Strongly norm-attaining Lipschitz maps

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Abstract

We study the possibility of approximating every Lipschitz map by Lipschitz maps which attain their Lipschitz constant. That is, we study the denseness of the set $\mathrm{LipSNA}(M,Y)$ of strongly norm-attaining Lipschitz maps in the space $\mathrm{Lip}_0(M,Y)$ of all Lipschitz maps from a (complete pointed) metric space M to a Banach space Y. A Lipschitz map $f\colon M \longrightarrow Y$ is said to strongly attain its (Lipschitz) norm if there are distinct points $p,q \in M$ satisfying

$$\|f(p)-f(q)\| = \mathrm{L}(f)\, d(p,q),$$

where $\mathrm{L}(f)$ is the Lipschitz constant of f defined by

$$\mathrm{L}(f) = \sup\left\{ \frac{\|f(p)-f(q)\|}{d_M(p,q)} : p,q \in M,\, p \neq q \right\}.$$

The leading question in this thesis is to study for which pointed metric spaces M and Banach spaces Y, the set $\mathrm{LipSNA}(M,Y)$ is dense.

We first recall the previously known results on the topic: that $\mathrm{LipSNA}(M,Y)$ is dense in $\mathrm{Lip}_0(M,Y)$ for every Y when the Lipschitz-free space $\mathcal{F}(M)$ has the Radon-Nikodým property (in particular, when the little Lipschitz space uniformly separates the points of M) and that $\mathrm{LipSNA}(M,\mathbb{R})$ is not dense in $\mathrm{Lip}_0(M,\mathbb{R})$ when M is a geodesic metric space (in particular, when it is a closed convex subset of a Banach space). Next, we provide new sufficient conditions on the metric space M to get that $\mathrm{LipSNA}(M,Y)$ is dense in $\mathrm{Lip}_0(M,Y)$ for every Banach space Y which deal with geometrical properties of the Lipschitz-free space $\mathcal{F}(M)$. For instance, we show that this holds when $\mathcal{F}(M)$ has property α or property quasi-α or when the unit ball of $\mathcal{F}(M)$ contains a norming subset of uniformly strongly exposed points. Besides, we characterize the above geometrical properties of $\mathcal{F}(M)$ in terms of the underlying metric space M. We also provide some more examples for which there is density of the strongly norm-attaining Lipschitz maps: arbitrary Hölder metric spaces and some particular subset of the plane which contain a segment (and so, their Lipschitz-free spaces fail the Radon-Nikodým property). The latter examples provide a negative answer to an open question proposed by G. Godefroy in 2015. We also show that none of the previously known and none of the newly introduced sufficient conditions to get the density of $\mathrm{LipSNA}(M,Y)$ is also necessary, and we provide some consequences of such a density on the geometry of $\mathcal{F}(M)$ which improve, in this ambient, results of Lindenstrauss and Bourgain. We also prove that $\mathrm{LipSNA}(M,\mathbb{R})$ is weakly dense in $\mathrm{Lip}_0(M,\mathbb{R})$ for all metric spaces M.

Next, we extend the previously known negative results on the density of strongly norm-attaining Lipschitz maps, showing that $\mathrm{LipSNA}(M,\mathbb{R})$ is not dense if M is a length metric spaces or if M is a compact subset of the real line with positive Lebesgue measure. Other interesting negative examples presented here include the unit sphere of the Euclidean plane and, more in general, every C^2-smooth curve.

Adapting a definition for bounded linear operators of 2008, we introduce and study the Lipschitz Bishop-Phelps-Bollobás property (Lip-BPB property in short) as a condition assuring that a Lipschitz map and a pair of points where the map almost attains its Lipschitz norm can be simultaneously approximated by, respectively, a strongly norm-attaining Lipschitz map and a pair of points where the map attains its Lipschitz norm. We study this property for finite metric spaces showing that, contrary to the case of the density of strongly norm-attaining Lipschitz maps, finiteness is not enough to get the Lip-BPB property. We also show that (M,Y) has the Lip-BPB property for every Banach space Y when M is a uniformly Gromov concave metric space (that is, when the whole set of molecules is a set of uniformly strongly exposed points). Examples of uniformly Gromov concave metric spaces include finite concave metric spaces, concave metric spaces for which $\mathcal{F}(M)$ has property α, Hölder metric spaces, and ultrametric spaces.

Finally, some stability properties of the density of strongly norm-attaining Lipschitz maps and of the Lip-BPB property are provided, and we also introduce and study versions of these two properties for Lipschitz compact maps.

Resumen

Esta tesis doctoral está dedicada a dar respuestas positivas y negativas a la siguiente pregunta:

> ¿Cuándo es posible aproximar cualquier función Lipschitziana por funciones Lipschitzianas que alcanzan fuertemente su norma?

Más concretamente, estudiamos la posible densidad del conjunto $\operatorname{LipSNA}(M,Y)$ de las aplicaciones Lipschitz que alcanzan su norma (Lipschitz) fuertemente, esto es, el conjunto de las aplicaciones Lipschitz f definidas en un espacio métrico (completo y punteado) M con valores en un espacio de Banach Y para las que existen puntos distintos $p, q \in M$ verificando que

$$\|f(p) - f(q)\| = \mathrm{L}(f)\, d(p,q),$$

donde $\mathrm{L}(f)$ denota la constante de Lipschitz de f definida por

$$\mathrm{L}(f) = \sup\left\{\frac{\|f(p) - f(q)\|}{d_M(p,q)} : p, q \in M,\ p \neq q\right\}.$$

Por tanto, la pregunta principal de esta tesis es para qué espacios métricos M y para qué espacios de Banach Y el conjunto $\operatorname{LipSNA}(M,Y)$ es denso en el espacio $\operatorname{Lip}_0(M,Y)$ de todas las aplicaciones Lipschitz de M en Y.

Comenzamos la tesis recordando los resultados previamente conocidos sobre el tema: que el conjunto $\operatorname{LipSNA}(M,Y)$ es denso en $\operatorname{Lip}_0(M,Y)$ para todo espacio de Banach Y siempre que el espacio Lipschitz-libre $\mathcal{F}(M)$ tiene la propiedad de Radon-Nikodým (en particular, cuando el espacio de las funciones Lipschitz localmente planas separa los puntos de M uniformemente) y que $\operatorname{LipSNA}(M,\mathbb{R})$ no es denso en $\operatorname{Lip}_0(M,\mathbb{R})$ cuando M es geodésico (en particular, cuando es un subconjunto convexo y cerrado de un espacio de Banach). Damos a continuación nuevas condiciones suficientes sobre el espacio métrico M para que $\operatorname{LipSNA}(M,Y)$ sea denso en $\operatorname{Lip}_0(M,Y)$ para todo espacio de llegada Y, que se escriben en términos de la geometría del espacio Lipschitz-libre $\mathcal{F}(M)$. En particular, probamos que este es el caso cuando $\mathcal{F}(M)$ tiene la propiedad α o la propiedad quasi-α o cuando su bola unidad contiene un subconjunto normante formado por puntos uniformemente fuertemente expuestos. Además, caracterizamos estas propiedades del espacio de Banach $\mathcal{F}(M)$ en términos de la geometría del espacio métrico M. También damos algunos ejemplos particulares para los que hay densidad de las funciones que alcanzan fuertemente su norma Lipschitz: esto pasa en cualquier espacio de Hölder y en algunos subconjuntos del plano que contienen al intervalo unidad (y, por tanto, los espacios Lipschitz-libres sobre ellos no tienen la propiedad de Radon-Nikodým). Estos últimos ejemplos dan respuesta negativa a una pregunta planteada por G. Godefroy en 2015. También demostramos que ninguna de las condiciones suficientes para tener densidad (ni las previamente conocidas ni las nuevas) es necesaria y damos consecuencias de tener densidad en la geometría de los espacios Lipschitz-libres que mejoran, en este ambiente, resultados de Lindenstrauss y de Bourgain.

Seguidamente, extendemos los resultados negativos conocidos sobre densidad de aplicaciones Lipschitz que alcanzan fuertemente su norma, demostrando que $\operatorname{LipSNA}(M,\mathbb{R})$ no es denso si M es un espacio métrico length o si M es un subconjunto compacto de la recta real. Otros ejemplos negativos interesantes que hemos conseguido son la esfera unidad del plano Euclídeo y, más en general, cualquier curva de clase C^2.

Adaptando una definición introducida en 2008 para operadores lineales y continuos, introducimos la propiedad de Bishop-Phelps-Bollobás Lipschitz (propiedad Lip-BPB) como una condición asegurando

que cualquier aplicación Lipschitz y cualquier par de puntos en los que dicha aplicación casi alcanza su norma pueden ser aproximados, simultáneamente, por una aplicación Lipschitz que alcanza su norma y, respectivamente, un par de puntos en los que la nueva aplicación alcanza su norma Lipschitz. Estudiamos primero esta propiedad para espacios métricos finitos demostrando que, a diferencia de lo que ocurre con la densidad, la finitud no es suficiente para conseguir la propiedad Lip-BPB. Demostramos también que los pares (M, Y) tienen la propiedad Lip-BPB para todos los espacios de llegada Y si M es uniformemente Gromov cóncavo (esto es, si el conjunto de todas las moléculas está uniformemente fuertemente expuesto). Ejemplos de espacios métricos uniformemente Gromov cóncavos son los espacios métricos finitos y cóncavos, los espacios métricos cóncavos para los que $\mathcal{F}(M)$ tiene la propiedad α, los espacios métricos de Hölder y los espacios ultramétricos.

Por último, damos algunas propiedades de estabilidad de la densidad de aplicaciones Lipschitz que alcanzan fuertemente su norma y de la propiedad Lip-BPB, e introducimos y estudiamos versiones de estas dos propiedades para aplicaciones Lipschitz compactas.

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Introduction

The concept of Lipschitz function was introduced in 1864 by the German mathematician Rudolf Lipschitz. Since then, Lipschitz functions have been an object widely studied for their applications in fields of mathematics such as Fourier Analysis and Differential Equations. Let us recall this concept.

Definition. Let (M, d_M) and (N, d_N) be metric spaces and let $f\colon M \longrightarrow N$ be a map. We say that f is Lipschitz if there exists a constant $k \geqslant 0$ such that

$$d_N(f(p), f(q)) \leqslant k d_M(p, q) \quad \forall\, p, q \in M.$$

The least constant satisfying the above inequality is called the *Lipschitz constant* of f and it is usually denoted by $\mathrm{L}(f)$:

$$\mathrm{L}(f) = \sup\left\{ \frac{d_N(f(p), f(q))}{d_M(p, q)} : p, q \in M, p \neq q \right\}. \tag{1}$$

With this notation, we can say that a map $f\colon M \longrightarrow N$ is Lipschitz if and only if $\mathrm{L}(f) < \infty$.

The appearance of Functional Analysis and, in particular, of the theory of Banach spaces, has made it possible to study Lipschitz maps not only as maps satisfying certain good behavior, but also as elements of some Banach space. Indeed, given a real Banach space Y, we can consider the vector space $\mathrm{Lip}(M, Y)$ of all Lipschitz maps from M to Y. Then, the function $\|\cdot\|_L\colon \mathrm{Lip}(M, Y) \longrightarrow \mathbb{R}$ that assigns to each Lipschitz map $f \in \mathrm{Lip}(M, Y)$ its Lipschitz constant, that is, $\|f\|_L = \mathrm{L}(f)$, is a seminorm that, after considering the quotient space under the subspace of the constant functions, becomes a complete norm.

Instead of considering the quotient space, it is more convenient to consider pointed metric spaces. A *pointed metric space* is simply a metric space M in which we distinguish an element, usually denoted by 0. Then, given a real Banach space Y, we can consider the space $\mathrm{Lip}_0(M, Y)$ of all Lipschitz maps from M to Y that vanish at 0. With this idea, the only constant map that we may find as element of this space is the zero map, so $\mathrm{Lip}_0(M, Y)$ endowed with the Lipschitz constant as norm, is a Banach space. Let us say that we are not losing generality restricting to pointed metric spaces since the only thing we do is to fix a point on it. Furthermore, if M_0 is a pointed metric space and M is its completion, then it is well known that every Lipschitz map with domain M_0 uniquely extends to M. Indeed, it is immediate to verify that the spaces $\mathrm{Lip}_0(M_0, Y)$ and $\mathrm{Lip}_0(M, Y)$ are isometrically isomorphic for any Banach space Y. Therefore, we do not lose generality if we restrict our study to complete metric spaces. For these reasons, all along this work every metric space will be pointed and complete.

The study of the geometric structure of these Banach spaces has been strongly developed during the last decades (good references for background on this are [41], [44], [51], and [65]). Moreover, this line of research has gained popularity due to the importance that Lipschitz maps have in the theory of Nonlinear Geometry of Banach spaces (see [42]).

Throughout this work, Banach spaces will be over the real scalars. Let us give a piece of notation. Given a metric space M, $p \in M$, and $r > 0$, the closed ball of center p and radius r is denoted by $B(p, r)$. Given a Banach space X, we will denote by B_X and S_X the closed unit ball and the unit sphere of X, respectively. We will denote by X^* the topological dual of X. If Y is another Banach space, we write $\mathcal{L}(X, Y)$ to denote the Banach space of all bounded linear operators from X to Y, endowed with the operator norm. We say that $T \in \mathcal{L}(X, Y)$ *attains its norm*, and write $T \in \mathrm{NA}(X, Y)$, if there is $x \in X$ with $\|x\| = 1$ such that $\|Tx\| = \|T\|$. The study of the density of norm-attaining linear operators has its roots in the classical Bishop-Phelps theorem, which states that $\mathrm{NA}(X, \mathbb{R})$ is dense in $X^* = \mathcal{L}(X, \mathbb{R})$ for every Banach space X (see [16]). After this result, in 1963 Lindenstrauss began the study of the vectorial case, that is, the study of norm-attaining linear operators, showing that in general the Bishop-Phelps theorem is not true in the vectorial case and giving some positive partial results (see [57]). Since then, the theory of norm-attaining linear operators has been intensely studied by many mathematicians. Relevant contributions were given by, among many others, J. Bourgain in the 1970's and W. Schachermayer in the 1980's. We refer to [1] for a detailed account on the subject. Recently, this theory has gained strength again with the introduction in 2008 of the *Bishop-Phelps-Bollobás property* (BPBp for short) [4]. This property tries to study for which pairs of Banach spaces (X, Y), we can approximate both any linear operator $T \in \mathcal{L}(X, Y)$ and any point $x \in S_X$ at which T is close to attain its norm by a norm-attaining linear operator $S \in \mathrm{NA}(X, Y)$ and a point $x' \in S_X$ at which S attains its norm.

In this work we will focus our attention on studying a natural question that appears as a combination of the two aforementioned research lines, Lipschitz maps and norm-attaining linear operators. The next definition constitutes our main object of study along this thesis.

Definition. Let M be a metric space and let Y be a Banach space. We will say that a Lipschitz map $f\colon M \longrightarrow Y$ *attains its Lipschitz constant* or *strongly attains its norm* when there exist two distinct points $p, q \in M$ for which the supremum in (1) is attained, that is,

$$\|f\|_L = \mathrm{L}(f) = \frac{\|f(p) - f(q)\|}{d(p, q)}.$$

The set of all strongly norm-attaining Lipschitz maps from M to Y is denoted by $\mathrm{LipSNA}(M, Y)$.

If M is a finite metric space, it is clear that every Lipschitz map strongly attains its norm. On the other hand, it is easy to show that for any infinite metric space M and Banach space Y, it is possible to find a Lipschitz function $f\colon M \longrightarrow Y$ which does not strongly attain its norm (see Corollary 3.46 in [65]). However, since any Lipschitz map from M to Y vanishing at 0 can be seen as an element of the Banach space $\mathrm{Lip}_0(M, Y)$, where we have a topology, it makes sense to wonder if given a Lipschitz map $f\colon M \longrightarrow Y$, we can find strongly norm-attaining Lipschitz maps as close to f as we want. The first negative answer to this question was given in [50, Example 2.1], where a Lipschitz function $f\colon [0, 1] \longrightarrow \mathbb{R}$ which cannot be approximated in norm by strongly norm-attaining Lipschitz functions is presented. In view of this, the following natural question appears:

Problem (Leading problem of the thesis)**.** For which metric spaces M and Banach spaces Y is it possible to approximate every Lipschitz map from M to Y by strongly norm-attaining Lipschitz maps? Equivalently, for which metric spaces M and Banach spaces Y is the set $\mathrm{LipSNA}(M, Y)$ norm-dense in $\mathrm{Lip}_0(M, Y)$?

This is the leading question for this work. Let us comment that this question has been asked by G. Godefroy in the 2015 survey paper [41] (see Problem 6.7) for the case when M is compact. There are several concrete questions there, some of which are studied in this thesis.

Although at first appearance this question seems to deal only with Lipschitz maps, we will show that there is an intimate connection with the theory of norm-attaining linear operators. Indeed, this connection will play an essential role throughout this work, but before being able to understand why, we need to learn more about Lipschitz spaces.

The first thing we need to know is that, for any pointed metric space M, the Lipschitz space $\mathrm{Lip}_0(M, \mathbb{R})$ is a dual Banach space, that is, there exists a Banach space X whose topological dual, denoted as usual by X^*, is isometrically isomorphic to $\mathrm{Lip}_0(M, \mathbb{R})$. In fact, an explicit expression of an isometric predual of $\mathrm{Lip}_0(M, \mathbb{R})$ can be given. Indeed, we denote by δ the canonical isometric embedding of M into $\mathrm{Lip}_0(M, \mathbb{R})^*$, which is given by

$$\langle f, \delta(p) \rangle = f(p) \quad \forall\, p \in M, \quad \forall\, f \in \mathrm{Lip}_0(M, \mathbb{R}).$$

We denote by $\mathcal{F}(M)$ the norm-closed linear span of $\delta(M)$ in the dual of $\mathrm{Lip}_0(M, \mathbb{R})$:

$$\mathcal{F}(M) := \overline{\mathrm{span}}\{\delta(p) \colon p \in M\} \subseteq \mathrm{Lip}_0(M, \mathbb{R})^*,$$

which is usually called the *Lipschitz-free space over M*. It is known that $\mathcal{F}(M)$ is an isometric predual of the Banach space $\text{Lip}_0(M,\mathbb{R})$ (actually, in [65] it is shown that it is the unique isometric predual when M is bounded or a geodesic space). For more background on Lipschitz-free spaces, we refer to the papers [41] and [43], and to the book [65] (where they are called *Arens-Eells spaces*).

The second thing we need to know is that we can identify every Lipschitz map from M to Y with a bounded and linear operator from $\mathcal{F}(M)$ to Y. This identification plays an essential role throughout this work since it is the bridge between strongly norm-attaining Lipschitz maps and norm-attaining linear operators. Given a Lipschitz map $f\colon M \longrightarrow Y$, we can consider the unique bounded linear operator $\widehat{f}\colon \mathcal{F}(M) \longrightarrow Y$ satisfying

$$\widehat{f}(\delta(p)) = f(p) \quad \forall\, p \in M.$$

The mapping $f \longmapsto \widehat{f}$ turns out to be an isometric isomorphism between the Lipschitz space $\text{Lip}_0(M,Y)$ and the space $\mathcal{L}(\mathcal{F}(M),Y)$ of all bounded and linear operators from $\mathcal{F}(M)$ to Y. Therefore, every time that we have a Lipschitz map $f\colon M \longrightarrow Y$, we can consider its associated linear operator $\widehat{f}\colon \mathcal{F}(M) \longrightarrow Y$, and vice versa. Indeed, we will constantly use this identification without specifying more details.

The third thing we need to introduce is the notion of molecule. Given a metric space M, a *molecule* of $\mathcal{F}(M)$ is just an element of the form

$$m_{p,q} = \frac{\delta(p) - \delta(q)}{d(p,q)}, \quad \text{where } p, q \in M,\ p \neq q.$$

We write $\text{Mol}(M)$ to denote the set of all molecules of $\mathcal{F}(M)$. As a consequence of Hahn-Banach theorem, it is easy to see that every molecule has norm one and that we can recover the unit ball of $\mathcal{F}(M)$ as the closed convex hull of the molecules, that is,

$$B_{\mathcal{F}(M)} = \overline{\text{co}}(\text{Mol}(M)).$$

Now, we are ready to state the connection between our main question and the theory of norm-attaining linear operators. Let M be a metric space, let Y be a Banach space, and let $f\colon M \longrightarrow Y$ be a Lipschitz map. Recall that f strongly attains its norm when there exist distinct points p, $q \in M$ such that

$$\|f\|_L = \frac{\|f(p) - f(q)\|}{d(p,q)}.$$

Then, using the identification between Lipschitz maps and their associated linear operators, we can reformulate this condition as follows:

$$\|\widehat{f}\| = \frac{\|\widehat{f}(\delta_p) - \widehat{f}(\delta_q)\|}{d(p,q)} = \left\|\widehat{f}\left(\frac{\delta_p - \delta_q}{d(p,q)}\right)\right\| = \|\widehat{f}(m_{p,q})\|.$$

Consequently, a Lipschitz map strongly attains its norm if, and only if, its associated linear operator attains its norm (in the classical sense) at a molecule of $\mathcal{F}(M)$. In view of this, the study of the density of the strongly norm-attaining Lipschitz maps can be understood as a non-linear generalization of the classical theory of norm-attaining linear operators. Indeed, our leading question can be reformulated in the following way:

For which metric spaces M and Banach spaces Y is the set of all bounded linear operators from $\mathcal{F}(M)$ to Y which attain their norm at some molecule dense in $\mathcal{L}(\mathcal{F}(M),Y)$?

Let us comment that this stronger version of density is not equivalent to the classic one. Indeed, it is clear that if $\text{LipSNA}(M,Y)$ is dense in $\text{Lip}_0(M,Y)$, then $\text{NA}(\mathcal{F}(M),Y)$ is dense in $\mathcal{L}(\mathcal{F}(M),Y)$. However, the reverse implication is not true in general: the Bishop-Phelps theorem implies that $\text{NA}(\mathcal{F}(M),\mathbb{R})$ is dense in $\mathcal{L}(\mathcal{F}(M),\mathbb{R})$ for every metric space M, but we have already commented that $\text{LipSNA}([0,1],\mathbb{R})$ is not dense in $\text{Lip}_0([0,1],\mathbb{R})$. This shows that attaining the norm at a molecule is a condition quite more restrictive than simply attaining the norm. We will see that molecules play an important role in the extremal structure of Lipschitz-free spaces and, under some assumptions, this will allow us to obtain positive partial results for our problem using classic results coming from the theory of norm-attaining linear operators.

After this background, we proceed to outline the content of the thesis.

Chapter 1. Preliminary results

First of all, we present a brief summary containing some background on Lipschitz spaces and Lipschitz-free spaces, as well as some preliminary results that we will need for the rest of the chapters. We start commenting on the main results that have appeared in the last twenty years about the extremal structure of Lipschitz-free spaces. We consider classical notions of extremal points such as extreme points, exposed points, preserved extreme points, denting points, and strongly exposed points, studying them in the particular case when the Banach space is a Lipschitz-free space. Then, from different recent papers devoted to studying the geometry of Lipschitz-free spaces, we collect some characterizations of these extremal points in terms of the underlying metric space, and analyze the relationship between molecules and the extremal structure of Lipschitz-free spaces. Finally, we introduce some metric notions and technical results that we will need later.

Chapter 2. Strong density. Positive results

The second chapter is devoted to obtaining positive results concerning strong density. In the first section we present the first positive results that we find in the literature. They are due to G. Godefroy and deal with the little Lipschitz space, which under some assumptions acts as an isometric predual for the Lipschitz-free space. Furthermore, we want to highlight the following result which appeared shortly after that.

Proposition ([36, Proposition 7.4])**.** Let M be a metric space such that $\mathcal{F}(M)$ has the Radon-Nikodým property. Then, $\mathrm{LipSNA}(M,Y)$ is norm-dense in $\mathrm{Lip}_0(M,Y)$ for every Banach space Y.

J. Bourgain showed in [18] that if a Banach space X has the Radon-Nikodým property (RNP for short), then X also has *Lindenstrauss property A*, that is, $\mathrm{NA}(X,Y)$ is dense in $\mathcal{L}(X,Y)$ for every Banach space Y. Motivated by this fact, in the second section of the chapter we study some other known sufficient conditions for Lindenstrauss property A. More concretely, we analyze the property of having a norming set of uniformly strongly exposed points, property α, and property quasi-α. We prove that if M is a metric space such that $\mathcal{F}(M)$ satisfy any of the previous properties, then $\mathrm{LipSNA}(M,Y)$ is norm-dense in $\mathrm{Lip}_0(M,Y)$ for every Banach space Y. Moreover, we translate these properties in terms of the underlying metric spaces and present some criteria that help us to verify if the Lipschitz-free space over some metric space satisfies any of them.

We devote the third section of the chapter to present other kind of examples of metric spaces for which there is strong density. The first family we present is that of Hölder metric spaces. Recall that a *Hölder metric space* is a metric space of the form (M,d^θ), where (M,d) is a metric space and $0<\theta<1$.

Proposition (Corollary 2.25)**.** Let M be a Hölder metric space. Then, $\mathrm{LipSNA}(M,Y)$ is dense in $\mathrm{Lip}_0(M,Y)$ for every Banach space Y.

Next, we present some concrete subset of the plane for which there is strong density but which contain the unit interval $[0,1]$.

Example (Part of Theorem 2.26)**.** Consider the subsets of $\mathbb{R}^2$ given by

$$A_n=\left\{\left(\frac{k}{2^n},\frac{1}{2^n}\right):k\in\{0,\dots,2^n\}\right\}\subseteq\mathbb{R}^2\quad\forall n\in\mathbb{N}\cup\{0\},$$

$$M_\infty=\bigcup_{n=0}^{\infty}A_n,\quad M=M_\infty\cup([0,1]\times\{0\}).$$

Let $\mathfrak{M}_p$ be the set M endowed with the distance inherited from $(\mathbb{R}^2,\|\cdot\|_p)$ for $p=1,2$. Then, $\mathrm{LipSNA}(\mathfrak{M}_p,Y)$ is dense in $\mathrm{Lip}_0(\mathfrak{M}_p,Y)$ for every Banach space Y and for $p=1,2$.

The examples above are the first known examples of metric space for which there is strong density and whose Lipschitz-free space fails the RNP. Indeed, for $p=1,2$, $\mathfrak{M}_p$ contains $[0,1]$, so $\mathcal{F}(\mathfrak{M}_p)$ contains $L_1[0,1]$ as a subspace and thus, $\mathcal{F}(\mathfrak{M}_p)$ fails the RNP. This provides a negative answer to a question proposed by G. Godefroy in 2015, see [41, p. 115].

Next, we dedicate a section to discuss the relationship between all the sufficient conditions for strong density that we have presented. Furthermore, we use the criteria that we have developed in the previous section to generate examples of metric spaces satisfying and failing strong density. This allows us to prove that none of the sufficient conditions are necessary to have strong density.

The fifth section of Chapter 2 is devoted to studying the consequences that the density of strongly norm-attaining Lipschitz maps produces on the geometry of the Lipschitz-free space. Among the results we have obtained, we want to spotlight the following one:

Theorem (Theorem 2.35)**.** Let M be a metric space for which $\mathrm{LipSNA}(M,\mathbb{R})$ is dense in $\mathrm{Lip}_0(M,\mathbb{R})$. Then, $B_{\mathcal{F}(M)}$ is the closed convex hull of its extreme molecules.

Moreover, for compact metric spaces we obtain an improvement of the above result.

Theorem (Theorem 2.48)**.** Let M be a compact metric space for which $\mathrm{LipSNA}(M,\mathbb{R})$ is dense in $\mathrm{Lip}_0(M,\mathbb{R})$. Then, $B_{\mathcal{F}(M)}$ is the closed convex hull of its strongly exposed molecules.

These two results somehow improve a result by Lindenstrauss in the Lipschitz setting (see [57, Theorem 2]). As a consequence of our results, we also get the following corollary.

Corollary (Corollary 2.43)**.** Let M be a compact metric space for which $\mathcal{F}(M)$ has the RNP. Then, for every Banach space Y, $\mathrm{LipSNA}(M,Y)$ (and so $\mathrm{NA}(\mathcal{F}(M),Y)$) contains a dense open subset.

This also somehow improves a result from J. Bourgain in the Lipschitz setting for compact metric spaces (see [18, Theorem 5]).

Finally, in the last section of this chapter we study the density of the set $\mathrm{LipSNA}(M,\mathbb{R})$, but for a different topology. Indeed, $\mathrm{Lip}_0(M,\mathbb{R})$ is a Banach space for any metric space M, so we may consider its weak topology. Consequently, it makes sense to wonder for which metric spaces M the set $\mathrm{LipSNA}(M,\mathbb{R})$ is weakly dense in $\mathrm{Lip}_0(M,\mathbb{R})$. We obtain a plenary satisfactory answer to this question. In fact, we prove that for any metric space M, the set $\mathrm{LipSNA}(M,\mathbb{R})$ is sequentially weakly dense in $\mathrm{Lip}_0(M,\mathbb{R})$. Actually, we prove more:

Theorem (Theorem 2.53)**.** Let M be a metric space. Then, $\mathrm{LipSNA}(M,\mathbb{R})$ is weakly sequentially dense in $\mathrm{Lip}_0(M,\mathbb{R})$. Moreover, for every $g \in \mathrm{Lip}_0(M,\mathbb{R})$ there is a sequence $\{g_n\} \subset \mathrm{LipSNA}(M,\mathbb{R})$ such that $g_n \xrightarrow{w} g$, $\|g_n\|_L \to \|g\|_L$, and $g_n \to g$ uniformly on bounded sets.

Chapter 3. Strong density. Negative results

The third chapter is devoted to obtaining negative results. In the first section we present the first negative results that we find in the literature. They can be found in [50] and we can see them as a generalization of the next example.

Example ([50, Example 2.1])**.** Consider $[0,1]$ endowed with the usual metric. Then, $\mathrm{LipSNA}([0,1],\mathbb{R})$ is not dense in $\mathrm{Lip}_0([0,1],\mathbb{R})$.

In the next section, we generalize this example in two directions. On the one side, we prove the following result concerning length metric spaces.

Theorem (Theorem 3.3)**.** Let M be a length metric space. Then, the set $\mathrm{LipSNA}(M,\mathbb{R})$ is not dense in $\mathrm{Lip}_0(M,\mathbb{R})$.

On the other side, as a consequence of our results, we characterize the closed subsets of $[0,1]$ for which we have strong density.

Corollary (Corollary 3.8)**.** Let M be a closed subset of $[0,1]$. Then, $\mathrm{LipSNA}(M,\mathbb{R})$ is dense in $\mathrm{Lip}_0(M,\mathbb{R})$ if and only if M has measure zero. Moreover, in such a case, we have that $\mathrm{LipSNA}(M,Y)$ is dense in $\mathrm{Lip}_0(M,Y)$ for every Banach space Y.

In the third section we study the case of the unit circle $\mathbb{T} \subseteq \mathbb{R}^2$ endowed with the Euclidean metric. The main result is the following.

Theorem (Theorem 3.10)**.** Let $\mathbb{T}$ be the unit sphere of the Euclidean plane endowed with the inherited Euclidean metric. Then, $\mathrm{LipSNA}(\mathbb{T},\mathbb{R})$ is not dense in $\mathrm{Lip}_0(\mathbb{T},\mathbb{R})$.

Let us comment that this result is somehow surprising, as $\mathbb{T}$ is Gromov concave (i.e. all the molecules are strongly exposed point of the unit ball of $\mathcal{F}(\mathbb{T})$), see Proposition 3.9. This shows that the later condition is not sufficient to get strong density.

The last section of the chapter is devoted to showing that, under some regularity assumptions, the previous result dealing with $\mathbb{T}$ can be extended to differentiable curves. More concretely, we prove the following result.

Theorem (Theorem 3.20)**.** Let E be a normed space, let $J \subseteq \mathbb{R}$ be an interval, let $\alpha\colon J \longrightarrow E$ be a curve, and let $\Gamma \subseteq E$ be its range. Assume that there is an interval $I \subseteq J$ for which $\alpha|_I : I \longrightarrow E$ is a C^2 curve parametrized by arc length and $\alpha(I)$ has nonempty interior with respect to Γ. Then,

$$\overline{\mathrm{LipSNA}(\Gamma,\mathbb{R})} \neq \mathrm{Lip}_0(\Gamma,\mathbb{R}).$$

Let us mention that in order to prove Theorem 3.20, several technical results are needed and the result is not merely a routinely extension (see Theorem 3.12 and Lemma 3.19).

Chapter 4. The Lipschitz Bishop-Phelps-Bollobás property

Recall that the Bishop-Phelps-Bollobás property was introduced in 2008 as a stronger version of the density of the set of norm-attaining linear operators (see [4]). This property tries to study for which pairs of Banach spaces (X, Y) we can approximate simultaneously both any linear operator $T \in \mathcal{L}(X, Y)$ and any point $x \in S_X$ at which T is close to attain its norm by a norm-attaining linear operator $S \in \mathrm{NA}(X, Y)$ and a point $x' \in S_X$ at which S attains its norm.

In the fourth chapter of the thesis we introduce an analogous property for Lipschitz maps, the Lipschitz Bishop-Phelps-Bollobás property (Lip-BPB property for short).

Definition (Definition 4.1)**.** Let M be a metric space and let Y be a Banach space. We say that the pair (M, Y) has the *Lipschitz Bishop-Phelps-Bollobás property* (*Lip-BPB property* for short), if given $\varepsilon > 0$ there is $\eta(\varepsilon) > 0$ such that for every norm-one $F \in \mathrm{Lip}_0(M, Y)$ and every $p, q \in M$, $p \neq q$ such that $\|F(p) - F(q)\| > \big(1 - \eta(\varepsilon)\big)d(p, q)$, there exist $G \in \mathrm{Lip}_0(M, Y)$ and $r, s \in M$, $r \neq s$, such that

$$\frac{\|G(r) - G(s)\|}{d(r, s)} = \|G\|_L = 1, \quad \|G - F\|_L < \varepsilon, \quad \frac{d(p, r) + d(q, s)}{d(p, q)} < \varepsilon.$$

If the previous definition holds for a class of linear operators from $\mathcal{F}(M)$ to Y, we will say that the pair (M, Y) has the Lip-BPB property for that class.

Let M be a metric space and Y be a Banach space. It is immediate to verify that the pair (M, Y) has the Lip-BPB property if and only if we can approximate both any Lipschitz map $F \in \mathrm{Lip}_0(M, Y)$ and any molecule $m \in \mathrm{Mol}(M)$ at which $\widehat{F}$ is close to attain its norm by a strongly norm-attaining Lipschitz map $G \in \mathrm{LipSNA}(M, Y)$ and molecule $m' \in \mathrm{Mol}(M)$ at which $\widehat{G}$ attains its norm. This observation makes clear that the Lip-BPB property is indeed a non-linear generalization of the classical Bishop-Phelps-Bollobás property.

In the first section we focus our attention on finite metric spaces. We study conditons to ensure that the pair (M, Y) has the Lip-BPB property for some Banach space Y. The next theorem is the main result of this kind.

Theorem (Theorem 4.2)**.** Let M be a finite metric space and let Y be a Banach space. If $(\mathcal{F}(M), Y)$ has the BPBp, then (M, Y) has the Lip-BPB property.

Moreover, we give some examples to show that in general it is not possible to remove any of the assumptions.

In the next section we give sufficient conditions on more general metric spaces M to guarantee that (M, Y) has the Lip-BPB property for every Banach space Y. Our main result in this section deals with the extremal structure of the Lipschitz-free space. As we will define later, a metric space M is said to be *uniformly Gromov concave* when the whole $\mathrm{Mol}(M)$ is a set of uniformly strongly exposed points (see Definition 1.12 for more details). In this family of metric spaces are included finite concave metric spaces, ultrametric spaces, and arbitrary Hölder metric spaces. For these spaces we obtain the following.

Theorem (Theorem 4.9)**.** Let M be a uniformly Gromov concave metric space. Then, (M, Y) has the Lip-BPB property for every Banach space Y.

Chapter 5. Stability results

In the first section we study the relationship between the Lip-BPB property for scalar Lipschitz functions and the Lip-BPB property for vector-valued Lipschitz maps. We give some conditions over the Banach space Y that allow us to pass from the Lip-BPB property of $(M, \mathbb{R})$ to the Lip-BPB property of (M, Y) for some classes of operators. Our main result in this section deals with the recent notion of ACK structure and Γ-flat operators from [21], see Definition 5.3.

Theorem (Theorem 5.6)**.** Let M be a metric space such that $(M, \mathbb{R})$ has the Lip-BPB property, let Y be a Banach space in ACK_ρ with associated norming set $\Gamma \subseteq B_{Y^*}$ of Definition 5.3, and let $\varepsilon > 0$. Then, there exists $\eta(\varepsilon, \rho) > 0$ such that if we take $\widehat{T} \in \mathcal{L}(\mathcal{F}(M), Y)$ a Γ-flat operator with $\|T\|_L = 1$ and $m \in \mathrm{Mol}(M)$ satisfying $\|\widehat{T}(m)\| > 1 - \eta(\varepsilon, \rho)$, then there exist an operator $\widehat{S} \in \mathcal{L}(\mathcal{F}(M), Y)$ and a molecule $u \in \mathrm{Mol}(M)$ such that

$$\|\widehat{S}(u)\| = \|S\|_L = 1, \quad \|m - u\| < \varepsilon, \quad \|T - S\|_L < \varepsilon.$$

Due to the generality of the last result, we obtain a series of consequences about the Lip-BPB property in the vector-valued case. These consequences include results concerning spaces of continuous functions, injective tensor products, and sequence spaces. Among them, we want to highlight the following corollary.

Corollary (Corollary 5.7)**.** Let M be a metric space such that $(M, \mathbb{R})$ has the Lip-BPB property. If Y is a Banach space having property β, then (M, Y) has the Lip-BPB property.

Let us comment that an analogous results for property quasi-β does not hold for the Lip-BPB property in general (see Example 5.14). Furthermore, we do a parallel study to obtain versions of these results for strong density. In fact, since we do not need to control the distance between molecules, for some of them we obtain improvements when we restrict to strong density. As an example of this, a version for strong density of Corollary 5.7 holds for property quasi-β.

Proposition (Proposition 5.15)**.** Let M be a metric space such that $\mathrm{LipSNA}(M, \mathbb{R})$ is norm dense in $\mathrm{Lip}_0(M, \mathbb{R})$ and let Y be a Banach space having property quasi-β. Then, we have that

$$\overline{\mathrm{LipSNA}(M, Y)} = \mathrm{Lip}_0(M, Y).$$

Finally, in the last section of this chapter we study stability properties of the Lip-BPB property and strong density under some operations that we can consider on the domain space, such as sums of metric spaces, or on the range space, such as absolute sums of Banach spaces. We also obtain more satisfactory results when we restrict our study to strong density. An example of this is the next result dealing with sum of metric spaces, which is not true for the Lip-BPB property as Example 5.19 shows.

Theorem (Theorem 5.20)**.** Let $\{M_i\}_{i \in I}$ be a family of metric spaces, consider the sum $M = \coprod_{i \in I} M_i$ and let Y be a Banach space. Then the following are equivalent:

(i) $\mathrm{LipSNA}(M_i, Y)$ is dense in $\mathrm{Lip}_0(M_i, Y)$ for every $i \in I$.

(ii) $\mathrm{LipSNA}(M, Y)$ is dense in $\mathrm{Lip}_0(M, Y)$.

Chapter 6. Lipschitz compact maps

This chapter is devoted to studying Lipschitz compact maps. Let M be a metric space, Y be a Banach space, and $F\colon M \longrightarrow Y$ be a Lipschitz map. We say that F is *Lipschitz compact* when its Lipschitz image, that is, the set

$$\left\{\frac{F(p)-F(q)}{d(p,q)} : p,q \in M,\ p \neq q\right\} \subseteq Y,$$

is relatively compact. We repeat the study of the strong density and the Lip-BPB property that we have done all along the previous chapters, but now restricted to Lipschitz compact maps. We see that most of our results are still valid when we restrict to this class. Moreover, in this setting we are able to improve some results and give new ones that are only valid for Lipschitz compact maps. For instance, given a metric space M and a Banach space Y, every compact operator with Y as codomain is Γ-flat for every $\Gamma \subseteq B_{Y^*}$. Therefore, we obtain results analogous to Theorem 5.6 and its consequences, but forgetting about the class of Γ-flat operators.

Proposition (Proposition 6.6)**.** Let M be a metric space such that $(M,\mathbb{R})$ has the Lip-BPB property and let Y be an ACK_ρ Banach space. Then, the pair (M,Y) has the Lip-BPB property for Lipschitz compact maps.

Moreover, an analogous result for strong density also holds. Let us also spotlight the following result concerning L_1-spaces, which is only valid when restricted to Lipschitz compact maps.

Proposition (Proposition 6.12)**.** Let M be a metric space such that $(M,\mathbb{R})$ has the Lip-BPB property. Let Y be a Banach space such that Y^* is isometrically isomorphic to an L_1-space. Then, (M,Y) has the Lip-BPB property for Lipschitz compact maps.

We also obtain an analogous result for the strong density in this case. Finally, following the previous chapters, we also present results dealing with the stability of the Lip-BPB property and the strong density for Lipschitz compact maps under some operations on the domain and range spaces. As before, we obtain some results that are only valid when restricted to Lipschitz compact maps.

Chapter 7. Conclusions and open problems

Finally, we present the conclusions of the thesis together with some open problems. Let us briefly comment on the open problems that we propose in Chapter 7.

The first problem asks if it is enough to have strong density for real valued functions to guarantee that we have strong density for every Banach space as range space:

Problem (Problem 7.1)**.** Let M be a metric space so that $\operatorname{LipSNA}(M,\mathbb{R})$ is dense in $\operatorname{Lip}_0(M,\mathbb{R})$. Is $\operatorname{LipSNA}(M,Y)$ dense in $\operatorname{Lip}_0(M,Y)$ for every Banach space Y?

The second one asks if the fact that the Banach space $\mathcal{F}(M)$ satisfies Lindenstrauss property A implies strong density for every Banach space:

Problem (Problem 7.2)**.** Let M be a metric space. Assume that $\mathcal{F}(M)$ has Lindenstrauss property A, that is, $\operatorname{NA}(\mathcal{F}(M),Y)$ is dense in $\mathcal{L}(\mathcal{F}(M),Y)$ for every Banach space Y. Is it true that $\operatorname{LipSNA}(M,Y)$ is dense in $\operatorname{Lip}_0(M,Y)$ for every Banach space Y?

The next problem is related to the extremal structure of Lipschitz-free spaces, but it is directly connected to strongly norm-attaining Lipschitz maps:

Problem (Problem 7.3)**.** Let M be a metric space so that $\operatorname{LipSNA}(M,\mathbb{R})$ is dense in $\operatorname{Lip}_0(M,\mathbb{R})$. Is $B_{\mathcal{F}(M)}$ the closed convex hull of its strongly exposed points?

Theorem 3.20 shows that strong density fails for C^2 curves. We wonder whether the same remains true for more general curves.

Problem (Problem 7.4)**.** Let $I \subset \mathbb{R}$ be an interval, let E be a normed space, and let $\alpha \colon I \longrightarrow E$ be a rectifiable curve. Is there a Lipschitz function $f \in \mathrm{Lip}_0(\alpha(I), \mathbb{R})$ that cannot be approximated by strongly norm-attaining Lipschitz functions?

For the next problem, we wonder whether strong density is stable under equivalent metrics. More specifically, we propose the following question:

Problem (Problem 7.5)**.** Let (M, d) be a metric space and let d' be an equivalent metric to d. Assume that $\mathrm{LipSNA}((M, d), \mathbb{R})$ is dense in $\mathrm{Lip}_0((M, d), \mathbb{R})$. Is $\mathrm{LipSNA}((M, d'), \mathbb{R})$ also dense in $\mathrm{Lip}_0((M, d'), \mathbb{R})$?

Finally, motivated by certain results of Bourgain and Huff, we propose the following two open problems.

Problem (Problem 7.6)**.** Let M be a metric space. Suppose $\overline{\mathrm{LipSNA}(M', Y)} = \mathrm{Lip}_0(M', Y)$ for every metric space M' bi-Lipschitz equivalent to M and every Banach space Y. Does $\mathcal{F}(M)$ have the RNP?

Problem (Problem 7.7)**.** Let M be a metric space. Suppose $\overline{\mathrm{LipSNA}(N, Y)} = \mathrm{Lip}_0(N, Y)$ for every closed subset N of M and every Banach space Y. Does $\mathcal{F}(M)$ have the RNP?

Chapter 1

Preliminary results

All along this work, metric spaces will be pointed and complete, and Banach spaces will be over the real scalars. It is well known that norm-attaining linear operators are strongly related to the extremal structure of a Banach space. Also, as we have already commented, there is an intimate connection between norm-attaining linear operators and strongly norm-attaining Lipschitz maps. In view of this, the extremal structure of the Lipschitz-free space over a metric space M is expected to play an important role when studying strongly norm-attaining Lipschitz maps from M to a Banach space Y. Indeed, all along this work we will show that there is a deep connection between them. Understanding the extremal structure of Lipschitz-free space gives deep knowledge about the behavior of strongly norm-attaining Lipschitz maps, and vice versa.

Thankfully, before the preparation of this work the extremal structure of Lipschitz-free spaces has been widely studied. Many of the classical notions of extremal points have been metrically characterized. Also, it has been studied the shape of those elements of the unit sphere of the Lipschitz-free space that are extremal points. Of course, there are plenty of open questions concerning the extremal structure of these Banach spaces, but their study has advanced enough to be able to give nontrivial results about strongly norm-attaining Lipschitz maps. We consider very convenient to start this work with a small summary of the study done in the last years.

Let X be a Banach space. A *slice* of the unit ball B_X is a non-empty intersection of an open half-space with B_X. All slices can be written in the form

$$S(B_X, f, \delta) = \{x \in B_X : f(x) > 1 - \delta\},$$

where $f \in S_{B_{X^*}}$ and $\delta > 0$. The *diameter* of a metric space M is $\operatorname{diam}(M) = \sup\{d(p, q) : p, q \in M\}$. Let us introduce the most studied notions of extremal points in the context of Lipschitz-free spaces.

Definition 1.1. Let X be a Banach space, $x \in S_X$. We say that x is:

(i) An *extreme point of* B_X if whenever $x = ty + (1-t)z$ with $y, z \in S_X$, $t \in [0, 1]$, we have that $x = z = y$.

(ii) An *exposed point of* B_X if there exists a functional attaining its norm only at the point x.

(iii) A *preserved extreme point of* B_X if it is an extreme point of $B_{X^{**}}$.

(iv) A *denting point of* B_X if there exist slices of B_X containing x of arbitrarily small diameter.

(v) A *strongly exposed point of* B_X if there exists a functional $f \in S_{X^*}$ so that $f(x) = \|x\|$ and whenever a sequence $\{x_n\}$ of elements of B_X satisfies that $f(x_n) \longrightarrow 1$, we have that $\{x_n\} \longrightarrow x$ (equivalently, $f(x) = 1$ and there are slices of B_X associated to f of arbitrarily small diameter).

The notations $\operatorname{ext}(B_X)$, $\exp(B_X)$, $\text{pre-ext}(B_X)$, $\operatorname{dent}(B_X)$, and $\text{str-exp}(B_X)$ stand for the sets of extreme points, exposed points, preserved extreme points, denting points, and strongly exposed points of B_X, respectively. We have the following relations between them:

$$\text{str-exp}(B_X) \subset \operatorname{dent}(B_X) \subset \text{pre-ext}(B_X) \subset \operatorname{ext}(B_X) \quad \text{and} \quad \text{str-exp}(B_X) \subset \exp(B_X) \subset \operatorname{ext}(B_X).$$

These inclusions are strict in general.

Let us begin this summary of the extremal structure of Lipschitz-free spaces by talking about preserved extreme points. The first thing that must be said about them is that not every element of the unit sphere of a Lipschitz-free space can be a preserved extreme point. Indeed, N. Weaver proved in 1999 the following important result.

Theorem 1.2 ([65, Corollary 3.44]). *Let M be a metric space. Let μ be a preserved extreme point of $B_{\mathcal{F}(M)}$. Then, μ is a molecule.*

In view of the relations between the different notions of extremal points described above, we obtain the following immediate consequence.

Corollary 1.3. *Let M be a metric space. Let μ be a strongly exposed point of $B_{\mathcal{F}(M)}$. Then, μ is a molecule.*

Consider M to be a metric space, Y to be a Banach space, and $f\colon M \longrightarrow Y$ to be a Lipschitz map. Notice that if $\widehat{f}\colon \mathcal{F}(M) \longrightarrow Y$ attains its norm, as a linear operator, at a strongly exposed point $\mu \in B_{\mathcal{F}(M)}$, by Corollary 1.3 we get that there are $p, q \in M$ with $p \neq q$ such that $\mu = m_{p,q}$. Consequently, $\widehat{f}$ attains its norm at $m_{p,q}$ or, equivalently, f attains its Lipschitz norm at the pair (p, q). This simple observation turns out to be essential for the study of strongly norm-attaining Lipschitz maps and we will make use of it all along this work.

Given a metric space M, it is clear that the Lipschitz-free space over M is a Banach space which is completely determined by M. For this reason, any geometrical aspect of $\mathcal{F}(M)$ should be able to be expressed as some metric aspect of M. Very recently it has been metrically characterized when a molecule of $\mathcal{F}(M)$ is a preserved extreme point of its unit ball.

Theorem 1.4 ([9, Theorem 4.1]). *Let M be a metric space and $p, q \in M$ be distinct points. Then, the following are equivalent:*

(i) *$m_{p,q}$ is a preserved extreme point of $B_{\mathcal{F}(M)}$.*

(ii) *For every $\varepsilon > 0$ there is $\delta > 0$ such that*

$$d(p, z) + d(z, q) - d(p, q) > \delta \quad \textit{whenever } z \in M \textit{ satisfies} \quad \min\{d(p, z), d(q, z)\} > \varepsilon.$$

As a consequence of this metric characterization, they obtain the following interesting result.

Corollary 1.5 ([9, Theorem 4.2]). *Let M be a compact metric space and $p, q \in M$ be distinct points. Assume that $m_{p,q}$ is an extreme point of $B_{\mathcal{F}(M)}$. Then, $m_{p,q}$ is a preserved extreme point of $B_{\mathcal{F}(M)}$.*

On the other hand, there is an equivalence between preserved extreme points and denting points of $B_{\mathcal{F}(M)}$.

Theorem 1.6 ([36, Theorem 2.4]). *Let M be a metric space. Then, every preserved extreme point of $B_{\mathcal{F}(M)}$ is actually a denting point of $B_{\mathcal{F}(M)}$.*

With respect to strongly exposed points, although there are some particular cases when denting points and strongly exposed points are the same, in general these two notions are distinct in the context of Lipschitz-free spaces. Given a metric space M, Theorem 5.4 in [37] gives a metric characterization of when a molecule of $\mathcal{F}(M)$ is a strongly exposed point of $B_{\mathcal{F}(M)}$ (recall that every strongly exposed point is a molecule by Corollary 1.3).

Theorem 1.7 ([37, Theorem 5.4]). *Let M be a metric space and $p, q \in M$ be distinct points. Then, the following are equivalent:*

(i) *$m_{p,q}$ is a strongly exposed point of $B_{\mathcal{F}(M)}$.*

(ii) *There is $\varepsilon > 0$ such that for every $z \in M \setminus \{p, q\}$ we have that*

$$d(p, z) + d(z, q) - d(p, q) > \varepsilon \min\{d(p, z), d(z, q)\}.$$

Actually, the authors of this result use a metric notion known as property (Z) to characterize the strongly exposed points of $B_{\mathcal{F}(M)}$. Let M be a metric space. A pair of distinct points (p, q) of M is said to satisfy *property (Z)* if for every $\varepsilon > 0$ there is $z \in M \setminus \{p, q\}$ such that

$$d(p, z) + d(z, q) - d(p, q) \leqslant \varepsilon \min\{d(p, z), d(z, q)\}.$$

The metric space M is said to have property (Z) when every pair of distinct points of M has property (Z). Under this notation, Theorem 1.7 states that a molecule $m_{p,q}$ is a strongly exposed point of $B_{\mathcal{F}(M)}$ if, and only if, the pair (p, q) fails property (Z). As a consequence of this, we obtain the following result.

Corollary 1.8. *Let M be a metric space. Then, the following are equivalent:*

(i) M has property (Z).

(ii) $B_{\mathcal{F}(M)}$ has no strongly exposed points.

Property (Z) was introduced in [45] in order to give a metric characterization of what the authors call *local metric spaces* in the compact case. After this paper, some other works appeared studying this property and obtaining interesting results about it. In order to comment this results, let us introduce two well-known notions. On the one hand, a Banach space X is said to have the *Daugavet property* if

$$\| \operatorname{Id} + T\| = 1 + \|T\|$$

for every rank-1 linear operator $T\colon X \longrightarrow X$. On the other hand, a metric space M is said to be *length* if given any two distinct points p, $q \in M$ and $\varepsilon > 0$, there exists a rectifiable curve $\Gamma \subseteq M$ joining p and q of length $d(p, q) + \varepsilon$. These metric spaces can be seen as "almost" geodesic spaces. Indeed, given two distict points p, $q \in M$, even if we are not able to find a curve joining p and q of length $d(p, q)$, we can find curves as close to satisfy it as we want.

It was shown in [45] that if M is a compact metric space, then the space $\operatorname{Lip}_0(M, \mathbb{R})$ has the Daugavet property if and only if M has property (Z). After that, this result was generalized by the main theorem of [13] together with Theorem 3.5 in [37], obtaining the following interesting result.

Theorem 1.9 ([13, Theorem 1.5])**.** *Let M be a metric space. The following are equivalent:*

(i) M is a length space.

(ii) M has property (Z).

(iii) $\operatorname{Lip}_0(M, \mathbb{R})$ *has the Daugavet property.*

(iv) $\mathcal{F}(M)$ has the Daugavet property.

Notice that from Theorem 1.4 and the definition of length metric space easily follows that if M is a length metric space, then $B_{\mathcal{F}(M)}$ has no preserved extreme points. In view of Corollary 1.8 and Theorem 1.9 it is obtained the following curious result.

Corollary 1.10 ([13, Theorem 1.5])**.** *Let M be a metric space. The following are equivalent:*

(i) $B_{\mathcal{F}(M)}$ has no preserved extreme points.

(ii) $B_{\mathcal{F}(M)}$ has no strongly exposed points.

There is no analogous result for extreme points. Indeed, Example 2.4 in [45] presents a metric space for which every molecule is an extreme point of $B_{\mathcal{F}(M)}$, but none of them are preserved extreme points.

In view of all these results, one can see that preserved extreme points, denting points and strongly exposed points of the unit ball of a Lipschitz-free space are very well described in terms of the underlying metric space. Let us give metric characterizations of when a molecule is an extreme point or a exposed point of the unit ball of Lipschitz-free spaces. Let M be a metric space and p, $q \in M$ be distinct points.

The *metric segment* $[p,q]$ is defined to be the set of all points that metrically lie between p and q, that is,

$$[p,q] = \{z \in M \colon d(p,z) + d(z,q) = d(p,q)\}.$$

In other words, $[p,q]$ is the set of points $z \in M$ for which the triangle inequality turns out to be an equality. Under this notation, the next result follows from Theorem 1.1 in [10], that generalizes one of the consequences of Theorem 4.2 in [9] and [62, Theorem 1].

Theorem 1.11 ([10, Theorem 1.1], [62, Theorem 1])**.** *Let M be a metric space and $p,\, q \in M$ be distinct points. Then, the following are equivalent:*

(i) $m_{p,q}$ *is an extreme point of* $B_{\mathcal{F}(M)}$.

(ii) $m_{p,q}$ *is an exposed point of* $B_{\mathcal{F}(M)}$.

(iii) $[p,q] = \{p,q\}$.

Let us now introduce some metric notions related to the extremal structure of Lipschitz-free spaces that we are going to use all along this work. A metric space M is said to be *concave* if every molecule of $\mathcal{F}(M)$ is a preserved extreme point of $B_{\mathcal{F}(M)}$. In view of Theorems 1.5 and 1.11, this property can be characterized in the compact case as following: a compact metric space M is concave if and only if

$$d(x,y) < d(x,z) + d(z,y)$$

for all distinct points $x,y,z \in M$. Indeed, this result is extended to the boundedly compact case in [65, Proposition 3.34] (recall that a metric space M is said to be *boundedly compact* if every bounded closed subset of M is compact). A strengthening of this concept is provided when we require all molecules to be strongly exposed points of the unit ball of $\mathcal{F}(M)$. By using the characterization given in Theorem 1.7, this property can be written in terms of the metric space. Moreover, we may also introduce a uniform version of it, that will be helpful in Chapter 2. We need some more notation. Given $x,y,z \in M$, the *Gromov product* of x and y at z is defined in [19, p. 410] as the following quantity:

$$(x,y)_z := \frac{1}{2}\big(d(x,z) + d(z,y) - d(x,y)\big) \geqslant 0.$$

It corresponds to the distance of z to the unique closest point b on the unique geodesic between x and y in any $\mathbb{R}$-tree into which $\{x,y,z\}$ can be isometrically embedded (such a tree always exists). We send the reader to the commentary before Theorem 3.6 for more details on this. Notice that

$$(x,z)_y + (y,z)_x = d(x,y) \quad \text{and that} \quad (x,y)_z \leqslant d(x,z),$$

facts which we will use without further comment.

Definition 1.12. Let M be a metric space.

(i) We say that M is *Gromov concave* if for every $x,y \in M$, $x \neq y$, there is $\varepsilon_{x,y} > 0$ such that

$$(x,y)_z > \varepsilon_{x,y} \min\{d(x,z), d(y,z)\}$$

for every $z \in M \setminus \{x,y\}$.

(ii) Let $A \subseteq \mathrm{Mol}(M)$. We say that A is *uniformly Gromov rotund* if there is $\varepsilon_0 > 0$ such that

$$(x,y)_z > \varepsilon_0 \min\{d(x,z), d(y,z)\}$$

for every distinct $x,y,z \in M$ such that $m_{x,y} \in A$.

(iii) We say that M is *uniformly Gromov concave* when $\mathrm{Mol}(M)$ is uniformly Gromov rotund.

As we have said, by Theorem 1.7, M is Gromov concave if and only if every molecule is a strongly exposed point of the unit ball of $\mathcal{F}(M)$. In the next chapter we will show that the notion of uniformly Gromov concave can be also characterized in terms of extremal points of $B_{\mathcal{F}(M)}$.

Moving away from the extremal structure of Lipschitz-free spaces, there are some other results that we are going to need. Let M be a metric space, Y be a Banach space, and $f\colon M \longrightarrow Y$ be a Lipschitz map. Recall that f attains its Lipschitz constant if and only if its associated linear operator $\widehat{f}$ attains its norm (in the classical sense) at some molecule of $\mathcal{F}(M)$. In view of these, it is clear that molecules are going to play an important role in this work. That is why we need to know some basic properties about these points.

Proposition 1.13 ([36, Proposition 2.9])**.** *Let M be a metric space. Then, $\overline{\mathrm{Mol}(M)}^{\omega} \subseteq \mathrm{Mol}(M) \cup \{0\}$. In particular,* $\mathrm{Mol}(M)$ *is norm-closed.*

The second result that we need is a technical lemma that appears in [35] and provides a useful estimate of the norm of the difference of two molecules.

Lemma 1.14 ([35, Lemma 4.13])**.** *Let M be a metric space and $x, y, u, v \in M$, with $x \neq y$ and $u \neq v$. Then,*

$$\|m_{x,y} - m_{u,v}\| \leqslant 2\frac{d(x,u) + d(y,v)}{\max\{d(x,y), d(u,v)\}}.$$

If, moreover, $\|m_{x,y} - m_{u,v}\| < 1$, then

$$\frac{\max\{d(x,u), d(y,v)\}}{\min\{d(x,y), d(u,v)\}} \leqslant \|m_{x,y} - m_{u,v}\|.$$

Finally, we will also need the following result coming from [52], which was not included in the final version of that paper [53]. This result gives a tool to construct new metric spaces by gluing metric spaces already constructed. It can also be used to decompose some metric space into smaller "pieces", so we can study its Lipschitz-free space by studying the Lipschitz-free space over each piece. Given a family $\{(X_\gamma, \|\cdot\|_\gamma) \colon \gamma \in \Gamma\}$ of Banach spaces, we will denote by $\left[\bigoplus_{\gamma\in\Gamma} X_\gamma\right]_{\ell_1}$ the ℓ_1-*sum* of the family, that is, the product space $X = \prod_{\gamma\in\Gamma} X_\lambda$ endowed with the norm $\|x\| = \sum_{\gamma\in\Gamma} \|x(\gamma)\|_\gamma$ for every $x \in X$.

Proposition 1.15 ([52, Proposition 5.1])**.** *Suppose that $M = \bigcup_{\gamma\in\Gamma} M_\gamma$ is a metric space with metric d, and suppose that there exists $0 \in M$ satisfying*

(i) $M_\gamma \cap M_\eta = \{0\}$ if $\gamma \neq \eta$, and

(ii) there exists $C \geqslant 1$ such that $d(x, 0) + d(y, 0) \leqslant Cd(x, y)$ for all $\gamma \neq \eta$, $x \in M_\gamma$ and $y \in M_\eta$.

Then, $\mathcal{F}(M)$ is isomorphic to $\left[\bigoplus_{\gamma\in\Gamma} \mathcal{F}(M_\gamma)\right]_{\ell_1}$. If $C = 1$ such an isomorphism can be chosen to be isometric.

The previous result motivates the following definition, which corresponds to the case when $C = 1$. We follow the terminology introduced by Definition 1.13 in [65].

