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Networked fusion estimation with multiple uncertainties and time-correlated channel noise

R. Caballero-Águila[a,*], A. Hermoso-Carazo[b], J. Linares-Pérez[b]

[a] Departamento de Estadística e I.O., Universidad de Jaén, Paraje Las Lagunillas s/n, Jaén, 23071, Spain
[b] Departamento de Estadística e I.O., Universidad de Granada, Campus Fuentenueva s/n, Granada, 18071, Spain

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Abstract

This paper is concerned with the fusion filtering and fixed-point smoothing problems for a class of networked systems with multiple random uncertainties in both the sensor outputs and the transmission connections. To deal with this kind of systems, random parameter matrices are considered in the mathematical models of both the sensor measurements and the data available after transmission. The additive noise in the transmission channel from each sensor is assumed to be sequentially time-correlated. By using the time-differencing approach, the available measurements are transformed into an equivalent set of observations that do not depend on the time- correlated noise. The innovation approach is then applied to obtain recursive distributed and centralized fusion estimation algorithms for the filtering and fixed-point smoothing estimators of the signal based on the transformed measurements, which are equal to the estimators based on the original ones. The derivation of the algorithms does not require the knowledge of the signal evolution model, but only the mean and covariance functions of the processes involved (covariance information). A simulation example illustrates the utility and effectiveness of the proposed fusion estimation algorithms, as well as the applicability of the current model to deal with different network-induced random phenomena.

1. Introduction

One of the most active challenges related to networked systems is how to optimize the use of multiple sets of information coming from different sensors or sources, which are sent over transmission channels to a processing center for the purpose of estimating the system state. Due to the great number of applications of networked systems in different fields, such as space and terrestrial exploration, target tracking and localization, guidance and navigation, remote diagnostics and troubleshooting, aircraft, communication, etc., the design of accurate and reliable fusion estimation algorithms that suit the singularities of real situations has become a hot research topic over the last decades. Depending on the data fusion architecture, the most significant fusion techniques are the centralized fusion scheme, in which the raw measured data from all the sensors are sent to a central processor that performs the fusion, and the distributed fusion scheme, in which the sensor measurements are firstly processed independently to obtain local estimators which are then sent to the fusion center. A thorough review of the most common data fusion techniques, not only aimed at the estimation problem but also at data association and decision problems, can be found in [1] or [2].

Usually, the states of modern industrial systems are not available and the measurement outputs are often subject to stochastic uncertainties, due to physical constraints, measurement costs or environmental complexities, among others; this kind of uncertainties in the measurements may include, for example, fading measurements, which are frequently related to sensor aging, and missing measurements or only-noise measurement outputs, which are often caused by temporal sensor failures. Besides these uncertainties in the sensor measured outputs, the data transmission through communication networks may also be impaired by limited communication bandwidth, imperfect communication channels or network congestion, among others, which produce unavoidable random uncertainties in the measurements available after transmission; for example, packet losses can occur as a consequence of communication failures; communication delays are frequently found in modern industrial systems due to the finite speed of amplifiers or information processing; and fading measurements are usually a consequence of imperfect communication channels transforming part of the signal energy into heat energy or absorbing it.

Motivated by these considerations, which show the simultaneous existence of different network-induced uncertainties in both the sensor measurements and the data transmission from the sensor nodes to the remote processing center, we propose to address the estimation problem in sensor networks susceptible to both kinds of uncertainties.

In general, conventional sensor networks, where the sensor measured outputs always contain information about the real signal or target to be estimated and perfect transmission connections are considered (see e.g. [3], [4] and [5]), are not appropriate in the above-mentioned real-world engineering problems where the observations, perturbed by additive noises, are also corrupted by other types of random errors caused by the existence of different network-induced uncertainties that can occur in both the sensor measurements and the data transmission. These random failures in the measurement and transmission mechanisms yield the degradation of the estimator performance and, consequently, the use of innovative observation models, suitable to describe these random phenomena, is an essential issue and the estimation problem with one or more network-induced uncertainties has attracted considerable attention (see e.g. [6], [7], [8], [9], [10] and [11]). A detailed overview of significant contributions on the estimation and fusion for networked systems with network-induced phenomena is presented in [12] and [13].

Another remarkable issue when dealing with networked systems is the occurrence of intermittent or even random faults in signals. Recently, the joint state and fault estimation problem has been investigated in [14] for a class of uncertain time-varying nonlinear systems with randomly occurring faults and sensor saturations, characterizing the phenomenon of randomly occurring faults by Bernoulli random variables with known probabilities.

Network-induced uncertainties in the sensor measurements. Sensor networks usually suffer intermittent failures or inaccuracy of the measurement devices which may cause, for instance, random observation losses [6], sensor gain degradation [9] or missing measurements [10]. A unified framework to model these random phenomena is provided by the use of random parameter matrices in the mathematical model of the sensor measured outputs. This fact has encouraged an increasing research interest in the estimation problem in networked systems with random measurement matrices, since they cover a great variety of different networked-induced uncertainties as the above-mentioned ones (see e.g. [15], [16], [17], [18], [19] and references therein).

Network-induced uncertainties in the transmission channel. Since perfect transmission is not always obtainable, especially in wireless sensor networks, random time-delays, packet dropouts or fading phenomena often occur during the data transmission from the sensors to the processing center. Several estimation algorithms have been proposed in multisensor systems with only transmission time-delays (see e.g. [20] and [21]) or packet dropouts (see e.g. [22] and [23]), as well as in systems influenced by both phenomena simultaneously (see e.g. [24], [25], and [26]). The estimation problem in systems with fading transmission channels has also drawn an increasing research interest over the last years (see e.g. [27] and [28]). Such systems with fading transmission channels must clearly include multiplicative noises that can be modelled by random matrices. Also, systems with random sensor delays and/or multiple packet dropouts are transformed into equivalent observation models with random measurement matrices (see, e.g., [24] and [26]). Hence, the use of random parameter matrices in the model of the data available after transmission, which will be used for the estimation, provides a comprehensive way to incorporate these random disturbances occurring during the transmission process (see e.g. [29] , [30] , [31], [32], [33] and references therein).

Transmission noise correlation. Another significant transmission impairment is the existence of noise in the communication channel. Most of the above-mentioned papers consider networked systems with noise-free transmission channels, or transmission channels whose additive noise is either white or correlated on a finite-time interval. These hypotheses about the channel noise can be generalized to the case of sequentially correlated noise and numerous papers have emerged during the last decade concerning the estimation problem under the assumption that the time-correlated channel noise is the output of a linear system model with white noise (see, e.g. [34], [35], [36], [37], [38] and references therein). The recursive filtering problem for discrete-time linear systems with fading measurements is investigated in [34] under the assumption that the time-correlated channel noise is the output of a linear system model with white noise. This approach is becoming an important focus of attention for the scientific community, as it is suitable to model not only time-correlation but also periodic step changes in the noise process caused by changes in the system (see [36]). In [35], the distributed fusion filtering problem for networked systems with fading measurements and time-correlated noise is addressed. Some of the most recent advances in this field are the results in [37], where the estimation problem for networked systems with time-correlated channel noises, correlated multiplicative noises and fading measurements is studied, considering that the time-correlated channel noises are described by a seemingly autoregressive moving average model. In [38], the Tobit Kalman filtering problem for discrete-time systems subject to non-Gaussian Lèvy and time-correlated additive measurement noises is investigated.

This paper is concerned with the least-squares centralized and distributed fusion estimation problems based on covariance information. Taking the previous considerations into account, these problems will be addressed in a class of multi-sensor networked systems with the following characteristics: (a) The sensor measured outputs, influenced by random parameter matrices, are sent over imperfect transmission channels to the processing center, yielding uncertainties in the observations received by the estimator, which are also modelled by random matrices; (b) the time-correlated channel noise is assumed to obey a dynamic linear equation perturbed by white noise (a first-order autoregressive model). By using the time-differencing approach, the original measurements are remodeled to obtain an equivalent set of transformed observations (linear combinations of the original ones), which do not depend on the time-correlated noise. Recursive algorithms are then obtained for the centralized and distributed fusion estimators of the signal based on the transformed measurements, for which the innovation approach is adopted.

The main distinctive features of the current work are summarized as follows: *(1)* The observation model includes time-varying random parameter matrices in both the sensor measurements and the data available after transmission, which provides a unified framework to deal with different network-induced phenomena, that depend on time explicitly, such as random observation losses, sensor gain degradation, fading or missing measurements, in both the sensor measured outputs and the transmissions. *(2)* Since usually the noise changes with the environment or system structure, the channel noise is assumed to be time-correlated, according to a linear system driven by white noise, which is suitable to depict the changes in the noise process caused by changes in the system. *(3)* The distributed and centralized fusion estimation problems are addressed without requiring full knowledge of the state-space model generating the signal process, but only the mean and covariance functions of the processes involved (covariance information), thus providing a general approach to deal with different kinds of signal processes. Actually, the proposed algorithms are also applicable to the conventional formulation using the state-space model, even in the presence of state-dependent multiplicative noise. *(4)* The innovation approach is used to simplify the derivation of the proposed filtering and fixed-point smoothing algorithms, which are recursive and computationally simple, thus being suitable for online implementation.

