and a segment", which allowed to identify the category "way of indicating the angle".

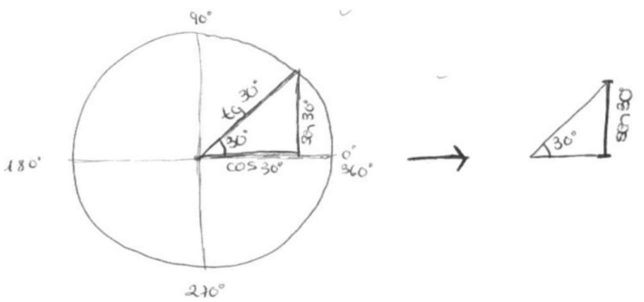


Figure 4.6. Production associated with the category "interior angle of a triangle"

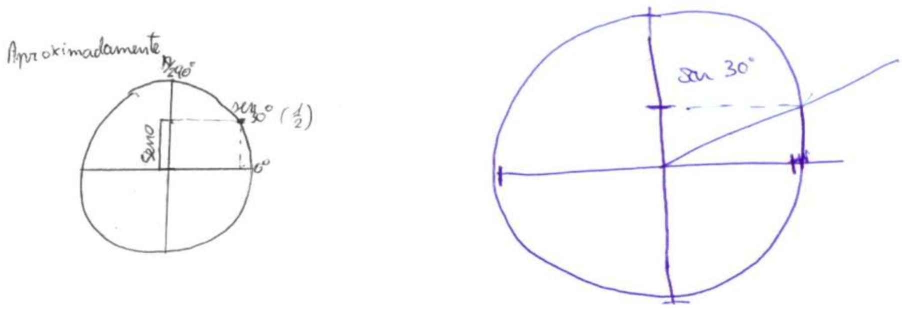


Figure 4.7. Production related to the subcategory "point in a circle and a segment" and "central angle"

Finally, interpretations of the sine and cosine of an angle given by the students were identified. It allows us to establish the category "way of indicating the trigonometric ratio". The identified subcategories were: "quotient", "cartesian coordinate", and "length".

Table 4.1. *Categorization of productions for the circumference theme*

Categories			Circumference
	Identification		Signalling
Elements of division of the circumference	Axis		With signalling X-Y Without signalling
	Segments	Diameter Radii Radii and diameter	
Way of indicating the angle	Central angle		The amplitude is indicated
	Interior angle of a triangle		The amplitude is indicated or not
	Point in a circle and a segment		
Way of indicating the trigonometric ratio	Quotient Cartesian coordinate Length	Catetus/ hypotenuse	Internal indication: coloured, underlined and segment External indication: arrow, bracket,

Regarding the triangle theme, two subcategories were revealed by the data, "right triangle" and "non-right triangle", both included in the category "type of triangle used". Similarly, to the way we analysed the theme circumference, we studied how the students interpreted the sine and cosine of an angle. Three subcategories were included under the category, "way of indicating the value of the trigonometric ratio": "quotient", "length" and "interior angle".

Table 4.2. *Categorization of productions for the triangle theme*

Triangle			
Categories	Identification		Signalling
Type of triangle used	Right triangle		With or without the right angle symbol
	Non-right triangle		
Way of indicating the value of the trigonometric ratio	Quotient	Catetus/ Hypotenuse Sides	Through vertices By sides
	Length	Side Catetus	Internal indication: coloured, underlined and segment
	Interior angle		External indication: arrow, bracket, auxiliary segment and dimension

4.6. Stage 2. Methodological aspects

4.6.1. Data collection tools

Once again, we designed a semantic questionnaire associated with the sine and cosine of an angle, consisting of 10 items, which sought to gather evidence of issues such as constructions of angles associated with a value of the sine or cosine, conversions and reasoning between some of the frequent trigonometric representation systems, reasoning and identification of students' mistakes, or how pre-service teachers make sense of the sine and cosine.

This study was designed following researches of Fi (2003), Brown (2005) and Martín-Fernández et al. (2019). We also consulted tasks used in other studies in order to design the questionnaire, and some items were taken or modified from them.

In addition, two specialists in the field of mathematics education were consulted about the adequacy of the tasks proposed, their order, their presentation, and their aims, with regards to our theoretical perspective. It was highlighted that components of the three semantic categories stated by Rico (2012, pp. 51-53) can be recognized in the responses to the questionnaire.

Afterwards, we analysed item 3, chosen as reactive question of the survey, as shown in figure below. It has been selected as a catalyst given that it has the potential to elicit relationships between the configurations of two partial trigonometry systems, as required by the core of this study. Additionally, this item was also chosen because of its complexity and its similarity to a task closely related to participants' abilities to answer related questions trigonometry relational system, as suggested by Brown (2005, p. 139). Thus, this dissertation includes only one task of the questionnaire aforementioned, related to the transitions between mathematical notions in a trigonometry relational system because of the importance of the conversions of representations in the understanding of trigonometry relational system.

Item 3. Point out in the trigonometric function the point which corresponds to the point P. Define the cosine of the angle corresponding to the point P.

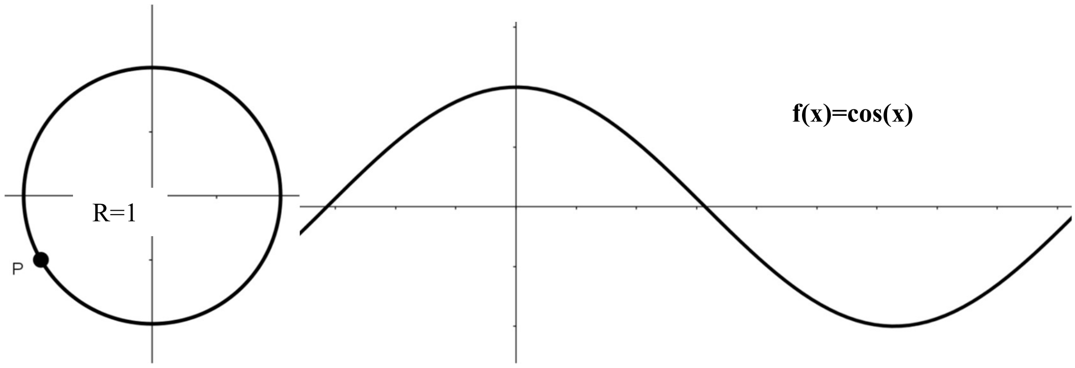


Figure 4.8. Analysed question of the questionnaire

The analysed task of the questionnaire asks students not only to convert a point P from the unit circle to the representation of its corresponding one on the trigonometric analytic function, but also to define the cosine of the related angle to P in one or both of the involved partial trigonometry systems depending on their ability. Obviously, the two systems are mainly linked by the concept of angle, which is the central concept in trigonometry relational system whose representation may be made by a point in both systems. The definition and the construction of the cosine operate as a support to identify the representation of the angle corresponding to P, since the cosine of the angle is a coordinate of the point P in both graphical representations. Implicitly, this task also requires students to process the point P in the first partial trigonometry system. Concretely, students should be able to process the point P either into the measure of an angle, into a number associated with it, into an ordered pair of coordinates, or into its cosine. Then, by means of converting the angle, its measure, its cosine, etc., between representation systems, students would point out the point in a trigonometric function. Normally, the mentioned conversions consist of going from angle as a measure of an arc in the goniometric circumference to angle measured as a numerical value (abscissa of a point in a function), and of going from cosine as an abscissa to a cosine as y-coordinate.

The chosen situation of the task is also based on the reconceptualization of the cosine of an angle, when the domain of the angle is expanded from the interval $[0,2\pi]$ to the real numbers field and its meaning changes. A field extension occurs in mathematics when a partial system is generalized and included into a broader system, as is the case of the nesting of the finite decimal numbers in the rational numbers (Feferman, 1989). A generalization is a new combination of concepts and procedures with different levels of extension, which focuses on certain essential notions that maintain their sense by their application to extended situations. Some of the notations and rules of the previous worked representation system are maintained; some are generalized to fit the enlarged concepts while others are not. Namely, the trigonometric representation systems are not bijective, emphasizing different properties and highlighting uniqueness by means of specific signs, which increases the difficulty when converting notions between them (Lakoff & Nunes, 2000). As a consequence, learners cope with the changes of meaning caused by conversions (Skemp, 1987, pp. 40-41; Chin, 2013, p. 44).

The chosen partial trigonometry systems have been selected due to the fact that the unit circle and the trigonometric function are essential contents for the systems of representation of angles, and for the values of their trigonometric lines, whose meanings are required as a prior basis for their understanding and for solving trigonometry relational system-related tasks (Koyunkaya, 2016, p. 1471). Furthermore, they are usually introduced in classroom following that order (Demir, 2012, p. 1).

4.6.2. Participants and settings

The participants that were selected for this stage were seventy-two graduate students following a Pre-service Secondary Mathematics Teachers training program at a large Spanish public university. The program has four modules: a generic educational module, a specific module (in which students take courses from the Department of Mathematics and from the Department of Mathematics Education), an elective module, and another module, which comprises a practicum, together with a final project (MEC, 2007). None of the participants had any previous teaching experience in a school. While 53% of the sample possessed a bachelor's degree in mathematics, the remaining possessed others bachelor's degrees such as: civil engineering, architecture, physic, electrical engineering, chemical engineering, and statistics. We highlighted the variety of bachelor's degrees within our sample. The participants had developed notions about mathematics as students in high school, and in college mathematics courses. Thus, the instructional experiences of the participants were multiple and varied previously to the study. This group of students might be considered as a convenient sample of pre-service secondary school teachers in Spain.

Since the participants in this study were Spanish students, and we work within a curricular framework, it is adequate to summarize the references to the teaching and learning trigonometry relational system in the Spanish educational setting as implemented prior to the university level. In 10^{th} grade, the evaluation criteria stated in the mathematics curriculum include: a) using the angular units of the sexagesimal and international metric system and relations and ratios of the partial elementary geometry system to solve trigonometric problems in real contexts. Next, in 11^{th} grade students encounter the trigonometric functions. At this level some of the evaluation criteria stated in the curriculum include: a) identifying elementary functions (trigonometric and their inverses) given through statements, tables or algebraic expressions that describe a real situation, and analysing qualitatively and quantitatively, their properties to represent them graphically and extract practical information that helps to interpret the phenomenon from which they are derived, b) studying and representing functions graphically.

As it has been mentioned, the reason why we work with pre-service teachers is partially strategic, given that previously we had studied the meaning of the sine and cosine of an angle in a group of secondary school students (Martín-Fernández et al., 2016; Martín-Fernández et al., 2019), and we utilise their results in order to shed light on pre-service teachers' findings. Furthermore, Chin (2013) states "it is definitely worthwhile to see how pre-service teachers who have studied more advanced concepts in trigonometry at university think about the forms of trigonometry that they may teach in school" (p. 33).

4.6.3. Implementation of the instrument

The questionnaire was implemented in the winter of the academic year 2016-2017. After finishing the subjects of the educational module, and while the participants were already pursuing subjects of the specific module, the participants answered the questionnaire. Concretely, the questionnaire was delivered and completed during an ordinary college class period of 60 minutes. The tasks were presented in a booklet which included ten open-ended tasks, some of which comprised more than one question.

The students were told to be creative and to answer the booklet with interest.

4.6.4. Data analysis

The respondents' answers to the items were analysed qualitatively. For this study, the grounded theory approach was utilized in order to establish patterns, procedures associated to the tasks, and explanations of participants' answers through analysis of data (Strauss & Corbin, 1994, 1998).

First of all, we coded each data source by participant and by the bachelor degree held by each of them. Subsequently, we examined the task proposed, and identified units of content in the responses. Afterwards, we defined criteria to organize the variety of knowledge associated with each response, and coded the data. After comparing our system of coding, it became clear that some of the criteria needed to be reviewed and refined. Once the revisions had been made, we arranged our system of organization by categorizing the criteria into themes by means of their contents and components which are identified below.

Finally, we produced contingency tables to find out the effect of the meaning of the concept angle corresponding to the point P in the goniometric circumference in the responses.

Table 4.3. *Analysis of the school mathematical content for the analysed question*

CONCEPTUAL KNOWLEDGE		PROCEDURAL KNOWLEDGE
First level: Facts		First level: Skills
Terms	Unit circle	Drawing angles
	Quadrants	Identifying angles
	Coordinates	Estimating angles
	Trigonometric function	Relating angles
Notations	Degree sexagesimal (º), Radians	Comparing the values of the cosine of some angles
	Sin, cos	Projecting points towards the axes
Conventions	Positive angle are represented anticlockwise	Calculating metric relationships
		Identifying the cosine of an angle in the unit circle
		Limiting the value of the cosine within the confines of an interval
		Labelling Cartesian axes, unit circle
		Bound the angle in an interval
		Identifying the cosine of an angle in the trigonometric function
		Estimating the cosine in the trigonometric function
		Identifying the negative value of the cosine in the unit circle and in the trigonometric function.
Results	The value of the cosine is between -1 and +1	
Second level: Abstractions and generalization		Second level: Reasoning
Concept	The cosine in the unit circle	Converting the cosine from the unit circle to the graph of the trigonometric function.
	The cosine in the trigonometric function	Converting the angle from the unit circle to the graph of the trigonometric function.
Third level: Structuring		Third level: Strategies
Strategies	The technique to solve the task	Solving geometric task utilizing different results

5.RESULTS OF THE FIRST STAGE

5.1. Introduction

"*Meaning in mathematics is the fruit of constructive activity*" (Thom, 1973, p.205)

This chapter describes the main results from the data analysis of the study conducted in this report concerning the first stage. We have divided this chapter in four different sections.

After the introduction, the next section begins linking the meanings shown by secondary school students, their teaching and the history of trigonometry relational system. Subsequently, we explain the obtained themes of each one of the semantic categories. The information obtained allows the identification of the typologies of meaning. Namely, in this chapter we identify and characterize the meanings of the sine and cosine of an angle shown by secondary school students.

5.2. The teaching related to the meanings shown by secondary school students

Once the categories and subcategories that organize the notions and representations obtained had been established for the questions 1 and 6, we analysed their relationships and links, and we elaborated the following conceptual map that shows the structure of the responses, their percentages of use and the interpretation of the students. We present it with up to four levels according to its complexity, revealing the sequence of contents used and inferred meanings.

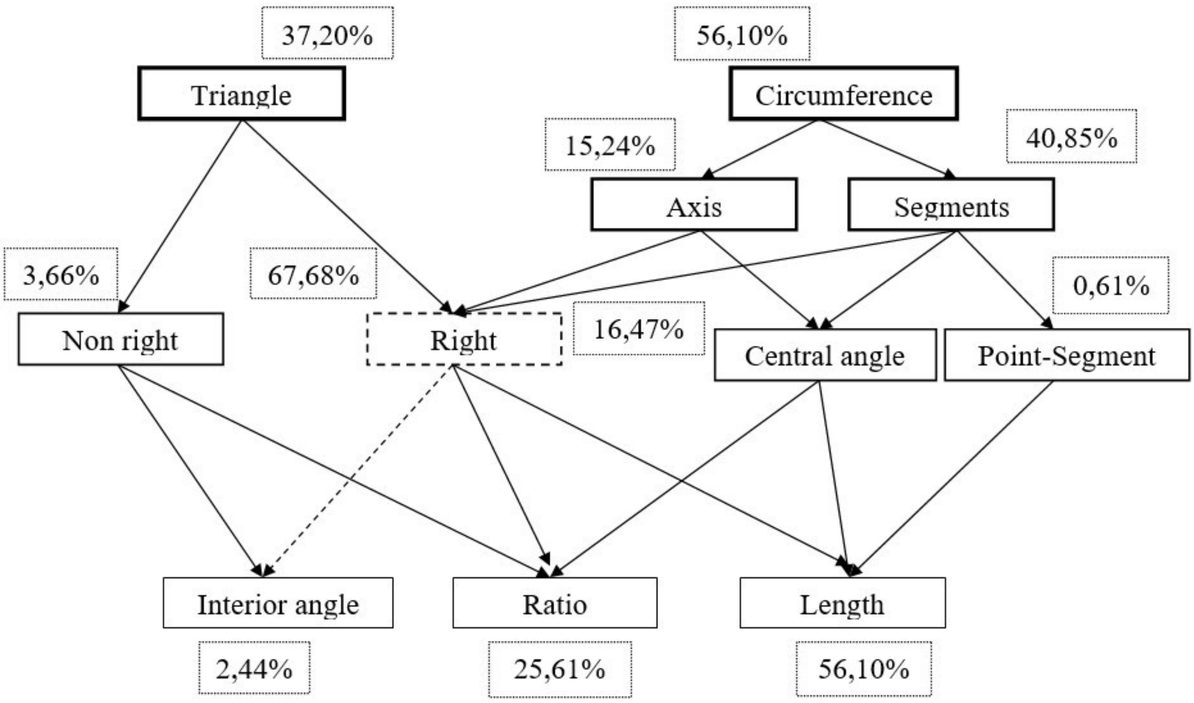


Figure 5.1. Relationship between themes, categories and subcategories for questions 1 and 6.

