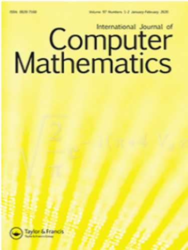

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Covariance-based least-squares filtering algorithm under Markovian measurement delays

M. J. García-Ligero, A. Hermoso-Carazo & J. Linares-Pérez

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Abstract

This paper addresses the least-squares linear filtering problem of signals from measurements which may be randomly delayed by one or two sampling times. The delays are modelled by a homogeneous discrete-time Markov chain to capture the dependence between them. Assuming that the evolution equation generating the signal is not available and that only the first- and second-order moments of the processes involved in the observation model are known, a recursive filtering algorithm is derived using an innovation approach. Recursive formulas for the filtering error variances are also obtained to measure the precision of the proposed estimators.

1. Introduction

An unavoidable problem in communication networks is the existence of delays in the arrival of measurements. In practical situations such as wireless communication channels or heavy network traffic, errors commonly occur during transmission, which can lead to delayed measurements. The delays may be deterministic or random, although in most practical cases, such as mobile communications or exploration seismology, the delays are considered to be random, and can be modelled by a stochastic process.

The problem of signal estimation in linear stochastic systems has been widely studied using different models of delayed observation. In this context, many researchers have focused on the case in which measurements are subject to a random delay which does not exceed one sample time, and when the observation available at any time is either the current or the previous one.

A common approach to deal with one-step delays is to consider a sequence of Bernoulli random variables that take values of 0 or 1 depending on whether the real observation is received on time or otherwise. Assuming the independence of these variables, the signal estimation problem has been addressed assuming that the state-space model is either completely known or not known, in which case the information provided by the covariance functions of the processes involved in the observation model is used. In [12], who considered linear unbiased state estimation for dynamic systems with a one-step sensor delay, the question was approached by reformulating the problem as a stochastic parameter estimation contained within the filtering problem, and full and reduced-order estimators were proposed. In later studies, new filtering algorithms, both full and reduced-order, were derived by [11] for stochastic dynamic systems with random one-step sensor delays, and [10] addressed the problem of robust filtering with variance constraints. In the context of multiple sensors, a centralised linear minimum variance unbiased filter was derived by [7], in a study in which delays were modelled by independent Bernoulli variables with different delay probabilities in each sensor.

All the papers cited above assume that the state-space model is known. However, the estimation problem from delayed measurements has also been addressed under a covariance information approach. For example, in [1, 2] recursive estimation algorithms based on covariance were derived assuming one-step random delays modelled by independent Bernoulli random variables. Similarly, [8] addressed the problem of filtering from one-and-two-step randomly delayed observations, and, recently, in [3], this delayed observation model was generalised to the case of measurements subject to bounded multiple-step random delays, and recursive estimation algorithms based on covariances were obtained.

Nevertheless, in real-world communication systems, current delays are usually correlated with previous ones; then, a reasonable way to model the dependence on the delays is to consider them as homogeneous Markov chains.

Assuming that the state-space model is known, optimal estimation problems in networked systems with correlated transmission delays are discussed in [5] and [6], where the delay process is modelled by a two-state Markov chain and by a multi-state Markov chain, respectively. In this respect, too, [9] investigated the H_∞ filtering problem in a class of network systems with random delays modelled by a Markov chain. Finally, random delay from sensor to controller was modelled by [13] using a Markov chain with partly-known transition probability matrix, covering completely known and completely unknown transition probabilities as special cases.

As in the case of delays modelled by independent random variables, estimation algorithms from Markovian delayed observations have also been deduced using only covariance information. Specifically, [4] proposed covariance-based recursive filtering and smoothing algorithms to estimate a signal from one-step delayed observations, assuming that the delay is modelled by a Markov chain whose statistical characteristics are known.

In the present study, the observation model presented in [4] is generalised by considering measurements that can be randomly delayed by one or two sampling times. Assuming that the state-space model of the signal is not available and that the delays are modelled by a homogeneous discrete-time Markov chain with three states, a recursive algorithm for the least-squares linear filtering problem is proposed. This algorithm uses only the information provided by the covariance functions of the processes involved in the measurements of the signal, together with the probability distribution of the Markov chain modelling the delays. The least-squares linear filter is obtained by an innovation approach which enables the estimation algorithm to be derived without too much difficulty. The proposed algorithm, based on covariances, is also applicable in situations based on the state-space model.