To the best of the authors' knowledge, the distributed and centralized fusion estimation problems in this framework where the sensor measured outputs and the measurement transmission are both perturbed by random parameter matrices, in the presence of time-correlated channel noises, have not yet been investigated, so it is an interesting research topic. Actually, this is a realistic and comprehensive assumption to deal with networked systems featuring simultaneous random failures in the sensor outputs and the measurements available after transmission. The existing results mainly focus on specific impairments in the measurement or transmission mechanisms (missing/fading observations, random delays/packet dropouts, etc.), while the consideration of random parameter matrices provides a global model to cover all these impairments at once. Hence, the proposed results are more general and realistic for practical applications.

The organization of the paper is summarized as follows. The problem is formulated in Section 2. Section 3 discusses the distributed fusion filtering and fixed-point smoothing problems. The centralized fusion estimation algorithms are presented in Section 4. Section 5 shows a simulation example in target tracking and some conclusions are drawn in Section 6. Finally, three appendices provide the mathematical proofs of the main theoretical results.

Notation: The following standard notation is used throughout the paper. $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ denote the n-dimensional Euclidean space and the set of all $m \times n$ real matrices, respectively. For a matrix A, A^T and A^{-1} denote its transpose and inverse, respectively. $I_{n\times n}$ denotes the $n \times n$ identity matrix. If the dimensions of a vector or a matrix are not explicitly stated, they are assumed to be compatible with algebraic operations. $E[\cdot]$ denotes the mathematical expectation and $P[A]$ is the probability of event A. For any function $G_{k,h}$, depending on the time instants k and h, we will write $G_k =$

$G_{k,k}$ for simplicity; analogously, $F^{(i)} = F^{(ii)}$ will be written for any function $F^{(ij)}$, depending on the sensors i and j. Finally, $\delta_{k,h}$ denotes the Kronecker delta function.

2. Problem formulation and observation model

The aim of this paper is to design recursive algorithms for the distributed and centralized fusion estimation problems using observations coming from a sensor network, supposing that the sensor measurements of the signal to be estimated are transmitted through unreliable communication channels and different uncertainties can randomly occur not only in the measured outputs, but also during the transmissions. Moreover, white additive noises in the sensor measurements and time-correlated additive noises in the transmissions are considered. A unifying framework to model multiple random phenomena (e.g. sensor gain degradation, missing or fading measurements, uncertainties caused by the presence of multiplicative noise, or both multiplicative noises and missing measurements) is provided by the use of random measurement matrices. In view of this consideration, the proposed observation model will be perturbed by random measurement matrices in both the measured outputs and the transmissions, thus allowing us to design general estimation algorithms that will be suitable to address all the aforementioned random phenomena comprehensively.

In the centralized estimation problem, the estimators are obtained by fusion of all the network observations at each sampling time, whereas in the distributed one, local estimators, based only on the observations of each individual sensor, are previously obtained and the distributed estimator is then calculated by fusion of the local ones. Our aim is to design recursive algorithms for the optimal linear distributed and centralized filters and fixed-point smoothers under the least-squares (LS) criterion, requiring only the first and second-order moments of the processes involved in the model that describes the observations coming from the different sensors (covariance information). To deal with the LS estimation problem, the signal to be estimated as well as the measurements and the noises in the observation model will be assumed to be described by second-order random vectors defined on a common probability space $(\Omega, \mathcal{A}, P)$.

2.1. Signal process

As it is well known, the Kalman filter is the most popular contribution to solve the estimation problem in gaussian linear stochastic systems and provides the optimal LS estimator when the state-space model generating the signal to be estimated is known. Since the estimation algorithms based on the state-space model depend on the signal evolution equation, if this equation is modified, new algorithms are needed. For example, the algorithms designed for stationary signals whose evolution model is given by (a) $x_{k+1} = Fx_k + w_k$, may not be applied to estimate non-stationary signals generated by the equation (b) $x_{k+1} = F_k x_k + w_k$ and, in turn, those obtained for (b) may not be applied to signals including perturbations described by multiplicative noises, as (c) $x_{k+1} = (F_k + G_k \varepsilon_k) x_k + w_k$. On the other hand, the evolution model of the signal to be estimated can be unknown in many practical applications and alternative information should be used; consequently, different algorithms, based on that new information, ought to be derived.

In this paper, fusion estimation algorithms are proposed without requiring the evolution model generating the signal process to be estimate, but only its mean and covariance functions (covariance-based approach). More precisely, the following hypothesis is assumed on the signal process:

(H1) *The n_x-dimensional signal $\{x_k\}_{k\geq 1}$ is a second-order zero-mean process whose covariance function is expressed in a separable form, $E[x_k x_h^T] = A_k B_h^T$, $h \leq k$, where $A_k, B_h \in \mathbb{R}^{n_x \times n}$ are known matrices.*

Remark 1. *Hypothesis (H1) on the signal covariance function is fulfilled by different kinds of signals and the estimation based on such hypothesis, instead of the state-space model, provides a unifying framework to obtain general algorithms which are applicable to a large number of practical situations. For example, assuming non-singular transition matrices:*

- *For a zero-mean stationary signal driven by a white noise which is independent of the initial state, (a) $x_{k+1} = Fx_k + w_k$, $k \geq 0$, the state at time k can be expressed in terms of the state at any previous time as $x_k = F^{k-h} x_h + \sum_{j=h}^{k-1} F^{k-j-1} w_j$, $h \leq k$. Hence, taking into account that*

$E[w_j x_h^T] = 0,\ j \geq h$, the covariances are expressed as

$$E[x_k x_h^T] = F^{k-h} E[x_h x_h^T] + \sum_{j=h}^{k-1} F^{k-j-1} E[w_j x_h^T] = F^{k-h} E[x_h x_h^T], \quad h \leq k,$$

and (H1) is satisfied just taking $A_k = F^k$ and $B_h = E[x_h x_h^T](F^{-h})^T$.

- *Similarly, the covariance function of the non-stationary signal (b) $x_{k+1} = F_k x_k + w_k,\ k \geq 0$, can be expressed as $E[x_k x_h^T] = \mathbf{F}_{k,h} E[x_h x_h^T],\ h \leq k$, where $\mathbf{F}_{k,h} = F_{k-1} \cdots F_h$, and (H1) is also fulfilled taking $A_k = \mathbf{F}_{k,0}$ and $B_h = E[x_h x_h^T](\mathbf{F}_{h,0}^{-1})^T$.*

Even when the transition matrix is singular, hypothesis (H1) is satisfied, although a different reasoning and factorization are needed (see e.g. [17]). Finally, hypothesis (H1) is also fulfilled by the class of signals (c) $x_{k+1} = (F_k + G_k \varepsilon_k) x_k + w_k,\ k \geq 0$, affected by multiplicative noise, that will be considered in Section 5.

In summary, hypothesis *(H1)* on the signal covariance function can be verified for different kinds of signals and the estimation based on such hypothesis, instead of the state-space model, provides a unifying framework to obtain general algorithms which are applicable to a large number of practical situations.

2.2. Multi-sensor observation model

Let us consider m sensors, which measure a discrete-time random signal $x_k \in \mathbb{R}^{n_x}$ and provide measured outputs perturbed by random parameter matrices and additive noises, according to the following model:

$$z_k^{(i)} = C_k^{(i)} x_k + v_k^{(i)}, \quad k \geq 1, \qquad i = 1, \ldots, m, \tag{1}$$

where $z_k^{(i)} \in \mathbb{R}^{n_z}$ is the output response of the i-th sensor at time k. The following hypotheses are required about the matrices $C_k^{(i)} \in \mathbb{R}^{n_z \times n_x}$ and the measurement noise vectors $v_k^{(i)} \in \mathbb{R}^{n_z}$:

(H2) *$\{C_k^{(i)}\}_{k \geq 1}$, $i = 1, \ldots, m$, are independent sequences of independent random parameter matrices with known first and second-order moments, $E\big[c_{pq}^{(i)}(k)\big]$ and $E\big[c_{pq}^{(i)}(k) c_{p'q'}^{(j)}(k)\big]$, for $p, p' = 1, \ldots, n_z$ and $q, q' = 1, \ldots, n_x$, where $c_{pq}^{(i)}(k)$ denotes the (p,q)-th entry of $C_k^{(i)}$. We denote $\overline{C}_k^{(i)} \equiv E\big[C_k^{(i)}\big]$.*

(H3) *The measurement noises* $\{v_k^{(i)}\}_{k\geq 1}$*,* $i=1,\dots,m$*, are second-order zero-mean white processes with* $E\big[v_k^{(i)}v_h^{(j)T}\big]=R_k^{(ij)}\delta_{k,h},\ \ i,j=1,\dots,m.$

It is assumed that the measured outputs of the different sensors, $z_k^{(i)}$, $i=1,\dots,m$, are transmitted through unreliable channels which cause stochastic impairments; on the one hand, sequentially correlated additive noises, $\eta_k^{(i)}$, and, on the other, transmission uncertainties described by random parameter matrices, $H_k^{(i)}$, are considered. The presence of multiplicative random parameter matrices in the model of the data available after transmission provides a unified framework to describe some random disturbances, which are often yielded by unreliable transmissions, such as gain degradation, missing or fading measurements, or presence of multiplicative noise, as it will be shown in Section 5. Specifically, the following model is assumed for the measurements, $\breve{z}_k^{(i)}$, available after transmission from the i-th sensor:

$$\breve{z}_k^{(i)}=H_k^{(i)}z_k^{(i)}+\eta_k^{(i)},\quad k\geq 1,\qquad i=1,\dots,m, \tag{2}$$

with

$$\eta_k^{(i)}=D_{k-1}^{(i)}\eta_{k-1}^{(i)}+\xi_{k-1}^{(i)},\quad k\geq 1,\qquad i=1,\dots,m, \tag{3}$$

where $D_k^{(i)}\in\mathbb{R}^{n_z\times n_z}$ are known matrices. The following hypotheses are imposed about the sequences $\{H_k^{(i)}\}_{k\geq 1}$, $\{\xi_k^{(i)}\}_{k\geq 0}$ and the random vectors $\eta_0^{(i)}$:

(H4) $\{H_k^{(i)}\}_{k\geq 1}$*,* $i=1,\dots,m$*, are independent sequences of independent random parameter matrices with known first an second-order moments,* $E\big[h_{pq}^{(i)}(k)\big]$ *and* $E\big[h_{pq}^{(i)}(k)h_{p'q'}^{(j)}(k)\big]$*, for* $p,q,p',q'=1,\dots,n_z$*, where* $h_{pq}^{(i)}(k)$ *denotes the* (p,q)*-th entry of* $H_k^{(i)}$*. We denote* $\overline{H}_k^{(i)}\equiv E[H_k^{(i)}]$*.*

(H5) *The noises* $\{\xi_k^{(i)}\}_{k\geq 0}$*,* $i=1,\dots,m$*, are second-order zero-mean white processes with* $E\big[\xi_k^{(i)}\xi_h^{(j)T}\big]=S_k^{(ij)}\delta_{k,h},\ \ i,j=1,\dots,m.$

(H6) *For* $i=1,\dots,m$*,* $\eta_0^{(i)}$ *are second-order zero-mean random vectors with* $E\big[\eta_0^{(i)}\eta_0^{(j)T}\big]=Q_0^{(ij)},\ \ i,j=1,\dots,m.$

(H7) *For* $i=1,\dots,m$*, the signal process* $\{x_k\}_{k\geq 1}$*, the vector* $\eta_0^{(i)}$ *and the processes* $\{C_k^{(i)}\}_{k\geq 1}$*,* $\{H_k^{(i)}\}_{k\geq 1}$*,* $\{v_k^{(i)}\}_{k\geq 1}$ *and* $\{\xi_k^{(i)}\}_{k\geq 0}$ *are mutually independent.*

Remark 2. *Clearly, from (1) and the above model hypotheses, the processes* $\{z_k^{(i)}\}_{k\geq 1}$, $i = 1,\dots,m$, *have zero mean and, taking into account the conditional expectation properties, the following expressions for the covariance matrices* $\Sigma_{k,h}^{z^{(ij)}} \equiv E\big[z_k^{(i)} z_h^{(j)T}\big]$ *and* $\Sigma_{k,h}^{Hz^{(ij)}} \equiv E\big[H_k^{(i)} z_k^{(i)} z_h^{(j)T} H_h^{(j)T}\big]$, $i,j = 1,\dots,m$, *are easily obtained:*

$$\begin{aligned}
\Sigma_{k,h}^{z^{(ij)}} &= E\big[C_k^{(i)} A_k B_h^T C_h^{(j)T}\big] + R_k^{(ij)}\delta_{k,h}, \quad h \leq k,\\
\Sigma_{k,h}^{Hz^{(ij)}} &= E\big[H_k^{(i)} \Sigma_{k,h}^{z^{(ij)}} H_h^{(j)T}\big], \quad h \leq k.
\end{aligned} \tag{4}$$

From the independence hypothesis, it is clear that, for $j \neq i$ *or* $h \neq k$, $E\big[C_k^{(i)} A_k B_h^T C_h^{(j)T}\big] = \overline{C}_k^{(i)} A_k B_h^T \overline{C}_h^{(j)T}$, $E\big[H_k^{(i)} \Sigma_{k,h}^{z^{(ij)}} H_h^{(j)T}\big] = \overline{H}_k^{(i)} \Sigma_{k,h}^{z^{(ij)}} \overline{H}_h^{(j)T}$; *otherwise, the entries of* $E\big[C_k^{(i)} A_k B_k^T C_k^{(i)T}\big]$ *and* $E\big[H_k^{(i)} \Sigma_k^{z^{(i)}} H_k^{(i)T}\big]$ *are calculated by using the following general formula, in which* $\mathbf{\Gamma} = \big(\Gamma_{pq}\big) \in \mathbb{R}^{n_1\times n_2}$ *and* $\mathbf{P} = \big(P_{pq}\big) \in \mathbb{R}^{n_2\times n_2}$ *are random and deterministic matrices, respectively:*

$$\big(E[\mathbf{\Gamma}\mathbf{P}\mathbf{\Gamma}^T]\big)_{pq} = \sum_{a,b=1}^{n_2} E[\Gamma_{pa}\Gamma_{qb}]P_{ab}, \quad p,q = 1,\dots,n_1.$$

3. Distributed fusion estimation problem

In *Subsection 3.1*, LS local signal estimators, $\widehat{x}_{k/L}^{(i)}$, $L \geq k$, are obtained by recursive algorithms and then, in *Subsection 3.2*, the distributed fusion signal estimators, $\widehat{x}_{k/L}^{(D)}$, $L \geq k$, are deduced as a matrix-weighted linear combination of the local ones by applying the LS criterion.

3.1. Recursive algorithms for the local LS linear estimators

Our aim is to obtain recursive algorithms for the local LS linear filter and fixed-point smoothers, by using an innovation approach. According to such approach, an observation process is transformed into an equivalent one, named *innovation process*, and the LS linear estimator, $\widehat{\zeta}_{k/L}$, of a random vector ζ_k based on a set of observations $\{y_h,\ 1 \leq h \leq L\}$, can be expressed as a linear combination of the innovations $\{\mu_h,\ 1 \leq h \leq L\}$, defined by $\mu_h = y_h - \widehat{y}_{h/h-1}$, as follows:

$$\widehat{\zeta}_{k/L} = \sum_{h=1}^{L} E[\zeta_k \mu_h^T]\big(E\big[\mu_h \mu_h^T\big]\big)^{-1}\mu_h, \quad k \geq 1; \tag{5}$$

consequently, the first key point is obtaining the innovation process.

Taking into account that the observations $\{\breve{z}_k^{(i)}\}_{k\geq 1}$ in (2) are influenced by the sequentially correlated noises $\{\eta_k^{(i)}\}_{k\geq 1}$ given in (3), we will work with linear transformations of them adopting, as is usual in such cases, the time-differencing approach [34].

3.1.1. Measurement differencing

To address the LS linear estimation problem of the signal, x_k, from the observations with time-correlated measurement noise given in (2), $\breve{z}_1^{(i)}, \dots, \breve{z}_L^{(i)}$, the measurement differencing approach is used to remove the noise $\eta_k^{(i)}$. Namely, the transformed measurements are obtained as follows:

$$y_k^{(i)} = \breve{z}_k^{(i)} - D_{k-1}^{(i)} \breve{z}_{k-1}^{(i)}, \quad k \geq 2; \qquad y_1^{(i)} = \breve{z}_1^{(i)}, \qquad i = 1, \dots, m. \tag{6}$$

Since the sets $\{y_1^{(i)}, \dots, y_L^{(i)}\}$ and $\{\breve{z}_1^{(i)}, \dots, \breve{z}_L^{(i)}\}$ can be obtained one from the other by linear transformations, the LS linear estimator of x_k based on $\breve{z}_1^{(i)}, \dots, \breve{z}_L^{(i)}$ is just that based on $y_1^{(i)}, \dots, y_L^{(i)}$.

Next, in order to look for an expression that allows us to obtain the innovation $\mu_k^{(i)} = y_k^{(i)} - \widehat{y}_{k/k-1}^{(i)}$ –or, equivalently, the one-step observation predictor $\widehat{y}_{k/k-1}^{(i)}$– in a simple way, we will rewrite (6).

Substituting (1)-(3) into (6), and taking into account that $H_{k-1}^{(i)}$ and $C_{k-1}^{(i)}$ are correlated with $y_{k-1}^{(i)}$, expression (6) is rewritten as follows:

$$\begin{aligned} y_k^{(i)} &= H_k^{(i)} C_k^{(i)} x_k - D_{k-1}^{(i)} \overline{H}_{k-1}^{(i)} \overline{C}_{k-1}^{(i)} x_{k-1} + V_{k-1}^{(i)}, \quad k \geq 2; \\ y_1^{(i)} &= H_1^{(i)} z_1^{(i)} + D_0^{(i)} \eta_0^{(i)} + \xi_0^{(i)}, \end{aligned} \tag{7}$$

where

$$V_k^{(i)} = \xi_k^{(i)} - D_k^{(i)} \big(H_k^{(i)} C_k^{(i)} - \overline{H}_k^{(i)} \overline{C}_k^{(i)} \big) x_k + H_{k+1}^{(i)} v_{k+1}^{(i)} - D_k^{(i)} H_k^{(i)} v_k^{(i)}, \quad k \geq 1. \tag{8}$$

Note that the original observations (1)–(3) have been transformed into the new observations (7) that do not depend on the time-correlated noise $\{\eta_k^{(i)}\}_{k\geq 1}$. To address the signal estimation problem from the observations given by (7), it is necessary to know the first and second-order properties of the involved processes, which are presented in the next lemma.