In a first level, we show the general themes, identified at the beginning of the content analysis described in the previous chapter, which were the circumference and the triangle. They provided a first organization for the information collected. A second level arises when comparing the different representations, observing their differences and similarities, including in the case of the circumference the geometric elements that are used to divide it, axes and segments. It should be noted that all the productions related to the "circumference" theme are included in these two subcategories. At a third level, a new step is collected in the representation sequence which, for the "circumference" theme, is the way of indicating an angle in the initially drawn circumference. We are mainly referring to the central angle and the point-segment. If we examine the topic "triangle," we place at this level the distinction between right and non-right triangle.

Finally, we identify a fourth level, which includes the meanings or interpretations and modes of use that students make of the sine and cosine of an angle: interior angle, ratio and length. It should be highlighted that the sine of an angle can only be interpreted as an interior angle if it starts from the triangle. That is the reason why that connection is shown as discontinuous.

Table 5.1. *Contingency table on units of analysis and content elements of the levels*

Content elements of the levels/ Basic units of analysis	Triangle	Circumference
Non-right triangle	Ratio/Interior angle	
Right triangle	Ratio/Length/Interior angle	Ratio / Length /Value
Central angle		Ratio / Length
Point-segment		Length

As a contribution of the conceptual analysis to the meaning analysis, we work with the organizer History of mathematics to interpret the information collected in a conceptual map (figure 5.2). This concept map shows the structure of the responses and the students' interpretation.

First of all, we consider the original problem from which trigonometry relational system arises: "Given an angle arc, find the length of the chord that connects the end points of the arc" (Van Brummelen, 2009, p. 41)

On a first level, the two historical ways of measuring angles are presented, which are related to the ways of teaching and understanding trigonometry relational system: the “ratio system”, using the triangle, and the “line system”, using the circumference.

On a second level appears the subdivision of the triangles into right triangles and non-right triangles, and the subdivision of the circumference into goniometric circumference (first used by Abu'l Wafa) and the circumference of radius $R \neq 1$. We must remember that Hipparchus and Ptolemy began using circumferences of radius 60 units. This radius was increased to obtain greater precision in the calculations.

With regard to the circumference, we find two types of angle measurements, by means of the length of the segment of the semicord (Aryabhata), and by its coordinates in the goniometric circumference (François Viète).

As far the triangle is concerned, we can distinguish between right triangles and non-right triangles. In the case of a non-right triangle, it would become a right triangle. Among the valid interpretations for the sine or cosine we find the ratio (Simon Klugel). Other interpretations emerging from the triangle and which are considered as misunderstandings are the identification of the sine or cosine with an interior angle, and with a distance in a right triangle whose hypotenuse is not one.

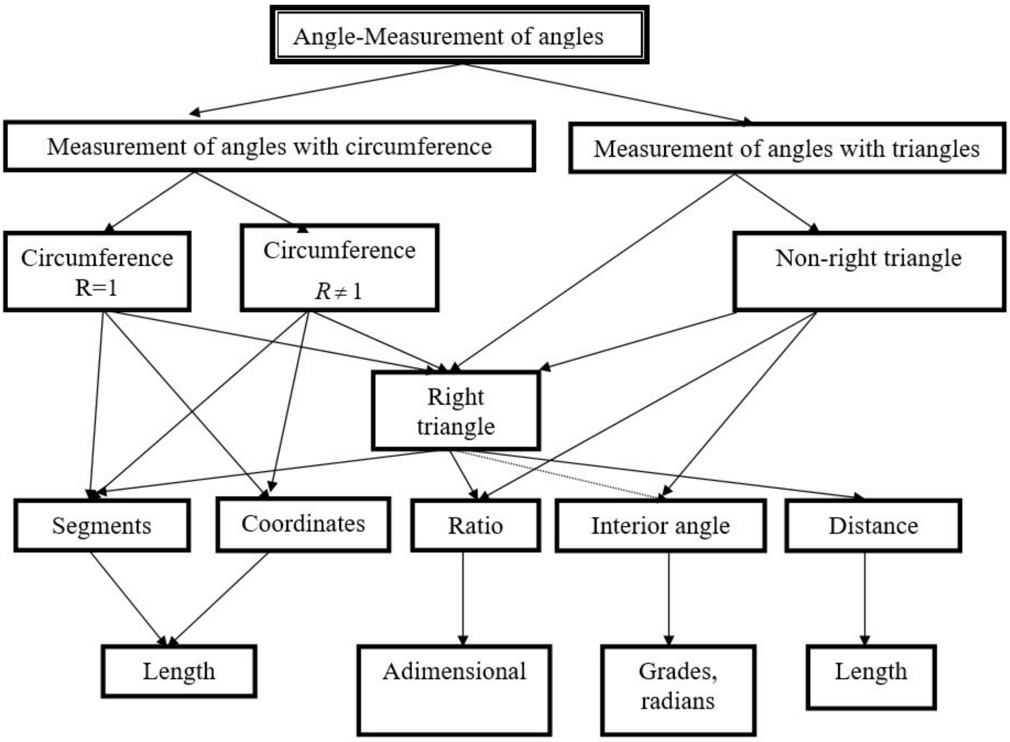


Figure 5.2. Conceptual map of the measurement of the angle

This conceptual map provides the meanings shown by the students of the sine and cosine of an angle.

5.3. Semantic triad of the sine and cosine of an angle

The meaning of the concept sine and cosine of an angle is presented for secondary school students, and it includes the conceptual structure, the representation systems, and the sense.

5.3.1. Conceptual structure of the concept of the sine and cosine of an angle

The questions 2-A and 2-B were designed to exemplify this subsection. The answers to these questions were classified together due to the similarity between the concepts sine and cosine. The table 5.2 displays the percentage of the types of responses associated with conceptual structure.

The first theme of response is associated to different ways of expressing a ratio such as: a formula, a relation, or the use of a variety of synonyms of the verb to divide up or to split. The second theme was determined basing on the students' reference to a length as a cathetus, a height, a base, a side of a triangle, a segment or a coordinate. The third kind of response is shown when students write down a numerical value, for example: "0.7"," 1⁄2"; when they point out that the sine or cosine of an angle is a "number" and when they express that it is a "value". In the fourth type of answer, students show the sine and cosine of an angle as a tool by means of employing expressions like: "to solve unknowns", "to work out triangles". Besides, there are answers that show the sine and cosine as a measurement, as can be recognized in the use of expressions such as "half of the radius", in which the unit (radius) and the measuring (half) are expressed. Finally, a response was interpreted in the theme angle because the studied concepts were described with expressions such as "side of the angle" or "adjacent side of an angle". Most students related the sine and cosine with the right angle triangle and almost nobody made mistakes when they named the different parts of it. The results show that the ratio is the main theme, and almost none of the participants considered the sine and cosine as a tool or as a proportion. Besides, forty percent of students failed the question. Finally, it should be noted that each response can involve diversity themes given that an answer includes various sentences.

Table 5.2. *Percentage of the types responses for questions 2A and 2B*

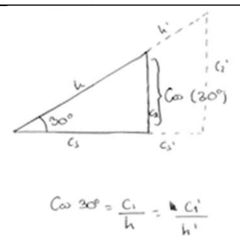
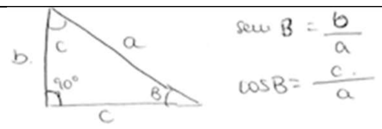
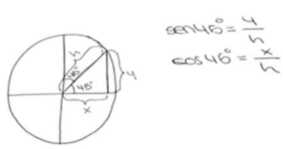
Themes	Subthemes	Percentage N= 101
Ratio	Proportion	1.98%
	Ratio definition of sine and cosine in a right-angle triangle	36.63%
	Relation of sides in a non-right triangle	4.95%
Length	Sides of a right-triangle with hypotenuse 1	0.99%
	Side of a right-triangle with hypotenuse different to 1	0.99%
	Sides of a triangle	0.99%
	Directed segments on a unit circle	9.90%
	Directed segments of a circumference	2.97%
	Cosine as the x-coordinate and the sine as y-coordinate of a point of the unit circle	0.99%
Value	Value of the sine and cosine	11.88%
Tool	As a tool for solving problems	1.98%
Measurement	Measurement	4.95%
Angle	Sides of an angle	8.91%
Others		11.88%

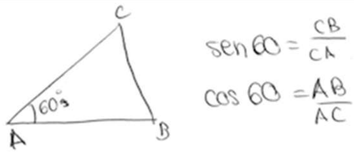
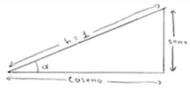
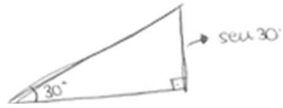
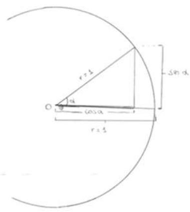
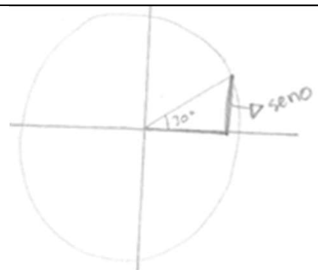
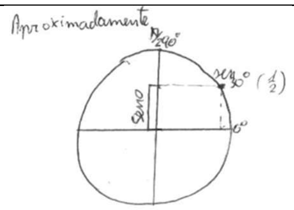
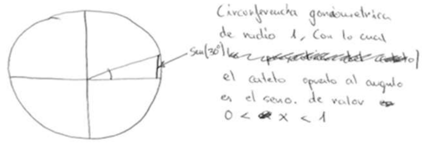
5.3.2. Systems of representation of the concept of the sine and cosine of an angle

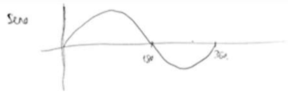
The first and sixth questions were designed to exemplify this category, which demands a graphic representation to show what students draw as the sine or cosine of an angle.

Although there is a wide variability of responses, and thus subthemes, finally, they mainly interpret the sine and cosine as ratio or length. Furthermore, responses seldom include the value of the sought sine and cosine. The interpretation as angle and function appears incidentally in productions. There are answers that involve more than one interpretation of the concepts studied.

Table 5.3. *Percentages of the types of response for questions 1 and 6*

Themes	Subthemes	Examples	Percentage N=184
	Ratios of lengths in right-angle triangles		1.09%
	Ratio definition of sine and cosine in a right-angle triangle		21.74%
Ratio	Ratio of lengths for a rotations of a general radius (on a circumference)		2.72%

Themes	Subthemes	Examples	Percentage N=184
	Ratio of lengths in any triangle		0.54%
	Sides of a triangle with hypotenuse 1		1.63%
	Sides of a right-angle triangle		7.07%
	Directed segments on a unit circle		15.76%
Length	Directed segment on a circumference		22.83%
	Cosine as the x-coordinate and the sine as y-coordinate of a point of the unit circle		1.63%
	Estimation of the sine or cosine		0.54%

Themes	Subthemes	Examples	Percentage N=184
Value	Value of the sine or cosine		9.24%
Function	Function graph		0.54%
Angle	Angle interior of a triangle		2.72%

The notion of circumference appears in most of the answers, although the most frequent correct answer response expresses the idea of ratio. In addition, the findings also show that subjects rarely refer to them as a function. Lastly, there are answers that involve more than two themes, as it can be seen for instance in the subtheme "Ratios of lengths in right-angle triangles", whose response can be included in the subtheme "Sides of a right-angle triangle" as well.

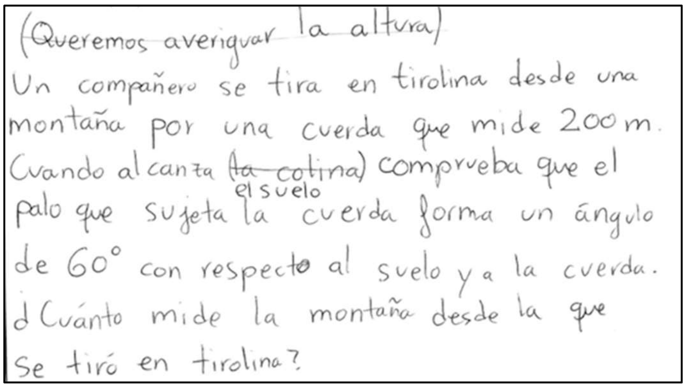
5.3.3. Sense of the concept of sine and cosine of an angle

The eighth question asked pupils to pose a problem where the sine or cosine of 60º must be used. We distinguished two main themes in the obtained responses.

Indirect measurement of a magnitude: A response belongs to this theme when the unknown quantity or quantities are a distance or an angle, for whose knowledge it is necessary to calculate the sine or cosine of an angle.

Although sine and cosine are used to calculate angles, volumes or areas, these applications barely appear. Besides, although some answers require calculating an angle, the cosine or sine are rarely used to find it out, and thus it is only required to add up the angles of a triangle and to consider that the angles in a triangle sum 180º.

Another point worth mentioning here is that in order to solve the problem, it is only necessary to know the definition of the trigonometric ratios in the right triangle. Thus, few of these problems are solved using the relations and properties of the sine or cosine. For example:


(Queremos averiguar la altura)
Un compañero se tira en tirolina desde una montaña por una cuerda que mide 200 m. Cuando alcanza ~~la cotina~~) el suelo comprueba que el palo que sujeta la cuerda forma un ángulo de 60° con respecto al suelo y a la cuerda. ¿Cuánto mide la montaña desde la que se tiró en tirolina?

A student went zip-lining from a mountain using a rope that measures 200 m. When he reached the ground, he checked that the stick that held the rope formed an angle of 60º with respect to the ground and to the rope. How high was the mountain from which he went zip-lining?

Figure 5.3. Example of response offered by a student

Computation of the sine and cosine of an angle: In this theme two subthemes appear: a) Calculate the trigonometric ratios of an angle, given the trigonometric ratio of another, and b) Find the value of an expression in which the studied concepts are involved.

A statement was interpreted as being in the sense a) if it includes a question in which the properties of trigonometric ratios are needed so as to find out a value. An example answer for this subtheme is "knowing the trigonometric ratios of the angle of 30º, find the cosine of 60º" (Figure 5.4). It should be noted that all responses of students were related to the angles of the first quadrant.

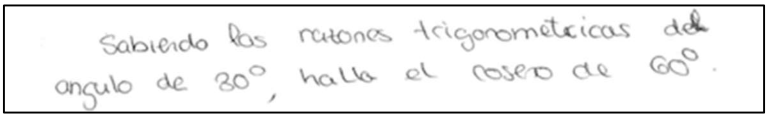


Figure 5.4. Answer about working out a trigonometric ratio

Option b) is associated with responses that involve substituting a parameter using an angle and remembering perfectly some values of the sine or cosine. It includes answers such as: "solve: sin2x where x=30º" (Figure 5.5). Similarly, in these productions only angles of the first quadrant appeared.

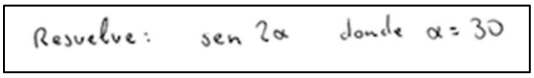


Figure 5.5. Answer related to find the value of a trigonometric expression.

The table below displays the percentage of the types of responses associated with the sense.

Table 5.4. *Percentage of themes expressed by the answers to question 8A and 8B*

Themes	Subthemes	Percentage N=77
	Angles	5.19%
	Length	70.13%
Indirect measure of a magnitude	Angles and length	6.49%
Computation of the sine and cosine of an angle		9.09%
Without answer		9.09%

All the responses to this question are related to a triangle. Thus, problems related to trigonometric functions or unit circle and their applications do not occur. We also indicate that students do not propose any problem in which it was required to solve an identity, or a trigonometric equation. Therefore, this kind of tasks may be considered somehow complicated from a mathematical point of view. This would imply that the students have not understood the variety of senses of the sine and cosine.

Finally, the majority of responses pose a question in an educational setting. Any response includes an angle of depression. Specific terms name objects that are related by means of well-known links between the objects. The terms used by pupils were: roof, post, firefighter, tower, lamppost, statue, tree, cities, ladder, building, house, balloon, lighthouse, mountain, rope, road, radio station, kite, and shadow.

Furthermore, we illustrate the most common situations (i.e. those with a percentage higher than 5%) that are included in the previous themes.