The rest of this paper is organised as follows. In Section 2, the observation model is described and the estimation problem is formulated using an innovation approach. In Section 3, the recursive filtering algorithm and the filtering error variances, which provide a measure of the filter performance, are derived. Section 4 presents a simulation example, in which a scalar signal is estimated from observations that can be delayed by one or two sample periods during the transmission; the effectiveness of the proposed estimator is measured by the error variances. Finally, some concluding remarks are made in Section 5.

2. Observation model

Consider an n-dimensional signal, x_k, whose scalar measured output, z_k, at each sampling time is perturbed by an additive noise:

$$z_k = H_k x_k + v_k, \quad k \geq 1, \tag{1}$$

where H_k is a known matrix and v_k is the measurement noise.

A common problem in communication theory is the existence of failures during the transmission, which can lead to delays in the arrivals of the measurements. In this paper, it is assumed that the measurements may be randomly delayed by one or two sampling times during the transmission; that is, the observed measurement at time k is $z_{k-a},\ a = 1, 2$, if there is delay, or z_k, if there is no delay. The delay is modelled by a homogeneous Markov chain, $\{\theta_k,\ k \geq 1\}$, that takes values in the state space $S = \{0, 1, 2\}$. If $\theta_k = a,\ a = 1, 2$, this means that the k-th measurement is delayed by a sampling periods; otherwise, if $\theta_k = 0$, there is no delay in arrival. This situation can be represented by the following observation model:

$$y_k = \sum_{a=0}^{(k-1)\wedge 2} \delta(\theta_k, a) z_{k-a}, \quad k \geq 1, \tag{2}$$

where $\delta(\cdot, \cdot)$ is the Kronecker delta function and $(k-1) \wedge 2$ represents the minimum of $k-1$ and 2.

The aim of this paper is to study the least-squares (LS) linear filtering problem of the signal, x_k, from the randomly delayed observations $y_1, \cdots, y_k$. This problem is addressed using the information provided by the covariance functions of the signal and the noises present in the observation model. For this purpose, the following hypotheses about the signal and noise processes are assumed:

(i) The signal process, $\{x_k,\ k \geq 1\}$, has zero mean and its covariance function is given by

$$E[x_k x_s^T] = A_k B_s^T, \quad s \leq k,$$

where A and B are known $n \times M$ matrix functions.

(ii) The measurement noise, $\{v_k,\ k \geq 1\}$, is a white process with zero mean and known variances $E[v_k^2] = R_k,\ \forall k \geq 1$.

(iii) $\{\theta_k,\ k \geq 1\}$ is a homogeneous Markov chain that takes values in $S = \{0, 1, 2\}$, with known probabilities $\pi_a^{(k)} = P(\theta_k = a),\ k \geq 1,\ a \in S$, and transition probability matrix $\mathbf{P} = \begin{pmatrix} p_{00} & p_{01} & p_{02} \\ p_{10} & p_{11} & p_{12} \\ p_{20} & p_{21} & p_{22} \end{pmatrix}$, where $p_{ab} = P\left(\theta_{h+1} = b/\theta_h = a\right),\ h \geq 1,\ \ a, b \in S$.

(iv) The signal process, $\{x_k,\ k \geq 1\}$, the noise, $\{v_k,\ k \geq 1\}$, and $\{\theta_k,\ k \geq 1\}$, are mutually independent.

Remark 1. Hypothesis (i) actually covers many practical situations; for example, when the state-space model is available, $x_k = \Phi_{k-1} x_{k-1} + w_{k-1}$, the covariance function can be expressed as $E[x_k x_s^T] = \Phi_{k,s} E[x_s x_s^T],\ s \leq k$, where $\Phi_{k,s} = \Phi_{k-1} \cdots \Phi_s$, and Hypothesis (i) is satisfied taking $A_k = \Phi_{k,0}$ and $B_s = E[x_s x_s^T](\Phi_{s,0}^{-1})^T$. Note also that, although a state-space model can be generated from covariances, when only this kind of information is available, it is preferable to address the estimation problem directly using covariances, thus obviating the need for previous identification of the state-space model.

Remark 2. From hypothesis (iii), denoting $p_{ab}^{(k)} = P(\theta_{h+k} = b/\theta_h = a),\ \ h, k \geq 1$, $a, b \in S$ $\left(p_{ab}^{(1)} = p_{ab}\right)$, the $k-$step transition probability from state a to state b, we have:

$$E[\delta(\theta_k, a)\delta(\theta_h, b)] = P(\theta_k = a/\theta_h = b)P(\theta_h = b) = p_{ba}^{(k-h)} \pi_b^{(h)}, \quad h < k.$$

Given the randomly delayed measurements up to time k, $y_1, \cdots, y_k$, our aim is to determine the LS linear estimator, $\widehat{x}_{k/k}$, of the signal, x_k. For this purpose an innovation approach is used.