Definition 1.16. Given a family of pointed metric spaces $\{(M_i, d_i)\}_{i\in I}$, the (metric) *sum* of the family is the disjoint union of all M_i's, identifying the base points, endowed with the following metric d: $d(x, y) = d_i(x, y)$ if both $x, y \in M_i$, and $d(x, y) = d_i(x, 0) + d_j(0, y)$ if $x \in M_i$, $y \in M_j$ and $i \neq j$. We write $\coprod_{i\in I} M_i$ to denote the sum of the family of metric spaces.

Chapter 2

Strong density. Positive results

Our goal in this work is to study the problem of finding for which metric spaces M and Banach spaces Y, the set $\mathrm{LipSNA}(M,Y)$ of those Lipschitz maps that strongly attain their norm is dense in $\mathrm{Lip}_0(M,Y)$. All along this chapter we will give positive results concerning this problem. More precisely, we will present sufficient conditions over $\mathcal{F}(M)$ or Y that guarantee that $\mathrm{LipSNA}(M,Y)$ is dense in $\mathrm{Lip}_0(M,Y)$. Moreover, we will try to reformulate these conditions in term of the metric space, obtaining criteria that make easier to show that $\mathcal{F}(M)$ satisfies any of these properties. We will also widely analyze the relationship between all these sufficient conditions. Finally, we will present some applications of the study of the strong density that allow us to obtain information about the geometric structure of Lipschitz-free spaces, as well as the metric structure of their underlying metric spaces.

The results obtained in this chapter come from the papers [20], [23], and [24]. They are collaborative works with Bernardo Cascales, Luis Carlos García Lirola, Miguel Martín, and Abraham Rueda Zoca.

2.1 Previous results

Let us present some of the known results about strongly norm-attaining Lipschitz maps.

The first positive examples are due to G. Godefroy and appeared in [41, §5]. They deal with the *little Lipschitz space over* M, denoted by $\mathrm{lip}_0(M,\mathbb{R})$, consisting of all Lipschitz functions f from M to $\mathbb{R}$ with $f(0)=0$ satisfying that for any $\varepsilon>0$ there exists $\delta>0$ such that if $d(p,q)<\delta$, then $|f(p)-f(q)|\leqslant \varepsilon d(p,q)$. We say that $\mathrm{lip}_0(M,\mathbb{R})$ *uniformly separates points of* M when there exists $a>0$ such that for all $(p,q)\in M\times M$ there exists $f\in \mathrm{lip}_0(M,\mathbb{R})$ such that $f(p)-f(q)=d(p,q)$ and $\|f\|_L\leqslant a$. We have the following result.

Proposition 2.1 ([41, Proposition 5.4])**.** *Let M be a compact metric space such that* $\mathrm{lip}_0(M,\mathbb{R})$ *uniformly separates points of M. Let Y be a Banach space. Let $f\colon M\longrightarrow Y$ be a Lipschitz map. Then, the following assertions are equivalent:*

(i) The map f strongly attains its norm.

(ii) The operator $\widehat{f}\colon \mathcal{F}(M)\longrightarrow Y$ attains its norm.

As a consequence of this, the following result is obtained.

Corollary 2.2 ([41, Proposition 5.5])**.** *Let M be a compact metric space such that* $\mathrm{lip}_0(M,\mathbb{R})$ *uniformly separates points of M. Let Y be a finite-dimensional normed space. Then,* $\mathrm{LipSNA}(M,Y)$ *is norm-dense in* $\mathrm{Lip}_0(M,Y)$.

Examples of metric spaces for which the uniformly separating condition is satisfied are: the usual middle-thirds Cantor set, metric spaces which are compact and countable, and compact Hölder metric spaces (see [41, §5] for more details).

After these first examples, a more general result dealing with the Radon-Nikodým property was given in [36]. Indeed, Theorem 4.38 in [65] states that if M is a boundedly compact metric space such that $\mathrm{lip}_0(M,\mathbb{R})$ uniformly separates points of M, then $\mathrm{lip}_0(M,\mathbb{R})^*$ is isometrically isomorphic to $\mathcal{F}(M)$. In particular, $\mathcal{F}(M)$ is a dual Banach space. Additionally, if M is separable, then $\mathcal{F}(M)$ is a separable Banach space, which is a dual space, so $\mathcal{F}(M)$ has the Radon-Nikodým property. Therefore, the following result from [36] generalizes Corollary 2.2.

Proposition 2.3 ([36, Proposition 7.4]). *Let M be a metric space such that $\mathcal{F}(M)$ has the Radon-Nikodým property. Then,* $\mathrm{LipSNA}(M,Y)$ *is norm-dense in* $\mathrm{Lip}_0(M,Y)$ *for every Banach space Y.*

The next result collects some examples of metric spaces M for which $\mathcal{F}(M)$ has the RNP.

Example 2.4. *Let M be a metric space. The space $\mathcal{F}(M)$ has the RNP in the following cases:*

(i) M is uniformly discrete (i.e. $\inf_{x\neq y} d(x,y) > 0$); [51, Proposition 4.4].

(ii) M is a countable compact metric space (since, in this case, $\mathcal{F}(M)$ is a separable dual Banach space); [31, Theorem 2.1]. Indeed, this result was generalized for countable boundedly compact metric spaces (see [65, Corollary 4.39]).

(iii) M is a boundedly compact Hölder metric space (since, in this case, $\mathcal{F}(M)$ is a separable dual Banach space); [65, Corollary 4.39].

(iv) M is a closed subset of $\mathbb{R}$ with measure 0 (since, in this case, $\mathcal{F}(M)$ is isometric to ℓ_1); [40].

2.2 Classical sufficient conditions for Lindenstrauss property A in the Lipschitz context

As we have already commented in the introduction, there is a connection between the study of the density of the strongly norm-attaining Lipschitz maps and the study of the norm-attaining linear operators. Recall that a Banach space X has *Lindenstrauss property A* when $\overline{\mathrm{NA}(X,Y)} = \mathcal{L}(X,Y)$ for every Banach space Y, where $\mathrm{NA}(X,Y)$ denotes the set of all norm-attaining linear operators that go from X to Y. We also commented that in 1977 it was shown by J. Bourgain that every Banach space with the Radon-Nikodým property also has Lindenstrauss property A (see [18]). Comparing this result with Proposition 2.3, one may wonder whether other classic results concerning norm-attaning linear operators may offer new positive results about strongly norm-attaining Lipschitz maps.

Motivated by this observation, we considered some other conditions implying Lindenstrauss property A that we find in the literature in order to check if indeed they provide new positive results on the density of strongly norm-attaining Lipschitz maps, analogous to Proposition 2.3. That is the case as we will show.

Let us briefly comment on the conditions that we study throughout the section. First, we start by introducing the following uniform notion of strongly exposed point.

Definition 2.5. Let X be a Banach space. A subset $S \subset S_X$ is said to be a *set of uniformly strongly exposed points* if there is a family of functionals $\{h_x\}_{x\in S}$ with $|h_x(x)| = \|h_x\| = 1$ for every $x \in S$ such that, given $\varepsilon > 0$ there is $\delta > 0$ satisfying that

$$\sup_{x\in S} \mathrm{diam}(S(B_X, h_x, \delta)) \leqslant \varepsilon.$$

Having a norming subset of uniformly strongly exposed points is the first condition that we are going to study. Indeed, Lindenstrauss proved in [57, Proposition 1] that if X is a Banach space with a set of uniformly strongly exposed points $S \subset S_X$ such that $B_X = \overline{\mathrm{co}}(S)$, then X has Lindenstrauss property A.

As a particular way in which a Banach space may contain a norming set of uniformly strongly exposed points, property α appears. It was introduced in [64] by W. Schachermayer and its main interest is that "many" Banach spaces (e.g. separable, reflexive, WCG. . .) can be equivalently renormed to have it.

Definition 2.6. A Banach space X is said to have *property* α if there exist a balanced subset $\{x_\lambda\}_{\lambda\in\Lambda}$ of X and a subset $\{x_\lambda^*\}_{\lambda\in\Lambda} \subseteq X^*$ such that

(i) $\|x_\lambda\| = \|x_\lambda^*\| = |x_\lambda^*(x_\lambda)| = 1$ for all $\lambda \in \Lambda$.

(ii) There exists $0 \leqslant \rho < 1$ such that

$$|x_\lambda^*(x_\mu)| \leqslant \rho \quad \forall\, x_\lambda \neq \pm x_\mu.$$

(iii) $\overline{\text{co}}\,(\{x_\lambda\}_{\lambda\in\Lambda}) = B_X$.

Let us note that the usual definition of property α does not require the set $\{x_\lambda\}_{\lambda\in\Lambda}$ to be balanced and it is actually taken "half" of the set; on the other hand, absolutely closed convex hull instead of closed convex hull is required in item (iii). Both definitions are clearly equivalent, but for our purposes, Definition 2.6 is more convenient.

It is immediate (see [64, Fact in p. 202]) that if X has property α witnessed by a set $\Gamma \subset S_X$, then Γ is a set of uniformly strongly exposed points and, by hypothesis, $\overline{\text{co}}(\Gamma) = B_X$. Consequently, any Banach space with property α satisfies Lindenstrauss property A.

Finally, in [27] it is defined a property which, in spite of being weaker than property α, still implies property A.

Definition 2.7. A Banach space X is said to have *property quasi-*α if there exist a balanced subset $\{x_\lambda\}_{\lambda\in\Lambda}$ of X, a subset $\{x_\lambda^*\}_{\lambda\in\Lambda} \subseteq X^*$, and $\rho\colon \Lambda \longrightarrow \mathbb{R}$ such that

a) $\|x_\lambda\| = \|x_\lambda^*\| = \|x_\lambda^*(x_\lambda)\| = 1$ for all $\lambda \in \Lambda$.

b) $|x_\lambda^*(x_\mu)| \leqslant \rho(\mu) < 1$ for all $x_\lambda \neq \pm x_\mu$.

c) For every $e \in \text{ext}\,(B_{X^{**}})$, there exists a subset $A_e \subseteq A$ such that either e or $-e$ belong to $\overline{A_e}^{\omega^*}$ and $r_e = \sup\{\rho(\mu)\colon x_\mu \in A_e\} < 1$.

Again, the definition above is not the one given in [27], but an equivalent one in which the set $\{x_\lambda\}_{\lambda\in\Lambda}$ is balanced. Property quasi-alpha will be the third sufficient condition that we will study in the section.

Diagram 2.1 shows the relations between these conditions for general Banach spaces. None of the implications reverses.

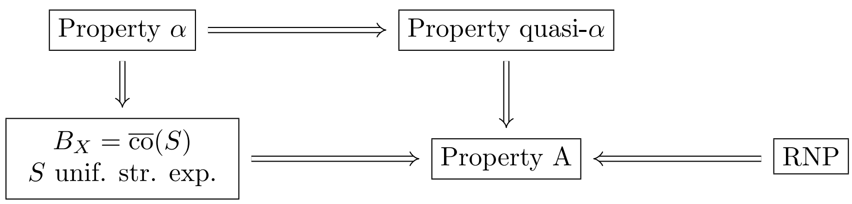

Figure 2.1: Relations between properties implying Lindenstrauss property A in general Banach spaces.

For a more extended background on norm-attaining linear operators, we refer to the survey paper [1].

Our goal now is to study the properties described above in the context of Lipschitz-free spaces. This will allow us to give new positive results on the density of the strongly norm-attaining Lipschitz maps. Moreover, we try to metrically reformulate these new properties in order to obtain criteria that help us to show whether the Lipschitz-free space over some metric space satisfies any of them.

2.2.1 Norming subset of uniformly strongly exposed points

We proceed by studying the property of having a subset of uniformly strongly exposed points first.

Proposition 2.8. *Let M be a metric space and assume that $B_{\mathcal{F}(M)}$ is the closed convex hull of a set of uniformly strongly exposed points. Then* $\mathrm{LipSNA}(M,Y)$ *is norm dense in* $\mathrm{Lip}_0(M,Y)$ *for every Banach space Y.*

Proof. Let S be a set of uniformly strongly exposed points so that $B_{\mathcal{F}(M)} = \overline{\mathrm{co}}(S)$. Then, having a look at the proof of Theorem 1 in [57] we see that, given a Banach space Y, the set

$$\{T \in \mathcal{L}(\mathcal{F}(M), Y) \colon T \text{ attains its norm at a point of } \overline{S}\}$$

is dense in $\mathcal{L}(\mathcal{F}(M), Y)$. By Corollary 1.2, we get $S \subseteq \mathrm{Mol}(M)$. Finally, Proposition 1.13 implies that $\overline{S} \subseteq \mathrm{Mol}(M)$, which finishes the proof. □

We continue with a characterization, inspired by [37, Theorem 5.4], of the existence of a norming subset of uniformly strongly exposed points in the unit ball of a Lipschitz free space, which depends only on the metric space M.

Having a look at Theorem 1.7, where strongly exposed points of $B_{\mathcal{F}(M)}$ were metrically characterized, we see that being uniformly Gromov concave corresponds to a uniform notion of the statement appearing in that result. In view of this, it is not surprising to obtain the following characterization:

Proposition 2.9. *Let M be a metric space and let A be a set of molecules in $\mathcal{F}(M)$. Then, the following statements are equivalent:*

(i) A is a set of uniformly strongly exposed points.

(ii) A is uniformly Gromov rotund.

Before giving the proof of Proposition 2.9, we need two technical results.

Lemma 2.10. *Let M be a metric space, let $A = \{m_{x,y}\}_{(x,y)\in\Lambda}$ be a family of molecules in $\mathcal{F}(M)$. Suppose that there is $\varepsilon_0 > 0$ such that*

$$(x,y)_z > \varepsilon_0 \min\{d(x,z), d(y,z)\}$$

whenever $m_{x,y} \in A$ and $z \in M \setminus \{x,y\}$. Then, there exists a family $B = \{h_{x,y}\}_{(x,y)\in\Lambda}$ in $S_{\mathrm{Lip}_0(M,\mathbb{R})}$ such that

(a) $\widehat{h}_{x,y}(m_{x,y}) = 1$ for every $(x,y) \in \Lambda$, and

(b) for every $\varepsilon > 0$ there is $\delta = \delta(\varepsilon, \varepsilon_0) > 0$ such that

$$\widehat{h}_{x,y}(m_{u,v}) > 1 - \delta \quad \textit{implies} \quad \|m_{x,y} - m_{u,v}\| < \varepsilon \tag{2.1}$$

for every $(x,y) \in \Lambda$ and every $u, v \in M$, $u \neq v$.

Proof. Fix $\varepsilon_1 > 0$ with $\frac{\varepsilon_1}{1-\varepsilon_1} < \frac{\varepsilon_0}{4}$. For $x, y \in M$ such that $m_{x,y}$ belongs to A, consider the Lipschitz function $g_{x,y}$ defined in [45, Proposition 2.8], namely

$$g_{x,y}(z) := \begin{cases} \max\left\{\frac{d(x,y)}{2} - (1-\varepsilon_1)d(z,x), 0\right\} & \text{if } d(z,y) \geqslant d(z,x), \\ & d(z,y) + (1-2\varepsilon_1)d(z,x) \geqslant d(x,y), \\ -\max\left\{\frac{d(x,y)}{2} - (1-\varepsilon_1)d(z,y), 0\right\} & \text{if } d(z,x) \geqslant d(z,y), \\ & d(z,x) + (1-2\varepsilon_1)d(z,y) \geqslant d(x,y). \end{cases}$$

It is well defined and satisfies that $\|g_{x,y}\|_L = \widehat{g}_{x,y}(m_{x,y}) = 1$, and

$$\widehat{g}_{x,y}(m_{u,v}) > 1 - \varepsilon_1 \quad \text{implies} \quad \max\{d(x,u), d(y,v)\} < \frac{d(x,y)}{4} \tag{2.2}$$

for any $u, v \in M$, $u \neq v$ (see the proof of Proposition 2.8 in [45]). Consider also the function defined by

$$f_{x,y}(t) := \frac{d(x,y)}{2} \frac{d(t,y) - d(t,x)}{d(t,y) + d(t,x)}$$

for every $t \in M$, and take $h_{x,y} = \frac{1}{2}(g_{x,y} + f_{x,y})$. Now, one can check that the family $B = \{h_{x,y}\}_{(x,y)\in\Lambda}$ does the work following word-by-word the proof of [37, Theorem 5.4]. □

We also need the following result.

Lemma 2.11. *Let M be a metric space. Let $x, y \in M$, $x \neq y$, and let $f \in \mathrm{Lip}_0(M, \mathbb{R})$ be such that $\|f\|_L = 1$ and $\widehat{f}(m_{x,y}) = 1$. Then, for every $z \in M \setminus \{x, y\}$ we have that*

$$\widehat{f}(m_{x,z}) \geqslant 1 - 2\frac{(x,y)_z}{d(x,z)} \quad \text{and} \quad \widehat{f}(m_{z,y}) \geqslant 1 - 2\frac{(x,y)_z}{d(y,z)}.$$

Proof. Note that

$$1 = \widehat{f}(m_{x,y}) = \widehat{f}\left(\frac{d(x,z)}{d(x,y)}m_{x,z} + \frac{d(z,y)}{d(x,y)}m_{z,y}\right) = \frac{d(x,z)}{d(x,y)}\widehat{f}(m_{x,z}) + \frac{d(z,y)}{d(x,y)}\widehat{f}(m_{z,y}).$$

Thus,

$$\begin{aligned} d(x,z) + d(z,y) - 2(x,y)_z = d(x,y) &= d(x,z)\widehat{f}(m_{x,z}) + d(z,y)\widehat{f}(m_{z,y}) \\ &\leqslant d(x,z)\widehat{f}(m_{x,z}) + d(z,y) \end{aligned}$$

and the conclusion follows. □

We can now present the proof of the metric characterization of when a set of molecules in $\mathcal{F}(M)$ is uniformly strongly exposed.

Proof of Proposition 2.9. (i)⇒(ii). Let $\{h_{x,y}\}_{m_{x,y}\in A} \subset S_{\mathrm{Lip}_0(M,\mathbb{R})}$ be a family which uniformly strongly exposes the family A. Take $\delta > 0$ such that

$$\sup_{m_{x,y}\in A} \mathrm{diam}(S(B_{\mathcal{F}(M)}, h_{x,y}, \delta)) < \frac{1}{2}.$$

Assume that A is not uniformly Gromov rotund. Then, there are $x, y \in M$, $x \neq y$, and $z \in M \setminus \{x, y\}$ such that

$$(x,y)_z < \frac{\delta}{2} \min\{d(x,z), d(y,z)\}.$$

By interchanging the roles of x and y if needed, we may assume that $d(x,z) \leqslant d(y,z)$ and so, $d(y,z) \geqslant \frac{1}{2}d(x,y)$. Now, Lemma 2.11 implies that

$$\widehat{h}_{x,y}(m_{x,z}) \geqslant 1 - 2\frac{(x,y)_z}{d(x,z)} > 1 - \delta.$$

From this and Lemma 1.14, it follows that

$$\frac{1}{2} \leqslant \frac{d(y,z)}{d(x,y)} \leqslant \|m_{x,y} - m_{x,z}\| < \frac{1}{2}$$

which is a contradiction.

(ii)⇒(i). By hypothesis, there is $\varepsilon_0 > 0$ such that

$$d(x,z) + d(z,y) > d(x,y) + \varepsilon_0 \min\{d(x,z), d(z,y)\}$$

whenever $m_{x,y} \in A$ and $z \in M \setminus \{x, y\}$. Let $B = \{h_{x,y}\}_{(x,y)\in\Lambda}$ be the set provided by Lemma 2.10. We claim that B uniformly strongly exposes A. Indeed, given $\varepsilon > 0$, take $0 < \delta < \varepsilon$ such that

$$\widehat{h}_{x,y}(m_{u,v}) > 1 - \delta \quad \text{implies} \quad \|m_{x,y} - m_{u,v}\| < \varepsilon$$

for every $(x, y) \in \Lambda$ and every $u, v \in M$, $u \neq v$. Thus,

$$\operatorname{diam}\big(S(B_{\mathcal{F}(M)}, \widehat{h}_{x,y}, \delta) \cap \operatorname{Mol}(M)\big) \leqslant 2\varepsilon.$$

Finally, note that

$$\operatorname{diam}(S(B_{\mathcal{F}(M)}, \widehat{h}_{x,y}, \delta^2) \leqslant 2 \operatorname{diam}\Big(S(B_{\mathcal{F}(M)}, \widehat{h}_{x,y}, \delta) \cap \operatorname{Mol}(M)\Big) + 4\delta \leqslant 8\varepsilon,$$

see e.g. Lemma 2.7 in [36]. □

As an immediate consequence of Propositions 2.8 and 2.9, we obtain the following corollary.

Corollary 2.12. *Let M be a metric space. If there exists a uniformly Gromov rotund subset $A \subseteq \operatorname{Mol}(M)$ such that $B_{\mathcal{F}(M)}$ is the closed convex hull of A, then* $\operatorname{LipSNA}(M, Y)$ *is dense in* $\operatorname{Lip}_0(M, Y)$ *for every Banach space Y.*

Let M be a metric space and p, $q \in M$ be distinct points. Recall that the molecule $m_{p,q}$ is a strongly exposed point of $B_{\mathcal{F}(M)}$ if and only if the pair (p, q) fails property (Z). Then, Corollary 2.12 is telling us that if we find a collection of molecules $\{m_{p_i,q_i}\}_{i\in I}$ such that $B_{\mathcal{F}(M)} = \overline{\operatorname{co}}(\{m_{p_i,q_i}\})$ and the pairs of points $\{(p_i, q_i)\}_{i\in I}$ fail property (Z) in a uniform way, then $\operatorname{LipSNA}(M, Y)$ will be dense in $\operatorname{Lip}_0(M, Y)$ for every Banach space Y. In view of Theorem 1.9, Corollary 2.12 shows that the failure of the Daugavet property in a very strong sense implies strong density for every Banach space Y. In contrast to this, in the next chapter of this work, we will see that if $\mathcal{F}(M)$ has the Daugavet property, then $\operatorname{LipSNA}(M, Y)$ is not dense in $\operatorname{Lip}_0(M, Y)$ for any (nontrivial) Banach space.

2.2.2 Property α

As we commented before, property α is just a particular way in which a Banach space may have a norming subset of uniformly strongly exposed points. Indeed, if a Banach space X satisfies Definition 2.6 with norming set $\Gamma = \{x_\lambda\}_{\lambda\in\Lambda}$, then Γ is a set of uniformly strongly exposed points. Therefore, as a consequence of Proposition 2.8 we obtain the following result.

Corollary 2.13. *Let M be a metric space such that $\mathcal{F}(M)$ has property α. Then,* $\operatorname{LipSNA}(M, Y)$ *is dense in* $\operatorname{Lip}_0(M, Y)$ *for every Banach space Y.*

Given a metric space M for which $\mathcal{F}(M)$ has property α, if $\Gamma \subset S_{\mathcal{F}(M)}$ is the norming set of Definition 2.6, then we know that Γ is made up of molecules. However, in the context of property α we can say something more. J. P. Moreno proved in [60, Proposition 3.6] that if a Banach space X has property α witnessed by $\Gamma \subset S_X$, then $\Gamma = \operatorname{dent}(B_X) = \operatorname{str-exp}(B_X)$. Indeed, if $x \in S_X$ is a denting point, then the slices of B_X containing x are a neighborhood basis of x for the norm topology in B_X. Since $\overline{\operatorname{co}}(\Gamma) = B_X$, we have that every slice of B_X intersects Γ. It follows that $x \in \overline{\Gamma}$. Finally, if X has property α, then the set Γ is obviously uniformly discrete, hence closed. Thus,

$$\operatorname{dent}(B_X) \subset \Gamma \subset \operatorname{str-exp}(B_X) \subset \operatorname{dent}(B_X).$$

From this and the fact that every preserved extreme point of $B_{\mathcal{F}(M)}$ is a denting point by Theorem 1.6, we get the following result.

Proposition 2.14. *Let M be a metric space and assume that $\mathcal{F}(M)$ has property α witnessed by $\Gamma \subset S_{\mathcal{F}(M)}$. Then,*

$$\Gamma = \operatorname{pre-ext}\left(B_{\mathcal{F}(M)}\right) = \operatorname{str-exp}\left(B_{\mathcal{F}(M)}\right).$$

In the case when the Banach space is a Lipschitz-free space, we want to reformulate property α in terms of the underlying metric space. In order to do so, we need the following elementary metric characterization of when a subset of molecules is uniformly discrete.

Lemma 2.15. *Let M be a metric space and consider $A \subset \mathrm{Mol}(M)$. Then, A is uniformly discrete if and only if there exists $\delta > 0$ such that*

$$d(x,u) + d(v,y) \geqslant \delta\, d(x,y) \tag{2.3}$$

whenever $m_{x,y}$ and $m_{u,v}$ are distinct elements of A.

Proof. If A is uniformly discrete, then there is $\delta > 0$ such that

$$2\delta \leqslant \|m_{x,y} - m_{u,v}\| \leqslant 2\frac{d(x,u) + d(y,v)}{d(x,y)},$$

where the last inequality follows from Lemma 1.14. Conversely, assume that the inequality (2.3) holds and pick $m_{x,y}, m_{u,v} \in A$ with $m_{x,y} \neq m_{u,v}$. If one has that $\|m_{x,y} - m_{u,v}\| < 1$ then, again by Lemma 1.14, we get that

$$\|m_{x,y} - m_{u,v}\| \geqslant \frac{\max\{d(x,u), d(u,v)\}}{d(x,y)} \geqslant \frac{1}{2}\frac{d(x,u) + d(u,y)}{d(x,y)} \geqslant \frac{\delta}{2}.$$

Thus, $\|m_{x,y} - m_{u,v}\| \geqslant \min\{1, \delta/2\}$. □

The following proposition characterizes Lipschitz-free spaces with property α in terms of the existence of a norming subset of molecules satisfying certain metrical conditions.

Proposition 2.16. *Let M be a metric space. The following are equivalent:*

(i) $\mathcal{F}(M)$ has property α.

(ii) There exists $\Lambda \subset \{(p,q) \in M \times M \colon p \neq q\}$ such that, writing $A = \{m_{x,y} \colon (x,y) \in \Lambda\} \subset \mathrm{Mol}(M)$, one has that:

- *there exists $\delta > 0$ such that $d(x,u) + d(y,v) \geqslant \delta d(x,y)$ for all $(x,y), (u,v) \in \Lambda$ with $(x,y) \neq (u,v)$ (equivalently, A is uniformly discrete);*
- *there is $\varepsilon > 0$ such that*
$$(x,y)_z > \varepsilon \min\{d(x,z), d(y,z)\}$$
whenever $(x,y) \in \Lambda$ and $z \in M \setminus \{x,y\}$ (equivalently, A is uniformly Gromov concave);
- *$\|f\|_L = \sup\left\{\frac{f(x)-f(y)}{d(x,y)} \colon (x,y) \in \Lambda\right\}$ for every $f \in \mathrm{Lip}_0(M,\mathbb{R})$ (equivalently, $B_{\mathcal{F}(M)} = \overline{\mathrm{co}}(A)$).*

Moreover, in such a case, the set A coincides with the whole set of strongly exposed points of $B_{\mathcal{F}(M)}$.

Proof. (i)⇒(ii). Let $A \subset S_{\mathcal{F}(M)}$ witnessing that $\mathcal{F}(M)$ has property α. Then $B_{\mathcal{F}(M)} = \overline{\mathrm{co}}(A)$. Moreover, it is clear that A is uniformly discrete and it is known that it is uniformly strongly exposed [64, Fact in p. 202], so Proposition 2.9 and Lemma 2.15 give the result.

(ii)⇒(i). Let $A = \{m_{x,y}\}_{(x,y)\in\Lambda}$ be a set of molecules satisfying the properties in the statement. Let $B = \{h_{x,y}\}_{(x,y)} \subset S_{\mathrm{Lip}_0(M,\mathbb{R})}$ be the family provided by Lemma 2.10. By Lemma 2.15,

$$\varepsilon = \inf\{\|m_{x,y} - m_{u,v}\| \colon m_{x,y}, m_{u,v} \in A,\, m_{x,y} \neq m_{u,v}\} > 0.$$

Take $\delta > 0$ such that (2.1) in Lemma 2.10 holds for that ε. Then,

$$\left|\widehat{h}_{x,y}(m_{u,v})\right| \leqslant 1 - \delta$$

whenever $m_{x,y}, m_{u,v} \in A$ and $m_{x,y} \neq \pm m_{u,v}$. Thus, $\mathcal{F}(M)$ has property α.

The last assertion follows from Proposition 2.14. □

We can provide an easier characterization in the bounded and uniformly discrete case.

Proposition 2.17. *Let M be a bounded and uniformly discrete metric space. The following are equivalent:*

(i) $\mathcal{F}(M)$ *has property* α.

(ii) The set $\text{str-exp}\left(B_{\mathcal{F}(M)}\right)$ *consists of uniformly strongly exposed points (equivalently, it is uniformly Gromov rotund).*

(iii) There is $\varepsilon > 0$ *such that for every* $x, y \in M$ *with* $x \neq y$,

$$\text{either} \quad \inf_{z\in M\setminus\{x,y\}} (x,y)_z = 0 \quad \text{or} \quad \inf_{z\in M\setminus\{x,y\}} (x,y)_z \geqslant \varepsilon.$$

Proof. Denote $D = \sup\{d(x,y) \colon x \neq y\} < \infty$ and $\theta = \inf\{d(x,y) \colon x \neq y\} > 0$.

(i)⇒(ii) follows from Propositions 2.14 and 2.16.

Next, assume that (ii) holds. Then, there is $\varepsilon > 0$ such that

$$(x,y)_z \geqslant \varepsilon \min\{d(x,z), d(z,y)\} \geqslant \varepsilon\theta$$

whenever $m_{x,y} \in \text{str-exp}\left(B_{\mathcal{F}(M)}\right)$. So, given $x, y \in M$, $x \neq y$, either $m_{x,y}$ is strongly exposed, and then $\inf_{z\in M\setminus\{x,y\}} (x,y)_z \geqslant \varepsilon\theta$, or $m_{x,y}$ is not strongly exposed, and then

$$\inf_{z\in M\setminus\{x,y\}} (x,y)_z \leqslant D \inf_{z\in M\setminus\{x,y\}} \frac{(x,y)_z}{\min\{d(x,z), d(y,z)\}} = 0.$$

This gives (iii).

Finally, assume that (iii) holds and let $A = \text{str-exp}\left(B_{\mathcal{F}(M)}\right)$. Then, for every $m_{x,y} \in A$ we have that $\inf_{z\in M\setminus\{x,y\}} (x,y)_z > 0$ and so

$$\inf_{z\in M\setminus\{x,y\}} (x,y)_z \geqslant \varepsilon \geqslant \frac{\varepsilon}{D} \min\{d(x,z), d(z,y)\}.$$

That is, A is uniformly Gromov concave. Moreover, $B_{\mathcal{F}(M)} = \overline{\text{co}}(A)$ since $\mathcal{F}(M)$ has the RNP. Finally,

$$d(x,u) + d(v,y) \geqslant \delta d(x,y),$$

for every distinct pairs of points $(x,y), (u,v) \in \{(p,q) \in M \times M \colon p \neq q\}$, where $\delta = 2\theta/D$. By Proposition 2.16, $\mathcal{F}(M)$ has property α, getting (i). □

Proposition 2.17 has very restrictive assumptions since we are considering a bounded and uniformly discrete metric space. However, property α is also a very restrictive property. Indeed, under an extra assumption over the metric space M, we will see that if $\mathcal{F}(M)$ has property α, then M must be bounded and uniformly discrete. For concave metric spaces, we can give an even simpler characterization.

Theorem 2.18. *Let M be a concave metric space. Then the following are equivalent:*

(i) $\mathcal{F}(M)$ *has property* α.

(ii) M is uniformly discrete and bounded, and there is $\varepsilon > 0$ *such that*

$$d(x,z) + d(z,y) - d(x,y) \geqslant \varepsilon$$

whenever x, y, z are distinct points in M.

Proof. Assume first that $\mathcal{F}(M)$ has property α with constant $\rho > 0$. By Proposition 2.14, the set $\Gamma \subset S_{\mathcal{F}(M)}$ witnessing property α coincides with $\text{pre-ext}\left(B_{\mathcal{F}(M)}\right)$, so $\Gamma = \text{Mol}(M)$ as M is concave. Now, take $m_{x,y}, m_{u,y} \in \text{Mol}(M) = \Gamma$ and let $g_{x,y} \in S_{\text{Lip}_0(M,\mathbb{R})}$ be the functional associated to $m_{x,y}$. Then, by Lemma 1.14, we have that

$$2\frac{d(x,u)}{d(x,y)} \geqslant \|_{x,y} - m_{u,y}\| \geqslant |\widehat{g}_{x,y}(m_{x,y} - m_{u,y})| \geqslant 1 - \rho.$$

From here, given $x, u \in M$ we have that

$$(1-\rho)\sup_{y\in M} d(x,y) \leqslant 2d(x,u).$$

So, it follows that M is bounded. Moreover, the following estimate holds:

$$(1-\rho)\operatorname{diam}(M) \leqslant 2(1-\rho)\sup_{y\in M} d(x,y) \leqslant 4d(x,u).$$

Since $x, u \in M$ were arbitrary, we conclude that M is uniformly discrete. Now, Proposition 2.17 provides $\varepsilon > 0$ such that $(x,y)_z \geqslant \varepsilon$ whenever $m_{x,y} \in \text{str-exp}\left(B_{\mathcal{F}(M)}\right)$ and $z \in M \setminus \{x,y\}$. Since every molecule is strongly exposed, the conclusion follows.

Finally, the converse statement follows from Proposition 2.17. □

As we said, the last result illustrates that metric spaces seem to need some discrete behavior in order for their Lipschitz-free spaces to satisfy property α. However, this is not always the case as Theorem 2.26 will show.

Let us present some examples of metric spaces for which its Lipschitz-free space has property α.

Example 2.19. *The space $\mathcal{F}(M)$ has property α in the following cases:*

(i) M is finite.

(ii) M is a compact subset of $\mathbb{R}$ with measure 0.

(iii) There exists a constant $1 \leqslant D < 2$ such that

$$1 \leqslant d(x,y) < D$$

holds for every pair of distinct points $x, y \in M$ (equivalently, up to rescaling, there are constants $C > 0$ and $1 \leqslant D < 2$ such that $C \leqslant d(x,y) < CD$ for all $x, y \in M$, $x \neq y$).

Proof. (i). Given $m_{x,y} \in \text{str-exp}\left(B_{\mathcal{F}(M)}\right)$, consider a strongly exposing functional $g_{x,y} \in S_{\text{Lip}_0(M,\mathbb{R})}$. Take ρ to be the maximum of the set

$$\{|\widehat{g}_{x,y}(m_{u,v})| \colon m_{x,y} \in \text{str-exp}\left(B_{\mathcal{F}(M)}\right), m_{u,v} \in \operatorname{Mol}(M), m_{x,y} \neq \pm m_{u,v}\}.$$

Then, $\rho < 1$ since M is finite. Moreover, $\mathcal{F}(M)$ is finite dimensional and so $B_{\mathcal{F}(M)}$ is the closed convex hull of its strongly exposed points. Thus, $\mathcal{F}(M)$ has property α.

(ii). $\mathcal{F}(M)$ is isometric to ℓ_1 by [40], so it clearly has property α.

(iii). Let $0 < \varepsilon < \frac{2}{D} - 1$. Observe that given $x, y, z \in M$, we get

$$\begin{aligned}\varepsilon \min\{d(x,z), d(y,z)\} &\leqslant \varepsilon D < 2 - D \leqslant d(x,z) + d(y,z) - D \\ &\leqslant d(x,z) + d(y,z) - d(x,y) = 2(x,y)_z.\end{aligned}$$

Consequently, if we define $\Lambda := \{(p,q) \in M \times M \colon p \neq q\}$, then Λ satisfies the condition (ii) in Proposition 2.16, and so $\mathcal{F}(M)$ has property α. □

As an application of Theorem 2.18, we may show that $D = 2$ is not possible in Example 2.19.iii.

Example 2.20. *Let $M = \{0, x_n, y_n \colon n \geqslant 2\} \subseteq c_0$, where $x_n := (2 - \frac{1}{n})e_n$ and $y_n := e_n + (1 + \frac{1}{n})e_1$ for every $n \geqslant 2$. It can be proved routinely that M is concave by using the characterization of the preserved extreme points given in Theorem 1.4. On the other hand, it is clear that the inequality*

$$1 \leqslant d(x,y) < 2$$

holds for every $x, y \in M$ with $x \neq y$. Nevertheless, one has that

$$d(0,y_n) + d(y_n,x_n) - d(0,x_n) = \frac{3}{n}$$

for every $n \geqslant 2$, so $\mathcal{F}(M)$ fails property α by Theorem 2.18.

2.2.3 Property quasi-α

Let X be a Banach space having property quasi-α. In this case, we cannot ensure the existence of a norming subset of uniformly strongly exposed points in B_X. However, we will see that property quasi-alpha is another sufficient condition that guarantees strong density for every Banach space Y. Indeed, if $\Gamma = \{x_\lambda\}_{\lambda\in\Lambda} \subseteq X$ is the subset appearing in Definition 2.7, then $B_X = \overline{\text{co}}(\Gamma)$. Moreover, the same argument that the one used for property α in [64, Fact in p. 202], shows that for every $\lambda \in \Lambda$, $\varepsilon > 0$, and $x \in B_X$, one has that

$$x^*_\lambda(x) > 1 - \varepsilon(1-\rho(\lambda)) \Longrightarrow \|x - x_\lambda\| < 2\varepsilon;$$

so each x_λ is strongly exposed in B_X by x^*_λ. But now, as $\sup_{\lambda\in\Lambda} \rho(\lambda)$ may be equal to one, we do not get that $\{x_\lambda\}_{\lambda\in\Lambda}$ is a set of uniformly strongly exposed points. Nevertheless, the proof of Proposition 2.1 in [27] shows that if $\mathcal{F}(M)$ has property quasi-α then the set

$$\mathcal{A} := \{T \in \mathcal{L}(\mathcal{F}(M), Y) \colon \|T\| = \|T(x_\lambda)\| \text{ for some } \lambda \in \Lambda\}$$

is norm-dense in $\mathcal{L}(\mathcal{F}(M), Y) \cong \text{Lip}_0(M, Y)$. Now, every x_λ is a strongly exposed point of $B_{\mathcal{F}(M)}$, and so, a molecule by Corollary 1.3. Thus, $\mathcal{A} \subseteq \text{LipSNA}(M, Y)$. We have proved the following.

Proposition 2.21. *Let M be a metric space such that $\mathcal{F}(M)$ has property quasi-α. Then,* $\text{LipSNA}(M, Y)$ *is dense in* $\text{Lip}_0(M, Y)$ *for every Banach space Y.*

We also want to give a metric criterion to know for which metric space we can have that its Lipschitz-free space has property quasi-α. An analogous argument to the one given in the proof of Proposition 2.14 shows the following:

Proposition 2.22. *Let M be a metric space such that $\mathcal{F}(M)$ has property quasi-α witnessed by a set $\Gamma \subset S_{\mathcal{F}(M)}$. Then,*

$$\text{pre-ext}\left(B_{\mathcal{F}(M)}\right) \subset \overline{\Gamma}.$$

As a consequence, we obtain the following result in the case when M is concave.

Proposition 2.23. *Let M be a concave metric space. If $\mathcal{F}(M)$ has property quasi-α, then the set of isolated points of M is dense in M.*

Proof. Assume that $\mathcal{F}(M)$ has property quasi-α witnessed by the sets $\Gamma \subset S_{\mathcal{F}(M)}$, $\Gamma^* \subset S_{\text{Lip}_0(M,\mathbb{R})}$, and the function $\rho\colon \Gamma \longrightarrow \mathbb{R}$. Take $m_{x,y} \in \Gamma$ and let $\widehat{g}_{x,y} \in \Gamma^*$ be its associated functional. Then,

$$\|m_{x,y} - m_{u,v}\| \geqslant |\widehat{g}_{x,y}(m_{x,y} - m_{u,v})| \geqslant 1 - \rho(m_{x,y}) \tag{2.4}$$

for every $m_{u,v} \in \Gamma$ with $m_{u,v} \neq m_{x,y}$. By Proposition 2.22,

$$\text{Mol}(M) = \text{pre-ext}\left(B_{\mathcal{F}(M)}\right) \subset \overline{\Gamma}$$

and so (2.4) holds also for every $m_{u,v} \in \text{Mol}(M) \setminus \{m_{x,y}\}$. Thus, by Lemma 1.14, we have that

$$2\frac{d(x,u)}{d(x,y)} \geqslant \|m_{x,y} - m_{u,y}\| \geqslant 1 - \rho(m_{x,y})$$

whenever $m_{x,y} \in \Gamma$ and $u \in M \setminus \{x, y\}$. In particular, the open ball centered at x of radius $\frac{1-\rho(m_{x,y})}{2} d(x,y)$ is a singleton whenever $m_{x,y} \in \Gamma$. This means that the set

$$A = \{x \in M \colon m_{x,y} \in \Gamma \text{ for some } y \in M \setminus \{x\}\}$$

is made up of isolated points. In order to prove that A is dense in M, consider the Lipschitz function $f(t) = d(t, A) - d(0, A)$ for every $t \in M$, which belongs to $\text{Lip}_0(M, \mathbb{R})$, and consider its canonical linear extension $\widehat{f}$ from $\mathcal{F}(M)$ to $\mathbb{R}$. Then, $\widehat{f}$ vanishes on the norming set Γ, so $\widehat{f} = 0$. Thus, $f = 0$, which yields that $\overline{A} = M$. □

Even when property quasi-α is weaker than property α, they are very similar. Therefore, in the same way as with Proposition 2.18, we still see that in order for $\mathcal{F}(M)$ to satisfy property quasi-α, it seems to be necessary for M to present a discrete behavior.

2.3 Other kind of examples

The first example that we want to present in this section is the class of Hölder metric spaces. Recall that if M is a boundedly compact Hölder metric space, then $\mathcal{F}(M)$ has the Radon-Nikodým property (see Example 2.4). Consequently, applying Proposition 2.3 we obtain that $\mathrm{LipSNA}(M,Y)$ is dense in $\mathrm{Lip}_0(M,Y)$ for every Banach space Y. We want to remove the hypothesis of boundedly compact, however it is not known whether for every Hölder metric space M, the space $\mathcal{F}(M)$ has the RNP, so we cannot apply Proposition 2.3 as before. We will show that the new sufficient conditions that we have introduced in this chapter allow us to get the result.

Proposition 2.24. *Every Hölder metric space is uniformly Gromov concave.*

Proof. Let (M,d) be a metric space and fix $0<\theta<1$. Consider $\varepsilon_0 = 1-2^\theta$ and let us show that (M,d^θ) is uniformly Gromov concave witnessed by $\frac{\varepsilon_0}{2}$. Indeed, given $t>0$ define $f_t\colon [0,t)\longrightarrow \mathbb{R}$ by

$$f_t(s) = \frac{t^\theta - s^\theta}{(t-s)^\theta} \quad \forall\, s\in[0,t).$$

It is easy to see that f_t is strictly decreasing. Besides, for every $t>0$ we have that

$$f_t\left(\frac{t}{2}\right) = \frac{t^\theta - (\frac{t}{2})^\theta}{(\frac{t}{2})^\theta} = 2^\theta - 1.$$

Take x,y,z distinct points of M. We may assume that $d(x,z)\leqslant d(y,z)$. Consequently, we have that $d(y,z)\geqslant \frac{d(x,y)}{2}$. We distinguish two cases:
(i): $d(x,y)>d(y,z)$. In this case, we estimate

$$\begin{aligned}\frac{d(x,z)^\theta + d(y,z)^\theta - d(x,y)^\theta}{d(x,z)^\theta} &= 1 - \frac{d(x,y)^\theta - d(y,z)^\theta}{d(x,z)^\theta}\\ &= 1 - \frac{d(x,y)^\theta - d(y,z)^\theta}{(d(x,y)-d(y,z))^\theta}\frac{(d(x,y)-d(y,z))^\theta}{d(x,z)^\theta}\\ &\geqslant 1 - f_{d(x,y)}(d(y,z))\frac{d(x,z)^\theta}{d(x,z)^\theta} \geqslant 1 - f_{d(x,y)}\left(\frac{d(x,y)}{2}\right) = 2-2^\theta.\end{aligned}$$

(ii): $d(x,y)\leqslant d(y,z)$. Here it is enough to note that

$$\frac{d(x,z)^\theta + d(y,z)^\theta - d(x,y)^\theta}{d(x,z)^\theta} \geqslant \frac{d(x,z)^\theta}{d(x,z)^\theta} = 1. \qquad \square$$

As a consequence of the last proposition, together with Proposition 2.8, we get the desired result.

Corollary 2.25. *Let M be a Hölder metric space. Then,* $\mathrm{LipSNA}(M,Y)$ *is dense in* $\mathrm{Lip}_0(M,Y)$ *for every Banach space Y.*

Indeed, in Chapter 4 we will see that a stronger notion of density holds for this family of metric spaces.

For the next family of examples, let us introduce the notion of non-local Lipschitz map. Let M be a metric space and Y be a Banach space. We say that a Lipschitz map $f\colon M\longrightarrow Y$ is *non-local* if we can find $\varepsilon>0$ such that

$$\sup\{\|\widehat{f}(m_{p,q})\|\colon 0<d(p,q)<\varepsilon\} < \|f\|_L - \varepsilon.$$

The utility to us of these functions can be inferred from the following observation: if M is compact, then every non-local Lipschitz map strongly attains its norm. Indeed, if $f\colon M\longrightarrow Y$ is a non-local Lipschitz map and $\{m_{p_n,q_n}\}$ is a sequence of molecules so that $\{\|\widehat{f}(m_{p_n,q_n})\|\}$ converges to $\|f\|_L$, then we must have that $\inf\{d(p_n,q_n)\colon n\in\mathbb{N}\}>0$. On the other hand, since M is compact we find subsequences $\{p_{n_k}\}$, $\{q_{n_k}\}$ converging to p^*, $q^*\in M$, respectively. Thus, we must have $d(p^*,q^*)>0$, and so $p^*\neq q^*$. Then, it makes sense to consider the molecule m_{p^*,q^*}, and it is clear that $\widehat{f}$ attains its norm at m_{p^*,q^*}, so f

strongly attains its norm. Moreover, Lemma 2.46 will show that, in the compact setting, any non-local Lipschitz map strongly attains its norm at some strongly exposed point of $B_{\mathcal{F}(M)}$.

In the following theorem we construct two metric spaces $\mathfrak{M}_p$, with $p = 1,2$, for which $\mathrm{LipSNA}(\mathfrak{M}_p, Y)$ is dense in $\mathrm{Lip}_0(\mathfrak{M}_p, Y)$ for every Banach space Y, but $\mathcal{F}(\mathfrak{M}_p)$ does not have the Radon-Nikodým property. Moreover, the metric space $\mathfrak{M}_1$ will have property α and, on the other hand, $\mathfrak{M}_2$ will not contain any norming subset of uniformly strongly exposed molecules.

Theorem 2.26. *Consider the subsets of $\mathbb{R}^2$ given by*

$$A_n = \left\{ \left(\frac{k}{2^n}, \frac{1}{2^n} \right) : k \in \{0, \ldots, 2^n\} \right\} \subseteq \mathbb{R}^2 \quad \forall\, n \in \mathbb{N} \cup \{0\},$$

$$M_\infty = \bigcup_{n=0}^{\infty} A_n, \quad M = M_\infty \cup ([0,1] \times \{0\}).$$

Let $\mathfrak{M}_p$ be the set M endowed with the distance inherited from $(\mathbb{R}^2, \|\cdot\|_p)$ for $p = 1, 2$. Then, the following assertions hold:

(i) $\mathrm{LipSNA}(\mathfrak{M}_p, Y)$ *is dense in* $\mathrm{Lip}_0(\mathfrak{M}_p, Y)$ *for every Banach space Y and for $p = 1, 2$.*

(ii) $\mathcal{F}(\mathfrak{M}_1)$ *has property α,*

(iii) The unit sphere of $\mathcal{F}(\mathfrak{M}_2)$ does not contain any subset of uniformly strongly exposed points which generates the ball by closed convex hull.

This theorem answers a question from [41, p. 115], where the author asks whether, given a compact metric space M, the density of $\mathrm{LipSNA}(M, \mathbb{R})$ in $\mathrm{Lip}_0(M, \mathbb{R})$ implies that $\mathrm{lip}_0(M, \mathbb{R})$ strongly separates the points of M (see [41, p. 110] for details). Note that the spaces $\mathfrak{M}_p$ provide a counterexample for $p = 1, 2$, because if $\mathrm{lip}_0(M, \mathbb{R})$ strongly separates the points of M then, in particular, $\mathcal{F}(M)$ is isometric to a dual Banach space. However, $\mathcal{F}(\mathfrak{M}_p)$ is not even isomorphic to any dual Banach space as it is a separable Banach space failing the Radon-Nikodým property. As we will comment later in this section, Theorem 2.26 also answers some questions asked in [20, Section 3.4].

We divide the proof of the theorem into several steps. We start by showing that $\mathrm{LipSNA}(\mathfrak{M}_p, Y)$ is dense in $\mathrm{Lip}_0(\mathfrak{M}_p, Y)$ for every Y. Actually, we will give a more general result.

Proposition 2.27. *Let M_∞, $M \subseteq \mathbb{R}^2$ be the sets defined in Theorem 2.26, and let $|\cdot|$ be a norm in $\mathbb{R}^2$ satisfying that $\|\cdot\|_\infty \leqslant |\cdot| \leqslant \|\cdot\|_1$. Consider now $\mathfrak{M}$ to be the set M endowed with the distance inherited from $(\mathbb{R}^2, |\cdot|)$. Then,* $\mathrm{LipSNA}(\mathfrak{M}, Y)$ *is dense in* $\mathrm{Lip}_0(\mathfrak{M}, Y)$ *for every Banach space Y.*

Proof. Let $f \in \mathrm{Lip}_0(\mathfrak{M}, Y)$ with $\|f\|_L = 1$. Our aim is to approximate f by strongly norm-attaining Lipschitz maps, so we may assume that f does not strongly attain its norm. In order to clarify the proof, let us introduce some notation. For every $n \in \mathbb{N} \cup \{0\}$ and $k \in \{0, \ldots, 2^n\}$, we denote by (n, k) the point $\left(\frac{k}{2^n}, \frac{1}{2^n}\right) \in M_\infty$. Given $n \in \mathbb{N} \cup \{0\}$ and $k \in \{0, \ldots, 2^n - 1\}$, we write $h_{n,k}$ to denote the molecule $m_{(n,k),(n,k+1)}$. We will say that

$$H = \{h_{n,k} \colon n \in \mathbb{N} \cup \{0\}, k \in \{0, \ldots, 2^n - 1\}\}$$

is the set of horizontal molecules. Given $n \in \mathbb{N} \cup \{0\}$ and $k \in \{0, \ldots, 2^n\}$, we write $v_{n,k}$ to denote the molecule $m_{(n,k),(n+1,2k)}$. We will say that

$$V = \{v_{n,k} \colon n \in \mathbb{N} \cup \{0\}, k \in \{0, \ldots, 2^n\}\}$$

is the set of vertical molecules. Finally, we define $\Gamma = \pm H \cup \pm V$.

Fix $\varepsilon > 0$ and let us distinguish two cases: First of all, assume that

$$\rho = \sup\{\|\widehat{f}(m)\| \colon m \in \Gamma\} < 1.$$

Since M_∞ is dense, we may find $u = (n_1, k_1)$, $v = (n_2, k_2) \in M_\infty$ such that

$$\frac{k_1}{2^{n_1}} \neq \frac{k_2}{2^{n_2}}, \quad n_1 \neq n_2 \quad \text{and} \quad \widehat{f}(m_{u,v}) > \frac{1 + \rho\varepsilon}{1 + \varepsilon}.$$

Let us write $n_3 = \max\{n_1, n_2\}$ and consider the set

$$N = \bigcup_{n=0}^{n_3} A_n.$$

Note that if we denote by φ_0 the restriction of f to A_{n_3}, we have $\|\varphi_0\|_L \leqslant \rho < 1$. Then, we may extend this function to a Lipschitz function $\varphi\colon [0,1] \longrightarrow Y$ with $\|\varphi\|_L \leqslant \rho < 1$ (we may define it affine in the gaps). We define $h\colon M_\infty \longrightarrow Y$ by

$$h((n,k)) = \begin{cases} f((n,k)) & \text{if } n \leqslant n_3; \\ \varphi\left(\frac{k}{2^n}\right) & \text{if } n > n_3. \end{cases}$$

By the way we have extended φ_0, it is clear that

$$\sup\left\{\|\widehat{h}(m)\| \colon m \in \Gamma\right\} \leqslant \sup\left\{\|\widehat{f}(m)\| \colon m \in \Gamma\right\} = \rho < 1.$$

Furthermore, $|(s,0)| = s \leqslant |(s,t)|$ and $|(0,t)| = t \leqslant |(s,t)|$ for every s, $t \in \mathbb{R}$. Consequently, if p, q are distinct points of $M_\infty \setminus N$, then we may find a molecule $m \in \Gamma$ such that $\|\widehat{h}(m_{p,q})\| \leqslant \|\widehat{h}(m)\|$. Indeed, given two different points $p = (\frac{k_1}{2^n}, \frac{1}{2^n})$ and $q = (\frac{k_2}{2^m}, \frac{1}{2^m})$ of $M_\infty \setminus N$, we assume with no loss of generality that $n \geqslant m$, define $q' := (\frac{2^{n-m}k_2}{2^n}, \frac{1}{2^n})$. By the assumptions on the norm we get that $|p - q'| \leqslant |p - q|$ and, since $\frac{k_2}{2^m} = \frac{2^{n-m}k_2}{2^n}$, we obtain that

$$\left\|\widehat{h}(m_{p,q})\right\| = \frac{\|\varphi(\frac{k_1}{2^n}) - \varphi(\frac{k_2}{2^m})\|}{|p-q|} \leqslant \frac{\|\varphi(\frac{k_1}{2^n}) - \varphi(\frac{2^{n-m}k_2}{2^n})\|}{|p-q'|} = \|\widehat{h}(m_{p,q'})\|,$$

and notice that $m_{p,q'} \in \mathrm{co}(\Gamma)$. Given $\varepsilon > 0$, let us define

$$g\colon M \longrightarrow Y, \qquad g = f + \varepsilon h. \tag{2.5}$$

It is clear that $\|g - f\|_L \leqslant \varepsilon$, so it will be enough to show that g strongly attains its norm. On the one hand, note that

$$\|\widehat{g}(m_{u,v})\| = \|\widehat{f}(m_{u,v}) + \varepsilon\widehat{h}(m_{u,v})\| > \frac{1+\rho\varepsilon}{1+\varepsilon}(1+\varepsilon) = 1 + \rho\varepsilon.$$

On the other hand, given p, q distinct points of $M_\infty \setminus N$, we have that

$$\|\widehat{g}(m_{p,q})\| \leqslant 1 + \varepsilon\|\widehat{h}(m_{p,q})\| \leqslant 1 + \varepsilon\rho < \|\widehat{g}(m_{u,v})\|.$$

Therefore, g cannot approximate its norm at points of $M_\infty \setminus N$. Since $M_\infty \setminus N$ is dense in $[0,1] \times \{0\}$, this implies that g cannot approximate its norm at arbitrarily close points, that is, g is non-local. Consequently, by compactness of M, we conclude that g must strongly attain its norm.

Secondly, assume that $\sup\{\|\widehat{f}(m)\| \colon m \in \Gamma\} = \|f\|_L$. In this case we need to define two kinds of functionals. By a density argument, it will be enough to define them on M_∞. First of all, we will define functionals associated to the vertical molecules. Fix $n \in \mathbb{N} \cup \{0\}$, $k \in \{0, \ldots, 2^n\}$. Then, we define $f_{n,k}\colon M_\infty \longrightarrow \mathbb{R}$ given by

$$f_{n,k}(p) = \begin{cases} \frac{1}{2^{n+1}} & \text{if } p = (n,k); \\ 0 & \text{if } p \neq (n,k). \end{cases}$$

Note that

$$\widehat{f}_{n,k}(v_{n,k}) = \frac{f_{n,k}((n,k)) - f_{n,k}((n+1,2k))}{|(n,k) - (n+1,2k)|} = \frac{1/2^{n+1}}{1/2^{n+1}} = 1.$$

Furthermore, if $(n',k') \in M$ is such that $m_{(n,k),(n',k')} \in \Gamma$ and $(n',k') \neq (n+1,2k)$, then we have that $|(n,k) - (n',k')| \geqslant \frac{3}{2^{n+2}}$, which implies that

$$\frac{|f_{n,k}((n,k)) - f_{n,k}((n',k'))|}{|(n,k) - (n',k')|} \leqslant \frac{1/2^{n+1}}{3/2^{n+2}} = \frac{2}{3}.$$

Since $f_{n,k}$ is null on the rest of the points, we obtain that $\widehat{f}_{n,k}(m) \leqslant \frac{2}{3}$ holds for every $m \in \Gamma$ with $m \neq \pm v_{n,k}$.

Next, we define the functionals associated to the horizontal molecules: fixed $n \in \mathbb{N} \cup \{0\}$ and fixed $k \in \{0, \dots, 2^n - 1\}$, let us define $\varphi_{n,k} \colon [0,1] \longrightarrow \mathbb{R}$ by

$$\varphi_{n,k}(x) = \begin{cases} \dfrac{3}{2^{n+2}} & \text{if } x \in [0, \frac{k}{2^n}]; \\ \dfrac{2k+3}{2^{n+2}} - \dfrac{x}{2} & \text{if } x \in [\frac{k}{2^n}, \frac{k+1}{2^n}]; \\ \dfrac{1}{2^{n+2}} & \text{if } x \in [\frac{k+1}{2^n}, 1]. \end{cases}$$

It is easy to see that $\varphi_{n,k}$ is a Lipschitz function with $\|\varphi_{n,k}\|_L = \frac{1}{2}$. Now, define $g_{n,k} \colon M_\infty \longrightarrow \mathbb{R}$ as follows

$$g_{n,k}((n',k')) = \begin{cases} \frac{1}{2^n} & \text{if } n' \leqslant n \text{ and } \frac{k'}{2^{n'}} \leqslant \frac{k}{2^n}; \\ 0 & \text{if } n' \leqslant n \text{ and } \frac{k'}{2^{n'}} > \frac{k}{2^n}; \\ \varphi_{n,k}(\frac{k'}{2^{n'}}) & \text{if } n' > n. \end{cases}$$

On the one hand, note that

$$\widehat{g}_{n,k}(h_{n,k}) = \frac{g_{n,k}((n,k)) - g_{n,k}((n,k+1))}{|(n,k) - (n,k+1)|} = \frac{1/2^n}{1/2^n} = 1.$$

On the other hand, let us show that for every $u \in \Gamma$ with $u \neq \pm h_{(n,k)}$ we have

$$|\widehat{g}_{n,k}(u)| \leqslant \frac{1}{2}.$$

For this, take any vertical molecule $v_{n',k'} \in V$. Note that we have $\widehat{g}_{n,k}(v_{n',k'}) = 0$ unless $n' = n$. On the one hand, if $k' \leqslant k$ we get

$$|\widehat{g}_{n,k}(v_{n,k'})| = \frac{|g_{n,k}((n,k')) - g_{n,k}((n+1,2k'))|}{|(n,k') - (n+1,2k')|} = \frac{1/2^n - 3/2^{n+2}}{1/2^{n+1}} = \frac{1}{2}.$$

On the other hand, if $k' \geqslant k+1$ we have

$$|\widehat{g}_{n,k}(v_{n,k'})| = \frac{|g_{n,k}((n,k')) - g_{n,k}((n+1,2k'))|}{|(n,k') - (n+1,2k')|} = \frac{1/2^{n+2}}{1/2^{n+1}} = \frac{1}{2}.$$

Finally, take any horizontal molecule $h_{n',k'}$ such that $(n',k') \neq (n,k)$. If $n' < n$, we have that

$$|\widehat{g}_{n,k}(h_{n',k'})| = \frac{|g_{n,k}((n',k')) - g_{n,k}((n',k'+1))|}{|(n',k') - (n',k'+1)|} \leqslant \frac{1/2^n}{1/2^{n-1}} = \frac{1}{2}.$$

If $n' = n$, the only horizontal molecule h such that $g_{n,k}(h) \neq 0$ is $h = h_{n,k}$, and if $n' > n$ we obtain

$$\widehat{g}_{n,k}(h_{n',k'}) = \widehat{\varphi}_{n,k}\,(h_{n',k'}) \leqslant \|\varphi_{n,k}\|_L = \frac{1}{2}.$$

Actually, notice that given a pair of different points $p, q \in M_\infty \setminus \bigcup_{j=0}^{n} A_j$ it follows that there exists a pair of different points $p', q' \in M_\infty \setminus \bigcup_{j=0}^{n} A_j$ such that $m_{p',q'} \in \Gamma$ and $|\widehat{g}_{n,k}(m_{p,q})| \leqslant |\widehat{g}_{n,k}(m_{p',q'})| \leqslant \frac{1}{2}$.