Table 5.5. *Situations of the types of responses*

Situations	Percentage N=77
The height of an object	14.29%
The distance to an object	19.48%
Finding unknowns of a triangle	14.29%
Distance of a shadow	5.19%
Distance between two cities	5.19%
Calculating the sine or cosine of an angle	7.79%

Finally, we distinguish between problems with one step or more steps. The majority of them are one-step problems (91.5%). Figure below illustrates a multiple-steps problem:

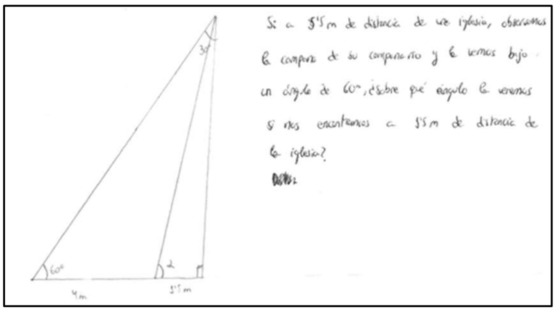
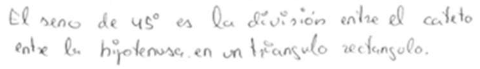
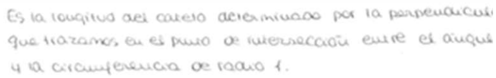


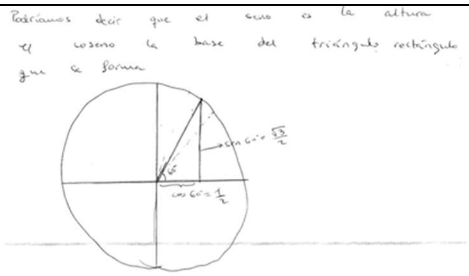
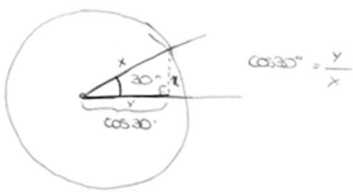
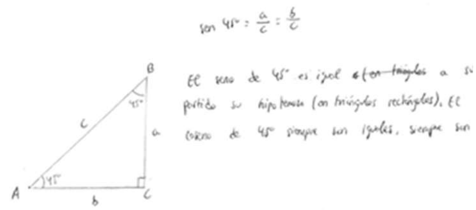
Figure 5.6. Answer related to problems of more than one step

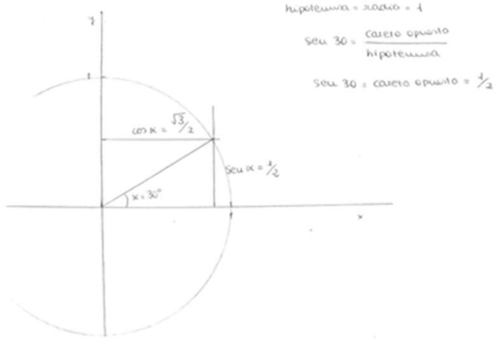
5.4. Combinations of themes in items 1, 6 and 2

In this section we classify the individual responses depending on the combinations of themes that appear in them. There are student answers that involve diverse themes. These productions may imply various meanings. The table below shows percentages for themes and their combinations, as expressed by students in their individual answers to questions 1, 6 and 2. Some combinations present a low percentage. We show in the table below those whose percentage is higher than 2%.

Table 5.6. *Percentage of responses which includes combinations for question 1, 6 and 2*

Number of themes	Themes and their blending	Example	Percentage N=222
One single theme	Quotient	 (The sine of 45º is the division between the opposite leg and the hypotenuse in a right triangle)	23.32%
	Length	 (It is the length of the leg determined by the perpendicular to the axis that we draw at the point of intersection between the angle and the circumference of radius 1)	28.70%

Number of themes	Themes and their blending	Example	Percentage N=222
	Angle		4.48%
	Length-Value	 (We could say that the sine is the height and the cosine is the base of the right triangle)	5.38%
Two themes together	Quotient-Length		9.42%
	Quotient-Value	 (The sine of 45º is equal to its opposite leg divided by its hypotenuse. The sine and cosine of 45º are always the same, they are $\frac{\sqrt{2}}{2}$)	6.28%

Number of themes	Themes and their blending	Example	Percentage N=222
Three themes		 (Hypotenuse= radius=1; sin30º=opposite leg/hypotenuse; sin30º=opposite leg= $\frac{1}{2}$)	2.24%

5.5. Typologies of meaning of the sine and cosine of an angle

After considering in isolation the themes and subthemes by categories, we have regarded their most frequent combinations and identified the triads presented jointly by each student. These combinations, empirically observed according to semantic categories, show a variety of meanings expressed by the student group for these concepts. To infer that information, we built a contingency table, which summarizes the relationships and interactions between the most frequent themes.

Table 5.7. *Table the contingency with the percentage of theme*

			Systems of representation		
Conceptual structure			Ratio	Length	Value
Ratio	Sense	Indirect measure of a magnitude	17.07%	19.51%	5.69%
		Computation of the sine and cosine	3.25%	1.62%	0.8%
Length	Sense	Indirect measure of a magnitude	4.06%	17.07%	2.43%
Value	Sense	Indirect measure of a magnitude	4.06%	6.50%	4.06%
		Computation of the sine and cosine	0.8%	0.8%	0.8%

Note. 11.39% (=14/123) of the responses are not included in table 6 because they do not consist of all elements of the semantic triad.

Each of these triads show a specific type of meaning, according to the facts and components collected in the responses. As it can be seen, most of the options obtain a very low percentage of answers. We will not consider triads with percentages lower than 15%, hence only three prevalent groupings, with highest percentages will be considered as prototypes of meaning, as shown by these student answers.

The themes of the contingency table are represented by the vertexes of a triangle and schematized as in figure below. They are identified as follow: top vertex –conceptual structure-, lower left vertex -systems of representation-, lower right vertex -sense.

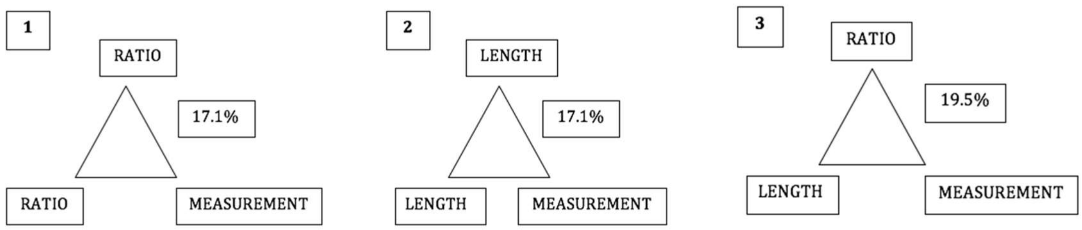


Figure 5.7. Prototypes of the meaning

6.RESULTS OF THE SECOND STAGE

6.1. Introduction

"In sciences, thinking is progressive: its most recent stages correct previous ones and include the truths that persist from these initial stages" (The Harvard Report, 1945)

This chapter describes the main results from the data analysis of the study conducted in this report concerning the second stage. We have divided this chapter in four different sections.

After the introductory section, the second and third sections correspond to the contents used as criteria to classify the types of participants' answers in the partial goniometry system and in the partial analytic geometry system respectively. The last section analyses jointly some criteria so as to determine the influence of the concept angle.

We distinguish six different content patterns chosen as criteria to classify the responses, three in each one of the partial trigonometry systems employed.

We describe each of these six criteria in detail, categorizing them into themes. In order to achieve this, we analysed the involved conceptual and procedural knowledge in the responses taking into account the facts, skills, concepts, reasoning, and strategies used by participants in their answers (Figure 4.3). We choose and display some examples of individual pre-service teachers works that are very revealing of how they think.

6.2. Criteria for the partial goniometry system

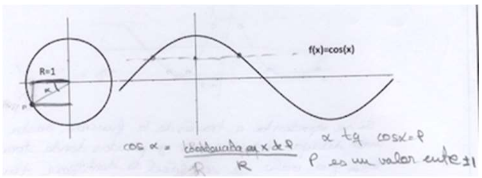
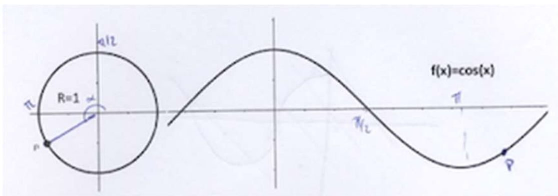
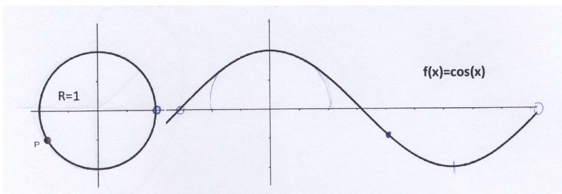
Related to the goniometric circumference, we recognize the following criteria:

- Identification of the angle corresponding to the point P,
- Strategies to build the cosine of the angle corresponding to P, and
- Meaning of the cosine corresponding to P in the unit circle.

6.2.1. Criterion one: Identification of the angle corresponding to the point P

The themes emerging from the responses can be described in terms of two different angle concepts: absolute geometric angle, and oriented goniometric angle, both didactically important according to Freudenthal (1973, p. 488-489). On the one hand, the absolute angle type corresponds to "elementary static angle" in the non-oriented plane, determined between 0ºy180º.On the other hand, the oriented angle type is considered as a dynamic version of angle, determined by an ordered pair of half-lines in the oriented plane or by means of a rotation higher than π (Hilbert, 1991; Russell, 1973, pp. 723-725; Freudenthal, 1973, pp. 488-489; Clements & Burns, 2000, p. 31). Therefore, the classification of the angle corresponding to P is based on the angle measure, and on the orientation of its pair of sides. Concretely, the theme "oriented goniometric angle" appears when the participants draw the angle in the third or first quadrant using sides, with origin in the positive x-axis or y-axis (only one participant), orientating it either clockwise (for positive values) or counter clockwise (for negative values, -a minority); when participants only label the unit circle so that the point P is included in the third quadrant, and finally if its measure is estimated and expressed by sexagesimals values higher than 180º. The theme "absolute geometric angle" is mainly related to angles whose measure is lower than 180º. Moreover, 15.27% of the participants do not identify the angle corresponding to the point P. Finally, one participant interprets P as an oriented angle and an absolute angle, and a small percentage of the participants represent two or more angles in the unit circle. The table below shows different types of responses under these themes, selected among those obtained.

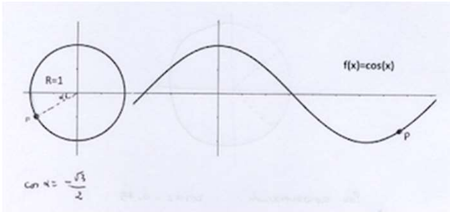
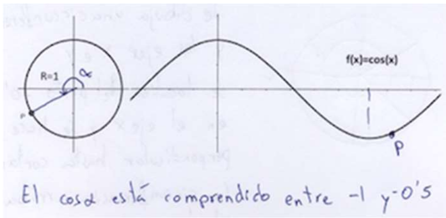
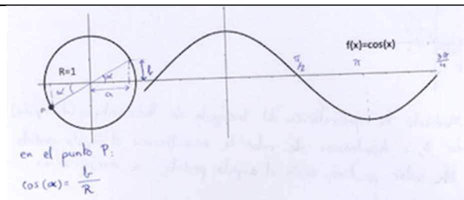
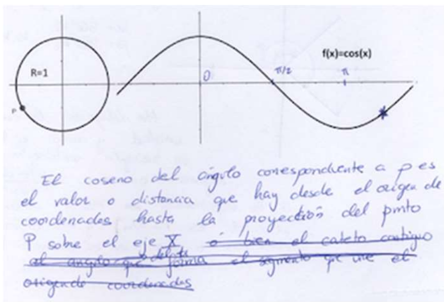
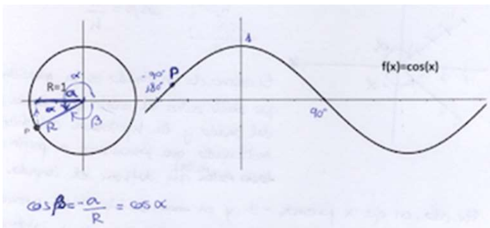
Table 6.1. *Percentage of types of answers for the criterion one*

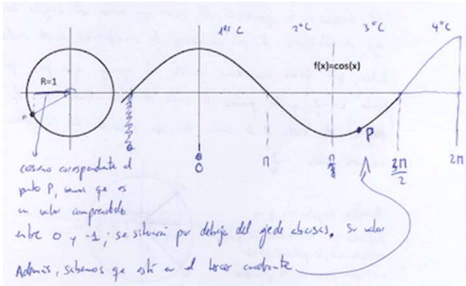
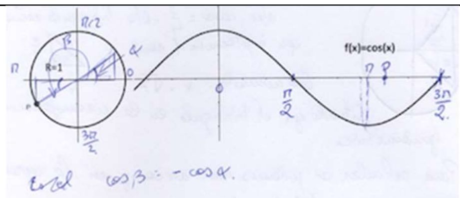
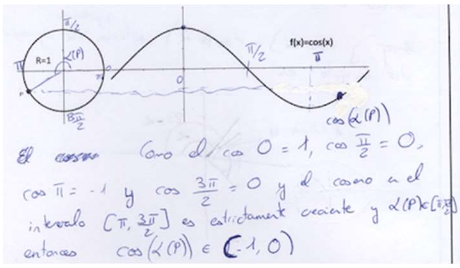
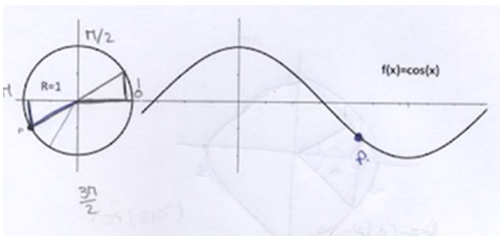
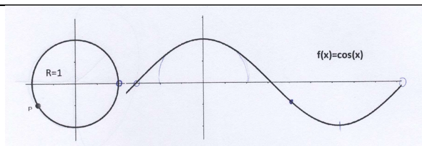
Classification	Examples	Percentage N=72
Elementary geometry angle		37.50%
Goniometry angle		45.83%
Do not identify		15.27%

6.2.2. Criterion two. Strategies to build the cosine of an angle corresponding to P

The emerged themes are: not build, estimation strategy, goniometric strategy, metric-goniometric strategy, metric strategy, and no apparent strategy. Estimation strategy is identified when participants give a numerical value (1) or limit the value within the confines of an interval without explanation (2). Metric strategy is developed when subjects calculate a ratio in a right angle triangle included in the unit circle but without expressing the negative value of the cosine (3) or when they only project the point P and define the cosine as a projection or distance (4). Metric-goniometric strategy is considered when participants project, and draw on some features of the unit circle by which, they consider the negative value of the cosine, and either use it when calculating metric relationships (5) or when estimating the value of the cosine of the angle (6). Goniometric strategy is utilized when subjects utilize relations between angles (7) or when they compare values of the cosine for certain angles in the unit circle (8). Moreover, some responses, which are not possible to know exactly what participants have considered, have been classified as no apparent strategy. Finally, there are answer that do not build the cosine, that we labelled “not build”.

Table 6.2. *Percentage of types the responses for the criterion two*

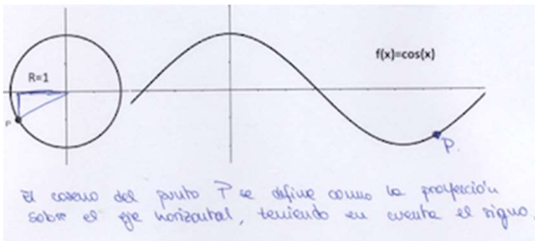
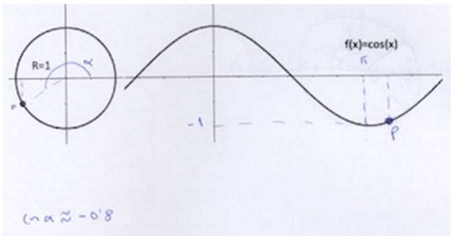
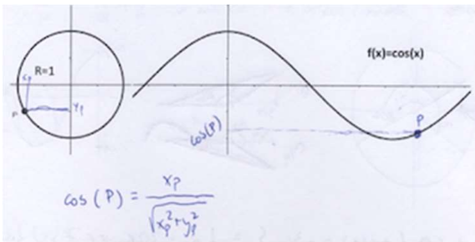
Themes	Subthemes	Examples	Percentage N=72
Estimation strategy	(1)		
	(2)		4.16%
Metric-strategy	(3)		
	(4)		25%
Metric-goniometric strategy	(5)		25%

Themes	Subthemes	Examples	Percentage N=72
	(6)		
Goniometric strategy	(7)		9.72%
	(8)		
No apparent strategy			8.33%
Not build			27.77%

6.2.3. Criterion three. Meaning of the cosine corresponding to P in the unit circle

The responses to the task show that the interpretation of the cosine of the angle in the unit circle was related to some of those from Brown (2005), and from Martín-Fernández et al. (2016; 2019). Concretely, the emerged themes from this criterion can be expressed as follows: length, value, ratio, and not build (Table 6.3). The first theme was based on participants' reference to a length, as a cathetus, a height, a base, or a projection. Three subthemes were found: sides of the triangle with hypotenuse 1 (2.77%), segment on a unit circle (8.33%), and cosine as a coordinate (12.5%). The second theme is showed when students write a numerical value (9.72%); when they limit the value of the cosine in an interval (6.94%), and when participants use a property related to the unit circle to give a value (4.16%). The third theme of the responses is connected to different ways of expressing a ratio such as a formula, or a relation. Finally, 36.11% of the answers do not interpret the cosine of the angle corresponding to P in the unit circle. It is highlighted that only a few answers reveal several meanings (8.33%).