The innovation approach is based on an orthogonalisation procedure by means of which the observation process $\{y_k;\ k \geq 1\}$ is transformed into an equivalent one termed innovation process, which is denoted by $\{\mu_k;\ k \geq 1\}$ and defined as $\mu_k = y_k - \widehat{y}_{k/k-1}$; that is, the innovation at time k is the difference between the observation y_k and its estimation from the previous ones $\widehat{y}_{k/k-1}$. The linear estimation problem is then approached by replacing the observation process with the innovation one, since both processes provide the same information. As the innovation process is white, the estimator calculated as a linear combination of innovations provides a simpler form of obtaining the algorithms than that obtained when it is expressed as a linear combination of observations. In order to apply this approach, the first step is to calculate the explicit expression for the innovation and its covariance matrix, and afterwards to determine the estimator expression.

In this approach, the LS linear estimator of a random vector w_k based on the observations $\{y_1, \dots, y_k\}$, denoted by $\widehat{w}_{k/k}$, is expressed as a linear combination of the innovations $\{\mu_1, \dots, \mu_k\}$; specifically:

$$\widehat{w}_{k/k} = \sum_{h=1}^{k} E[w_k \mu_h] \Pi_h^{-1} \mu_h, \quad k \geq 1, \tag{3}$$

where $\Pi_h = E[\mu_h^2]$ denotes the innovation variance.

3. Filtering algorithm

Our purpose is to find the LS linear estimator of the signal x_k based on the observations $\{y_1, \dots, y_k\}$. Specifically, using an innovation approach our aim is to obtain a recursive algorithm for the filter $\widehat{x}_{k/k}$.

3.1. Notation

In order to simplify the expressions of the filtering algorithm, the following notations are used:

$$\mathbb{A}_k = \begin{cases} (H_1A_1 \mid 0 \mid 0)\,(\mathbf{P} \otimes I_M)^T, & k = 1, \\ (H_2A_2 \mid H_1A_1 \mid 0)\,(\mathbf{P}^2 \otimes I_M)^T, & k = 2, \\ (H_kA_k \mid H_{k-1}A_{k-1} \mid H_{k-2}A_{k-2})\,(\mathbf{P}^k \otimes I_M)^T, & k \geq 3, \end{cases}$$

$$\mathbb{B}_k = \begin{cases} \left(\pi_0^{(1)} H_1B_1 \mid 0 \mid 0\right)(\mathbf{P}^{-1} \otimes I_M)^T, & k = 1, \\ \left(\pi_0^{(2)} H_2B_2 \mid \pi_1^{(2)} H_1B_1 \mid 0\right)(\mathbf{P}^{-2} \otimes I_M)^T, & k = 2, \\ \left(\pi_0^{(k)} H_kB_k \mid \pi_1^{(k)} H_{k-1}B_{k-1} \mid \pi_2^{(k)} H_{k-2}B_{k-2}\right)(\mathbf{P}^{-k} \otimes I_M)^T, & k \geq 3, \end{cases}$$

$$\mathbb{F}_k = \begin{cases} p_{01}\pi_0^{(1)} R_1, & k = 1, \\ p_{02}\pi_0^{(k)} H_{k-1}(B_{k-1}A_k^T - A_{k-1}B_k^T)H_k^T + p_{01}\pi_0^{(k)} R_k + p_{12}\pi_1^{(k)} R_{k-1}, & k \geq 2, \end{cases}$$

$$\mathbb{G}_k^{(2)} = p_{02}^{(2)} \pi_0^{(k)} R_k, \quad k \geq 1,$$

where $\otimes$ denotes the Kronecker product and I_M is the $M \times M$ identity matrix.

3.2. Innovation process

As commented above, the estimation problem is addressed by using an innovation approach; thus, in order to determine the filter, our first aim is to derive an explicit formula for the innovation and for its variance.