Lemma 1. *Under the model hypotheses, the following properties are satisfied:*

(a) The observation processes $\{y_k^{(i)}\}_{k\geq 1}$, $i=1,\dots,m$, *given in (7), have zero-mean and their covariance matrices,* $\Sigma_k^{y^{(ij)}} \equiv E\big[y_k^{(i)}y_k^{(j)T}\big]$, *satisfy*

$$\begin{aligned}\Sigma_k^{y^{(ij)}} &= \Sigma_k^{Hz^{(ij)}} - \Sigma_{k,k-1}^{Hz^{(ij)}} D_{k-1}^{(j)T} - D_{k-1}^{(i)}\Sigma_{k-1,k}^{Hz^{(ij)}} + S_{k-1}^{(ij)} \\ &\quad + D_{k-1}^{(i)}\Sigma_{k-1}^{Hz^{(ij)}} D_{k-1}^{(j)T}, \quad k\geq 2; \\ \Sigma_1^{y^{(ij)}} &= \Sigma_1^{Hz^{(ij)}} + D_0^{(i)}Q_0^{(ij)}D_0^{(j)T} + S_0^{(ij)}.\end{aligned} \tag{9}$$

(b) The signal process $\{x_k\}_{k\geq 1}$ *and the processes* $\{y_k^{(i)}\}_{k\geq 1}$, $i=1,\dots,m$, *are correlated and, for* $i=1,\dots,m$, *we have*

$$E\big[x_k y_h^{(i)T}\big] = \begin{cases} A_k\Psi_{B_h}^{(i)T}, & h\leq k, \\ B_k\Psi_{A_h}^{(i)T}, & h>k, \end{cases} \tag{10}$$

where $\Psi_{G_k}^{(i)}$, *for* $G_k = A_k, B_k$, *are given by*

$$\begin{aligned}\Psi_{G_k}^{(i)} &\equiv \overline{H}_k^{(i)}\overline{C}_k^{(i)}G_k - D_{k-1}^{(i)}\overline{H}_{k-1}^{(i)}\overline{C}_{k-1}^{(i)}G_{k-1}, \quad k\geq 2; \\ \Psi_{G_1}^{(i)} &\equiv \overline{H}_1^{(i)}\overline{C}_1^{(i)}G_1.\end{aligned} \tag{11}$$

(c) The processes $\{V_k^{(i)}\}_{k\geq 1}$, $i=1,\dots,m$, *given in (8), satisfy* $E\big[V_k^{(i)}y_h^{(j)T}\big] = \mathcal{V}_k^{(ij)}\delta_{k,h}$, *for* $h\leq k$, *where*

$$\mathcal{V}_k^{(ij)} = D_k^{(i)}\big(\overline{H}_k^{(i)}\overline{C}_k^{(i)}A_kB_k^T\overline{C}_k^{(j)T}\overline{H}_k^{(j)T} - \Sigma_k^{Hz^{(ij)}}\big), \quad k\geq 1. \tag{12}$$

The matrices $\Sigma_{k,k-1}^{Hz^{(ij)}}$ *and* $\Sigma_k^{Hz^{(ij)}}$ *in (9) and (12) are given in (4).*

Proof. See Appendix A. □

3.1.2. One-stage observation predictor: $\widehat{y}_{k/k-1}^{(i)}$, $i=1,\dots,m$.

From *(c)* in Lemma 1, $E[V_k^{(i)}\widehat{y}_{h/h-1}^{(i)T}] = 0$, for $h<k$, and $E\big[V_k^{(i)}\mu_k^{(i)T}\big] = E\big[V_k^{(i)}y_k^{(i)T}\big] = \mathcal{V}_k^{(i)}$; then, using (5) and denoting $\Pi_k^{(i)} = E\big[\mu_k^{(i)}\mu_k^{(i)T}\big]$, it is

easy to see that $\widehat{V}^{(i)}_{k/k} = \mathcal{V}^{(i)}_k \Pi^{(i)-1}_k \mu^{(i)}_k, \ \ k \geq 1$, and, from (7) for $y^{(i)}_k$, according to the projection theory, the one-stage observation predictor $\widehat{y}^{(i)}_{k/k-1}$, satisfies:

$$\begin{array}{l} \widehat{y}^{(i)}_{k/k-1} = \overline{H}^{(i)}_k \overline{C}^{(i)}_k \widehat{x}^{(i)}_{k/k-1} - D_{k-1} \overline{H}^{(i)}_{k-1} \overline{C}^{(i)}_{k-1} \widehat{x}^{(i)}_{k-1/k-1} + \mathcal{V}^{(i)}_{k-1} \Pi^{(i)-1}_{k-1} \mu^{(i)}_{k-1}, \ k \geq 2; \\ \widehat{y}^{(i)}_{1/0} = 0. \end{array} \tag{13}$$

Hence, besides the last innovation, the one-stage predictor and the filter of the signal are required.

Since from (5), the signal estimators are given by $\widehat{x}^{(i)}_{k/L} = \sum_{h=1}^{L} \mathcal{X}^{(i)}_{k,h} \Pi^{(i)-1}_h \mu^{(i)}_h$, with $\mathcal{X}^{(i)}_{k,h} = E\big[x_k \mu^{(i)T}_h\big]$, we proceed to calculate the required coefficients for the predictor and filter, $\mathcal{X}^{(i)}_{k,h} = E\big[x_k y^{(i)T}_h\big] - E\big[x_k \widehat{y}^{(i)T}_{h/h-1}\big], \ 1 \leq h \leq k$.

Using expression (13) for $\widehat{y}^{(i)}_{h/h-1}$, together with (5) for $\widehat{x}^{(i)}_{h/h-1}$ and $\widehat{x}^{(i)}_{h-1/h-1}$, we have that

$$\begin{aligned} E\big[x_k \widehat{y}^{(i)T}_{h/h-1}\big] = & \sum_{j=1}^{h-1} \mathcal{X}^{(i)}_{k,j} \Pi^{(i)-1}_j \big(\overline{H}^{(i)}_h \overline{C}^{(i)}_h \mathcal{X}^{(i)}_{h,j} + D^{(i)}_{h-1} \overline{H}^{(i)}_{h-1} \overline{C}^{(i)}_{h-1} \mathcal{X}^{(i)}_{h-1,j}\big)^T \\ & - \mathcal{X}^{(i)}_{k,h-1} \Pi^{(i)-1}_{h-1} \mathcal{V}^{(i)T}_{h-1}, \quad 2 \leq h \leq k, \end{aligned}$$

which, using (10) for $E\big[x_k y^{(i)T}_h\big], \ h \leq k$, guarantees that $\mathcal{X}^{(i)}_{k,h} = A_k J^{(i)}_h, \ 1 \leq h \leq k$, with $J^{(i)}_h$ given by

$$J^{(i)}_h = \Psi^{(i)T}_{B_h} - \sum_{j=1}^{h-1} J^{(i)}_j \Pi^{(i)-1}_j J^{(i)T}_j \Psi^{(i)T}_{A_h} - J^{(i)}_{h-1} \Pi^{(i)-1}_{h-1} \mathcal{V}^{(i)T}_{h-1}, \ h \geq 2; \quad J^{(i)}_1 = \Psi^{(i)T}_{B_1}.$$

Then, by defining the vectors

$$O^{(i)}_L = \sum_{l=1}^{L} J^{(i)}_l \Pi^{(i)-1}_l \mu^{(i)}_l, \ L \geq 1, \tag{14}$$

we have that the signal predictors and filter are given by $\widehat{x}^{(i)}_{k/L} = A_k O^{(i)}_L$, $L \leq k$, and, from (13), the following expression for the observation one-step predictor is obtained:

$$\widehat{y}^{(i)}_{k/k-1} = \Psi^{(i)}_{A_k} O^{(i)}_{k-1} + \mathcal{V}^{(i)}_{k-1} \Pi^{(i)-1}_{k-1} \mu^{(i)}_{k-1}, \quad k \geq 2. \tag{15}$$

3.1.3. Local filtering and fixed-point smoothing algorithms

Recursive algorithm for the local LS linear filter, $\widehat{x}^{(i)}_{k/k}$, and smoothers, $\widehat{x}^{(i)}_{k/k+N}$, at the fixed point k, for any $N \geq 1$, are presented in the following theorem. The proof, based on the above results, is deduced without much difficulty, so the details are omitted here.