Table 6.3. *Percentage of the types of responses for criterion three*

Themes	Examples	Percentages N=72
Length		23.60%
Value		20.82%
Ratio		6.94%

6.3. Criteria for the partial analytic geometry system

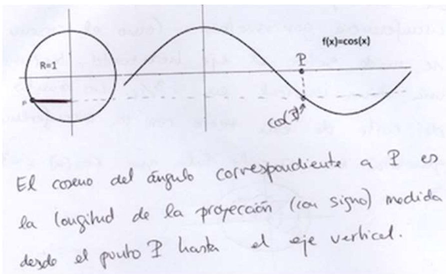
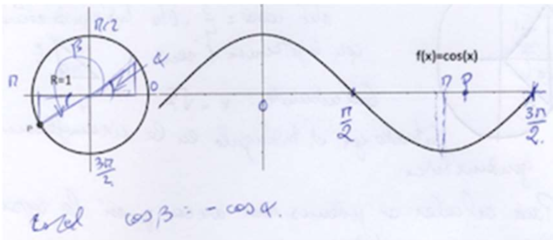
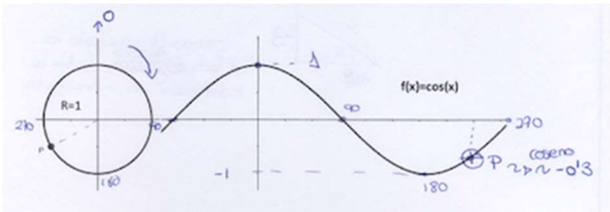
The second partial trigonometry system associated with the analytical function is characterized by the contents given for:

- Angle corresponding to P in the analytical function,
- Strategies to represent P in the graph of the analytical function, and
- Meaning of the cosine of the angle corresponding to P in the analytical function.

6.3.1. Criterion four. Angle corresponding to P in the analytical function

After analysing the responses, from this criterion emerged four themes: as P, as an angle and as P, as an angle, and not identified (Table 6.4). It is considered that when students mark the point P in the x-axis of the analytical function, they point out as P the value of the angle corresponding to P. There are some responses in which students label the x-axis as well (mostly in radians). Then, the value of the angle corresponding to P is indicated as an angle and as P. If participants label the x-axis and draw an auxiliary vertical line (8.33%), if they mark the x-axis using a typical sign of an angle (5.55%) and if they limit or bound somehow the value of the angle corresponding to P (6.94%), we can state that they express the angle corresponding to P as an angle. Finally, it is suggested that participants do not identify the angle corresponding to P in the analytical function when none of the above conditions are found in the responses (68.05%).

Table 6.4. *Percentage of responses for criterion four*

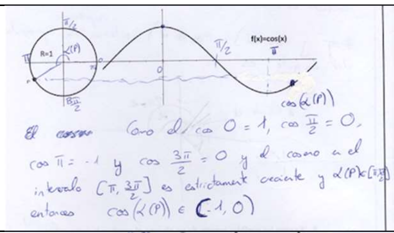
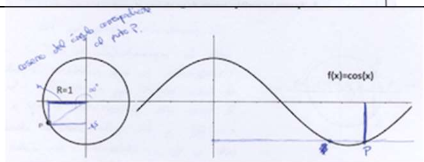
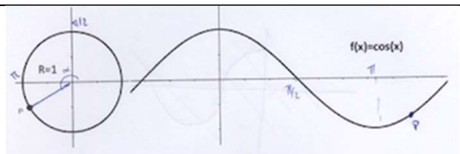
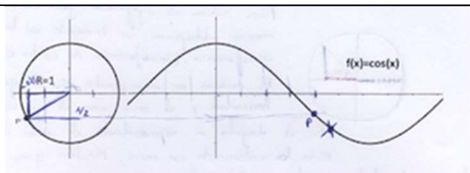
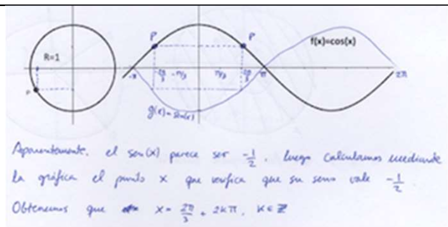
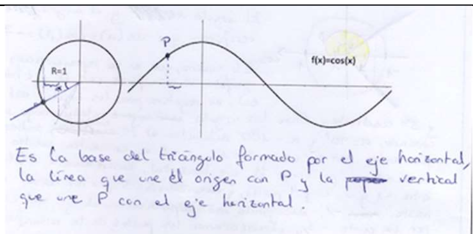
Themes	Examples	Percentage N=72
As P		5.55%
As P and as an angle		5.55%
As an angle		20.82%

6.3.2. Criterion five. Strategies to represent P in the graph of the analytical function

We identify the chosen strategy to answer the task. Seven themes were identified in the productions: using the angle and the value, using the angle and the ordinate, using the angle, using the ordinate, using the value, not build, and no apparent strategy.

Using the angle and the value is a strategy based on drawing the angle related to P in the unit circle and/or projecting the point P towards the Cartesian axes. After that, respondents usually identify the value of the cosine in the unit circle. Eventually, all of them convert it to the graph of the trigonometric function taking into consideration its associated angle, determining the point P. If participants point out or express the cosine of the angle as a length and convert it to the second partial system as ordinate taking into account the associated angle, we consider that this strategy is based on the angle and on the ordinate. Using the angle is another strategy which involves identifying the angle associated with P in the unit circle, and its subsequent use in the second partial system or conversion to the Cartesian axes of the graph of the trigonometric function. Then, based on this angle, the point P is determined with regards to the trigonometric function. Subjects utilize the strategy of using the ordinate when they perform a parallel line to the x-axis to represent a point or mark in the analytical function -they confuse in the second partial system the cosine with the sine- (22.22%), and when they identify the cosine in the unit circle as a length, converting it to the graph of the trigonometric function determining the point P without expressing any information about the angle (1.39%). Basing on the value means that participants identify points in the second partial system considering only the value of the sine or cosine associated to P in the first one. Furthermore, 12.5% of the responses are categorized as "not built" given that participants do not represent a point in the analytical function. Finally, the impossibility to infer how some subjects have solved the task make us codify their responses into the theme "no apparent strategy".

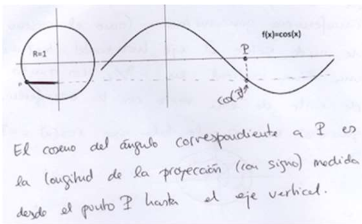
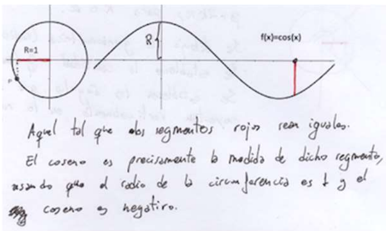
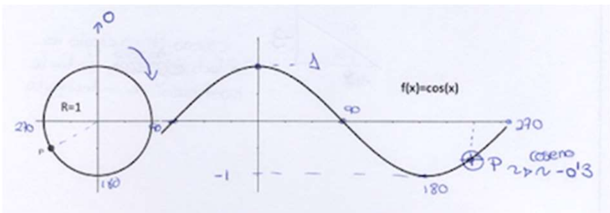
Table 6.5. *Classification of the criterion*

Themes	Examples	Percentage N=72
Using the angle and the value		18.05%
Using the angle and the ordinate		4.16%
Using the angle		27.77%
Using the ordinate		23.61%
Based on the value		1.39%
No apparent strategy		13.88%

6.3.3. Criterion six. Meaning of the cosine of the angle in the analytical function

Analogous to criterion three, the determination of the themes is grounded on the same principles. However, there is another theme that arises: point. It is considered that this theme appears when participants identify the cosine of the angle as a point in the analytical function. Therefore, the themes identified are the following: point, length, value, and not built (Table below). It is remarked that 54.17% of the responses belong to the last theme.

Table 6.6. *Classification of criterion six*

Themes	Examples	Percentages N=72
Point		5.56%
Length		16.67%
Value		23.61%

6.4. Influence of the angle concept corresponding to P in the unit circle

We produce contingency tables between criteria to study the influence that the angle concept in the unit circle exerts on the responses. As it does not provide relevant information, the row "not identified" is not included.

Table 6.7. *Contingency table with percentage for the themes of criteria one and two*

	Do not build	Estimation strategy	Goniometric strategy	Metric-goniometric strategy	Metric strategy	No apparent strategy
Absolute angle	5.48%	1.37%	1.37%	6.84%	17.81%	5.48%
Oriented angle	13.70%	2.74%	8.22%	15.07%	5.48%	2.74%

Note. N=73 and the percentage total related to the absolute angle and the oriented angle is 38.35% and 47.95% respectively.

The results suggest that the participants who draw absolute angles and solve the task predominantly use a metric strategy. Additionally, almost 50%of the students who draw an oriented angle utilise a goniometric strategy or a metric-goniometry strategy.

Table 6.8. *Contingency table with percentage for the themes of criteria one and three*

	As a numerical value	As a ratio	As a length	Do not build	Total (N=80)
Absolute angle	3.75%	6.25%	12.50%	13.75%	36.25%
Oriented angle	22.50%	6.25%	11.25%	11.25%	51.25%

As it is shown in the table above, whereas the given answers suggest that the oriented angle is connected with a "numerical value", the absolute angle is mostly associated with "length" and with "not built".

Table 6.9. *Contingency table with percentage for the themes of criteria one and four*

	Do not identify	As an angle	As an angle and as P	As P	Total (N=73)
Absolute angle	34.24%	1.37%	1.37%	1.37%	38.35%
Oriented angle	24.66%	17.81%	4.11%	1.37%	47.95%

These results show that participants who perform an absolute angle cope with its conversion to the second partial system. However, subjects who consider the angle as an oriented angle are more likely to convert the angle properly.

Table 6.10. *Contingency table with percentage for the themes of criteria one and five (N=73)*

	Using value	Using ordinate	Using angle and value	Using angle and ordinate	Using angle	Do not build	No apparent strategy
Absolute angle	0.00%	13.70%	2.74%	2.74%	4.11%	5.48%	9.58%
Oriented angle	0.00%	5.48%	12.33%	1.37%	21.92%	2.74%	4.11%

From this table, it can be considered that participants who use the absolute angle mainly utilize the ordinate to solve the task. It is related to the fact that many participants confuse the sine with the cosine in the trigonometric function and draw a parallel to the x-axis to determine the point P in the second partial system. Additionally, almost three quarters of the subjects who use an oriented angle utilize at least an angle to solve the task. Thus, whereas more than half of respondents who draw an oriented angle used a possible strategy to solve the task, 75% of the subjects who draw an absolute angle do not perform a valid one.

Table 6.11. *Contingency table with percentage for the themes of criteria one and six*

	As a numerical value	As a point	As a length	Do not build	Total (N=73)
Absolute angle	2.74%	1.37%	4.11%	30.13%	38.35%
Oriented angle	19.18%	2.74%	8.22%	17.81%	47.95%

As we can see, the majority of the participants that consider the angle as an absolute angle do not build the cosine in the trigonometric function. However, more than fifty percent of the students who have drawn an oriented angle build the cosine of the angle corresponding to the point P.

7. DISCUSSION AND CONCLUSIONS

7.1. Introduction

"A story has no beginning or end: arbitrarily one chooses that moment of experience from which to look back or from which to look ahead (Graham Greene, *The End of the Affair*)

This chapter presents an overview and discussion of the results that address the research aims. We justify the achievement of the aims supported by the empirical contributions obtained. This is followed by a discussion of the limitations of the study. The chapter concludes with suggestions for further research.

7.2. Achievement of specific aims

In this section, we discuss the degree of achievement of the specific aims established in the study.

7.2.1 Achievement of aims 1 and 2

Aims 1 and 2. To build a valid and reliable instrument to identify and collect the meanings shown by the secondary school students and by teachers in training, following established methodological criteria.

Chapter 4 indicates the main characteristics of a semantic questionnaire. In addition, the design of both questionnaires is presented. This design is based on the three semantic categories: conceptual structure, representation systems and sense.

The results obtained allow us to conclude that the implemented instrument has made it possible to characterize the meanings of the secondary school students and the teachers in training. This is the reason why we consider that the aim has been achieved. Furthermore, the versatility of the instrument employed has facilitated the emergence of unforeseen information.

7.2.2. Achievement of aim 3

Aim 3. To identify, describe and interpret the meanings about the sine and cosine of an angle that schoolchildren show when high school students respond to tasks strongly connected with each of the categories of meaning according to the perspective of Rico (2013).

As it can be seen in the empirical study, secondary school students interpret the sine and cosine of an angle in a variety of ways. To characterize their meanings, students' responses to different questions were analysed. Item 2 is related to the conceptual structure; items 1 and 6 are associated with the representation systems, and, finally, item 8 refers to the sense. We discuss the results of the analysis to the students' answers.

As it has already been mentioned, the data corresponding to the first meaning category are collected from the answers to the second item. These responses are not consistent with Brown's (2005) results, given that students do not only express the sine and cosine of an angle as length or ratio, but also as value, measuring, tool and angle. Most participants understand these notions as a ratio, since most answers are related to derivative terms of "divide". Thus, these findings are not aligned with those of Kamber and Takaci (2018), for whom the sine was defined by a minority of students as a ratio in a right angle triangle. Besides, some responses included numeric values and terms such as "value" or "result". It appears that the range of values of the sine and cosine is known because all the values proposed by students were ranged between -1 and 1. There are also a small number of productions that have been linked to an instrumental character given that they include a reflection of the usefulness of studied concepts. Besides, a minority of responses are related to a measurement or an angle. Finally, the participants do not identify these concepts with the idea of proportion, which appears incidentally, nor with the notion of coordinate of the sinusoidal function. Obviously, if its conceptual structure is not well understood by students, it will be hard to teach other concepts related to them.

Knowing the history of trigonometry education will serve as a partial guide for mathematics educators and researchers of mathematics education to suggest actions (Sickle, 2011). Therefore, based on the history of the teaching of trigonometry, we conjecture the understanding of this first category by an appropriate use of five fundamental notions: angle, ratio, right triangle, circumference and trigonometric functions. These notions are linked among them and originate various relationships (Martín-Fernández et al., 2016). Angles, ratios and circumference are related on account of the "line system", which defines the trigonometric ratios of an angle as line segments of a circumference, and it was originated from the ancient Greek and Arab conceptions of trigonometry (Maor, 1998). By the turn of the twentieth century, in secondary schools, trigonometric ratios were defined using real numbers arguments, "the ratio system", in which angles, ratios and right triangle are associated and trigonometric ratios are expressed as an a-dimensional ratio of sides of a right triangle (Sickle, 2011). In many countries, the most commonly used approach to introduce the trigonometric ratios is currently the “ratio system”. However, from a curricular perspective, if we define trigonometry basing on how this term is currently used, we must establish that trigonometry refers to the study of measuring the pertaining angles, triangles or functions (Fanning, 2016).

According to the second category of meaning posed in the first and sixth questions, and in contrast to the first category, most of the surveyed students mainly represent the sine and cosine of an angle as a length. Therefore, different items may imply and emphasize different themes and inferences about the understanding of the involved notions.

The themes for the representation system category are similar to those just mentioned for the previous category. Students understand how to represent the studied concepts by means of a ratio, length, value, and angle, but that students were apparently unable to relate the sine and cosine to the trigonometric functions as well. On the other hand, a great number of students tend to represent the unit circle without measurement, making mistakes. Moreover, the unit circle generally acts as a fixed icon, which at the end was not used to represent and understand the sine and cosine of an angle. Consistent to Challenger (2009), we interpret that students are not aware of the variety of the representation systems of the studied concepts. Students should understand their implications, advantages and drawbacks.

Related to the third category, students express problems and tasks associated with two different senses: indirect measure of a magnitude and computation of the sine and cosine of an angle. The sense of the studied concepts is mainly understood as a device to solve triangles; in particular, student responses aim to find out the distance between two cities, the height of a statue, the length of a flagpole, etc., involving fundamentally distances, angle of elevation, and height. Nevertheless, they are limited to apply the “ratio system” to solve triangles at the first quadrant so that the variety of problems posed by students was scarce. Additionally, few students used the law of sine and cosine. It seems that multi-step problems, calculating the measurement of an angle given the measurement of two sides, modeling periodic phenomena such as vibrations and planetary orbits, are not conceived as problems by students. In contrast to Kamber and Takaci's (2018) study, student responses stay at triangle trigonometry. This might be explained by noting that the definition of the sine and cosine using the unit circle is more generalizable and it requires more abstraction. Hence, the connections between the "ratio system" and "line system" are not adequately established. These answers can also be due to the instruction received by the students, even though the syllabus includes the identification, study, representation and interpretation of real tasks of trigonometric functions. It follows that it is compulsory to present tasks more related to the modern sciences in order to position trigonometry correctly within secondary education. Furthermore, sense is the main missing category considering the 12% of responses that does not include any reference to it. Thus, we detect a gap between the understanding of both concepts and we conclude that more attention to this issue is required. When participants were unable to express verbal problems, they utilized drawings to clarify their statements visually. This result corresponds with Sevimli and Delice´s investigation (2012).