Theorem 3.1. *Under hypotheses (i)-(iv) set out in Section 2, the innovation, μ_k, is given by*

$$\mu_k = y_k - \mathbb{A}_k O^y_{k-1} - \sum_{j=1}^{(k-1)\wedge 2} \mathbb{G}^{(j)}_{k-j} \Pi^{-1}_{k-j} \mu_{k-j}, \quad k \geq 2; \qquad \mu_1 = y_1, \tag{4}$$

where the vectors O^y_k are recursively calculated as

$$O^y_k = O^y_{k-1} + J^y_k \Pi^{-1}_k \mu_k, \quad k \geq 1; \qquad O^y_0 = 0, \tag{5}$$

with

$$J^y_k = \mathbb{B}^T_k - r^y_{k-1} \mathbb{A}^T_k - \sum_{j=1}^{(k-1)\wedge 2} J^y_{k-j} \Pi^{-1}_{k-j} \mathbb{G}^{(j)}_{k-j}, \quad k \geq 2; \qquad J^y_1 = \mathbb{B}^T_1 \tag{6}$$

and where $\mathbb{G}^{(1)}_k$ is given by

$$\mathbb{G}^{(1)}_k = \mathbb{F}_k + \mathbb{G}^{(2)}_{k-1} \Pi^{-1}_{k-1} (\mathbb{A}_k J^y_{k-1} + \mathbb{G}^{(1)}_{k-1}), \quad k \geq 2; \qquad \mathbb{G}^{(1)}_1 = \mathbb{F}_1. \tag{7}$$

The matrices $r^y_k = E[O^y_k O^{yT}_k]$ are obtained by

$$r^y_k = r^y_{k-1} + J^y_k \Pi^{-1}_k J^{yT}_k, \quad k \geq 1; \qquad r^y_0 = 0. \tag{8}$$

The innovation variance, Π_k, is given by

$$\begin{aligned}\Pi_k = & \sum_{a=0}^{(k-1)\wedge 2} \pi^{(k)}_a \left(H_{k-a} A_{k-a} B^T_{k-a} H^T_{k-a} + R_{k-a} \right) - \mathbb{A}_k \left(\mathbb{B}^T_k - J^y_k \right) \\ & - \sum_{j=1}^{(k-1)\wedge 2} \mathbb{G}^{(j)}_{k-j} \Pi^{-1}_{k-j} \left(\mathbb{A}_k J^y_{k-j} + \mathbb{G}^{(j)}_{k-j} \right), \quad k \geq 2; \\ \Pi_1 = & \, \pi^{(1)}_0 \left(H_1 A_1 B^T_1 H^T_1 + R_1 \right).\end{aligned} \tag{9}$$

Proof. In order to obtain the expression (4) for the innovation, $\mu_k = y_k - \widehat{y}_{k/k-1}$, it is necessary to calculate the one-stage observation predictor, $\widehat{y}_{k/k-1}$. From (3) and noting $T_{k,h} = E[y_k \mu_h]$, the predictor is expressed as

$$\widehat{y}_{k/k-1} = \sum_{h=1}^{k-1} T_{k,h} \Pi^{-1}_h \mu_h, \quad k \geq 2; \qquad \widehat{y}_{1/0} = 0. \tag{10}$$

Then, it is necessary to calculate the coefficients $T_{k,h} = E[y_k y_h] - E[y_k \widehat{y}_{h/h-1}]$, for $k \geq 2$ and $1 \leq h \leq k-1$. Using (10) for $\widehat{y}_{h/h-1}$, it is clear that

$$T_{k,h} = \begin{cases} E[y_k y_1], & h = 1, \\ E[y_k y_h] - \displaystyle\sum_{j=1}^{k-1} T_{k,j} \Pi^{-1}_j T_{h,j}, & 2 \leq h \leq k-1. \end{cases} \tag{11}$$

First, we obtain the expectations $E[y_k y_h]$, for $k \geq 2$ and $1 \leq h \leq k-1$.

- For $h = 1$, using (2) for the observations and the independence of $\{\theta_k,\ k \geq 1\}$ and $\{z_k,\ k \geq 1\}$, we have

$$E[y_k y_1] = \sum_{a=0}^{(k-1)\wedge 2} p_{0a}^{(k-1)} \pi_0^{(1)} E[z_{k-a} z_1].$$

  Now, taking into account hypotheses (i) and (iii), it is clear that, for $a \leq (k-1)\wedge 2$, we have $E[z_{k-a}z_1] = H_{k-a}A_{k-a}B_1^T H_1^T + R_1\delta_{k-a,1}$, and it is then apparent that

$$E[y_k y_1] = \begin{cases} \mathbb{A}_2\mathbb{B}_1^T + \mathbb{F}_1, & k = 2, \\ \mathbb{A}_3\mathbb{B}_1^T + \mathbb{G}_1^{(2)}, & k = 3, \\ \mathbb{A}_k\mathbb{B}_1^T, & k > 3. \end{cases}$$