Finally, let us consider $\delta > 0$ satisfying

$$\left(1 + \frac{\varepsilon}{2}\right)(1-\delta) > 1 + \frac{\varepsilon}{3}.$$

Since $\|f\|_L = \sup\left\{\|\widehat{f}(m)\| \colon m \in \Gamma\right\}$, we may find $m \in \Gamma$ such that $\|\widehat{f}(m)\| > 1 - \delta$. If $m \in H \cup V$, then consider $\widehat{f_m}$ the functional associated to m, and if $-m \in H \cup V$, consider the same functional but multiplied by -1. Now, let us define

$$g \colon M \longrightarrow Y, \qquad \widehat{g}(x) = \widehat{f}(x) + \frac{\varepsilon}{2}\widehat{f}_m(x)\widehat{f}(m) \quad \forall\, x \in \mathcal{F}(M). \tag{2.6}$$

It is clear that $\|f-g\|_L \leqslant \frac{\varepsilon}{2}$, so it remains to prove that g strongly attains its norm. On the one hand, note that

$$\|\widehat{g}(m)\| = \left(1+\frac{\varepsilon}{2}\right)\|\widehat{f}(m)\| > \left(1+\frac{\varepsilon}{2}\right)(1-\delta).$$

On the other hand, if $m = m_{p_0,q_0}$ for suitable $p_0 \in A_{n_{p_0}}, q_0 \in A_{n_{q_0}}$ we have, by the properties of the functionals $f_{n,k}$ and $g_{n,k}$, that

$$|\widehat{f}_m(m_{p,q})| \leqslant \frac{2}{3}$$

if p and q does not belong to $\bigcup_{i=0}^{j} A_i$ for $j = \max\{n_{p_0}, n_{q_0}\}$. Consequently, for $p, q \notin \bigcup_{i=0}^{j} A_i$ we get that

$$\|\widehat{g}(m_{p,q})\| = \|\widehat{f}(m_{p,q}) + \frac{\varepsilon}{2}\widehat{f}_m(m_{p,q})\widehat{f}(m)\| \leqslant \left(1+\frac{\varepsilon}{3}\right) < \left(1+\frac{\varepsilon}{2}\right)(1-\delta) < \|\widehat{g}(m)\|,$$

which implies that g cannot approximate its norm at arbitrarily close points, that is, it is non-local. By compactness, we deduce that g strongly attains its norm. □

Remark 2.28. Note that, in the above proof, the map g defined in both cases by, respectively, formulas (2.5) and (2.6), is non-local.

Next, we show that $\mathcal{F}(\mathfrak{M}_1)$ has property α, giving the proof of assertion (ii) of Theorem 2.26.

Proof of assertion (ii) of Theorem 2.26. Since the metric d consists of summing vertical and horizontal coordinates, and M_∞ is dense in M, it is clear that the set $\Gamma = \pm H \cup \pm V$ considered in the proof of Proposition 2.27 verifies that $B_{\mathcal{F}(M)} = \overline{\text{co}}(\Gamma)$. To see this, it is enough to note that given $(n_1, k_1), (n_2, k_2) \in M$, with $n_1 < n_2$, we will have that

$$d((n_1,k_1),(n_2,k_2)) = d((n_1,k_1),(n_2,2^{n_2-n_1}k_1)) + d((n_2,2^{n_2-n_1}k_1),(n_2,k_2)).$$

Therefore, we need to find a set of functionals Γ^* associated to Γ satisfying the definition of property α. In view of the proof of Proposition 2.27, it will be enough to consider the sets

$$H^* = \{f_{n,k} \colon n \in \mathbb{N}\cup\{0\}, k \in \{0,\ldots,2^n-1\}\},$$
$$V^* = \{g_{n,k} \colon n \in \mathbb{N}\cup\{0\}, k \in \{0,\ldots,2^n\}\},$$

and $\Gamma^* = \pm H^* \cup \pm V^*$, to obtain that the pair $(\Gamma, \Gamma^*) \subseteq \mathcal{F}(M) \times \text{Lip}_0(M,\mathbb{R})$ satisfies the statements of property α with constant $\frac{2}{3}$. □

Assertion (iii) of Theorem 2.26 is contained in the next proposition.

Proposition 2.29. *Let $\mathfrak{M}_2$ be the metric space given in Theorem 2.26. If $\Gamma \subseteq \text{Mol}(\mathfrak{M}_2)$ is a subset satisfying that $\overline{\text{co}}(\Gamma) = B_{\mathcal{F}(\mathfrak{M}_2)}$, then Γ is not a uniformly strongly exposed set.*

Proof. Pick such a subset Γ. By the paragraph below Corollary 2.13 , it follows that dent $\left(B_{\mathcal{F}(\mathfrak{M}_2)}\right) \subseteq \overline{\Gamma}$. Now, for every $n \in \mathbb{N}$, consider the points

$$x_n := \left(0, \frac{1}{2^n}\right) \quad \text{and} \quad y_n := \left(1, \frac{1}{2^{n+1}}\right).$$

It is clear that the pair (x_n, y_n) fails property (Z), so m_{x_n,y_n} is a strongly exposed point. Furthermore, Lemma 1.14 implies that m_{x_n,y_n} is an isolated point in $\text{Mol}(\mathfrak{M}_2)$, so $m_{x_n,y_n} \in \Gamma$. We will prove that the set $\{m_{x_n,y_n} \colon n \in \mathbb{N}\}$ is not uniformly strongly exposed. To do so, we will use the criterium given in Proposition 2.9. Let $z_n = (\frac{1}{2}, \frac{1}{2^{n+1}})$. Note that

$$\min\{d(x_n,z_n), d(z_n,y_n)\} = \frac{1}{2}$$

and

$$2(x_n,y_n)_{z_n} = \left(\frac{1}{4}+\frac{1}{2^{2n+2}}\right)^{1/2} + \frac{1}{2} - \left(1+\frac{1}{2^{2n+2}}\right)^{1/2} \longrightarrow 0$$

as $n \to \infty$. Now, Proposition 2.9 finishes the proof. □

Remark 2.30. *Note that the Lipschitz-free space over the metric space $\mathfrak{M}_p$ in Theorem 2.26 fails to have the Radon-Nikodým property for $p = 1, 2$ since $\mathfrak{M}_p$ contains an isometric copy of $[0, 1]$ and so, $L_1[0, 1]$ embeds into $\mathcal{F}(\mathfrak{M}_p)$*. Even more, there is a 1-Lipschitz retraction $r\colon \mathfrak{M}_p \longrightarrow [0, 1]$, and this implies that $\mathcal{F}(\mathfrak{M}_p)$ even contains a complemented copy of $L_1[0, 1]$.

In [20, Section 3.4] it is stated to be unknown whether the density of $\mathrm{LipSNA}(M, Y)$ in $\mathrm{Lip}_0(M, Y)$, for every Banach space Y, implies that at least one of the following properties holds:

(i) $\mathcal{F}(M)$ has the RNP.

(ii) $B_{\mathcal{F}(M)} = \overline{\mathrm{co}}(S)$, where S is a set of uniformly strongly exposed points.

(iii) $\mathcal{F}(M)$ has property quasi-α.

This is not the case, as the following example shows.

Example 2.31. *There is a metric space M satisfying that $\mathrm{LipSNA}(M, Y)$ is dense in $\mathrm{Lip}_0(M, Y)$ for every Banach space Y and such that $\mathcal{F}(M)$ fails the RNP, property quasi-α, and it does not contain any norming subset of uniformly strongly exposed points.*

We need the following easy result, which we prove since we have not been able to find a reference.

Lemma 2.32. *Let X, Y be two Banach spaces and write $Z := X \oplus_1 Y$.*

(i) If Z has property quasi-α, then X has property quasi-α.

(ii) Assume that S_Z contains a uniformly strongly exposed subset Γ such that $\overline{\mathrm{co}}(\Gamma) = B_Z$. Then, S_X contains a uniformly strongly exposed subset Δ such that $\overline{\mathrm{co}}(\Delta) = B_X$.

Proof. (i). Let $A_Z := \big\{(x_\lambda, y_\lambda)\colon \lambda \in \Lambda_Z\big\}$, $\big\{(x^*_\lambda, y^*_\lambda)\colon \lambda \in \Lambda_Z\big\}$ and $\rho_Z : \Lambda_Z \longrightarrow \mathbb{R}$ be the sets and the function given by the definition of property quasi-α. Since

$$A_Z \subseteq \mathrm{ext}\,(B_Z) = (\mathrm{ext}\,(B_X) \times \{0\}) \cup (\{0\} \times \mathrm{ext}\,(B_Y))\,,$$

we may consider

$$A_X := A_Z \cap (B_X \times \{0\}) \equiv \{x_\lambda\colon \lambda \in \Lambda_X\}$$

for convenient non-empty subset Λ_X of Λ_Z. Let us see that X has property quasi-α witnessed by the sets A_X and $\{x^*_\lambda\colon \lambda \in \Lambda_X\}$ and the function $\rho_X := \rho_Z|_{\Lambda_X} : \Lambda_X \longrightarrow \mathbb{R}$. Indeed:

- For every $\lambda \in \Lambda_X$, we have that
$$x^*_\lambda(x_\lambda) = (x^*_\lambda, y^*_\lambda)(x_\lambda, 0) = 1.$$

- For $\mu \neq \lambda$, we have that
$$|x^*_\lambda(x_\mu)| = |(x^*_\lambda, y^*_\lambda)(x_\mu, 0)| \leqslant \rho_Z(\lambda) = \rho_X(\lambda) < 1.$$

- Given $e^{**} \in \mathrm{ext}\,(B_{X^{**}})$, then $(e^{**}, 0) \in \mathrm{ext}\,(B_{Z^{**}})$, so we can find $A_{(e^{**},0)} \subset A_Z$ and $\omega \in \{-1, 1\}$ such that
$$\omega(e^{**}, 0) \in \overline{J_Z(A_{(e^{**},0)})}^{w^*}$$
and $\sup\{\rho_Z(\lambda)\colon (x_\lambda, y_\lambda) \in A_{(e^{**},0)}\} < 1$; we define $A_{e^{**}} = \pi(A_{(e^{**},0)})$ (where $\pi\colon Z \longrightarrow X$ denotes the natural projection) and observe that
$$\begin{aligned}\omega e^{**} = \omega\pi^{**}(e^{**}, 0) \in \pi^{**}\Big(\overline{J_Z(\Lambda_{(e^{**},0)})}^{w^*}\Big) &\overset{\diamondsuit}{\subseteq} \overline{[\pi^{**} \circ J_Z](\Lambda_{(e^{**},0)})}^{w^*}\\ = \overline{[J_X \circ \pi](\Lambda_{(e^{**},0)})}^{w^*} &= \overline{J_X(\Lambda_{e^{**}})}^{w^*},\end{aligned}$$
where the inclusion $\diamondsuit$ follows from the weak-star continuity of π^{**}. Now, it is clear that
$$\sup\{\rho_X(\lambda)\colon x_\lambda \in A_{e^{**}}\} \leqslant \sup\{\rho_Z(\lambda)\colon (x_\lambda, y_\lambda) \in A_{(e^{**},0)}\} < 1.$$

(ii). Since Γ is made of strongly exposed points of B_Z, then every element $(x, y) \in \Gamma$ satisfies that either $\|x\| = 1$ and $y = 0$ or $x = 0$ and $\|y\| = 1$. Define

$$\Delta := \{x \in S_X : (x, 0) \in \Gamma\}.$$

Given $(x, 0) \in \Gamma$, the definition of uniformly strongly exposing set yields a strongly exposing functional $(f_x, g_x) \in S_{Z^*}$ associated to $(x, 0)$. Notice that $\|f_x\| = 1$ since $1 = \langle (f_x, g_x), (x, 0)\rangle = f_x(x)$. It is clear that Δ is a uniformly strongly exposed set by making use of the fact that it is identified with a subset of Γ which is a uniformly strongly exposed set. The fact that $\overline{\text{co}}(\Delta) = B_X$ follows from the fact that $\overline{\text{co}}(\Gamma) = B_Z$ and the shape of the unit ball of an ℓ_1-sum. □

Proof of Example 2.31. Let us consider the metric space

$$M := \mathfrak{M}_2 \coprod [0, 1]^{\frac{1}{2}}.$$

By Proposition 1.15 we get that $\mathcal{F}(M) \cong \mathcal{F}(\mathfrak{M}_2) \oplus_1 \mathcal{F}([0, 1]^{\frac{1}{2}})$. In Chapter 5 we will prove that the strong density is stable under sums of metric spaces (see Theorem 5.20). Then, by Proposition 2.27, assertion (iii) of Example 2.4, and Theorem 5.20, we get that $\text{LipSNA}(M, Y)$ is dense in $\text{Lip}_0(M, Y)$ for every Banach space Y. Also $\mathcal{F}(M)$ fails property quasi-α because $\mathcal{F}([0, 1]^{\frac{1}{2}})$ fails property quasi-α (see Example 2.33) and we may use assertion (i) of Lemma 2.32. Further, $\mathcal{F}(M)$ fails the RNP because it contains an isometric copy of $L_1[0, 1]$. Finally, there is no uniformly strongly exposed set $\Gamma \subseteq S_X$ such that $\overline{\text{co}}(\Gamma) = B_{\mathcal{F}(M)}$ by Proposition 2.29 and assertion (ii) of Lemma 2.32. □

2.4 Relationship between the sufficient conditions

The purpose of this section is to study how the distinct sufficient conditions for Lindenstrauss property A that we have studied on this work are related to each other. Diagram 2.2, presented in [20], describes the relationship between them.

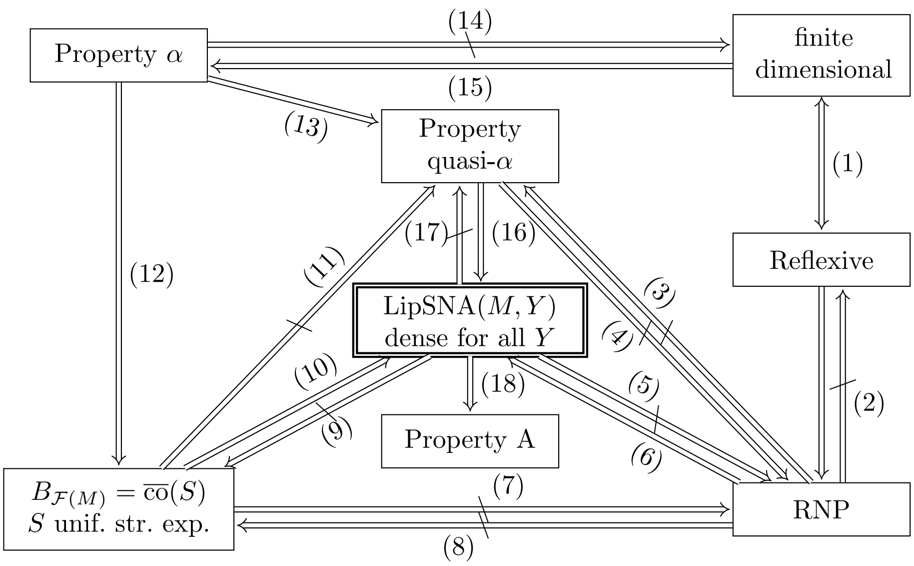

Figure 2.2: Relations between the sufficient conditions for Lindenstrauss property A in Lipschitz-free spaces

Let us discuss why the numbered implications and non-implications hold.

(1). It follows since every infinite-dimensional Lipschitz-free space contains an isomorphic copy of ℓ_1 (we refer to [30], where more is proved).

(2). It is well known that a reflexive Banach space has the RNP. With respect to the reciprocal arrow, $\mathcal{F}(\mathbb{N})$ is isometrically isomorphic to ℓ_1, so it has the RNP but it is not reflexive.

(3). It follows from the following example:

Example 2.33. *Let (M,d) be a boundedly compact metric space and $0<\theta<1$. Then $\mathcal{F}(M,d^\theta)$ has the RNP (see Example 2.4). Moreover, Proposition 2.24 implies that M is concave. By Proposition 2.23, as long as the isolated points of M do not form a dense set, we have that $\mathcal{F}(M)$ does not satisfy property quasi-α.*

(4). Notice that Theorem 2.26 provides a metric space $\mathfrak{M}_1$ such that $\mathcal{F}(\mathfrak{M}_1)$ satisfies property α (and so property quasi-α), but it fails to have the Radon-Nikodým property.

(5). Similarly to (4), it follows from Theorem 2.26.

(6). It follows from Proposition 2.3.

(7). Notice that Theorem 2.26 provides a metric space $\mathfrak{M}_1$ for which $\mathcal{F}(\mathfrak{M}_1)$ satisfies property α (and so $B_{\mathcal{F}(\mathfrak{M}_1)}$ is the closed convex hull of a set of uniformly strongly exposed points), but it fails to have the Radon-Nikodým property.

(8). It follows from the following example.

Example 2.34. *For every $n\in\mathbb{N}$, consider $M_n=\{0,x_n,y_n\}$, where*

$$d(0,x_n)=d(0,y_n)=1+1/n \qquad \textit{and} \qquad d(x_n,y_n)=2$$

for each $n\in\mathbb{N}$, and let $M=\coprod_{i=1}^\infty M_i$ be its sum. Then, $\mathcal{F}(M)$ has the RNP, but $B_{\mathcal{F}(M)}$ is not the closed convex hull of any set of uniformly strongly exposed points.

Proof. First, $\mathcal{F}(M)$ has the RNP as it is the ℓ_1-sum of finite-dimensional Banach spaces by Proposition 1.15. Suppose that $B_{\mathcal{F}(M)}=\overline{\text{co}}(A)$. We claim that $m_{x_n,y_n}\in A\cup(-A)$ for every $n\in\mathbb{N}$. Indeed, assume that $m_{x_n,y_n}\notin A\cup(-A)$. Consider $f\colon M\longrightarrow\mathbb{R}$ given by

$$f(0)=f(x_m)=f(y_m)=0 \text{ if } m\neq n, \quad f(x_n)=-1, \quad \text{and} \quad f(y_n)=1.$$

Clearly, $\|f\|_L=1$. Moreover, we have that

$$|\widehat{f}(m_{x,y})|<(1+1/n)^{-1} \qquad \text{for every } m_{x,y}\in \text{Mol}(M)\setminus\{m_{x_n,y_n},m_{y_n,x_n}\}.$$

Thus, $A\cup(-A)$ is not norming, a contradiction.

Now, note that

$$d(x_n,0)+d(y_n,0)-d(x_n,y_n)=\frac{2}{n}$$

goes to 0 as n goes to ∞, and so A is not uniformly Gromov concave. □

(9.) Notice that Theorem 2.26 provides a metric space $\mathfrak{M}_2$ for which $\text{LipSNA}(\mathfrak{M}_2,Y)$ is dense in $\text{Lip}_0(\mathfrak{M}_2,Y)$ for every Banach space Y, but $B_{\mathcal{F}(\mathfrak{M}_2)}$ is not generated by the closed convex hull of any set of uniformly strongly exposed points.

(10). It follows from Proposition 2.8, whose proof is based on [57, Proposition 1], where it is proved that the existence of such a set S implies Lindenstrauss property A.

(11). It follows from Example 2.33 together with Proposition 2.24.

(12). It follows from Definition 2.6 (see [64] for more details).

(13). It is obvious from the very definitions.

(14). $\mathcal{F}(\mathbb{N})$ is isometrically isomorphic to ℓ_1, that has property α.

(15). See Example 2.19.

(16). It follows from Proposition 2.21.

(17). It follows from Example 2.31.

(18). It is obvious.

There are some reversed implications not considered in the diagram. These ones are not known in the context of Lipschitz-free spaces. We find particularly interesting the case of whether the converse of (18) holds, that is, whether Lindenstrauss property A of $\mathcal{F}(M)$ is sufficient to get that $\text{LipSNA}(M,Y)$ is dense in $\text{Lip}_0(M,Y)$ for every Banach space Y.

2.5 Consequences of the strong density on the metric space

As we have already mentioned, there are deep connections between the extremal structure of a Banach space and the behavior of its norm-attaining linear operators. For instance, as a consequence of a result by Lindenstrauss [57, Theorem 2], if X is a separable Banach space satisfying Lindenstrauss property A, then B_X is the closed convex hull of its strongly exposed points. This shows that the density of norm-attaining linear operators has important consequences in the extremal structure of the unit ball of a Banach space. On the other hand, we know that the density of the set of strongly norm-attaining Lipschitz maps from a metric space M to a Banach space Y is stronger than the density of $\mathrm{NA}(\mathcal{F}(M), Y)$ as, for instance, $\mathrm{NA}(\mathcal{F}(M), \mathbb{R})$ is always dense by the Bishop-Phelps theorem but, as we will see in the next chapter, there are many metric spaces M for which $\mathrm{LipSNA}(M, \mathbb{R})$ is not dense. In view of this, it is reasonable to expect that the density of strongly norm-attaining Lipschitz functions may have important consequences on the extremal structure of the Lipschitz-free space. Our aim in this section is to deepen in this line. In particular, we will show that some results of Lindenstrauss and Bourgain can be somehow improved in the setting of Lipschitz-free spaces.

First, let us present the first main result of this section.

Theorem 2.35. *Let M be a metric space. If $\mathrm{LipSNA}(M, \mathbb{R})$ is dense in $\mathrm{Lip}_0(M, \mathbb{R})$, then*

$$B_{\mathcal{F}(M)} = \overline{\mathrm{co}}\Big(\mathrm{ext}\left(B_{\mathcal{F}(M)}\right) \cap \mathrm{Mol}(M)\Big).$$

Notice that in Theorem 2.35 we just need to assume density of the strongly norm-attaining Lipschitz functions from M to $\mathbb{R}$. Actually, it is enough to assume that there exists a Banach space Y for which we have strong density. This follows from the following easy result.

Proposition 2.36. *Let M be a metric space. Suppose that there exists a Banach space $Y \neq 0$ such that $\mathrm{LipSNA}(M, Y)$ is norm-dense in $\mathrm{Lip}_0(M, Y)$. Then,*

$$\overline{\mathrm{LipSNA}(M, \mathbb{R})} = \mathrm{Lip}_0(M, \mathbb{R}).$$

Proof. Fix $\varepsilon > 0$ and consider $f \in \mathrm{Lip}_0(M, \mathbb{R})$, which we may assume to have norm one. Define $F \in \mathrm{Lip}_0(M, Y)$ by

$$F(p) = f(p)y_0 \quad \forall\, p \in M.$$

Then, we have that $\|F\|_L = 1$. Thus, by hypothesis, there exist $G \in \mathrm{LipSNA}(M, Y)$ and $m \in \mathrm{Mol}(M)$ satisfying

$$\|\widehat{G}(m)\| = \|G\|_L = 1, \quad \|F - G\|_L < \frac{\varepsilon}{2}.$$

Now, take $y^* \in S_{Y^*}$ such that $y^*\big(\widehat{G}(m)\big) = 1$ and note that

$$\|y^*(y_0)f - y^* \circ G\|_L = \|y^* \circ F - y^* \circ G\|_L \leqslant \|y^*\|\|F - G\|_L < \frac{\varepsilon}{2}.$$

This implies that

$$y^*(y_0) \geqslant y^*(y_0)\widehat{f}(m) \geqslant y^*(\widehat{G}(m)) - |y^*(y_0)\widehat{f}(m) - y^*(\widehat{G}(m))| \geqslant 1 - \frac{\varepsilon}{2}.$$

Therefore, writing $g = y^* \circ G \in \mathrm{Lip}_0(M, \mathbb{R})$, we have that

$$|\widehat{g}(m)| = \|g\|_L = 1, \qquad \|g - f\|_L \leqslant \|g - y^*(y_0)f\|_L + \|y^*(y_0)f - f\|_L < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$ □

In view of this, the next corollary follows from Theorem 2.35 and from the fact that a molecule is an extreme point if and only if it is an exposed point (see Theorem 1.11).

Corollary 2.37. *Let M be a metric space. Assume that $\mathrm{LipSNA}(M, Y)$ is dense in $\mathrm{Lip}_0(M, Y)$ for some Banach space Y. Then,*

$$B_{\mathcal{F}(M)} = \overline{\mathrm{co}}(\exp\left(B_{\mathcal{F}(M)}\right)).$$

Compare this result with the following one by Lindenstrauss [57, Theorem 2.i]: if X is a Banach space which admits a strictly convex renorming (for instance, if X is separable) such that $\mathrm{NA}(X,Y)$ is dense in $\mathcal{L}(X,Y)$ for *all* Banach spaces Y, then $B_X = \overline{\mathrm{co}}(\exp(B_X))$.

Let us prove Theorem 2.35. In order to do it, we need the next two technical results, which will be the key to get all the goals of this section.

Lemma 2.38. *Let M be a metric space, let $f \in \mathrm{LipSNA}(M,\mathbb{R})$, and let $m_{p,q} \in \mathrm{Mol}(M)$ such that $\widehat{f}(m_{p,q}) = \|f\|_L$. Consider the set*

$$F_{p,q} := \{(x,y) \in M^2 \colon x \neq y,\ d(p,q) = d(p,x) + d(x,y) + d(y,q)\}.$$

Then, either there is $(x,y) \in F_{p,q}$ such that $m_{x,y} \in \mathrm{ext}\left(B_{\mathcal{F}(M)}\right)$ or there is an isometric embedding $\phi\colon [0, d(p,q)] \longrightarrow M$ for which $\phi(0) = p$ and $\phi(d(p,q)) = q$.

Proof. First, note that $(p,q) \in F_{p,q}$ and so $F_{p,q}$ is not empty. Assume that $m_{x,y}$ is not an extreme point whenever $(x,y) \in F_{p,q}$. By [10, Theorem 1.1], for every $(x,y) \in F_{p,q}$ there is $z \in M$ such that $d(x,z) + d(z,y) = d(x,y)$.

The rest of the proof is just a small modification of the one of Proposition 4.1 in [37]. Our aim is to show that there is an isometry $\phi\colon [0, d(p,q)] \longrightarrow M$ such that $\phi(0) = p$ and $\phi(d(p,q)) = q$. Consider the set $\mathcal{A}$ of all (A,ψ), where $\{0, d(p,q)\} \subset A \subset [0, d(p,q)]$ is closed and $\psi\colon A \longrightarrow M$ is an isometry such that $\psi(0) = p$, $\psi(d(p,q)) = q$, and $(\psi(t), \psi(s)) \in F_{p,q}$ for every $t,\ s \in A$ with $t < s$. Consider the following partial order "$\leqslant$" on $\mathcal{A}$: $(A,\psi) \leqslant (B,\xi)$ if $A \subset B$ and $\xi|_A = \psi$. Clearly $\mathcal{A} \neq \emptyset$.

We claim that every chain in $\mathcal{A}$ has an upper bound. Indeed, let $(A_i,\psi_i)_{i\in I}$ be a chain in $\mathcal{A}$. Take $A = \overline{\bigcup_{i\in I} A_i}$ and $\psi(x) := \psi_i(x)$ if $x \in A_i$. By completeness, we can extend ψ uniquely to an isometry defined on A. Moreover, let $t, s \in A$, $t < s$. Then there are sequences $\{t_n\}, \{s_n\}$ in $\bigcup_{i\in I} A_i$ such that $t_n < s_n$, $t_n \to t$ and $s_n \to s$. Then

$$\left|\widehat{f}(m_{\psi(t),\psi(s)})\right| = \lim_n \left|\widehat{f}(m_{\psi(t_n),\psi(s_n)})\right| = \|f\|_L$$

since $(\psi(t_n), \psi(s_n)) \in F_{p,q}$ for every n. Thus $(\psi(t), \psi(s)) \in F_{p,q}$. This means that $(A,\psi) \in \mathcal{A}$.

Now, let (A,ϕ) be a maximal element in $\mathcal{A}$. Assume that there are $a, b \in A$, $a < b$ such that $(a,b) \cap A = \emptyset$. Since $(\phi(a), \phi(b)) \in F_{p,q}$, we have that $m_{\phi(a),\phi(b)}$ is not an extreme point. Consequently, there is $z \in [\phi(a), \phi(b)] \setminus \{\phi(a), \phi(b)\}$. Then, we extend ϕ defining $\phi(a + d(\phi(a), z)) := z$. Let us show that this map contradicts the maximality of (A,ϕ). It is clear that ϕ is still an isometry with $\phi(0) = p$ and $\phi(d(p,q)) = q$. It remains to prove that $(\phi(t), \phi(s)) \in F_{p,q}$ for every $t \in A$ with $t < s$. Clearly, we may assume that either $\phi(s) = z$ or $\phi(t) = z$. Let's assume the first case holds, since the other one is similar. Since $t \in A$ and $t \leqslant \phi^{-1}(z)$, we have $t \leqslant a$. Then, we have that $\phi(a) \in [\phi(t), z]$ and it is clear that $\phi(b) \in [z, q]$. Joining these two equalities we obtain

$$\begin{aligned} &d(p,\phi(t)) + d(\phi(t), z) + d(z,q) \\ &\qquad = d(p,\phi(t)) + d(\phi(t),\phi(a)) + d(\phi(a), z) + d(z,\phi(b)) + d(\phi(b), q). \end{aligned}$$

Recall that $z \in [\phi(a), \phi(b)]$ and $\phi(a) \in [\phi(t), \phi(b)]$, so we have that

$$d(p,\phi(t)) + d(\phi(t), z) + d(z,q) = d(p,\phi(t)) + d(\phi(t),\phi(b)) + d(\phi(b), q) = d(p,q),$$

since $(\phi(t), \phi(b)) \in F_{p,q}$. This means that $(\phi(t), z) \in F_{p,q}$. □

Lemma 2.39. *Let M be a metric space. Let Γ be a balanced subset of $S_{\mathcal{F}(M)}$ and denote*

$$N_\Gamma(M) = \left\{ f \in \mathrm{Lip}_0(M,\mathbb{R}) \colon \sup_{m\in\Gamma} |\widehat{f}(m)| = \|f\|_L \right\}.$$

Suppose that the set

$$\left\{ f \in \mathrm{Lip}_0(M,\mathbb{R}) \colon \widehat{f}(m_{x,y}) = \|f\|_L \text{ for some } m_{x,y} \in \mathrm{Mol}(M) \cap \mathrm{ext}\left(B_{\mathcal{F}(M)}\right) \right\}$$

is contained in $N_\Gamma(M)$ and that $\mathrm{LipSNA}(M,\mathbb{R})$ *is dense in* $\mathrm{Lip}_0(M,\mathbb{R})$*. Then,*

$$\mathrm{LipSNA}(M,\mathbb{R}) \subset N_\Gamma(M)$$

and so, $B_{\mathcal{F}(M)} = \overline{\mathrm{co}}(\Gamma)$.

Proof. Assume that $h_1 \in \mathrm{LipSNA}(M,\mathbb{R}) \setminus N_\Gamma(M)$, with $\|h_1\|_L = 1$. Take $0 < \delta < 1$ in such a way that $\sup_{m\in\Gamma} |\widehat{h}_1(m)| = 1-\delta$. Now, if h_1 strongly attains its norm at a molecule $m_{p,q}$, by applying Lemma 2.38 and taking into account that h_1 does not attain its norm at any extreme molecule, we find an isometry $\phi\colon [0, d(p,q)] \longrightarrow M$ satisfying $\phi(0) = p$ and $\phi(d(p,q)) = q$. Consider $u_0\colon \phi([0, d(p,q)]) \longrightarrow [0, d(p,q)]$ its inverse map and an extension $u\colon M \longrightarrow [0, d(p,q)]$ of u_0 such that $\|u\|_L = 1$. Note that such extension exists thanks to McShane extension theorem. On the other hand, let $C \subseteq [0, d(p,q)]$ be a "fat Cantor set", that is, a measurable closed subset C with $\lambda(C) > (1-\delta)d(p,q)$ such that for each nontrivial interval $I \subseteq [0, d(p,q)]$ there exists a nontrivial interval $J \subseteq I$ such that $J \cap C = \emptyset$. Let us consider $\varphi\colon [0, d(p,q)] \longrightarrow \mathbb{R}$ given by

$$\varphi(t) = -\int_0^t \chi_C(s)\, ds \quad \forall\, t \in [0, d(p,q)].$$

We define $h_2\colon M \longrightarrow \mathbb{R}$ by $h_2 = \varphi \circ u$ and $f\colon M \longrightarrow \mathbb{R}$ by $f = \frac{1}{2}(h_1 + h_2)$. It is clear that

$$\|f\|_L \leqslant \frac{1}{2}(\|h_1\|_L + \|h_2\|_L) = 1.$$

Moreover,

$$\|f\|_L \geqslant \widehat{f}(m_{p,q}) = \frac{1}{2}(\widehat{h}_1(m_{p,q}) + \widehat{h}_2(m_{p,q})) = \frac{1}{2}\left(1 + \frac{\lambda(C)}{d(p,q)}\right) > 1 - \frac{\delta}{2}.$$

On the other hand,

$$\sup_\Gamma \widehat{f} \leqslant \frac{1}{2}(\sup_\Gamma \widehat{h}_1 + \sup_\Gamma \widehat{h}_2) = \frac{1}{2}(2-\delta) = 1 - \frac{\delta}{2}.$$

Therefore, $f \notin N_\Gamma(M)$. Take $\varepsilon > 0$ such that $\|f\|_L - \varepsilon > 1 - \frac{\delta}{2}$. Since $N_\Gamma(M)$ is closed and $\mathrm{LipSNA}(M,\mathbb{R})$ is dense there is $g \in \mathrm{LipSNA}(M,\mathbb{R}) \setminus N_\Gamma(M)$ such that $\|g\|_L = \|f\|_L$ and $\|f - g\|_L < \varepsilon$. Consider $m_{x,y} \in \mathrm{Mol}(M)$ for which $\widehat{g}(m_{x,y}) = \|g\|_L$. Since $g \notin N_\Gamma(M)$, we have that $\widehat{g}$ does not attain its norm at any extreme molecule of $B_{\mathcal{F}(M)}$. In particular, by applying Lemma 2.38 we obtain that there exists an isometry $\phi'\colon [0, d(x,y)] \longrightarrow M$ satisfying $\phi'(0) = x$ and $\phi'(d(x,y)) = y$. Notice that

$$\widehat{h}_2(m_{x,y}) \geqslant 2\widehat{f}(m_{x,y}) - 1 \geqslant 2(\widehat{g}(m_{x,y}) - \varepsilon) - 1 \geqslant 2(\|f\|_L - \varepsilon) - 1 > 1 - \delta,$$

from where $u(x) \neq u(y)$. Hence, we can find different points a, b of $\phi'([0, d(x,y)]) \subseteq M$ such that $d(x,y) = d(x,a) + d(a,b) + d(b,y)$ and $[u(a), u(b)] \cap C = \emptyset$. Since g attains its norm at $m_{x,y}$ it follows that $\widehat{g}(m_{a,b}) = \|g\|_L$ and so, $\widehat{f}(m_{a,b}) > \|f\|_L - \varepsilon$. As before, this implies that $\widehat{h}_2(m_{a,b}) > 1 - \delta$, whereas the fact that $[u(a), u(b)] \cap C = \emptyset$ implies that $\widehat{h}_2(m_{a,b}) = 0$, leading to a contradiction.

This shows that $\mathrm{LipSNA}(M,\mathbb{R}) \subset N_\Gamma(M)$ and the Hahn-Banach theorem gives $B_{\mathcal{F}(M)} = \overline{\mathrm{co}}(\Gamma)$. □

Now, the proof of Theorem 2.35 is immediate.

Proof of Theorem 2.35. Apply Lemma 2.39 with $\Gamma = \mathrm{Mol}(M) \cap \mathrm{ext}\left(B_{\mathcal{F}(M)}\right)$. □

The next result deals with strongly norm-attaining vector-valued Lipschitz maps. In the case of real-valued maps, it improves Theorem 2.35 for metric spaces not containing isometric copies of the unit interval.

Proposition 2.40. *Let M be a metric space which does not contain any isometric copy of $[0,1]$ and let Y be a Banach space. Then,* $\mathrm{LipSNA}(M,Y)$ *coincides with the set*

$$\left\{f \in \mathrm{Lip}_0(M,Y)\colon \|\widehat{f}(m_{x,y})\| = \|f\|_L \text{ for some } m_{x,y} \in \mathrm{Mol}(M) \cap \mathrm{ext}\left(B_{\mathcal{F}(M)}\right)\right\}.$$

In particular, if $\mathrm{LipSNA}(M,Y)$ *is dense in* $\mathrm{Lip}_0(M,Y)$*, then so is the set*

$$\left\{T \in \mathcal{L}(\mathcal{F}(M),Y)\colon T \text{ attains its norm at some element of } \mathrm{ext}\left(B_{\mathcal{F}(M)}\right)\right\}$$

in $\mathcal{L}(\mathcal{F}(M),Y)$.

Proof. Pick $f \in \mathrm{LipSNA}(M,Y)$. Hence there exists $u, v \in M, u \neq v$ and $y^* \in S_{Y^*}$ such that $[y^* \circ \widehat{f}](m_{u,v}) = \|f\|_L$. Since M does not contain any isometric copy of $[0,1]$, then Lemma 2.38 applies to get a molecule $m_{x,y} \in \operatorname{ext}\left(B_{\mathcal{F}(M)}\right)$ such that $\left[y^* \circ \widehat{f}\right](m_{x,y}) = \|f\|_L$. From here it is clear that f strongly attains its norm at the pair (x,y), and we are done. □

Using Proposition 2.40 we can show that the assumption that $\mathrm{LipSNA}(M,\mathbb{R})$ is dense cannot be removed from the statement of Theorem 2.35. Indeed, let M be a fat Cantor set in $[0,1]$, which clearly does not contain any isometric copy of $[0,1]$; then, $B_{\mathcal{F}(M)} \neq \overline{\mathrm{co}}\big(\exp\left(B_{\mathcal{F}(M)}\right)\big)$. Indeed, it is known that $\mathcal{F}(M) \cong L_1[0,1] \oplus_1 \ell_1$ [40, pp. 4315], but $L_1[0,1]$ has no extreme points (and in particular exposed points). Hence $\overline{\mathrm{co}}\big(\exp\left(B_{\mathcal{F}(M)}\right)\big) \subseteq \{0\} \oplus_1 \ell_1$. This in particular proves that $\mathrm{LipSNA}(M,\mathbb{R})$ is not dense in $\mathrm{Lip}_0(M,\mathbb{R})$. We will study this situation in the next chapter in more detail.

Let X, Y be Banach spaces. Following J. Bourgain, we say that an operator $T \in \mathcal{L}(X,Y)$ is *absolutely strongly exposing* if there exists $x \in S_X$ such that for every sequence $\{x_n\} \subseteq B_X$ with $\lim_n \|Tx_n\| = \|T\|$, there is a subsequence $\{x_{n_k}\}$ which converges either to x or $-x$. In this situation, it is clear that T attains its norm at the point x and that x is a strongly exposed point of B_X. Indeed, let $y^* \in S_{Y^*}$ such that $y^*(Tx) = \|T\|$ and consider $x^* \in S_{X^*}$ such that $\|T\|x^* = T^*(y^*)$; if $\{x_n\}$ is a sequence in B_X such that $x^*(x_n) \longrightarrow 1 = x^*(x)$, then

$$\|T(x_n)\| \geqslant y^*(Tx_n) = \|T\|x^*(x_n) \longrightarrow \|T\|,$$

so there is a subsequence $\{x_{n_k}\}$ converging to x (it cannot converge to $-x$), showing that x is strongly exposed by x^*. A famous result of J. Bourgain [18, Theorem 5] says that if X is a Banach space with the RNP and Y is any Banach space, then the set of absolutely strongly exposing operators from X to Y is a G_δ-dense subset of $\mathcal{L}(X,Y)$ (in particular, the space X has Lindenstrauss property A). Our goal is to give an improvement of this result in the context of Lipschitz maps. In order to do it, we will make use of the notion of non-local Lipschitz map.

Let us exhibit the second main result of this subsection.

Theorem 2.41. *Let M be a compact metric space which does not contain any isometric copy of* $[0,1]$ *and let Y be a Banach space. Then, the following assertions are equivalent:*

(i) $\mathrm{LipSNA}(M,Y)$ *is dense in* $\mathrm{Lip}_0(M,Y)$.

(ii) The set of absolutely strongly exposing operators from $\mathcal{F}(M)$ to Y is dense in $\mathcal{L}(\mathcal{F}(M),Y)$.

(iii) The set of non-local Y-valued Lipschitz maps is dense in $\mathrm{Lip}_0(M,Y)$.

Before proving the result, let us present its main consequence, which follows immediately from the fact that the set of non-local Y-valued Lipschitz maps is an open set (indeed, if $f \in \mathrm{Lip}_0(M,Y)$ is a non-local Lipschitz map, then taking $\varepsilon > 0$ such that

$$\sup\{\|\widehat{f}(m_{p,q})\| \colon 0 < d(p,q) < \varepsilon\} < \|f\|_L - \varepsilon$$

we have that the whole ball $B(f, \frac{\varepsilon}{3})$ is made of non-local Lipschitz maps).

Corollary 2.42. *Let M be a compact metric space which does not contain any isometric copy of* $[0,1]$ *and let Y be a Banach space. If* $\mathrm{LipSNA}(M,Y)$ *is dense in* $\mathrm{Lip}_0(M,Y)$*, then* $\mathrm{LipSNA}(M,Y)$ *(and, in particular,* $\mathrm{NA}(\mathcal{F}(M),Y)$*) contains an open dense subset.*

In the case when $\mathcal{F}(M) \cong \ell_1$ or, more generally, when $\mathcal{F}(M)$ has property α witnessed by a set $\Gamma \subset S_{\mathcal{F}(M)}$, it is easy to see the result from the proof of [64, Proposition 1.3.a]: indeed, it is proved there that the set of those operators $T \colon \mathcal{F}(M) \longrightarrow Y$ for which there is $x \in \Gamma$ such that

$$\sup\{\|Ty\| \colon y \in \Gamma \setminus \{\pm x\}\} < \|Tx\| = \|T\|$$

is dense and, on the other hand, it is clearly open as $\overline{\mathrm{co}}(\Gamma) = B_{\mathcal{F}(M)}$.

A specially interesting particular case of Corollary 2.42 is the one in which $\mathcal{F}(M)$ has the RNP. In this case, M does not contain copies of $[0,1]$ (otherwise, $L_1[0,1]$ would be a subspace of $\mathcal{F}(M)$) and $\mathrm{LipSNA}(M,Y)$ is dense in $\mathrm{Lip}_0(M,Y)$ by Proposition 2.3.

Corollary 2.43. *Let M be a compact metric space for which $\mathcal{F}(M)$ has the RNP. Then, for every Banach space Y,* $\mathrm{LipSNA}(M,Y)$ *(and so* $\mathrm{NA}(\mathcal{F}(M),Y)$*) contains a dense open subset.*

Compare the result above with the one by Bourgain [18, Theorem 5]: if X is a Banach space with the RNP, then for every Banach space Y, $\mathrm{NA}(X,Y)$ contains a dense G_δ subset of $\mathcal{L}(X,Y)$. Actually, by the cited results of Bourgain [18], $\mathrm{NA}(X,\mathbb{R})$ contains a dense G_δ subset of X^* whenever X has the RNP. Moreover, in this case, $X^* = \mathrm{NA}(X,\mathbb{R}) - \mathrm{NA}(X,\mathbb{R})$ (see the proof of [15, Proposition 2.23], for instance). But this is far from implying that $\mathrm{NA}(X,\mathbb{R})$ contains an open set. Let us comment that the result in Corollary 2.43 is somehow unexpected, even for functionals, as the following remark shows.

Remark 2.44. The presence of open subsets in the set of norm-attaining operators or even functionals is a rare phenomenon.

(i) If X is a non-reflexive Banach space, then there always exists an equivalent renorming $\tilde{X}$ of X such that $\mathrm{NA}(\tilde{X},\mathbb{R})$ has empty interior (see [6]). Therefore, the RNP is not enough to get that the set of norm-attaining operators (or even functionals) has non empty interior.

(ii) Even for the Lipschitz-free norm, the hypothesis of density of the strongly norm-attaining Lipschitz functions is important to get that the set of norm-attaining functionals has non-empty interior, as the following example shows: *For $M=[0,1]$, the interior of the set $\mathrm{NA}(\mathcal{F}(M),\mathbb{R})$ is empty.* Indeed, recall that $\mathcal{F}([0,1]) \cong L_1[0,1]$ (c.f. e.g. [41, Example 2.1]). Now, the result follows from the fact proved in [3, Theorem 2.7] that $L_1[0,1]^* \setminus \mathrm{NA}(L_1[0,1],\mathbb{R})$ is dense in $L_1[0,1]^*$.

On the other hand, we do not know whether the hypothesis that M does not contain isometric copies of $[0,1]$ can be dropped in Theorem 2.41. The only metric spaces M which we know that contain $[0,1]$ and for which $\mathrm{LipSNA}(M,\mathbb{R})$ is dense in $\mathrm{Lip}_0(M,\mathbb{R})$ are the metric spaces $\mathfrak{M}_p$ given in Theorem 2.26. As a matter of facts, it is immediate to check that the three assertions of Theorem 2.41 and the thesis of Corollary 2.43 hold for them. Indeed, it was shown that the set of non-local Lipschitz maps is dense, and it is an open subset of the set of strongly norm-attaining Lipschitz maps since the metric spaces are compact.

Let us now prove Theorem 2.41. We need a number of preliminary results which could be of independent interest. First, we prove the abundance of non-local Lipschitz maps when the set of strongly norm-attaining maps is dense, in the compact setting.

Lemma 2.45. *Let M be a compact metric space, let Y be a Banach space, and $f \in S_{\mathrm{Lip}_0(M,Y)}$. Assume that there exists $m_{x,y} \in \mathrm{ext}\left(B_{\mathcal{F}(M)}\right)$ such that $\|\widehat{f}(m_{x,y})\| = 1$. Then, for every $\varepsilon > 0$, there exists a non-local Lipschitz map $\phi : M \longrightarrow Y$ such that $\|f-\phi\|_L < \varepsilon$.*

Proof. Since $\|\widehat{f}(m_{x,y})\| = 1$ then we can find $y^* \in S_{Y^*}$ such that $[y^* \circ \widehat{f}](m_{x,y}) = 1$. By assumption, $m_{x,y}$ is an extreme point. Hence, by [9, Theorem 4.2] it is a preserved extreme point or, equivalently by [36, Theorem 2.4], $m_{x,y}$ is a denting point. Fix $0 < \delta < \frac{1}{2}$ and find a slice $S = S(B_{\mathcal{F}(M)}, \widehat{h}, \beta)$ with $h \in S_{\mathrm{Lip}_0(M,\mathbb{R})}$ and $\beta > 0$, containing $m_{x,y}$ and such that $\mathrm{diam}(S) < \delta$. Select $z \in S_Y$ such that $y^*(z)\widehat{h}(m_{x,y}) > 1-\beta$ and define

$$\widehat{\phi} := \widehat{f} + \varepsilon \widehat{h} \otimes z,$$

where $\widehat{h} \otimes z(m_{u,v}) := \widehat{h}(m_{u,v})z$ for every $u,v \in M, u \neq v$. It is clear that $\|f-\phi\|_L < \varepsilon$. Let us now prove that ϕ is not local. To begin with, notice that

$$\|\phi\|_L \geqslant [y^* \circ \widehat{f}](m_{x,y}) + \varepsilon \widehat{h}(m_{x,y}) y^*(z) > 1 + \varepsilon(1-\beta).$$

Now, given $u,v \in M, u \neq v$ such that $\|\widehat{\phi}(m_{u,v})\| > 1+\varepsilon(1-\beta)$, it follows that

$$1+\varepsilon(1-\beta) < \|\widehat{f}(m_{u,v})\| + \varepsilon|\widehat{h}(m_{u,v})| \leqslant 1 + \varepsilon|\widehat{h}(m_{u,v})|,$$

from where we get that $\widehat{h}(m_{u,v}) > 1-\beta$ or $\widehat{h}(-m_{u,v}) = \widehat{h}(m_{v,u}) > 1-\beta$. Assume that $\widehat{h}(m_{u,v}) > 1-\beta$ (the other case runs similarly). This implies that $m_{u,v} \in S$, hence $\|m_{u,v} - m_{x,y}\| < \delta$. Now, by using Lemma 1.14 we obtain that

$$\frac{\max\{d(x,u), d(y,v)\}}{d(x,y)} \leqslant \|m_{x,y} - m_{u,v}\| < \delta,$$

so $\max\{d(x,u), d(y,v)\} < \delta d(x,y)$. Hence

$$d(u,v) \geqslant d(x,y) - d(x,u) - d(y,v) > (1-2\delta)d(x,y),$$

from where we deduce that ϕ does not approximate its Lipschitz constant at arbitrarily close points, as desired. □

Next, we also need the following lemma, whose proof is encoded in [45, Proposition 2.8.b] for the real-valued case.

Lemma 2.46. *Let M be a compact metric space, Y be a Banach space, and $f \in S_{\mathrm{Lip}_0(M,Y)}$ be a non-local Lipschitz map. Then, there exists a strongly exposed point $m_{x,y} \in \mathcal{F}(M)$ such that $\|\widehat{f}(m_{x,y})\| = 1$.*

Proof. Since f is non-local, then an easy compactness argument yields that we can find a pair of different points $x, y \in M$ such that not only $\|\widehat{f}(m_{x,y})\| = 1$, but also that $\|\widehat{f}(m_{u,v})\| < 1$ if $0 < d(u,v) < d(x,y)$. We claim that the pair (x,y) fails property (Z). Indeed, assume by contradiction that (x,y) has property (Z). Pick $y^* \in S_{Y^*}$ such that $[y^* \circ \widehat{f}](m_{x,y}) = 1$. Then, for every $n \in \mathbb{N}$, there exists a point $z_n \in M \setminus \{x,y\}$ satisfying that

$$d(x,z_n) + d(y,z_n) \leqslant d(x,y) + \frac{1}{n}\min\{d(x,z_n), d(y,z_n)\}.$$

Up to taking a subsequence, we may assume that $d(z_n,x) \leqslant d(z_n,y)$ for every $n \in \mathbb{N}$. Also, up to taking a further subsequence, we may assume by compactness that $\{z_n\} \longrightarrow z \in M$. Now, we have two possibilities:

- If $x \neq z$ then it is clear that $d(x,z) + d(y,z) = d(x,y)$, which implies that $[y^* \circ \widehat{f}](m_{x,z}) = 1$ and, in particular, f strongly attains its norm at the pair (x,z). However, notice that

$$d(x,z) \leqslant \frac{1}{2}(d(x,z) + d(y,z)) = \frac{1}{2}d(x,y),$$

  which contradicts the minimality condition on $d(x,y)$.

- If $x = z$, then

$$\begin{aligned}
\|\widehat{f}(m_{x,z_n})\| &\geqslant [y^* \circ \widehat{f}](m_{x,z_n}) \\
&= [y^* \circ \widehat{f}](m_{x,y})\frac{d(x,y)}{d(x,z_n)} - [y^* \circ \widehat{f}](m_{z_n,y})\frac{d(z_n,y)}{d(x,z_n)} \\
&\geqslant \frac{d(x,y) - d(z_n,y)}{d(x,z_n)} \geqslant 1 - \frac{1}{n},
\end{aligned}$$

  which entails a contradiction with the assumption that f is not local.

Consequently, we get that the pair (x,y) fails property (Z), so $m_{x,y}$ is a strongly exposed point by [37, Theorem 5.4]. □

The last preliminary result we present on the way to proving Theorem 2.41 deals with norm-attaining operators on general Banach spaces.

Proposition 2.47. *Let X and Y be Banach spaces. The following assertions are equivalent:*

(i) The set $\{T \in \mathcal{L}(X,Y) \colon T$ attains its norm at a strongly exposed point$\}$ is dense in $\mathcal{L}(X,Y)$.

(ii) The set $\{T \in \mathcal{L}(X,Y) \colon T$ is absolutely strongly exposing operator$\}$ is dense in $\mathcal{L}(X,Y)$.

Proof. (ii) ⇒ (i). This implication follows from the fact that if T is an absolutely strongly exposing operator for $x \in S_X$, then x is strongly exposed.

(i) $\Rightarrow$ (ii). Pick an operator $T \in \mathcal{L}(X,Y)$ which attains its norm at a strongly exposed point x, and let us find an absolutely strongly exposing operator S such that $\|T-S\| < \varepsilon$. For this, pick a strongly exposing functional f_x for x. Define

$$S := T + \varepsilon f_x \otimes T(x),$$

which satisfies that $\|S-T\| < \varepsilon$ obviously. Let us prove that S is absolutely strongly exposing. To this end, it is clear that $\|S\| \leqslant 1+\varepsilon$. Also, we get that

$$1+\varepsilon = (1+\varepsilon)\|T(x)\| = \|S(x)\|,$$

so $\|S\| = 1+\varepsilon$. Pick a sequence $\{x_n\} \in S_X$ such that $\|S(x_n)\| \longrightarrow 1+\varepsilon$. Since $S = T + \varepsilon f_x \otimes T(x)$ this implies that $|f_x(x_n)| \longrightarrow 1$ from where we can find a subsequence $\{x_{n_k}\}$ such that $f_x(x_{n_k}) \longrightarrow 1$ or $f_x(x_{n_k}) \longrightarrow -1$. Making use of the fact that f_x strongly exposes x, we get that $\{x_{n_k}\} \longrightarrow x$ or $\{x_{n_k}\} \longrightarrow -x$. By definition, S is an absolutely strongly exposing operator, so we are done. $\square$

We are now able to present the pending proof.

Proof of Theorem 2.41. (i)$\Rightarrow$(iii) follows from Proposition 2.40 and Lemma 2.45. (iii)$\Rightarrow$(ii) follows by Lemma 2.46 and Proposition 2.47. Finally, (ii)$\Rightarrow$(i) follows from the fact that every absolutely strongly exposing operator attains its norm at a strongly exposed point, so at a molecule of $\mathcal{F}(M)$. $\square$

As a consequence of the techniques involved in the proofs of Theorems 2.35 and 2.41, we get the third main result of this section, which improves Theorem 2.35 in the compact case.

Theorem 2.48. *Let M be a compact metric space. If* $\operatorname{LipSNA}(M,\mathbb{R})$ *is dense in* $\operatorname{Lip}_0(M,\mathbb{R})$*, then*

$$B_{\mathcal{F}(M)} = \overline{\operatorname{co}}\big(\text{str-exp}\left(B_{\mathcal{F}(M)}\right)\big).$$

Proof. Let $\Gamma = \text{str-exp}\left(B_{\mathcal{F}(M)}\right)$. Assume that $f \in \operatorname{Lip}_0(M,\mathbb{R})$ is such that $\widehat{f}$ attains its norm at an element of $\operatorname{Mol}(M) \cap \operatorname{ext}\left(B_{\mathcal{F}(M)}\right)$. By Lemmata 2.45 and 2.46, $\widehat{f}$ can be approximated by elements in $\mathcal{L}(\mathcal{F}(M),\mathbb{R})$ attaining their norms on Γ. Therefore, $\sup_{m\in\Gamma} |\widehat{f}(m)| = \|f\|_L$. Now, Lemma 2.39 gives that $B_{\mathcal{F}(M)} = \overline{\operatorname{co}}(\Gamma)$, as desired. $\square$

Theorem 2.48 somehow improves, in the case of Lipschitz-free spaces over compact metric spaces, another result by Lindenstrauss appearing in [57]. Let M be a compact metric space. If $\operatorname{LipSNA}(M,Y)$ is dense in $\mathcal{L}(\mathcal{F}(M),Y)$ for *some* Banach space Y, then

$$B_{\mathcal{F}(M)} = \overline{\operatorname{co}}\big(\text{str-exp}\left(B_{\mathcal{F}(M)}\right)\big).$$

Indeed, this follows from Theorem 2.48, as the density of $\operatorname{LipSNA}(M,Y)$ in $\operatorname{Lip}_0(M,Y)$ for some Y implies the density of $\operatorname{LipSNA}(M,\mathbb{R})$ in $\operatorname{Lip}_0(M,\mathbb{R})$ by Proposition 2.36.

Compare this result with the following one by Lindenstrauss [57, Theorem 2.ii]: if X is a Banach space which admits a LUR renorming (for instance, if X is separable) such that $\operatorname{NA}(X,Y)$ is dense in $\mathcal{L}(X,Y)$ for *all* Banach spaces Y, then $B_X = \overline{\operatorname{co}}(\text{str-exp}\,(B_X))$.

Let us comment that in [41, Problem 6.7] it is proposed to study for which compact metric spaces M and Banach spaces Y one has that $\operatorname{LipSNA}(M,Y)$ is dense in $\operatorname{Lip}_0(M,Y)$. Note that a necessary condition is that $B_{\mathcal{F}(M)} = \overline{\operatorname{co}}\big(\text{str-exp}\left(B_{\mathcal{F}(M)}\right)\big)$, according to the previous remark. However, we will show in the next chapter that this is not a sufficient condition (see Theorem 3.10).

Our goal now is to generalize some previous results to a more general setting. Notice that techniques similar to those of Lemma 2.45 can be used in the locally compact case to get the following result.

Proposition 2.49. *Let M be a locally compact metric space and let Y be a Banach space. Then, the following assertions are equivalent:*

(i) The set $\left\{f \in \operatorname{Lip}_0(M,Y)\colon \widehat{f} \text{ attains its norm at a denting point}\right\}$ *is dense in* $\operatorname{Lip}_0(M,Y)$.

(ii) The set of absolutely strongly exposing operators from $\mathcal{F}(M)$ to Y is dense in $\mathcal{L}(\mathcal{F}(M),Y)$.

(iii) $\operatorname{LipSNA}(M,Y)$ *contains the open dense set B of the Lipschitz maps $f\colon M \longrightarrow Y$ with the following property: there are $\eta > 0$, $x, y \in M$ with $x \neq y$ and $r > 0$ such that*

- *$B(x,r)$ and $B(y,r)$ are compact and disjoint, and,*
- *$\|\widehat{f}(m_{u,v})\| \leqslant \|f\|_L - \eta$ if $(u,v) \notin (B(x,r)\times B(y,r)) \cup (B(y,r)\times B(x,r))$.*

In particular, in such a case, $B_{\mathcal{F}(M)} = \overline{\operatorname{co}}\big(\text{str-exp}\,(B_{\mathcal{F}(M)})\big)$.

In particular, for locally compact metric spaces whose Lipschitz-free space has the RNP, the proposition above gives the following corollary, which extends Corollary 2.43, since the set of absolutely strongly exposing operators from $\mathcal{F}(M)$ to Y is dense in $\mathcal{L}(\mathcal{F}(M),Y)$ by [18, Theorem 5].

Corollary 2.50. *Let M be a locally compact metric space for which $\mathcal{F}(M)$ has the RNP and let Y be a Banach space. Then,* $\operatorname{LipSNA}(M,Y)$ *(and so* $\operatorname{NA}(\mathcal{F}(M),Y)$*) contains an open dense set.*

Observe that this applies to the main examples in the literature of metric spaces M for which it is known that $\mathcal{F}(M)$ has the RNP, for example the class of uniformly discrete metric spaces or the class of boundedly compact Hölder metric spaces.

Proof of Proposition 2.49. Assume that

$$A := \{f \in \operatorname{Lip}_0(M,Y)\colon \widehat{f} \text{ attains its norm at a denting point}\}$$

is dense in $\operatorname{Lip}_0(M,Y)$. Pick $f \in A$ with $\|f\|_L = 1$ and find a denting point $m_{x,y} \in \mathcal{F}(M)$ and an element $y^* \in S_{Y^*}$ such that $[y^* \circ \widehat{f}](m_{x,y}) = 1$. Fix $0 < \delta < \frac{1}{2}$ and find a slice $S = S(B_{\mathcal{F}(M)}, \widehat{h}, \beta)$ containing $m_{x,y}$ and such that $\operatorname{diam}(S) < \delta$. Select $z \in S_Y$ such that $y^*(z)\widehat{h}(m_{x,y}) > 1-\beta$ and define

$$\widehat{\phi} := \widehat{f} + \varepsilon \widehat{h} \otimes z.$$

It is clear that $\|f-\phi\|_L \leqslant \varepsilon$. Also, the proof of Lemma 2.45 reveals that, given $u, v \in M, u \neq v$ then

$$\begin{aligned}&\|\widehat{\phi}(m_{u,v})\| > 1 + \varepsilon(1-\beta)\\ &\qquad \Longrightarrow (u,v) \in (B(x,\delta d(x,y)) \times B(y,\delta d(x,y))) \cup (B(y,\delta d(x,y)) \times B(x,\delta d(x,y))).\end{aligned}$$

Taking into account that $B(x,\delta d(x,y))$ and $B(y,\delta d(x,y))$ are compact and disjoint for a small enough δ, we derive that $\phi \in B$. This proves that the set B is dense. To get (iii) let us prove that B enjoys the following properties:

(a) *B is open.* Indeed, given a map $f \in B$, consider $\eta > 0$, $x, y \in M$ with $x \neq y$ and $r > 0$ for which $B(x,r)$ and $B(y,r)$ are compact and disjoint, and $\|\widehat{f}(m_{u,v})\| \leqslant \|f\| - \eta$ if $(u,v) \notin (B(x,r) \times B(y,r)) \cup (B(y,r)\times B(x,r))$. Pick $0 < \delta < \frac{\eta}{2}$ and let us prove that $B(f,\delta) \subseteq B$. To this end take $g \in \operatorname{Lip}_0(M,Y)$ with $\|f-g\|_L < \delta$. Now, if $(u,v) \notin (B(x,r)\times B(y,r)) \cup (B(y,r)\times B(x,r))$ then $\|\widehat{f}(m_{u,v})\| \leqslant \|f\|_L - \eta$, from where

$$\|\widehat{g}(m_{u,v})\| \leqslant \delta + \|\widehat{f}(m_{u,v})\| \leqslant \delta + \|f\|_L - \eta \leqslant \|g\|_L + 2\delta - \eta,$$

which proves that $g \in B$, as desired.

(b) *Every map in B attains its norm at a strongly exposed point.* To see this, take $f \in B$ and, by definition, consider $\eta > 0$, $x, y \in M$ with $x \neq y$ and $r > 0$ for which $B(x,r)$ and $B(y,r)$ are compact and disjoint, and $\|\widehat{f}(m_{u,v})\| \leqslant \|f\|_L - \eta$ if $(u,v) \notin (B(x,r)\times B(y,r)) \cup (B(y,r)\times B(x,r))$. Notice that f strongly attains its norm because the set $B(x,r) \cup B(y,r)$ is a compact set and from the fact that f cannot approximate its norm at arbitrarily close points. The previous fact even provides a pair of different points $u \in B(x,r)$ and $v \in B(y,r)$ with the property that $\|\widehat{f}(m_{u,v})\| = \|f\|_L$. The pair (u,v) fails property (Z). Indeed, if otherwise there is a sequence $\{z_n\}$ such that $\frac{(u,v)_{z_n}}{\min\{d(u,z_n),d(v,z_n)\}} \longrightarrow 0$, it follows from Lemma 2.11 that $z_n \in B(x,r) \cap B(y,r) = \emptyset$ for large n, getting a contradiction. Equivalently, the molecule $m_{u,v}$ is strongly exposed point. This proves that (iii) implies (ii) by Proposition 2.47.