Consistent with Thompson´s (2007) and Kamber and Takaci´s (2018) studies, we recognize a scarce connection between the studied concepts with real life and everyday mathematics of the lived-in experiences of children. In contrast with Allen (1977), navigation, surveying, carpentry and ballistics are not the predominant topics in trigonometry at the introductory levels of trigonometry in our data. The findings show that calculating the distance to an object is the most popular phenomenon among secondary school pupils. Nevertheless, they should perceive the need of this topic for their future profession, so that they would be more motivated in the study of trigonometric concepts.

On the whole, this analysis provides useful information to design and plan the didactic materials in order to implement them in school mathematics. Each category includes a variety of themes and elements of content which ought to be taken into account by teachers. The predominance of some of them gives teachers information to enrich their students' meanings. Noticeably, if students master a higher variety of themes, their meanings will be more complete and coherent.

After the identification of the main themes that emerge in the responses, we combine all of them to describe the meanings of the participants. To achieve this, a contingency table with the different themes is constructed. The three most representative meanings are considered as prototypes of meaning (Figure 5.7).

Historically, two ways of measuring angles have been used and preferably developed: the “ratio system” that uses ratio length, and the “line system”, using circumference (Sickle, 2011). Each of these meanings shows a proper way of understanding the sine and cosine notions, which is described by means of their themes.

With regard to the first meaning, schematized in triangle 1 (Figure 5.7), the sine and cosine of an angle are defined and represented as a ratio. This ratio is used to calculate an indirect measure of a magnitude. Ratio notion prevails in this trigonometric meaning prototype, which might be mainly related to the ratio system (Sickle, 2011).

As far as the second meaning (Triangle 2, Figure 5.7) is concerned, it requires that a length, related to the calculation of an indirect measure of a magnitude, characterizes the analysed concepts. Length of a bounded line, or segment, prevails in this trigonometric meaning prototype, which might be chiefly associated with the line system (Sickle, 2011).

In the third prototype meaning (Figure 5.7), students emphasize the connections between the “ratio system” (definition) and “line system” (representation) (Sickle, 2011) and it is also utilized to obtain an indirect measure of a magnitude.

Other meanings expressed by the students are localized in the contingency table by means of the triads of identified categories, together with their percentages. There are also students who leave questions unanswered, which are not analysed in this thesis.

7.2.3. Achievement of aim 4

Aim 4. To identify the meanings about the angle concept and its cosine shown by secondary school teachers in training. To describe the conceptual and procedural content when moving between the partial goniometry system and the partial analytic geometry system.

To study what meanings the students show, we follow the characterization of Freudenthal (1973, p. 479). From an instrumental point of view, he describes and highlights the main definitions of angles throughout history. Thus, he establishes three subsystems fundamentally within the trigonometry relational system: the elementary geometry system, the goniometry system, and the analytical geometry system.

To analyse the meanings shown by the teachers in training of the sine and cosine of an angle, we rely on results of the first stage. This part of the research is therefore a deepening of the study of the meanings of the sine and cosine of an angle. Knowing how subjects who have studied advanced mathematics think and handle trigonometric content is relevant and may be important to know its impact on the teaching of the trigonometry relational system.

Specifically, in the goniometric circumference, the identification of the angle corresponding to a point P is interpreted by a geometric absolute angle (37.50%) and by a goniometric oriented angle (45.83%). In the trigonometric function, the angle corresponding to a point P on the goniometric circumference is considered a point (5.55%), a point and an angle (5.55%), and an angle (20.82%).

Regarding the meanings of the sine and cosine considered by the teachers in training in the goniometric circumference, the following themes are observed: length (23.60%), value (20.82%), and ratio (6.94%). It is important to note that more than a third of the participants do not interpret the cosine of an angle corresponding to point P and that very few put various meanings into play.

In relation to the meanings of the cosine of an angle in the analytic function, it should be noted that more than half of the answers did not interpret the cosine in the analytic function. The themes that emerge are the following: point (5.56%), length (16.67%) and value (23.61%).

With regard to the meanings of the teachers in training on the cosine of an angle, it should be highlighted that almost all the productions are included within those already found in secondary school students. However, a new theme appears: point. This theme appears incidentally when participants identify the cosine of the angle as a point in the analytic function.

Finally, the Table 4.3 gathers the conceptual and procedural content utilised by students when moving between the involved partial systems. By means of these contents themes and subthemes are codified.

7.2.4. Achievement of aim 5

Aim 5. To investigate the understanding of secondary school students and teachers in training on these contents by means of the characterization of their meanings and their components.

We used the meaning framework to study the subjects' understanding of the trigonometric contents. Thus, the description of the meaning of a content allows us to know how the study participants understand them.

The semantic triad works as a three-component tool for identifying meanings of a school mathematical concept. Students with the elements of their productions have shown several meanings that we have interpreted and characterized in terms of its themes within semantic categories. Accordingly, this research has developed arguments and has provided reasons in order to improve our knowledge of mathematics concept understanding by students, and it has also contributed to establishing a method for its characterization. Our conjecture holds that when students have been in contact with specific mathematical notions, and have received training about them, they understand and learn them, strongly or weakly; they also use them and express them with a meaningful intention. Different students reach different developments, but their interpretative richness in each case may be identified from its content elements, and it can be described by the themes and structured by the meaning categories and their relationships. In fact, the combinations of the meaning categories expressed by students provide us with information about what each one has understood and what should be improved. Secondary school students are not aware of the basic elements of the studied concepts and their relationships. An incomplete meaning of a concept may involve difficulties for its understanding, and thus for teaching other related mathematical topics and disciplines.

Eighty-eight percent of the students' responses have expressed their understanding of the concepts sine and cosine of an angle, based on a coherent meaning notion, supported by the chosen three categories as semantic framework. Fifty-three percent of answers have revealed their meaning utilizing ratio, length and indirect measure of a magnitude. Most of them have explained their meanings using the themes of ratio and length, both balanced. Some students understand these concepts of several ways.

Taking into account the prototypes of meaning, we may conclude that the group of secondary school students do not possess a prevalent meaning of the studied concepts. In prototypes of the first and second meaning, they use as representation the one coincident with the conceptual structure in the definition; only in the third prototype there is a percentage of students that clearly distinguish between definition and representation systems, which provides a greater richness of meaning and shows a better understanding of these notions. We believe that these results are due to the lack of the connection between the "ratio system" and "line system", and our syllabus. In our syllabus, both systems are taught in mathematics in two school years and the teachers who participated in this study introduced them at the same time. Consequently, "ratio system" and "line system" are predominant in student responses, and the same percentage is obtained in the first and second prototype of meaning. However, more lessons should focus on the connections between both systems, so that students can be provided with more opportunities to enrich the meaning of the studied concepts. Emphasis on connections between "ratio system" and "line systems" may avoid fragmented understanding of the sine and cosine. Finally, all the prototypes concentrate on the indirect measurement of a magnitude. It seems that even although "line system" is predominant in a prototype of meaning, the kind of problems posed by students are typical of the "ratio system". Extra attention should be given to the connections with problems involving other themes in math classes. Finally, supporting understanding of the analised concepts appears to be not only an issue of supporting understanding of concepts such as angles, periodicity, properties learnt by heart, or connecting different periodic representations (Kamber & Takaci, 2018, p. 174; DeJarnette, 2018, p. 417) but also an issue of connecting categories of meaning and possessing a variety of meanings. Providing such relationships and a diversity of meanings has the potential to allow the understanding of school mathematical concepts.

The results also indicate that the students showed little knowledge about the sine and cosine as a coordinate. In most of the productions related to circumference (56.10%), their division elements were segments (40.85%). An incorrect expression of the coordinate axes was observed by the students, when using segments instead of lines or rays. Furthermore, the incidental character of the circumference was revealed in most of the answers. In general, although they start from a circumference (56.10%), in more than half of the cases the students construct a right triangle to identify the sine and the cosine of an angle.

Incorrect connections between the «line system» and «ratio system» were evidenced because, although the sine and cosine as segments are originally associated with the «line system» and should fundamentally arise from the circumference theme, they also arose from the triangle theme. Additionally, in most of the productions the students used a circumference of radius $R \neq 1$ to indicate the sine and cosine. Conversely, participants identified the sine and cosine with segments, ignoring that, in this case, such values are incorrect since they must be considered as a ratio.

In relation to the goniometric circumference, a high percentage of the students revealed a poor understanding of it. These students drew the circumference without indicating the value of its radius, which led them to making mistakes. This also points out that some students lack knowledge about the range of possible values of the sine or cosine of an angle.

Most of the students did not identify the sine or cosine of an angle either with the notion of proportion or with the values of a trigonometric function. The idea of proportion and the ordinate of the sinusoidal function appeared incidentally.

Another specific result was achieved when the representation of the angle and its cosine, and how surveyed pre-service teachers give meaning to them were revealed. Taking into account the interpretative richness and relations of the contents, pre-service teachers reason differently and show a determined progress. The modes were differentiated by the selected themes. Therefore, this research helps improve the theoretical knowledge about mathematical concepts understanding by prospective teachers and contributes to their characterization. Indeed, the different criteria provide us with information about what each one has understood and what should be enhanced.

It is clear that the construction and the meaning of the cosine of the angle is a key aspect of the task in order to understand how pre-service teachers convert trigonometric notions. As seen in Table 6.2, 50% of pre-service teachers utilize metric strategy somehow in order to build the cosine. Consequently, interpretations that emphasize metric aspects (length and ratio) predominate over those that emphasize analytical aspects (value) in the unit circle (Table 6.3). Additionally, there are few combinations of meanings of the cosine of the angle in the first partial system, and more than one third of participants do not build the cosine of the angle properly (Table 6.2). The emphasis on metric aspects and the scarcity of combinations of meanings of the cosine in the responses in the first partial system may involve a lack of connections with the second one. Indeed, while 36.11% of the participants do not interpret the cosine of the angle in the first partial system (Table 6.3), this percentage increases to 54.17% in the second one (Table 6.6). Therefore, the findings of this study shed light on the fragmented understanding of the cosine of the angle of the participants by the difficulty in building and linking meanings of the cosine of the angle when participants convert notions between trigonometric partial systems (Even, 1998, p. 109; Brown, 2005; Martín-Fernández, 2019).

The difficulties of the participants partially stem from impoverished connections between similar notions involved in different partial systems that constitute the core of the trigonometry relational system (Akkoc, 2008; Moore, 2016). The findings of this study also indicate that meanings and contents were not paid enough attention in the training of the pre-service teachers, and they were unable to convert notions and to move between partial systems. We stated that an underlying difficulty when trying to master and comprehend trigonometry relational system is that surveyed subjects are required to reason on the absolute angle in the partial elementary geometry system, on the oriented angle in the partial goniometry system, and on the analytic angle in the partial analytic geometry system. In other words, participants must link different meanings of the angle's concept, using contents of the trigonometry relational system. These meanings cause troubles to participants. Indeed, participants use a type of angle and contents associated with a representation model, even though the task is situated in another one. We believe that the root of this problem is the treatment given for the teaching of the concept of angle, which imposes an unnecessary division in the trigonometry relational system. It is clear that the manner in which the angle is introduced, in which situations and contexts, together with its order of appearance in the teaching are the keys for how participants understand the trigonometry relational system. We hold the view that oriented angles and absolute angles must be taught simultaneously, emphasizing their differences. We sustain that this task, the wrong use of the angle concept, the scarcity of combining meanings of the angle concept and of its cosine illustrate why this division is unnecessary from a trigonometry relational system perspective. Thus, we found evidence that the oriented angle must be reinforced in the global trigonometry relational system to facilitate the learning of this mathematic relational system. As Moore (2016) highlights, an underdeveloped angle measure understanding contributes to pre-service teachers' difficulties with trigonometry relational system, but we have also proved that so does an underdeveloped angle concept. Therefore, we should prevent future teachers from creating divisions on the teaching or learning of the trigonometry relational system (Martín-Fernández et al., 2019).

Regarding understanding concept of oriented angle by pre-service teachers, the results were similar to the responses of participants in Chaar's study (2015, p. 62, p. 64). Although our results come from advanced students, this concept can be considered problematic. In fact, drawing angles in the unit circle is a skill scarcely known to pre-service teachers; although they could have used the unit circle, only 45.83% of the participants show the ability to draw oriented angles in the unit circle (Table 6.1).

Consistent with Fi (2003), teachers in training appear to have broad understanding of clockwise and counter clockwise rotations given that only one participant does not draw an angle oriented from the positive x-axis; nevertheless, the low number of responses using negatives angles may reveal that pre-service teachers' understanding of them is not as solid as it could have been expected. Furthermore, even though the use of absolute angles does not allow students to approximate the cosine of a given angle, determine the quadrant in which the angle is included, and graph the trigonometric function on the Cartesian plane, nearly forty percent of participants draw this type of angle. Thus, the findings of this study also align with Fi (2003, p. 198), Chaar (2015, p. 125), and Martínez-Planell and Cruz Delgado, (2016, p. 130), who emphasize the lack of understanding of the properties of the unit circle and of the advantages of its use by adult learners. In other words, as secondary school students, the unit circle remains as a scarcely used iconic form to draw angles (Martín-Fernández et al., 2019). On the other hand, with regard to the representation of an angle in the analytic function, as it can be seen in Table 6.4, only 26.37% of the subjects consider an angle in the x-axis associated with the marked point in the analytic function. This fact and the low percentage of participants who represent two angles in the unit circle may be caused because the participants do not consider that drawing angles is a key aspect for solving the task. This results in their inability of linking not only the graph with the Cartesian plane (Marchi, 2012, p. 10), but also with the partial systems. Furthermore, participants generally consider radians when they work in the second partial trigonometry system; in short, apparently the preference for using degree measure over radian measure depends not only on the appearance of π

(Akkoc, 2008, p. 860; Chaar, 2015, p. 277), but also on the partial system in which learners work.

Consistent with previous studies (Marchi, 2012; Chin, 2013), this research reveals that a high percentage of pre-service teachers could neither reason about the flexibility of trigonometric representation system, nor combine adequately the partial trigonometry systems in order to process and convert trigonometric notions to solve the task. We highlight that the proposed task was feasible by the participants given that the majority of them tried to solve it. Contrary to Marchi (2012, p. 217), a great part of the participants does not identify the x-coordinate value as the measure of the angle corresponding to P in the analytic function. Indeed, as seen in Table 6.4, the majority of the subjects do not identify the angle corresponding to P as an angle in the analytic function. Furthermore, given that only 22.21% of the subjects base their responses on the angle and on the ordinate, or in the angle and on the value of its cosine according to Table 6.5, consistent with Marchi, (2012, p. 216), it is argued that there is no solid evidence that the subjects understand what the point P in the analytic function means (x, cos (x)). In other words, it is not evident that participants understand that the y-value on the graph of the analytical function is the output for the formula y = cos (x). Besides, it is not clear whether the participants are able connect the point corresponding to P in the analytical function as a point with two coordinates. It is our conviction that it is the consequence of the poor perception of the point P as an oriented angle in the unit circle (Table 6.1), of the great percentage of participants that do not build the cosine of the angle (Table 6.3), and of the types and scarcity of combinations of meanings of the cosine of the angle in the unit circle.

Aligned with Marchi (2012, p. 212), we remark that participants also incorrectly recalled information and made false connections when trying to connect the graph for y=cos (x) with the unit circle. Indeed, 22.22% of the subjects confuse the graph for y= sin (x) with the graph for y=cos (x). Additionally, more than half of the participants use an invalid strategy to solve the task (Table 6.5), and few students argue their answers properly. Furthermore, only a low percentage of participants correctly convert the angle between partial systems (Table 6.4). Thus, consistent with Chaar (2007, p. 267-268), it seems that the unit circle has not been taught to convert trigonometric notions. Besides, in contrast to Steckroth (2007, p. 173), there is no evidence that the subjects who have connected partial systems consider the value of reference angles as vital to their link. In fact, using the angle (Table 6.5) is the most common strategy to solve the task. In brief, most of the students neither convert notions from one trigonometric representation system to another, nor are able to draw on foundational notions such as the analytical function, the unit circle, and the oriented angle concept with a coherent meaning. Thus, this result shows that the majority of the students only make slight connections between the two trigonometric partial systems.

7.2.5. Achievement of aim 6

Aim 6. To examine the influence of interpreted meanings for the concept of angle by pre-service teachers.

Starting from the meanings of an angle shown by the subjects in the partial goniometry system, contingency tables were made relating them with the strategies used by the pre-service teachers to carry out the task, and with their meanings for the angle concept. This allows describing how a specific meaning of a trigonometric notion (specifically, the angle concept), leads to the generation of other meanings.