Theorem 1. *Under hypotheses (H1)-(H7), for each $i = 1, \dots, m$, the local LS linear filtering estimators, $\widehat{x}^{(i)}_{k/k}$, and the corresponding error covariance matrices, $\Sigma^{(i)}_{k/k} \equiv E\big[(x_k - \widehat{x}^{(i)}_{k/k})(x_k - \widehat{x}^{(i)}_{k/k})^T\big]$, are given by*

$$
\begin{aligned}
\widehat{x}^{(i)}_{k/k} &= A_k O^{(i)}_k, \quad k \geq 1, \\
\Sigma^{(i)}_{k/k} &= A_k \big(B_k - A_k r^{(i)}_k\big)^T, \quad k \geq 1,
\end{aligned}
$$

where

$$
\begin{aligned}
O^{(i)}_k &= O^{(i)}_{k-1} + J^{(i)}_k \Pi^{(i)-1}_k \mu^{(i)}_k, \quad k \geq 1; \quad O^{(i)}_0 = 0, \\
r^{(i)}_k &= r^{(i)}_{k-1} + J^{(i)}_k \Pi^{(i)-1}_k J^{(i)T}_k, \quad k \geq 1; \quad r^{(i)}_0 = 0, \\
J^{(i)}_k &= \Psi^{(i)T}_{B_k} - r^{(i)}_{k-1} \Psi^{(i)T}_{A_k} - J^{(i)}_{k-1} \Pi^{(i)-1}_{k-1} \mathcal{V}^{(i)T}_{k-1}, \quad k \geq 2; \quad J^{(i)}_1 = \Psi^{(i)T}_{B_1}.
\end{aligned}
$$

The innovations, $\mu^{(i)}_k$, and their covariance matrices, $\Pi^{(i)}_k = E\big[\mu^{(i)}_k \mu^{(i)T}_k\big]$, are obtained by

$$
\begin{aligned}
\mu^{(i)}_k &= y^{(i)}_k - \Psi^{(i)}_{A_k} O^{(i)}_{k-1} - \mathcal{V}^{(i)}_{k-1} \Pi^{(i)-1}_{k-1} \mu^{(i)}_{k-1}, \quad k \geq 2; \quad \mu^{(i)}_1 = y^{(i)}_1, \\
\Pi^{(i)}_k &= \Sigma^{y^{(i)}}_k - \Psi^{(i)}_{A_k} (\Psi^{(i)T}_{B_k} - J^{(i)}_k) - \mathcal{V}^{(i)}_{k-1} \Pi^{(i)-1}_{k-1} \left(\Psi^{(i)}_{A_k} J^{(i)}_{k-1} + \mathcal{V}^{(i)}_{k-1}\right)^T, \quad k \geq 2; \\
\Pi^{(i)}_1 &= \Sigma^{y^{(i)}}_1,
\end{aligned}
$$

where $y^{(i)}_k$, $\Sigma^{y^{(i)}}_k$, $\Psi^{(i)}_{A_k}$, $\Psi^{(i)}_{B_k}$ and $\mathcal{V}^{(i)}_k$ are given in (6), (9), (11) and (12), respectively.

Moreover, at any sampling time $k \geq 1$, by starting from the filter, $\widehat{x}^{(i)}_{k/k}$, and its error covariance matrix, $\Sigma^{(i)}_{k/k}$, the local LS linear smoothers, $\widehat{x}^{(i)}_{k/k+N}$, and their error covariances, $\Sigma^{(i)}_{k/k+N} \equiv E\big[(x_k - \widehat{x}^{(i)}_{k/k+N})(x_k - \widehat{x}^{(i)}_{k/k+N})^T\big]$, are recursively calculated as follows:

$$
\begin{aligned}
\widehat{x}^{(i)}_{k/k+N} &= \widehat{x}^{(i)}_{k/k+N-1} + \mathcal{X}^{(i)}_{k,k+N} \Pi^{(i)-1}_{k+N} \mu^{(i)}_{k+N}, \quad N \geq 1, \\
\Sigma^{(i)}_{k/k+N} &= \Sigma^{(i)}_{k/k+N-1} - \mathcal{X}^{(i)}_{k,k+N} \Pi^{(i)-1}_{k+N} \mathcal{X}^{(i)T}_{k,k+N}, \quad N \geq 1,
\end{aligned}
$$

where

$$
\begin{aligned}
&\mathcal{X}^{(i)}_{k,k+N} = \big(B_k - E^{(i)}_{k,k+N-1}\big)\Psi^{(i)T}_{A_{k+N}} - \mathcal{X}^{(i)}_{k,k+N-1}\Pi^{(i)-1}_{k+N-1}\mathcal{V}^{(i)T}_{k+N-1}, \quad N \geq 1;\\
&\mathcal{X}^{(i)}_{k,k} = A_k J^{(i)}_k,
\end{aligned}
$$

and $E^{(i)}_{k,k+N} \equiv E[\widehat{x}^{(i)}_{k/k+N} O^{(i)}_{k+N}]$ *is given by*

$$
E^{(i)}_{k,k+N} = E^{(i)}_{k,k+N-1} + \mathcal{X}^{(i)}_{k,k+N}\Pi^{(i)-1}_{k+h}J^{(i)T}_{k+N}, \quad N \geq 1; \quad E^{(i)}_{k,k} = A_k r^{(i)}_k.
$$

□

Remark 3. *The simultaneous consideration of random parameter matrices to model the uncertainties in both the sensor measured outputs and the transmissions, as well as the presence of sequentially time-correlated noises in the transmission channels, entail some difficulties. They are mainly related to the choice of expression (7), which describes the measurements transformed by differentiation,* $y^{(i)}_k$*, and allows us to obtain the one-step observation predictor* $\widehat{y}^{(i)}_{k/k-1}$ *and, consequently, the innovation,* $\mu^{(i)}_k = y^{(i)}_k - \widehat{y}^{(i)}_{k/k-1}$*, in a simple way. Actually, expression (7) has been the starting point for the derivation of the local algorithms presented in Theorem 1 and it will be also essential to obtain the proposed distributed and centralized algorithms in the following sections.*

3.2. Distributed LS fusion linear estimators

The aim of this section is to derive distributed fusion estimators $\widehat{x}^{(D)}_{k/k+N}$, $k \geq 1$, $N \geq 0$, as matrix-weighted linear combinations of the corresponding local estimators, $\widehat{x}^{(i)}_{k/k+N}$, $i = 1, \ldots, m$, where the weight matrices are computed by minimizing the mean squared estimation error. These estimators require the cross-covariance matrices between any two local estimators, $K^{(ij)}_{k/k+N} \equiv E[\widehat{x}^{(i)}_{k/k+N}\widehat{x}^{(j)T}_{k/k+N}]$, which are recursively obtained as shown below by starting from those of the filters, $K^{(ij)}_{k/k}$.

The algorithms to obtain $K^{(ij)}_{k/k}$, $k \geq 1$, and $K^{(ij)}_{k/k+N}$, $N \geq 1$, for fixed $k \geq 1$, are presented in theorems 2 and 3, respectively, and the distributed fusion estimators $\widehat{x}^{(D)}_{k/k+N}$, $k \geq 1$, $N \geq 0$, are given in Theorem 4. The assumptions and notation in these theorems are the same as those of the previous sections.

Theorem 2. *For $i,j=1,\dots,m$, the cross-covariance matrices between any two local filters, $K^{(ij)}_{k/k}=E\big[\widehat{x}^{(i)}_{k/k}\widehat{x}^{(j)T}_{k/k}\big]$, are given by*

$$K^{(ij)}_{k/k}=A_k r^{(ij)}_k A^T_k, \quad k\geq 1, \tag{16}$$

where $r^{(ij)}_k \equiv E\big[O^{(i)}_k O^{(j)T}_k\big]$ are obtained from the following algorithm, in which $J^{(ij)}_{h,k}\equiv E[O^{(i)}_h \mu^{(j)T}_k]$ and $\Pi^{(ij)}_{h,k}\equiv E[\mu^{(i)}_h\mu^{(j)T}_k]$, for $h=k-1,\ k$:

$$r^{(ij)}_k=r^{(ij)}_{k-1}+J^{(ij)}_{k-1,k}\Pi^{(j)-1}_k J^{(j)T}_k+J^{(i)}_k\Pi^{(i)-1}_k J^{(ji)T}_k, \quad k\geq 1; \qquad r^{(ij)}_0=0, \tag{17}$$

$$J^{(ij)}_k=J^{(ij)}_{k-1,k}+J^{(i)}_k\Pi^{(i)-1}_k\Pi^{(ij)}_k, \quad k\geq 1, \tag{18}$$

$$\begin{aligned}&J^{(ij)}_{k-1,k}=\big(r^{(i)}_{k-1}-r^{(ij)}_{k-1}\big)\Psi^{(j)T}_{A_k}+J^{(i)}_{k-1}\Pi^{(i)-1}_{k-1}\mathcal{V}^{(ji)T}_{k-1}-J^{(ij)}_{k-1}\Pi^{(j)-1}_{k-1}\mathcal{V}^{(j)T}_{k-1}, \quad k\geq 2;\\ &J^{(ij)}_{0,1}=0,\end{aligned} \tag{19}$$

$$\begin{aligned}\Pi^{(ij)}_k=&\ \Sigma^{y^{(ij)}}_k-\Psi^{(i)}_{A_k}\Big(r^{(j)}_{k-1}\Psi^{(j)T}_{A_k}+J^{(j)}_{k-1}\Pi^{(j)-1}_{k-1}\mathcal{V}^{(j)T}_{k-1}+J^{(ij)}_{k-1,k}\Big)\\ &-\mathcal{V}^{(ij)}_{k-1}\Pi^{(j)-1}_{k-1}\big(\Psi^{(j)}_{A_k}J^{(j)}_{k-1}+\mathcal{V}^{(j)}_{k-1}\big)^T-\mathcal{V}^{(i)}_{k-1}\Pi^{(i)-1}_{k-1}\Pi^{(ij)}_{k-1,k}, \quad k\geq 2;\\ \Pi^{(ij)}_1=&\ \Sigma^{y^{(ij)}}_1,\end{aligned} \tag{20}$$

$$\Pi^{(ij)}_{k-1,k}=\big(J^{(i)}_{k-1}-J^{(ji)}_{k-1}\big)^T\Psi^{(j)T}_{A_k}+\mathcal{V}^{(ji)T}_{k-1}-\Pi^{(ij)}_{k-1}\Pi^{(j)-1}_{k-1}\mathcal{V}^{(j)T}_{k-1}, \quad k\geq 2. \tag{21}$$