Now, the previous two facts prove that (i) implies (iii). Finally, (ii) implies (i) is trivial, which finishes the proof. □

Apart from Corollary 2.50, Proposition 2.49 also applies to another large class of metric spaces.

Example 2.51. *Let M be a locally compact metric space, let $0 < \theta < 1$, and consider $M^\theta := (M, d^\theta)$. Then,* $\operatorname{LipSNA}(M^\theta, Y)$ *contains an open dense subset. Indeed, M^θ is locally compact and* $\operatorname{LipSNA}(M^\theta, Y)$ *is dense in* $\operatorname{Lip}_0(M^\theta, Y)$ *for every Banach space Y by Corollary 2.25. Moreover, Proposition 2.24 implies that every molecule of $\mathcal{F}(M^\theta)$ is a strongly exposed point, so Proposition 2.49 applies.*

Let us end this section by giving a generalization of Theorem 2.48.

Corollary 2.52. *Let M be a boundedly compact metric space. If* $\operatorname{LipSNA}(M, \mathbb{R})$ *is dense in* $\operatorname{Lip}_0(M, \mathbb{R})$*, then*

$$B_{\mathcal{F}(M)} = \overline{\operatorname{co}}\big(\text{str-exp}\left(B_{\mathcal{F}(M)}\right)\big).$$

Proof. As in Theorem 2.48, let $\Gamma = \text{str-exp}\left(B_{\mathcal{F}(M)}\right)$ and suppose $f \in \operatorname{Lip}_0(M, \mathbb{R})$ is such that $\widehat{f}$ attains its norm at an element $m_{p,q} \in \operatorname{ext}\left(B_{\mathcal{F}(M)}\right)$. Since M is a boundedly compact metric space, by using the techniques involved in the proof of Theorem 4.2 in [9] and Theorem 2.4 in [36], we obtain that $m_{p,q}$ is a denting point of $B_{\mathcal{F}(M)}$. Now, it follows from the proof of Proposition 2.49 that $\widehat{f}$ can be approximated by elements of $\mathcal{L}(\mathcal{F}(M), \mathbb{R})$ attaining their norms on Γ. Therefore, $\sup_{m\in\Gamma} |\widehat{f}(m)| = \|f\|_L$. Now, Lemma 2.39 does the work. □

2.6 Weak density of LipSNA$(M, \mathbb{R})$

Up to now, we have studied for which metric spaces M and Banach spaces Y, the set $\operatorname{LipSNA}(M, Y)$ of strongly norm-attaining Lipschitz maps is dense in $\operatorname{Lip}_0(M, Y)$ with respect to the norm topology. However, there is another interesting topology that we may consider: the weak topology. We will finish the present chapter by studying the density of the strongly norm-attaining Lipschitz functions with respect to the weak topology.

We have seen in the previous sections that the fact that $\operatorname{LipSNA}(M, \mathbb{R})$ is norm-dense in $\operatorname{Lip}_0(M, \mathbb{R})$ imposes severe restrictions on the metric space M (see Theorems 3.3, 2.35, and 2.48). Moreover, we will see in Chapter 3 that there are many metric spaces M for which it is possible to find Lipschitz maps that cannot be approximated by strongly norm-attaining Lipschitz maps. However, that is not the case if we replace norm-density with weak density, as the main result of this section shows:

Theorem 2.53. *Let M be a metric space. Then,* $\operatorname{LipSNA}(M, \mathbb{R})$ *is weakly sequentially dense in* $\operatorname{Lip}_0(M, \mathbb{R})$*. Moreover, for every $g \in \operatorname{Lip}_0(M, \mathbb{R})$ there is a sequence $\{g_n\} \subset \operatorname{LipSNA}(M, \mathbb{R})$ such that $g_n \xrightarrow{w} g$, $\|g_n\|_L \to \|g\|_L$, and $g_n \to g$ uniformly on bounded sets.*

This result extends [50, Theorem 2.6], where it was proved under the assumption of M being a length metric space.

In order to prove our result we need some preliminary lemmata.

The next lemma provides a criterion to get weak convergence of sequences of Lipschitz functionals and maps, for which the weak topology does not have any easy description. It is inspired by [50, Lemma 2.4], improves [50, Corollary 2.5] and will be the key to prove the main result of this section.

Lemma 2.54. *Let M be a metric space, let Y be a Banach space, and let $\{f_n\}$ be a sequence of functions in the unit ball of* $\operatorname{Lip}_0(M, Y)$*. For each $n \in \mathbb{N}$, we write $U_n := \{x \in M \colon f_n(x) \neq 0\}$ for the support of f_n. If $U_n \cap U_m = \emptyset$ for every $n \neq m$, then the sequence $\{f_n\}$ is weakly null.*

Proof. We will show that for every finite collection of reals $\{a_j\}_{j=1}^n$, we have

$$\left\| \sum_{j=1}^{n} a_j f_j \right\|_L \leqslant 2 \max_j |a_j|$$

and so $Te_n := f_n$ defines a bounded linear operator from c_0 to $\mathrm{Lip}_0(M,Y)$.

To this end, denote $f = \sum_{j=1}^n a_j f_j$. Take $x, y \in M$ with $x \neq y$, and let us give an upper estimate for $\frac{\|f(x)-f(y)\|}{d(x,y)}$. Since the supports of the functions $\{f_n\}$ are pairwise disjoint, there are $j_1, j_2 \in \{1,\dots,n\}$ such that $\{x,y\} \cap U_j = \emptyset$ if $j \in \{1,\dots,n\} \setminus \{j_1, j_2\}$. Therefore,

$$\begin{aligned}\frac{\|f(x)-f(y)\|}{d(x,y)} &= \frac{\|a_{j_1}(f_{j_1}(x)-f_{j_1}(y)) + a_{j_2}(f_{j_2}(x)-f_{j_2}(y))\|}{d(x,y)} \\ &\leqslant |a_{j_1}| + |a_{j_2}| \leqslant 2\max_j |a_j|.\end{aligned}$$

This shows that the operator T defined above is bounded. Thus, it is also weak-to-weak continuous and the conclusion follows. □

Next, the following result is implicitly proved in [50, Theorem 2.6] under the assumption of M being a length space, but thanks to Lemma 2.54, we can show that the same argument works in a much more general setting.

Lemma 2.55. *Let M be a metric space. Assume that there exists a sequence $\{B(x_n, r_n)\}_{n\in\mathbb{N}}$ of disjoint balls of M and a sequence $\{y_n\}_{n\in\mathbb{N}}$ of points of M such that $0 < d(x_n,y_n)/r_n \to 0$ and $r_n \to 0$. Then for every $g \in \mathrm{Lip}_0(M,\mathbb{R})$ there is a sequence $\{g_n\} \subset \mathrm{LipSNA}(M,\mathbb{R})$ such that $g_n \xrightarrow{w} g$, $\|g_n\|_L \longrightarrow \|g\|_L$ and $g_n \longrightarrow g$ uniformly.*

Proof. Given $g \in S_{\mathrm{Lip}_0(M,\mathbb{R})}$, just follow the proof of Theorem 2.6 in [50] to construct a sequence $\{g_n\}$ in $\mathrm{LipSNA}(M,\mathbb{R})$ with $\mathrm{supp}(g_n - g) \subset B(x_n, r_n)$, $g_n(y_n) = g(y_n)$ and $\|g_n\|_L = 1 + 2\frac{d(x_n,y_n)}{r_n} \to 1$. Then, $\{g_n\} \xrightarrow{w} g$ by Lemma 2.54. Moreover,

$$\begin{aligned}|g_n(x) - g(x)| &\leqslant |g_n(x) - g_n(y_n)| + |g(y_n) - g(x)| \leqslant (\|g_n\|_L + \|g\|_L) d(y_n, x) \\ &\leqslant \Big(2 + 2\frac{d(x_n,y_n)}{r_n}\Big)(r_n + d(x_n,y_n))\end{aligned}$$

whenever $x \in B(x_n, r_n)$. Since $r_n \to 0$, $d(x_n,y_n)/r_n \to 0$ and $\mathrm{supp}(g_n - g) \subset B(x_n, r_n)$, it follows that $g_n \to g$ uniformly. □

The following technical lemma will allow us to apply Lemma 2.55 in the case of M being discrete but not uniformly discrete.

Lemma 2.56. *Let M be a metric space. Assume that M is discrete but not uniformly discrete. Then, for every $k \geqslant 2$ and every $\varepsilon > 0$, there exist $x, y \in M$ such that $0 < d(x,y) \leqslant \varepsilon$ and the set $M \setminus B(x, k\,d(x,y))$ is not uniformly discrete.*

Proof. Assume that there exist $k \geqslant 2$ and $\varepsilon > 0$ such that

$$\alpha(x,y) := \inf\{d(u,v)\colon u, v \in M \setminus B(x, kd(x,y)),\ u \neq v\} > 0$$

whenever $0 < d(x,y) \leqslant \varepsilon$. Since M is not uniformly discrete, one can construct inductively two sequences $\{x_n\}$ and $\{y_n\}$ in M such that $0 < d(x_1,y_1) \leqslant \varepsilon$ and $0 < d(x_{n+1}, y_{n+1}) \leqslant \min\{\alpha(x_n,y_n), 2^{-n-1}\varepsilon\}$ for every $n \in \mathbb{N}$. It follows that either $x_{n+1} \in B(x_n, kd(x_n,y_n))$ or $y_{n+1} \in B(x_n, kd(x_n,y_n))$. In any case,

$$x_{n+1} \subset B(x_n, kd(x_n,y_n) + d(x_{n+1},y_{n+1})) \subset B(x_n, 2^{-n}\varepsilon(k+1/2)).$$

Thus, $\{x_n\}$ is Cauchy and so it has a limit in M, say x. Moreover, it is clear that $\{y_n\}$ also converges to x. Since M is discrete, we conclude the existence of $n \in \mathbb{N}$ such that $x_n = y_n$, a contradiction. □

Proof of Theorem 2.53. We distinguish several cases depending on the properties of the set of cluster points M'. If M' is infinite, then Lemma 2.55 applies and so $\mathrm{LipSNA}(M,\mathbb{R})$ is weakly sequentially dense. Indeed, in such case it is not difficult to construct an infinite sequence of disjoint balls centered at (different) cluster points; as the centers are cluster points, we may also get the sequence $\{y_n\}_{n\in\mathbb{N}}$.

If M' is empty, then we distinguish two more cases:

- If M is uniformly discrete, then $\mathcal{F}(M)$ has the RNP (see Example 2.4), and so $\operatorname{LipSNA}(M,\mathbb{R})$ is indeed norm-dense in $\operatorname{Lip}_0(M,\mathbb{R})$ by Proposition 2.3. Note that if $\|g_n - g\|_L \to 0$ then $g_n \to g$ uniformly on bounded sets.
- If M is discrete but not uniformly discrete, then we can inductively apply Lemma 2.56 to find sequences $\{x_n\}$, $\{y_n\}$ in M such that, for every $n \in \mathbb{N}$, the space $M \setminus \bigcup_{m=1}^n B(x_m, 2md(x_m, y_m))$ is discrete but not uniformly discrete, $x_{n+1}, y_{n+1} \in M \setminus \bigcup_{m=1}^n B(x_m, 2md(x_m, y_m))$ and
$$d(x_{n+1}, y_{n+1}) \leqslant \min\left\{\frac{n}{n+1} d(x_n, y_n), n^{-2}\right\}.$$
It is easy to check that the balls $\{B(x_n, nd(x_n, y_n))\}$ are pairwise disjoint and satisfy the requirement of Lemma 2.55. The conclusion follows.

It remains to consider the case when M' is non-empty and finite, say $M' = \{a_1, \dots, a_k\}$. Moreover, we may assume that $a_1 = 0$. Given $\varepsilon > 0$, we denote $E_\varepsilon := \bigcup_{i=1}^k \left[M \setminus B(a_i, \varepsilon)\right]$. If E_ε is finite for every $\varepsilon > 0$, then M is compact and countable. Then, $\mathcal{F}(M)$ has the RNP (see Example 2.4) and the conclusion follows. Thus, we may and do assume that there is $0 < \varepsilon_0 < \frac{1}{4} \min_{i \neq j}\{d(a_i, a_j)\}$ such that E_{ε_0} is infinite. Moreover, note that E_ε is discrete for every $\varepsilon > 0$. If there is $0 < \varepsilon \leqslant \varepsilon_0$ such that E_ε is not uniformly discrete in M, then the same argument as above provides a sequence of disjoint balls such that Lemma 2.55 applies. Thus, we may also suppose that E_ε is infinite and uniformly discrete in M for every $0 < \varepsilon \leqslant \varepsilon_0$. By rescaling the metric space, we may assume that $\varepsilon_0 = 2^{-1}$. For $n \in \mathbb{N}$ and $i \in \{1, \dots, k\}$, let us denote $C_n^i := E_{(n+1)^{-1}} \cap B(a_i, n^{-1})$ and
$$\alpha_n^i := \inf\{d(x, M \setminus \{x\}) \colon x \in C_n^i\},$$
with the convention that $\inf \emptyset = +\infty$. Note, by passing, that
$$M = E_{2^{-1}} \cup \bigcup_{n,k} C_n^i.$$
Now, we distinguish two cases.

Case 1: assume that there is $i \in \{1, \dots, k\}$ such that $\liminf_{n\to\infty} n\alpha_n^i = 0$. Then we claim that it is possible to find a sequence $\{j_n\}$ in $\mathbb{N}$, and sequences $\{x_n\}$ and $\{y_n\}$ in M, such that:

(i) $3nd(x_n, y_n) < (j_n + 1)^{-1}$ for every n;

(ii) $4j_{n+1}^{-1} < (j_n + 1)^{-1} - 3nd(x_n, y_n)$ for every n;

(iii) $x_n \in C_{j_n}^i$.

Indeed, take $j_1 \geqslant 1$ such that $6j_1\alpha_{j_1}^i < 1$. Then there is $x_1 \in C_{j_1}^i$ such that
$$3d(x_1, M \setminus \{x_1\}) < 2^{-1}j_1^{-1} \leqslant (j_1 + 1)^{-1}.$$
Thus, there is $y_1 \in M$ with $3d(x_1, y_1) < (j_1 + 1)^{-1}$. Now, assume that we have defined x_n, y_n and j_n, and let us define x_{n+1}, y_{n+1} and j_{n+1}. By condition (i), we can take $j_{n+1} \in \mathbb{N}$ such that
$$4j_{n+1}^{-1} < (j_n + 1)^{-1} - 3nd(x_n, y_n) \quad \text{and} \quad 6nj_{n+1}\alpha_{j_{n+1}}^i < 1.$$
Then, there are $x_{n+1} \in C_{j_{n+1}}^i$ and $y_{n+1} \in M$ such that
$$3nd(x_{n+1}, y_{n+1}) < 2^{-1}j_{n+1}^{-1} \leqslant (j_{n+1} + 1)^{-1}.$$
This completes the construction of the sequences $\{x_n\}$, $\{y_n\}$ and $\{j_n\}$. Now, we claim that
$$B(x_n, 3nd(x_n, y_n)) \cap B(x_m, 3md(x_m, y_m)) = \emptyset$$
whenever $n \neq m$. Indeed, assume that $n < m$. It follows from (i) and (ii) that
$$B(x_n, 3nd(x_n, y_n)) \cap B(a_i, j_{n+1}^{-1}) = \emptyset.$$

Moreover, from (i) and (iii) it follows that

$$\begin{aligned}B(x_m, 3md(x_m, y_m)) &\subset B(x_m, 3(j_m+1)^{-1})\\ &\subset B(a_i, 3(j_m+1)^{-1} + j_m^{-1}) \subset B(a_i, 4j_m^{-1}).\end{aligned}$$

Finally, note that $4j_m^{-1} \leqslant 4j_{n+1}^{-1} < (j_n+1)^{-1}$. Thus $B(x_m, 3md(x_m, y_m))$ is contained in $B(a_i, 4j_{n+1}^{-1})$ and so it does not intersect $B(x_n, 3nd(x_n, y_n))$. Therefore, we can apply Lemma 2.55 to get that LipSNA$(M,\mathbb{R})$ is weakly sequentially dense. This completes the proof in the first case.

Case 2: assume now that there is a constant $C > 0$ such that $C \leqslant n\alpha_n^i$ for every $n \in \mathbb{N}$ and $i \in \{1, \dots, k\}$. We will show that in this case $\mathcal{F}(M)$ has the RNP. To this end, we will apply Proposition 1.15 several times in order to decompose $\mathcal{F}(M)$ as an ℓ_1-sum of spaces with the RNP. Let us denote $E = E_{2^{-1}} \cup \{0\}$ and $N = \bigcup_{i=1}^k B(a_i, 1/2)$. We claim that $\mathcal{F}(M)$ is isomorphic to $\mathcal{F}(E) \oplus_1 \mathcal{F}(N)$. Note that N is bounded and so, $R = \sup\{d(x,0)\colon x \in N\} < +\infty$. Moreover, note also that E is uniformly discrete in M and so,

$$\alpha := \inf\{d(x,y)\colon x \in E, y \in N, x \neq y\} > 0.$$

Thus, given $x \in E$ and $y \in N$, we have that

$$d(x,0) + d(y,0) \leqslant d(x,y) + 2d(y,0) \leqslant \left(1 + 2\frac{R}{\alpha}\right) d(x,y).$$

By applying Proposition 1.15, we get the claim.

Now, for $i \in \{1, \dots, k\}$, denote $\tilde{C}_0^i = \{0, a_i\}$, $\tilde{C}_n^i := C_n^i \cup \{a_i\}$ if $n \geqslant 1$ and $\tilde{C}^i := \bigcup_{n=0}^\infty \tilde{C}_n^i$. Note that $N = \bigcup_{i=1}^k \tilde{C}^i$ and $\tilde{C}^i \cap \tilde{C}^j = \{0\}$ if $i \neq j$. We claim that there is a constant $L > 0$ such that

$$d(x,0) + d(y,0) \leqslant L\, d(x,y)$$

whenever $x \in \tilde{C}^i$ and $y \in \tilde{C}^j$ with $i \neq j$. Take such an x and y. Note that

$$\begin{aligned}d(x,y) &\geqslant d(a_i,a_j) - d(x,a_i) - d(y,a_j) \geqslant \frac{\min_{i\neq j} d(a_i,a_j)}{2}\\ &\geqslant \frac{\min_{i\neq j} d(a_i,a_j)}{4\operatorname{diam}(N)}(d(x,0) + d(y,0)).\end{aligned}$$

Therefore, $L = \frac{4\operatorname{diam}(N)}{\min_{i\neq j} d(a_i,a_j)}$ does the work. This shows that

$$\mathcal{F}(M) \approx \mathcal{F}(E) \oplus_1 \mathcal{F}(N) \approx \mathcal{F}(E) \oplus_1 \mathcal{F}(\tilde{C}^1) \oplus_1 \cdots \oplus_1 \mathcal{F}(\tilde{C}^k).$$

Finally, we will show that $\mathcal{F}(\tilde{C}^i) \approx \left[\bigoplus_{n=0}^\infty \mathcal{F}(\tilde{C}_n^i)\right]_{\ell_1}$ for every $i \in \{1, \dots, k\}$. To this end, consider a_i as the distinguished point in $\tilde{C}^i$ and notice that $\tilde{C}_n^i \cap \tilde{C}_m^i = \{a_i\}$ if $n \neq m$. Fix $n, m \in \mathbb{N} \cup \{0\}$ with $n < m$, take $x \in \tilde{C}_n^i$ and $y \in \tilde{C}_m^i$ with $x \neq y$. Then

$$d(y, a_i) \leqslant \frac{1}{m} \leqslant \frac{\alpha_m^i}{C} \leqslant \frac{d(x,y)}{C}$$

by definition of α_m^i, and so

$$d(x,a_i) + d(y,a_i) \leqslant d(x,y) + 2d(y,a_i) \leqslant (1 + 2C^{-1})d(x,y).$$

Thus, we can apply Proposition 1.15 to get that $\mathcal{F}(\tilde{C}^i) \approx \left[\bigoplus_{n=0}^\infty \mathcal{F}(\tilde{C}_n^i)\right]_{\ell_1}$. Therefore,

$$\mathcal{F}(M) \approx \mathcal{F}(E) \oplus_1 \left[\bigoplus_{n,i} \mathcal{F}(\tilde{C}_n^i)\right]_{\ell_1}$$

where each one of the summands has the RNP as they are the Lipschitz-free space over a uniformly discrete metric space (see Example 2.4). □

Let us finish the chapter with the following observation. If we identify $\mathrm{Lip}_0(M,\mathbb{R}) \equiv \mathcal{L}(\mathcal{F}(M),\mathbb{R}) \equiv \mathcal{F}(M)^*$, the Bishop-Phelps theorem gives that the set of those elements in $\mathrm{Lip}_0(M,\mathbb{R})$ which attain their norm *as elements of the dual of* $\mathcal{F}(M)$ is always norm dense. On the other hand, $\mathrm{LipSNA}(M,\mathbb{R})$ is the set of elements in $\mathcal{F}(M)^*$ which attain their norm at a molecule. As the unit ball of $\mathcal{F}(M)$ is the closed convex hull of $\mathrm{Mol}(M)$, one may wonder whether Theorem 2.53 actually follows from these facts, that is, if whenever a subset A of a Banach space X satisfies that $B_X = \overline{\mathrm{co}}(A)$, then the set of elements of X^* which attain their norms at a point of A is weakly dense on X^*. This is not true in general, as the following example shows.

Example 2.57. *Let $X = c_0\widehat{\otimes}_\pi Y$ be the projective tensor product of c_0 and Y, where Y is an equivalent renorming of ℓ_1 such that Y^* is strictly convex (see e.g. [34, Theorem II.2.6]). We consider the subset of B_X given by*

$$A := \{x \otimes y \colon x \in B_{c_0},\ y \in B_Y\}$$

which satisfies that $B_X = \overline{\mathrm{co}}(A)$ (see e.g. [63, Proposition 2.2]). Next, observe that if an element of $X^* \equiv \mathcal{L}(c_0, Y^*)$ attains its norm at a point of A then, in particular, it attains its norm as an operator from c_0 to Y^*, that is, the set of elements of X^* attaining their norms at a point of A is contained in $\mathrm{NA}(c_0, Y^*)$*. However, this set is not weakly dense since it is contained in the space of compact operators $\mathcal{K}(c_0, Y^*)$ by [57, Proposition 4] and there are non-compact operators from c_0 to Y^*.*

Chapter 3

Strong density. Negative results

Even though we presented in the previous chapter many conditions on a metric space M guaranteeing that the set $\mathrm{LipSNA}(M, Y)$ is dense in $\mathrm{Lip}_0(M, Y)$ for every Banach space Y, this density is a very restrictive property. There are many examples of metric spaces for which not every Lipschitz map can be approximated by strongly norm-attaining Lipschitz maps. This chapter will be devoted to studying precisely those metric spaces. We will present some properties that imply absence of strong density, as well as some examples that complement the positive results that we have already obtained.

The results obtained in this chapter come from the papers [20] and [23]. They were collaborative works with Bernardo Cascales, Luis Carlos García Lirola, Miguel Martín, and Abraham Rueda Zoca.

3.1 The previously known negative examples

The first negative examples are due to V. Kadets, M. Martín, and M. Soloviova and can be found in [50]. The authors show that if we consider $[0, 1]$ with its usual distance as the metric space, then it is possible to find a Lipschitz function $f\colon [0, 1] \longrightarrow \mathbb{R}$ that cannot be approximated by strongly norm-attaining Lipschitz functions. In other words, they obtain:

Example 3.1 ([50, Example 2.1])**.** $\mathrm{LipSNA}([0, 1], \mathbb{R})$ *is not norm-dense in* $\mathrm{Lip}_0([0, 1], \mathbb{R})$.

Actually, Proposition 2.36 implies that $\mathrm{LipSNA}([0, 1], Y)$ is not norm-dense in $\mathrm{Lip}_0([0, 1], Y)$ for any Banach space Y.

Example 3.1 can be a little shocking since $[0, 1]$ with its usual distance is one of the most natural and simplest metric spaces that we can consider, but it turns out that strong density fails for it. In other to clarify why this happens, we consider convenient to give a proof of this example:

Proof of Example 3.1. It is well known that every Lipschitz function from $[0, 1]$ to $\mathbb{R}$ is differentiable almost everywhere in $[0, 1]$. Actually, the derivative operator $\Phi\colon \mathrm{Lip}_0([0, 1], \mathbb{R}) \longrightarrow \mathrm{L}_\infty[0, 1]$ given by

$$\Phi(f) = f' \quad \forall\, f \in \mathrm{Lip}_0([0, 1], \mathbb{R})$$

is an isometric isomorphism. Now, pick $g \in \mathrm{LipSNA}([0, 1], \mathbb{R})$. Then, there are $s, t \in [0, 1]$ with $t < s$ such that $\widehat{g}$ attains its norm at the molecule $m_{s,t}$ and, replacing g by $-g$ if necessary, we may suppose that actually $\widehat{g}(m_{s,t}) = \|g\|_L$. Now, consider $p, q \in (t, s)$ with $p > q$ and notice that

$$d(s, t) = d(s, p) + d(p, q) + d(q, t),$$

so

$$m_{s,t} = \frac{d(s, p)}{d(s, t)} m_{s,p} + \frac{d(p, q)}{d(s, t)} m_{p,q} + \frac{d(q, t)}{d(s, t)} m_{q,t}.$$

Therefore, $m_{s,t}$ can be written as a convex combination of the molecules $m_{s,p}$, $m_{p,q}$, $m_{q,t}$. Consequently, we must also have that $\widehat{g}(m_{p,q}) = \|g\|_L$. Notice that this argument is valid for any two distinct points

$p, q \in (t, s)$ with $p > q$. As a consequence of this, if $p \in (t, s)$ is a point for which the derivative of g exists, then

$$g'(p) = \lim_{h\to 0} \frac{g(p+h) - g(p)}{h} = \lim_{h\to 0} \widehat{g}(m_{p+h,p}) = \lim_{h\to 0} \|g\|_L = \|g\|_L.$$

Therefore, $g'(p) = \|g\|_L$ for almost every point $p \in (t, s)$. The conclusion of this argument is that if $g \in \operatorname{LipSNA}([0,1], \mathbb{R})$, then there exists an interval (t, s) where g' is either equal to $\|g\|_L$ or $-\|g\|_L$ almost everywhere. Let now C be a nowhere dense measurable subset of $[0, 1]$ whose Lebesgue measure is positive (e.g. any so-called "fat" Cantor set). Then, C has the property that for every interval $(t, s) \subseteq [0, 1]$ there exists a subinterval $(u, v) \subseteq (t, s)$ such that $(u, v) \cap C = \emptyset$. Let us define $f\colon [0, 1] \longrightarrow \mathbb{R}$ by

$$f(p) = \int_0^p \chi_C(x)\, dx \quad \forall\, p \in [0, 1].$$

It is clear that f is Lipschitz and $\|f\|_L = 1$ (indeed, $f = \Phi^{-1}(\chi_C)$, so $\|f\|_L = \|\chi_C\|_\infty = 1$). Now, showing that this Lipschitz function cannot be approximated by strongly norm-attaining Lipschitz maps is easy. Indeed, let $g \in \operatorname{LipSNA}([0,1], \mathbb{R})$. Since we are trying to approximate f, we may assume that $\|g\|_L = 1$. Then, we know that there exists an interval (t, s) where $|g'| = 1$ almost everywhere. On the other hand, by the shape of the Cantor set C, there exists a subinterval $(u, v) \subseteq (t, s)$ so that $\chi_C = 0$ in (u, v). Consequently, $\|\chi_C - g'\|_\infty \geqslant 1$, but applying the isometry Φ^{-1} we obtain that $\|f - g\|_L \geqslant 1$. Since this is true for every norm-one strongly norm-attaining Lipschitz function g, we conclude that $f \notin \overline{\operatorname{LipSNA}([0,1], \mathbb{R})}$. Hence, $\operatorname{LipSNA}([0,1], \mathbb{R})$ cannot be dense in $\operatorname{Lip}_0([0,1], \mathbb{R})$. □

Let us comment that with the tools that we have developed in Chapter 2, we can give an alternative proof of Example 3.1. Indeed, if $s, t \in [0, 1]$, say $t < s$, and $p \in (s, t)$, we have that $d(t, s) = d(t, p) + d(p, s)$ and so

$$m_{t,s} = \frac{d(t,p)}{d(t,s)} m_{t,p} + \frac{d(p,s)}{d(t,s)} m_{p,s}.$$

Then, $m_{t,s}$ is not an extreme point of $B_{\mathcal{F}([0,1])}$. Since this is valid for every molecule of $\mathcal{F}([0, 1])$, we conclude that $\operatorname{ext}\left(B_{\mathcal{F}([0,1])}\right) \cap \operatorname{Mol}([0, 1]) = \emptyset$. Finally, Theorem 2.35 implies that there is no strong density.

Observe that, in essence, both proofs use the that for $M = [0, 1]$, given $p, q \in M$ with $p \neq q$, there are too many points in the metric segments $[p, q] = \{z \in M : d(p, q) = d(p, z) + d(z, q)\}$. Both proofs extend routinely to convex subset of normed spaces. Actually, it is possible to go further. Recall that a metric space (M, d) is said to be *geodesic* if for every pair of distinct points $p, q \in M$, there exists a rectifiable curve joining p and q of length $d(p, q)$. The authors of [50] generalized Example 3.1 obtaining the following result.

Theorem 3.2 ([50, Theorem 2.3])**.** *Let M be a geodesic metric space. Then,* $\operatorname{LipSNA}(M, \mathbb{R})$ *is not dense in* $\operatorname{Lip}_0(M, \mathbb{R})$.

3.2 New negative examples I. Abundance of metrically aligned points

Let us recall that a metric space (M, d) is said to be a *length metric space* if $d(p, q)$ is equal to the infimum of the length of the rectifiable curves joining p and q for every pair of distinct points p, $q \in M$. It is clear that every geodesic space is a length space, but Example 2.4 in [45] shows that the converse is not true. Let us comment that Theorem 1.9 proves that M has property (Z) if and only if M is length. Also, Corollary 1.8 states that M has property (Z) if and only if $B_{\mathcal{F}(M)}$ has no strongly exposed points. In contrast, we have seen in Chapter 2 that a strong presence of strongly exposed points inside $B_{\mathcal{F}(M)}$ implies density (see Proposition 2.8 for instance). On the other hand, Theorem 1.9 also states that $\mathcal{F}(M)$ has the Daugavet property if and only if M is length. In contrast, Proposition 2.3 states that the Radon-Nikodým property implies strongly density, and the Daugavet property is usually considered as nearly opposite to the RNP. These observations make natural that the following result holds.

Theorem 3.3. *Let M be a length metric space. Then, the set* $\operatorname{LipSNA}(M, \mathbb{R})$ *is not dense in* $\operatorname{Lip}_0(M, \mathbb{R})$.

Theorem 2.48 states that if M is a compact metric space, for which $\mathrm{LipSNA}(M,\mathbb{R})$ is dense, then $B_{\mathcal{F}(M)}$ is the closed convex hull of its strongly exposed points. We do not know if this result holds when the assumption of compactness is removed. However, as a consequence of Theorem 3.3 we have the next result.

Corollary 3.4. *Let M be a metric space. Assume that* $\mathrm{LipSNA}(M,\mathbb{R})$ *is dense in* $\mathrm{Lip}_0(M,\mathbb{R})$. *Then, there exists at least one strongly exposed point in $B_{\mathcal{F}(M)}$.*

Proof. Assume there are no strongly exposed points in $B_{\mathcal{F}(M)}$. Then, by Corollary 1.8 we have that M has property (Z). In view of Theorem 1.9, M must be length, but then Theorem 3.3 leads to a contradiction. □

We now proceed to prove Theorem 3.3. In order to do it, we need the next easy lemma.

Lemma 3.5. *Let M be a metric space, let $f \in \mathrm{LipSNA}(M,\mathbb{R})$ which attains its norm at a pair (p,q) of distinct points of M, let $\varepsilon > 0$, and let α_ε be a rectifiable curve in M joining p, q such that*

$$\mathrm{length}(\alpha_\varepsilon) \leqslant d(p,q) + \varepsilon.$$

Then, we have that

$$|f(z_1) - f(z_2)| \geqslant \|f\|_L (d(z_1,z_2) - \varepsilon) \quad \forall\, z_1, z_2 \in \alpha_\varepsilon.$$

Proof. Fix z_1, $z_2 \in \alpha_\varepsilon$. By the definition of length of a curve, we have that

$$d(p,q) \leqslant d(p,z_1) + d(z_1,z_2) + d(z_2,q) \leqslant \mathrm{length}(\alpha_\varepsilon) \leqslant d(p,q) + \varepsilon.$$

Consequently,

$$\begin{aligned} |f(z_1) - f(z_2)| &= |(f(p) - f(q)) - ((f(p) - f(z_1)) + (f(z_2) - f(q)))| \\ &\geqslant |f(p) - f(q)| - |f(p) - f(z_1)| - |f(z_2) - f(q)| \\ &\geqslant |f(p) - f(q)| - \|f\|_L d(p,z_1) - \|f\|_L d(z_2,q) \\ &= \|f\|_L (d(p,q) - d(p,z_1) - d(z_2,q)) \geqslant \|f\|_L (d(z_1,z_2) - \varepsilon). \end{aligned}$$ □

We are now ready to prove the desired result.

Proof of Theorem 3.3. Fix $\delta > 0$, $x_0 \in M \setminus \{0\}$. Let us consider a curve

$$\gamma_\delta \colon [0,(1+\delta)d(0,x_0)] \longrightarrow M$$

joining 0 and x_0.

Now, let us consider a Lipschitz function $u_0 \colon \gamma_\delta([0,(1+\delta)d(0,x_0)]) \longrightarrow \mathbb{R}$ such that $u_0(0) = 0$, $u_0(x_0) = 1$. Since $\gamma_\delta([0,(1+\delta)d(0,x_0)])$ is compact and connected, we have that $u_0(\gamma_\delta([0,(1+\delta)d(0,x_0)]))$ is a compact connected subset of $\mathbb{R}$, i.e. $u_0(\gamma_\delta([0,(1+\delta)d(0,x_0)])) = [a_0,b_0]$ for certain a_0, $b_0 \in \mathbb{R}$. We will write

$$a = \frac{a_0}{\|u_0\|_L}, \quad b = \frac{b_0}{\|u_0\|_L}, \quad \frac{u_0}{\|u_0\|_L} \colon \gamma_\delta([0,(1+\delta)d(0,x_0)]) \longrightarrow [a,b].$$

We can apply McShane's extension theorem to $\frac{u_0}{\|u_0\|_L}$ to get a surjective function $u \colon M \longrightarrow [a,b]$ satisfying that $\|u\|_L = 1$. Let $A \subseteq [a,b]$ be a nowhere dense closed set of positive Lebesgue measure. Consider $g \in \mathrm{Lip}_0([a,b],\mathbb{R})$ the function whose derivate equals χ_A (characteristic function of A). We define $h = g \circ u \colon M \longrightarrow \mathbb{R}$. It is clear that $h(0) = g(u(0)) = g(0) = 0$ and $\|h\|_L = \|g\|_L = 1$. Therefore, $h \in \mathrm{Lip}_0(M,\mathbb{R})$. Now, take $f \in \mathrm{LipSNA}(M,\mathbb{R})$. We will show that $\|h - f\|_L \geqslant \frac{1}{2}$. To this end, assume the contrary, that is,

$$\|f - h\|_L < \frac{1}{2}.$$

In particular, note that $\|f\|_L > \frac{1}{2}$. We know that there exist $p, q \in M$ with $p \neq q$ such that

$$\|f\|_L = \frac{|f(p) - f(q)|}{d(p,q)}.$$

Suppose that $u(p) = u(q)$, hence $h(p) = h(q)$ and we have that

$$\|h - f\|_L \geqslant \frac{|(h-f)(p) - (h-f)(q)|}{d(p,q)} = \frac{|f(p) - f(q)|}{d(p,q)} = \|f\|_L > \frac{1}{2},$$

a contradiction. Therefore, $u(p) \neq u(q)$. We can assume that $u(p) < u(q)$ without any loss of generality. By the construction of g, there exist $c,\, d \in \mathbb{R}$ such that the interval $[c, d]$ is contained in $(u(p), u(q))$ and that g is constant in $[c, d]$. Take $\varepsilon_0 > 0$ satisfying

$$0 < \varepsilon_0 < |d - c| \frac{\|f\|_L - \|h - f\|_L}{\|f\|_L} \tag{3.1}$$

and a rectifiable curve α_{ε_0} joining p and q such that

$$\operatorname{length}(\alpha_{\varepsilon_0}) \leqslant d(p,q) + \varepsilon_0.$$

Note that such a curve exists because M is a length space. Let us write $\Lambda = \alpha_{\varepsilon_0}([0, d(p,q) + \varepsilon]) \subseteq M$ and observe that

$$[c, d] \subseteq (u(p), u(q)) \subseteq u(\Lambda),$$

so there exist $\tilde{z}_1,\, \tilde{z}_2 \in \Lambda$ such that $c = u(\tilde{z}_1)$, $d = u(\tilde{z}_2)$. Moreover, we have

$$|d - c| = |u(\tilde{z}_2) - u(\tilde{z}_1)| \leqslant d(\tilde{z}_2, \tilde{z}_1).$$

Hence, if $z_1,\, z_2$ are different points of Λ, using Lemma 3.5 we get that

$$\begin{aligned} |h(z_1) - h(z_2)| &\geqslant |f(z_1) - f(z_2)| - \|h - f\|_L d(z_1, z_2) \\ &\geqslant \|f\|_L d(z_1, z_2) - \|f\|_L \varepsilon_0 - \|h - f\|_L d(z_1, z_2) \\ &= \left(\|f\|_L - \|h - f\|_L - \frac{\varepsilon_0 \|f\|_L}{d(z_1, z_2)} \right) d(z_1, z_2). \end{aligned}$$

Taking $z_1 = \tilde{z}_1$, $z_2 = \tilde{z}_2$ and applying the above inequality, we have

$$\begin{aligned} |h(\tilde{z}_1) - h(\tilde{z}_2)| &\geqslant \left(\|f\|_L - \|h - f\|_L - \frac{\varepsilon_0 \|f\|_L}{d(\tilde{z}_1, \tilde{z}_2)} \right) d(\tilde{z}_1, \tilde{z}_2) \\ &\overset{(3.1)}{>} (\|f\|_L - \|h - f\|_L - (\|f\|_L - \|h - f\|_L)) d(\tilde{z}_1, \tilde{z}_2) = 0. \end{aligned}$$

This implies that $h(\tilde{z}_1) \neq h(\tilde{z}_2)$ and so $g(c) \neq g(d)$, getting a contradiction with the fact that g is constant in $[c, d]$. □

The last theorem can be seen as a generalization of the fact that $\operatorname{LipSNA}([0,1], \mathbb{R})$ is not dense in $\operatorname{Lip}_0([0,1], \mathbb{R})$. Our next goal is to generalize that fact but in a different way. We are interested in characterizing those closed subsets M of $[0,1]$ for which $\operatorname{LipSNA}(M, \mathbb{R})$ is dense. This new generalization will allow us to produce examples of metric spaces M with very different geometric and topological properties for which $\operatorname{LipSNA}(M, \mathbb{R})$ is still not dense in $\operatorname{Lip}_0(M, \mathbb{R})$. In order to do that, we need to introduce a class of metric spaces M, the so-called $\mathbb{R}$-trees. An $\mathbb{R}$-*tree* is a metric space T satisfying:

(i) for any points $x,\, y \in T$, there exists a unique isometry ϕ from the closed interval $[0, d(x,y)]$ into T such that $\phi(0) = x$ and $\phi(d(x,y)) = y$. Such isometry will be denoted by ϕ_{xy};

(ii) any one-to-one continuous mapping $\varphi\colon [0,1] \longrightarrow T$ has the same range as the isometry ϕ associated to the points $x = \varphi(0)$ and $y = \varphi(1)$.

Let us introduce some more notation, coming from [40]. Given points x, y in an $\mathbb{R}$-tree T, it is usual to write $[x, y]$ to denote the range of ϕ_{xy}, which is called a segment. We say that a subset A of T is *measurable* whenever $\phi_{xy}^{-1}(A)$ is Lebesgue measurable for any $x,\, y \in T$. If A is measurable and S is a segment $[x, y]$, we write $\lambda_S(A)$ for $\lambda(\phi_{xy}^{-1}(A))$, where λ is the Lebesgue measure on $\mathbb{R}$. We denote by $\mathcal{R}$ the set of those subsets of T which can be written as a finite union of disjoint segments, and for $R = \bigcup_{k=1}^{n} S_k$ (with disjoint S_k) in $\mathcal{R}$, we put

$$\lambda_R(A) = \sum_{k=1}^{n} \lambda_{S_k}(A).$$

Now, we can define the *length measure* of a measurable subset A of T by

$$\lambda_T(A) = \sup_{R\in\mathcal{R}} \lambda_R(A).$$

$\mathbb{R}$-trees were considered in [40] in order to characterize the metric spaces M for which $\mathcal{F}(M)$ is isometric to a subspace of L_1 as those which isometrically embed into an $\mathbb{R}$-tree. Let us say that it was proved in [11, Corollary 4.5] that if M is a complete subset of an $\mathbb{R}$-tree, then every extreme point of the unit ball of $\mathcal{F}(M)$ is a preserved extreme point. In particular, every extreme point of $B_{\mathcal{F}(M)}$ is a molecule.

The next result generalizes Example 3.1 in the announced direction.

Theorem 3.6. *Let T be an $\mathbb{R}$-tree and let M be a closed subset of T containing the origin. If M has positive length measure, then* $\operatorname{LipSNA}(M,\mathbb{R})$ *is not dense in* $\operatorname{Lip}_0(M,\mathbb{R})$.

Proof. Note that, as M has positive length measure, we can find a segment $S = [x_0, y_0] \subseteq T$ such that $\lambda_T(M\cap S) > 0$. We distinguish two cases:
First, assume that there exists a segment $[x_1, y_1] \subseteq M \cap S$. By Theorem 2.3 in [50] we know that there exists a function $f \in \operatorname{Lip}_0([x_1, y_1],\mathbb{R})$ such that $\|f\|_L = 1$ and $\|f - g\|_L \geqslant \frac{1}{2}$ holds for all $g \in \operatorname{LipSNA}([x_1, y_1])$. Consider $\pi_1 \colon T \longrightarrow [x_1, y_1]$ the metric projection, which satisfies that

$$d(x, y) = d(x, \pi_1(x)) + d(\pi_1(x), y) \quad \forall\, x \in T, y \in [x_1, y_1]$$

(c.f. e.g. [19, Chapter II.2]). Define the norm-one Lipschitz function $\tilde{f}\colon M \longrightarrow \mathbb{R}$ by $\tilde{f}(p) = [f \circ \pi_1](p)$ for every $p \in M$, and suppose that there exists $g \in \operatorname{LipSNA}(M,\mathbb{R})$ such that $\|\tilde{f} - g\|_L < \frac{1}{2}$. If we take x, $y \in M$ with $x \neq y$ such that $\widehat{g}(m_{x,y}) = \|g\|_L$, we get

$$\frac{1}{2} > \frac{|f(\pi_1(x)) - f(\pi_1(y)) - (g(x) - g(y))|}{d(x, y)} \geqslant \|g\|_L - \frac{|f(\pi_1(x)) - f(\pi_1(y))|}{d(x, y)},$$

so $\pi_1(x) \neq \pi_1(y)$. Using that $\widehat{g}(m_{x,y}) = \|g\|_L$, Lemma 2.2 in [50] gives that $\widehat{g}(m_{\pi_1(x),\pi_1(y)}) = \|g\|_L$. Hence, $g|_{[x_1,x_2]} \in \operatorname{LipSNA}([x_1, y_1])$. It follows from this that

$$\|\tilde{f} - g\|_L \geqslant \|f - g|_{[x_1,y_1]}\|_L \geqslant \frac{1}{2},$$

a contradiction.
Now, assume that no segment is contained in $M\cap S$. Define the norm-one Lipschitz function $f\colon S \longrightarrow \mathbb{R}$ by

$$f(t) = \int_{[x_0,t]} \chi_M(x)\, dx = \lambda_T([x_0, t] \cap M)$$

for all $t \in [x_0, y_0]$. As above, define $\tilde{f}\colon M \longrightarrow \mathbb{R}$ by $\tilde{f}(p) = [f \circ \pi_2](p)$ for every $p \in M$, where $\pi_2\colon M \longrightarrow S$ is the metric projection onto S. Again, assume that there exists $g \in \operatorname{LipSNA}(M,\mathbb{R})$ such that $\|g - \tilde{f}\|_L < \frac{1}{2}$. Take $x, y \in M$ such that $x \neq y$ and $\widehat{g}(m_{x,y}) = \|g\|_L$. Then, using the same argument as above, we deduce that $\pi_2(x) \neq \pi_2(y)$. Now, since $[\pi_2(x), \pi_2(y)] \nsubseteq M \cap S$ by the assumption, we can find distinct points x_2, $y_2 \in M$ such that $[x_2, y_2] \subseteq]\pi_2(x), \pi_2(y)[\setminus (M \cap S)$. Recall that $\widehat{g}(m_{x,y}) = \|g\|_L$ and this implies that $\widehat{g}(m_{x_2,y_2}) = \|g\|_L$ by Lemma 2.2 in [50]. On the other hand, note that

$$\tilde{f}(x_2) = f(x_2) = \lambda_T([x_0, x_2] \cap M) = \lambda_T([x_0, y_2] \cap M) = f(y_2) = \tilde{f}(y_2).$$

Therefore, we obtain

$$\frac{1}{2} > \|g - \tilde{f}\|_L \geqslant \frac{(g - \tilde{f})(x_2) - (g - \tilde{f})(y_2)}{d(x_2, y_2)} = \widehat{g}(m_{x_2,y_2}) = \|g\|_L > \frac{1}{2},$$

getting again a contradiction. Consequently, the set $\operatorname{LipSNA}(M,\mathbb{R})$ is not dense in $\operatorname{Lip}_0(M,\mathbb{R})$. $\square$

Corollary 3.7. *Let M be a compact subset of an $\mathbb{R}$-tree containing* 0*. Then,* $\mathrm{LipSNA}(M,\mathbb{R})$ *is dense in* $\mathrm{Lip}_0(M,\mathbb{R})$ *if and only if $\lambda_T(M)=0$. Moreover, in such a case we have that* $\mathrm{LipSNA}(M,Y)$ *is dense in* $\mathrm{Lip}_0(M,Y)$ *for every Banach space Y.*

Proof. First, if M has positive measure, then Theorem 3.6 implies that $\mathrm{Lip}_0(M,\mathbb{R})$ is not dense in $\mathrm{Lip}_0(M,\mathbb{R})$. Now, if M is a compact subset of an $\mathbb{R}$-tree such that $\lambda_T(M)=0$, then $\mathcal{F}(M)$ is isometric to a subspace of ℓ_1 [32, Proposition 8], so $\mathcal{F}(M)$ has the RNP. Then, Proposition 2.3 applies. □

As a particular case, we obtain the desired characterization of the subsets of $[0,1]$.

Corollary 3.8. *Let M be a closed subset of* $[0,1]$*. Then,* $\mathrm{LipSNA}(M,\mathbb{R})$ *is dense in* $\mathrm{Lip}_0(M,\mathbb{R})$ *if and only if M has measure zero. Moreover, in such a case we have that* $\mathrm{LipSNA}(M,Y)$ *is dense in* $\mathrm{Lip}_0(M,Y)$ *for every Banach space Y.*

Notice that the examples of metric spaces M such that $\mathrm{LipSNA}(M,\mathbb{R})$ is not dense in $\mathrm{Lip}_0(M,\mathbb{R})$ provided by Theorem 3.3 (and so by [50, Theorem 2.3]) have very strong topological properties. For instance, it is clear that length metric spaces are arc-connected and, in particular, do not have isolated points. Nevertheless, Corollary 3.8 produces quite different kind of such examples. For instance, let C be the Cantor set considered in the proof of Example 3.1. Corollary 3.8 implies that $\mathrm{LipSNA}(C,\mathbb{R})$ is not dense in $\mathrm{Lip}_0(C,\mathbb{R})$, and C is totally disconnected.

3.3 New negative examples II. The unit sphere of the Euclidean plane

Our main goal in this section is to show that when we consider as the metric space the unit sphere of $\mathbb{R}^2$, $\mathbb{T}=\{x\in\mathbb{R}^2\colon \|x\|_2=1\}$, endowed with the Euclidean metric, then $\mathrm{LipSNA}(\mathbb{T},\mathbb{R})$ is not dense in $\mathrm{Lip}_0(\mathbb{T},\mathbb{R})$.

We have already commented the close relationship between Linsdenstrauss property A and the extremal structure of a Banach space. Furthermore, this intimate relationship still exists when studying the density of the strongly norm-attaining Lipschitz maps and the extremal structure of Lipschitz-free spaces. For instance, Theorem 2.35 states that if M is a metric space for which $\mathrm{LipSNA}(M,\mathbb{R})$ is dense in $\mathrm{Lip}_0(M,\mathbb{R})$, then $B_{\mathcal{F}(M)}$ is the closed convex hull of its extreme molecules. Even more, Theorem 2.48 states that if we also assume that M is compact, then $B_{\mathcal{F}(M)}$ is the closed convex hull of its strongly exposed molecules.

As an application of these results, we have easily proved in the first section of this chapter that $\mathrm{LipSNA}([0,1],\mathbb{R})$ is not dense in $\mathrm{Lip}_0([0,1],\mathbb{R})$. The case of the unit sphere of the Euclidean plane $\mathbb{T}$ is, however, quite more delicate. This is because the curvature of $\mathbb{T}$ suggests an abundance of strongly exposed points in $B_{\mathcal{F}(\mathbb{T})}$ which could help to get density of $\mathrm{LipSNA}(\mathbb{T},\mathbb{R})$ (for instance, such density would be obtained if $B_{\mathcal{F}(\mathbb{T})}$ were the closed convex hull of a set of *uniformly* strongly exposed points according to Proposition 2.8). Even when $\mathbb{T}$ is compact, this abundance of strongly exposed points does not allow us to use Theorem 2.48. Indeed, take a look to the following result.

Proposition 3.9. $\mathbb{T}$ *endowed with the Euclidean metric is Gromov concave, that is, every molecule of* $\mathcal{F}(\mathbb{T})$ *is a strongly exposed point of* $B_{\mathcal{F}(\mathbb{T})}$*.*

Proof. Consider $m_{x,y}$ a molecule of $\mathcal{F}(\mathbb{T})$. Clearly, we may assume that $y=1$ and $x=\mathrm{e}^{it}$ with $t\in(0,\pi]$ since isometries of $\mathbb{T}$ can be used to carry strongly exposed molecules to strongly exposed molecules by Theorem 1.7. Let us define the continuous function $\phi\colon[-\pi+t/2,t/2]\setminus\{0\}\longrightarrow\mathbb{R}$ given by

$$\phi(s):=\frac{1}{8}\left|\frac{\mathrm{e}^{is}-1}{|\,\mathrm{e}^{is}-1|}-\frac{\mathrm{e}^{it}-1}{|\,\mathrm{e}^{it}-1|}\right|^2.$$

A simple calculation shows that $\varepsilon:=\inf\bigl\{\phi(s)\colon s\in[-\pi+t/2,t/2]\setminus\{0\}\bigr\}>0$. We claim that

$$(x,y)_z\geqslant\varepsilon\min\{d(x,z),d(y,z)\}$$

for every $z \in \mathbb{T} \setminus \{x, y\}$ and so the pair (x, y) fails property (Z). Indeed, let $z = \mathrm{e}^{is}$. By symmetry, we may assume that $s \in [-\pi + t/2, t/2] \setminus \{0\}$ and so,

$$\min\{d(x, z), d(y, z)\} = d(y, z).$$

Now, Clarkson's inequality [28, Theorem 3] yields that

$$|\,\mathrm{e}^{it} - 1| \leqslant (1 - 2\delta(\alpha_1))|\,\mathrm{e}^{it} - \mathrm{e}^{is}\,| + (1 - 2\delta(\alpha_2))|\,\mathrm{e}^{is} - 1|,$$

where

$$\alpha_1 = \left| \frac{\mathrm{e}^{it} - \mathrm{e}^{is}}{|\,\mathrm{e}^{it} - \mathrm{e}^{is}\,|} - \frac{\mathrm{e}^{it} - 1}{|\,\mathrm{e}^{it} - 1|} \right|, \quad \alpha_2 = \left| \frac{\mathrm{e}^{is} - 1}{|\,\mathrm{e}^{is} - 1|} - \frac{\mathrm{e}^{it} - 1}{|\,\mathrm{e}^{it} - 1|} \right|,$$

and $\delta(u) = 1 - (1 - u^2/4)^{1/2} \geqslant u^2/8$ is the modulus of uniform convexity of $\mathbb{R}^2$. Thus,

$$\begin{aligned} \frac{(x, y)_z}{d(y, z)} &\geqslant \frac{\delta(\alpha_1)|\,\mathrm{e}^{it} - \mathrm{e}^{is}\,| + \delta(\alpha_2)|\,\mathrm{e}^{is} - 1|}{|\,\mathrm{e}^{is} - 1|} \\ &\geqslant \frac{1}{8}\alpha_1^2 \frac{|\,\mathrm{e}^{it} - \mathrm{e}^{is}\,|}{|\,\mathrm{e}^{is} - 1|} + \frac{1}{8}\alpha_2^2 \geqslant \frac{1}{8}\alpha_2^2 = \phi(s) \geqslant \varepsilon, \end{aligned}$$

as desired. □

This proposition makes $\mathbb{T}$ an interesting example to study. On the one hand, if the strong density fails for $\mathbb{T}$, then it would be the first example of a Gromov concave metric space failing to have strong density. This shows that the reciprocal of Theorems 2.35 and 2.48 does not hold. On the other hand, by the last proposition again, the Banach space $\mathcal{F}(\mathbb{T})$ satisfies that the strongly exposed points of its unit ball generate it by closed convex hull. However, if strong density fails, then the set of all strongly exposing functionals cannot be dense in $\mathcal{F}(\mathbb{T})^*$, since each strongly exposing functional of $\mathcal{F}(\mathbb{T})$ is identified with a strongly norm-attaining Lipschitz function. This phenomenon is pretty interesting by itself.

Let us present the main result of the section.

Theorem 3.10. *Let $\mathbb{T}$ be the unit sphere of the Euclidean plane endowed with the inherited Euclidean metric. Then,* $\operatorname{LipSNA}(\mathbb{T}, \mathbb{R})$ *is not dense in* $\operatorname{Lip}_0(\mathbb{T}, \mathbb{R})$.

In order to prove Theorem 3.10, we will need the following key result, which has been suggested to us by F. Nazarov.

Lemma 3.11. *Let $M = ([0, 1], d)$, where $d(x, y) := |\,\mathrm{e}^{ix} - \mathrm{e}^{iy}\,| = \sqrt{2(1 - \cos(x - y))}$. Then, there exists a compact subset C inside the open interval $]0, 1[$ such that the function $f \in \operatorname{Lip}_0(M, \mathbb{R})$ defined by*

$$f(x) := \int_0^x \chi_C(t)\, dt \quad \forall\, x \in [0, 1]$$

is a norm-one Lipschitz function which does not belong to $\operatorname{LipSNA}(M, \mathbb{R})$.

Proof. Consider a Cantor set $C = \bigcap_{n=0}^{\infty} C_n$, where $C_0 = [1/4, 3/4]$ and C_{n+1} is obtained by removing an interval of length $\lambda(I)^2$ at the middle of each connected component I of C_n. Note that C_n has 2^n connected components, all of them with the same length $\frac{\lambda(C_n)}{2^n}$. By construction, $\lambda(C_n \setminus C_{n+1}) = 2^n \left(\frac{\lambda(C_n)}{2^n}\right)^2$. Taking into account that $\lambda(C_n) < \frac{1}{4}$ for $n \geqslant 1$, it follows that

$$\lambda(C) = \frac{1}{2} - \sum_{n=0}^{\infty} \lambda(C_n \setminus C_{n+1}) = \frac{1}{2} - \sum_{n=0}^{\infty} 2^n \left(\frac{\lambda(C_n)}{2^n}\right)^2 > \frac{1}{2} - \sum_{n=0}^{\infty} \frac{1}{4}\frac{1}{2^n} = 0.$$

Consider the Lipschitz function $f\colon ([0, 1], d) \longrightarrow \mathbb{R}$ given by $f(x) = \int_0^x \chi_C(t)\, dt$ for every $x \in [0, 1]$. Note that $\|f\|_L \neq 0$ since $\lambda(C) > 0$. We claim that f does not attain its Lipschitz norm. Indeed, assume that there are $x, y \in [0, 1]$, $x < y$, such that

$$\|f\|_L = \frac{|f(y) - f(x)|}{d(x, y)} = \frac{f(y) - f(x)}{d(x, y)}.$$

Clearly, $x, y \in [1/4, 3/4]$. We claim that $x, y \in C$. Indeed, assume that $x \notin C$. Then there is $0 < \varepsilon < y-x$ such that $(x, x+\varepsilon) \cap C = \emptyset$. Thus, $f(x) = f(x+\varepsilon)$. Then,

$$\|f\|_L = \frac{f(y)-f(x)}{d(x,y)} < \frac{f(y)-f(x+\varepsilon)}{d(x+\varepsilon,y)},$$

a contradiction. So $x \in C$. Analogously, we get that $y \in C$. Now, let n be the maximum integer such that x and y belong to the same connected component I of C_n. Since x and y do not belong to the same connected component of C_{n+1}, there are u, v such that $(u, v) \subset C_n \setminus C_{n+1}$ and $|u-v| = \lambda(I)^2 \geqslant |x-y|^2$. Note also that $(u, v) \cap C = \emptyset$ and so $f(u) = f(v)$. We have

$$\|f\|_L = \frac{f(y)-f(x)}{d(x,y)} = \frac{f(y)-f(v)+f(u)-f(x)}{d(x,y)} \leqslant \|f\|_L \frac{d(y,v)+d(u,x)}{d(x,y)}$$

and so $d(x,y) \leqslant d(y,v) + d(u,x)$. One can routinely check that

$$t - \frac{t^3}{24} \leqslant \sqrt{2(1-\cos(t))} \leqslant t \quad \forall t \in [0,1].$$

Thus,

$$\begin{aligned}(y-x) - \frac{(y-x)^3}{24} &\leqslant \sqrt{2(1-\cos(y-x))} = d(x,y) \leqslant d(y,v) + d(u,x) \\ &= \sqrt{2(1-\cos(y-v))} + \sqrt{2(1-\cos(u-x))} \\ &\leqslant y - v + u - x \leqslant y - x - (y-x)^2.\end{aligned}$$

Therefore, $(y-x)^2 \leqslant \frac{(y-x)^3}{24}$, a contradiction. Thus, f does not attain its Lipschitz norm. It remains to show that $\|f\|_L = 1$. First, note that

$$\|f\|_L = \sup_{x,y\in[0,1]} \frac{|f(x)-f(y)|}{d(x,y)} \geqslant \sup_{x,y\in[0,1]} \frac{|f(x)-f(y)|}{|x-y|} = \|f'\|_\infty = 1.$$

Now, pick a pair of sequences $\{x_n\}, \{y_n\}$ with $x_n \neq y_n$ for every n and such that $\frac{f(x_n)-f(y_n)}{d(x_n,y_n)} \longrightarrow \|f\|_L$. Observe that $d(x_n, y_n) \longrightarrow 0$. Otherwise, we could extract, by compactness, subsequences $\{x_{n_k}\}$ and $\{y_{n_k}\}$ converging to different points x,y in $[0,1]$, so f would attains its Lipschitz norm at the pair (x, y), a contradiction. Consequently, $\frac{|y_n-x_n|}{d(x_n,y_n)} \longrightarrow 1$ and then

$$\lim_{n\to\infty} \widehat{f}(m_{x_n,y_n}) = \lim_{n\to\infty} \frac{f(x_n)-f(y_n)}{d(x_n,y_n)} = \lim_{n\to\infty} \frac{f(x_n)-f(y_n)}{|y_n-x_n|}\frac{|y_n-x_n|}{d(x_n,y_n)} \leqslant 1.$$

Hence, we conclude that $\|f\|_L = 1$. □

We are now able to give the proof of the theorem.