Following the results, there is evidence that angle concept plays an important role for solving trigonometric problems. On the one hand, participants who draw absolute angles stress metric strategies and meanings (Table 6.7). It is highlighted that more than one third of the participants who draw absolute angles were not able to build the cosine of the angle (Table 6.8). Additionally, almost all the participants who draw absolute angles were unable to identify an angle in the second partial system. In other words, they do not convert angles between trigonometric representation systems (Table 6.9). As far as the strategy utilized by participants drawing absolute angles is concerned, the majority do not use a valid strategy to solve the task, which suggests that their knowledge was not enough to answer the task (Table 6.10). This idea is reinforced by the fact that a high percentage of participants who give this type of answer do not build the cosine of the angle in the second partial system (Table 6.11). On the other hand, participants who draw oriented angles mostly utilize a metric-goniometric strategy for building the cosine of the angle (Table 6.7). As a consequence, they chiefly interpret the cosine of the angle as a numerical value. Moreover, we remark that the relative percentage of the participants who draw an oriented angle and do not identify the angle in the second partial system is much lower than those that utilize an absolute angle (Table 6.9). With regard to the strategy used by subjects who draw oriented angles, a large percentage of their responses utilize a valid approach in order to solve the task (Table 6.10). Finally, less than half of the participants who draw oriented angles do not interpret the cosine of the angle in the second partial system (Table 6.11). Therefore, the findings of this study are also aligned with Brown's suggestion (2005) that conceptualizing angles as oriented ones could improve students' connecting to the goniometric circumference, and subsequently, to graph trigonometric functions.

7.3. Limitations of the study and suggestions for further research

Firstly, we refer to the samples used, which were chosen intentionally for their availability. Regarding the sample of secondary school students, it should be emphasized that the students to whom the pilot questionnaire and the final questionnaire were applied had different sociodemographic characteristics. This choice allows the results to show a wider range by having saturated the data from the definitive study with those from the pilot study. However, we are aware that the results are not generalizable, although the consideration of having implemented the questionnaires in several secondary schools provides a greater relevance to the study. In relation to the sample of the teachers in training, we emphasize the variability of grades studied by the participants and the diversity of their experiences as students that allow the results to have greater amplitude as well.

Moreover, the data collected came from a single semantic questionnaire. We are aware that other questionnaires with different tasks will generate different information; however, the tasks of the questionnaire are based on the semantic triad developed by Rico (2013), focusing on the meanings of the mathematical contents.

The methodologies used, the design of the questionnaire, the collection, organization, and categorization of the data provide our research with internal validity, being supported by grounded theory and content analysis.

Another limitation of our study refers to the fact that the results are mediated by the instruction received by secondary school students. Although the three teachers had considerable experience and belong to different generations, given that students belong to the same school, we can consider that the instruction is somewhat homogeneous. We mentioned that the questionnaire was applied after the implementation of the didactic unit of the trigonometry relational system, which allowed all the items to be accessible to the participants.

In relation to the future lines of research we distinguish:

This study emphasizes the relevance of semantic questionnaires and indicates some of their basic characteristics. However, some questions arise such as: When is the questionnaire used? How is it used and what kinds of questions will it answer? What is the most appropriate data collection? What type of data analysis is the most appropriate? How is reliability and validity ensured with this method? Was the topic covered well enough to get the meanings shown for the students? Therefore, an in-depth study of semantic questionnaires and the application of several semantic questionnaires to a group is a future line of research.

Designing and implementing a proposal for curricular innovation on the trigonometry relational system that is based on the theoretical study carried out on the meaning of the concepts of the sine and cosine of an angle. This curricular proposal could also include other methodologies already implemented in undergraduate students in other countries and their possible application to participants from secondary schools. However, this innovation will focus on promoting school meanings of trigonometric contents in the different partial systems of trigonometry.

Because the sense of high school students is fundamentally the calculation of magnitudes, designing and implementing tasks that involve other meanings is an obligation.

This study starts from a sample with high school students. After that, we analyse the meanings of pre-service teachers. In this way, another future line of research that we propose is delving into the meanings of teachers in service when solving tasks that involve the sine and cosine of an angle. In fact, there are studies that indicate that the knowledge of teachers in training differs from that of teachers.

Characterizing the meaning that high school students attribute to the notion of the tangent, when they explain and represent said notion. In this study we have relied on the sine and the cosine, but there is a research gap on the concept of the tangent.

Investigating partial trigonometry systems along with the relationships and dependencies between the meanings of their notions. The current study could be expanded using tasks that link and convert notions between other partial systems.

Studying the understanding of other trigonometric concepts or exploring the relationships between the semantic categories expressed by the subjects in order to know the level of importance of each one.

Finally, it would be valuable to investigate the incomplete meanings shown by secondary school students and examine the remaining tasks of the questionnaires.

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9.SUMMARY

9.1. Introducción

La planificación, el diseño, la realización y la puesta en práctica en el aula de una unidad didáctica sobre el sistema relacional de la trigonometría para 4º de ESO, dentro de un programa de formación inicial de profesores de enseñanza secundaria, fue el germen de la presente memoria, y la primera fase de la aproximación al problema de investigación en el año 2010-2011. La investigación se vertebra en torno a la perspectiva teórica del análisis didáctico (Rico, Lupiáñez y Molina, 2013).

En esta primera fase del estudio se realizaron distintos análisis de los contenidos didácticos implicados: análisis conceptual, análisis de significados, análisis cognitivo, análisis de instrucción y análisis de actuación o evaluativo.

Durante la realización del Máster de Investigación en Didáctica de la Matemática de la Universidad de Granada en el curso 2012-2013 se diseñó y realizó un estudio con estudiantes de 1º curso de Bachillerato. Nuestro foco se encontraba en las tres categorías semánticas del análisis de significados (estructura conceptual, sistemas de representación y sentido). Estas categorías forman una tríada, dando lugar al triángulo semántico que establece el significado de un contenido escolar. Para identificar la diversidad de significados para el seno y coseno de un ángulo se diseñó un cuestionario semántico fundamentado en la tríada semántica mencionada. Además, se realizó un estudio de la evolución histórica de este tópico que permitió distinguir distintos tipos de instrucción. Mediante un análisis de contenido llegamos a definir temas y subtemas, y finalmente identificamos y caracterizamos los significados atribuidos por estudiantes de secundaria al seno y coseno de un ángulo. Una síntesis de parte del trabajo realizado y de sus resultados fue presentada en el XVIII simposio SEIEM como comunicación, y al 38th PME como comunicación corta, ambos en 2014. Este trabajo se presenta con más profundidad en dos artículos publicados en las revistas: *Enseñanza de la Ciencias* y *Eurasia Journal of Mathematics, Science and Technology Education*.

Utilizando los resultados obtenidos anteriormente, diseñamos un segundo cuestionario semántico para analizar los significados y el contenido evocado por profesores en formación sobre el seno y coseno de un ángulo. Nosotros caracterizamos en detalle distintos conceptos de ángulo en el plano según modalidades de sus componentes semánticas; singularizamos tres tipos principales: ángulos absolutos, ángulos orientados, y ángulos analíticos, cada uno caracterizado por sus sistemas de representación correspondientes. Establecemos sus medidas, y estudiamos los procesos de conversión entre ellos, delimitando un sistema relacional trigonométrico constituido por esos tipos de ángulos, sus procesos de conversión, la construcción de sus líneas trigonométricas y las relaciones entre ellas.

9.2. Planteamiento del problema

El sistema relacional de la trigonometría es un tópico interesante, y unificador de la asignatura de matemáticas en la escuela secundaria. "Es conceptualmente rico y contiene conexiones con otras ideas y estructuras matemáticas" (Fi, 2003, p. 13). El sistema relacional de la trigonometría utiliza nociones provenientes de varias partes de las matemáticas que permite que los estudiantes desarrollen destrezas, razonamientos y estrategias para resolver problemas (Sarac & Aslan-Tutak, 2017, p. 70). De hecho, Tuna (2013) afirmó que "la trigonometría es importante en términos de la mejora de las habilidades de razonamiento de los estudiantes" (p. 1). Es más, "la enseñanza de teoremas y conceptos de trigonometría es importante para desarrollar las habilidades de pensamiento creativo, lógico y analítico de los estudiantes" (Dündar, 2015, p. 1380). Además, según la Asociación Matemática de Inglaterra, "la trigonometría fusiona aritmética, álgebra, geometría y mecánica" (MA, 1950, p. 3), lo que nos da una idea de la relevancia de este tópico en la escuela secundaria. Finalmente, el sistema relacional de la trigonometría tiene multitud de aplicaciones en distintas disciplinas (Army, 1991).

A pesar de su importancia, la trigonometría es una parte de las matemáticas que es difícil de entender para los estudiantes (De Kee et al., 1996; Tuna, 2011). Son muchos los factores relacionados con este hecho como su complejidad conceptual, sus diversos subsistemas relacionales, la conexión entre ellos, las diversas aproximaciones a los mismos, las formas de abordar y representar sus nociones básicas, la gran diversidad de contextos, modos de uso y fenómenos en los que participa, etc.

Sin embargo, hay poca investigación sobre lo que hace difícil a la trigonometría, y sobre las ideas intuitivas que los estudiantes tienen sobre los conceptos trigonométricos (Brown, 2005, p. 10). Weber (2005) afirma que la investigación que nos dice cómo superar las dificultades de los estudiantes en trigonometría es escasa y dispersa. Del mismo modo, Chin (2013) afirma que: "sería provechoso investigar la transición de la trigonometría goniométrica a la trigonometría analítica en investigaciones futuras" (p. 253).

Debido al necesario conocimiento del sistema relacional de la trigonometría para otros temas matemáticos, y a la luz de las oportunidades de razonamiento entre representaciones que involucran el sistema relacional de la trigonometría, se debe prestar más atención a la comprensión de las nociones trigonométricas. Además, para mejorar la formación inicial de los profesores de secundaria en este sistema relacional, es conveniente investigar las conexiones entre distintos sistemas parciales trigonométricos.

9.3. Objetivos de la investigación

El objetivo general de la investigación es explorar y describir los significados puestos de manifiesto por estudiantes y profesores de secundaria en formación sobre el seno y coseno de un ángulo, al evocar conocimientos previamente estudiados.

Además, se plantean como objetivos específicos los siguientes:

Objetivo 1. Construir un instrumento válido y fiable para evaluar los significados puestos de manifiesto por estudiantes, siguiendo unos criterios metodológicos establecidos.

Objetivo 2. Construir un instrumento válido y fiable para evaluar los significados puestos de manifiesto por profesores de secundaria en formación, siguiendo unos criterios metodológicos establecidos.

Objetivo 3. Identificar, describir e interpretar los significados del seno y coseno de un ángulo que muestran los escolares cuando los estudiantes de secundaria responden a tareas fuertemente conectadas con cada una de las categorías de significado según la perspectiva de Rico (2013).

Objetivo 4. Identificar los significados sobre el concepto de ángulo y su coseno que muestran los profesores de secundaria en formación y describir el contenido conceptual y procedimental al moverse entre el sistema goniométrico parcial y el sistema analítico parcial.

Objetivo 5. Indagar la comprensión de los estudiantes y profesores en formación sobre estos contenidos por medio de la caracterización de sus significados y componentes.

Objetivo 6. Examinar la influencia de los significados interpretados para el concepto ángulo en los profesores en formación.

9.4 Revisión de la literatura

Las investigaciones previas examinan el significado del seno y el coseno de los estudiantes de secundaria, pero se concentran principalmente en una de las categorías semánticas de nuestro marco. Mientras que algunos de estos estudios se centran en examinar las representaciones trigonométricas y cómo los estudiantes de secundaria las vinculan (Marchi, 2012), otros examinan cómo los estudiantes de secundaria entienden los conceptos involucrados (Brown, 2005; Demir, 2012). Finalmente, otros investigan la presencia de la trigonometría en las ciencias actuales y su relación con las prácticas educativas y los currículos modernos (Hertel, 2013). De este modo, no hemos encontrado investigaciones previas sobre las conexiones entre estas tres categorías semánticas en estudiantes de secundaria. Un objetivo de este estudio es contribuir a superar esa carencia.

Por otro lado, aunque existen algunos estudios sobre estrategias de instrucción, no existe consenso sobre la mejor estrategia para enseñar el sistema relacional de trigonometría (Kendal & Stacey, 1997; Weber, 2008; Moore, 2014; Demir, 2012; Fanning, 2016). Por lo tanto, se necesita más investigación para arrojar luz sobre cómo superar las dificultades de las personas involucradas en la educación sobre el sistema relacional de trigonometría.

Finalmente, la revisión de la literatura sobre educación matemática sobre conversión de nociones entre sistemas de representación revela la existencia de solo un pequeño número de estudios que se enfocan en esta capacidad o que enfatizan su importancia (e.g. Brown, 2005; Challenger, 2009; Marchi, 2012; Demir, 2012; Chin, 2013; Martinez-Planell & Cruz Delgado, 2016). Además, hay pocas investigaciones que se concentran en la comprensión de los conceptos del sistema relacional de trigonometría por parte de los profesores de matemáticas en formación (e.g., Akkoc, 2008; Çekmez, 2020; Chaar, 2015; Fi, 2003; Hertel & Cullen, 2011; Moore et al., 2016; Paoletti et al., 2015). Finalmente, todos ellos difieren en su alcance, en su enfoque metodológico y en los marcos de referencia utilizados en comparación con los del presente trabajo.

9.5. Marco teórico

Nuestro estudio se basa en un enfoque curricular y en un marco general denominado Análisis Didáctico, propuesto por Rico y Ruiz-Hidalgo (2018).

Actualmente, el análisis didáctico de un contenido matemático se considera "un método para profundizar, estructurar y esclarecer el contenido curricular con miras a su programación e implementación" (Rico & Ruiz-Hidalgo, 2018, p. 7).

Es por ello que los currículos escolares y los planes de formación del profesorado de matemáticas se convierten en su foco de estudio (Rico, 2013, pp. 19-20).

Un currículo determinado está marcado por una serie de objetivos que se pueden clasificar en: conceptuales, cognitivos, formativos y sociales. Permiten establecer dimensiones: cultural/conceptual, cognitiva, ética/formativa y social que delimitan el currículo tal como lo entendemos.

Concretamente, nuestro estudio se centra en la dimensión cultural/ conceptual que tiene como objetivo principal los significados de los contenidos matemáticos escolares. Es necesario mencionar que esta dimensión también está relacionada con la historia de las nociones involucradas.

Estudios recientes sobre el significado de los conceptos matemáticos escolares demuestran que el enfoque semántico es una aproximación sólida para investigar la enseñanza y el aprendizaje de las matemáticas. Un fuerte argumento a favor de este enfoque sostiene que el significado del contenido matemático proporciona un marco esencial para explicar los fundamentos del conocimiento matemático de los alumnos, describir su comprensión y aclarar el fundamento de las decisiones para la orientación e instrucción de los estudiantes (Thompson, 2016, pág. 438; Rico, 2019). De hecho, "el conocimiento matemático que más importa a los profesores reside en el significado matemático" (Thompson, 2016, p. 437). Además, el significado juega un papel fundamental en la organización del contenido (Kumar, 2017, p. 559). En consecuencia, el significado debe ser el pilar para el futuro aprendizaje de los contenidos matemáticos de los estudiantes (Castro-Rodríguez et al., 2016; Thompson, 2016, p. 461).

Existen varias propuestas, en las que se utiliza un marco de tres categorías semánticas, para investigar cómo dotar de significado a un contenido matemático (Vergnaud, 1990; Radford, 2003; Sáenz-Ludlow, 2003; Biehler, 2005; Steinbring, 2006).

Siguiendo a Frege (1962), tomamos la noción de significado de un concepto matemático escolar desarrollada por Rico (2012), basado en referencia, signo y sentido. Mediante estas categorías se identifica, expresa y utiliza un concepto matemático. De este modo, la tríada semántica utilizada en este trabajo para definir el significado de un concepto matemático escolar es: la estructura conceptual, los sistemas de representaciones y el sentido.

Concretamente, en esta tesis, la estructura conceptual comprende nociones, conceptos, propiedades, proposiciones, procedimientos y sus relaciones implícitas en un concepto matemático. Establece prioridades y vínculos; muestra las trayectorias para organizar las expectativas de aprendizaje; proporciona referencias para establecer la verdad o la falsedad de una proposición.

Para caracterizar la estructura conceptual, consideramos dos campos para clasificar los contenidos matemáticos: conceptual y procedimental. Complementariamente, diferenciamos tres niveles cognitivos de complejidad en cada uno de ellos, con lo que se estructuran los diferentes contenidos en los campos considerados.

En este trabajo, entendemos por sistema de representación aquellas combinaciones de signos y reglas cuyo uso transmite contenidos matemáticos específicos de forma estructurada, operativa y coherente (Rico, 2009).

Finalmente, el sentido incluye aquellos contextos (son las preguntas que el concepto matemático da respuesta), situaciones (los escenarios en los que se involucran) y términos (los términos relacionados con el concepto) que dan sentido a las ideas matemáticas (Rico & Moreno, 2016, p. 139).