- For $2 \leq h \leq k-1$, an analogous procedure leads us, after some manipulations, to the following expressions:

$$E[y_k y_h] = \begin{cases} \mathbb{A}_k\mathbb{B}_h^T, & h < k-2, \\ \mathbb{A}_k\mathbb{B}_{k-2}^T + \mathbb{G}_{k-2}^{(2)}, & h = k-2, \\ \mathbb{A}_k\mathbb{B}_{k-1}^T + \mathbb{F}_{k-1}, & h = k-1. \end{cases}$$

Substituting the above expectations into (11), we obtain that, for $k \geq 2$ and $1 \leq h \leq k-1$, the coefficients $T_{k,h}$ can be expressed by

$$T_{k,h} = \begin{cases} \mathbb{A}_k J_h^y, & h < k-2, \\ \mathbb{A}_k J_{k-2}^y + \mathbb{G}_{k-2}^{(2)}, & h = k-2, \\ \mathbb{A}_k J_{k-1}^y + \mathbb{G}_{k-1}^{(1)}, & h = k-1, \end{cases} \tag{12}$$

where $\mathbb{G}_k^{(1)} = \mathbb{F}_k + \mathbb{G}_{k-1}^{(2)}\Pi_{k-1}^{-1}T_{k,k-1}$, and J_h^y is a function satisfying

$$J_h^y = \begin{cases} \mathbb{B}_1^T, & h = 1, \\ \mathbb{B}_h^T - \displaystyle\sum_{j=1}^{h-1} J_j^y \Pi_j^{-1} T_{h,j}, & h \geq 2. \end{cases} \tag{13}$$

Then, substituting (12) in (10) and defining $O_k^y = \sum_{h=1}^{k} J_h^y \Pi_h^{-1}\mu_h,\ k \geq 1$, with $O_0^y = 0$, the observation predictor is given by

$$\widehat{y}_{k/k-1} = \mathbb{A}_k O_{k-1}^y + \sum_{j=1}^{(k-1)\wedge 2} \mathbb{G}_{k-j}^{(j)} \Pi_{k-j}^{-1} \mu_{k-j},\ k \geq 2,$$

and expression (4) for innovation is obtained. From the definition of O_k^y, the recursive relation (5) is clear. Now, by substituting (12) in (13) for $h = k$ and defining $r_k^y =$

$\sum_{h=1}^{k} J_h^y \Pi_h^{-1} J_h^{yT}$, $k \geq 1$, with $r_0^y = 0$, the expression (6) is obtained. Using (12) for $T_{k,k-1}$, the formula (7) for $\mathbb{G}_k^{(1)}$ is obtained immediately, and the recursive relation (8) is clear from the definition of r_k^y.

Next, we prove (9) for the innovation variance, $\Pi_k = E[(y_k)^2] - E[y_k \widehat{y}_{k/k-1}]$.

- Clearly, from (2) and the model hypotheses

$$E[(y_k)^2] = \begin{cases} \pi_0^1 \left(H_1 A_1 B_1^T H_1^T + R_1 \right), & k = 1, \\ \displaystyle\sum_{a=0}^{(k-1)\wedge 2} \pi_a^{(k)} \left(H_{k-a} A_{k-a} B_{k-a}^T H_{k-a}^T + R_{k-a} \right), & k \geq 2. \end{cases}$$

- To derive $E[y_k \widehat{y}_{k/k-1}]$, we use (10) for the observation predictor, (12) for $T_{k,h}$ and (13) for $\sum_{h=1}^{k-1} J_h^y \Pi_h^{-1} T_{k,h}$; then

$$E[y_k \widehat{y}_{k/k-1}] = \mathbb{A}_k \left(\mathbb{B}_k^T - J_k^y \right) - \sum_{j=1}^{(k-1)\wedge 2} \mathbb{G}_{k-j}^{(j)} \Pi_{k-j}^{-1} \left(\mathbb{A}_k J_{k-j}^y + \mathbb{G}_{k-j}^{(j)} \right), \quad k \geq 2.$$

From the two expectations, expression (9) for Π_k is thus obtained. □

3.3. Recursive filtering algorithm

In the following theorem, an expression for the optimal LS linear signal filter is derived, which together with those of the innovations and their variances, given in Theorem 3.1, constitute the recursive filtering algorithm.