Proof of Theorem 3.10. Let $A \subseteq \mathbb{T}$ be the following arc of $\mathbb{T}$:

$$A = \{e^{it} : t \in [0,1]\}.$$

Let us first show that $\overline{\mathrm{LipSNA}(A,\mathbb{R})} \neq \mathrm{Lip}_0(A,\mathbb{R})$. In order to do so, remember that $\mathrm{Lip}_0([0,1],\mathbb{R})$ is isometrically isomorphic to $L_\infty[0,1]$, where the isometry is given by the derivative operator, and observe that $\Phi\colon \mathrm{Lip}_0(A,\mathbb{R}) \longrightarrow \mathrm{Lip}_0([0,1],\mathbb{R})$ given by

$$[\Phi(f)](t) = f(e^{it}) \quad \forall\, f \in \mathrm{Lip}_0(A,\mathbb{R}), \quad \forall\, t \in [0,1]$$

defines a linear isomorphism. Consequently, a Lipschitz function g will be close to f if, and only if, $\Phi(g)' \in L_\infty[0,1]$ is close to $\Phi(f)'$. Furthermore, we know that there exists a constant $0 < K < 1$ such that

$$K|u-v| \leqslant |\, e^{iu} - e^{iv}\,| \leqslant |u-v| \quad \forall\, u, v \in [0,1].$$

Now, let C be the set given by Lemma 3.11. We define $f \in \mathrm{Lip}_0(A,\mathbb{R})$ by

$$f(\mathrm{e}^{ix}) = \int_0^x \chi_C(t)\,dt \quad \forall\, t \in [0,1].$$

Let us consider $0 < \delta < \frac{K}{2}$ and define $h \in \mathrm{Lip}_0(A,\mathbb{R})$ such that

$$\Phi(h)'(x) = \begin{cases} 1 & \text{if } x \in C, \\ -\delta & \text{if } x \notin C. \end{cases}$$

We will show that if $g \in \mathrm{Lip}_0(A,\mathbb{R})$ verifies that $\|\Phi(g)' - \Phi(h)'\|_\infty < \delta$ and $\|\Phi(g)\|_L = \|\Phi(h)\|_L = 1$, then g does not attain its Lipschitz norm. Firstly, note that if $\|\Phi(g)\|_L = 1$ then $\|g\|_L \geqslant 1$ and $\|\Phi(g)'\|_\infty = 1$, and so

$$\Phi(g)'(x) \in \begin{cases} (1-\delta, 1) & \text{if } x \in C, \\ (-2\delta, 0) & \text{if } x \notin C. \end{cases}$$

Let us prove that $\widehat{g}(m_{\mathrm{e}^{iu},\mathrm{e}^{iv}}) < 1$ for every molecule $m_{\mathrm{e}^{iu},\mathrm{e}^{iv}} \in \mathrm{Mol}(A)$. Let us distinguish two cases:
Case 1*:* $u < v$. In this case we have that

$$\begin{aligned} g(\mathrm{e}^{iu}) - g(\mathrm{e}^{iv}) &= -\int_u^v \Phi(g)'(t)\,dt \leqslant \int_{[u,v]\cap C} \delta - 1\,dt + \int_{[u,v]\setminus C} 2\delta\,dt \\ &\leqslant 2\delta|u-v| < K|u-v| < |\,\mathrm{e}^{iu} - \mathrm{e}^{iv}\,|, \end{aligned}$$

so $\widehat{g}$ cannot attain its norm at the molecule $m_{\mathrm{e}^{iu},\mathrm{e}^{iv}}$.
Case 2*:* $u > v$. In this case we have that

$$\begin{aligned} g(\mathrm{e}^{iu}) - g(\mathrm{e}^{iv}) &= \int_v^u \Phi(g)'(t)\,dt \leqslant \int_{[v,u]\cap C} 1\,dt + \int_{[v,u]\setminus C} 0\,dt \\ &= \int_v^u \chi_C(t)\,dt = \int_v^u \Phi(f)'(t)\,dt \\ &= f(\mathrm{e}^{iu}) - f(\mathrm{e}^{iv}) < \|f\|_L |\,\mathrm{e}^{iu} - \mathrm{e}^{iv}\,| = |\,\mathrm{e}^{iu} - \mathrm{e}^{iv}\,|, \end{aligned}$$

since f does not attain its Lipschitz norm by Lemma 3.11. Consequently $\|g\|_L = 1$ and $\widehat{g}$ does not attain its norm at the molecule $m_{\mathrm{e}^{iu},\mathrm{e}^{iv}}$. By the arbitrariness of u and v, we get that $g \notin \mathrm{LipSNA}(A,\mathbb{R})$.

Consequently, $h \notin \overline{\mathrm{LipSNA}(A,\mathbb{R})}$. Now let us consider an extension of h, say φ, satisfying that $\varphi \notin \overline{\mathrm{LipSNA}(\mathbb{T},\mathbb{R})}$. In order to do so, pick $0 < \eta < 1$ and define

$$\varphi(\mathrm{e}^{ix}) := \begin{cases} h(\mathrm{e}^{ix}) & x \in [0,1], \\ h(\mathrm{e}^{i}) & x \in [1, 1+\eta], \\ -\frac{x}{2} + h(\mathrm{e}^{i}) + \frac{1+\eta}{2} & x \in [1+\eta, 2h(\mathrm{e}^{i}) + 1 + \eta], \\ 0 & x \in [2h(\mathrm{e}^{i}) + 1 + \eta, 2\pi]. \end{cases}$$

It is clear from the definition that $\varphi \in \mathrm{Lip}_0(\mathbb{T},\mathbb{R})$ with $\|\varphi\|_L = \|h\|_L = 1$ and satisfies that, for every sequence of molecules $\{m_{\mathrm{e}^{it_n},\mathrm{e}^{is_n}}\}$ such that $\widehat{\varphi}(m_{\mathrm{e}^{it_n},\mathrm{e}^{is_n}}) \longrightarrow 1$ there exists a natural number m such that $t_n, s_n \in]0,1[$ for all $n \geqslant m$. From this and the fact that $h \notin \overline{\mathrm{LipSNA}(A,\mathbb{R})}$, it follows immediately that $\varphi \notin \overline{\mathrm{LipSNA}(\mathbb{T},\mathbb{R})}$, as desired. □

Before finishing the section, let us say that we do not know if there exists a distance d' on $[0,1]$, equivalent to the usual one, such that $\mathrm{LipSNA}(([0,1],d'),\mathbb{R})$ is dense in $\mathrm{Lip}_0(([0,1],d'),\mathbb{R})$. Observe that Lemma 3.11 and the proof of Theorem 3.10 provide a particular equivalent distance d on $[0,1]$ for which every molecule of $\mathcal{F}([0,1],d)$ is a strongly exposed point, but $\mathrm{LipSNA}(([0,1],d),\mathbb{R})$ is still not dense in $\mathrm{Lip}_0(([0,1],d),\mathbb{R})$. On the other hand, any Hölder distance on $[0,1]$ provides the density (see Corollary 2.25). Although, Hölder distances are not equivalent to the original one.

3.4 New negative examples III. Generalization to C^2 curves

The last section was devoted to proving that for the unit sphere of $\mathbb{R}^2$, endowed with the Euclidean metric, there are Lipschitz functions that cannot be approximated by strongly norm-attaining Lipschitz functions (see Theorem 3.10). The idea behind the proof is that locally, the unit sphere behaves as the segment $[0, 1]$, endowed with the usual metric, and Example 3.1 showed that there is no strong density for this metric space. Indeed, in such an example it is proved that if C is a fat Cantor set and we define the function $f\colon [0, 1] \longrightarrow \mathbb{R}$ by

$$f(t) = \int_0^t \chi_C(s)\, ds \quad \forall\, t \in [0, 1],$$

then it cannot be approximated for strongly norm-attaining Lipschitz functions. When we parametrize $\mathbb{T}$ by arc length, we have that the Euclidean metric is strictly less than the arc length metric, which corresponds to the metric of the interval. Hence, we may find fat Cantor sets for which the function f defined above strongly attains its norm when the Euclidean metric is considered. To solve this problem, we constructed a Cantor set for which the gaps where big enough to beat the curvature of $\mathbb{T}$, and so the function f does not strongly attain its norm. Next, we slightly modified the function to obtain one that cannot be approximated by strongly norm-attaining Lipschitz functions.

The porpoise of this section is to generalize Theorem 3.10 to general curves of enough regularity. We dedicate the first part of this section to present a result that generates Cantor sets depending on some parameters. Then, the generalization of Theorem 3.10 will follow as a direct application of this result.

3.4.1 Density of measurable subsets of $[0, 1]$

Let C be a measurable subset of $[0, 1]$ with positive measure and consider $I \subseteq [0, 1]$ an interval. We are interested in determining how "dense" the set C is in I, that is, determining the value of the quotient

$$\frac{|C \cap I|}{|I|}.$$

It is clear that the value of this quotient depends on the interval. Even if I and I' are intervals of the same length, C may present different density in each of these intervals. However, we want to get a upper bound for the density of the measurable set in terms of the length of the interval, so we can ensure that for an interval I of a fixed length, the density of C in I will be less than some number, no matter where the interval is centered at. For this reason, we consider the function $\phi_C\colon (0, 1] \longrightarrow \mathbb{R}$ given by

$$\phi_C(s) = \sup\left\{ \frac{|C \cap I|}{|I|} \colon I \subseteq [0, 1] \text{ is an interval of length } s \right\} \quad \forall\, s \in (0, 1].$$

First, it is clear that $\phi_C(s) \leqslant 1$ for every $s \in (0, 1]$. On the other hand, it is immediate to verify that $\inf\{\phi_C(s) \colon s \in (0, 1]\} > 0$. Furthermore, this function also satisfies

$$\lim_{s \to 0} \phi_C(s) = 1.$$

This is a consequence of the celebrated Lebesgue's density theorem, which states that for almost every point $x \in C$ we have that

$$\lim_{\delta \to 0} \frac{|C \cap [x - \delta, x + \delta]|}{2\delta} = 1.$$

Our goal is to give a quantitative version of this theorem. In some sense, we want to study how slow this convergence can be. The whole subsection is dedicated to proving the following result.

Theorem 3.12. *Let $f\colon [0, 1] \longrightarrow [0, 1]$ be any function satisfying:*

(i) $f(0) = \lim_{s \to 0} f(s) = 1$.

(ii) $\inf\{f(s) \colon s \in [0, 1]\} > 0$.

(iii) There exists $M > 0$ such that $\frac{1-f(s)}{s} \leqslant M$ for every $s \in (0,1]$.

Then, there exists a measurable set $C \subseteq [0,1]$ of positive measure such that $\phi_C(s) < f(s)$ for every $s \in (0,1]$.

In other words, given a function $f\colon [0,1] \longrightarrow [0,1]$, Theorem 3.12 gives a measurable set $C \subseteq [0,1]$ of positive measure whose density is bounded above by f. In view of the observations that we have made about the function ϕ_C, it is clear that the first two hypotheses of Theorem 3.12 are, in fact, necessary conditions for f to bound the function ϕ_C. Unfortunately, we cannot say the same about the last hypothesis. We found this assumption convenient in order to make Theorem 3.12 useful in the study of strongly norm-attaining Lipschitz functions that we do in subsection 3.4.2. However, we could weaken the condition. Instead of setting the quotient $\frac{|f(s)-1|}{s}$ being bounded, it would be sufficient to have some control on the speed of divergence (see Remark 3.16). Actually, we do not know if the third assumption can be completely removed from the theorem.

In order to prove Theorem 3.12, we will introduce a family of measurable sets depending on some parameters. Furthermore, we will need to understand how they behave with respect to the notion of density that we have introduced. This study will be broken up into several lemmata. To begin with, let us introduce the family of measurable sets.

Consider a sequence of real numbers $\{\lambda_n\}_{n\in\mathbb{N}}$ with $0 < \lambda_n < 1$ for every $n \in \mathbb{N}$. Associated to this sequence, we are going to construct a Cantor set. Consider $C_0 = [0,1]$. Divide C_0 into two pieces: $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$, and consider intervals I_1, I_2, of length $\frac{|C_0|}{2}(1-\lambda_1)$ starting at the starting point of each piece. Then, define $C_1 = I_1 \cup I_2$, that is, $C_1 = [0, \frac{1}{2}(1-\lambda_1)] \cup [\frac{1}{2}, \frac{1}{2} + \frac{1}{2}(1-\lambda_1)]$. Now, divide each connected component of C_1 into two new pieces of the same length and consider in each of them two intervals of length $\frac{|C_1|}{4}(1-\lambda_2)$ starting at the same point as the one of each of the two pieces. Then, define C_2 to be the union of the new intervals that we have constructed. Repeating this process, we construct $C_n \subseteq [0,1]$ as a finite union of closed intervals, for every $n \in \mathbb{N}$. Finally, we define our Cantor set as

$$C = \bigcap_{n\in\mathbb{N}} C_n.$$

See the figure below to get an idea of the shape of this Cantor set.

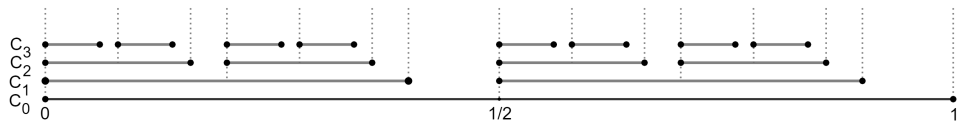

Figure 3.1: Shape of the Cantor set C on its first levels

In order to study the structure of these Cantor sets, let us introduce some notation. For any $n \in \mathbb{N} \cup \{0\}$, it is clear that C_n is the union of 2^n closed intervals of the same length. Define I_n to be the first interval composing C_n. Also, let us write $r_n = |I_n|$, that is, $I_n = [0, r_n]$. Moreover, for each $n \in \mathbb{N}$ let us denote by g_n the length of the gap we find between the interval I_n and the next one inside C_n.

The following lemma gives us some measurements that we are going to need.

Lemma 3.13. *Let $\{\lambda_n\} \subseteq (0,1)$ be a sequence of numbers and let C be the Cantor set associated to $\{\lambda_n\}$. Then,*

(i) $r_n = \frac{1}{2^n}\prod_{i=1}^{n}(1-\lambda_i)$ and $|C_n| = 2^n r_n$ for every $n \in \mathbb{N} \cup \{0\}$, understanding $r_0 = 1$.

(ii) $|C| = \prod_{i=1}^{\infty}(1-\lambda_i)$.

(iii) $g_n = \frac{1}{2} r_{n-1} \lambda_n$.

Proof. To prove (1), let us work by induction over n. For $n = 1$ we have $r_1 = \frac{1}{2}(1-\lambda_1)$, which is true by construction. Now, assume the statement is true for $n \in \mathbb{N}$. By the way we have defined the Cantor set, to obtain I_{n+1} we have to divide I_n into two pieces of the same length, and then consider I_{n+1} to be the interval starting at 0 of length $\frac{|I_n|}{2}(1-\lambda_{n+1})$, that is,

$$I_{n+1} = \left[0, \frac{|I_n|}{2}(1-\lambda_{n+1})\right].$$

Therefore,

$$r_{n+1} = |I_{n+1}| = \frac{|I_n|}{2}(1-\lambda_{n+1}) = \frac{r_n}{2}(1-\lambda_{n+1}) = \frac{1}{2^{n+1}}\prod_{i=1}^{n+1}(1-\lambda_i).$$

Now, notice that C_n is composed of 2^n closed intervals of the same length, and we have just calculated the length of I_n. In view of this, it is clear that

$$|C_n| = 2^n r_n = \prod_{i=1}^{n}(1-\lambda_i) \quad \forall\, n \in \mathbb{N}.$$

To show (2) we just need to observe that $C_{n+1} \subseteq C_n$ for every $n \in \mathbb{N} \cup \{0\}$. Moreover, $|C_0| < \infty$. Hence,

$$|C| = \lim_{n\to\infty} |C_n| = \prod_{i=1}^{\infty}(1-\lambda_i).$$

Finally, to prove (3) notice that $r_n + g_n + r_n + g_n = r_{n-1}$ for any $n \in \mathbb{N}$, from where we deduce that

$$\begin{aligned} g_n &= \frac{1}{2}(r_{n-1} - 2r_n) = \frac{1}{2}\left(\frac{1}{2^{n-1}}\prod_{i=1}^{n-1}(1-\lambda_i) - \frac{2}{2^n}\prod_{i=1}^{n}(1-\lambda_i)\right) \\ &= \frac{1}{2}\left(\frac{1}{2^{n-1}}\left(\prod_{i=1}^{n-1}(1-\lambda_i)\right)(1-(1-\lambda_n))\right) = \frac{1}{2}r_{n-1}\lambda_n. \end{aligned}$$ □

Given a sequence of numbers $\{\lambda_n\} \subseteq (0,1)$ and its associated Cantor set $C \subseteq [0,1]$, recall that we want to study the function $\phi_C \colon (0,1] \longrightarrow \mathbb{R}$ given by

$$\phi_C(s) = \sup\left\{\frac{|C \cap I|}{|I|} : I \subseteq [0,1] \text{ is an interval of length } s\right\} \quad \forall\, s \in (0,1].$$

Notice that if we take $s = r_n$, then we clearly have that

$$\phi(r_n) \geqslant \frac{|C \cap I_n|}{|I_n|} = \frac{|C|}{2^n}\frac{1}{|I_n|} = \prod_{i=n+1}^{\infty}(1-\lambda_i) \quad \forall\, n \in \mathbb{N}.$$

The next lemma gives a very explicit formula for the function ϕ_C. As a consequence of it, the above inequality is an equality.

Lemma 3.14. *Let $\{\lambda_n\} \subseteq (0,1)$ be a sequence of numbers and let C be the Cantor set associated to $\{\lambda_n\}$. Then,*

$$\phi_C(s) = \frac{|C \cap [0,s]|}{s} \quad \forall\, s \in (0,1].$$

In particular, ϕ_C is a continuous function.

Proof. If $s = 1$, there is nothing to prove. Fix $s \in (0,1)$ and consider $n \in \mathbb{N}$ so that $r_n \leqslant s < r_{n-1}$, understanding $r_0 = 1$. We want to show that the supremum is attained at the interval $[0,s]$, that is,

$$\frac{|C \cap I|}{s} \leqslant \frac{|C \cap [0,s]|}{s},$$

for any interval $I \subseteq [0,1]$ of length s. Equivalently, we have to show that $|C \cap I| \leqslant |C \cap [0,s]|$.

Claim: Let $s \in (0,1)$, let $I \subseteq [0,1]$ be an interval of length s, and pick $n \in \mathbb{N}$ such that $r_n \leqslant s < r_{n-1}$. Then, there exists an interval I^* of length $s^* < r_n$ satisfying

$$|C \cap I| - |C \cap [0,s]| \leqslant |C \cap I^*| - |C \cap [0,s^*]|.$$

Assume the above claim is true. Then, we can apply it k times to obtain an interval I_k^* of length $s_k^* < r_{n+k-1}$ satisfying

$$|C \cap I| - |C \cap [0,s]| \leqslant |C \cap I_k^*| - |C \cap [0,s_k^*]|.$$

Now, notice that $|C \cap I_k^*| - |C \cap [0,s_k^*]| \leqslant |C \cap I_k^*| < r_{n+k-1}$, from where we deduce that

$$|C \cap I| - |C \cap [0,s]| < r_{n+k-1}.$$

Since $\{r_n\} \longrightarrow 0$ as n goes to infinity, we conclude that $|C \cap I| \leqslant |C \cap [0,s]|$.

Let us prove the claim. We distinguish four cases:

- Case 1: I does not intersect any connected component of C_n. This means that I lies on a gap, and so $C \cap I = \emptyset$. Therefore, $|C \cap I| = 0$, so we can take I^* to be any interval of length smaller than r_n.
- Case 2: I intersects exactly one connected component of C_n. Let us denote by J such connected component of C_n. Then, we must have that $I \setminus J$ lies on a gap. Recall that $I_n = [0, r_n]$ and we are assuming $s \geqslant r_n$, which implies that

  $$|C \cap I| = |C \cap (I \cap J)| \leqslant |C \cap J| = |C \cap I_n| \leqslant |C \cap [0,s]|.$$

  Therefore, $|C \cap I| - |C \cap [0,s]| \leqslant 0$. In view of this, we can set $I^* = I_{n+1} = [0, r_{n+1}]$.
- Case 3: I intersects exactly two connected components of C_n. Going from left to right, denote such intervals by J_1 and J_2. Also, let us write $I \cap J_1 = [a_1, b_1]$ and $I \cap J_2 = [a_2, b_2]$. By the shape of the Cantor set C, observe that we may identify $I \cap J_2$ with a subinterval of I_n starting at 0. More precisely,

  $$|C \cap (I \cap J_2)| = |C \cap [a_2, b_2]| = |C \cap [0, b_2 - a_2]|,$$

  Analogously, we may identify $I \cap J_1$ with a subinterval of I_n ending at r_n. We just need to notice that

  $$|C \cap (I \cap J_1)| = |C \cap [a_1, b_1]| = |C \cap [r_n - (b_1 - a_1), r_n]|.$$

  Let us write $l = (b_1 - a_1) + (b_2 - a_2)$. Then, notice that if $l < r_n$ we have that $I \cap J_1$ and $I \cap J_2$ can be identified with two subintervals of I_n which do not overlap. Therefore, we would have

  $$\begin{aligned} |C \cap I| &= |C \cap (I \cap J_1)| + |C \cap (I \cap J_2)| \\ &= |C \cap [0, b_2 - a_2]| + |C \cap [r_n - (b_1 - a_1), r_n]| \\ &\leqslant |C \cap I_n| = |C \cap [0, r_n]| \leqslant |C \cap [0,s]|, \end{aligned}$$

  from where we deduce that $|C \cap I| - |C \cap [0,s]| \leqslant 0$, so we can set $I^* = I_{n+1}$ as in case 2. Moreover, in the case $l = r_n$ we can identify $I \cap J_1$ and $I \cap J_2$ with two subintervals of I_n that overlap only at one point, so we reach the same conclusion. Then, we may assume that $l > r_n$, so the intervals $[0, b_2 - a_2]$ and $[r_n - (b_1 - a_1), r_n]$ overlap. Let us write

  $$I_0^* = [0, b_2 - a_2] \cap [r_n - (b_1 - a_1), r_n] = [r_n - (b_1 - a_1), b_2 - a_2].$$

  Set $s^* = (b_2 - a_2) - (r_n - (b_1 - a_1)) = l - r_n$ to be the length of the above interval. We know that $s \geqslant l + g_n$, and $l > r_n$. Then, we have $s - (r_n + g_n) \geqslant s^*$. If $s^* \geqslant r_n$ then we would have that $s - (r_n + g_n) \geqslant r_n$, and so $s \geqslant 2r_n + g_n$. However, this implies that

  $$|C \cap [0,s]| \geqslant 2|C \cap I_n| \geqslant |C \cap I|.$$

  Therefore, $|C \cap I| - |C \cap [0,s]| \leqslant 0$, so we can set $I^* = I_{n+1}$. Consequently, we may suppose $s^* < r_n$. One the one hand, notice that

  $$|C \cap I| = |C \cap (I \cap J_1)| + |C \cap (I \cap J_2)| = |C \cap I_n| + |C \cap I_0^*|.$$

On the other hand,

$$|C \cap [0,s]| = |C \cap I_n| + |C \cap [r_n + g_n, s]| = |C \cap I_n| + |C \cap [0, s - (r_n + g_n)]|.$$

Moreover, since $s - (r_n + g_n) \geqslant s^*$, we have

$$|C \cap [0,s]| \geqslant |C \cap I_n| + |C \cap [0, s^*]|,$$

from where we deduce that

$$|C \cap I| - |C \cap [0,s]| \leqslant |C \cap I_0^*| - |C \cap [0, s^*]|.$$

Therefore, we can set $I^* = I_0^*$.

- Case 4: I intersects three or more connected components of C_n. First, I cannot intersect four connected components, since in such case we would have that

$$s \geqslant r_n + g_n + r_n + g_n = r_{n-1}.$$

Hence we know that I intersects exactly three connected components of C_n. Going from left to right, denote such intervals by J_1, J_2, and J_3. If the intersection with any of them is just a point, then in terms of Lebesgue measure, we may assume that I actually intersects only two intervals, case that we have already studied. Let us write $I \cap J_1 = [a_1, b_1]$, $I \cap J_3 = [a_3, b_3]$. It is clear that $I \cap J_2 = J_2$ and so we can decompose as follows:

$$\begin{aligned} |C \cap I| &= |C \cap (I \cap J_1)| + |C \cap (I \cap J_2)| + |C \cap (I \cap J_3)| \\ &= |C \cap (I \cap J_1)| + |C \cap I_n| + |C \cap (I \cap J_3)|. \end{aligned}$$

Let us write $l = (b_1 - a_1) + (b_3 - a_3)$. Then we must have that $l < r_n$. In fact, notice that

$$s \geqslant (b_1 - a_1) + g_n + r_n + g_n + (b_3 - a_3) = l + r_n + 2g_n.$$

If we suppose $l \geqslant r_n$, then $s \geqslant 2r_n + 2g_n = r_{n-1}$, which is a contradiction. In view of this, notice that we can identify $I \cap J_3$ with a subinterval of $[r_n + g_n, 2r_n + g_n]$ of the same length as $I \cap J_3$ starting at $r_n + g_n$. More precisely, we have

$$|C \cap (I \cap J_3)| = |C \cap [a_3, b_3]| = |C \cap [r_n + g_n, r_n + g_n + (b_3 - a_3)]|,$$

Analogously, notice that we can identify $I \cap J_1$ with a subinterval of I_n ending at r_n of the same length as $I \cap J_1$, that is,

$$|C \cap (I \cap J_1)| = |C \cap [a_1, b_1]| = |C \cap [r_n - (b_1 - a_1), r_n]|.$$

Then, if we consider a new interval $I^* = [r_n - (b_1 - a_1), r_n + g_n + (b_3 - a_3)]$ we will have that

$$|C \cap (I \cap J_1)| + |C \cap (I \cap J_3)| = |C \cap I^*|,$$

from where we deduce that

$$|C \cap I| = |C \cap I_n| + |C \cap I^*|.$$

On the other hand, since I intersects more than two connected components of C_n, we clearly have $s \geqslant r_n + g_n$. Thus,

$$|C \cap [0,s]| = |C \cap I_n| + |C \cap [r_n + g_n, s]| = |C \cap I_n| + |C \cap [0, s - (r_n + g_n)]|.$$

We claim that $s - (r_n + g_n) \geqslant |I^*|$. Indeed, if we suppose $s - (r_n + g_n) < |I^*|$ we would have

$$s - (r_n + g_n) < (r_n + g_n + (b_3 - a_3)) - (r_n - (b_1 - a_1)),$$

from where we obtain

$$s < r_n + 2g_n + l,$$

but we have already seen that in fact $s \geqslant r_n + 2g_n + l$. Hence,

$$|C \cap [0, s]| \geqslant |C \cap I_n| + |C \cap [0, |I^*|]|.$$

Consequently, we obtain that

$$|C \cap I| - |C \cap [0, s]| \leqslant |C \cap I^*| - |C \cap [0, |I^*|]|.$$

Since $l < r_n$ then I^* is an interval which only intersects two connected components, so this case reduces to case 3, so the claim is proved.

Once we know that $\phi_C(s)$ can be expressed as $\frac{|C\cap[0,s]|}{s}$, it is trivial to verify that ϕ_C is a continuous function. It follows from the fact that the function $s \longmapsto |C \cap [0, s]|$ is Lipschitz, since

$$|C \cap [0, s]| = \int_0^s \chi_C(t)\, dt \quad \forall\, s \in [0, 1].$$ □

Notice that the last results do not need any assumptions on the sequence of numbers $\{\lambda_n\}$. We need more information about the behavior of the function ϕ_C, but in order to obtain it we need to pick the sequence $\{\lambda_n\}$ satisfying some conditions. However, we will see that these conditons are not very restrictive.

We are interested in the case when the Cantor set C associated to a sequence $\{\lambda_n\} \subseteq (0, 1)$ has positive measure. In view of Lemma 3.13, this will happen when $\prod_{n=1}^{\infty}(1 - \lambda_n) > 0$. Equivalently, it will happen when $\sum_{n=1}^{\infty} \lambda_n < \infty$. In particular, the sequence $\{\lambda_n\}$ must converge to zero. For the next lemma we need to assume that such a convergence is monotone.

Lemma 3.15. *Let $\{\lambda_n\} \subseteq (0, 1)$ be a decreasing sequence of numbers and let C be the Cantor set associated to $\{\lambda_n\}$. Pick $s \in (0, 1)$, and consider $n \in \mathbb{N}$ so that $r_n \leqslant s < r_{n-1}$. Then,*

$$\phi_C(s) \leqslant \phi_C(r_n).$$

Proof. In view of Lemma 3.14, it will be enough to show that

$$\frac{|C \cap [0, s]|}{s} \leqslant \frac{|C \cap [0, r_n]|}{r_n}.$$

In order to do so, it will be enough to verify this for s so that $s \notin C$, since $[0, 1] \setminus C$ is dense in $[0, 1]$ and ϕ_C is a continuous function. Pick $s \in (0, 1) \setminus C$. Then, take the smallest integer $k \in \mathbb{N}$ so that $s \notin C_k$. Let us consider $n \in \mathbb{N}$ so that $r_n \leqslant s < r_{n-1}$. Clearly, we must have that $k \geqslant n$. Moreover, we know that I_n is formed by 2^{k-n} copies of I_k and gaps, so we have

$$|C \cap I_n| = |C \cap [0, r_n]| = 2^{k-n}|C \cap I_k|. \tag{3.2}$$

Furthermore, since

$$r_{n+i} = 2r_{n+i+1} + 2g_{n+i+1} = 4r_{n+i+2} + 4g_{n+i+2} + 2g_{n+i+1} = \dots$$

we obtain that

$$r_{n+i} = 2^{k-(n+i)}r_k + \sum_{j=i+1}^{k-n} 2^{j-i} g_{n+j}. \quad \forall\, i \in \{0, \dots, k-n\}.$$

Let us write

$$G_{n+i} = \sum_{j=i+1}^{k-n} 2^{j-i} g_{n+j} \quad \forall\, i \in \{0, \dots, k-n\}$$

and notice that it represents the measure of all gaps in I_{n+i} of order less or equal than k, understanding $G_k = 0$. On the other hand, we know that s lies on a gap of C_k. Going from left to right, let $J = [a, b]$ be the interval of C_k that we find just before such a gap. Clearly, if $s' \in (0, 1)$ lies on the closure of the same gap as s we will have that $|C \cap [0, s]| = |C \cap [0, s']|$, so the worst case that we may consider is when

$s' = b$, since in such a case we also have $|C \cap [0,s]| = |C \cap [0,b]|$ and clearly $b \leqslant s$, so $\phi_C(b) \geqslant \phi_C(s)$. In this case, we can write b as

$$b = r_k + \sum_{i=0}^{k-n} \theta_i (r_{n+i} + g_{n+i}) \tag{3.3}$$

for some $\theta_i \in \{0,1\}$ for $i = 0, \ldots, k-n$. Notice that if $b = r_n$ there is nothing to prove, so we may assume $b > r_n$ and consequently $\theta_0 = 1$. Now, using the last equality we obtain

$$\begin{aligned} b &= r_k + \sum_{i=0}^{k-n} \theta_i (r_{n+i} + g_{n+i}) = r_k + \sum_{i=0}^{k-n} \theta_i \left(2^{k-(n+i)} r_k + G_{n+i} + g_{n+i}\right) \\ &= r_k + r_k \left(\sum_{i=0}^{k-n} \theta_i 2^{k-(n+i)}\right) + \sum_{i=0}^{k-n} \theta_i (G_{n+i} + g_{n+i}). \end{aligned}$$

Hence, we can decompose the interval $[0,b]$ as following:

$$[0,b] = [0, r_n + g_n] \bigcup_{i=1}^{k-n-1} \left[\sum_{j=1}^{i} \theta_j (r_{n+j} + g_{n+j}), \sum_{j=1}^{i+1} \theta_j (r_{n+j} + g_{n+j})\right] \cup \left[\sum_{j=1}^{k-n} \theta_j (r_{n+j} + g_{n+j}), b\right].$$

In view of this, $[0, r_{n+i} + g_{n+i}]$ has $2^{k-(n+i)}$ copies of the interval I_k for every $i = 0, \ldots, k-n$, we deduce that

$$|C \cap [0,b]| = |C \cap I_k| \left(1 + \sum_{i=0}^{k-n} \theta_i 2^{k-(n+i)}\right). \tag{3.4}$$

Recall that we want to show that

$$r_n |C \cap [0,b]| \leqslant b |C \cap [0, r_n]|.$$

Using equalities (3.2), (3.3), and (3.4) and simplifying the obtained expression, we can rewrite this inequality as

$$G_n \left(1 + \sum_{i=0}^{k-n} \theta_i 2^{k-(n+i)}\right) \leqslant \left(\sum_{i=0}^{k-n} \theta_i (G_{n+i} + g_{n+i})\right) 2^{k-n}.$$

Equivalently, it will be enough to show that

$$\sum_{i=0}^{k-n} 2^{k-(n+i)} \theta_i (2^i (G_{n+i} + g_{n+i}) - G_n) \geqslant G_n.$$

Notice that $G_m = 2G_{m+1} + 2g_{m+1}$ for every integer m between n and k. Therefore,

$$G_n = 2^i G_{n+i} + \sum_{j=1}^{i} 2^j g_{n+j}, \quad \forall\, i \in \{1, \ldots, k-n\}.$$

From here, for any $i \in \{1, \ldots, k-n\}$ we deduce that

$$\begin{aligned} 2^i (G_{n+i} + g_{n+i}) - G_n &= 2^i (G_{n+i} + g_{n+i} - G_{n+i}) - \sum_{j=1}^{i} 2^j g_{n+j} \\ &= 2^i g_{n+i} - \sum_{j=1}^{i} 2^j g_{n+j} = -\sum_{j=1}^{i-1} 2^j g_{n+j}. \end{aligned}$$

Consequently, we need to show that

$$2^{k-n} g_n - \sum_{i=1}^{k-n} \left(2^{k-(n+i)} \theta_i \left(\sum_{j=1}^{i-j} 2^j g_{n+j}\right)\right) \geqslant G_n.$$

In view of this, it will be enough to study the case when $\theta_i = 1$ for every $i = 0, \dots, k-n$. In such case, we have that $b = r_{n-1} - \sum_{i=0}^{k-n} g_{n+i}$ and $|C \cap [0,b]| = |C \cap I_{n-1}|$. Recall that we need to show that

$$r_n |C \cap [0,b]| \leqslant b |C \cap [0, r_n]|,$$

which in this case can be written as

$$r_n r_{n-1} \prod_{i=n}^{\infty} (1-\lambda_i) \leqslant \left(r_{n-1} - \sum_{i=0}^{k-n} g_{n+i} \right) r_n \prod_{i=n+1}^{\infty} (1-\lambda_i).$$

Equivalently, we need to show that

$$r_{n-1}(1-\lambda_n) \leqslant r_{n-1} - \sum_{i=0}^{k-n} g_{n+i},$$

but this is the same as

$$\sum_{i=0}^{k-n} g_{n+i} \leqslant \lambda_n r_{n-1}.$$

To end the proof, note that by Lemma 3.13 we have $g_i = \lambda_i \frac{r_{i-1}}{2}$ for every $i \in \mathbb{N}$. Therefore, since $\lambda_{i+1} \leqslant \lambda_i$ for every $i \in \mathbb{N}$, we get

$$\frac{g_{i+1}}{g_i} = \frac{\lambda_{i+1}}{\lambda_i} \frac{r_i}{r_{i-1}} \leqslant \frac{r_i}{r_{i-1}} = \frac{\frac{1}{2^i} \prod_{j=1}^{i} (1-\lambda_j)}{\frac{1}{2^{i-1}} \prod_{j=1}^{i-1} (1-\lambda_j)} = \frac{1}{2}(1-\lambda_i) < \frac{1}{2}.$$

As a consequence of this, we deduce that

$$\sum_{i=0}^{k-n} g_{n+i} < \sum_{i=0}^{\infty} g_{n+i} \leqslant \sum_{i=0}^{\infty} \frac{1}{2^i} g_n = 2 g_n = \lambda_n r_{n-1},$$

as we wanted to prove. □

We are now able to present the proof of Theorem 3.12.

Proof of Theorem 3.12. Let us first prove the theorem in the case when f is decreasing.

Consider the sequence $\lambda_n = \frac{1}{(n+1)^2}$ for every $n \in \mathbb{N}$. Then, we can consider the Cantor set C associated to such a sequence. Since $\{\lambda_n\}_{n=1}^{\infty}$ decreases to 0, we are in the situation of Lemma 3.15. Moreover, since $\sum_{n=1}^{\infty} \lambda_n < \infty$, by Lemma 3.13 we know that $|C| > 0$. Associated to the sequence $\{\lambda_n\}_{n=1}^{\infty}$, we have the sequence $\{r_k\}_{k=0}^{\infty}$ that satisfies

$$\phi_C(r_k) = \prod_{n=k+1}^{\infty} (1-\lambda_n) = \prod_{n=k+1}^{\infty} \frac{(n+1)^2 - 1}{(n+1)^2} = \prod_{n=k+1}^{\infty} \frac{(n+2)n}{(n+1)^2} = \frac{k+1}{k+2} \quad \forall\, k \in \mathbb{N}.$$

Now, we claim that the quotient $\frac{1-\phi_C(r_k)}{r_k}$ diverges as k goes to ∞. Indeed, notice that

$$\frac{1-\phi_C(r_k)}{r_k} = \frac{\frac{1}{k+2}}{\frac{1}{2^k} \prod_{n=1}^{k} (1-\lambda_n)} \geqslant \frac{2^k}{k+2} \quad \forall\, k \in \mathbb{N}.$$

Therefore, we may pick $k_0 \in \mathbb{N}$ such that $\frac{1-\phi_C(r_k)}{r_k} > 4M$ for every $k \geqslant k_0$. Then, we have that $\phi_C(r_k) < 1 - 4Mr_k$ for every $k \geqslant k_0$. On the other hand, observe that

$$\frac{r_k}{r_{k-1}} = \frac{1}{2}(1-\lambda_k) > \frac{1}{2}\frac{1}{2} = \frac{1}{4} \quad \forall\, k \in \mathbb{N}.$$

Then, for every $k \geqslant k_0$ we obtain that $\phi_C(r_k) < 1 - 4Mr_k < 1 - Mr_{k-1}$. On the other hand, by hypothesis we have

$$\frac{1 - f(r_{k-1})}{r_{k-1}} \leqslant M \quad \forall\, k \in \mathbb{N},$$

from where we deduce that $f(r_{k-1}) \geqslant 1 - Mr_{k-1}$. Hence, $\phi_C(r_k) < f(r_{k-1})$ for every $k \geqslant k_0$.

Now, by hypothesis $\inf\{f(s)\colon s \in [0,1]\} > 0$. Let us denote such an infimum by θ. Then, pick $\delta = \theta r_{k_0}$ and define $C' = C \cap [0,\delta]$. Let us show that C' a measurable set for which the statement of the theorem is satisfied. First, it is clear that $|C'| > 0$. Also, since $C' \subseteq C$, it is clear that $\phi_{C'} \leqslant \phi_C$. Thus, for $k \geqslant k_0$ we have that $\phi_{C'}(r_k) \leqslant \phi_C(r_k) < f(r_{k-1})$. Now, for $k < k_0$ we have

$$\phi_{C'}(r_k) \leqslant \frac{\delta}{r_k} < \frac{\delta}{r_{k_0}} = \theta \leqslant f(r_{k-1}).$$

In conclusion, we have that $\phi_{C'}(r_k) < f(r_{k-1})$ for every $k \in \mathbb{N}$. Now, pick $s \in (0,1]$. If $s = 1$, we have already seen that $\phi_{C'}(1) < \theta \leqslant f(1)$, so we may assume that $s \neq 1$. Then, we find $n \in \mathbb{N}$ so that $r_n \leqslant s < r_{n-1}$. We just need to notice that

$$\phi_{C'}(s) \leqslant \phi_{C'}(r_n) < f(r_{n-1}) \leqslant f(s),$$

where the first inequality comes from Lemma 3.15 and the last inequality comes from the fact that f is decreasing.

We now proceed to prove the general case. Consider g to be any function satisfying the hypotheses of the theorem. Define a function $h\colon [0,1] \longrightarrow \mathbb{R}$ by

$$h(s) = \inf\{g(x)\colon x \in [0,s]\} \quad \forall\, s \in [0,1].$$

Then, it is clear that h is decreasing. Moreover, h also satisfies the hypotheses of the theorem. Indeed, it is clear that $\lim_{s\to 0} h(s) = \lim_{s\to 0} g(s) = 1$. Also,

$$\inf\{h(s)\colon s \in [0,1]\} = h(1) = \inf\{g(s)\colon s \in [0,1]\} > 0.$$

Finally, given $s \in (0,1]$ there exists $x \in (0,s]$ such that $|h(s) - g(x)| \leqslant s$. Then, notice that

$$\frac{1-h(s)}{s} = \frac{|(h(s)-g(x)) + (g(x)-1)|}{s} \leqslant \frac{|h(s)-g(s)|}{s} + \frac{1-g(x)}{x} \leqslant 1 + M.$$

Therefore, we can apply the last case of the proof to the function h to obtain a Cantor set $C' \subseteq [0,1]$ of positive measure so that $\phi_{C'}(s) < h(s)$ for every $s \in (0,1]$. To end the proof we just have to notice that $h(s) \leqslant g(s)$ for every $s \in [0,1]$. □

Remark 3.16. Let us make an observation. Essentially, we are considering only one Cantor set, the one associated to the sequence $\lambda_n = \frac{1}{(n+1)^2}$ for every $n \in \mathbb{N}$. This is because we are assuming that the quotient $\frac{1-f(s)}{s}$ is bounded in $(0,1]$, so it was sufficient for us to find a sequence $\{\lambda_n\}$ for which the quotient $\frac{1-\phi_C(r_k)}{r_k}$ diverges as k goes to infinity. In that way, we could guarantee that for r_k small enough we have

$$\frac{1-\phi_C(r_k)}{r_k} > \frac{2}{(1-\lambda_1)} \frac{1-f(r_{k-1})}{r_{k-1}}, \tag{3.5}$$

from here, together with the fact that $\lambda_k \leqslant \lambda_1$, we deduce that

$$\phi_C(r_k) < 1 - \frac{2r_k}{(1-\lambda_1)} \frac{1-f(r_{k-1})}{r_{k-1}} \leqslant f(r_{k-1}).$$

However, assuming that $\frac{1-f(s)}{s}$ is bounded is very strong. We would obtain the same result as long as (3.5) is satisfied. For this reason, we believe that it is possible to get a stronger result than Theorem 3.12. Indeed, even if the quotient $\frac{1-f(s)}{s}$ is not bounded, in many cases should be possible to find a sequence $\{\lambda_n\}$ (depending on f) decreasing to 0 slow enough to obtain that (3.5) holds, which was the key point of the proof. In fact, maybe this argument can be made for any function f satisfying only the first two conditions of Theorem 3.12, which are indeed necessary conditions.

3.4.2 Strong density on C^2 curves

Our main goal in this subsection is to give a generalization of Theorem 3.10 for curves of enough regularity. More concretely, we prove that if $\alpha\colon J \longrightarrow E$ is any C^2 curve parametrized by arc length, for some normed space E and some interval $J \subseteq \mathbb{R}$, and $\Gamma = \{\alpha(t)\colon t \in \mathbb{R})\}$ is its range, then $\operatorname{LipSNA}(\Gamma, \mathbb{R})$ is not dense in $\operatorname{Lip}_0(\Gamma, \mathbb{R})$. In fact, we will show that it is sufficient to assume that there exists an interval $I \subseteq J$ such that if we restrict the curve α to I then it is C^2. This result broadly expands the metric spaces for which it is known that strong density fails.

In order to prove it, we will need some previous lemmata. The next easy result is a basic fact about C^2 maps. We include the proof since we were not able to find a reference for it.

Lemma 3.17. *Let E be a normed space, let $I \subseteq \mathbb{R}$ be an interval, and let $f\colon I \longrightarrow E$ be a C^2 map. Then, the mapping $\Phi\colon I \times I \longrightarrow \mathbb{R}^d$ defined by*

$$\Phi(t,s) = \begin{cases} \frac{f(t)-f(s)}{t-s} & \text{if } t,s \in I \text{ with } t \neq s; \\ f'(t) & \text{if } t,s \in I \text{ with } t = s, \end{cases}$$

is continuous.

Proof. Let us denote $\Lambda = \{(t,t) \in I \times I\colon t \in I\}$. Since f is differentiable, it is clearly continuous, so Φ is trivially continuous at every point of $I \times I \setminus \Lambda$. Now, pick $t_0 \in I$. In order to show that Φ is continuous at the point (t_0, t_0), consider a sequence $\{(t_n, s_n)\} \subseteq I \times I$ converging to (t_0, t_0). Since continuity at t_0 is a local property, we may assume that the interval I is compact. First, assume that $t_n \neq s_n$ eventually. Let us write $p_n = \frac{t_n+s_n}{2}$ for every $n \in \mathbb{N}$. Let $M = \sup\{\|f'(t)\|\colon t \in I\}$. Then, applying Taylor's formula we have that

$$f(p_n + h) = f(p_n) + hf'(p_n) + R_1(h),$$

as long as $p_n + h$ lies in I, where the reminder satisfies $\|R_1(h)\| \leqslant \frac{M}{2}|h|^2$. Applying this formula with $h_1 = (t_n - p_n)$ and $h_2 = (s_n - p_n)$ we get

$$f(t_n) - f(s_n) = (t_n - s_n)f'(p_n) + R_1(t_n - p_n) - R_1(s_n - p_n).$$

Then, notice that

$$\begin{aligned} \|\Phi(t_n, s_n) - f'(t_0)\| &= \left\| \frac{f(t_n) - f(s_n)}{t_n - s_n} - f'(t_0) \right\| \leqslant \left\| \frac{f(t_n) - f(s_n)}{t_n - s_n} - f'(p_n) \right\| + \|f'(p_n) - f'(t_0)\| \\ &= \frac{\|R_1(t_n - p_n) - R_1(s_n - p_n)\|}{|t_n - s_n|} + \|f'(p_n) - f'(t_0)\| \\ &\leqslant \frac{M|t_n - p_n|^2}{|t_n - s_n|} + \|f'(p_n) - f'(t_0)\| = \frac{M}{4}|t_n - s_n| + \|f'(p_n) - f'(t_0)\|. \end{aligned}$$

The result follows from the continuity of f' together with the fact that $\{t_n - s_n\} \longrightarrow 0$. Now, assume $t_n = s_n$ for large enough $n \in \mathbb{N}$. Then, we have

$$\Phi(t_n, s_n) = f'(t_n),$$

that converges to $f'(t_0)$ again since f' is continuous. □

We will also need the following result in order to apply Theorem 3.12.

Lemma 3.18. *Let E be a normed space, let $r > 0$, and let $\alpha\colon [0, r] \longrightarrow E$ be a C^2 curve parametrized by arc length. Consider the function $g\colon (0, r] \longrightarrow \mathbb{R}$ given by*

$$g(x) = \inf \left\{ \frac{\|\alpha(t) - \alpha(s)\|}{|t - s|} \colon t, s \in [0, r], |t - s| = x \right\} \quad \forall\, x \in (0, r].$$

Then, $\lim_{x\to 0} g(x) = 1$.

Proof. First, it is clear that $|g(x)| \leqslant 1$ for every $x \in (0, r]$. Now, assume the statement is not true. Then, we find $\varepsilon > 0$ and sequences $\{t_n\}$, $\{s_n\} \subseteq [0, r]$ with $t_n \neq s_n$ so that $\lim_{n\to\infty} |t_n - s_n| = 0$ and

$$\frac{\|\alpha(t_n) - \alpha(s_n)\|}{|t_n - s_n|} < 1 - \epsilon \quad \forall n \in \mathbb{N}.$$

However, this contradicts the fact that $\|\alpha'(t)\| = 1$ for every $t \in [0, r]$. Indeed, since $[0, r]$ is compact and $|t_n - s_n|$ goes to 0, we find partial sequences $\{t_{\sigma(n)}\}$, $\{s_{\sigma(n)}\}$ converging to a common point $t_0 \in [0, r]$. On the other hand, by Lemma 3.17 we have that the map $\Phi\colon [0, r] \longrightarrow E$ given by

$$\Phi(t,s) = \begin{cases} \frac{\alpha(t)-\alpha(s)}{t-s} & \text{if } t,s \in [0,r] \text{ with } t \neq s; \\ \alpha'(t) & \text{if } t,s \in [0,r] \text{ with } t = s, \end{cases}$$

is continuous. Consequently, $\|\Phi(\cdot,\cdot)\|$ is also continuous, but $\|\Phi(t_0, t_0)\| = \|\alpha'(t_0)\| = 1$, so we must have

$$\lim_{n\to\infty} \frac{\|\alpha(t_{\sigma(n)}) - \alpha(s_{\sigma(n)})\|}{|t_n - s_n|} = 1,$$

which contradicts the assumption. □

Let us make an observation. We have seen that if $\alpha\colon \mathbb{R} \longrightarrow \mathbb{R}^d$ is a C^2 curve parametrized by arc length, then for every $\varepsilon > 0$ there exists an interval $[0, r]$ such that

$$\inf\left\{\frac{\|\alpha(t) - \alpha(s)\|}{|t - s|} : t, s \in [0, r], t \neq s\right\} > 1 - \varepsilon.$$

Therefore, given $\gamma \in (0, 1)$, we find an interval $[0, r]$ so that

$$\gamma|t - s| \leqslant \|\alpha(t) - \alpha(s)\| \leqslant |t - s| \quad \forall\, t, s \in [0, r].$$

Now, we can consider the linear operator $\Phi\colon \mathrm{Lip}_0([0, r], \mathbb{R}) \longrightarrow \mathrm{Lip}_0(\Gamma, \mathbb{R})$ given by

$$\Phi(f)(\alpha(t)) = f(t) \quad \forall\, t \in [0, r], \quad \forall\, f \in \mathrm{Lip}_0([0, r], \mathbb{R}),$$

where $\Gamma = \{\alpha(t) : t \in [0, r]\}$. Let us verify that Φ is a linear isomorphism. First, it is clear that Φ is linear. Also, notice that

$$\|\Phi(f)\|_L = \sup\left\{\frac{|f(t) - f(s)|}{\|\alpha(s) - \alpha(t)\|} : s, t \in [0, r]\right\} \leqslant \frac{1}{\gamma} \sup\left\{\frac{|f(t) - f(s)|}{|t - s|} : s, t \in [0, r]\right\} = \frac{1}{\gamma}\|f\|_L,$$

from where we deduce that Φ is continuous. It is easy to verify that $\|f\|_L \leqslant \|\Phi(f)\|_L$ also holds. Furthermore, it is clear that the inverse of Φ is $\Phi^{-1}\colon \mathrm{Lip}_0(\Gamma, \mathbb{R}) \longrightarrow \mathrm{Lip}_0([0, r], \mathbb{R})$ given by

$$\Phi^{-1}(g)(t) = g(\alpha(t)) \quad \forall\, t \in [0, r], \quad \forall\, g \in \mathrm{Lip}_0(\Gamma, \mathbb{R}).$$

Hence, Φ is a linear isomorphism.

The next result makes possible to localize the condition of satisfying the strong density. It shows that, under some assumptions, it is enough to find a subset failing to have strong density to guarantee that the whole metric space does. It will be very useful when proving our main result in this section, since we can restrict our curve to a smaller curve having a simpler behavior.

Lemma 3.19. *Let M be a metric space and let N be subset of M with nonempty interior. Assume that there exist $f_0 \in \mathrm{Lip}_0(N, \mathbb{R})$ with $\|f_0\|_L = 1$ and p_0 an interior of N such that $f_0 \notin \overline{\mathrm{LipSNA}(N, \mathbb{R})}$ and the restriction of f_0 to the ball $B(p_0, r)$ has norm one for every $r \in \mathbb{R}^+$. Then,* $\mathrm{LipSNA}(M, \mathbb{R})$ *is not dense in* $\mathrm{Lip}_0(M, \mathbb{R})$.

Proof. We may and do assume that the base point of our metric spaces N and M is the point p_0 given by the hypothesis. Let $f_0 \in \mathrm{Lip}_0(N, \mathbb{R})$ be the function given by the hypothesis. Pick $\delta > 0$ such that for every $g_0 \in \mathrm{Lip}_0(N,\mathbb{R})$ satisfying $\|f_0 - g_0\|_L < \delta$, we have $g_0 \notin \overline{\mathrm{LipSNA}(N,\mathbb{R})}$. Consider $r_0 \in \mathbb{R}^+$ such that $B(p_0, r_0) \subseteq N$ and pick $\varepsilon > 0$ satisfying

$$\varepsilon < \frac{12\delta}{5r_0}.$$

Let us define a function $\varphi\colon M \longrightarrow \mathbb{R}$ by

$$\varphi(p) = \begin{cases} 1 & \text{if } p \in B(p_0, \frac{r_0}{4}); \\ 1 - \varepsilon(d(p,p_0) - \frac{r_0}{4}) & \text{if } p \in B(p_0, \frac{r_0}{3}) \setminus B(p_0, \frac{r_0}{4}); \\ 1 - \frac{\varepsilon r_0}{12} & \text{if } p \notin B(p_0, \frac{r_0}{3}). \end{cases}$$

It is immediate to verify that φ is Lipschitz and $\|\varphi\|_L \leqslant \varepsilon$. Now, consider $f\colon M \longrightarrow \mathbb{R}$ to be an extension of f_0 via McShane preserving the norm $\|\cdot\|_L$. Let us define $g\colon M \longrightarrow \mathbb{R}$ by $g(p) = f(p)\varphi(p)$ for every $p \in M$. It is clear that g is Lipschitz. In fact, if $p \in B(p_0, \frac{r_0}{3})$ and $q \in M$, notice that

$$\begin{aligned} |(g-f)(m_{p,q})| &= \frac{|(g(p)-f(p)) - (g(q)-f(q))|}{d(p,q)} \\ &= \frac{|(\varphi(p)-\varphi(q))f(p) + (\varphi(q)f(p) - f(p)) - (\varphi(q)f(q) - f(q))|}{d(p,q)} \\ &\leqslant |f(p)||\varphi(m_{p,q})| + \frac{|(f(p)-f(q)) - (\varphi(q)f(p) - \varphi(q)f(q))|}{d(p,q)} \\ &= |f(p)||\varphi(m_{p,q})| + \frac{|(f(p)-f(q))(1-\varphi(q))|}{d(p,q)} \\ &\leqslant \frac{r_0}{3}\varepsilon + \frac{\varepsilon r_0}{12} = \frac{5r_0}{12}\varepsilon < \delta. \end{aligned}$$

By symmetry, the same happens if we assume $p \in M$, $q \in B(p_0, \frac{r_0}{3})$. Finally, if neither p nor q belong to $B(p_0, \frac{r_0}{3})$, then we simply have that

$$|(g-f)(m_{p,q})| = \left| \left(1 - \frac{\varepsilon r_0}{12}\right) f(m_{p,q}) - f(m_{p,q}) \right| \leqslant \frac{r_0}{12}\varepsilon < \delta.$$

Consequently, $\|g - f\|_L < \delta$, which implies that g is Lipschitz. Moreover, denoting by g_0 the restriction of g to N, we obtain that $g_0 \notin \overline{\mathrm{LipSNA}(N,\mathbb{R})}$. Indeed, we claim that $g \notin \overline{\mathrm{LipSNA}(M,\mathbb{R})}$. In order to see it, let us suppose that there exist sequences $\{h_n\} \subseteq \mathrm{Lip}_0(M,\mathbb{R})$, $\{p_n\}$, $\{q_n\} \subseteq M$ with $p_n \neq q_n$ and $h_n(m_{p_n,q_n}) = \|h_n\|_L = \|g\|_L$ for every $n \in \mathbb{N}$ such that $\{\|h_n - g\|_L\}$ converges to 0. We will distinguish three cases:

- Case 1. p_n, $q_n \in B(p_0, r_0)$ eventually. Recall that $B(p_0, r_0) \subseteq N$, so this assumption implies that restrictions of h_n to N strongly attain their norms eventually, which is impossible since $g_0 \notin \overline{\mathrm{LipSNA}(N,\mathbb{R})}$.

- Case 2. p_n, $q_n \notin B(p_0, \frac{r_0}{3})$ eventually. Fix $n \in \mathbb{N}$ and note that

$$\begin{aligned} |(h_n - g)(m_{p_n,q_n})| &= \|g\|_L - g(m_{p_n,q_n}) = \|g\|_L - \left(1 - \frac{\varepsilon r_0}{12}\right) f(m_{p_n,q_n}) \\ &\geqslant \|g\|_L - \left(1 - \frac{\varepsilon r_0}{12}\right)\|f\|_L \geqslant \|f\|_L - \left(1 - \frac{\varepsilon r_0}{12}\right)\|f\|_L \\ &= \frac{\varepsilon r_0}{12}. \end{aligned}$$

  Therefore, $\|h_n - g\|_L \geqslant \frac{\varepsilon r_0}{12}$ for every $n \in \mathbb{N}$, which contradicts the assumption.

- Case 3. $p_n \in B(p_0, \frac{r_0}{3})$, $q_n \notin B(p_0, r_0)$ eventually.

In this case, for a fixed $n \in \mathbb{N}$ we have that

$$\begin{aligned}
g(m_{p_n,q_n}) &= \frac{g(p_n)-g(q_n)}{d(p_n,q_n)} = \frac{\varphi(p_n)f(p_n)-(1-\frac{\varepsilon r_0}{12})f(q_n)}{d(p_n,q_n)} \\
&\leqslant \frac{\varphi(p_n)f(p_n)-(1-\frac{\varepsilon r_0}{12})(f(p_n)-d(p_n,q_n))}{d(p_n,q_n)} \\
&= \frac{\left(\varphi(p_n)-\left(1-\frac{\varepsilon r_0}{12}\right)\right)f(p_n)}{d(p_n,q_n)} + \left(1-\frac{\varepsilon r_0}{12}\right) \\
&\leqslant \frac{\left(1-\left(1-\frac{\varepsilon r_0}{12}\right)\right)\frac{r_0}{3}}{d(p_n,q_n)} + \left(1-\frac{\varepsilon r_0}{12}\right) \\
&\leqslant \frac{\frac{\varepsilon r_0}{12}\frac{r_0}{3}}{\frac{2r_0}{3}} + \left(1-\frac{\varepsilon r_0}{12}\right) = \frac{\varepsilon r_0}{24} + \left(1-\frac{\varepsilon r_0}{12}\right) \\
&\leqslant 1-\frac{\varepsilon r_0}{24}.
\end{aligned}$$

Since $\|g\|_L \geqslant 1$, we conclude that $\{h_n\}$ cannot converge to g in this case either. The case when $p_n \notin B(p_0, r_0)$ and $q_n \in B(p_0, \frac{r_0}{3})$ is completely analogous to the last one. □

Now, we are ready to present the main result of this section.

Theorem 3.20. *Let E be a normed space, let $J \subseteq \mathbb{R}$ be an interval, let $\alpha\colon J \longrightarrow E$ be a curve, and let $\Gamma \subseteq E$ be its range. Assume that there is an interval $I \subseteq J$ for which $\alpha|_I : I \longrightarrow E$ is a C^2 curve parametrized by arc length and $\alpha(I)$ has nonempty interior with respect to Γ. Then,*

$$\overline{\mathrm{LipSNA}(\Gamma,\mathbb{R})} \neq \mathrm{Lip}_0(\Gamma,\mathbb{R}).$$

Proof. Let us consider an interval $I_0 \subseteq \mathbb{R}$ satisfying the hypotheses of the theorem. Let I be a subinterval of I_0 still satisfying that $\alpha(I)$ has nonempty interior and small enough so that

$$\frac{1}{2}|t-s| \leqslant \|\alpha(t)-\alpha(s)\| \leqslant |t-s| \quad \forall\, t,s \in I. \tag{3.6}$$

Up to a change of variables we can write $I = [0, \rho]$ for some $\rho > 0$. Observe that Lemma 3.18 guarantees the existence of such a constant ρ. Let us consider the function $g\colon I \longrightarrow \mathbb{R}$ given by

$$g(x) = \inf\left\{\frac{\|\alpha(t)-\alpha(s)\|}{|t-s|} : t,s \in I, |t-s| = x\right\} \quad \forall\, x \in I.$$

In order to apply Theorem 3.12, we need to verify that the function g satisfies the hypotheses.

First, it is clear that $g(I) \subseteq [0,1]$ and Lemma 3.18 gives us that $\lim_{s\to 0} g(s) = 1$. Moreover, since

$$\frac{1}{2}|t-s| \leqslant \|\alpha(t)-\alpha(s)\| \leqslant |t-s| \quad \forall\, t,s \in I,$$

we have that $\inf\{g(x)\colon x \in I\} \geqslant \frac{1}{2}$. Finally, let us verify that the quotient $\frac{1-g(x)}{x}$ is bounded in $I \setminus \{0\}$. Consider two points $t > s$ in I and write $p = \frac{t+s}{2}$. Then, notice that

$$\frac{\left|\frac{\|\alpha(t)-\alpha(s)\|}{t-s}-1\right|}{t-s} = \frac{\left|\left\|\frac{\alpha(t)-\alpha(s)}{t-s}\right\| - \|\alpha'(p)\|\right|}{t-s} \leqslant \frac{\left\|\frac{\alpha(t)-\alpha(s)}{t-s} - \alpha'(p)\right\|}{t-s}.$$

Now, by Taylor's theorem we can write

$$\alpha(x) = \alpha(p) + (x-p)\alpha'(p) + R_1(x-p) \quad \forall\, x \in I,$$

where it is known that the reminder $R_1(x-p)$ satisfies

$$\|R_1(x-p)\| \leqslant \frac{(x-p)^2}{2}\sup\{\|\alpha''(t)\|\colon t \in I\}.$$

Hence, we deduce that

$$\alpha(t) - \alpha(s) = (t-s)\alpha'(p) + R_1(t-p) - R_1(s-p).$$

In view of this, we obtain that

$$\frac{\left|\frac{\|\alpha(t)-\alpha(s)\|}{t-s} - 1\right|}{t-s} \leqslant \frac{\|R_1(t-p) - R_1(s-p)\|}{(t-s)^2} \leqslant \frac{(t-p)^2}{(t-s)^2} \sup\{\|\alpha''(x)\| \colon x \in I\}$$
$$= \frac{1}{4} \sup\{\|\alpha''(x)\| \colon x \in I\}.$$

Finally, since $\alpha \in C^2$ and I is compact, we conclude that the above quotient is bounded. Consequently, $\frac{1-g(x)}{x}$ is also bounded in $I \setminus \{0\}$.