Así, en este estudio utilizamos también tres categorías para analizar el conocimiento matemático y el significado mostrado por los participantes de nuestro estudio: estructura conceptual, sistemas de representación y sentido. Las categorías de significado nos permiten interpretar el valor y la adecuación de los conocimientos de los estudiantes cuando intentan definir, representar o utilizar los conceptos matemáticos considerados en la escuela secundaria. Es evidente que este sistema está organizado para lograr el significado de un contenido matemático escolar. Además, el sistema de categorías, campos y componentes es sencillo para su interpretación y permite conjeturar una explicación teórica sobre el conocimiento local de un alumno singular o de un grupo de ellos, y ayuda a detectar cómo se ha entendido en un momento determinado. Para lograr esto, el marco elegido identifica, organiza, sintetiza, analiza e interpreta elementos de contenido, sus relaciones y reglas de procesamiento y conversión. Todo esto, juega un papel significativo dentro de las categorías utilizadas para llevar a cabo el análisis de significados de los contenidos matemáticos escolares (Rico & Ruiz-Hidalgo, 2018, p. 1-6).

Comprender un contenido matemático escolar es dotarle de significado, es decir, definirlo, representarlo, identificar sus operaciones, relaciones y propiedades, sus modos de uso, su interpretación y aplicación.

En la comprensión de los conceptos matemáticos escolares, las representaciones de las nociones matemáticas y sus vínculos juegan un papel importante. Las representaciones dan sentido dentro de una estructura matemática (Rico, 2009). Por lo tanto, una forma de ampliar la comprensión de conceptos y procedimientos incluye usar y combinar diferentes sistemas de representación en la resolución de problemas para convertir y procesar una representación en otra diferente.

Cabe destacar que hemos identificado y elegido tres términos diferentes: convertir, transformar una representación en otra entre diferentes sistemas de representación; procesar, transformar expresiones dentro de un sistema de representación; y mover, transformar una representación en otra entre sistemas parciales.

Nuestro trabajo versa sobre el sistema relacional de trigonometría. Simbólicamente definimos el sistema relacional de trigonometría como una N-upla ($A, R_1, R_2, R_3, \ldots, R_k$) formada principalmente por un conjunto A de conceptos trigonométricos y k relaciones establecidas entre ellos.

En la segunda mitad del siglo XX varios autores revisaron la definición del sistema relacional de trigonometría para dar respuesta a una variedad de problemas teóricos y prácticos derivados de la medición de ángulos. El más relevante para nuestro estudio es el desarrollado por Freudenthal (1973, p. 479) que describe la forma instrumental en la que se han medido y organizado los ángulos, destacando sus principales definiciones y cambios a lo largo de la historia de las matemáticas. Él establece los siguientes subsistemas dentro del sistema relacional de trigonometría global: geometría elemental, goniometría y geometría analítica.

9.6. Fundamentación metodológica

Esta investigación implica un estudio descriptivo y explicativo basado en el método de encuesta. Distinguimos dos etapas en nuestro estudio. En ambas etapas, dependiendo de su disponibilidad, los participantes fueron elegidos deliberadamente. El análisis de este estudio se basa en dos perspectivas metodológicas: teoría fundamentada y análisis de contenido.

En una primera etapa, diseñamos dos cuestionarios semánticos (Blok, 2014), con ocho tareas cada uno, presentados como dos opciones diferentes, A y B. Los ítems se diseñaron teniendo en cuenta otras investigaciones (Fi, 2003; Weber, 2005; Brown, 2005; Dominic, 2012) y algunos libros escolares (Ibañes et al., 1998; Arias & Maza, 2008; Bescós & Pena, 2010). Todas las preguntas del cuestionario A fueron las mismas que las del cuestionario B, con la excepción de que el cuestionario A se refería al seno de un ángulo menor a 90º y el cuestionario B se refería al coseno de un ángulo menor a 90º. El sexto ítem fue el mismo en ambos cuestionarios.

En esta primera etapa se realizó un estudio piloto a 27 alumnos de 1º de Bachillerato, de los cuales 13 respondieron al cuestionario A y 14 respondieron al cuestionario B. La versión final del cuestionario se implementó en 74 alumnos de 1º de Bachillerato. Los participantes de esta etapa estaban cursando asignaturas orientadas a las Matemáticas de Ciencia y Tecnología.

En cuanto a la segunda etapa, una vez más, diseñamos un cuestionario semántico (Blok, 2014) asociado al seno y coseno de un ángulo con 10 ítems. Los ítems se diseñaron teniendo en cuenta a otras investigaciones entre las cuales se encuentran las de Fi (2003), Brown (2005) y Martín-Fernández et al. (2016; 2019). Sin embargo, utilizando este cuestionario, buscamos recopilar evidencias sobre los futuros profesores de secundaria en diferentes temas como: construcciones de ángulos relacionados con un valor del coseno o el seno de un ángulo, transiciones y conversiones entre sistemas de representación, interpretaciones de los profesores en formación sobre respuestas de los estudiantes de secundaria a preguntas sobre trigonometría y cómo los profesores en formación dan sentido al seno y al coseno de un ángulo.

Los participantes en esta etapa fueron setenta y dos profesores en formación que estaban cursando una asignatura como parte del programa de formación de profesores de matemáticas de secundaria.

En ambas etapas utilizamos un cuadernillo para la implementación de los cuestionarios. Además, los participantes del estudio los cumplimentaron en una clase ordinaria de 60 minutos. Finalmente, el primer cuestionario se implementó en el año 2012-2013, mientras que el segundo se aplicó el año 2016-2017.

En esta investigación hemos utilizado estas dos teorías emergentes: el análisis de contenido y la teoría fundamentada, que están estrechamente relacionadas con el surgimiento de conceptos y relaciones durante el análisis, las cuales exigen una gran apertura por parte del investigador (Strauss & Corbin, 1998). Estos dos métodos se basan en los datos y permiten un mayor conocimiento y comprensión de las producciones (Strauss & Corbin, 1998).

Sin embargo, mientras que el análisis de contenido se enfoca en describir el significado del material cualitativo y enumerar categorías o temas, la teoría fundamentada va más allá (Cho & Lee, 2014), colocando los datos en estructuras interpretativas y explicativas, generando una teoría.

9.7. Resultados de la primera etapa

9.7.1 La enseñanza relacionada con los significados mostrados por los estudiantes de secundaria

Establecidas las categorías y subcategorías que organizan las nociones y representaciones obtenidas, observamos sus relaciones y vinculaciones, y elaboramos un mapa conceptual que relaciona las producciones de las tareas 1 y 6 con la historia y enseñanza de la trigonometría (Figura 5.2). Para ello, se parte del problema original por el cual surge la trigonometría: «Dado un arco de ángulo encuentra la longitud de la cuerda que conecta los puntos finales del arco» (Van Brummelen, 2009, p. 41).

Los alumnos utilizan para representar el seno y el coseno la circunferencia y el triángulo. Se observan así, en un primer nivel, las dos formas históricas de medir ángulos, que se presentan relacionadas con las formas de enseñar y entender la trigonometría. En primer lugar, el «ratio system», que usa un triángulo e interpreta la medida de un ángulo como una razón, y el «line system», que utiliza la circunferencia y entiende la medida del ángulo como un segmento (Sickle, 2011).

En un segundo nivel aparece la subdivisión de los triángulos en rectángulos y no rectángulos y la subdivisión de la circunferencia, entre la circunferencia goniométrica y la circunferencia de radio R.

A partir de aquí y considerando solo la circunferencia, encontramos dos tipos de medidas del ángulo, mediante la longitud de un segmento y mediante la consideración de sus coordenadas en una circunferencia goniométrica.

En relación con el triángulo, este puede ser rectángulo y no rectángulo. En el caso del triángulo no rectángulo, para la medida del ángulo se transformaría en triángulo rectángulo. Entre las interpretaciones válidas para el seno o el coseno aparece la de razón. Otras interpretaciones que aparecen en las producciones con origen en el triángulo y consideradas errores de comprensión son la identificación del seno o coseno con el ángulo interior y con una distancia en un triángulo rectángulo de hipotenusa diferente de uno.

9.7.2 Estructura conceptual del concepto del seno y coseno de un ángulo

Los resultados para esta categoría semántica provienen fundamentalmente de la pregunta 2 del cuestionario aplicado a estudiantes de secundaria (Tabla 5.2).

Al analizar las producciones de los estudiantes aparecen diversos tipos de respuesta o temas. El primer tipo de respuesta está asociado a razón. Las respuestas dentro de este tema son aquellas que utilizan una fórmula o una variedad de sinónimos del verbo dividir o partir.

El tema valor se muestra cuando los estudiantes escriben un valor numérico, por ejemplo: "0,7", "1⁄2"; cuando los alumnos señalan que la razón trigonométrica es un “número” y cuando expresan que es un “valor”.

El tercer tipo de respuesta es longitud, y se incluyen en este tema aquellas producciones que hacen referencia a una longitud como, por ejemplo: un cateto, una altura, una base, un lado de un triángulo, un segmento o una coordenada.

Además, hay otras respuestas que muestran el seno y coseno de un ángulo como una herramienta. Este tema es observado cuando se emplean expresiones como: “resolver incógnitas”, “resolver triángulos”.

Un cuarto tema se denota por medida, y se reconoce cuando aparecen expresiones como "la mitad del radio", en las que se expresa la unidad (radio) y la medida (mitad).

Finalmente, el tema ángulo aparece asociado a expresiones como "lado de un ángulo" y "lado adyacente de un ángulo".

9.7.3 Sistemas de representación del concepto del seno y coseno de un ángulo

Para indagar los significados relacionados con los sistemas de representación se examinaron las respuestas dadas por los estudiantes a las preguntas 1 y 6.

Se realizaron distintos análisis para la clasificación de las respuestas. En un primer análisis, se consideraron niveles en función de los elementos de contenido utilizados. En un primer nivel, se muestran los temas generales, identificados al inicio del análisis de contenido, que fueron la circunferencia y el triángulo. Un segundo nivel surge al comparar las distintas representaciones, incluyendo en el caso de la circunferencia los elementos geométricos que se utilizan para dividirla, ejes y segmentos. Cabe destacar que todas las producciones relacionadas con el tema «circunferencia» quedan incluidas en estas dos subcategorías. En un tercer nivel se recoge un nuevo paso en la secuencia de representación que, para el tema «circunferencia», es el modo de indicar un ángulo en la circunferencia inicialmente dibujada. Nos estamos refiriendo al ángulo interior de un triángulo rectángulo, al ángulo central y al punto-segmento. Si examinamos el tema «triángulo», situamos en este nivel la distinción entre triángulo rectángulo y no rectángulo. Finalmente, identificamos un cuarto nivel, el cual incluye los sentidos o interpretaciones y modos de uso que los alumnos hacen del seno y coseno de un ángulo (Figura 5.1).

Un segundo análisis nos permite establecer los significados puestos de manifiesto. Para ello, identificamos todas las interpretaciones de los alumnos en sus respuestas para el coseno y seno de un ángulo. Los temas utilizados por los alumnos son: razón, longitud, valor, función y ángulo. En este análisis, no sólo se considera el principal, sino que hay repuestas que implican más de una interpretación de los conceptos estudiados (Tabla 5.3).

9.7.4 Sentido del concepto del seno y coseno de un ángulo

Finalmente, el sentido se estudió utilizando la pregunta 8 del primer cuestionario. El sentido involucra los contextos y modos de usos, las situaciones y los términos.

Nosotros observamos dos principales contextos y modos de uso: el cálculo indirecto de una magnitud y el cálculo del seno o coseno de un ángulo.

Una respuesta pertenece al primer contexto cuando las incógnitas son una longitud y/o un ángulo para cuyo conocimiento es necesario calcular una razón trigonométrica. Dos temas aparecen relacionados con el segundo contexto: a) Calcular la razón trigonométrica de un ángulo, dada la razón de otro, y b) Averiguar el valor de una expresión en la cual los conceptos estudiados estén involucrados.

Otros aspectos a tener en cuenta en el sentido, son las situaciones y los términos que se utilizan. Concretamente todas las producciones de los estudiantes presentan situaciones educativas tales como, el cálculo de la altura de un objeto, el seno o coseno de un ángulo o de elementos desconocidos en un triángulo, la distancia a un objeto, de una sombra o entre ciudades, etc. Los términos usados nombran objetos como: techo, poste, bombero, torre, farola, estatua, árbol, ciudades y sombra.

Para la definición de los significados de los participantes consideramos que el sentido se ejemplifica por los contextos y modos de uso.

9.7.5 Tipologías de significado del seno y coseno de un ángulo

Una vez que han emergido los temas en las diferentes categorías de significado, se realizan entre las categorías, todas las posibles combinaciones de los temas para obtener y describir los significados de los participantes. Para ello, se elabora una tabla de contingencia con los diferentes temas. Los tres significados más representativos se consideran como prototipos de significado. Para representar estos prototipos, consideramos el triángulo como una figura icónica. Los vértices del triángulo están asociados con los temas de la tabla de contingencia. El vértice superior está relacionado con la estructura conceptual, el vértice inferior izquierdo está asociado con los sistemas de representación y el vértice inferior derecho representa el sentido (Figura 5.7).

9.8. Resultados de la segunda etapa

En esta sección se presentan los contenidos utilizados como criterio para clasificar los tipos de respuestas de los participantes en el sistema de goniometría parcial, y en el sistema de geometría analítica parcial respectivamente, del ítem analizado del segundo cuestionario. Además, se analizan conjuntamente algunos criterios para determinar la influencia del ángulo conceptual en las producciones.

Distinguimos seis patrones de contenido diferentes elegidos como criterio para clasificar las respuestas, tres en cada uno de los sistemas de trigonometría parcial empleados.

En relación con la circunferencia goniométrica, reconocemos los siguientes criterios:

- Identificación del ángulo correspondiente al punto P,
- Estrategias para construir el coseno del ángulo correspondiente a P, y
- Significado del coseno correspondiente a P en el círculo unitario.

El segundo sistema de trigonometría parcial asociado a la función analítica se caracteriza por los siguientes criterios:

• Ángulo correspondiente a P en la función analítica,

• Estrategias para representar P en el gráfico de la función analítica, y

• Significado del coseno del ángulo correspondiente a P en la función analítica.

Con respecto a la identificación del ángulo correspondiente al punto P en el sistema goniometrico parcial, los temas que surgen de las respuestas pueden describirse en términos de dos conceptos de ángulo que son didácticamente importantes según Freudenthal (1973, p. 488): ángulo absoluto y ángulo goniométrico (Tabla 6.1).

En relación al sistema analítico parcial, el ángulo correspondiente a P da lugar a los siguientes tres temas: como P, como ángulo y como P, y como ángulo (Tabla 6.4).

Se considera que cuando los estudiantes marcan el punto P en el eje x de la función analítica, identifican como P el valor del ángulo correspondiente a P. Hay algunas respuestas en las que los estudiantes también etiquetan el eje x (en su mayoría en radianes). Entonces, el valor del ángulo correspondiente a P se indica como un ángulo y como P. Si los participantes etiquetan el eje x y dibujan una línea vertical auxiliar (8.33%), si marcan el eje x usando un signo típico de un ángulo (5.55%) y si limitan o acotan de alguna manera el valor del ángulo correspondiente a P (6.94%), podemos afirmar que expresan el ángulo correspondiente a P como un ángulo. Finalmente, se considera que los participantes no identifican el ángulo correspondiente a P en la función analítica cuando ninguna de las condiciones anteriores se encuentra en las respuestas (68.05%).

Otro de los criterios considerados es la estrategia para construir el coseno de un ángulo en la circunferencia unidad. Los temas que emergen son: no construyen, estrategia de estimación, estrategia goniométrica, estrategia métrica-goniométrica, estrategia métrica y ninguna estrategia aparente (Tabla 6.2).

La estrategia de estimación se identifica cuando los participantes dan un valor numérico o limitan el valor dentro de los límites de un intervalo sin explicación. La estrategia métrica se desarrolla cuando los sujetos calculan una razón en un triángulo rectángulo incluido en el círculo unitario, pero sin expresar el valor negativo del coseno o cuando solo proyectan el punto P y definen el coseno como una proyección o distancia. Los participantes utilizan la estrategia métrico-goniométrica cuando proyectan y se basan en algunas características del círculo unidad mediante las cuales consideran el valor negativo del coseno y lo utilizan al calcular relaciones métricas o al estimar el valor del coseno. del ángulo. La estrategia goniométrica se utiliza cuando los sujetos utilizan relaciones entre ángulos o cuando comparan valores del coseno para ciertos ángulos en el círculo unitario. Además, algunas respuestas, en las que no es posible saber exactamente qué han considerado los participantes, se han clasificado como "sin estrategia aparente". Finalmente, las respuestas que no construyen el coseno, las etiquetamos como "no construye".