Theorem 3.2. *Under hypotheses (i)-(iv) of the model, the filter, $\widehat{x}_{k/k}$, of the signal x_k is obtained by*

$$\widehat{x}_{k/k} = A_k O_k^x, \quad k \geq 1, \tag{14}$$

where the vectors O_k^x are recursively calculated as

$$O_k^x = O_{k-1}^x + J_k^x \Pi_k^{-1} \mu_k, \quad k \geq 1; \qquad O_0^x = 0, \tag{15}$$

with

$$\begin{aligned} J_k^x &= \sum_{a=0}^{(k-1)\wedge 2} \pi_a^{(k)} B_{k-a}^T H_{k-a}^T - r_{k-1}^{xy} \mathbb{A}_k^T - \sum_{j=1}^{(k-1)\wedge 2} J_{k-j}^x \Pi_{k-j}^{-1} \mathbb{G}_{k-j}^{(j)}, \quad k \geq 2; \\ J_1^x &= \pi_0^{(1)} B_1^T H_1^T. \end{aligned} \tag{16}$$

The matrices $r_k^{xy} = E[O_k^x O_k^{yT}]$ *are obtained by*

$$r_k^{xy} = r_{k-1}^{xy} + J_k^x \Pi_k^{-1} J_k^{yT}, \quad k \geq 1; \quad r_0^{xy} = 0. \tag{17}$$

The innovation μ_k*, its variance* Π_k*,* J_k^y *and* $\mathbb{G}_k^{(1)}$ *are given in Theorem 3.1.*

Proof. From the general estimator expression of the estimator (3), to derive the LS linear filter of the signal, x_k, the coefficients $\mathcal{X}_{k,h} = E[x_k \mu_h] = E[x_k y_h] - E[x_k \widehat{y}_{h/h-1}]$, $h \leq k$, must be calculated. For this purpose, from expression (10) for $\widehat{y}_{k/k-1}$, we have

$$\mathcal{X}_{k,h} = \begin{cases} E[x_k y_1], & h = 1, \\ E[x_k y_h] - \displaystyle\sum_{j=1}^{k-1} \mathcal{X}_{k,j} \Pi_j^{-1} T_{h,j}, & 2 \leq h \leq k. \end{cases}$$

Using (2) for the observations, taking into account the model hypotheses and that for $h - a \leq k$, $E[x_k z_{h-a}] = A_k B_{h-a}^T H_{h-a}^T$, it is clear that

$$\mathcal{X}_{k,h} = \begin{cases} \pi_0^{(1)} A_k B_1^T H_1^T, & h = 1, \\ A_k \displaystyle\sum_{a=0}^{(h-1)\wedge 2} \pi_a^{(h)} B_{h-a}^T H_{h-a}^T - \sum_{j=1}^{h-1} \mathcal{X}_{k,j} \Pi_j^{-1} T_{h,j}, & 2 \leq h \leq k. \end{cases}$$

Next, introducing the function

$$J_h^x = \begin{cases} \pi_0^{(1)} A_k B_1^T H_1^T, & h = 1, \\ \displaystyle\sum_{a=0}^{(k-1)\wedge 2} \pi_a^{(h)} B_{h-a}^T H_{h-a}^T - \sum_{j=1}^{h-1} J_j^x \Pi_j^{-1} T_{h,j}, & 2 \leq h \leq k, \end{cases}$$

we have

$$\mathcal{X}_{k,h} = A_k J_h^x, \quad 1 \leq h \leq k,$$

and after defining $O_k^x = \displaystyle\sum_{h=1}^{k} J_h^x \Pi_h^{-1} \mu_h$, with $O_0^x = 0$, the filter expression (14) is obtained. The recursive relation (15) is immediate from the definition of O_k^x. By reasoning analogous to that used in deriving J_k^y in Theorem 3.1, but now defining $r_k^{xy} = \displaystyle\sum_{i=1}^{k} J_i^x \Pi_i^{-1} J_i^{yT}$, $k \geq 1$, with $r_0^{xy} = 0$, we have expression (16) for J_k^x. Finally, (17) for r_k^{xy} is deduced directly from its definition. □

3.4. Filtering error variances

The performance of the filter can be measured by the estimation errors, $x_k - \widehat{x}_{k/k}$, and, more specifically, by the variances of these errors,

$$\Sigma_{k/k} = E[(x_k - \widehat{x}_{k/k})(x_k - \widehat{x}_{k/k})^T], \quad k \geq 1.$$