Therefore, if we extend the function g from $I = [0, \rho]$ to $[0,1]$ by $g(t) = g(\rho)$ for $t \in [\rho, 1]$, then we can apply Theorem 3.12 to this function to obtain a Cantor set $C \subseteq [0,1]$ satisfying that $|C| > 0$ and $\phi_C(t) < g(t)$ for every $t \in I \setminus \{0\}$. Now, let us define $f \colon I \longrightarrow \mathbb{R}$ by

$$f(t) = \int_0^t \chi_C(s)\, ds \quad \forall\, t \in I.$$

Clearly, $f \in \mathrm{Lip}_0(I, \mathbb{R})$ with $\|f\|_L = 1$. Let us write $\Gamma_0 = \{\alpha(t) \colon t \in I\}$. Consider the linear isomorphism $\Phi \colon \mathrm{Lip}_0(I, \mathbb{R}) \longrightarrow \mathrm{Lip}_0(\Gamma_0, \mathbb{R})$ given by

$$\Phi(f)(\alpha(t)) = f(t) \quad \forall\, t \in I, \quad \forall\, f \in \mathrm{Lip}_0(I, \mathbb{R}).$$

Also, let us write $F = \Phi(f)$. Consider $p,\, q \in \Gamma_0$ with $p \neq q$. Then, pick $t,\, s \in I$ with $\alpha(t) = p$, $\alpha(s) = q$. Notice that from (3.6) we deduce that $t \neq s$ and $\|\Phi^{-1}\| \leqslant 1$. Hence, we have $\|F\|_L \geqslant \|f\|_L = 1$. We claim that F does not attain its Lipschitz norm. Indeed, if we suppose that there are $t,\, s \in I$ such that $|\widehat{F}(m_{\alpha(t),\alpha(s)})| \geqslant 1$. Then we have

$$\phi_C(|t-s|) \geqslant \frac{|f(t) - f(s)|}{|t-s|} \geqslant \frac{\|\alpha(t) - \alpha(s)\|}{|t-s|} \geqslant g(|t-s|),$$

which is a contradiction. Hence, F does not attain its Lipschitz norm and $\|F\|_L = 1$.

Now, pick $0 < \theta < \frac{1}{4}$ and define a function $h \colon I \longrightarrow \mathbb{R}$ by

$$h(t) = \int_0^t \chi_C(s) - \theta \chi_{I \setminus C}(s)\, ds \quad \forall\, t \in I.$$

As before, write $H = \Phi(h)$. We claim that $H \notin \overline{\mathrm{LipSNA}(\Gamma_0, \mathbb{R})}$. In order to prove it, let us assume the opposite. Then, we find a sequence of strongly norm-attaining Lipschitz functions $\{L_n\}$ converging to H. Since, Φ is an isomorphism, this is the same as saying that $\{l_n\}$ converges to h in $\mathrm{Lip}_0(I, \mathbb{R})$, where $l_n = \Phi^{-1}(L_n)$ for every $n \in \mathbb{N}$. Then, we may assume that $\|l_n\|_L = \|h\|_L$ for every $n \in \mathbb{N}$. Now, recall that $\mathrm{Lip}_0(I, \mathbb{R})$ is isometrically isomorphic to $\mathrm{L}_\infty(I)$, where the isometry is given by the derivative operator. Therefore, we also have $\|l_n'\|_\infty = \|h'\|_\infty = 1$ for every $n \in \mathbb{N}$. Take $n \in \mathbb{N}$ large enough so that $\|L_n - H\|_L < \theta$. Then,

$$\|l_n' - h'\|_\infty = \|l_n - h\|_L = \|\Phi^{-1}(L_n - H)\|_L \leqslant \|L_n - H\|_L < \theta.$$

Since $\|l_n'\|_\infty = 1$, we conclude that $l_n' \leqslant \chi_C$ almost everywhere in I. Also, $\|L_n\|_L \geqslant \|\Phi^{-1}(L_n)\|_L = \|l_n\|_L = 1$. Let us show that L_n cannot attain its Lipschitz norm, which leads to a contradiction. Pick two points $t > s \in I$. First, assume that $\widehat{L_n}(m_{\alpha(t),\alpha(s)}) \geqslant 1$. Then, we have

$$\widehat{F}(m_{\alpha(t),\alpha(s)}) = \frac{\int_s^t \chi_C(x)\, dx}{\|\alpha(t) - \alpha(s)\|} \geqslant \frac{\int_s^t l_n'(x)\, dx}{\|\alpha(t) - \alpha(s)\|} = \widehat{L_n}(m_{\alpha(t),\alpha(s)}) \geqslant 1,$$

which contradicts the fact that $\|F\|_L = 1$ and it does not attain its Lipschitz norm. Finally, observe that

$$\widehat{L_n}(m_{\alpha(s),\alpha(t)}) = \frac{-\int_s^t l_n'(x)\, dx}{\|\alpha(t) - \alpha(s)\|} \leqslant \frac{-\int_s^t h'(x) - \theta\, dx}{\|\alpha(t) - \alpha(s)\|} < 2\theta \frac{t-s}{\|\alpha(t) - \alpha(s)\|} < 1.$$

In conclusion, L_n does not attain its Lipschitz norm. Consequently, $H \notin \overline{\mathrm{LipSNA}(\Gamma_0, \mathbb{R})}$.

Finally, it is clear that we may apply Lemma 3.19 to the subset Γ_0 and the function H to obtain that $\mathrm{LipSNA}(\Gamma, \mathbb{R})$ is not dense in $\mathrm{Lip}_0(\Gamma, \mathbb{R})$. Indeed, if $t \in (0, \rho)$ is any Lebesgue point of density of the Cantor set C, then we would have $\{\widehat{h}(m_{t-\frac{1}{n}, t+\frac{1}{n}})\} \longrightarrow 1$, from where we deduce that $\{\widehat{H}(m_{\alpha(t-\frac{1}{n}), \alpha(t+\frac{1}{n})})\} \longrightarrow 1$. Hence, the Lipschitz function H restricted to $B(\alpha(t), r)$ has norm one for every $r \in \mathbb{R}^+$. □

Remark 3.21. Let us make an observation. Theorem 3.20 assumes that the curve α is C^2 in some interval. However, we can give a weaker condition for which the result remains true. In fact, the regularity of the curve has been used to apply Lemma 3.18, which needs only C^1, and Taylor's theorem. Hence, in order to get a nice bound for the reminder, it is sufficient if α is twice differentiable on some interval I and $\sup\{\|\alpha''(t)\| \colon t \in I\}$ is finite. It is not necessary to assume continuity of the second derivative of α.

Notice that in order to apply Theorem 3.20 we need the curve α to be parametrized by arc length. We needed that hypothesis in order to verify that the quotient $\frac{1-g(x)}{x}$ is bounded, so we can apply Theorem 3.12. Indeed, it was necessary to use that $\|\alpha'(p)\| = 1$ at some step of the argument. This hypothesis could seem to be unnecessary since at the end the condition $\overline{\mathrm{LipSNA}(\Gamma, \mathbb{R})} \neq \mathrm{Lip}_0(\Gamma, \mathbb{R})$, where Γ is the range of α, does not depend on the parametrization. However, the last condition does depend on the norm $\|\cdot\|$ that we consider on E. If $\alpha \colon I \longrightarrow E$ is a C^2 curve, it could be possible to lose regularity when we parametrize it by arc length. Indeed, consider $\alpha \colon I \longrightarrow E$ a C^2 curve and assume that $\alpha'(t) \neq 0$ for every $t \in I$. Then, we can parametrize α by arc length. More concretely, we can consider the function $h \colon I \longrightarrow \mathbb{R}$ given by

$$h(t) = \int_a^t \|\alpha'(s)\|\, ds \quad \forall\, t \in I.$$

Since α is C^2, then $s \longmapsto \|\alpha'(s)\|$ is continuous, so h is C^1. Moreover, $h'(t) > 0$ for every $t \in I$, so h is strictly increasing, from where we deduce that h^{-1} exists and it is a C^1-diffeomorphism. Then, $\beta = \alpha \circ h^{-1}$ is C^1 curve parametrized by arc length. However, in order to get that β is C^2, we need the application $t \longmapsto \|\alpha'(t)\|$ to be a C^1 map. For instance, this is the case when we consider Hilbert spaces.

Corollary 3.22. *Let H be a Hilbert space, let $J \subseteq \mathbb{R}$ be an interval, let $\alpha \colon J \longrightarrow H$, and let $\Gamma \subseteq H$ be its range. Assume that there is an interval $I \subseteq J$ for which $\alpha|_I \colon I \longrightarrow H$ is C^2 and $\alpha(I)$ has nonempty interior with respect to Γ. Then,*

$$\overline{\mathrm{LipSNA}(\Gamma, \mathbb{R})} \neq \mathrm{Lip}_0(\Gamma, \mathbb{R}).$$

Proof. Let $I_0 \subseteq \mathbb{R}$ be an interval satisfying the hypotheses of the theorem. First, if $\alpha'(t) = 0$ for every $t \in I_0$, then $\alpha(t)$ is a straight line and the result follows from [50, Theorem 2.3] together with Lemma 3.19. Indeed, one can verify that the Lipschitz function given in the proof of [50, Theorem 2.3] that is far from all the strongly norm-attaining Lipschitz functions satisfies the hypotheses of Lemma 3.19. Hence, we may assume that $\alpha'(t_0) \neq 0$ for some $t_0 \in I_0$. Therefore, we can pick a subinterval $I \subseteq I_0$ such that $\inf\{\|\alpha'(t)\| \colon t \in I\} > 0$. Now, the mapping $t \longmapsto \|\alpha'(t)\| = \sqrt{(\alpha'(t), \alpha'(t))}$ is C^1, where $(\cdot, \cdot)$ denotes the inner product of H. Hence, we can parametrize α by arc length preserving its regularity. Now, we can apply Theorem 3.20. □

Chapter 4

The Lipschitz Bishop-Phelps-Bollobás property

The results obtained in this chapter can be found in the papers [24] and [25]. They were collaborative works with Miguel Martín.

In the previous chapters, we have studied for which metric spaces M and Banach spaces Y the set $\operatorname{LipSNA}(M, Y)$ of those Lipschitz maps that strongly attain their norm is dense in the Lipschitz space $\operatorname{Lip}_0(M, Y)$. As we have already discussed, this study corresponds to a non-linear generalization of the classical study of norm-attaining linear operators between Banach spaces. This research line was initiated by Lindenstrauss [57] in the 1960's, trying to extend to operators the Bishop-Phelps theorem [16], which states that the set of functionals which attain their norm on a Banach space X is always dense in X^*.

An extension of the Bishop-Phelps theorem was given by Bollobás [17] in 1970, who showed that one is always able to make a simultaneous approximation of a functional f and a vector x at which f almost attains its norm by a functional g and a vector y such that g attains its norm at y. To study the validity of this result for operators, a property was introduced in 2008. A pair of Banach spaces (X, Y) has the *Bishop-Phelps-Bollobás property* (*BPBp* in short) [4] if given $\varepsilon > 0$, there is $\eta(\varepsilon) > 0$ such that for every norm-one $T \in \mathcal{L}(X, Y)$ and every $x \in S_X$ such that $\|T(x)\| > 1 - \eta(\varepsilon)$, there exist $u \in S_X$ and $S \in \mathcal{L}(X, Y)$ satisfying

$$\|S(u)\| = \|S\| = 1, \quad \|T - S\| < \varepsilon, \quad \|x - u\| < \varepsilon.$$

If an analogous definition is valid for operators T and S belonging to a subspace $\mathcal{M} \subseteq \mathcal{L}(X, Y)$, then we say that (X, Y) has the BPBp for operators from $\mathcal{M}$. There is a vast literature about this topic as, for instance, the cited seminal paper [4] and [12, 21, 26, 33].

Our aim in this chapter is to extend the Bishop-Phelps-Bollobás property to the Lipschitz context in a natural way. Let M be a metric space and let Y be a Banach space. The role of the norm-attaining operator S will be played by a strongly norm-attaining Lipschitz map or, equivalently, by an element of $\mathcal{L}(\mathcal{F}(M), Y)$ attaining its norm at a molecule. Recall that $\operatorname{Mol}(M)$ is closed in norm (see Proposition 1.13), so the only elements in the unit sphere of $\mathcal{F}(M)$ that can be approximated by molecules are molecules. Thus, we restrict the point x to be a molecule. Our generalization reads as follows.

Definition 4.1. Let M be a metric space and let Y be a Banach space. We say that the pair (M, Y) has the *Lipschitz Bishop-Phelps-Bollobás property* (*Lip-BPB property* for short), if given $\varepsilon > 0$ there is $\eta(\varepsilon) > 0$ such that for every norm-one $F \in \operatorname{Lip}_0(M, Y)$ and every $p, q \in M$, $p \neq q$ such that $\|F(p) - F(q)\| > \big(1 - \eta(\varepsilon)\big) d(p, q)$, there exist $G \in \operatorname{Lip}_0(M, Y)$ and $r, s \in M$, $r \neq s$, such that

$$\frac{\|G(r) - G(s)\|}{d(r, s)} = \|G\|_L = 1, \quad \|G - F\|_L < \varepsilon, \quad \frac{d(p, r) + d(q, s)}{d(p, q)} < \varepsilon.$$

If the previous definition holds for a class of linear operators from $\mathcal{F}(M)$ to Y, we will say that the pair (M, Y) has the Lip-BPB property for that class.

Observe that the quantity $\frac{d(p,r)+d(q,s)}{d(p,q)}$ in the definition above measures the nearness of the pair (p,q) to the pair (r,s) modulated by the distance of p to q, so the smallness of it represents that the two pairs are "relatively" near one to the other.

Notice that from a straightforward application of Lemma 1.14, analogous to what is done in Lemma 2.15, we can give the next reformulation of the Lip-BPB property. Indeed, if M is a metric space and Y is a Banach space, then the pair (M,Y) has the Lip-BPB property if and only if given $\varepsilon > 0$ there is $\eta(\varepsilon) > 0$ such that for every norm-one $\widehat{F} \in \mathcal{L}(\mathcal{F}(M),Y)$ and every $m \in \mathrm{Mol}(M)$ such that $\|\widehat{F}(m)\| > 1-\eta(\varepsilon)$, there exist $\widehat{G} \in \mathcal{L}(\mathcal{F}(M),Y)$ and $u \in \mathrm{Mol}(M)$ such that

$$\|\widehat{G}(u)\| = \|G\|_L = 1, \quad \|\widehat{F}-\widehat{G}\| < \varepsilon, \quad \|m-u\| < \varepsilon.$$

We will use both equivalent formulations without giving any explicit reference.

In the case when M is a Banach space, notice also that we get the (classical) BPBp of the pair (M,Y) if for every linear and bounded operator F satisfying the assumptions of Definition 4.1 we actually get a linear and bounded operator G with the desired properties.

In the same way as strong density in the setting of Lipschitz maps is a non-linear generalization of the density of norm-attaining linear operators, the last two observations show that the study of the Lip-BPB property is both a non-linear generalization of the (classical) BPBp and a particular case of the BPBp, where the domain space is a Lipschitz-free space and the concept of norm-attainment is stronger than the usual one. Let us start the study of this property.

4.1 Finite metric spaces

In this section we will focus our attention in studying the case when the metric space M is finite, that is, M has only finitely many elements. Let us present the main result of this section.

Theorem 4.2. *Let M be a finite metric space and let Y be a Banach space. If $(\mathcal{F}(M),Y)$ has the BPBp, then (M,Y) has the Lip-BPB property.*

Proof. Notice that Example 2.19 shows that $\mathcal{F}(M)$ has property α. Let $\Gamma = \{x_\lambda\}_{\lambda\in\Lambda} \subseteq \mathcal{F}(M)$ be the set given by the statement (ii) of Definition 2.6. Then, we have that $\|x_\lambda - x_\mu\| \geqslant |x_\lambda^*(x_\lambda) - x_\lambda^*(x_\mu)| \geqslant 1-\rho$ when $x_\lambda \neq \pm x_\mu$. Therefore, as M is finite and so $B_{\mathcal{F}(M)}$ is compact, Γ must be a finite set:

$$\Gamma = \{x_k \colon k = 1,\dots,n\}.$$

Moreover, as $B_{\mathcal{F}(M)} = \overline{\mathrm{co}}(\Gamma) = \mathrm{co}(\Gamma)$, every molecule $m_{p,q} \in \mathrm{Mol}(M)$ can be written as a convex combination of these points. Let us take

$$\delta = \min\left\{\min\left\{\lambda_k \colon m_{p,q} = \sum\nolimits_{k=1}^{n} \lambda_k x_k, \quad \lambda_k > 0\right\} \colon m_{p,q} \in \mathrm{Mol}(M)\right\} > 0.$$

Now, fix $0 < \varepsilon < \min\left\{\frac{1}{2}, (1-\rho)\delta\right\}$ and take $\eta(\varepsilon)$ the constant associated to the BPBp of the pair $(\mathcal{F}(M),Y)$. Consider $F \in \mathrm{Lip}_0(M,Y)$ with $\|F\|_L = 1$ and $m \in \mathrm{Mol}(M)$ such that $\|\widehat{F}(m)\| > 1-\eta(\varepsilon)$. By hypothesis, there exist $G \in \mathrm{Lip}_0(M,Y)$ and $\xi \in B_{\mathcal{F}(M)}$ satisfying

$$\|\widehat{G}(\xi)\| = \|G\|_L = 1, \quad \|F-G\|_L < \varepsilon, \quad \|m-\xi\| < \varepsilon.$$

Note that we can write

$$m = \sum_{k=1}^{n} \lambda_k x_k, \quad \xi = \sum_{k=1}^{n} \theta_k x_k, \quad \sum_{k=1}^{n} \lambda_k = \sum_{k=1}^{n} \theta_k = 1, \quad \lambda_k, \theta_k \geqslant 0$$

for every $k = 1,\dots,n$. We claim that $\lambda_k = 0$ whenever $\theta_k = 0$. Indeed, if we suppose that there exists $k \in \{1,\dots,n\}$ satisfying that $\lambda_k \neq 0$ but $\theta_k = 0$, then it makes sense to take the constant $\delta_{\xi,m}$ given by

$$\delta_{\xi,m} = \min\left\{\lambda_k \colon \lambda_k \neq 0,\, \theta_k = 0,\, k = 1,\dots,n\right\}.$$

Let us consider $j \in \{1, \ldots, n\}$ such that $\lambda_j = \delta_{\xi,m}$, so $\theta_j = 0$ and we obtain that

$$\begin{aligned}\|m - \xi\| \geqslant x_j^*(m) - x_j^*(\xi) &= \sum_{k=1}^{n} \lambda_k x_j^*(x_k) - \sum_{k=1}^{n} \theta_k x_j^*(x_k) \\ &= \lambda_j + \sum_{k \neq j} \lambda_k x_j^*(x_k) - \sum_{k \neq j} \theta_k x_j^*(x_k) \\ &= \lambda_j - \sum_{k \neq j} (\theta_k - \lambda_k) x_j^*(x_k) \geqslant \lambda_j - \rho \sum_{k \neq j} (\theta_k - \lambda_k) \\ &= \lambda_j - \rho(1 - (1 - \lambda_j)) = (1 - \rho)\lambda_j = (1 - \rho)\delta_{x,m} \geqslant (1 - \rho)\delta > \varepsilon,\end{aligned}$$

a contradiction. Now, taking $y^* \in S_{Y^*}$ such that $y^*(\widehat{G}(\xi)) = 1$, we have that

$$1 = y^*(\widehat{G}(\xi)) = \sum_{k=1}^{n} \theta_k y^*(\widehat{G}(x_k)) \leqslant \sum_{k=1}^{n} \theta_k = 1.$$

Then, $y^*(\widehat{G}(x_k)) = 1$ for every $k = 1, \ldots, n$ such that $\theta_k \neq 0$. By our assumption, this also happens for every $k = 1, \ldots, n$ such that $\lambda_k \neq 0$. Consequently, we have that

$$\|\widehat{G}(m)\| \geqslant y^*(\widehat{G}(m)) = \sum_{\lambda_k \neq 0} \lambda_k y^*(\widehat{G}(x_k)) = \sum_{\lambda_k \neq 0} \lambda_k = 1.$$

That is, $\widehat{G}$ attains its norm at the molecule $m \in \mathrm{Mol}(M)$. □

It is shown in [4, Proposition 2.4] that if X and Y are finite-dimensional Banach spaces, then (X, Y) has the BPBp. Consequently, we obtain the following corollary.

Corollary 4.3. *Let M be a finite metric space and let Y be a finite-dimensional Banach space. Then, (M, Y) has the Lip-BPB property.*

In particular, we obtain the following.

Corollary 4.4. *Let M be a finite metric space. Then, $(M, \mathbb{R})$ has Lip-BPB property.*

In [4, Theorem 2.2] it is also shown that if a Banach space Y has property β, then the pair (X, Y) has the BPBp for every Banach space X. Note that, by using Theorem 4.2, we obtain that given a finite metric space M and a Banach space Y having property β, the pair (M, Y) will have the Lip-BPB property. In this way we could give more corollaries. However, we will give in Chapter 5 a stronger result which generalizes all of them.

One can think that the hypotheses appearing in the statement of Theorem 4.2 are very restrictive. However, we will show that satisfying the Lip-BPB property is also a very restrictive condition. Indeed, the next example shows that we cannot remove the hypothesis of $(\mathcal{F}(M), Y)$ having the BPBp from Theorem 4.2. It is an adaption of [12, Lemma 3.2].

Example 4.5. *Let $M = \{0, 1, 2\} \subseteq \mathbb{R}$ with the usual metric and let Y be a strictly convex Banach space which is not uniformly convex. Then, (M, Y) fails the Lip-BPB property.*

Proof. Observe that $\mathcal{F}(M)$ is two-dimensional and that $m_{0,2} = \frac{1}{2} m_{0,1} + \frac{1}{2} m_{1,2}$, so

$$B_{\mathcal{F}(M)} = \overline{\mathrm{co}}\{\pm m_{0,1}, \pm m_{1,2}\}$$

is a square. On the other hand, as Y is not uniformly convex, there exist sequences $\{x_n\}, \{y_n\} \subseteq S_Y$ and $\varepsilon_0 > 0$ such that

$$\lim_{n \to \infty} \|x_n + y_n\| = 2 \quad \text{and} \quad \|x_n - y_n\| > \varepsilon_0 \quad \forall\, n \in \mathbb{N}.$$

Fix $0 < \varepsilon < \frac{\varepsilon_0}{2}$ and assume that (M, Y) has the Lip-BPB property witnessed by the function $\varepsilon \longmapsto \eta(\varepsilon) > 0$. Take $m \in \mathbb{N}$ such that

$$\|x_m + y_m\| > 2 - 2\eta(\varepsilon)$$

and define the linear operator $\widehat{F} \in \mathcal{L}(\mathcal{F}(M), Y)$ by

$$\widehat{F}(m_{0,1}) = x_m, \qquad \widehat{F}(m_{1,2}) = y_m.$$

It is clear, by the shape of the unit ball of $\mathcal{F}(M)$, that $\|\widehat{F}\| = 1$. Furthermore, note that

$$\|\widehat{F}(m_{0,2})\| = \left\|\widehat{F}\left(\frac{m_{0,1} + m_{1,2}}{2}\right)\right\| = \frac{1}{2}\|x_m + y_m\| > 1 - \eta(\varepsilon).$$

Therefore, there exist a linear operator $\widehat{G}\colon \mathcal{F}(M) \longrightarrow Y$ and a molecule $u \in \mathrm{Mol}(M)$ such that

$$\|\widehat{G}(u)\| = \|\widehat{G}\| = 1, \quad \|\widehat{F} - \widehat{G}\| < \varepsilon, \quad \|m_{0,2} - u\| < \varepsilon.$$

A straightforward application of Lemma 1.14 shows that

$$\|m_{0,2} - m_{0,1}\|, \|m_{0,2} - m_{1,2}\| \geqslant 1,$$

hence $u = m_{0,2}$. Now, note that

$$1 = \|\widehat{G}(m_{0,2})\| = \left\|\frac{1}{2}\widehat{G}(m_{0,1}) + \frac{1}{2}\widehat{G}(m_{1,2})\right\|.$$

Since Y is strictly convex, it follows that $\widehat{G}(m_{0,1}) = \widehat{G}(m_{1,2})$, which implies that

$$\begin{aligned}\|x_m - y_m\| &= \|\widehat{F}(m_{0,1}) - \widehat{F}(m_{1,2})\| \\ &\leqslant \|\widehat{F}(m_{0,1}) - \widehat{G}(m_{0,1})\| + \|\widehat{F}(m_{1,2}) - \widehat{G}(m_{1,2})\| \leqslant \varepsilon + \varepsilon < \varepsilon_0,\end{aligned}$$

a contradiction. □

On the other hand, the following example shows that the finiteness of the metric space is also necessary in Theorem 4.2.

Example 4.6. $(\mathbb{N}, \mathbb{R})$ *does not have the Lip-BPB property.*

Proof. Fix $0 < \varepsilon < \frac{1}{2}$ and suppose that $(\mathbb{N}, \mathbb{R})$ has the Lip-BPB property witnessed by a function $\varepsilon \longmapsto \eta(\varepsilon) > 0$ which we can suppose satisfies $\eta(\varepsilon) < \frac{1}{2}$.

Take $n \in \mathbb{N}$ such that $n > \frac{1}{2\eta(\varepsilon)}$ and define $f\colon \mathbb{N} \longrightarrow \mathbb{R}$ by

$$f(p) = \begin{cases} p - 1 & \text{if } p \leqslant 2n \\ p - 2 & \text{if } p > 2n \end{cases}$$

It is clear that $f \in \mathrm{Lip}_0(\mathbb{N}, \mathbb{R})$ with $\|f\|_L = 1$. Besides,

$$\widehat{f}(m_{3n,n}) = \frac{f(3n) - f(n)}{3n - n} = \frac{2n - 1}{2n} = 1 - \frac{1}{2n} > 1 - \eta(\varepsilon).$$

Now, given $p < q \in \mathbb{N}$, if $\widehat{g} \in \mathcal{L}(\mathcal{F}(\mathbb{N}), \mathbb{R})$ with $\|g\|_L = 1$ attains its norm at a molecule $m_{q,p}$ such that $\|m_{q,p} - m_{3n,n}\| < \varepsilon$, Lemma 1.14 implies that $[2n, 2n+1] \subseteq [p, q]$. Indeed, if we assume that $p > 2n$ or $q < 2n + 1$, then by applying that lemma we obtain

$$\|m_{q,p} - m_{3n,n}\| \geqslant \frac{\max\{|q - 3n|, |p - n|\}}{\min\{|q - p|, 2n\}} \geqslant \frac{n}{2n} = \frac{1}{2},$$

which is a contradiction since $\varepsilon < \frac{1}{2}$. According to [50, Lemma 2.2], g attains its norm at the molecule $m_{2n+1,2n}$. In view of this, it is enough to note that

$$\|g - f\|_L \geqslant \widehat{g}(m_{2n+1,2n}) - \widehat{f}(m_{2n_0+1,2n}) = 1 - 0 = 1,$$

which is a contradiction. □

Before finishing this section, let us say that Example 4.6 actually holds for any Banach space, that is, $(\mathbb{N}, Y)$ does not have the Lip-BPB property for any Banach space Y. The reason of this is because an analogous result to Proposition 2.36 can be given in the context of the Lip-BPB property. Hence, by the next result, anytime we are interested in proving that a pair (M, Y) fails to have the Lip-BPB property, it will be enough to show that $(M, \mathbb{R})$ fails to have it.

Proposition 4.7. *Let M be a metric space. Suppose that there exists a Banach space $Y \neq 0$ such that (M, Y) has the Lip-BPB property. Then, $(M, \mathbb{R})$ has the Lip-BPB property.*

Proof. Let Y be a Banach space such that (M, Y) has the Lip-BPB property. Fix $\varepsilon > 0$ and consider $\eta(\varepsilon)$ the constant associated to the Lip-BPB property of (M, Y). Let us consider $f \in \mathrm{Lip}_0(M, \mathbb{R})$ with $\|f\|_L = 1$ and $m \in \mathrm{Mol}(M)$ such that $\widehat{f}(m) > 1 - \eta(\frac{\varepsilon}{2})$. Pick $y_0 \in S_Y$ and define $F \in \mathrm{Lip}_0(M, Y)$ by

$$F(p) = f(p)y_0 \quad \forall\, p \in M.$$

Then, we have that $\|F\|_L = 1$ and $\|\widehat{F}(m)\| > 1 - \eta(\frac{\varepsilon}{2})$. So, by hypothesis, there exist $G \in \mathrm{Lip}_0(M, Y)$ and $u \in \mathrm{Mol}(M)$ satisfying that

$$\|\widehat{G}(u)\| = \|G\|_L = 1, \quad \|F - G\|_L < \frac{\varepsilon}{2}, \quad \|m - u\| < \frac{\varepsilon}{2}.$$

Now, take $y^* \in S_{Y^*}$ such that $y^*\big(\widehat{G}(u)\big) = 1$ and note that

$$\|y^*(y_0)f - y^* \circ G\|_L = \|y^* \circ F - y^* \circ G\|_L \leqslant \|y^*\|\|F - G\|_L < \frac{\varepsilon}{2}.$$

This implies that

$$y^*(y_0) \geqslant y^*(y_0)\widehat{f}(u) \geqslant y^*(\widehat{G}(u)) - |y^*(y_0)\widehat{f}(u) - y^*(\widehat{G}(u))| \geqslant 1 - \frac{\varepsilon}{2}.$$

Therefore, writing $g = y^* \circ G \in \mathrm{Lip}_0(M, \mathbb{R})$, we have that

$$|\widehat{g}(u)| = \|g\|_L = 1, \qquad \|g - f\|_L \leqslant \|g - y^*(y_0)f\|_L + \|y^*(y_0)f - f\|_L < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

As we already know that $\|m - u\| < \varepsilon$, we have that $(M, \mathbb{R})$ has the Lip-BPB property. □

4.2 Uniformly Gromov concave metric spaces

Our goal in this section is to obtain positive results for more general metric spaces than finite metric spaces. We will present some sufficient conditions over the metric space M for which we can ensure that the pair (M, Y) satisfies the Lip-BPB property for every Banach space Y.

Let us recall the notions of Gromov concave and uniformly Gromov concave metric spaces.

Definition 4.8 (Definition 1.12)**.** Let M be a metric space.

(i) We say that M is *Gromov concave* if for every $x, y \in M$, $x \neq y$, there is $\varepsilon_{x,y} > 0$ such that

$$(x, y)_z > \varepsilon_{x,y} \min\{d(x, z), d(y, z)\}$$

for every $z \in M \setminus \{x, y\}$.

(ii) Let $A \subseteq \mathrm{Mol}(M)$. We say that A is *uniformly Gromov rotund* if there is $\varepsilon_0 > 0$ such that

$$(x, y)_z > \varepsilon_0 \min\{d(x, z), d(y, z)\}$$

for every distinct $x, y, z \in M$ such that $m_{x,y} \in A$.

(iii) We say that M is *uniformly Gromov concave* when $\mathrm{Mol}(M)$ is uniformly Gromov rotund.

Remember that by Theorem 1.7, a metric space M is Gromov concave if and only if every molecule of $\mathcal{F}(M)$ is a strongly exposed point of $B_{\mathcal{F}(M)}$. On the other hand, by Proposition 2.9, M is uniformly Gromov concave if and only if $\mathrm{Mol}(M)$ is a set of uniformly strongly exposed points.

The following theorem is the main result of this section.

Theorem 4.9. *Let M be a uniformly Gromov concave metric space. Then, (M, Y) has the Lip-BPB property for every Banach space Y.*

Proof. Fix $0 < \varepsilon < 1$. Since $\mathrm{Mol}(M)$ is a set of uniformly strongly exposed points, there exists $0 < \delta < 1$ such that

$$\mathrm{diam}\big(S(B_{\mathcal{F}(M)}, \widehat{f}_m, \delta)\big) < \varepsilon \quad \forall\, m \in \mathrm{Mol}(M), \tag{4.1}$$

where $\{\widehat{f}_m\}_{m\in \mathrm{Mol}(M)}$ are the functionals which uniformly strongly expose the molecules of M. We take $\eta > 0$ satisfying

$$\left(1 + \frac{\varepsilon}{4}\right)(1-\eta) > 1 + \frac{\varepsilon(1-\delta)}{4}.$$

Now, consider $F \in \mathrm{Lip}_0(M, Y)$ with $\|F\|_L = 1$ and a molecule $m \in \mathrm{Mol}(M)$ such that $\|\widehat{F}(m)\| > 1 - \eta$. Then, we define $\widehat{G}_0 \in \mathcal{L}(\mathcal{F}(M), Y)$ given by

$$\widehat{G}_0(x) = \widehat{F}(x) + \frac{\varepsilon}{4}\widehat{f}_m(x)\widehat{F}(m) \quad \forall\, x \in \mathcal{F}(M).$$

It is clear that $\|\widehat{F} - \widehat{G}_0\| \leqslant \frac{\varepsilon}{4}$. In addition, note that

$$\|\widehat{G}_0(m)\| = \left(1 + \frac{\varepsilon}{4}\right)\|\widehat{F}(m)\| \geqslant \left(1 + \frac{\varepsilon}{4}\right)(1-\eta).$$

On the other hand, if $x \notin \pm S(B_{\mathcal{F}(M)}, \widehat{f}_m, \delta)$ and $\|x\| \leqslant 1$, then we will have that

$$\|\widehat{G}_0(x)\| = \left\|\widehat{F}(x) + \frac{\varepsilon}{4}\widehat{f}_m(x)\widehat{F}(m)\right\| \leqslant 1 + \frac{\varepsilon}{4}|\widehat{f}_m(x)| \leqslant 1 + \frac{\varepsilon(1-\delta)}{4}.$$

Therefore, $\|\widehat{G}_0(x)\| \geqslant \|\widehat{G}_0(m)\|$ implies that $x \in \pm S(B_{\mathcal{F}(M)}, \widehat{f}_m, \delta)$. By defining $\widehat{G} = \frac{\widehat{G}_0}{\|\widehat{G}_0\|}$, we have that

$$\|\widehat{F} - \widehat{G}\| \leqslant \|\widehat{F} - \widehat{G}_0\| + \|\widehat{G}_0 - \widehat{G}\| = \|\widehat{F} - \widehat{G}_0\| + \big|\|\widehat{G}_0\| - 1\big| \leqslant \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}.$$

Note that if $\widehat{G}$ attains its norm at the molecule m, then we have finished. Otherwise, we may take ε' with $0 < \varepsilon' < \min\{\frac{\varepsilon}{2}, \|\widehat{G}\| - \|\widehat{G}(m)\|\}$. Now, thanks to Proposition 2.8, $\mathrm{LipSNA}(M, Y)$ is dense in $\mathrm{Lip}_0(M, Y)$. Hence, there exist $\widehat{H} \in \mathcal{L}(\mathcal{F}(M), Y)$ and $u \in \mathrm{Mol}(M)$ satisfying

$$\|\widehat{H}\| = \|\widehat{H}(u)\| = 1 \quad \text{and} \quad \|\widehat{G} - \widehat{H}\| < \varepsilon'.$$

Next, we note that

$$\|\widehat{G}(u)\| \geqslant \|\widehat{H}(u)\| - \|\widehat{H} - \widehat{G}\| \geqslant \|\widehat{H}\| - \varepsilon' \geqslant \|\widehat{H}\| - (\|\widehat{G}\| - \|\widehat{G}(m)\|) = \|\widehat{G}(m)\|,$$

which implies that $\|\widehat{G}_0(u)\| \geqslant \|\widehat{G}_0(m)\|$, hence $u \in \pm S(B_{\mathcal{F}(M)}, \widehat{f}_m, \delta)$. It follows from (4.1) that

$$\|m - u\| < \varepsilon \quad \text{or} \quad \|m + u\| < \varepsilon.$$

Finally, note that

$$\|\widehat{F} - \widehat{H}\| \leqslant \|\widehat{F} - \widehat{G}\| + \|\widehat{G} - \widehat{H}\| < \varepsilon.$$ □

Notice that Proposition 2.8 states that if M is a metric space for which we find a norming set $A \subseteq \mathrm{Mol}(M)$ of uniformly strongly exposed points, then $\mathrm{LipSNA}(M, Y)$ is dense in $\mathrm{Lip}_0(M, Y)$ for every Banach space Y. However, this assumption is not enough to guarantee the Lip-BPB property, even in the case $Y = \mathbb{R}$. Observe that $\mathcal{F}(\mathbb{N}) \cong \ell_1$, which has property α, so it has a norming subset of strongly exposed points. However, Example 4.6 shows that $(\mathbb{N}, \mathbb{R})$ fails to have the Lip-BPB property.

From Theorem 4.9 we extract a series of interesting corollaries. First, if M is concave and $\mathcal{F}(M)$ has property α (see Definition 2.6), then M is uniformly Gromov concave by Theorem 2.18. Therefore, we obtain the next corollary.

Corollary 4.10. *Let M be a concave metric space such that $\mathcal{F}(M)$ has property α. Then, (M,Y) has the Lip-BPB property for every Banach space Y.*

Note that the concavity hypothesis in the previous result is necessary as Example 4.6 shows.

Since for every finite metric space, $\mathcal{F}(M)$ has property α (see Example 2.19), we obtain the following interesting particular case.

Corollary 4.11. *Let M be a concave finite metric space. Then, (M,Y) has the Lip-BPB property for every Banach space Y.*

Another class of metric spaces for which Theorem 4.9 is applicable is the one of ultrametric spaces. A metric space is said to be *ultrametric* if the inequality

$$d(x,y) \leqslant \max\big\{d(x,z), d(z,y)\big\}$$

holds for all $x,y,z \in M$. This class of metric spaces has been deeply studied due to its relations with the problem of finding good embedding of metric spaces, see [59] and references therein, for instance. Properties on the Lipschitz-free space over an ultrametric space can be found in [29] and references therein, for instance. It readily follows that every ultrametric space is uniformly Gromov concave, so we get the following consequence of Theorem 4.9.

Corollary 4.12. *Let M be an ultrametric space. Then, (M,Y) has the Lip-BPB property for every Banach space Y.*

Finally, we may also obtain a large class of metric spaces, which includes some connected metric spaces, for which the Lip-BPB property is satisfied for every Banach space Y: the class of Hölder metric spaces. Indeed, Proposition 2.24 states that every Hölder metric space is uniformly Gromov concave. We refer the reader to the paper [51] and the book [65] as good references on Hölder metric spaces.

Corollary 4.13. *Let M be a Hölder metric space. Then, (M,Y) has the Lip-BPB property for every Banach space Y.*

Examples 4.5 and 4.6 show that it seems like the Lip-BPB property does not hold when the metric space has many nontrivial metric segments. For this reason, and in view of Corollary 4.10, we could believe that if the metric space is concave or even Gromov concave, (M,Y) may have the Lip-BPB property for all Banach spaces Y. However, the next example shows that this does not always happen, even for scalar Lipschitz functions.

Example 4.14. *There exists a Gromov concave metric space M such that $\mathcal{F}(M)$ has the RNP and $(M,\mathbb{R})$ fails the Lip-BPB property.*

Proof. Let us consider $M = \big\{\big(n, \frac{1}{n^2}\big) : n \in \mathbb{N}\big\} \subseteq \mathbb{R}^2$ with the Euclidean metric. This metric space is boundedly compact and every metric segment is trivial, so M is concave by [65, Proposition 3.34]. Furthermore, since M is uniformly discrete, Proposition 5.3 in [36] gives that M is Gromov concave. In addition, uniformly discreteness also implies that $\mathcal{F}(M)$ has the RNP [51, Proposition 4.4]. We will write $\overline{n}$ to refer to the point $\big(n, \frac{1}{n^2}\big)$ for every $n \in \mathbb{N}$. Fix $0 < \varepsilon < \frac{1}{3}$ and suppose that $(M,\mathbb{R})$ has the Lip-BPB property witnessed by the function $\varepsilon \longmapsto \eta(\varepsilon)$, which we may suppose satisfies $0 < \eta(\varepsilon) < \frac{1}{3}$.

For every $n \in \mathbb{N}$, we define $f_n \colon M \longrightarrow \mathbb{R}$ by

$$f_n(\overline{p}) = \begin{cases} p-1 & \text{if } p \leqslant 2n \\ p-2 & \text{if } p > 2n \end{cases}$$

It is clear that $f_n \in \operatorname{Lip}_0(M,\mathbb{R})$ and $\|f_n\|_L \leqslant 1$. Furthermore, given $k > 2n$ we have that

$$\widehat{f_n}(m_{\overline{k+1},\overline{k}}) = \frac{f_n(\overline{k+1}) - f_n(\overline{k})}{d(\overline{k+1},\overline{k})} = \frac{1}{\sqrt{1+\Big(\frac{1}{k^2} - \frac{1}{(k+1)^2}\Big)^2}},$$

from which we deduce that $\lim_{k\to\infty} \widehat{f_n}(m_{\overline{k+1},\overline{k}}) = 1$ and so $\|f_n\|_L = 1$. Now, let us estimate the value of $\widehat{f_n}$ at the molecule $m_{\overline{3n},\overline{n}}$:

$$\widehat{f_n}(m_{\overline{3n},\overline{n}}) = \frac{f_n(\overline{3n}) - f_n(\overline{n})}{d(\overline{3n},\overline{n})} = \frac{2n-1}{2n} \frac{2n}{\sqrt{(2n)^2 + \left(\frac{1}{n^2} - \frac{1}{(3n)^2}\right)^2}}.$$

Therefore, there exists $n_0 \in \mathbb{N}$ such that for every $n \geqslant n_0$ we have that $\widehat{f_n}(m_{\overline{3n},\overline{n}}) > 1 - \eta(\varepsilon)$. Now, the Lip-BPB property of $(M,\mathbb{R})$ gives $g_n \in \mathrm{Lip}_0(\mathbb{N},\mathbb{R})$ and a molecule $m_{\overline{p_n},\overline{q_n}}$ such that

$$\|g_n\|_L = |\widehat{g_n}(m_{p_n,q_n})| = 1, \quad \|f_n - g_n\|_L < \varepsilon, \quad \|m_{\overline{p_n},\overline{q_n}} - m_{\overline{3n},\overline{n}}\| < \varepsilon.$$

Note that since f_n is increasing, p_n must be greater than q_n. As we did in the proof of Example 4.6, by applying Lemma 1.14 we obtain that $[2n, 2n+1] \subseteq [q_n, p_n]$. On the one hand, we have that

$$\widehat{f_n}(m_{\overline{2n+1},\overline{2n}}) = 0.$$

On the other hand, from Lemma 2.11 it follows that

$$\widehat{g_n}(m_{\overline{p_n},\overline{2n+1}}) \geqslant 1 - 2\frac{(\overline{p_n},\overline{q_n})_{\overline{2n+1}}}{d(\overline{2n+1},\overline{p_n})} \geqslant 1 - \frac{1}{(2n+1)^2 d(\overline{2n+1},\overline{p_n})},$$

which implies that

$$\begin{aligned}
\widehat{g_n}(m_{\overline{2n+1},\overline{2n}}) &= \widehat{g_n}(m_{\overline{p_n},\overline{2n+1}})\frac{d(\overline{2n+1},\overline{p_n})}{d(\overline{2n},\overline{2n+1})} - \widehat{g_n}(m_{\overline{p_n},\overline{2n}})\frac{d(\overline{2n},\overline{p_n})}{d(\overline{2n},\overline{2n+1})} \\
&\geqslant \widehat{g_n}(m_{\overline{p_n},\overline{2n+1}})\frac{d(\overline{2n+1},\overline{p_n})}{d(\overline{2n},\overline{2n+1})} - \frac{d(\overline{2n},\overline{p_n})}{d(\overline{2n},\overline{2n+1})} \\
&\geqslant \frac{(2n+1)^2 d(\overline{2n+1},\overline{p_n}) - 1 - (2n+1)^2 d(\overline{2n},\overline{p_n})}{(2n+1)^2 d(\overline{2n},\overline{2n+1})} \\
&\geqslant \frac{d(\overline{2n+1},\overline{p_n}) - d(\overline{2n},\overline{p_n})}{d(\overline{2n},\overline{2n+1})} - \frac{1}{(2n+1)^2}.
\end{aligned}$$

A simple calculation shows that we may take $n_1 > n_0 \in \mathbb{N}$ such that $\widehat{g_n}(m_{\overline{2n+1},\overline{2n}}) \geqslant \frac{1}{2}$ for every $n \geqslant n_1$. Finally, for $n \geqslant n_1$ observe that

$$\|g_n - f_n\|_L \geqslant \widehat{g_n}(m_{\overline{2n+1},\overline{2n}}) - \widehat{f_n}(m_{\overline{2n+1},\overline{2n}}) \geqslant \frac{1}{2} - 0 = \frac{1}{2},$$

a contradiction. □

Before finishing the chapter, let us ask a natural question: does there exist any relationship between the BPBp of the pair $(\mathcal{F}(M), Y)$ and the Lip-BPB property of the pair (M, Y)? Example 4.6 partially answers this question in a negative way. Note that, since the Bishop-Phelps-Bollobás theorem is valid for every Banach space, we know that the pair $(\mathcal{F}(\mathbb{N}), \mathbb{R})$ has the BPBp. However, in that example it is shown that $(\mathbb{N}, \mathbb{R})$ fails the Lip-BPB property, so the BPBp of $(\mathcal{F}(M), Y)$ does not imply the Lip-BPB property of (M, Y) in general. Conversely, as a consequence of Corollary 4.4 and the next result, we can show that the Lip-BPB property of (M, Y) does not imply the BPBp of $(\mathcal{F}(M), Y)$ either.

Proposition 4.15. *Let M be a finite metric space with more than two points. Then, there exists a Banach space Y such that $(\mathcal{F}(M), Y)$ fails the BPBp.*

Proof. Assume that $(\mathcal{F}(M), Y)$ has the BPBp for every Banach space Y. Being finite-dimensional, $\mathcal{F}(M)$ is isomorphic to a strictly convex Banach space. Then, by [12, Corollary 3.5], the set of extreme points of $B_{\mathcal{F}(M)}$ is dense in $S_{\mathcal{F}(M)}$. However, we have that $B_{\mathcal{F}(M)} = \mathrm{co}\big(\mathrm{Mol}(M)\big)$ since $\mathrm{Mol}(M)$ is finite hence compact. Moreover, Theorem 1.1 in [10] tells us that every extreme point of $B_{\mathcal{F}(M)}$ has to be contained in $\mathrm{Mol}(M)$. But, being finite, the set $\mathrm{Mol}(M)$ cannot be dense in $S_{\mathcal{F}(M)}$ if M contains more than two points. □

We can now present the desired example.

Example 4.16. *If we consider a concave finite metric space M, then (M, Y) has the Lip-BPB property for every Banach space Y by Corollary 4.11, while we may consider a Banach space Y such that $(\mathcal{F}(M), Y)$ fails the BPBp thanks to Proposition 4.15.*

Chapter 5

Stability results

Throughout this chapter, we will make a parallel study of the stability behavior of the Lip-BPB property and of the density of strongly norm-attaining Lipschitz maps. It is clear that if the pair (M,Y) has the Lip-BPB property, then $\mathrm{LipSNA}(M,Y)$ is dense in $\mathrm{Lip}_0(M,Y)$. Therefore, it is natural to think that results dealing with the Lip-BPB property may give us results concerning strong density. But we will see that not all of our results regarding the density of strongly norm-attaining Lipschitz maps are valid for the Lip-BPB property. First, we will study conditions which allow to pass from the Lip-BPB property for $(M,\mathbb{R})$ to (M,Y), also discussing analogous conditions for the density of strongly norm-attaining Lipschitz maps. Next, we will study stability properties of the Lip-BPB property and strong density. More concretely, we will analyze some operations that we can consider on metric spaces and Banach spaces for which they are stable.

The results obtained in this chapter come from the papers [24] and [25]. They were collaborative works with Miguel Martín.

5.1 From scalar functions to vector-valued maps

Given a metric space M and a Banach space Y, we have already seen that if the pair (M,Y) satisfies the Lip-BPB property, then so does the pair $(M,\mathbb{R})$ (see Proposition 4.7). Analogously, if $\mathrm{LipSNA}(M,Y)$ is dense, so is $\mathrm{LipSNA}(M,\mathbb{R})$ (see Proposition 2.36). Our goal in this section is to present conditions over the Banach space Y that allow us to pass from the Lip-BPB property of $(M,\mathbb{R})$ to the Lip-BPB property of (M,Y) for some class of operators, and from the density of $\mathrm{LipSNA}(M,\mathbb{R})$ to the density of $\mathrm{LipSNA}(M,Y)$.

The main results of this section will deal with the notions of Γ*-flat operators* and *ACK structure.* These two notions were introduced and deeply studied in [21], where they are analyzed to study the BPBp in a very general setting. We will follow [21] in order to get Lipschitz versions of their results.

First of all, we need to introduce some necessary definitions.

Definition 5.1. Let A be a topological space and (M,d) be a metric space. A function $f\colon A\longrightarrow M$ is said to be *openly fragmented*, if for every nonempty open subset $U\subset A$ and every $\varepsilon>0$ there exists a nonempty open subset $V\subset U$ with $\mathrm{diam}(f(V))<\varepsilon$.

It is clear that every continuous function $f\colon A\longrightarrow M$ is openly fragmented. In particular, if A is a discrete topological space, then every $f\colon A\longrightarrow M$ is openly fragmented.

Definition 5.2. Let X, Y be Banach spaces and $\Gamma\subset Y^*$. An operator $T\in\mathcal{L}(X,Y)$ is said to be Γ*-flat*, if $T^*|_\Gamma\colon(\Gamma,\omega^*)\longrightarrow(X^*,\|\cdot\|_{X^*})$ is openly fragmented. In other words, if for every ω^*-open subset $U\subseteq Y^*$ with $U\cap\Gamma\neq\emptyset$ and every $\varepsilon>0$ there exists a w^*-open subset $V\subset U$ with $V\cap\Gamma\neq\emptyset$ such that $\mathrm{diam}(T^*(V\cap\Gamma))<\varepsilon$. The set of all Γ-flat operators in $\mathcal{L}(X,Y)$ will be denoted by $\mathrm{Fl}_\Gamma(X,Y)$.

In [21] it is shown that every *Asplund operator* $T\in\mathcal{L}(X,Y)$ is Γ-flat for every $\Gamma\subseteq B_{Y^*}$. Consequently, every compact operator is Γ-flat for every $\Gamma\subseteq B_{Y^*}$. In addition, it is shown that if (Γ,ω^*) is discrete then every bounded operator $T \in \mathcal{L}(X,Y)$ is Γ-flat. Let us also comment that the recently introduced notion of *dentable map* [38] implies Γ-flatness.

Finally, they introduce the notion of ACK_ρ structure, which has the structural properties of $C(K)$ and its uniform subalgebras that are essential for the BPBp to hold. Let us recall that a subset Γ of the unit ball of the dual of a Banach space Y is *norming* if the absolutely weak-star closed convex hull of Γ equals the whole of B_{Y^*} or, equivalently, if $\|y\| = \sup\{|f(y)|\colon f \in \Gamma\}$ for every $y \in Y$.

Definition 5.3. We say that a Banach space Y has *ACK structure* with parameter ρ, for some $\rho \in [0,1)$ ($Y \in ACK_\rho$ for short) whenever there exists a norming set $\Gamma \subset B_{Y^*}$ such that for every $\varepsilon > 0$ and every nonempty relatively ω^*-open subset $U \subset \Gamma$, there exist a nonempty subset $V \subset U$, vectors $y_1^* \in V$, $e \in S_X$, and an operator $F \in \mathcal{L}(Y,Y)$ with the following properties:

(i) $\|Fe\| = \|F\| = 1$;

(ii) $y_1^*(Fe) = 1$;

(iii) $F^* y_1^* = y_1^*$;

(iv) denoting $V_1 = \{y^* \in \Gamma\colon \|F^*y^*\| + (1-\varepsilon)\|(\mathrm{I}_{Y^*} - F^*)(y^*)\| \leqslant 1\}$, then $|v^*(Fe)| \leqslant \rho$ for every $v^* \in \Gamma \setminus V_1$;

(v) $d(F^*y^*, \operatorname{aco}\{0, V\}) < \varepsilon$ for every $y^* \in \Gamma$; and

(vi) $|v^*(e) - 1| \leqslant \varepsilon$ for every $v^* \in V$.

The Banach space Y has *simple ACK structure* ($X \in ACK$) if $V_1 = \Gamma$ (and so ρ is redundant).

The following statement is a compilation of results that can be found in [21]. We introduce some notation. Given a Banach space Y, we write $c_0(Y,w)$ to denote the Banach space of all weakly null sequences in Y; if K is a compact Hausdorff topological space, $C_w(K,Y)$ is the Banach space of all Y-valued weakly continuous functions from K to Y. Also, we need the next definition which was introduced in [57] by Lindenstrauss.

Definition 5.4. A Banach space Y has property β if there is a set $\{(y_\lambda^*, y_\lambda)\colon \lambda \in \Lambda\} \subset Y^* \times Y$, and a constant $0 \leqslant \rho < 1$ satisfying

(i) $\|y_\lambda^*\| = \|y_\lambda\| = y_\lambda^*(y_\lambda) = 1$ for every $\lambda \in \Lambda$.

(ii) $|y_\lambda^*(y_\mu)| \leqslant \rho$ for every $\lambda \neq \mu \in \Lambda$.

(iii) $\|y\| = \sup\{|y_\lambda^*(y)|\colon \lambda \in \Lambda\}$ for every $y \in Y$.

If Y is a Banach space with property β, Lindenstrauss proved in [57] that $\mathrm{NA}(X,Y)$ is dense in $\mathcal{L}(X,Y)$ for every Banach space X. Examples of Banach spaces with property β are finite-dimensional spaces whose unit ball is a polyhedron and those spaces Y such that $c_0 \subset Y \subset \ell_\infty$ (canonical copies). Besides, J. Partington proved in [61] that every Banach space can be renormed to satisfy property β. It is convenient to comment that this property β is somehow dual to property α (see Definition 2.6). This fact can be seen in [64, Proposition 1.4].

Proposition 5.5 ([21])**.** *The following statements hold.*

(i) If Y is a Banach space having property β, then $Y \in ACK_\rho$ for a discrete norming set Γ.

(ii) $C(K)$ has simple ACK structure for every compact Hausdorff topological space K.

(iii) Finite injective tensor products of Banach spaces which have ACK_ρ structure also have ACK_ρ structure.

(iv) Given a compact Hausdorff topological space K, if $Y \in ACK_\rho$ then $C(K,Y) \in ACK_\rho$.

(v) Let Y be a Banach space having ACK_ρ structure. Then $c_0(Y)$, $\ell_\infty(Y)$, and $c_0(Y,w)$ have ACK_ρ structure.

(vi) Given a compact Hausdorff topological space K, if $Y \in ACK_\rho$, then $C_w(K,Y)$ has ACK_ρ structure.

The main result of this section is the following one.

Theorem 5.6. *Let M be a metric space such that $(M,\mathbb{R})$ has the Lip-BPB property, let Y be a Banach space in ACK_ρ with associated norming set $\Gamma \subseteq B_{Y^*}$ of Definition 5.3, and let $\varepsilon > 0$. Then, there exists $\eta(\varepsilon,\rho) > 0$ such that if we take $\widehat{T} \in \mathcal{L}(\mathcal{F}(M),Y)$ a Γ-flat operator with $\|T\|_L = 1$ and $m \in \operatorname{Mol}(M)$ satisfying $\|\widehat{T}(m)\| > 1 - \eta(\varepsilon,\rho)$, then we may find an operator $\widehat{S} \in \mathcal{L}(\mathcal{F}(M),Y)$ and a molecule $u \in \operatorname{Mol}(M)$ such that*

$$\|\widehat{S}(u)\| = \|S\|_L = 1, \quad \|m-u\| < \varepsilon, \quad \|T-S\|_L < \varepsilon.$$

Prior to give the proof, we present the main consequences of Theorem 5.6.

Corollary 5.7. *Let M be a metric space such that $(M,\mathbb{R})$ has the Lip-BPB property. The following statements hold.*

(i) If Y is a Banach space having property β, then (M,Y) has the Lip-BPB property.

(ii) For every compact Hausdorff topological space K, the pair $(M,C(K))$ has the Lip-BPB property for Γ-flat operators, where Γ is the norming set given by Definition 5.3 for $C(K)$.

(iii) Let Z be a finite injective tensor product of Banach spaces which have ACK_ρ structure. Then, (M,Z) has the Lip-BPB property for Γ-flat operators, where Γ is the norming set given by Definition 5.3 for Z.

(iv) Let K be a compact Hausdorff topological space. If $Y \in ACK_\rho$, then $(M,C(K,Y))$ and $(M,C_w(K,Y))$ have the Lip-BPB property for Γ-flat operators, where Γ is the norming set given by Definition 5.3 for $C(K,Y)$ and $(M,C_w(K,Y))$, respectively.

(v) Let $Y \in ACK_\rho$. Then, $(M,c_0(Y))$, $(M,\ell_\infty(Y))$, and $(M,c_0(Y,w))$ have the Lip-BPB property for Γ-flat operators, where Γ is the corresponding norming set given by Definition 5.3 for each case.

Proof. The proof of assertion (i) follows from Theorem 5.6 and the fact that if Y has property β, then $Y \in ACK_\rho$ for a discrete norming set Γ. Indeed, it is clear that in such a case, every operator $T \in \mathcal{L}(\mathcal{F}(M),Y)$ is Γ-flat, and Theorem 5.6 applies. The rest of the assertions follow immediately from Theorem 5.6 and Proposition 5.5. □

Let us give some comments on assertion (ii) of Corollary 5.7. First, the set Γ of Definition 5.3 for the case $Y = C(K)$ is just $\Gamma = \{\delta_t : t \in K\} \subset S_{C(K)^*}$ (this follows from the results in the paper [21]), so given $T \in \mathcal{L}(X,C(K))$, $T^*|_\Gamma$ is just the usual representation function of the operator T, that is, $\mu_T : K \longrightarrow X^*$ given by $\mu_T(t) = T^*(\delta_t)$ for all $t \in K$. This procedure actually gives an identification between $\mathcal{L}(X,C(K))$ and the space of those weak-star continuous functions $\mu : K \longrightarrow X^*$. Norm continuous functions correspond to compact operators (which are Γ-flat). We do not know which functions are openly fragmented or, equivalently, which functions correspond to Γ-flat operators, but there is an intermediate condition which has been studied widely in the literature: quasi-continuous functions. A function $\mu : K \longrightarrow X^*$ is *quasi-continuous* if for every non-empty open subset $U \subset K$, every $s \in U$, and every neighborhood V of $\mu(s)$, there exists a non-empty open subset $W \subset U$ such that $\mu(W) \subset V$. This is a classical notion which is still investigated, see the paper [14] and references therein, for instance. Quasi-continuous functions are openly fragmented and they form a class more general than the one of continuous functions.

Let us now prepare the way for the proof of Theorem 5.6 by presenting some preliminary results.

Lemma 5.8. *Let M be a metric space and let $\varepsilon > 0$. Suppose that $(M,\mathbb{R})$ has the Lip-BPB property witnessed by the function $\varepsilon \longmapsto \eta(\varepsilon) > 0$. Then, given $f \in \operatorname{Lip}_0(M,\mathbb{R})$ with $\|f\|_L \leqslant 1$ and $m \in \operatorname{Mol}(M)$ such that $|\widehat{f}(m)| > 1 - \eta(\varepsilon)$, there exist $g \in \operatorname{Lip}_0(M,\mathbb{R})$ with $\|g\|_L = 1$ and $u \in \operatorname{Mol}(M)$ satisfying*

$$|\widehat{g}(u)| = 1, \quad \|f-g\|_L < \varepsilon + \eta(\varepsilon), \quad \|m-u\| < \varepsilon.$$

Proof. If $\|f\|_L = 1$ then it is enough to apply the Lip-BPB property. If $\|f\|_L < 1$, by applying the Lip-BPB property, we know that there exist $g \in S_{\mathrm{Lip}_0(M,\mathbb{R})}$ and $u \in \mathrm{Mol}(M)$ satisfying

$$\left\| g - \frac{f}{\|f\|} \right\|_L < \varepsilon, \quad \|u - m\| < \varepsilon.$$

Then, note that

$$\|g - f\|_L \leqslant \left\| g - \frac{f}{\|f\|} \right\|_L + \left\| \frac{f}{\|f\|} - f \right\|_L < \varepsilon + |1 - \|f\|_L| \leqslant \varepsilon + \eta(\varepsilon). \qquad \square$$

Lemma 5.9. *Let M be a metric space such that $(M,\mathbb{R})$ has the Lip-BPB property, let Y be a Banach space, and let $\Gamma \subseteq B_{Y^*}$ be a norming set. Fix $\varepsilon > 0$ and consider $\eta(\varepsilon)$ the constant given by the Lip-BPB property of $(M,\mathbb{R})$. Let $\widehat{T} \in \mathrm{Fl}_\Gamma(\mathcal{F}(M), Y)$ be a Γ-flat operator with $\|T\|_L = 1$ and $m \in \mathrm{Mol}(M)$ such that*

$$\|\widehat{T}(m)\| > 1 - \eta(\varepsilon).$$

Then, for every $r > 0$ there exist:

(i) a ω^*-open subset $U_r \subset V$ with $U_r \cap \Gamma \neq \emptyset$,

(ii) $\widehat{f}_r \in S_{\mathcal{F}(M)^*}$ and $u_r \in \mathrm{Mol}(M)$ satisfying

$$\widehat{f}_r(u_r) = 1, \quad \|m - u_r\| \leqslant \varepsilon, \quad \|\widehat{T}^* z^* - \widehat{f}_r\| \leqslant r + \varepsilon + \eta(\varepsilon) \quad \forall\, z^* \in U_r \cap \Gamma.$$

Proof. We just have to repeat the proof of Lemma 2.9 in [21] using Lemma 5.8 instead of Proposition 2.11 in [21]. Since Γ is norming, we can pick $y_0^* \in \Gamma$ such that

$$|\widehat{T}^*(y_0^*)(m)| = |y_0^*(\widehat{T}(m))| > 1 - \eta(\varepsilon).$$

Set $U = \{y^* \in Y^* \colon |\widehat{T}^*(y^*)(m)| > 1 - \eta(\varepsilon)\}$. We have that $y_0^* \in U \cap \Gamma \subseteq B_{Y^*}$. Since U is ω^*-open in Y^* and $U \cap \Gamma \neq \emptyset$, according to Definition 5.2, for every $r > 0$ there is a ω^*-open set $U_r \subseteq U$ with $U_r \cap U \neq \emptyset$ such that $\mathrm{diam}(\widehat{T}^*(U_r \cap \Gamma)) < r$. $\square$

Now, we are able to prove the main result of this section.