Proof. See Appendix B. □

Theorem 3. *For $i,j=1,\dots,m$, and fixed $k\geq 1$, the cross-covariance matrices between any two local smoothers, $K^{(ij)}_{k/k+N}=E\big[\widehat{x}^{(i)}_{k/k+N}\widehat{x}^{(j)T}_{k/k+N}\big]$, are obtained by*

$$\begin{aligned}K^{(ij)}_{k/k+N}=&\ K^{(ij)}_{k/k+N-1}+\Phi^{(ij)}_{k,k+N}\Pi^{(j)-1}_{k+N}\mathcal{X}^{(j)T}_{k,k+N}\\ &+\mathcal{X}^{(i)}_{k,k+N}\Pi^{(i)-1}_{k+N}\left(\Phi^{(ji)}_{k,k+N}+\mathcal{X}^{(j)}_{k,k+N}\Pi^{(j)-1}_{k+N}\Pi^{(ji)}_{k+N}\right)^T, \quad N\geq 1,\end{aligned} \tag{22}$$

where the initial condition, $K^{(ij)}_{k/k}$, is given in Theorem 2 and the matrices $\Phi^{(ij)}_{k,k+N}\equiv E\big[\widehat{x}^{(i)}_{k/k+N-1}\mu^{(j)T}_{k+N}\big]$ are calculated as follows:

$$\begin{aligned}\Phi^{(ij)}_{k,k+N}=&\left(E^{(i)}_{k,k+N-1}-E^{(ij)}_{k,k+N-1}\right)\Psi^{(j)T}_{A_{k+N}}+\mathcal{X}^{(i)}_{k,k+N-1}\Pi^{(i)-1}_{k+N-1}\mathcal{V}^{(ji)T}_{k+N-1}\\ &-\left(\mathcal{X}^{(i)}_{k,k+N-1}\Pi^{(i)-1}_{k+N-1}\Pi^{(ij)}_{k+N-1}+\Phi^{(ij)}_{k,k+N-1}\right)\Pi^{(j)-1}_{k+N-1}\mathcal{V}^{(j)T}_{k+N-1}, \ N\geq 1;\\ \Phi^{(ij)}_k=&\ A_k J^{(ij)}_{k-1,k},\end{aligned} \tag{23}$$

with $E^{(ij)}_{k,k+N} \equiv E\big[\widehat{x}^{(i)}_{k/k+N} O^{(j)T}_{k+N}\big]$ *given by*

$$
\begin{aligned}
E^{(ij)}_{k,k+N} &= E^{(ij)}_{k,k+N-1} + \Phi^{(ij)}_{k,k+N} \Pi^{(j)-1}_{k+N} J^{(j)T}_{k+N} \\
&\qquad + \mathcal{X}^{(i)}_{k,k+N} \Pi^{(i)-1}_{k+N} \left(J^{(ji)}_{k+N-1,k+N} + J^{(j)}_{k+N} \Pi^{(j)-1}_{k+N} \Pi^{(ji)}_{k+N} \right)^T, \quad N \geq 1; \\
E^{(ij)}_{k} &= A_k r^{(ij)}_k .
\end{aligned}
\tag{24}
$$

Proof. See Appendix C. □

Remark 4. *Since, from the Orthogonal Projection Lemma (OPL),* $J^{(i)}_{k-1,k} = 0$ *and* $\Pi^{(i)}_{k-1,k} = 0$*, it is easy to verify that expressions (17)-(21) in Theorem 2 are consistent with their homologous ones in the single-sensor case. Consequently, the recursive formula (23) leads to* $\Phi^{(i)}_{k,k+N} = 0$ *for* $N \geq 0$*, as it is directly derived from the OPL, and (24) is also consistent with formula of* $E^{(i)}_{k,k+N}$ *given in Theorem 1. So, for* $j = i$*, expressions (16) and (22) for the covariances* $K^{(i)}_{k/k+N} = E\big[\widehat{x}^{(i)}_{k/k+N} \widehat{x}^{(i)T}_{k/k+N}\big]$*,* $N \geq 0$*, are reduced to*

$$
\begin{aligned}
&K^{(i)}_{k/k+N} = K^{(i)}_{k/k+N-1} + \mathcal{X}^{(i)}_{k,k+N} \Pi^{(i)-1}_{k+N} \mathcal{X}^{(i)T}_{k,k+N}, \quad N \geq 1, \\
&K^{(i)}_{k/k} = A_k r^{(i)}_k A^T_k ,
\end{aligned}
$$

with $r^{(i)}_k$*,* $\Pi^{(i)}_{k+N}$ *and* $\mathcal{X}^{(i)}_{k,k+N}$ *given in Theorem 1.*

Once the local LS linear estimators and their cross-covariance matrices have been calculated, in the following theorem the distributed fusion estimators are designed as the matrix-weighted linear combination of the local ones that minimizes the mean squared error.

Theorem 4. *From the local estimators* $\widehat{x}^{(i)}_{k/k+N}$*,* $i = 1, \ldots, m$*, calculated by the algorithm in Theorem 1, and their cross-covariance matrices,* $K^{(ij)}_{k/k+N}$*, given in theorems 2 and 3, the distributed filtering and smoothing estimators are given by*

$$
\widehat{x}^{(D)}_{k/k+N} = \mathbf{\Xi}_{k/k+N} \left(\boldsymbol{K}_{k/k+N} \right)^{-1} \widehat{X}_{k/k+N}, \quad k \geq 1, \quad N \geq 0,
$$

where $\widehat{X}_{k/k+N} = \left(\widehat{x}^{(1)T}_{k/k+N}, \ldots, \widehat{x}^{(m)T}_{k/k+N} \right)^T$*,* $\boldsymbol{K}_{k/k+N} = \left(K^{(ij)}_{k/k+N} \right)_{i,j=1,\ldots,m}$ *and* $\mathbf{\Xi}_{k/k+N} = \left(K^{(1)}_{k/k+N}, \ldots, K^{(m)}_{k/k+N} \right)$.

The error covariance matrices of the distributed estimators are given by

$$\Sigma^{(D)}_{k/k+N} = A_k B_k^T - \Xi_{k/k+N} \boldsymbol{K}^{-1}_{k/k+N} \Xi^T_{k/k+N}, \quad k \geq 1, \quad N \geq 0.$$

Proof. This proof is standard and it can be seen in [17]. □

4. Centralized fusion estimators

This section is concerned with the design of recursive LS linear filtering and fixed-point smoothing algorithms under the centralized fusion approach; namely, our aim is to obtain recursive algorithms for the LS linear estimators of x_k based on the observations $\{y_1^{(i)}, \dots, y_{k+N}^{(i)}, \ N \geq 0, \ \ i = 1, \dots, m\}$. Such estimators will be denoted by $\widehat{x}^{(C)}_{k/k+N}$. To address this problem, in which the observations of the different sensors are jointly processed at each sampling time, the following vectors and matrices are defined:

$$z_k = \begin{pmatrix} z_k^{(1)} \\ \vdots \\ z_k^{(m)} \end{pmatrix}, \quad v_k = \begin{pmatrix} v_k^{(1)} \\ \vdots \\ v_k^{(m)} \end{pmatrix}, \quad \xi_k = \begin{pmatrix} \xi_k^{(1)} \\ \vdots \\ \xi_k^{(m)} \end{pmatrix}, \quad \eta_0 = \begin{pmatrix} \eta_0^{(1)} \\ \vdots \\ \eta_0^{(m)} \end{pmatrix}, \quad C_k = \begin{pmatrix} C_k^{(1)} \\ \vdots \\ C_k^{(m)} \end{pmatrix},$$

$$y_k = \begin{pmatrix} y_k^{(1)} \\ \vdots \\ y_k^{(m)} \end{pmatrix}, \quad H_k = \begin{pmatrix} H_k^{(1)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & H_k^{(m)} \end{pmatrix}, \quad D_k = \begin{pmatrix} D_k^{(1)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & D_k^{(m)} \end{pmatrix}.$$

Then, denoting $\overline{H}_k \equiv E[H_k]$ and $\overline{C}_k \equiv E[C_k]$, we obtain the following stacked version of the observation model (7) and (8):

$$\begin{aligned} y_k &= H_k C_k x_k - D_{k-1} \overline{H}_{k-1} \overline{C}_{k-1} x_{k-1} + V_{k-1}, \quad k \geq 2; \\ y_1 &= H_1 z_1 + D_0 \eta_0 + \xi_0, \\ V_k &= \xi_k - D_k \big(H_k C_k - \overline{H}_k \overline{C}_k \big) x_k + H_{k+1} v_{k+1} - D_k H_k v_k, \quad k \geq 1. \end{aligned} \tag{25}$$

From hypotheses *(H2)-(H7)*, the following properties of the processes involved in (25) are easily inferred:

- $\{H_k\}_{k\geq1}$ and $\{C_k\}_{k\geq1}$ are sequences of independent random matrices with known means, $\overline{H}_k$ and $\overline{C}_k$, $\ k \geq 1$.