Since the estimation error is orthogonal to the estimator, taking into account Hypothesis (i),

$$\Sigma_{k/k} = A_k B_k^T - E[\widehat{x}_{k/k}\widehat{x}_{k/k}^T], \quad k \geq 1.$$

From (14) for $\widehat{x}_{k/k}$, and defining $r_k^x = E[O_k^x O_k^{xT}]$, the filtering error variance is given by

$$\Sigma_{k/k} = A_k[B_k^T - r_k^x A_k^T], \quad k \geq 1.$$

Using the recursive relation (15) for O_k^x and taking into account that O_{k-1}^x and μ_k are uncorrelated, the following recursive relation is obtained for r_k^x

$$r_k^x = r_{k-1}^x + J_k^x \Pi_k^{-1} J_k^{xT}, \quad k \geq 1; \qquad r_0^x = 0.$$

4. Numerical simulation results

In this section, the efficiency of the proposed filtering algorithm is illustrated by a numerical example. For the simulation, a zero-mean scalar signal $\{x_k,\ k \geq 0\}$ with covariance function

$$E[x_k x_s] = 1.025641 \times 0.95^{k-s}, \quad s \leq k,$$

is considered. Clearly this covariance function, according to Hypothesis (i), can be factorised taking

$$A_k = 1.025641 \times 0.95^k \text{ and } B_s = 0.95^{-s}.$$

The measured outputs are affected by a white noise, $\{v_k,\ k \geq 1\}$, with zero mean and variances $R_k = 0.9, \forall k$.

According to the proposed observation model, it is assumed that the available measurements of the signal can be delayed by one or two sample periods during the transmission; that is, the processed observations are modeled by

$$y_k = \sum_{a=0}^{(k-1)\wedge 2} \delta(\theta_k, a) z_{k-a}, \quad k \geq 1.$$

As in Hypothesis (iii), it is assumed that $\{\theta_k,\ k \geq 1\}$ is a homogeneous Markov chain with initial distribution $\pi_0^{(1)} = 1,\ \pi_1^{(1)},\ \pi_2^{(1)} = 0$, (the first observation is not delayed) and transition probability matrix $\mathbf{P} = \begin{pmatrix} 0.99 & 0.006 & 0.004 \\ 0.15 & 0.98 & 0.005 \\ 0.002 & 0.028 & 0.97 \end{pmatrix}$.

Moreover, the signal and noise processes are assumed to be mutually independent.

For simulation purposes, the signal is assumed to be generated from the following first-order autoregressive model,

$$x_{k+1} = 0.95x_k + w_k$$

where $\{w_k,\ k \geq 0\}$ is a zero-mean white Gaussian noise with $E[w_k^2] = 0.1,\ \forall k$.

Figure 1 shows a simulated signal together with the filtering estimates, $\widehat{x}_{k/k}$. This figure highlights the close relation between the evolution of filtering estimates and the signal, reflecting the good performance of the proposed estimators. In addition,

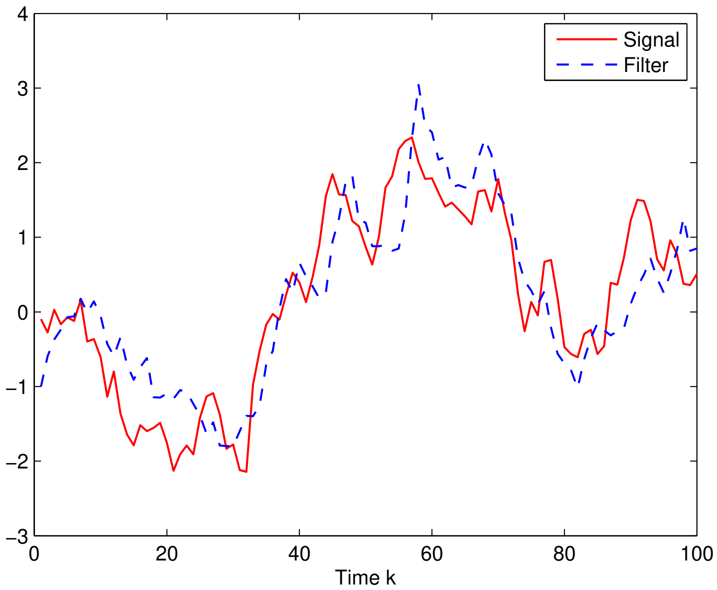

Figure 1. Simulated signal, filtering estimates

we calculated the filtering error variances assuming that the delay was modelled by different Markov chains. Specifically, we assume the same initial distribution (the first observation is not delayed) and the following transition probability matrices:

$$\mathbf{P_1} = \begin{pmatrix} 0.95 & 0.03 & 0.02 \\ 0.05 & 0.89 & 0.06 \\ 0.03 & 0.07 & 0.9 \end{pmatrix}, \quad \mathbf{P_2} = \begin{pmatrix} 0.9 & 0.04 & 0.06 \\ 0.07 & 0.87 & 0.06 \\ 0.05 & 0.06 & 0.89 \end{pmatrix}.$$

The properties of the Markov chains lead us to conclude that the no-delay probabilities converge to constant values; in our case these values are $0.58, 0.44$, and 0.37, for the different transition probability matrices considered, $\mathbf{P}, \mathbf{P_1}$ and $\mathbf{P_2}$, respectively. Figure 2 shows the filtering error variances for these models; in this figure, it can be seen that as the limit probability of no delay increases, the filtering error variances become smaller and, consequently, the performance of the estimator improves.

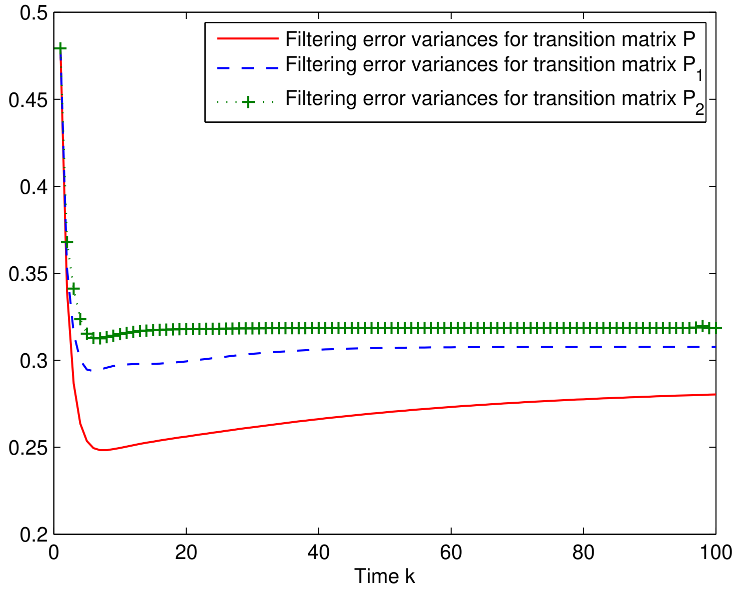

Figure 2. Filtering error variances for different transition probability matrix

5. Conclusions

In this paper, we propose a recursive algorithm for the least-squares linear filter from measurements which can be randomly delayed by one or two sampling times. The delays are modelled by a homogeneous discrete-time Markov chain, which reflects more realistic situations in communications systems than does the independence assumption. Specifically, a Markov chain with three states is considered to model the possibility of measurements with one or two delays as well as no-delay. Assuming that the evolution equation generating the signal is not available and that only the first and second-order moments of the processes involved in the observation model are known, a recursive filtering algorithm is derived using an innovation approach. Furthermore, in order to provide a measure of the goodness of the proposed filter, a recursive expression to calculate the filtering error variances is also derived. The main contributions of this paper are: a) the proposed algorithm, based on the knowledge of the autocovariance function of the signal and its expression in a semidegenerated kernel form, is also applicable for the conventional formulation of the estimation problem, using the state-space model; in this situation, the autocovariance function of the signal is also known and admits the factorisation assumed in this paper; b) the observation model considered generalises that in [4], admitting the possibility of a delay of two sampling periods, and also generalises the models with delays described by independent random variables.

An interesting area for future research with this kind of system could be to extend the results presented here to the case of systems with correlation in the measurement noise. The modelling of random delays, and of more general situations where delays are bounded and driven by a finite-state Markov process, could also be considered. Another challenging topic would be to use the current approach to manage transmission delays in the case of sensor network systems, by investigating problems of centralised and distributed fusion estimation.

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ORCID

García-Ligero, M.J. http://orcid.org/0000-0001-5604-5599
Hermoso-Carazo, A. http://orcid.org/0000-0001-8120-2162
Linares-Pérez, J. http://orcid.org/0000-0002-6853-555X

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