Proof of Theorem 5.6. Given $\varepsilon > 0$, let $\widehat{\eta}(\varepsilon) > 0$ be the constant associated to the Lip-BPB property of $(M,\mathbb{R})$. Fix $0 < \varepsilon_0 < \varepsilon$ and take $\varepsilon_1 > 0$ such that

$$\max\left\{\varepsilon_1, 2\left((\varepsilon_1 + \eta(\varepsilon_1)) + \frac{2(\varepsilon_1 + \eta(\varepsilon_1))}{1 - \rho + (\varepsilon_1 + \eta(\varepsilon_1))}\right)\right\} \leqslant \varepsilon_0.$$

Take $r > 0$ and $0 < \varepsilon_2 < \frac{2}{3}$. Consider $\widehat{T} \in \mathcal{L}(\mathcal{F}(M), Y)$ a Γ-flat operator with $\|T\|_L = 1$ and a molecule $m \in \mathrm{Mol}(M)$ such that $\|\widehat{T}(m)\| > 1 - \widehat{\eta}(\varepsilon)$. Then, applying Lemma 5.9 with Y, Γ, r and ε_1, we obtain an ω^*-open subset $U_r \subseteq Y^*$ with $U_r \cap \Gamma \neq \emptyset$, and $\widehat{f}_r \in S_{\mathcal{F}(M)^*}$, $u_r \in \mathrm{Mol}(M)$ satisfying

$$\widehat{f}_r(u_r) = 1, \quad \|m - u\| \leqslant \varepsilon_1, \quad \|\widehat{T}^* z^* - \widehat{f}_r\| \leqslant r + \varepsilon_1 + \eta(\varepsilon_1) \quad \forall\, z^* \in U_r \cap \Gamma.$$

On the other hand, since $U_r \cap \Gamma \neq \emptyset$, by applying the definition of ACK_ρ structure to $U = U_r \cap \Gamma$ and ε_2, we obtain a nonempty subset $V \subseteq U$, points $y_1^* \in V$ and $e \in S_Y$, an operator $F \in \mathcal{L}(Y, Y)$, and a subset $V_1 \subseteq \Gamma$ satisfying the properties of Definition 5.3.

Let us define the linear operator $\widehat{S} \colon \mathcal{F}(M) \longrightarrow Y$ by

$$\widehat{S}(x) = \widehat{f}_r(x) F e + (1 - \delta)(\mathrm{Id}_Y - F)\widehat{T}(x),$$

where $\delta \in [\varepsilon_2, 1)$. We will choose δ so that $\|\widehat{S}\| \leqslant 1$. In order to estimate $\|\widehat{S}\|$, recall that since Γ is a norming set, we have that

$$\|\widehat{S}\| = \|\widehat{S}^*\| = \sup\left\{\left\|\widehat{S}^* y^*\right\| \colon y^* \in \Gamma\right\}.$$

Therefore, we take $y^* \in \Gamma$ and estimate

$$\|\widehat{S}^* y^*\| = \big\|y^*(Fe)\widehat{f}_r + (1-\delta)\widehat{T}^*(\mathrm{Id}_{Y^*} - F^*)(y^*)\big\|.$$

If $y^* \in V_1$, then that $\|\widehat{S}^* y^*\| \leqslant 1$ follows from the property (iv) of Definition 5.3. Therefore, we have to consider only the case when $y^* \in \Gamma \setminus V_1$. As before, by Definition 5.3, for every $y^* \in \Gamma$ there exists a point $v^* = \sum_{k=1}^n \lambda_k v_k^*$ satisfying

$$\{v_1^*, \dots, v_n^*\} \subseteq V, \quad \sum_{k=1}^n |\lambda_k| \leqslant 1, \quad \|F^* y^* - v^*\| < \varepsilon_2.$$

Consequently,

$$\begin{aligned}
\|v^*(e)\widehat{f}_r - \widehat{T}^* v^*\| &\leqslant \sum_{k=1}^n |\lambda_k| \|v_k^*(e)\widehat{f}_r - \widehat{T}^* v_k^*\| \\
&\leqslant \sum_{k=1}^n |\lambda_k| (\|v_k^*(e)\widehat{f}_r - \widehat{f}_r\| + \|\widehat{f}_r - \widehat{T}^* v_k^*\|) \\
&\leqslant \varepsilon_2 + \sum_{k=1}^n |\lambda_k| \|\widehat{f}_r - \widehat{T}^* v_k^*\| \leqslant \varepsilon_2 + r + \varepsilon_1 + \eta(\varepsilon_1).
\end{aligned}$$

Now, for every $y^* \in \Gamma \setminus V_1$ we have that

$$\begin{aligned}
\|\widehat{S}^* y^*\| &\leqslant \delta|y^*(Fe)| + (1-\delta)\|y^*(Fe)\widehat{f}_r + \widehat{T}^* y^* - \widehat{T}^* F^* y^*\| \\
&\leqslant \delta\rho + (1-\delta)\|\widehat{T}^* y^*\| + (1-\delta)\|(F^* y^*)(e)\widehat{f}_r - \widehat{T}^* F^* y^*\| \\
&\leqslant \delta\rho + (1-\delta) + 2\varepsilon_2(1-\delta) + (1-\delta)\|v^*(e)\widehat{f}_r - R^* v^*\| \\
&\leqslant \delta\rho + (1-\delta) + 2\varepsilon_2(1-\delta) + (1-\delta)(\varepsilon_2 + r + \varepsilon_1 + \eta(\varepsilon_1)) \\
&\leqslant \delta\rho + (1-\delta)(1 + 3\varepsilon_2 + r + \varepsilon_1 + \eta(\varepsilon_1)).
\end{aligned}$$

Therefore, if we choose

$$\delta = \frac{3\varepsilon_2 + r + \varepsilon_1 + \eta(\varepsilon_1)}{1 - \rho + 3\varepsilon_2 + r + \varepsilon_1 + \eta(\varepsilon_1)} \in [2/3, 1) \subseteq [\varepsilon_2, 1),$$

then we will have that $\|\widehat{S}\| \leqslant 1$. In this case,

$$1 = |\widehat{f}_r(u)| = |y_1^*(\widehat{f}_r(u)Fe)| = |y_1^*(\widehat{S}(u))| \leqslant \|\widehat{S}(u)\| \leqslant 1,$$

from which we deduce that $\|\widehat{S}\| = 1$ and $\widehat{S}$ attains its norm at the molecule u, which we already knew satisfies $\|m - u\| \leqslant \varepsilon_1 \leqslant \varepsilon_0 < \varepsilon$.

Finally, let us estimate $\|\widehat{S} - \widehat{T}\|$. First,

$$\begin{aligned}
\|\widehat{S} - \widehat{T}\| = \|\widehat{S}^* - \widehat{T}^*\| &= \sup\big\{\big|\widehat{S}^* y^* - \widehat{T}^* y^*\big| : y^* \in \Gamma\big\} \\
&\leqslant 2\delta + \sup\Big\{\|y^*(Fe)\widehat{f}_r - \widehat{T}^* F^* y^*\| : y^* \in \Gamma\Big\}.
\end{aligned}$$

Second,

$$\|(F^* y^*)(e)\widehat{f}_r - \widehat{T}^* F^* y^*\| \leqslant 2\varepsilon_2 + \|v^*(e)\widehat{f}_r - \widehat{T}^* v^*\| \leqslant 3\varepsilon_2 + r + \varepsilon_1 + \eta(\varepsilon_1).$$

Therefore, we obtain that

$$\|\widehat{S} - \widehat{T}\| \leqslant 2\delta + 3\varepsilon_2 + r + \varepsilon_1 + \eta(\varepsilon_1).$$

Since ε_2 and r were arbitrary, by taking these constants satisfying $3\varepsilon_2 + r \leqslant \varepsilon_1 + \eta(\varepsilon_1)$, we will have that

$$\begin{aligned}
\|\widehat{S} - \widehat{T}\| &\leqslant 2(\varepsilon_1 + \eta(\varepsilon_1) + \delta) \\
&\leqslant 2\left((\varepsilon_1 + \eta(\varepsilon_1)) + \frac{2(\varepsilon_1 + \eta(\varepsilon_1))}{1 - \rho + \varepsilon_1 + \eta(\varepsilon_1)}\right) \leqslant \varepsilon_0 < \varepsilon.
\end{aligned}$$

□

Let us now discuss when the density of $\operatorname{LipSNA}(M,\mathbb{R})$ implies the density of $\operatorname{LipSNA}(M,Y)$. First, it is possible to give a result analogous to Theorem 5.3, but for the density of $\operatorname{LipSNA}(M,Y)$. We just have to repeat the proof of Theorem 5.6, using that $\operatorname{LipSNA}(M,\mathbb{R})$ is dense in $\operatorname{Lip}_0(M,\mathbb{R})$ instead of the Lip-BPB property of $(M,\mathbb{R})$ and forgetting about the estimation of the distance between molecules.

Theorem 5.10. *Let M be a metric space such that $\operatorname{LipSNA}(M,\mathbb{R})$ is dense in $\operatorname{Lip}_0(M,\mathbb{R})$, let Y be a Banach space in ACK_ρ, and let $\Gamma \subseteq B_{Y^*}$ be the norming set given by Definition 5.3. Then, we have that*

$$\operatorname{Fl}_\Gamma(\mathcal{F}(M),Y) \subseteq \overline{\operatorname{LipSNA}(M,Y)}.$$

As before, from this result we obtain a series of consequences.

Corollary 5.11. *Let M be a metric space such that $\operatorname{LipSNA}(M,\mathbb{R})$ is dense in $\operatorname{Lip}_0(M,\mathbb{R})$. Then, the following statements hold.*

(i) If Y is a Banach space having property β, then $\operatorname{LipSNA}(M,Y)$ is dense in $\operatorname{Lip}_0(M,Y)$.

(ii) Let K be a compact Hausdorff topological space and $\Gamma = \{\delta_t : t \in K\}$. Then $\operatorname{Fl}_\Gamma(\mathcal{F}(M),C(K)) \subseteq \overline{\operatorname{LipSNA}(M,C(K))}$.

(iii) Let Z be a finite injective tensor product of Banach spaces which have ACK_ρ structure. Then, if Γ is the norming set given by Definition 5.3, we have $\operatorname{Fl}_\Gamma(\mathcal{F}(M),Z) \subseteq \overline{\operatorname{LipSNA}(M,Z)}$.

(iv) Let K be a compact Hausdorff topological space. If $Y \in ACK_\rho$ and Γ is the norming set given by Definition 5.3, then

$$\operatorname{Fl}_\Gamma(\mathcal{F}(M),C(K,Y)) \subseteq \overline{\operatorname{LipSNA}(M,C(K,Y))}.$$

(v) Let $Y \in ACK_\rho$. If Γ is the norming set given by Definition 5.3, then

$$\operatorname{Fl}_\Gamma(\mathcal{F}(M),c_0(Y)) \subseteq \overline{\operatorname{LipSNA}(M,c_0(Y))},$$
$$\operatorname{Fl}_\Gamma(\mathcal{F}(M),\ell_\infty(Y)) \subseteq \overline{\operatorname{LipSNA}(M,\ell_\infty(Y))},$$
$$\operatorname{Fl}_\Gamma(\mathcal{F}(M),c_0(Y,w)) \subseteq \overline{\operatorname{LipSNA}(M,c_0(Y,w))}.$$

(vi) Let K be a compact Hausdorff topological space. If $Y \in ACK_\rho$ and Γ is the norming set given by Definition 5.3, then

$$\operatorname{Fl}_\Gamma(\mathcal{F}(M),C_w(K,Y)) \subseteq \overline{\operatorname{LipSNA}(M,C_w(K,Y))}.$$

Let us comment that, as happened in Corollary 5.7, since the norming set Γ is discrete when Y has property β, in such a case every operator is Γ-flat. Consequently, we obtain strong density for the whole space $\operatorname{Lip}_0(M,Y)$.

This observation about property β motivates the study of the relationship between property quasi-β and the Lip-BPB property or strong density. Property quasi-β was introduced in [2] and it is a property on a Banach space X weaker than property β that is still a sufficient condition to guarantee that $\overline{\operatorname{NA}(X,Y)} = \mathcal{L}(X,Y)$ for every Banach space X.

Definition 5.12. We say that a Banach space Y has *property quasi-β* if there exist a subset $A \subset S_{Y^*}$, a mapping $\sigma\colon A \longrightarrow S_Y$, and a real-valued function ρ on A satisfying the following conditions:

(i) $y^*(\sigma(y)) = 1$ for every $y^* \in A$.

(ii) $|z^*(\sigma(y^*))| \leqslant \rho(y^*) < 1$ for every $y^*, z^* \in A$, $y^* \neq z^*$.

(iii) For every extreme point e^* in the unit ball of Y^*, there is a subset A_{e^*} of A and a scalar t with $|t| = 1$ such that te^* lies in the w^*-closure of A_{e^*} and $\sup\{\rho(y^*) : y^* \in A_{e^*}\} < 1$.

Every Banach space having property β has property quasi-β. Moreover, property quasi-β is stable under c_0-sums (see [2, Proposition 4]), so c_0-sums of Banach spaces having property β have property quasi-β, but they may have not property β. In addition, there are finite-dimensional Banach spaces having property quasi-β but not β (see [2, Example 5]).

Notice that assertion (i) of Corollary 5.7 states that property β is a sufficient condition for the Banach space Y to ensure that (M, Y) has the Lip-BPB property when assuming that the pair $(M, \mathbb{R})$ also does. It is very natural to wonder if property quasi-β is still enough to guarantee that we can pass from the Lip-BPB property of $(M, \mathbb{R})$ to the Lip-BPB property of (M, Y). Let us show the answer to this question is negative. In order to prove it, we need the following preliminary result, which is based on [12, Proposition 2.3]. For the reader's convenience, we include a sketch of the proof.

Lemma 5.13. *Let M be a metric space, Y be a Banach space and Y_1 be an ℓ_∞ summand of Y. If the pair (M, Y) has the Lip-BPB property with a function $\eta(\varepsilon)$, then (M, Y_1) also has the Lip-BPB property with the same function.*

Proof. Let E denote the isometric embedding of Y_1 into Y, and let P denote the natural projection from Y onto Y_1. Fix $\varepsilon > 0$, let $\eta(\varepsilon) > 0$ be the constant given by the Lip-BPB property of (M, Y), and consider $\widehat{F_1} \in \mathcal{L}(\mathcal{F}(M), Y_1)$ with $\|F_1\|_L = 1$ and $m \in \operatorname{Mol}(M)$ such that $\|\widehat{F_1}(m)\| > 1 - \eta(\varepsilon)$. Let us consider the linear operator $\widehat{F} \in \mathcal{L}(\mathcal{F}(M), Y)$ given by $\widehat{F} = E \circ \widehat{F_1}$. It is immediate to check that $\|\widehat{F}\| = \|\widehat{F_1}\|$ and $\|\widehat{F}(m)\| > 1 - \eta(\varepsilon)$. Hence, since the pair $(\mathcal{F}(M), Y)$ has the Lip-BPB property, we find an operator $\widehat{G} \in \mathcal{L}(\mathcal{F}(M), Y)$ and a molecule $u \in \operatorname{Mol}(M)$ such that

$$\|\widehat{G}(u)\| = \|G\|_L = 1, \quad \|F - G\|_L < \varepsilon, \quad \|m - u\| < \varepsilon.$$

Finally, consider the linear operator $G_1 = P \circ G \in \mathcal{L}(\mathcal{F}(M), Y_1)$. Then, since $\widehat{F_1} = P \circ \widehat{F}$ and $\widehat{F} = E \circ \widehat{F_1}$, we must have that

$$\|\widehat{G_1}(u)\| = \|G_1\|_L = 1, \quad \|F_1 - G_1\|_L < \varepsilon, \quad \|m - u\| < \varepsilon. \qquad \square$$

The last result will be improved in the next section, where we study the stability behavior of the Lip-BPB property under some operations that we can consider for metric and Banach spaces. The following example is based on [12, Example 4.1].

Example 5.14. *For each $k \in \mathbb{N}$ with $k \geqslant 2$, consider $Y_k = \mathbb{R}^2$ endowed with the norm*

$$\|(x, y)\| = \max\left\{|x|, |y| + \frac{1}{k}|x|\right\} \qquad \forall\, (x, y) \in \mathbb{R}^2.$$

Observe that B_{Y_k} is the absolutely convex hull of the set $\{(0,1), (1, 1 - \frac{1}{k}), (-1, 1 - \frac{1}{k})\}$, so each Y_k is polyhedral. Consequently, Y_k has property β (see [57]). Now, consider the metric space $M = \{0, 1, 2\}$ with the usual metric. By Corollary 4.4, we know that the pair $(M, \mathbb{R})$ has the Lip-BPB property. Besides, $Y = [\oplus_{k \in \mathbb{N}} Y_k]_{c_0}$ has property quasi-β by [2, Proposition 4] (as it is a c_0-sum of Banach spaces with property β). However, the pair (M, Y) fails the Lip-BPB property.

Proof. Fix $0 < \varepsilon < \frac{1}{2}$ and assume that there exists $\eta(\varepsilon) > 0$ such that (M, Y_k) has the Lip-BPB property with this function for that ε for every $k \in \mathbb{N}$ with $k \geqslant 2$, that is, for every $\widehat{F_k} \in \mathcal{L}(\mathcal{F}(M), Y_k)$ with $\|\widehat{F_k}\| = 1$ and every $m_k \in \operatorname{Mol}(M)$ such that $\|\widehat{F_k}(m_k)\| > 1 - \eta(\varepsilon)$, there exist $\widehat{G_k} \in \mathcal{L}(\mathcal{F}(M), Y_k)$ and $u_k \in \operatorname{Mol}(M)$ such that

$$\|\widehat{G}_k(u_k)\| = \|G_k\|_L = 1, \quad \|\widehat{F}_k - \widehat{G}_k\| < \varepsilon, \quad \|m_k - u_k\| < \varepsilon,$$

for every $k \in \mathbb{N}$ with $k \geqslant 2$. Recall that $\mathcal{F}(M)$ is two-dimensional and that $m_{0,2} = \frac{1}{2}m_{0,1} + \frac{1}{2}m_{1,2}$, so $B_{\mathcal{F}(M)} = \overline{\operatorname{co}}\{\pm m_{0,1}, \pm m_{1,2}\}$ is a square. For every $k \in \mathbb{N}$ with $k \geqslant 2$, define $\widehat{F}_k \colon \mathcal{F}(M) \longrightarrow Y_k$ by

$$\widehat{F}_k(m_{0,1}) = \left(-1, 1 - \frac{1}{k}\right) \quad \text{and} \quad \widehat{F}_k(m_{1,2}) = \left(1, 1 - \frac{1}{k}\right).$$

Clearly $\|F_k\|_L = 1$ and $\widehat{F}_k(m_{0,2}) = \widehat{F}_k\left(\frac{1}{2}m_{0,1} + \frac{1}{2}m_{1,2}\right) = \left(0, 1 - \frac{1}{k}\right)$. Hence, $\|\widehat{F}_k(m_{0,2})\| = 1 - \frac{1}{k}$. Then, for every $k \in \mathbb{N}$ such that $1 - \frac{1}{k} > 1 - \eta(\varepsilon)$, we may find $\widehat{G}_k \colon \mathcal{F}(M) \longrightarrow Y_k$ and $u_k \in \operatorname{Mol}(M)$ such that

$$\|\widehat{G}_k(u_k)\| = \|G_k\|_L = 1 \quad \|F_k - G_k\|_L < \varepsilon \quad \|u_k - m_{0,2}\| < \varepsilon.$$

A straightforward application of Lemma 1.14 shows that

$$\|m_{0,2} - m_{0,1}\|, \|m_{0,2} - m_{1,2}\| \geqslant 1.$$

Hence, $u_k = m_{0,2}$ for every $k \in \mathbb{N}$ such that $1 - \frac{1}{k} > 1 - \eta(\varepsilon)$. As $u_k = m_{0,2} = \frac{1}{2}m_{0,1} + \frac{1}{2}m_{1,2}$ and $\|\widehat{G}_k(u_k)\| = 1$, it follows that the whole interval $[\widehat{G}_k(m_{0,1}), \widehat{G}_k(m_{1,2})]$ lies on S_{Y_k}, so $\widehat{G}_k(m_{0,1})$ and $\widehat{G}_k(m_{1,2})$ belong to the same face of B_{Y_k}. As a consequence, by the shape of B_{Y_k}, we obtain that $\|\widehat{G}_k(m_{0,1}) - \widehat{G}_k(m_{1,2})\| \leqslant 1$. Furthermore, since $\|F_k - G_k\|_L < \varepsilon$, we have that

$$\|\widehat{F}_k(m_{0,1}) - \widehat{G}_k(m_{1,2})\| \leqslant \|\widehat{F}_k(m_{0,1}) - \widehat{G}_k(m_{0,1})\| + \|\widehat{G}_k(m_{0,1}) - \widehat{G}_k(m_{1,2})\| < \varepsilon + 1 < \frac{3}{2}.$$

On the other hand, since $\|\widehat{F}_k(m_{0,1}) - \widehat{F}_k(m_{1,2})\| = 2$,

$$\|\widehat{F}_k(m_{0,1}) - \widehat{F}_k(m_{0,2})\| \geqslant \|\widehat{F}_k(m_{0,1}) - \widehat{F}_k(m_{0,2})\| - \|\widehat{F}_k(m_{1,2}) - \widehat{G}_k(m_{1,2})\| > 2 - \varepsilon > \frac{3}{2},$$

which is a contradiction. Note that $Y = [\oplus_{k\in\mathbb{N}} Y_k]_{c_0}$, so Lemma 5.13 implies that (M, Y) does not have the Lip-BPB property. □

The next result shows that property quasi-β is a sufficient condition on Y to get density of $\mathrm{LipSNA}(M, Y)$ from the density of $\mathrm{LipSNA}(M, \mathbb{R})$. Its proof is an adaptation for Lipschitz maps of the one given in [2, Theorem 2].

Proposition 5.15. *Let M be a metric space such that $\mathrm{LipSNA}(M, \mathbb{R})$ is norm dense in $\mathrm{Lip}_0(M, \mathbb{R})$ and let Y be a Banach space having property quasi-β. Then, we have that*

$$\overline{\mathrm{LipSNA}(M, Y)} = \mathrm{Lip}_0(M, Y).$$

Proof. First, we use a result of V. Zizler in [66] which states that the set

$$\{T \in \mathcal{L}(X, Y) \colon T^* \in \mathrm{NA}(Y^*, X^*)\}$$

is dense in $\mathcal{L}(X, Y)$ for every Banach spaces X and Y. Therefore, it will be enough to show that for every $\widehat{F} \in \mathcal{L}(\mathcal{F}(M), Y)$ with $\|F\|_L = 1$ in this set and $\varepsilon > 0$ there exist $\widehat{G} \in \mathcal{L}(\mathcal{F}(M), Y)$ and $u \in \mathrm{Mol}(M)$ such that

$$\|\widehat{G}(u)\| = \|G\|_L = 1 \quad \text{and} \quad \|F - G\|_L < \varepsilon.$$

By a result of T. Johannesen (see [56, Theorem 5.8]), we know that $\widehat{F}^*$ attains its norm at an extreme point e^* of B_{Y^*}, and the definition of property quasi-β gives us a set $A_{e^*} \subseteq A$ and a scalar t with $|t| = 1$ such that te^* lies in the w^*-closure of A_{e^*} and

$$r = \sup\{\rho(y^*) \colon y^* \in A_{e^*}\} < 1.$$

Let us fix $0 < \gamma < \frac{\varepsilon}{2}$ satisfying

$$1 + r\left(\frac{\varepsilon}{2} + \gamma\right) < \left(1 + \frac{\varepsilon}{2}\right)(1 - \gamma)$$

and take $y_1^* \in A_{e^*}$ such that $\|\widehat{F}^* y_1^*\| > 1 - \gamma$. By hypothesis, there exist $\widehat{g} \in \mathcal{F}(M)^*$ and $u \in \mathrm{Mol}(M)$ such that

$$\|\widehat{g}(u)\| = \|\widehat{g}\| = \|\widehat{F}^*(y_1^*)\| > 1 - \gamma \quad \text{and} \quad \|\widehat{g} - \widehat{F}^*(y_1^*)\| < \gamma.$$

Define the operator $\widehat{G} \in \mathcal{L}(\mathcal{F}(M), Y)$ by

$$\widehat{G}(x) = \widehat{F}(x) + \left[\left(1 + \frac{\varepsilon}{2}\right)\widehat{g}(x) - \widehat{F}^*(y_1^*)(x)\right] y_1 \quad \forall\, x \in \mathcal{F}(M),$$

where $y_1 = \sigma(y_1^*)$. Then, we have that

$$\|\widehat{G} - \widehat{F}\| \leqslant \frac{\varepsilon}{2}\|\widehat{g}\| + \|\widehat{g} - \widehat{F}^*(y_1^*)\| \leqslant \frac{\varepsilon}{2} + \gamma < \varepsilon.$$

Therefore, it is enough to show that $\widehat{G}$ attains its norm at a molecule of M. Since for every $y^* \in Y^*$ one has

$$\widehat{G}^*(y^*) = \widehat{F}^*(y^*) + y^*(y_1)\left(\frac{\varepsilon}{2}\widehat{g} + \widehat{g} - \widehat{F}^*(y_1^*)\right),$$

given $y^* \in A \setminus \{y_1^*\}$, we have that

$$\|\widehat{G}^* y^*\| \leqslant 1 + \rho(y_1^*)\left(\frac{\varepsilon}{2} + \gamma\right) \leqslant 1 + r\left(\frac{\varepsilon}{2} + \gamma\right).$$

On the other hand, for $y^* = y_1^*$ we get that $\widehat{G}^*(y_1^*) = \left(1 + \frac{\varepsilon}{2}\right)\widehat{g}$, so

$$\|\widehat{G}^*(y_1^*)\| = \left(1 + \frac{\varepsilon}{2}\right)\|\widehat{g}\| > \left(1 + \frac{\varepsilon}{2}\right)(1 - \gamma) > 1 + r\left(\frac{\varepsilon}{2} + \gamma\right).$$

Consequently, $\|\widehat{G}^*\| = \|\widehat{G}^*(y_1^*)\|$, but $\widehat{G}^*(y_1^*)$ is a multiple of $\widehat{g}$, so it attains its norm as a functional on $\mathcal{F}(M)$ at u, hence $\widehat{G}$ attains its norm at the molecule $u \in \mathrm{Mol}(M)$, as desired. □

5.2 Stability behavior under operations

In this section we analyze some operations that we can consider on metric spaces or on Banach spaces for which the Lip-BPB property or the density of strongly norm-attaining Lipschitz maps is stable.

5.2.1 Domain spaces

First, we will study operations on the domain. Let us recall the notion of sum of metric spaces.

Definition 5.16 (Definition 1.16)**.** Given a family of pointed metric spaces $\{(M_i, d_i)\}_{i\in I}$, the (metric) *sum* of the family is the disjoint union of all M_i's, identifying the base points, endowed with the following metric d: $d(x, y) = d_i(x, y)$ if both $x, y \in M_i$, and $d(x, y) = d_i(x, 0) + d_j(0, y)$ if $x \in M_i$, $y \in M_j$ and $i \neq j$. We write $\coprod_{i\in I} M_i$ to denote the sum of the family of metric spaces.

This notion of sum of metric spaces is analogous to the ℓ_1-sum of Banach spaces. Indeed, as a consequence of Proposition 1.15, we have that if $M = \coprod_{i\in I} M_i$ for some family of metric spaces $\{M_i\}_{i\in I}$, then

$$\mathcal{F}(M) \cong \left[\bigoplus_{i\in I} \mathcal{F}(M_i)\right]_{\ell_1}.$$

Now, the following result shows the good behavior of sums of metric spaces with respect to the Lip-BPB property.

Proposition 5.17. *Let $M = M_1 \coprod M_2$ be the sum of two metric spaces and let Y be a Banach space. If the pair (M, Y) has the Lip-BPB property, then so does (M_1, Y) and (M_2, Y).*

Proof. Fix $0 < \varepsilon < 1$ and let $\eta(\varepsilon)$ be the constant given by the Lip-BPB property of (M, Y), which we may suppose that satisfies $\eta(\varepsilon) < \varepsilon$. Let $\widehat{F_1} \in \mathcal{L}(\mathcal{F}(M_1), Y)$ with $\|F_1\|_L = 1$ and $m \in \mathrm{Mol}(M_1)$ such that $\|\widehat{F_1}(m)\| > 1 - \eta(\varepsilon)$. Now, let us define $\widehat{F} \in \mathcal{L}(\mathcal{F}(M), Y)$ by

$$F(p) = \begin{cases} F_1(p) & \text{if } p \in M_1, \\ 0 & \text{if } p \in M_2. \end{cases}$$

It is easy to see that $\|F\|_L = 1$ and $\|\widehat{F}(m)\| > 1 - \eta(\varepsilon)$, where we see m as a molecule of $\mathcal{F}(M)$. By hypothesis, there exist $\widehat{G} \in \mathcal{L}(\mathcal{F}(M), Y)$ and a molecule $u \in \mathrm{Mol}(M)$ such that

$$\|\widehat{G}(u)\| = \|G\|_L = 1, \quad \|m - u\| < \varepsilon, \quad \|F - G\|_L < \varepsilon.$$

Consider $\widehat{G}_1 \in \mathcal{L}(\mathcal{F}(M_1), Y)$ to be the restriction of $\widehat{G}$ to the subspace $\mathcal{F}(M_1)$. Then, it is clear that

$$\|G_1\|_L \leqslant \|G\|_L = 1 \quad \text{and} \quad \|F_1 - G_1\|_L \leqslant \|F - G\|_L < \varepsilon.$$

Hence, it will be enough to show that $\widehat{G}_1$ attains its norm at a molecule close enough to m. Let us write

$$u = \frac{\delta_p - \delta_q}{d(p,q)},$$

where p, $q \in M$, $p \neq q$. We distinguish four cases:

(i) $p, q \in M_1$: In this case u can be seen as a molecule of $\mathcal{F}(M_1)$ and so $\widehat{G}_1$ attains its norm at u.

(ii) $p, q \in M_2$: Then, note that

$$\widehat{F}(u) = \frac{F(p) - F(q)}{d(p,q)} = 0,$$

from where we deduce that $\|\widehat{G}(u)\| < \varepsilon$, a contradiction.

(iii) $p \in M_1, q \in M_2$: Let us write u as the following convex combination:

$$\begin{aligned} u = \frac{\delta_p - \delta_q}{d(p,q)} &= \frac{\delta_p - \delta_0}{d(p,0)} \frac{d(p,0)}{d(p,q)} + \frac{\delta_0 - \delta_q}{d(0,q)} \frac{d(0,q)}{d(p,q)} \\ &= m_{p,0} \frac{d(p,0)}{d(p,q)} + m_{0,q} \frac{d(0,q)}{d(p,q)}. \end{aligned}$$

Since $\widehat{G}$ attains its norm at u, then it also attains its norm at $m_{p,0} \in \mathrm{Mol}(M_1)$. Hence, $\widehat{G}_1$ attains its norm at $m_{p,0}$. Also, note that

$$\|\widehat{F}(u)\| = \frac{d(p,0)}{d(p,q)} \|\widehat{F}(m_{p,0})\| \leqslant \frac{d(p,0)}{d(p,q)}.$$

On the other hand, $\|\widehat{F}(m)\| > 1 - \eta(\varepsilon)$ and $\|m - u\| < \varepsilon$. Therefore, $\|\widehat{F}(u)\| > 1 - \eta(\varepsilon) - \varepsilon$ and so $\frac{d(p,0)}{d(p,q)} > 1 - \eta(\varepsilon) - \varepsilon$. Consequently, $\frac{d(0,q)}{d(p,q)} < \eta(\varepsilon) + \varepsilon$. Now, note that

$$\begin{aligned} \|m - m_{p,0}\| &= \left\| (m-u) + \left(\frac{d(p,0)}{d(p,q)} - 1 \right) m_{p,0} + \frac{d(0,q)}{d(p,q)} m_{0,q} \right\| \\ &\leqslant \|m - u\| + 2\frac{d(0,q)}{d(p,q)} \leqslant \|m-u\| + 2\eta(\varepsilon) + 2\varepsilon \\ &< 2\eta(\varepsilon) + 3\varepsilon < 5\varepsilon. \end{aligned}$$

(iv) $p \in M_2, q \in M_1$: We just have to repeat the previous argument.

Consequently, we conclude that (M_1, Y) has the Lip-BPB property. Since the situation is symmetric, we also get that (M_2, Y) has the Lip-BPB property. □

Observe that from this result we obtain the next corollary. We just have to notice that for every $j \in I$, we have that $\coprod_{i \in I} M_i \equiv M_j \coprod Z$ for some metric space Z.

Corollary 5.18. *Let $M = \coprod_{i \in I} M_i$ be the sum of a family $\{M_i\}_{i \in I}$ of metric spaces and let Y be a Banach space. If the pair (M, Y) has the Lip-BPB property, then so does (M_i, Y) for every $i \in I$.*

The converse result of Proposition 5.17 is false, as the next example shows.

Example 5.19. *Let $M_1 = \{0, 1\}$ and $M_2 = \{1, 2\}$ viewed as subsets of $\mathbb{R}$ with the usual metric and consider 1 as base point for both spaces. First, observe that $M = M_1 \coprod M_2$ is isometric to the subset $\{0, 1, 2\}$ of $\mathbb{R}$ with the usual metric. Now, the pairs (M_i, Y) has the Lip-BPB property for $i = 1, 2$ and every Banach space Y (this is obvious as the spaces $\mathcal{F}(M_1)$ and $\mathcal{F}(M_2)$ are one-dimensional), but for every strictly convex Banach space Y which is not uniformly convex, the pair (M, Y) fails the Lip-BPB property as Example 4.5 shows.*

In the case of the density of $\mathrm{LipSNA}(M, Y)$, the result obtained is more satisfactory.

Theorem 5.20. *Let $\{M_i\}_{i\in I}$ be a family of metric spaces, consider the sum $M = \coprod_{i\in I} M_i$ and let Y be a Banach space. Then the following are equivalent:*

(i) $\mathrm{LipSNA}(M_i, Y)$ *is dense in* $\mathrm{Lip}_0(M_i, Y)$ *for every* $i \in I$.

(ii) $\mathrm{LipSNA}(M, Y)$ *is dense in* $\mathrm{Lip}_0(M, Y)$.

Proof. (i) $\Rightarrow$ (ii) Consider the natural embeddings $E_i\colon \mathcal{F}(M_i) \longrightarrow \mathcal{F}(M)$ and the natural projections $P_i\colon \mathcal{F}(M) \longrightarrow \mathcal{F}(M_i)$ for every $i \in I$. Fix $\varepsilon > 0$ and take $\widehat{F} \in L(\mathcal{F}(M), Y) \cong \mathrm{Lip}_0(M, Y)$. Without loss of generality, we may assume that $\|F\|_L = 1$. Using that $\|F\|_L = \sup\{\|\widehat{F}E_i\|\colon i \in I\}$ we can find $h \in I$ such that $\|\widehat{F}E_h\| > \|F\|_L - \varepsilon$. By hypothesis, we can find $G_h \in \mathrm{LipSNA}(M_h, Y)$ satisfying $\|G_h\|_L = \|\widehat{F}E_h\|$ and $\|\widehat{G}_h - \widehat{F}E_h\| \leqslant \varepsilon$. Let us define $\widehat{G} \in \mathcal{L}(\mathcal{F}(M), Y)$ by

$$\widehat{G}E_i = (1-\varepsilon)\widehat{F}E_i \ \text{ for } i \in I, i \neq h \quad \text{ and } \quad \widehat{G}E_h = \widehat{G}_h.$$

Then, $\|G\|_L = \sup\{\|\widehat{G}E_i\|\colon i \in I\} = \|G_h\|_L$ and

$$\|G - F\|_L = \sup\{\|(\widehat{G} - \widehat{F})E_i\|\colon i \in I\} \leqslant \varepsilon.$$

Moreover, note that if we take a molecule $m_{p_h,q_h} \in \mathcal{F}(M_h)$ such that $\|\widehat{G}_h(m_{p_h,q_h})\| = \|G_h\|_L$ then, if we consider the molecule $E_h(m_{p_h,q_h}) \in \mathcal{F}(M)$, we will have that

$$\|\widehat{G}(E_h(m_{p_h,q_h}))\| = \|\widehat{G}_h(m_{p_h,q_h})\| = \|G_h\|_L = \|G\|_L.$$

Hence, $G \in \mathrm{LipSNA}(M, Y)$.

(ii) $\Rightarrow$ (i) Fix $\varepsilon > 0$, $h \in I$ and take $\widehat{F}_h \in \mathrm{Lip}_0(M_h, Y)$. As above, we may assume that $\|F_h\|_L = 1$. Let us define $\widehat{F}\colon \mathcal{F}(M) \longrightarrow Y$ by $\widehat{F} = \widehat{F}_h P_h$. Then, it is clear that $\|F\|_L = \|F_h\|_L = 1$. By hypothesis, we can find $G \in \mathrm{Lip}_0(M, Y)$ such that $\|G\|_L = 1$ and $\|G - F\|_L \leqslant \varepsilon$. Now, define $\widehat{G}_h\colon \mathcal{F}(M_h) \longrightarrow Y$ by $\widehat{G}_h = \widehat{G}E_h$. Then, $\|G_h\|_L \leqslant 1$ and

$$\|\widehat{G}_h - \widehat{F}_h\| = \|\widehat{G}E_h - \widehat{F}E_h\| \leqslant \|G - F\|_L \leqslant \varepsilon,$$

so we just have to see that $G_h \in \mathrm{LipSNA}(M_h, Y)$. To this end, consider a molecule $m_{p,q} \in \mathcal{F}(M)$ such that $\|\widehat{G}(m_{p,q})\| = \|G\|_L = 1$. We claim that $P_h(m_{p,q})$ is a molecule of $\mathcal{F}(M_h)$. Then, we would have that

$$\|\widehat{G}_h(P_h(m_{p,q}))\| = \|\widehat{G}(m_{p,q})\| = \|G\|_L = \|G_h\|_L = 1.$$

Hence, $G_h \in \mathrm{LipSNA}(M_h, Y)$ and the result would be proved. Indeed, assume that $P_h(m_{p,q})$ is not a molecule of $\mathcal{F}(M_h)$. Then, either $p \notin M_h$ or $q \notin M_h$. If we assume $q \notin M_h$, we will have that $P_h(\delta_q) = 0$, but $\widehat{G}_h$ attains its norm at $P_h(m_{p,q})$, so $P_h(m_{p,q}) \neq 0$, which implies that $p \in M_h$. Finally, observe that

$$\begin{aligned}\|\widehat{G}_h(P_h(m_{p,q}))\| &= \frac{\widehat{G}_h(P_h(\delta_p)) - \widehat{G}_h(P_h(\delta_q))}{d(p,q)} \\ &= \frac{\widehat{G}_h(\delta_p) - \widehat{G}_h(\delta_0)}{d(p,q)} < \frac{\widehat{G}_h(\delta_p) - \widehat{G}_h(\delta_0)}{d(p,0)} \leqslant \|G_h\|_L,\end{aligned}$$

a contradiction. The case $p \notin M_h$ is analogous to the above one. $\square$

5.2.2 Range spaces

Here we study the stability of the Lip-BPB property and the strong density under some operations on the range space. We need to introduce the notion of absolute sum of two Banach spaces.

Definition 5.21. An *absolute norm* is a norm $|\cdot|_a$ in $\mathbb{R}^2$ such that

$$|(1,0)|_a = |(0,1)|_a = 1 \quad \text{and} \quad |(s,t)|_a = |(|s|,|t|)|_a \text{ for every } s,t \in \mathbb{R}.$$

Given two Banach spaces W and Z and an absolute norm $|\cdot|_a$, the *absolute sum* of W and Z with respect to $|\cdot|_a$, denoted by $W \oplus_a Z$, is the Banach space $W \times Z$ endowed with the norm

$$\|(w,z)\|_a = |(\|w\|, \|z\|)|_a \quad \forall\, w \in W, \quad \forall\, z \in Z.$$

A closed subspace Y_1 of a Banach space Y is said to be an *absolute summand* of Y whenever there exists a closed subspace Z of Y and an absolute norm $|\cdot|_a$ in $\mathbb{R}^2$ such that $Y \cong Y_1 \oplus_a Z$.

We will need the next easy lemma, whose proof can be found in the below reference.

Lemma 5.22 ([39, Lemma 2.2])**.** *Let W and Z be Banach spaces and $|\cdot|_a$ be any absolute norm in $\mathbb{R}^2$. If $(w,z) \in S_{W\oplus_a Z}$ and $(w^*,z^*) \in S_{W^*\oplus_{a^*} Z^*}$ are such that $\langle (w,z),(w^*,z^*)\rangle = 1$, then*

$$w^*(w) = \|w^*\|\|w\| \quad \text{and} \quad z^*(z) = \|z^*\|\|z\|.$$

Our first result is the following lifting of the Lip-BPB property from a space to its absolute summands.

Proposition 5.23. *Let M be a metric space, Y be a Banach space, and Y_1 be an absolute summand of Y. If the pair (M,Y) has the Lip-BPB property with a function $\varepsilon \longmapsto \eta(\varepsilon)$, then so does (M,Y_1) with the function $\eta(\frac{\varepsilon}{3})$.*

This proposition is an extension of Lemma 5.14 and its proof is based on [26, Theorem 2.1].

Proof. Fix $0<\varepsilon<1$ and consider $\widehat{F_1} \in \mathcal{L}(\mathcal{F}(M),Y_1)$ with $\|F_1\|_L = 1$ and $m \in \operatorname{Mol}(M)$ satisfying that

$$\|\widehat{F_1}(m)\| > 1-\eta\left(\frac{\varepsilon}{3}\right).$$

Let us define the operator $\widehat{F} \in \mathcal{L}(\mathcal{F}(M),Y)$ by $\widehat{F}(x) = (\widehat{F_1}(x),0)$ for all $x \in \mathcal{F}(M)$, and note that it satisfies that $\|F\|_L = 1$ and

$$\|\widehat{F}(m)\| = \|(\widehat{F_1}(m),0)\|_a = \|\widehat{F_1}(m)\| > 1-\eta\left(\frac{\varepsilon}{3}\right).$$

Now, since the pair $(\mathcal{F}(M),Y)$ has the Lip-BPB property with function η, we find $\widehat{G} \in \mathcal{L}(\mathcal{F}(M),Y)$ and $u \in \operatorname{Mol}(M)$ such that

$$\|\widehat{G}(u)\| = \|G\|_L = 1, \quad \|G-F\|_L < \frac{\varepsilon}{3}, \quad \|m-u\| < \frac{\varepsilon}{3}.$$

Let us write $\widehat{G} = (\widehat{G}_1, \widehat{G}_2)$, where $\widehat{G}_i$ are the summands of G. Since $\|\cdot\|_a$ is an absolute norm, for $x \in S_{\mathcal{F}(M)}$ we have that

$$\|\widehat{G}_1(x) - \widehat{F}_1(x), \widehat{G}_2(x)\|_\infty \leqslant \|\widehat{G}_1(x) - \widehat{F}_1(x), \widehat{G}_2(x)\|_a \leqslant \|G-F\|_L < \frac{\varepsilon}{3}.$$

Consequently, $\|F_1 - G_1\|_L < \frac{\varepsilon}{3}$ and $\|G_2\|_L < \frac{\varepsilon}{3}$. Now, consider $y^* = (y_1^*, y_2^*) \in Y^*$ of norm one such that

$$1 = \|\widehat{G}(u)\| = y^*(\widehat{G}(u)) = y_1^*(\widehat{G}_1(u)) + y_2^*(\widehat{G}_2(u)).$$

Then, Lemma 5.22 gives that $y_1^*(\widehat{G}_1(u)) = \|y_1^*\|\|\widehat{G}_1(u)\|$ and $y_2^*(\widehat{G}_2(u)) = \|y_2^*\|\|\widehat{G}_2(u)\|$. Since

$$\|y_1^*\|\|\widehat{G}_1(u)\| = y_1^*(\widehat{G}_1(u)) = 1 - y_2^*(\widehat{G}_2(u)) \geqslant 1 - \|G_2\|_L > 0,$$

we have that $y_1^* \neq 0$ and $\|\widehat{G}_1(u)\| \neq 0$. Then, we can define the operator $\widehat{H}_1 \in \mathcal{L}(\mathcal{F}(M),Y_1)$ by

$$\widehat{H}_1(x) = \|y_1^*\|\widehat{G}_1(x) + y_2^*(\widehat{G}_2(x))\frac{\widehat{G}_1(u)}{\|\widehat{G}_1(u)\|} \quad \forall\, x \in \mathcal{F}(M).$$

Then, for every $x \in B_{\mathcal{F}(M)}$ we have that

$$\begin{aligned}\|\widehat{H}_1(x)\| &\leqslant \|y_1^*\|\|\widehat{G}_1(x)\| + \|y_2^*\|\|\widehat{G}_2(x)\| \leqslant |(\|\widehat{G}_1(x)\|, \|\widehat{G}_2(x)\|)|_a |(\|y_1^*\|, \|y_2^*\|)|_{a^*} \\ &= \|(\widehat{G}_1(x), \widehat{G}_2(x))\|_a \|(y_1^*, y_2^*)\|_{a^*} = \|\widehat{G}(x)\|_a \|(y_1^*, y_2^*)\|_{a^*} \leqslant \|G\|_L \|y^*\|_{a^*} = 1.\end{aligned}$$

Consequently, $\|H_1\|_L \leqslant 1$. On the other hand,

$$\|\widehat{H}_1(u)\| \geqslant \frac{y_1^*}{\|y_1^*\|}\left(\|y_1^*\|\widehat{G}_1(u) + y_2^*(\widehat{G}_2(u))\frac{\widehat{G}_1(u)}{\|\widehat{G}_1(u)\|}\right) = y_1^*(\widehat{G}_1(u)) + y_2^*(\widehat{G}_2(u)) = 1.$$

This shows that $\|\widehat{H}_1(u)\| = \|H_1\|_L = 1$, so it remains to prove that $\|H_1 - F_1\|_L < \varepsilon$. Indeed, since

$$1 - \|y_1^*\| \leqslant 1 - y_1^*(\widehat{G}_1(u)) = y_2^*(\widehat{G}_2(u)) \leqslant \|G_2\|_L < \frac{\varepsilon}{3},$$

for any $x \in B_{\mathcal{F}(M)}$ we have

$$\|\widehat{H}_1(x) - \widehat{F}_1(x)\| \leqslant \|y_1^*\|\|\widehat{G}_1(x) - \widehat{F}_1(x)\| + (1 - \|y_1^*\|)\|\widehat{F}_1(x)\| + \|G_2\|_L < \varepsilon. \qquad \square$$

Notice that, as it is proved in the above proposition, essentially the same function η from the Lip-BPB property of (M, Y) works for the Lip-BPB property of (M, Y_1). This is the key fact to obtain the following consequence.

Corollary 5.24. *Let M be a metric space such that (M, Y) has the Lip-BPB property for all Banach spaces Y. Then, there exists a function $\eta_M(\varepsilon)$, which depends only on M, such that the pair (M, Y) has the Lip-BPB property witnessed by the function $\eta_M(\varepsilon)$ for every Banach space Y.*

Proof. Suppose this is not the case. Then there is a sequence Y_n of Banach spaces such that whenever each pair (M, Y_n) has the Lip-BPB property witnessed by a function $\eta_n(\varepsilon) > 0$, one has that $\inf_n \eta_n(\varepsilon) = 0$ for every $0 < \varepsilon < 1$. Then, consider the space $Y = \left[\bigoplus_{n\in\mathbb{N}} Y_n\right]_{c_0}$ and observe that, by hypothesis, the pair (M, Y) has the Lip-BPB property witnessed by a function $\varepsilon \longmapsto \eta(\varepsilon) > 0$. As each Y_n is clearly an absolute summand of Y, it follows from Proposition 5.23 that for every $n \in \mathbb{N}$, each pair (M, Y_n) has the Lip-BPB property witnessed by the function $\varepsilon \longmapsto \eta(\frac{\varepsilon}{3}) > 0$, a contradiction to our assumption. $\square$

We can give a reciprocal of Proposition 5.23 for some particular cases. Let M be a metric space, let $\{Y_i\}_{i\in I}$ be a family of Banach spaces, and let $Y = [\bigoplus_{i\in I} Y_i]_{c_0}$ or $Y = [\bigoplus_{i\in I} Y_i]_{\ell_\infty}$ be the c_0-sum or ℓ_∞-sum of $\{Y_i\}_{i\in I}$, respectively. By Proposition 5.23, if the pair (M, Y) has the Lip-BPB property, then all the pairs (M, Y_i) have the Lip-BPB property witnessed by the same function. The next proposition gives us the reversed result.

Proposition 5.25. *Let M be a metric space, let $\{Y_i\}_{i\in I}$ be a family of Banach spaces, and set $Y = [\bigoplus_{i\in I} Y_i]_{c_0}$ or $Y = [\bigoplus_{i\in I} Y_i]_{\ell_\infty}$. Assume that (M, Y_i) has the Lip-BPB property witnessed by a function $\eta_i(\varepsilon)$ for every $i \in I$. If $\inf\{\eta_i(\varepsilon) \colon i \in I\} > 0$ for every $\varepsilon > 0$, then (M, Y) has the Lip-BPB property.*

Proof. Fix $\varepsilon > 0$, take $\eta(\varepsilon) := \inf\{\eta_i(\varepsilon) \colon i \in I\} > 0$ and note that we have $\eta_i(\varepsilon) \geqslant \eta(\varepsilon)$ for every $i \in I$. Consider $Q_i \colon Y \longrightarrow Y_i$ the natural projection and $E_i \colon Y_i \longrightarrow Y$ the natural embedding for every $i \in I$. Take $\widehat{F} \in \mathcal{L}(\mathcal{F}(M), Y)$ with $\|F\|_L = 1$ and $m \in \operatorname{Mol}(M)$ such that

$$\|\widehat{F}(m)\| > 1 - \eta(\varepsilon).$$

Then, there exists $k \in I$ so that $\|Q_k\widehat{F}(m)\| > 1 - \eta(\varepsilon)$. By hypothesis, there exist $\widehat{G}_k \in \mathcal{L}(\mathcal{F}(M), Y_k)$ and $u \in \operatorname{Mol}(M)$ satisfying

$$\|\widehat{G}_k(u)\| = \|G_k\|_L = 1, \quad \|Q_k\widehat{F} - \widehat{G}_k\| < \varepsilon, \quad \|m - u\| < \varepsilon.$$

Now, let us define $\widehat{G} \colon \mathcal{F}(M) \longrightarrow Y$ given by

$$\widehat{G}(x) = \sum_{i\neq k} E_i(Q_i(\widehat{F}))(x) + E_k\widehat{G}_k(x) \quad \forall\, x \in \mathcal{F}(M).$$

Then, we have that $\|G\|_L \leqslant 1$ and $\|\widehat{G}(u)\| \geqslant \|\widehat{G}_k(u)\| = 1$. Therefore, $\widehat{G}$ attains its norm at $u \in \mathrm{Mol}(M)$. Finally, notice that

$$\|F - G\|_L = \sup\{\|Q_i(\widehat{F} - \widehat{G})\| \colon i \in I\} = \|Q_k(\widehat{F} - \widehat{G})\| < \varepsilon,$$

that is, (M, Y) has the Lip-BPB property. □

With respect to the density of strongly norm-attaining Lipschitz maps density, the next result follows by repeating word-by-word the proof of Proposition 5.23, using the hypothesis of the density of $\mathrm{LipSNA}(M, \mathbb{R})$ instead of the Lip-BPB property of $(M, \mathbb{R})$ and forgetting about the estimation of the distance between the molecules.

Proposition 5.26. *Let M be a metric space, let Y be a Banach space, and let Y_1 be an absolute summand of Y. If $\mathrm{LipSNA}(M, Y)$ is dense in $\mathrm{Lip}_0(M, Y)$, then $\mathrm{LipSNA}(M, Y_1)$ is dense in $\mathrm{Lip}_0(M, Y_1)$.*

We can also get a converse of this result in the particular case when the absolute norm is the c_0-sum or the ℓ_∞-sum.

Proposition 5.27. *Let M be a metric space, let $\{Y_i\}_{i\in I}$ be a family of Banach spaces, and set $Y = [\bigoplus_{i\in I} Y_i]_{c_0}$ or $Y = [\bigoplus_{i\in I} Y_i]_{\ell_\infty}$. If $\overline{\mathrm{LipSNA}(M, Y_i)} = \mathrm{Lip}_0(M, Y_i)$ for every $i \in I$, then*

$$\overline{\mathrm{LipSNA}(M, Y)} = \mathrm{Lip}_0(M, Y).$$

Proof. For each $i \in I$, consider $Q_i \colon Y \longrightarrow Y_i$ the natural projection and $E_i \colon Y_i \longrightarrow Y$ the natural embedding. Fix $\varepsilon > 0$ and $\widehat{F} \in \mathcal{L}(\mathcal{F}(M), Y)$ with $\|F\|_L = 1$. There exists $k \in I$ so that $\|Q_k\widehat{F}\| > 1 - \frac{\varepsilon}{2}$. Then, since $\overline{\mathrm{LipSNA}(M, Y_k)} = \mathrm{Lip}_0(M, Y_K)$ we may find $G_k \in \mathrm{Lip}_0(M, Y_k)$ and $u \in \mathrm{Mol}(M)$ such that

$$\|\widehat{G}_k(u)\| = \|G_k\|_L = 1, \quad \|\widehat{G}_k - Q_k\widehat{F}\| < \varepsilon.$$

Now, let us define $\widehat{G} \colon \mathcal{F}(M) \longrightarrow Y$ given by

$$\widehat{G}(x) = \sum_{i\neq k} E_i(Q_i(\widehat{F}))(x) + E_k\widehat{G}_k(x) \quad \forall\, x \in \mathcal{F}(M).$$

Then, we have that $\|G\|_L \leqslant 1$ and $\|\widehat{G}(u)\| \geqslant \|\widehat{G}_k(u)\| = 1$. Therefore, $\widehat{G}$ attains its norm at u. Finally, notice that

$$\|F - G\|_L = \sup\{\|Q_i(\widehat{F} - \widehat{G})\| \colon i \in I\} = \|Q_k(\widehat{F} - \widehat{G})\| < \varepsilon.$$ □

To finish this chapter, let us present a couple of results in the same direction. The first one is based on Proposition 2.8 in [12].

Proposition 5.28. *Let M be a metric space, Y be a Banach space, and K be a compact Hausdorff topological space. If $(M, C(K, Y))$ has the Lip-BPB property witnessed by a function $\eta(\varepsilon)$, then (M, Y) has the Lip-BPB property witnessed by the same function.*

Proof. Fix $\varepsilon > 0$ and take $\eta(\varepsilon)$ the constant from the Lip-BPB property of $(M, C(K, Y))$. Consider $\widehat{F}_1 \in \mathcal{L}(\mathcal{F}(M), Y)$ with $\|\widehat{F}_1\| = 1$ and $m \in \mathrm{Mol}(M)$ satisfying

$$\|\widehat{F}_1(m)\| > 1 - \eta(\varepsilon).$$

Let us define $\widehat{F} \colon \mathcal{F}(M) \longrightarrow C(K, Y)$ given by

$$[\widehat{F}(x)](t) = \widehat{F}_1(x) \quad \text{for every } x \in \mathcal{F}(M),\ t \in K.$$

Then, it is clear that $\|\widehat{F}\| = \|\widehat{F}_1\| = 1$. Furthermore, $\|\widehat{F}(m)\| > 1 - \eta(\varepsilon)$. By the assumption, there exist $\widehat{G} \in \mathcal{L}(\mathcal{F}(M), C(K, Y))$ and $u \in \mathrm{Mol}(M)$ such that

$$\|\widehat{G}(u)\| = \|\widehat{G}\| = 1, \quad \|\widehat{F} - \widehat{G}\| < \varepsilon, \quad \|m - u\| < \varepsilon.$$

Moreover, since K is compact, there is $t_1 \in K$ such that

$$1 = \|\widehat{G}(u)\| = \|[\widehat{G}(u)](t_1)\|.$$

Now, let us define $\widehat{G}_1 \colon \mathcal{F}(M) \longrightarrow Y$ by $\widehat{G}_1(x) = [\widehat{G}(x)](t_1)$ for every $x \in \mathcal{F}(M)$. Note that

$$\|\widehat{G}_1\| = \sup_{x \in B_{\mathcal{F}(M)}} \big\|[\widehat{G}(x)](t_1)\big\| = \big\|[\widehat{G}(u)](t_1)\big\| = \big\|\widehat{G}_1(u)\big\| = 1.$$

In addition, we have that

$$\begin{aligned}\|G_1 - F_1\|_L &= \sup_{x \in B_{\mathcal{F}(M)}} \Big\{\big\|[\widehat{G}(x)](t_1) - [\widehat{F}(x)](t_1)\big\|\Big\} \\ &\leqslant \sup_{x \in B_{\mathcal{F}(M)}} \Big\{\big\|\widehat{G}(x) - \widehat{F}(x)\big\|\Big\} = \big\|\widehat{G} - \widehat{F}\big\| < \varepsilon.\end{aligned}$$

As we already know that $\|m - u\| < \varepsilon$, we obtain that (M, Y) has the Lip-BPB property witnessed by the function $\eta(\varepsilon)$. □

The second one is just its analogous version for the density of $\mathrm{LipSNA}(M, Y)$.

Proposition 5.29. *Let M be a metric space, let Y be a Banach space, and let K be a compact Hausdorff topological space. Assume that* $\mathrm{LipSNA}(M, C(K, Y))$ *is dense in* $\mathrm{Lip}_0(M, C(K, Y))$*. Then,* $\mathrm{LipSNA}(M, Y)$ *is dense in* $\mathrm{Lip}_0(M, Y)$*.*

Proof. Given $\varepsilon > 0$, consider $\widehat{F}_1 \in \mathcal{L}(\mathcal{F}(M), Y)$ with $\|F_1\|_L = 1$. Let us define $\widehat{F}$ as in the proof of Proposition 5.28. By hypothesis, there exist $\widehat{G} \in \mathcal{L}(\mathcal{F}(M), C(K, Y))$ and $u \in \mathrm{Mol}(M)$ such that

$$\|\widehat{G}(u)\| = \|\widehat{G}\| = 1 \text{ and } \quad \|\widehat{G} - \widehat{F}\| < \varepsilon.$$

Since K is compact, there is $t_1 \in K$ such that $1 = \|\widehat{G}(u)\| = \|[\widehat{G}(u)](t_1)\|$. Now, let us define the linear and bounded operator $\widehat{G}_1 \colon \mathcal{F}(M) \longrightarrow Y$ given by $\widehat{G}_1(x) = [\widehat{G}(x)](t_1)$ for every $x \in \mathcal{F}(M)$. By repeating the argument in Proposition 5.28, we obtain that $\widehat{G}_1$ attains its norm at $u \in \mathrm{Mol}(M)$ and $\|\widehat{G}_1 - \widehat{F}_1\| \leqslant \|\widehat{G} - \widehat{F}\| < \varepsilon$. Consequently, we obtain that $\mathrm{LipSNA}(M, Y)$ is dense in $\mathrm{Lip}_0(M, Y)$. □

Chapter 6

Lipschitz compact maps

In this chapter, we will introduce the class of Lipschitz compact maps. From now on, we will restrict our study of the Lip-BPB property and the strong density to this family of Lipschitz maps. Since all the results that we have studied in the previous chapters are valid for (general) Lipschitz maps, we can restrict them to obtain consequences for Lipschitz compact maps. Moreover, we will show that some of these results can be improved in the setting of Lipschitz compact maps. Finally, we will present new results that are only valid when restricting to this family of Lipschitz maps.

The results obtained in this chapter come from the papers [24] and [25]. They were collaborative works with Miguel Martín.

To start the study, let us introduce some notation. Let M be a metric space, Y be a Banach space, and $F\colon M \longrightarrow Y$ be a Lipschitz map. The *Lipschitz image* of F is the set

$$\left\{\frac{F(p)-F(q)}{d(p,q)} : p,q\in M,\ p\neq q\right\}\subseteq Y.$$

We say that F is *Lipschitz compact* when its Lipschitz image is relatively compact. We denote by $\mathrm{Lip}_{0K}(M,Y)$ the space of Lipschitz compact maps from M to Y that vanishes at 0. Some results related to this notion appear in [47]. Let us make two comments. First, observe that if Y is finite-dimensional, then all Lipschitz maps are Lipschitz compact. Second, it is immediate that a Lipschitz map $F\colon M \longrightarrow Y$ is Lipschitz compact if, and only if, its associated linear operator $\widehat{F}\colon \mathcal{F}(M) \longrightarrow Y$ is compact (that is, $\widehat{F}(B_{\mathcal{F}(M)})$ is relatively compact).

We denote by $\mathrm{LipSNA}_K(M,Y)$ the set of those Lipschitz compact maps from M to Y (vanishing at 0) which strongly attain their norm, that is,

$$\mathrm{LipSNA}_K(M,Y) = \mathrm{LipSNA}(M,Y)\cap \mathrm{Lip}_{0K}(M,Y).$$

We are interested in studying when the set $\mathrm{LipSNA}_K(M,Y)$ is dense in $\mathrm{Lip}_{0K}(M,Y)$ and which pairs (M,Y) have the Lip-BPB property for Lipschitz compact maps.