- $\{v_k\}_{k\geq1}$ and $\{\xi_k\}_{k\geq0}$ are zero-mean noise processes with $E[v_k v_s^T] = R_k \delta_{k,s}$ and $E[\xi_k \xi_s^T] = S_k \delta_{k,s}$, with $R_k = \big(R_k^{(ij)}\big)_{i,j=1,\dots,m}$ and $S_k = \big(S_k^{(ij)}\big)_{i,j=1,\dots,m}$, respectively.

- η_0 is a zero-mean random vector with $E[\eta_0\eta_0^T] = Q_0 = \big(Q_0^{(ij)}\big)_{i,j=1,\dots,m}$.
- The signal process $\{x_k\}_{k\geq 1}$, the vector η_0 and the processes $\{H_k\}_{k\geq 1}$, $\{C_k\}_{k\geq 1}$, $\{v_k\}_{k\geq 1}$ and $\{\xi_k\}_{k\geq 0}$ are mutually independent.

Equation (25) and the above properties on the involved processes yield LS linear filtering and fixed-point smoothing algorithms based on the stacked observations, $\{y_k\}_{k\geq 1}$, whose structure is similar to that of the local ones given in Theorem 1. Therefore, we will just indicate the computational procedure to obtain the centralized estimators, which is also valid for the local ones.

Computational procedure. The computational procedure of the proposed centralized fusion estimators is summarized as follows:

1) *Previous matrices.* From the matrices $\Sigma_k^{y^{(ij)}}$, $\Psi_{A_k}^{(i)}$, $\Psi_{B_k}^{(i)}$ and $\mathcal{V}_k^{(ij)}$, given in (9)-(12), we obtain $\Sigma_k^y = \big(\Sigma_k^{y^{(ij)}}\big)_{i,j=1,\dots,m}$, $\Psi_{G_k} = \big(\Psi_{G_k}^{(1)T}, \dots, \Psi_{G_k}^{(m)T}\big)^T$, for $G_k = A_k, B_k$, and $\mathcal{V}_k = \big(\mathcal{V}_k^{(ij)}\big)_{i,j=1,\dots,m}$. These matrices are obtained from the model hypotheses and, hence, they can be calculated before the observations are available.

2) *Centralized filtering recursive algorithm.*

2a) *Initial conditions:*

∗ Compute $J_1 = \Psi_{B_1}^T$, $\Pi_1 = \Sigma_1^y$ and $r_1 = J_1\Pi_1^{-1}J_1^T$; then, the error covariance matrix, $\Sigma_{1/1}^{(C)} = A_1\big(B_1 - A_1 r_1\big)^T$, is obtained.

∗ When $y_1 = \big(\breve{z}_1^{(1)T}, \dots, \breve{z}_1^{(m)T}\big)^T$ is available, we get the innovation, $\mu_1 = y_1$; we calculate $O_1 = J_1\Pi_1^{-1}\mu_1$ and, then, the centralized filter $\widehat{x}_{1/1}^{(C)} = A_1 O_1$ is obtained.

2b) At any sampling time $k \geq 2$, starting with the prior knowledge of the $(k-1)$-th iteration, which provides J_{k-1}, Π_{k-1}, r_{k-1}, μ_{k-1} and O_{k-1}, the proposed centralized filtering algorithm operates as follows:

∗ Compute $J_k = \Psi_{B_k}^T - r_{k-1}\Psi_{A_k}^T - J_{k-1}\Pi_{k-1}^{-1}\mathcal{V}_{k-1}^T$ and, from it,

$$\Pi_k = \Sigma_k^y - \Psi_{A_k}(\Psi_{B_k}^T - J_k) - \mathcal{V}_{k-1}\Pi_{k-1}^{-1}\left(\Psi_{A_k}J_{k-1} + \mathcal{V}_{k-1}\right)^T.$$

Then, we calculate $r_k = r_{k-1} + J_k\Pi_k^{-1}J_k^T$ and, from it, the error covariance matrix, $\Sigma_{k/k}^{(C)} = A_k\big(B_k - A_k r_k\big)^T$ is obtained.

* When the new measurement $\breve{z}_k = \left(\breve{z}_k^{(1)T}, \dots, \breve{z}_k^{(m)T}\right)^T$ is available, we get $y_k = \breve{z}_k - D_{k-1}\breve{z}_{k-1}$, and the innovation is calculated by

$$\mu_k = y_k - \Psi_{A_k} O_{k-1} - \mathcal{V}_{k-1}\Pi_{k-1}^{-1}\mu_{k-1}.$$

  Finally, $O_k = O_{k-1} + J_k \Pi_k^{-1}\mu_k$ is computed and the centralized filter, $\widehat{x}_{k/k}^{(C)} = A_k O_k$, is obtained.

3) *Centralized fixed-point smoothing recursive algorithm.*

3a) *Initial conditions:* For each fixed sampling point, $k \geq 1$, the initial conditions of the smoothing algorithm are the centralized filter, $\widehat{x}_{k/k}^{(C)} = A_k O_k$, the filtering error covariance matrix, $\Sigma_{k/k}^{(C)} = A_k\left(B_k - A_k r_k\right)^T$, and $\mathcal{X}_{k,k} = A_k J_k$, $E_{k,k} = A_k r_k$.

3b) At the sampling time $k+N$, with $N = 1, 2, \dots$, run the filtering algorithm until time $k+N$; then, by starting with the initial conditions, the proposed centralized smoothing algorithm operates as follows:

* For each fixed $k \geq 1$ and $N = 1, 2, \dots$, compute

$$\mathcal{X}_{k,k+N} = \left(B_k - E_{k,k+N-1}\right)\Psi_{A_{k+N}}^T - \mathcal{X}_{k,k+N-1}\Pi_{k+N-1}^{-1}\mathcal{V}_{k+N-1}^T$$

  and, from it, the smoother, $\widehat{x}_{k/k+N}^{(C)}$, and its error covariance matrix, $\Sigma_{k/k+N}^{(C)}$, are obtained by

$$\begin{aligned}\widehat{x}_{k/k+N}^{(C)} &= \widehat{x}_{k/k+N-1}^{(C)} + \mathcal{X}_{k,k+N}\Pi_{k+N}^{-1}\mu_{k+N},\\ \Sigma_{k/k+N}^{(C)} &= \Sigma_{k/k+N-1}^{(C)} - \mathcal{X}_{k,k+N}\Pi_{k+N}^{-1}\mathcal{X}_{k,k+N}^T.\end{aligned}$$

* For the next step, $E_{k,k+N} = E_{k,k+N-1} + \mathcal{X}_{k,k+N}\Pi_{k+h}^{-1}J_{k+N}^T$ is then calculated.

5. Numerical simulation example

In this section, the applicability of the filtering and fixed-point algorithms proposed in the current paper is illustrated by a simulation example. Consider the following target tracking system [5]:

$$x_{k+1} = \left(F_1 + \varepsilon_k F_2\right)x_k + \Upsilon w_k, \qquad k \geq 1,$$

where $F_1 = \begin{pmatrix} 0.95 & 0.01 \\ 0 & 0.95 \end{pmatrix}$, $F_2 = \begin{pmatrix} 0.01 & 0 \\ 0 & 0.01 \end{pmatrix}$ and $\Upsilon = \begin{pmatrix} 0.8 \\ 0.6 \end{pmatrix}$. The sequences $\{\varepsilon_k\}_{k\geq 1}$ and $\{w_k\}_{k\geq 1}$ are standard white gaussian scalar noises, and the initial signal x_0 is an standard gaussian two-dimensional random vector. The first and second component of x_k are the position and velocity of the target, respectively.

Assuming that the initial signal, x_0, and the sequences $\{\varepsilon_k\}_{k\geq 1}$ and $\{w_k\}_{k\geq 1}$ are mutually independent, the signal covariance function is given by $E[x_k x_s^T] = F_1^{k-s} E[x_s x_s^T],\ s \leq k$, where $E[x_s x_s^T]$ is recursively obtained by:

$$E[x_s x_s^T] = F_1 E[x_{s-1} x_{s-1}^T] F_1^T + F_2 E[x_{s-1} x_{s-1}^T] F_2^T + \Upsilon\Upsilon^T,\ s \geq 1,$$

with $E[x_0 x_0^T] = I_{2\times 2}$; hence, assumption *(H1)* is satisfied just taking $A_k = F_1^k$ and $B_s^T = F_1^{-s} E[x_s x_s^T]$.

Let us consider that this target is measured by four sensors, whose measured outputs, $z_k^{(i)},\ k \geq 1,\ i = 1,2,3,4$, are expressed by model (1) with $C_k^{(1)} = \beta_k^{(1)} (0.74,\ \ 0.75)$, $C_k^{(2)} = \beta_k^{(2)} (0.75,\ \ 0.70)$, $C_k^{(3)} = \beta_k^{(3)} (0.80,\ \ 0.75)$ and $C_k^{(4)} = \beta_k^{(4)} (0.84,\ \ 0.85)$, where $\{\beta_k^{(i)}\}_{k\geq 1},\ i = 1,2,3,4$, are mutually independent white processes with the following time-invariant probability distributions:

- $\beta_k^{(i)}$, $i = 1,4$, are uniformly distributed over $[0.2, 0.8]$ and $[0.3, 0.9]$, respectively.
- $P[\beta_k^{(2)} = 0.9] = 0.8, \quad P[\beta_k^{(2)} = 0.1] = 0.2.$
- $\beta_k^{(3)}$ are Bernoulli random variables with $P[\beta_k^{(3)} = 1] = \overline{\beta}$.