Para la clasificación de las respuestas otro criterio considerado es la estrategia para convertir un punto P de la circunferencia goniomérica a un punto de la función analítica. Concretamente, se identificaron siete temas en las producciones: usar el ángulo y el valor, usar el ángulo y la ordenada, usar el ángulo, usar la ordenada, usar el valor, no construyen y sin estrategia aparente (Tabla 6.5).

Usar el ángulo y el valor es una estrategia basada en dibujar el ángulo relacionado con P en el círculo unitario y/o proyectar el punto P hacia los ejes cartesianos. A continuación, los encuestados mayoritariamente identifican el valor del coseno en el círculo unitario. Finalmente, todos lo convierten a la gráfica de la función trigonométrica tomando en consideración su ángulo asociado, determinando el punto P. Si los participantes señalan o expresan el coseno del ángulo como una longitud y lo convierten al segundo sistema parcial como ordenada teniendo en cuenta el ángulo asociado, consideramos que esta estrategia se basa en el ángulo y en la ordenada. Usar el ángulo es otra estrategia que implica identificar el ángulo asociado con P en el círculo unitario y su posterior uso en el segundo sistema parcial o conversión, a los ejes cartesianos de la gráfica de la función trigonométrica. Luego, partiendode este ángulo, se determina el punto P con respecto a la función trigonométrica. Los sujetos utilizan la estrategia de utilizar la ordenada cuando realizan una línea paralela al eje x para representar un punto o marca en la función analítica -en el segundo sistema parcial confunden el coseno con el seno-, y cuando identifican el coseno en el círculo unitario como una longitud, convirtiéndolo en la gráfica de la función trigonométrica que determina el punto P sin expresar ninguna información sobre el ángulo. Basarse en el valor significa que los participantes identifican puntos en el segundo sistema parcial considerando solo el valor del seno o coseno asociado a P en el primer sistema parcial. Además, el 12,5% de las respuestas se categorizan como "no construyen" dado que los participantes no representan un punto en la función analítica. Finalmente, la imposibilidad de inferir cómo algunos sujetos han resuelto la tarea hace que codifiquemos sus respuestas en el tema "sin estrategia aparente".

En relación a las interpretaciones de los profesores en formación sobre el coseno de un ángulo, destacar que casi todas las producciones se incluyen dentro de los temas ya encontrados en los estudiantes de secundaria. Sin embargo, un tema aparece: punto. Éste aparece incidentalmente, cuando los participantes identifican el coseno del ángulo como un punto en la función analítica.

Finalmente, realizamos tablas de contingencia entre criterios para estudiar la influencia que el concepto ángulo en la circunferencia unidad ejerce sobre las respuestas. Las distintas tablas de contingencia permiten observar la influencia decisiva del concepto ángulo en las respuestas de los participantes.

9.9. Discusión y conclusiones

9.9.1 Grado de consecución de los objetivos

A continuación, se indica el grado de consecución de los objetivos específicos establecidos en el estudio.

Objetivos 1 y 2. Construir un instrumento válido y fiable para identificar y recoger los significados mostrados por los estudiantes de secundaria, y por los docentes en formación, siguiendo unos criterios metodológicos establecidos.

El capítulo 4 indica las principales características que posee un cuestionario semántico. Además, se presenta el diseño de ambos cuestionarios. Este diseño se basa en las tres categorías semánticas: estructura conceptual, sistemas de representación y sentido.

Los resultados obtenidos permiten concluir que el instrumento implementado ha permitido caracterizar los significados de los estudiantes de secundaria y de los docentes en formación. Ésta es la razón por la que consideramos el objetivo alcanzado. Además, la versatilidad del instrumento utilizado ha facilitado la aparición de información no prevista.

Objetivo 3. Identificar, describir e interpretar los significados del seno y coseno de un ángulo que muestran los escolares cuando los estudiantes de secundaria responden a tareas fuertemente conectadas con cada una de las categorías de significado según la perspectiva de Rico (2013)

Como se ha mencionado anteriormente, los estudiantes de secundaria interpretan el seno y el coseno de un ángulo de diversas formas. Para caracterizar sus significados, se analizaron las respuestas de los estudiantes a diferentes preguntas relacionadas con cada una de las categorías de significado.

Concretamente en relación a la estructura conceptual las producciones no son consistentes con Brown (2005) ya que los estudiantes no sólo utilizan una longitud o una razón para definir el seno o coseno de un ángulo, sino también valor, medida, herramienta y ángulo. Tampoco estos resultados están en línea con los de Kamber y Takaci (2018) ya que el tema predominante en su investigación fue longitud y en la nuestra razón.

Como una aportación de la historia de la enseñanza de la trigonometría, nosotros conjeturamos la comprensión de esta primera categoría mediante un uso correcto de cinco nociones fundamentales: ángulo, razón, triángulo rectángulo, circunferencia y funciones trigonométricas. Estas nociones se vinculan entre sí y dan origen a diversas relaciones (Martín-Fernández et al., 2016). Los ángulos, la razón y la circunferencia están relacionados debido al "line system", que define las razones trigonométricas de un ángulo como segmentos de líneas de una circunferencia, y se originó a partir de las concepciones griegas y árabes de la trigonometría (Maor, 1998). A principios del siglo XX, en las escuelas secundarias, las razones trigonométricas se definieron utilizando argumentos de números reales, "ratio system", en el que el ángulo, la razón y el triángulo rectángulo están asociados y las razones trigonométricas se expresan como una razón adimensional de lados de un triángulo rectángulo (Sickle, 2011). Hoy en día, en muchos países, el enfoque más comúnmente utilizado para introducir las razones trigonométricas es el "ratio system". Sin embargo, desde una perspectiva curricular, si definimos trigonometría en función de cómo se usa actualmente este término, la trigonometría se refiere al estudio de la medición de los ángulos, triángulos o funciones correspondientes (Fanning, 2016).

Respecto a la segunda categoría semántica, los temas son similares a la primera categoría semántica, pero en esta categoría semántica el tema dominante es longitud.

Finalmente, en la categoría sentido, el seno y el coseno de un ángulo se utilizan para el cálculo de magnitudes. En contraste con el estudio de Kamber y Takaci (2018), los estudiantes permanecen en el sistema parcial geométrico. Esto puede ser debido a varias causas entre las que situamos: la instrucción recibida y que la definición del seno y coseno en la circunferencia goniométrica es más generalizable, más abstracta.

De acuerdo con los estudios de Thompson (2007) y Kamber y Takaci (2018), reconocemos una escasa conexión entre los conceptos estudiados con la vida real y las matemáticas cotidianas de las experiencias vividas por los estudiantes. A diferencia de los resultados de Allen (1977), la navegación, la topografía, la carpintería y la balística no son los temas predominantes en los niveles introductorios de la trigonometría en nuestros datos. Los resultados muestran que calcular la distancia a un objeto es el fenómeno más popular entre los alumnos de secundaria. Éstos deben percibir la necesidad de este tópico para su futura profesión, de modo que estén más motivados para el estudio de conceptos trigonométricos. De este modo, es necesario presentar tareas más relacionadas con las ciencias modernas, los fenómenos periódicos, etc.

Una vez identificados los principales temas que surgen en las respuestas, los combinamos para describir los significados de los participantes. Para lograrlo, se elabora una tabla de contingencia con los diferentes temas. Las tres combinaciones más representativas se consideran prototipos de significado (Figura 5.7).

Mediante la interpretación de los resultados, se revelan las dos formas que históricamente se han utilizado para enseñar el sistema relacional de trigonometría. De este modo, el primer prototipo de significado se encuentra relacionado con el "ratio system", el segundo con el "line system", y el último enfatiza las conexiones entre el "ratio system" y el "line system".

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Objetivo 4. Identificar los significados sobre el concepto de ángulo y su coseno que muestran los profesores de secundaria en formación y describir el contenido conceptual y procedimental al moverse entre el sistema goniométrico parcial y el sistema analítico parcial.

Para estudiar qué significados muestran los profesores en formación sobre el concepto ángulo, seguimos la caracterización de Freudenthal (1973, p. 479). Él establece tres subsistemas fundamentalmente dentro del sistema relacional de trigonometría: el sistema de geometría elemental, el sistema goniométrico y el sistema analítico geométrico.

Para analizar los significados mostrados por los profesores en la formación del seno y coseno de un ángulo, nos basamos en los resultados obtenidos con los estudiantes de secundaria, en la primera fase del estudio.

En concreto, en la circunferencia goniométrica, la identificación del ángulo correspondiente a un punto P se interpreta por un ángulo absoluto (37,50%) y por un ángulo orientado (45,83%). En la función trigonométrica, el ángulo correspondiente a un punto P en la circunferencia goniométrica se considera: un punto (5,55%), un punto y un ángulo (5,55%), y un ángulo (20,82%).

En cuanto a los significados del coseno considerados por los docentes en formación en la circunferencia goniométrica, se observan los siguientes temas: longitud (23,60%), valor (20,82%), y razón (6,94%).

En relación a los significados del coseno de un ángulo en la función analítica, los temas que emergen son los siguientes: punto (5,56%), longitud (16,67%), y valor (23,61%).

El contenido movilizado es descrito en la tabla 4.3, y se ha utilizado para la clasificación de los temas y subtemas en las respuestas.

Objetivo 5. Indagar la comprensión de los estudiantes de secundaria y los profesores en formación sobre estos contenidos por medio de la caracterización de sus significados y componentes.

Usamos el marco de significado para estudiar la comprensión de los sujetos de los contenidos trigonométricos. Así, el significado de un contenido nos permite saber cómo lo comprenden los participantes del estudio. Además, teniendo en cuenta la riqueza interpretativa y las relaciones de los contenidos, los participantes razonan de manera diferente y muestran un determinado desarrollo. Los modos se diferenciaron por los temas seleccionados. Por tanto, esta investigación ayuda a mejorar el conocimiento teórico sobre la comprensión de conceptos matemáticos por parte de los futuros profesores y estudiantes de secundaria y contribuye a su caracterización.

Respecto a los estudiantes de secundaria, se observa que la idea de proporción apenas aparece en el sistema elemental geométrico parcial. Además, en el sistema goniométrico parcial se pone de manifiesto una pobre comprensión de las características de la circunferencia unidad. De hecho, la circunferencia aparece de modo incidental con normalmente, una incorrecta expresión de los ejes coordenados. En este sistema parcial se presenta también un escaso conocimiento del seno y coseno como coordenada. Todo ello, se traduce en una conexión deficiente entre el "line system" y el "ratio system". Finalmente, la idea de ordenada de la función sinusoidal emergió en muy pocas respuestas.

Concretamente, el 88% de las respuestas han expresado la comprensión de los conceptos seno y coseno de un ángulo, a partir de una noción de significado coherente, sustentada en las tres categorías elegidas como marco semántico. El 53% de las respuestas han revelado su significado utilizando la razón, la longitud y la medida de una magnitud. La mayoría de ellos han explicado sus significados utilizando los temas de razón y longitud, ambos equilibrados. Algunos estudiantes comprenden estos conceptos de varias formas.

En cuanto a los docentes en formación, los diferentes criterios nos brindan información sobre lo que cada uno ha entendido y lo que conviene potenciar. Se observa una falta de conexión entre el primer y el segundo sistema parcial que se traduce en la incapacidad para convertir nociones entre sistemas y vincular sus significados. Varias son las causas. En primer lugar, sus interpretaciones ponen énfasis en aspectos métricos (longitud y razón), sobre aquellos que enfatizan aspectos analíticos (valor). En segundo lugar, hay pocas combinaciones de significados del coseno de un ángulo en el sistema de goniometría parcial y en el sistema analítico parcial. Finalmente, se observan conexiones empobrecidas entre nociones similares involucradas.

En relación al contenido movilizado, dibujar ángulos orientados en la circunferencia goniométrica es una destreza que no es considerada importante por los participantes y quizás por ello es escasamente utilizada (45,83%, Tabla 6.1). Es decir, el uso del ángulo absoluto en vez del ángulo orientado hace que la circunferencia se convierta en una figura icónica y escasamente usada aspecto que coincide con lo expresado en las respuestas por los estudiantes de secundaria. De hecho, la estrategia utilizada por los sujetos se basa minoritariamente en el ángulo y la ordenada o en el ángulo y el valor (22,21%, tabla 6.5), y los participantes no identificaron normalmente las coordenadas x de la función analítica con la medida del ángulo correspondiente a P.

Nosotros suponemos que esto se debe a una pobre percepción del punto P como ángulo orientado en la circunferencia unidad, al gran porcentaje de participantes que no construyen el coseno del ángulo y a la escasez de combinaciones de significados del coseno del ángulo en la circunferencia goniométrica.

Todo ello hace que un alto porcentaje de estudiantes, recuerden incorrectamente información y hagan falsas conexiones entre el sistema analítico parcial y el sistema goniométrico parcial. Esto se pone de manifiesto en el gran número de participantes que confunden el gráfico del seno con el coseno (22,21%).

Finalmente, concluimos que los profesores en formación ni utilizan una estrategia válida para resolver la tarea, ni convierten nociones, ni parece que el círculo unidad les fue enseñado para moverse entre sistemas parciales. Además, no hay evidencia de que consideren valores de referencia como elementos fundamentales para el vínculo entre los sistemas parciales.

Objetivo 6. Examinar la influencia de los significados interpretados para el concepto ángulo por los profesores en formación.

Los profesores en formación que dibujan un ángulo absoluto tienden a utilizar estrategias métricas (longitud y razón). Además, generalmente no convierten ángulos entre sistemas parciales, ni utilizan una estrategia adecuada para realizar la tarea, ni construyen el coseno del ángulo en el segundo sistema parcial. Sin embargo, los participantes que dibujan ángulos orientados utilizan principalmente una estrategia métrica-goniométrica (valor). Además, utilizan una estrategia válida para resolver la tarea y tienden en mayor medida a interpretar el coseno del ángulo en el sistema analítico parcial. Por lo tanto, la promoción del ángulo como orientado podría no solo mejorar la comprensión de los estudiantes de la circunferencia goniométrica y sus características, sino también facilitar la representación las funciones trigonométricas. Por lo tanto, se concluye que el concepto ángulo considerado por los participantes tiene una influencia decisiva en la construcción y en el significado de su coseno, y en el movimiento entre sistemas parciales.

9.9.2. Limitaciones del estudio y sugerencias para futuras investigaciones

En primer lugar, las muestras utilizadas fueron elegidas intencionalmente por su disponibilidad. Sin embargo, somos conscientes de que los resultados no son generalizables.

Las metodologías empleadas, el diseño del cuestionario, la recogida, organización y categorización de los datos dan validez interna a nuestra investigación sustentada en la teoría fundamentada y el análisis de contenido.

Otra limitación de nuestro estudio se refiere a que los resultados están mediados por la instrucción recibida por los estudiantes de secundaria. Si bien los tres profesores tenían una experiencia considerable y pertenecen a distintas generaciones, dado que los alumnos pertenecen al mismo centro, podemos considerar que su instrucción fue homogénea.

Este estudio enfatiza la relevancia de los cuestionarios semánticos e indica algunas de sus características básicas. Sin embargo, surgen algunas preguntas como: ¿Cuándo se usa el cuestionario? ¿Cómo se usa y qué tipo de preguntas responderá? ¿Cuál es la recogida de datos más adecuada? ¿Qué tipo de análisis de datos es el más adecuado? ¿Cómo se asegura la fiabilidad y la validez con este método? ¿Se cubrió el tema suficientemente como para obtener los significados mostrados de los participantes? Todas estas preguntas nos muestran que un estudio en profundidad de los cuestionarios semánticos y la aplicación de varios cuestionarios semánticos a un mismo grupo analizando sus resultados, es una línea de futura investigación.

Otra línea de investigación es el diseño e implementarción de una propuesta de innovación curricular sobre el sistema relacional de la trigonometría que se base en el estudio teórico realizado sobre el significado de los conceptos de seno y coseno de un ángulo. Esta propuesta curricular también podría incluir otras metodologías ya implementadas en estudiantes universitarios de otros países y su posible aplicación a participantes no universitarios. Sin embargo, esta innovación se centraría en promover los significados escolares de los contenidos trigonométricos en los diferentes sistemas parciales de trigonometría.

Debido a que el significado de los estudiantes de secundaria es fundamentalmente el cálculo de magnitudes, diseñar e implementar tareas que involucren otros significados es una obligación.

Profundizar en los significados del profesor de secundaria en servicio a la hora de resolver tareas que involucran el seno y el coseno de un ángulo es otra línea de investigación futura.

Caracterizar el significado que los estudiantes de secundaria atribuyen a la noción de tangente, cuando explican y representan dicha noción. En este estudio nos hemos basado en el seno y el coseno, pero existe un hueco en la investigación sobre el concepto de tangente.

Investigar las relaciones y dependencias entre significados y sistemas de trigonometría parcial. El estudio actual podría ampliarse utilizando tareas que vinculen y conviertan nociones entre otros sistemas parciales.

Finalmente, quedan preguntas sin analizar en los cuestionarios semánticos. Claramente, el análisis de las respuestas de los estudiantes a otras preguntas es otra vía de continuación de la investigación.

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