6.1 Conditions on the metric space

Let M be a metric space and let Y be a Banach space. In this section we will focus our attention on studying conditions on the metric space M that ensure that $\mathrm{LipSNA}_K(M,Y)$ is dense in $\mathrm{Lip}_{0K}(M,Y)$ or that (M,Y) has the Lip-BPB property for Lipschitz compact maps, regardless of Y.

First, we show that all sufficient conditions that we studied in Chapter 2 are still valid when restricted to Lipschitz compact maps.

Proposition 6.1. *Let M be a metric space. Then,*

(i) If $\mathcal{F}(M)$ has the RNP, then $\mathrm{LipSNA}_K(M,Y)$ is dense in $\mathrm{Lip}_{0K}(M,Y)$ for every Banach space Y.

(ii) *If there exists a norming subset of uniformly strongly exposed points in* $B_{\mathcal{F}(M)}$*, then* $\mathrm{LipSNA}_K(M,Y)$ *is dense in* $\mathrm{Lip}_{0K}(M,Y)$ *for every Banach space* Y*.*

(iii) *If* $\mathcal{F}(M)$ *has property* α*, then* $\mathrm{LipSNA}_K(M,Y)$ *is dense in* $\mathrm{Lip}_{0K}(M,Y)$ *for every Banach space* Y*.*

(iv) *If* $\mathcal{F}(M)$ *has property quasi-*α*, then* $\mathrm{LipSNA}_K(M,Y)$ *is dense in* $\mathrm{Lip}_{0K}(M,Y)$ *for every Banach space* Y*.*

Proof. To prove the statements above, we just have to analyze carefully the proof of the original version of each result to see that if we fix a compact operator, then the operator that approximates it can be taken to be also compact.

(i) Bourgain proved in [18, Theorem 5] that if a Banach space X has the RNP, then the set A of absolutely strongly exposing operators is a G_δ-subset of $\mathcal{L}(X,Y)$ for every Banach space Y. Moreover, if $T \in \mathcal{L}(X,Y)$ and $\varepsilon > 0$ are given, there is $S \in A$ such that $\|T-S\| < \varepsilon$ and $T-S$ is a compact operator. Clearly, if T is an absolutely strongly exposing operator, then T attains its norm at the point x appearing at the definition; it is easy to show that such point $x \in S_X$ is a strongly exposed point (indeed, let $y^* \in S_{Y^*}$ such that $y^*(Tx) = \|T\|$ and consider $x^* \in S_{X^*}$ such that $\|T\|x^* = T^*(y^*)$; if $\{x_n\}$ is a sequence in B_X such that $x^*(x_n) \longrightarrow 1 = x^*(x)$, then

$$\|T(x_n)\| \geqslant y^*(Tx_n) = \|T\|x^*(x_n) \longrightarrow \|T\|,$$

so there is a subsequence $\{x_{n_k}\}$ converging to x (it cannot converge to $-x$), showing that x is strongly exposed by x^*). Then, as a consequence of Corollary 1.3, when $X = \mathcal{F}(M)$ for some metric space M, every absolutely strongly exposing operator attains its norm at a molecule, that is, the associated Lipschitz map strongly attains its norm. Now, if $\mathcal{F}(M)$ has the RNP, $T \in \mathcal{L}(\mathcal{F}(M),Y)$ is compact, and we pick $\varepsilon > 0$, Bourgain's result above provides with $S \in \mathcal{L}(\mathcal{F}(M),Y)$ attaining its norm at a molecule, with $\|T-S\| < \varepsilon$, and such that $T-S$ is compact, hence S is compact and we are done.

(ii) One can check the proof of Theorem 1 in [57] to see that for any $\varepsilon > 0$, if $T \in \mathcal{L}(\mathcal{F}(M),Y)$ is a compact operator, then the sequence of operators $\{T_k\}$ constructed in such a proof is a sequence of compact operators that converges to an operator $\hat{T}$ satisfying $\|\hat{T} - T\| < \varepsilon$, so $\hat{T}$ must be compact too. Now, the result follows in an analogous way as Proposition 2.8 does.

(iii) Since property α is just a particular way in which a Banach space can have a norming subset of uniformly strongly exposed points, the result follows from (ii).

(iv) One can check the proof of Proposition 2.1 in [27] to see that if $T \in \mathcal{L}(\mathcal{F}(M),Y)$ is a compact operator, then the constructed operator S that approximates T is also compact. Now, the result follows from the observation previous to Proposition 2.21. □

Example 4.6 shows that none of the assertions in Proposition 6.1 can be used to get results for the Lip-BPB property, even if we restrict our study to Lipschitz compact maps. Indeed, $\mathcal{F}(\mathbb{N}) \cong \ell_1$ has the four above properties, but $(\mathbb{N},\mathbb{R})$ fails to have the Lip-BPB property. However, we can get a result analogous to Theorem 4.9 for Lipschitz compact maps.

Proposition 6.2. *Let* M *be a uniformly Gromov concave metric space. Then,* (M,Y) *has the Lip-BPB property for Lipschitz compact maps for every Banach space* Y*.*

Proof. It is enough to note that if we take a Lipschitz compact map $F \in \mathrm{Lip}_{0K}(M,Y)$ with $\|F\|_L = 1$, then the Lipschitz maps G_0 and H which appear in the proof of Theorem 4.9 will also be Lipschitz compact. This is because $\widehat{G_0} - \widehat{F}$ is a rank-one operator and $\widehat{H}$ is obtained, following the proof of [57, Theorem 1], as the limit of a sequence of compact operators. □

As in Chapter 4, this proposition has the following interesting corollaries.

Corollary 6.3. *Let* M *be a concave metric space such that* $\mathcal{F}(M)$ *has property* α*. Then, for every Banach space* Y *the pair* (M,Y) *has the Lip-BPB property for Lipschitz compact maps.*

Corollary 6.4. *Let M be a Hölder metric space. Then, for every Banach space Y the pair (M,Y) has the Lip-BPB property for Lipschitz compact maps.*

Corollary 6.5. *Let M be an ultrametric space. Then, for every Banach space Y the pair (M,Y) has the Lip-BPB property for Lipschitz compact maps.*

In this case, it does not make sense to give an analogous result to Corollary 4.4, because if M is a finite metric space then $\mathrm{Lip}_0(M,Y) = \mathrm{Lip}_{0K}(M,Y)$ and so the result is the same.

6.2 Conditions on the range space

Let M be a metric space and let Y be a Banach space. Our aim in this section is to give conditions on the Banach space Y guaranteeing that $\mathrm{LipSNA}_K(M,Y)$ is dense in $\mathrm{Lip}_{0K}(M,Y)$ when $\mathrm{LipSNA}_K(M,\mathbb{R})$ is and, analogously, conditions on Y assuring that the pair (M,Y) has the Lip-BPB property when $(M,\mathbb{R})$ does.

Observe that one of the disadvantages of Theorems 5.6 and 5.10 and their consequences (Corollaries 5.7 and 5.11) is the necessity of dealing with Γ-flat operators. However, when we consider Lipschitz compact maps this requirement disappears: given a Banach space Y, every compact operator with Y as codomain is Γ-flat for every $\Gamma \subseteq B_{Y^*}$ (see [21, Example A]). Moreover, notice that if we take a compact operator $\widehat{T}$ in the proof of Theorem 5.6, then the operator $\widehat{S}$ that approximates $\widehat{T}$ will be also compact. Consequently, we obtain the following result.

Proposition 6.6. *Let M be a metric space such that $(M,\mathbb{R})$ has the Lip-BPB property and let Y be an ACK_ρ Banach space. Then, the pair (M,Y) has the Lip-BPB property for Lipschitz compact maps.*

Again, in view of Proposition 5.5, we obtain a series of implications.

Corollary 6.7. *Let M be a metric space such that $(M,\mathbb{R})$ has the Lip-BPB property. Then, the following statements hold.*

(i) Let Y be a Banach space having property β. Then, (M,Y) has the Lip-BPB property for Lipschitz compact maps.

(ii) Let K be a compact Hausdorff topological space. Then, $(M,C(K))$ has the Lip-BPB property for Lipschitz compact maps.

(iii) Let Z be a finite injective tensor product of Banach spaces which have ACK_ρ structure. Then, (M,Z) has the Lip-BPB property for Lipschitz compact maps.

(iv) Let K be a compact Hausdorff topological space. If $Y \in ACK_\rho$, then the pairs $(M,C(K,Y))$ and $(M,C_w(K,Y))$ have the Lip-BPB property for Lipschitz compact maps.

(v) Let $Y \in ACK_\rho$. Then, $(M,c_0(Y))$, $(M,\ell_\infty(Y))$, and $(M,c_0(Y,w))$ have the Lip-BPB property for Lipschitz compact maps.

The following example, which is just a rewriting of Example 4.5, shows that in general the Banach space Y needs to satisfy some hypotheses to ensure that the pair (M,Y) has the Lip-BPB property for Lipschitz compact maps.

Example 6.8. *Consider the metric space $M = \{0,1,2\}$ with the usual metric and let Y be a strictly convex Banach space which is not uniformly convex. Then, $(M,\mathbb{R})$ has the Lip-BPB property (for Lipschitz compact maps), but (M,Y) fails the Lip-BPB property for Lipschitz compact maps.*

Furthermore, recall that Example 5.14 gave us a finite metric space M and a Banach space Y having property quasi-β such that (M,Y) fails the Lip-BPB property. The Lip-BPB property and Lip-BPB property for Lipschitz compact maps are equivalent in this case, so we conclude that property quasi-β is not enough to guarantee the Lip-BPB property for Lipschitz compact maps in general.

As we did in Proposition 6.6, the proof of Theorem 5.10 can be easily adapted to the density of Lipschitz compact maps.

Proposition 6.9. *Let M be a metric space such that* $\mathrm{LipSNA}(M,\mathbb{R})$ *is dense in* $\mathrm{Lip}_0(M,\mathbb{R})$ *and* $Y \in ACK_\rho$ *be a Banach space. Then, the set* $\mathrm{LipSNA}_K(M,Y)$ *is dense in* $\mathrm{Lip}_{0K}(M,Y)$.

As before, this result has many consequences.

Corollary 6.10. *Let M be a metric space such that* $\mathrm{LipSNA}(M,\mathbb{R})$ *is dense in* $\mathrm{Lip}_0(M,\mathbb{R})$. *Then, the following statements hold.*

(i) *If Y is a Banach space having property β, then* $\mathrm{LipSNA}_K(M,Y)$ *is dense in* $\mathrm{Lip}_{0K}(M,Y)$.

(ii) $\mathrm{LipSNA}_K(M,C(K))$ *is dense in* $\mathrm{Lip}_{0K}(M,C(K))$ *for every compact Hausdorff topological space K.*

(iii) *Let Z be a finite injective tensor product of Banach spaces which have ACK_ρ structure. Then,* $\mathrm{LipSNA}_K(M,Z)$ *is dense in* $\mathrm{Lip}_{0K}(M,Z)$.

(iv) *Let K be a compact Hausdorff topological space. If $Y \in ACK_\rho$, then* $\mathrm{LipSNA}_K(M,C(K,Y))$ *and* $\mathrm{LipSNA}_K(M,C_w(K,Y))$ *are dense in* $\mathrm{Lip}_{0K}(M,C(K,Y))$ *and* $\mathrm{Lip}_{0K}(M,C_w(K,Y))$, *respectively.*

(v) *Let $Y \in ACK_\rho$. Then,* $\mathrm{LipSNA}_K(M,c_0(Y))$, $\mathrm{LipSNA}_K(M,\ell_\infty(Y))$, *and* $\mathrm{LipSNA}_K(M,c_0(Y,w))$ *are dense in* $\mathrm{Lip}_{0K}(M,c_0(Y))$, $\mathrm{Lip}_{0K}(M,\ell_\infty(Y))$, *and* $\mathrm{Lip}_{0K}(M,c_0(Y,w))$, *respectively.*

Let us consider again property quasi-β. If we analyze the proof of Proposition 5.15, we see that $\widehat{G}-\widehat{F}$ is a rank-one operator, so $\widehat{G}$ will be compact if $\widehat{F}$ is. Consequently, we obtain a more general result than the one given in the first assertion of Corollary 6.10.

Proposition 6.11. *Let M be a metric space such that* $\mathrm{LipSNA}(M,\mathbb{R})$ *is dense in* $\mathrm{Lip}_0(M,\mathbb{R})$ *and let Y be a Banach space having property quasi-β. Then, we have that*

$$\overline{\mathrm{LipSNA}_K(M,Y)} = \mathrm{Lip}_{0K}(M,Y).$$

The rest of the results of the section are not merely adaptations to Lipschitz compact maps of general results. Even more, their proofs use heavily the fact that the envolved maps are compact. The first result of this kind concerns preduals of L_1-spaces and it is based on [5, Theorem 4.2].

Proposition 6.12. *Let M be a metric space such that $(M,\mathbb{R})$ has the Lip-BPB property. Let Y be a Banach space such that Y^* is isometrically isomorphic to an L_1-space. Then, (M,Y) has the Lip-BPB property for Lipschitz compact maps.*

Proof. Fix $\varepsilon > 0$. Consider $\eta(\varepsilon)$ the function given by the Lip-BPB property of $(M,\mathbb{R})$. Since ℓ_∞^n has property β for every $n \in \mathbb{N}$, we may apply the first assertion of Corollary 6.7 to obtain that the pair (M,ℓ_∞^n) has the Lip-BPB property for Lipschitz compact maps witnessed by the function $\eta(\varepsilon)$ for every $n \in \mathbb{N}$. We take

$$\eta' = \min\left\{\frac{\varepsilon}{4}, \eta\left(\frac{\varepsilon}{2}\right)\right\} > 0.$$

Now, consider $F \in \mathrm{Lip}_{0K}(M,Y)$ with $\|F\|_L = 1$ and $m \in \mathrm{Mol}(M)$ such that $\|\widehat{F}(m)\| > 1-\eta'$. Let us take $0 < \delta < \frac{1}{4}\min\left\{\frac{\varepsilon}{4}, \|\widehat{F}(m)\| - 1 + \eta\left(\frac{\varepsilon}{2}\right)\right\}$ and let $\{y_1,\ldots,y_n\}$ be a δ-net of $\widehat{F}(B_{\mathcal{F}(M)})$. In view of [55, Theorem 3.1], we can find a subspace $E \subset Y$ isometric to ℓ_∞^m for some natural $m \in \mathbb{N}$ and such that $d(y_i,E) < \delta$ for every $i \in \{1,\ldots,n\}$. Let $P\colon Y \longrightarrow Y$ be a norm-one projection onto E. We claim that $\|P\widehat{F} - \widehat{F}\| < 4\delta$. In order to show it, fix $x \in B_{\mathcal{F}(M)}$. Then, there exists $i \in \{1,\ldots,n\}$ such that $\|\widehat{F}(x) - y_1\| < \delta$. Let $e \in E$ be such that $\|e - y_i\| < \delta$. Then, we have that

$$\begin{aligned}\|\widehat{F}(x) - P\widehat{F}(x)\| &\leqslant \|\widehat{F}(x) - y_i\| + \|y_i - e\| + \|e - P\widehat{F}(x)\| \leqslant 2\delta + \|P(e) - P\widehat{F}(x)\| \\ &\leqslant 2\delta + \|e - \widehat{F}(x)\| \leqslant 2\delta + \|e - y_i\| + \|y_i - \widehat{F}(x)\| < 4\delta.\end{aligned}$$

So $\|P\widehat{F}\| > \|\widehat{F}\| - 4\delta = 1 - 4\delta > 0$, which implies that

$$\|P\widehat{F}(m)\| > \|\widehat{F}(m)\| - 4\delta > 1 - \eta\left(\frac{\varepsilon}{2}\right).$$

Hence, the operator $\widehat{R} = \frac{P\widehat{F}}{\|P\widehat{F}\|}$ verifies that $\|\widehat{R}(m)\| > 1 - \eta\left(\frac{\varepsilon}{2}\right)$. Since the pair (M, ℓ_∞^m) has the Lip-BPB property for Lipschitz compact maps witnessed by the function $\eta(\varepsilon)$ and $E \subset Y$ is isometrically isomorphic to ℓ_∞^m, we can find a Lipschitz compact map $G \in \operatorname{Lip}_{0K}(M, E) \subseteq \operatorname{Lip}_{0K}(M, Y)$ and $u \in \operatorname{Mol}(M)$ such that

$$\|\widehat{G}(u)\| = \|G\|_L = 1, \quad \|\widehat{G} - \widehat{R}\| < \frac{\varepsilon}{2}, \quad \|m - u\| < \frac{\varepsilon}{2}.$$

Finally, we have that

$$\|\widehat{G} - \widehat{F}\| \leqslant \|\widehat{G} - \widehat{R}\| + \|\widehat{R} - P\widehat{F}\| + \|P\widehat{F} - \widehat{F}\| < \frac{\varepsilon}{2} + 1 - \|P\widehat{F}\| + 4\delta < \frac{\varepsilon}{2} + 8\delta < \varepsilon. \qquad \square$$

Repeating the above proof word by word, using the density of $\operatorname{LipSNA}_K(M, \mathbb{R})$ instead of the Lip-BPB property of the pair $(M, \mathbb{R})$, and forgetting about the estimation of the distance between the molecules, we obtain the following result.

Proposition 6.13. *Let M be a metric space such that $\operatorname{LipSNA}(M, \mathbb{R})$ is dense in $\operatorname{Lip}_0(M, \mathbb{R})$ and let Y be a Banach space such that Y^* is isometrically isomorphic to an L_1-space. Then, $\operatorname{LipSNA}_K(M, Y)$ is dense in $\operatorname{Lip}_{0K}(M, Y)$.*

For the next result we need some notation. A Banach space Y is said to have the (Grothendieck) *approximation property* if for every compact set $K \subset Y$ and every $\varepsilon > 0$, there is a finite-rank operator $R \in \mathcal{L}(Y, Y)$ such that $\|y - R(y)\| < \varepsilon$ for every $y \in K$. This is known to be equivalent to the fact that every compact linear operator whose range is Y can be approximated by finite-rank operators. This is a classical result which goes back to A. Grothendieck and can be found in Theorem 18.3.1 of [46], for instance. We send the interested reader to the cited book [46] for background.

The following preliminary result, based on [58, Proposition 4.4], is completely elemental.

Proposition 6.14. *Let M be a metric space and let Y be a Banach space with the approximation property. Suppose that for every finite-dimensional subspace W of Y, there exists a closed subspace Z such that $W \leqslant Z \leqslant Y$ and satisfying that $\overline{\operatorname{LipSNA}_K(M, Z)} = \operatorname{Lip}_{0K}(M, Z)$. Then, $\overline{\operatorname{LipSNA}_K(M, Y)} = \operatorname{Lip}_{0K}(M, Y)$*

If Y is a polyhedral Banach space (i.e. the unit ball of every finite-dimensional subspace of Y is the convex hull of finitely many points), then every finite-dimensional subspace of Y has property β, so Proposition 6.14 and Proposition 5.15 give us the following result.

Corollary 6.15. *Let M be a metric space and let Y be a polyhedral Banach space with the approximation property. If $\operatorname{LipSNA}(M, \mathbb{R})$ is dense in $\operatorname{Lip}_0(M, \mathbb{R})$, then*

$$\overline{\operatorname{LipSNA}_K(M, Y)} = \operatorname{Lip}_{0K}(M, Y).$$

Let us comment that Example 5.14 shows that the above corollary does not hold for the Lip-BPB property. Indeed, a family $\{Y_k : k \geqslant 2\}$ of two-dimensional polyhedral Banach spaces is constructed there such that if we consider the metric space $M = \{0, 1, 2\}$ with the usual metric and write $Y = [\oplus_{k \in \mathbb{N}} Y_k]_{c_0}$, then the pair (M, Y) fails the Lip-BPB property. However, Y is polyhedral as it is a c_0-sum of finite-dimensional polyhedral spaces, and $(M, \mathbb{R})$ has the Lip-BPB property by Corollary 4.4.

6.3 Stability behavior under operations

To finish the chapter, we will present some stability results for the Lip-BPB property for Lipschitz compact maps and for the density of $\operatorname{LipSNA}_K(M, Y)$. Let us start by studying sums of metric spaces.

Proposition 6.16. *Let $M = M_1 \coprod M_2$ be the sum of two metric spaces and let Y be a Banach space. If the pair (M, Y) has the Lip-BPB property for Lipschitz compact maps, then so do (M_1, Y) and (M_2, Y).*

Proof. The result follows by repeating the proof of Proposition 5.17 for a Lipschitz compact map F_1 observing that, in such a case, the strongly norm-attaining Lipschitz map which approximates F_1 is Lipschitz compact too. $\square$

From this result we obtain the following corollary.

Corollary 6.17. *Let $M = \coprod_{i\in I} M_i$ be the sum of metric spaces and let Y be a Banach space. If the pair (M, Y) has the Lip-BPB property for Lipschitz compact maps, then so does (M_i, Y) for every $i \in I$.*

In the same way as in the general case, the converse of Proposition 6.16 is not true, as Example 5.19 shows. Again, the analogous result for the density of $\operatorname{LipSNA}_K(M, Y)$ is more satisfactory.

Proposition 6.18. *Let $\{M_i\}_{i\in I}$ be a family of metric spaces, let Y be a Banach space, and consider $M = \coprod_{i\in I} M_i$. Then the following are equivalent:*

(i) $\operatorname{LipSNA}_K(M_i, Y)$ *is dense in* $\operatorname{Lip}_{0K}(M_i, Y)$ *for every* $i \in I$.

(ii) $\operatorname{LipSNA}_K(M, Y)$ *is dense in* $\operatorname{Lip}_{0K}(M, Y)$.

Proof. It is enough to note that the operators $\widehat{G}$ and $\widehat{G}_h$ defined in the proof of Theorem 5.20 are compact when the operators $\widehat{F}$ and $\widehat{F}_h$ are. □

We proceed now by studying some operations that we can consider for the range spaces. A cautious inspection of the proof of the results in subsection 5.2.2 shows that if one starts with a Lipschitz compact map, then one also gets a Lipschitz compact map in each case. Then, every result has its own version for the Lip-BPB property for Lipschitz compact maps and for the density of the strongly norm-attaining Lipschitz compact maps. We summarize all the results in the following proposition.

Proposition 6.19. *Let M be a metric space and let Y be a Banach space.*

(i) Let Y_1 be an absolute summand of Y. If the pair (M, Y) has the Lip-BPB property for Lipschitz compact maps with a function $\varepsilon \longmapsto \eta(\varepsilon)$, then so does (M, Y_1).

(ii) If for all Banach spaces Z the pair (M, Z) has the Lip-BPB property for Lipschitz compact maps, then there exists a function $\eta(\varepsilon)$, which depends only on M, such that for every Banach space Z the pair (M, Z) has the Lip-BPB property for Lipschitz compact maps witnessed by the function $\eta(\varepsilon)$.

(iii) Let Y_1 be an absolute summand of Y. If the set $\operatorname{LipSNA}_K(M, Y)$ *is dense in* $\operatorname{Lip}_{0K}(M, Y)$*, then* $\operatorname{LipSNA}_K(M, Y_1)$ *is dense in* $\operatorname{Lip}_{0K}(M, Y_1)$.

(iv) If for some compact Hausdorff space K the pair $(M, C(K, Y))$ has the Lip-BPB property for Lipschitz compact maps witnessed by a function $\eta(\varepsilon)$, then (M, Y) has the Lip-BPB property for Lipschitz compact maps witnessed by the same function.

(v) If $\operatorname{LipSNA}_K(M, C(K, Y))$ *is dense in* $\operatorname{Lip}_{0K}(M, C(K, Y))$ *for some compact Hausdorff space K, then* $\operatorname{LipSNA}_K(M, Y)$ *is dense in* $\operatorname{Lip}_{0K}(M, Y)$.

Proof. (i) Looking at the proof of Proposition 5.23, we see that if the operator $\widehat{F}_1$ is compact, then the operator $\widehat{T} \in \mathcal{L}(\mathcal{F}(M), Y_1)$ given by $\widehat{T}(x) = (\widehat{F}_1(x), 0)$ for all $x \in \mathcal{F}(M)$ is also compact. Then, we just have to follow that proof using that (M, Y) has the Lip-BPB property for Lipschitz compact maps.

(ii) This is a direct consequence of the fact that the same function η from the Lip-BPB property for Lipschitz compact maps of (M, Y) works for the Lip-BPB property for Lipschitz compact maps of (M, Y_1) for every summand Y_1 of Y.

(iii) We just have to proceed as in part (a), but disregarding the distance between molecules.

(iv) Observe that if the operator $\widehat{F}_1$ considered in the proof of Proposition 5.28 is compact, then so is the operator $\widehat{F}$. Following that proof we see that if we apply our assumption, then the operator $\widehat{G}$ is compact. Consequently, the operator $\widehat{G}_1$, which satisfies the conditions we wanted, is also compact.

(v) This is a slight modification of the previous item. We just have to disregard the distance between molecules. □

Moreover, a sight to the proofs of Propositions 5.25 and 5.27 shows that their corresponding versions for Lipschitz compact maps are also valid.

Proposition 6.20. *Let M be a metric space, let $\{Y_i\}_{i\in I}$ be a family of Banach spaces, and set $Y = [\bigoplus_{i\in I} Y_i]_{c_0}$ or $Y = [\bigoplus_{i\in I} Y_i]_{\ell_\infty}$. Assume that for each $i \in I$ the pair (M, Y_i) has the Lip-BPB property for Lipschitz compact maps witnessed by a function $\eta_i(\varepsilon)$. If $\inf\{\eta_i(\varepsilon) \colon i \in I\} > 0$ for every $\varepsilon > 0$, then (M, Y) has the Lip-BPB property for Lipschitz compact maps.*

Proposition 6.21. *Let M be a metric space, let $\{Y_i\}_{i\in I}$ be a family of Banach spaces, and set $Y = [\bigoplus_{i\in I} Y_i]_{c_0}$ or $Y = [\bigoplus_{i\in I} Y_i]_{\ell_\infty}$. If $\operatorname{LipSNA}_K(M, Y_i)$ is dense in $\operatorname{Lip}_{0K}(M, Y_i)$ for every $i \in I$, then the set $\operatorname{LipSNA}_K(M, Y)$ is dense in $\operatorname{Lip}_{0K}(M, Y)$.*

Finally, let us present some more results in the same line. They will be useful tools in order to carry the Lip-BPB property for Lipschitz compact maps from some sequence spaces to function spaces and, analogously, to carry the density of strongly norm-attaining Lipschitz compact maps from some sequence spaces to function spaces. Let us start with the result for the Lip-BPB property for Lipschitz compact maps.

Proposition 6.22. *Let M be a metric space and let Y be a Banach space. Suppose that there exists a net of norm-one projections $\{Q_\lambda\}_{\lambda\in\Lambda} \subset \mathcal{L}(Y, Y)$ such that $\{Q_\lambda(y)\} \longrightarrow y$ in norm for every $y \in Y$. If there is a function $\eta\colon \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that for every $\lambda \in \Lambda$, the pair $(M, Q_\lambda(Y))$ has the Lip-BPB property for Lipschitz compact maps witnessed by the function η , then the pair (M, Y) has the Lip-BPB property for Lipschitz compact maps.*

Proof. The proof is based on [33, Proposition 2.5]. Fix $\varepsilon > 0$ and write $\eta'(\varepsilon) = \frac{1}{2}\min\{\eta\left(\frac{\varepsilon}{2}\right), \varepsilon\}$. Consider $F \in \operatorname{Lip}_{0K}(M, Y)$ with $\|F\|_L = 1$ and $m \in \operatorname{Mol}(M)$ such that

$$\|\widehat{F}(m)\| > 1 - \eta'(\varepsilon).$$

Since $\widehat{F}(B_{\mathcal{F}(M)})$ is compact, we may find $y_1, \ldots, y_m \in Y$ such that

$$\min\left\{\|\widehat{F}(x) - y_j\| \colon j = 1, \ldots, m\right\} < \frac{\eta'(\varepsilon)}{3} \quad \forall\, x \in B_{\mathcal{F}(M)}.$$

By hypothesis, there is $\lambda \in \Lambda$ such that

$$\|Q_\lambda(y_j) - y_j\| < \frac{\eta'(\varepsilon)}{3} \quad \forall\, j = 1, \ldots, m.$$

Now, for every $x \in \mathcal{F}(M)$ we have that

$$\begin{aligned} \|\widehat{F}(x) - Q_\lambda(\widehat{F}(x))\| &\leqslant \min\{\|\widehat{F}(x) - y_j\| + \|y_j - Q_\lambda(y_j)\| + \|Q_\lambda(y_j) - Q_\lambda(\widehat{F}(x))\| \colon j = 1, \ldots, m\} \\ &< \min\{2\|\widehat{F}(x) - y_j\| + \frac{\eta'(\varepsilon)}{3} \colon j = 1, \ldots, m\} < \eta'(\varepsilon). \end{aligned}$$

Consequently, we have that $\|(Q_\lambda \circ \widehat{F}) - \widehat{F}\| \leqslant \eta'(\varepsilon)$. Notice that $Q_\lambda \circ \widehat{F}$ is a compact operator from $\mathcal{F}(M)$ to $Q_\lambda(Y)$ and $\|Q_\lambda \circ \widehat{F}\| \leqslant 1$. Moreover, from the previous observation we deduce that

$$\|(Q_\lambda \circ \widehat{F})(m)\| \geqslant \|\widehat{F}(m)\| - \|(Q_\lambda \circ \widehat{F}) - \widehat{F}\| > 1 - 2\eta'(\varepsilon) \geqslant 1 - \eta\left(\frac{\varepsilon}{2}\right).$$

Then, there exist a compact operator $\widehat{G}_\lambda$ from $\mathcal{F}(M)$ to $Q_\lambda(Y)$ and a molecule $u \in \operatorname{Mol}(M)$ such that

$$\|\widehat{G}_\lambda(u)\| = \|\widehat{G}_\lambda\| = 1, \quad \|(Q_\lambda \circ \widehat{F}) - \widehat{G}_\lambda\| < \frac{\varepsilon}{2}, \quad \|m - u\| < \frac{\varepsilon}{2}.$$

Define $G \in \operatorname{Lip}_{0K}(M, Y)$ so that its associated linear operator $\widehat{G}$ is the operator $\widehat{G}_\lambda$ viewed as an operator with range in Y. Then, clearly we have $\|\widehat{G}(u)\| = \|G\|_L = 1$ and

$$\|G - F\|_L \leqslant \|\widehat{G}_\lambda - (Q_\lambda \circ \widehat{F})\| + \|(Q_\lambda \circ \widehat{F}) - \widehat{F}\| < \frac{\varepsilon}{2} + \eta'(\varepsilon) \leqslant \varepsilon.$$

□

The following result collects several consequences of the previous proposition. Observe that item (i) below extends item (ii) and part of item (iv) of our Corollary 6.7.

Corollary 6.23. *Let M be a metric space and let Y be a Banach space such that the pair (M, Y) has the Lip-BPB property for Lipschitz compact maps.*

(i) For every compact Hausdorff topological space K, $(M, C(K, Y))$ has the Lip-BPB property for Lipschitz compact maps.

(ii) For $1 \leqslant p < \infty$, if the pair $(M, \ell_p(Y))$ has the Lip-BPB property for Lipschitz compact maps, then so does $(M, L_p(\mu, Y))$ for every positive measure μ.

(iii) For every σ-finite positive measure μ, the pair $(M, L_\infty(\mu, Y))$ has the Lip-BPB property for Lipschitz compact maps.

Proof. This proof is based on the one of Theorem 3.15 in [33]. To prove (i), following the proof of Theorem 4 in [48], by using peak partitions of the unit we can find a net $\{Q_\lambda\}_{\lambda\in\Lambda}$ of norm-one projections on $C(K, Y)$ such that $\{Q_\lambda(f)\} \longrightarrow f$ in norm for every $f \in C(K, Y)$ and $Q_\lambda(C(K, Y))$ is isometrically isomorphic to $\ell_\infty^m(Y)$. Consequently, (i) follows from Propositions 6.20 and 6.22.

In order to prove (ii), fix $1 \leqslant p < \infty$. If $L_p(\mu)$ is finite-dimensional, the result is a consequence of assertion (i) of Proposition 6.19 since in that case $L_p(\mu, Y)$ is an absolute summand of $\ell_p(Y)$. Otherwise, by using Lemma 3.12 in [33], we may find a net $\{Q_\lambda\}_{\lambda\in\Lambda}$ of norm-one projections on $L_p(\mu, Y)$ such that $\{Q_\lambda\} \longrightarrow f$ in norm for every $f \in L_p(\mu, Y)$ and $Q_\lambda(L_p(\mu, Y))$ is isometrically isomorphic to $\ell_p(Y)$. Therefore, it is enough to apply Proposition 6.22.

As before, if $L_\infty(\mu)$ is finite-dimensional, the result is a consequence of Proposition 6.20. Otherwise, if $L_\infty(\mu)$ is infinite-dimensional, we may suppose that the measure is finite by using Proposition 1.6.1 in [22]. Then, Lemma 3.12 in [33] provides a net $\{Q_\lambda\}_{\lambda\in\Lambda}$ of norm-one projections on $L_\infty(\mu, Y)$ such that $\{Q_\lambda\} \longrightarrow f$ in norm for every $f \in L_\infty(\mu, Y)$ and $Q_\lambda(L_p(\mu, Y))$ is isometrically isomorphic to $\ell_\infty(Y)$. Consequently, the result follows from Propositions 6.20 and 6.22. □

We can give a result analogous to Proposition 6.22 for the density of $\mathrm{LipSNA}_K(M, Y)$ whose proof is just a slight modification of its proof.

Proposition 6.24. *Let M be a metric space and Y be a Banach space. Suppose that there exists a net of norm-one projections $\{Q_\lambda\}_{\lambda\in\Lambda} \subset \mathcal{L}(Y, Y)$ such that $\{Q_\lambda(y)\} \longrightarrow y$ in norm for every $y \in Y$. If $\mathrm{LipSNA}_K(M, Q_\lambda(Y))$ is dense in $\mathrm{Lip}_{0K}(M, Q_\lambda(Y))$ for every $\lambda \in \Lambda$, then*

$$\overline{\mathrm{LipSNA}_K(M, Y)} = \mathrm{Lip}_{0K}(M, Y).$$

Finally, by using this proposition instead of Proposition 6.22 and replacing the necessary results by their analogous versions with respect to the density of $\mathrm{LipSNA}_K(M, Y)$, the proof of Corollary 6.23 can be easily adapted to get the following results about the density of the strongly norm-attaining Lipschitz compact maps. Let us notice that item (i) below extends item (ii) and part of item (iv) of our Corollary 6.10.

Corollary 6.25. *Let M be a metric space and let Y be a Banach space such that $\mathrm{LipSNA}_K(M, Y)$ is dense in $\mathrm{Lip}_{0K}(M, Y)$.*

(i) $\mathrm{LipSNA}_K(M, C(K, Y))$ is dense in $\mathrm{Lip}_{0K}(M, C(K, Y))$ for every compact Hausdorff topological space K.

(ii) For $1 \leqslant p < \infty$, if $\mathrm{LipSNA}_K(M, \ell_p(Y))$ is dense in $\mathrm{Lip}_{0K}(M, \ell_p(Y))$, then $\mathrm{LipSNA}_K(M, L_p(\mu, Y))$ is dense in $\mathrm{Lip}_{0K}(M, L_p(\mu, Y))$ for every positive measure μ.

(iii) $\mathrm{LipSNA}_K(M, L_\infty(\mu, Y))$ is dense in $\mathrm{Lip}_{0K}(M, L_\infty(\mu, Y))$ for every σ-finite positive measure μ.

Proof. We proceed as in the proof of Corollary 6.23. To prove (i), Theorem 4 in [48] shows that we can find a net $\{Q_\lambda\}_{\lambda\in\Lambda}$ of norm-one projections on $C(K, Y)$ such that $\{Q_\lambda(f)\} \longrightarrow f$ in norm for every $f \in C(K,Y)$ and $Q_\lambda(C(K,Y))$ is isometrically isomorphic to $\ell_p(Y)$. Consequently, we can apply Propositions 6.21 and 6.24 to obtain the result.

In order to prove (ii), fix $1 \leqslant p < \infty$. If $L_p(\mu)$ is finite-dimensional, the result is a consequence of assertion (iii) of Proposition 6.19 since in that case $L_p(\mu,Y)$ is an absolute summand of $\ell_p(Y)$. Otherwise, using Lemma 3.12 in [33] we find a net $\{Q_\lambda\}_{\lambda\in\Lambda}$ of norm-one projections on $L_p(\mu,Y)$ such that $\{Q_\lambda\} \longrightarrow f$ in norm for every $f \in L_p(\mu,Y)$ and $Q_\lambda(L_p(\mu,Y))$ is isometrically isomorphic to $\ell_p(Y)$. Consequently, we can apply Proposition 6.24.

Finally, if $L_\infty(\mu)$ is finite-dimensional, the result follows from Proposition 6.21. If $L_\infty(\mu)$ is infinite-dimensional, we may suppose that the measure is finite by using Proposition 1.6.1 in [22]. Then, Lemma 3.12 in [33] provides a net $\{Q_\lambda\}_{\lambda\in\Lambda}$ of norm-one projections on $L_\infty(\mu,Y)$ such that $\{Q_\lambda\} \longrightarrow f$ in norm for every $f \in L_\infty(\mu,Y)$ and $Q_\lambda(L_p(\mu,Y))$ is isometrically isomorphic to $\ell_\infty(Y)$. Consequently, we can apply Propositions 6.21 and 6.24 to get the result. $\square$

Chapter 7

Conclusions and open problems

To finish this work, we consider convenient to dedicate a chapter to summarize all results we have obtained and propose some open problems.

Chapter 1 is a summary which contains some background about Lipschitz spaces and Lipschitz-free spaces. We presented there some definitions and preliminary results needed for the rest of the chapters. Among the results presented, we want to highlight those ones which study the extremal structure of the unit ball of the Lipschitz-free space. Indeed, we considered classical notions of extremal points such as extreme points, exposed points, preserved extreme points, denting points, and strongly exposed points. Then, we presented the wide study that recently has been done to show the relationship between these notions in the context of the Lipschitz-free space and metric characterizations of them, that is, metric conditions that a point of the unit sphere of the Lipschitz-free space must satisfy to be an extremal point of a certain type.

In Chapter 2 we presented positive results that we have obtained concerning strong density of Lipschitz maps. As we commented in the introduction, J. Bourgain showed in [18] that if a Banach space X has the RNP, then X also has Lindenstrauss property A. Proposition 2.3 states the stronger result that if M is a metric space for which $\mathcal{F}(M)$ has the RNP, then $\mathrm{LipSNA}(M,Y)$ is dense in $\mathrm{Lip}_0(M,Y)$ for every Banach space Y. Motivated by this fact, we considered some other properties from the theory of norm-attaining linear operators that also imply Lindenstrauss property A and we showed that all of them actually implies that strongly norm-attaining Lipschitz maps are dense. Concretely, we proved that if M is a metric space such that $\mathcal{F}(M)$ either has a norming set of uniformly strongly exposed points, satisfies property α, or satisfies property quasi-α, then $\mathrm{LipSNA}(M,Y)$ is dense in $\mathrm{Lip}_0(M,Y)$ for every Banach space Y. Moreover, we translated these properties in terms of the geometry of the underlying metric spaces and presented some criteria that are useful to verify if the Lipschitz-free space over some metric space satisfies any of them.

We provided in the third section of the chapter different kinds of examples of metric spaces for which strongly norm-attaining Lipschitz maps are dense. In particular, we showed that this is the case of arbitrary Hölder metric spaces and of some particular metric subspaces of the plane which contain the unit interval (see Theorem 2.26). These latter spaces provided the first examples of metric spaces whose Lipschitz free spaces do not have the RNP but there is strong density, solving in the negative an open problem proposed by G. Godefroy in 2015.

Next, in the fourth section we discussed the relationship between all the sufficient conditions implying strong density that we presented. Using the criteria that we previously developed, we generated examples of metric spaces satisfying and failing strong density. This allowed us to show that none of the sufficient conditions presented before is necessary to have strong density.

All examples known of metric spaces for which strongly norm-attaining Lipschitz scalar functions are dense actually satisfy that strong density also holds for vector-valued functions, regardless of the range space. This motivates the following natural question.

Problem 7.1. Let M be a metric space so that $\mathrm{LipSNA}(M,\mathbb{R})$ is dense in $\mathrm{Lip}_0(M,\mathbb{R})$. Is it $\mathrm{LipSNA}(M,Y)$ dense in $\mathrm{Lip}_0(M,Y)$ for every Banach space Y?

Notice that Proposition 2.36 stated that if $\mathrm{LipSNA}(M,Y)$ is dense in $\mathrm{Lip}_0(M,Y)$ for some nonzero Banach space Y, then $\mathrm{LipSNA}(M,\mathbb{R})$ is dense in $\mathrm{Lip}_0(M,\mathbb{R})$.

Unfortunately, we are inclined to think that Problem 7.1 will have a negative answer, and the reason of why it is not solved yet is due to the current lack of examples of metric spaces M for which we know that there is strong density. Indeed, as we have already commented, Proposition 2.3 states that if $\mathcal{F}(M)$ has the Radon-Nikodým property, then $\mathrm{LipSNA}(M,Y)$ is dense in $\mathrm{Lip}_0(M,Y)$ for every Banach space Y. On the other hand, as far as we know the only examples of metric spaces having strong density which does not have the Radon-Nikodým property are the ones constructed in Theorem 2.26 and its consequences. They all satisfy strong density for every Banach space Y, but we believe that it should be possible to find counterexamples for Problem 7.1. We just need to get a better understanding of the density of strongly norm-attaining Lipschitz maps.

The next problem corresponds to the reverse of implication (18) in the diagram 2.2 that we presented in Section 2.4.

Problem 7.2. Let M be a metric space. Assume that $\mathcal{F}(M)$ has Lindenstrauss property A, that is, $\mathrm{NA}(\mathcal{F}(M),Y)$ is dense in $\mathcal{L}(\mathcal{F}(M),Y)$ for every Banach space Y. Is it true that $\mathrm{LipSNA}(M,Y)$ is dense in $\mathrm{Lip}_0(M,Y)$ for every Banach space Y?

As before, we suspect that the biggest obstacle in order to answer this problem is the lack of examples.

After this study of the relationship between the sufficient conditions, we focus on the consequences that the presence of strong density produces on the geometry of the Lipschitz-free space. Among the results that we obtained, we want to highlight Theorem 2.35, which states that if M is a metric space for which $\mathrm{LipSNA}(M,\mathbb{R})$ is dense in $\mathrm{Lip}_0(M,\mathbb{R})$, then $B_{\mathcal{F}(M)}$ is the closed convex hull of its extreme molecules. In the case of a compact metric space M, we actually got from the density of $\mathrm{LipSNA}(M,\mathbb{R})$ that $B_{\mathcal{F}(M)}$ is the closed convex hull of its strongly exposed points (Theorem 2.48). We do not know what happens in the general case.

Problem 7.3. Let M be an arbitrary metric space so that $\mathrm{LipSNA}(M,\mathbb{R})$ is dense in $\mathrm{Lip}_0(M,\mathbb{R})$. Is $B_{\mathcal{F}(M)}$ the closed convex hull of its strongly exposed points?

Let us comment that Lindenstrauss proved in [57, Theorem 2.ii] that if X is a Banach space which admits a LUR renorming (for instance, if X is separable) such that $\mathrm{NA}(X,Y)$ is dense in $\mathcal{L}(X,Y)$ for all Banach spaces Y, then B_X is the closed convex hull of its strongly exposed points. Therefore, the cited Theorem 2.48 is an improvement of this result in the case of Lipschitz-free spaces on compact metric spaces.

We do not know if the solution to Problem 7.3 is positive outside of the compact case. Let us comment the idea of the argument in order to understand where are the difficulties. First of all, for a compact metric space M, Lemma 2.45 tells us that if $\widehat{f} \in \mathcal{L}(\mathcal{F}(M),\mathbb{R})$ attains its norm at a molecule $m_{p,q} \in \operatorname{ext}\left(B_{\mathcal{F}(M)}\right)$, then we can approximate f by a non-local Lipschitz function. In order to get this, we are using compactness to apply Corollary 1.5 and get that all extreme molecules are preserved extreme molecules. Then, we can use Theorem 1.6 to get that they are actually denting points. Now, using geometric properties of these points, we get the result. However, we really need compactness to identify extreme molecules and preserved extreme molecules. On the other hand, Lemma 2.38 tells us that we are not losing too much strength if we assume that $\widehat{f}$ attains its norm at a extreme molecule, but we do not have a similar result for denting points. Second, in view of Lemma 2.45, if we assume that $\mathrm{LipSNA}(M,\mathbb{R})$ is dense in $\mathrm{Lip}_0(M,\mathbb{R})$, then every Lipschitz function can be approximated by non-local Lipschitz functions. Now, Lemma 2.46 states that, if M is compact, for every non-local Lipschitz function f there exists a strongly exposed molecule $m_{p,q} \in \mathrm{Mol}(M)$ such that $|\widehat{f}(m_{p,q})| = \|f\|_L$. From this fact is not too difficult to verify that the unit ball of $\mathcal{F}(M)$ is generated by its strongly exposed points. However, the assumption of compactness is again essential. Indeed, we cannot even ensure that a non-local Lipschitz function strongly attains its norm if M is not compact. For instance, if M is an infinite uniformly discrete metric space, then every Lipschitz function from M to $\mathbb{R}$ is non-local, but we know that there must be Lipschitz functions that do not strongly attain their norm since $\mathcal{F}(M)$ is not a reflexive Banach space.

For the previous reasons, we could not remove the compactness assumption from Theorem 2.48. Let us also notice that a positive answer for Problem 7.1 would solve Problem 7.3 for separable metric spaces.

Indeed, assume M is a metric space such that $\mathrm{LipSNA}(M,\mathbb{R})$ is dense in $\mathrm{Lip}_0(M,\mathbb{R})$. If this implies that $\mathrm{LipSNA}(M,Y)$ is dense in $\mathrm{Lip}_0(M,Y)$ for every Banach space Y, in particular, we would obtain that $\mathrm{NA}(\mathcal{F}(M),Y)$ is dense in $\mathcal{L}(\mathcal{F}(M),Y)$ for every Banach space Y. Now, using a result of Lindenstrauss (see [57, Theorem 2.ii]) we obtain that $B_{\mathcal{F}(M)} = \overline{\mathrm{co}}(\text{str-exp}\left(B_{\mathcal{F}(M)}\right))$.

Finally, in the last section of Chapter 2 we studied the density of the set $\mathrm{LipSNA}(M,\mathbb{R})$ in the weak topology. In this case, the problem is completely solved: we proved that for every metric space M, the set $\mathrm{LipSNA}(M,\mathbb{R})$ is (sequentially) weakly dense in $\mathrm{Lip}_0(M,\mathbb{R})$ (see Theorem 2.53).

Chapter 3 was devoted to presenting negative results concerning strong density. We started the chapter discussing the following important example from [50, Example 2.1]: $\mathrm{LipSNA}([0,1],\mathbb{R})$ is not dense in $\mathrm{Lip}_0([0,1],\mathbb{R})$. Analyzing the proof of this example we saw that the reason of why strong density fails is because all points are metrically aligned, that is, for any three points $x < z < y \in [0,1]$ we have $|x-y| = |x-z|+|z-y|$. With this idea in mind, we were able to generalize this example in two directions. First, we proved in Theorem 3.3 that if M is a length metric space, then $\mathrm{LipSNA}(M,\mathbb{R})$ is not dense in $\mathrm{Lip}_0(M,\mathbb{R})$. Second, we characterized in Corollary 3.8 for which closed subsets of $[0,1]$ we have strong density. We proved that if $M \subseteq [0,1]$ is closed, then $\mathrm{LipSNA}(M,\mathbb{R})$ is dense in $\mathrm{Lip}_0(M,\mathbb{R})$ if and only if M has measure zero.

Next, we dedicated a section to prove that the unit circle $\mathbb{T} \subseteq \mathbb{R}^2$ for the Euclidean metric also fails to have strong density (see Theorem 3.10). This shows that the reciprocal statement of Problem 7.3 is not true, as we also showed that every molecule of $\mathcal{F}(\mathbb{T})$ is a strongly exposed point. The idea behind the proof of Theorem 3.10 is that an arc of the unit circle has a similar local behavior to the segment $[0,1]$. Studying the density of the Cantor sets on $[0,1]$, in the last section of Chapter 3 we were able to generalize this result to C^2 curves (see Theorem 3.20). This leads to the following question.

Problem 7.4. Let $I \subset \mathbb{R}$ be an interval, let E be a Banach space, and let $\alpha\colon I \longrightarrow E$ be a rectifiable curve. Is there a Lipschitz function $f \in \mathrm{Lip}_0(\alpha(I),\mathbb{R})$ that cannot be approximated by strongly norm-attaining Lipschitz functions?

The next proposed problem is related to equivalent metrics. Let (M,d) be a metric space. We say that a metric d' defined on M is *equivalent* to d if there exist constants A, $B > 0$ such that

$$A\, d(p,q) \leqslant d'(p,q) \leqslant B\, d(p,q) \quad \forall\, p,q \in M.$$

Let us consider $[0,1]$ endowed with the usual metric. Notice that Theorem 3.20 provides plenty of equivalent metrics d' for which $([0,1],d')$ fails to have density. A natural question is whether this is true for every equivalent distance.

Problem 7.5. Let (M,d) be a metric space and let d' be an equivalent metric to d. Assume that the set $\mathrm{LipSNA}((M,d),\mathbb{R})$ is dense in $\mathrm{Lip}_0((M,d),\mathbb{R})$. Is necessarily the set $\mathrm{LipSNA}((M,d'),\mathbb{R})$ dense in $\mathrm{Lip}_0((M,d'),\mathbb{R})$?

It was not difficult to show that $[0,1]$, endowed with the usual metric $|\cdot|$, fails to have strong density. However, notice that if Problem 7.5 turns out to have a positive answer, it would be stronger than Theorem 3.20, for which several deep technical results has been necessary.

Let us recall that for almost every metric space M for which we know that there is strong density, we have that $\mathcal{F}(M)$ has the RNP. Furthermore, if d and d' are equivalent metrics on M, then the identity $\mathrm{Id}\colon (M,d) \longrightarrow (M,d')$ is a bi-Lipschitz map, that is, a bijective Lipschitz map whose inverse is also Lipschitz. Then, Id induces a linear isomorphism between the Banach spaces $\mathcal{F}((M,d))$ and $\mathcal{F}((M,d'))$. Since the RNP is stable under isomorphisms, if $\mathcal{F}((M,d))$ has the RNP, then $\mathcal{F}((M,d'))$ also does, so $\mathrm{LipSNA}((M,d'),\mathbb{R})$ is dense in $\mathrm{Lip}_0((M,d'),\mathbb{R})$ by Proposition 2.3. In view of this, with our current knowledge it is difficult to find a counterexample to give a negative answer to Problem 7.5.

In Chapter 4 we introduced a Lipschitz version of the classical Bishop-Phelps-Bollobás property. We saw that this stronger version of density is quite more restrictive than the usual strong density. Indeed, Example 4.6 showed that the pair $(\mathbb{N},\mathbb{R})$ fails the Lip-BPB property. In contrast, $\mathcal{F}(\mathbb{N})$ satisfies every sufficient condition studied in Chapter 2 that implies strong density for every Banach space Y. We gave some conditions that guarantee that the Lip-BPB property is satisfied for some pair (M,Y). Our main result in this chapter deals with the extremal structure of the Lipschitz-free space. Recall that a metric space M is said to be uniformly Gromov concave when the whole $\mathrm{Mol}(M)$ is a set of uniformly strongly exposed points. With this notation, Theorem 4.9 says that when M is a uniformly Gromov concave metric space, then (M, Y) has the Lip-BPB property for every Banach space Y. Recall that among uniformly Gromov concave spaces are included the concave metric spaces, the ultrametric spaces, and the Hölder metric spaces.

We continued the study of the Lip-BPB property in Chapter 5, where we focused our attention on the stability behavior of this property. In the first section we studied the relationship between the Lip-BPB property for scalar Lipschitz functions and the Lip-BPB property for vector-valued Lipschitz maps. More concretely, we gave some conditions over the Banach space Y that allowed us to pass from the scalar-valued case to the vector-valued case. Our main result in this section dealt with the notion of ACK structure and Γ-flat operators (see Theorem 5.6) and it is very general. We were able to obtain several consequences of this result for spaces of continuous functions, injective tensor products, sequence spaces, Banach spaces with property beta, etc.

On the other hand, we saw that it is possible to give versions of Theorem 5.6 for the strong density. Actually, we saw that in some cases the results for strong density that we obtained improved the analogous versions for the Lip-BPB property. As an example of this, Proposition 5.15 shows that if M is a metric space such that $\mathrm{LipSNA}(M, \mathbb{R})$ is dense in $\mathrm{Lip}_0(M, \mathbb{R})$, and Y is a Banach space satisfying property quasi-β, then $\mathrm{LipSNA}(M, Y)$ is dense in $\mathrm{Lip}_0(M, Y)$. In contrast, Example 5.14 shows that the Lip-BPB property of $(M, \mathbb{R})$ and property quasi-β of Y is not enough to guarantee that the pair (M, Y) has the Lip-BPB property.

Next, we observed that the Lip-BPB property is stable under some operations that we can consider on the domain space, such as sum of metric spaces, or on the range space, such as absolute sums. We also obtained versions of these results for strong density, which were more satisfactory in some cases (see Theorem 5.20).

Finally, we dedicated Chapter 6 to study Lipschitz compact maps. First, we saw that it is possible to give versions for this family of every sufficient condition implying strong density that we studied in Chapter 2. Indeed, we showed that most of our results concerning strong density or the Lip-BPB property are still valid when we restrict to Lipschitz compact maps, obtaining improvements in some of them (see Propositions 6.6 and 6.9) due to the fact that compact operators are always Γ-flat. Let us also mention that in Chapter 6 we also obtained results that are only valid for Lipschitz compact maps, as Propositions 6.12 and 6.13 which deals with L_1-preduals spaces.

As usual, we also obtained an analogous result for the strong density in this case. Finally, following Chapter 5, we also presented results dealing with the stability of the Lip-BPB property and strong density for Lipschitz compact maps under some operations on the domain and range spaces. As before, we obtained some results that are only valid when restricted to Lipschitz compact maps.

Finally, let us propose two more problems and give a couple of interesting remarks. Classical results coming from the theory of norm-attaining linear operators (due to Bourgain [18] and Bourgain and Huff [49]) state that each of the following properties characterizes the RNP of a Banach space X:

(a) for every bounded closed convex subset $C \subset X$, the set

$$\left\{T \in \mathcal{L}(X, Y)\colon \sup_{x \in C} \|Tx\| = \max_{x \in C} \|Tx\|\right\}$$

is dense in $\mathcal{L}(X, Y)$;

(b) every equivalent renorming of X has Lindenstrauss property A.

We may wonder whether there are Lipschitz versions of these characterizations, and so we propose the following open problems.

Problem 7.6. Let M be a metric space. Suppose that $\overline{\mathrm{LipSNA}(M', Y)} = \mathrm{Lip}_0(M', Y)$ for every metric space M' bi-Lipschitz equivalent to M and every Banach space Y. Does $\mathcal{F}(M)$ have the RNP?

Problem 7.7. Let M be a metric space. Suppose that $\overline{\mathrm{LipSNA}(N,Y)} = \mathrm{Lip}_0(N,Y)$ for every closed subset N of M and every Banach space Y. Does $\mathcal{F}(M)$ have the RNP?

We also want to present two remarks on possible variation of the above two problems. First, there is a classical result stating that the following property is also equivalent to the fact that the Banach space X has the RNP:

(c) the unit ball of every equivalent renorming of every closed subspace of X is dentable.

Hence, we may wonder whether a Lipschitz version of this result could characterize the RNP of the Lipschitz-free space, that is, whether for a given metric space M, $\mathcal{F}(M)$ has the RNP provided that the unit ball of the Lipschitz-free space over every metric space bi-Lipschitz equivalent to a closed subset of M is dentable. The following example shows that this is not the case. Actually, it shows that the latter property does not imply the density of strongly norm attaining Lipschitz functions defined on M.

Example. Let M be a nowhere dense closed subset of $[0,1]$ whose Lebesgue measure is positive. Then, given any metric space N' bi-Lipschitz equivalent to a closed subspace of M, it follows that N' is not length (because it is disconnected). Consequently, $B_{\mathcal{F}(N')}$ has strongly exposed points by Corollary 1.8 and Theorem 1.9. Thus, it is dentable. However, Corollary 3.8 shows that $\mathrm{LipSNA}(M,\mathbb{R})$ is not dense in $\mathrm{Lip}_0(M,\mathbb{R})$.

Let us next consider another characterization of the fact that X has the RNP which easily follows from (a) and (b) above:

(d) every Banach space Z' isomorphic to a closed subspace of X has Lindenstrauss property A.

We also may wonder whether there is a Lipschitz version of this result, that is, whether $\mathcal{F}(M)$ has the RNP provided that $\overline{\mathrm{LipSNA}(N',Y)} = \mathrm{Lip}_0(N',Y)$ for every metric space N' bi-Lipschitz equivalent to a closed subset of M and for every Banach space Y. Now, the answer is yes, and it is follows from a very recently announced result of R. Aliaga, C. Gartland, C. Petitjean, and A. Procházka [7, 8].

Remark. Let M be a complete pointed metric space. Then, the following are equivalent:

(i) $\mathcal{F}(M)$ has the Radon-Nikodým property.

(ii) $\mathrm{LipSNA}(N',Y)$ is dense in $\mathrm{Lip}_0(N',Y)$ for every metric space N' bi-Lipschitz equivalent to a closed subset of M and every Banach space Y.

(ii) $\mathrm{LipSNA}(N',\mathbb{R})$ is dense in $\mathrm{Lip}_0(N',\mathbb{R})$ for every metric space N' bi-Lipschitz equivalent to a closed subset of M.

Let us quickly present the argument. Indeed, (i)⇒(ii) follows from [36, Proposition 7.4] and (ii)⇒(iii) is immediate. Now, given a complete pointed metric space M, it has been very recently announced by R. Aliaga, C. Gartland, C. Petitjean, and A. Procházka [8] (see the talk given by R. Aliaga on January 22nd, 2021, at the *Banach spaces webinars* [7, 8]), that if the space $\mathcal{F}(M)$ fails the RNP then M contains a "curve fragment", that is, a subset N of M that is bi-Lipschitz equivalent to a compact subset K of the real line with positive Lebesgue measure. We just have to use Theorem 3.6 (or Corollary 3.8) to get that $\mathrm{LipSNA}(K,\mathbb{R})$ is not dense in $\mathrm{Lip}_0(K,\mathbb{R})$.

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Glossary

Concepts

List of Symbols

$C_\omega(K,Y)$: Space of weakly continuous functions from K to Y, 74
$\overline{\text{co}}$: Closed convex hull, xiii
δ: Cannonical isometric embedding of M into $\text{Lip}_0(M,\mathbb{R})$, xii
$\text{dent}\,(B_X)$: Set of denting points of B_X, 1
$\text{diam}(M)$: Diameter of a metric space M, 1
$\exp\,(B_X)$: Set of exposed points of B_X, 1
$\text{ext}\,(B_X)$: Set of extreme points of B_X, 1
$\widehat{f}$: Linear operator associated to the Lipschitz map f, xiii
$\text{Fl}_\Gamma(X,Y)$: Set of Γ-flat operators from X to Y, 73
$\mathcal{F}(M)$: Lipschitz-free space over M, xii
$(x,y)_z$: Gromov product of x and y at z, 4
$[\bigoplus_{\gamma\in\Gamma} X_\gamma]_{\ell_1}$: ℓ_1-sum of Banach spaces, 5
$\coprod_{i\in I} M_i$: Sum of metric spaces, 5
$\mathcal{L}(X,Y)$: Space of bounded linear operators from X to Y, xi
$\ell_\infty(Y)$: Space of bounded sequences on Y, 74
$[\bigoplus_{i\in I} Y_i]_{\ell_\infty}$: ℓ_∞-sum of Banach spaces, 85
$\text{L}(f)$: Lipschitz constant of f, xi
Lip-BPBp: Lipschitz Bishop-Phelps-Bollobás property, 63
$\text{Lip}(M,Y)$: Space of Lipschitz functions from M to Y, xi
$\text{Lip}_0(M,Y)$: Space of Lipschitz functions from M to Y vanishing at 0, xi
$\text{Lip}_{0K}(M,Y)$: Space of Lipschitz compact maps from M to Y vanishing at 0, 89
$\text{lip}_0(M,\mathbb{R})$: Little Lipschitz space over M, 7
$\text{LipSNA}(M,Y)$: Set of strongly norm-attaining Lipschitz maps from M to Y, xii
$\text{LipSNA}_K(M,Y)$: Set of strongly norm-attaining Lipschitz compact maps from M to Y, 89
$[p,q]$: Metric segment of points p and q, 4
M, N: Symbols used to denote pointed complete metric spaces, xi
$m_{p,q}$: Molecule of points (p,q), xiii
$\text{Mol}(M)$: Set of molecules of $\mathcal{F}(M)$, xiii
$\text{NA}(X,Y)$: Set of norm-attaining linear operators from X to Y, xi
$\text{pre-ext}\,(B_X)$: Set of preserved extreme points of B_X, 1
$X\widehat{\otimes}_\pi Y$: Projective tensor product of X and Y, 37
RNP: Radon-Nikodým property, xiv
S_X: Unit sphere of X, xi
$S(B_X,f,\delta)$: Slice of B_X with functional f and parameter δ, 1
$\text{str-exp}\,(B_X)$: Set of strongly exposed points of B_X, 1
$\mathbb{T}$: Unit circle of the Euclidean plane, xv
T: Symbol used to denote $\mathbb{R}$-trees, 42
λ_T: length measure on an $\mathbb{R}$-tree, 43
ω: Weak topology, xv