According to these probability distributions, the random coefficients $C_k^{(i)}$, $i = 1,2,3,4$, model *continuous gain degradation* in sensors 1 and 4, *discrete gain degradation* in sensor 2 and *missing measurements* in sensor 3.

The additive noise processes $\{v_k^{(i)}\}_{k\geq 1}$, $i = 1,2,3,4$, are defined by $v_k^{(i)} = c_i \rho_k,\ i = 1,2,3,4$, where $c_1 = c_3 = 0.5$, $c_2 = c_4 = 0.75$ and $\{\rho_k\}_{k\geq 1}$ is a zero-mean gaussian white process with variance 0.5.

After the transmission, the received information can be expressed by the model (2)-(3) assuming the following characteristics [19]:

- For $i = 1, 2, 3$, $H_k^{(i)} = \lambda_k^{(i)}$ and $H_k^{(4)} = \lambda_k^{(4)}\big(1 + 0.95\theta_k\big)$, where $\{\theta_k\}_{k\geq 1}$ is a standard gaussian white process, the sequences $\{\theta_k\}_{k\geq 1}$ and $\{\lambda_k^{(i)}\}_{k\geq 1}$, $i = 1, 2, 3, 4$, are mutually independent, and $\{\lambda_k^{(i)}\}_{k\geq 1}$, $i = 1, 2, 3, 4$, are white processes with the following time-invariant probability distributions:

  - $\lambda_k^{(1)}$ is uniformly distributed over $[0.1, 0.9]$.
  - $P[\lambda_k^{(2)} = 0] = 0.3, \quad P[\lambda_k^{(2)} = 0.5] = 0.3, \quad P[\lambda_k^{(2)} = 1] = 0.4.$
  - For $i = 3, 4$, $\lambda_k^{(i)}$ are Bernoulli random variables with $P[\lambda_k^{(i)} = 1] = \overline{\lambda}$.

  Consequently, the random coefficients $H_k^{(i)}$, $i = 1, 2, 3, 4$, model *continuous and discrete gain degradation* in transmissions from sensors 1 and 2, respectively, *missing measurements* in transmissions from sensor 3, and both *missing measurements and multiplicative noise* in transmissions from sensor 4.

- The noise processes $\{\eta_k^{(i)}\}_{k\geq 1}$, $i = 1, 2, 3, 4$, are defined by $\eta_k^{(i)} = D^{(i)}\eta_{k-1}^{(i)} + \xi_{k-1}^{(i)}$, with $D^{(1)} = D^{(3)} = 0.95$, $D^{(2)} = D^{(4)} = 0.75$, and $\{\xi_k^{(i)}\}_{k\geq 0}$ are defined by $\xi_k^{(i)} = a_i\xi_k$, with $a_1 = a_3 = 0.75$, $a_2 = a_4 = 1.25$, with $\{\xi_k\}_{k\geq 0}$ a zero-mean Gaussian white process with variance 1.5. Finally, for $i = 1, 2, 3, 4$, $\eta_0^{(i)} = \eta_0$, where η_0 is an standard gaussian variable.

First, to compare the effectiveness of the proposed distributed and centralized filtering and fixed-point smoothing estimators, one hundred iterations of the respective algorithms have been performed, considering constant values of the probabilities $\overline{\beta} = 0.5$ and $\overline{\lambda} = 0.5$. Figure 1 displays the local filtering error variances and the distributed and centralized filtering and fixed-point smoothing error variances of the first and second signal components. For each signal component, this figure shows, on the one hand, that the error variances of the distributed fusion filtering estimators are lower than those of every local estimator, but greater than those of the centralized ones. On the other hand, it is observed that the error variances corresponding to the smoothers are less than those of the filters. Also, it is deduced that the smoothers at each fixed-point k become more accurate as the number of available observations, $k + N$, increases; this fact is more evident in the case of the centralized smoothers, since for the distributed ones the difference is practically negligible for $N \geq 2$.

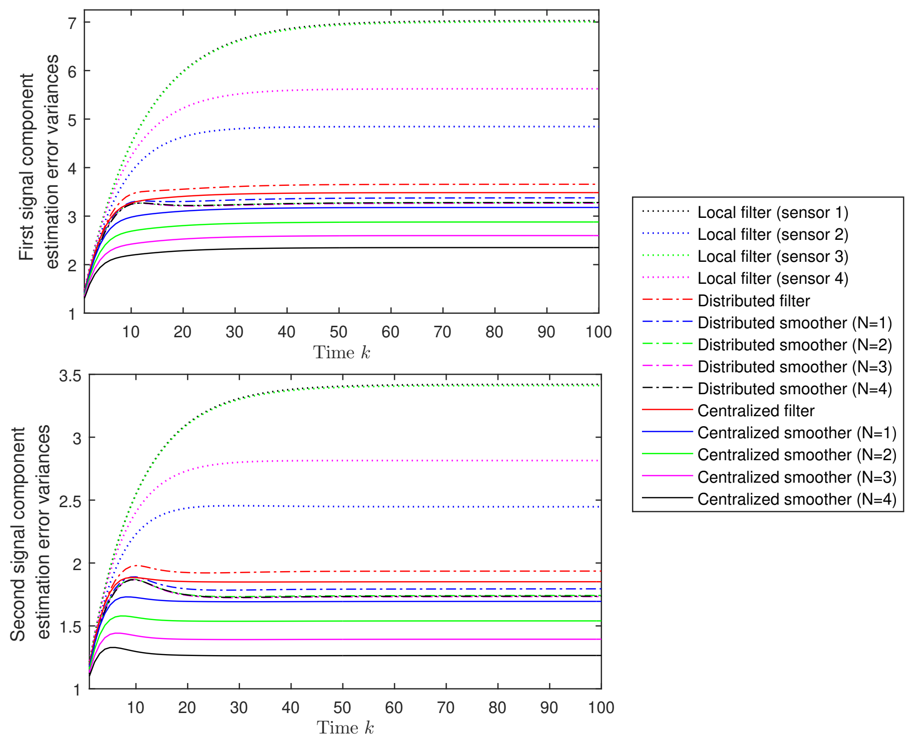

Figure 1: Error variance comparison of the local, distributed and centralized estimators.

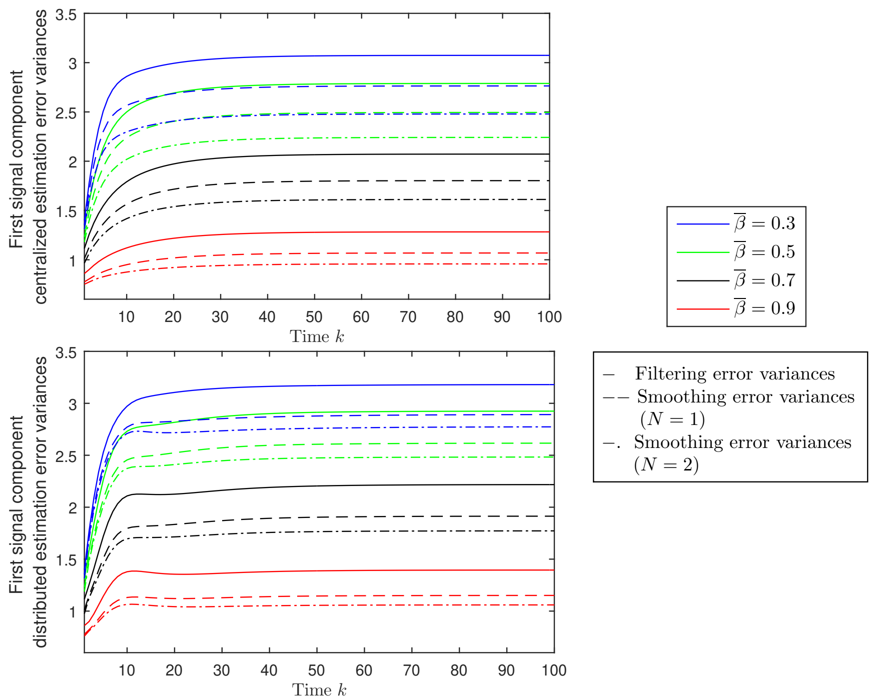

Figure 2: First signal component error variances for the distributed and centralized estimators when $\overline{\lambda} = 0.9$ and $\overline{\beta} = 0.3,\ 0.5,\ 0.7,\ 0.9$.

Next, in order to show the effect of the missing measurements phenomenon in sensor 3, the distributed and centralized estimation error variances of the first signal component are plotted in Figure 2 for different values of the probability $\overline{\beta}$. It can be seen that the performance of the estimators is indeed influenced by this probability and, as expected, the estimation error variances decrease as the probability $\overline{\beta}$ increases; hence, the performance of both, the distributed and centralized estimators, improves when $1 - \overline{\beta}$, the probability of missing measurements, decreases. Also, as in Figure 1, this figure shows that the error variances corresponding to the smoothers are less than those of the filters and the smoother with lag $N = 2$ is better than that with $N = 1$. Similar results and, therefore, the same conclusions are inferred for the second signal component.

Finally, in order to show how the estimation accuracy is influenced by the effect of missing measurements in the transmissions of the sensors 3 and

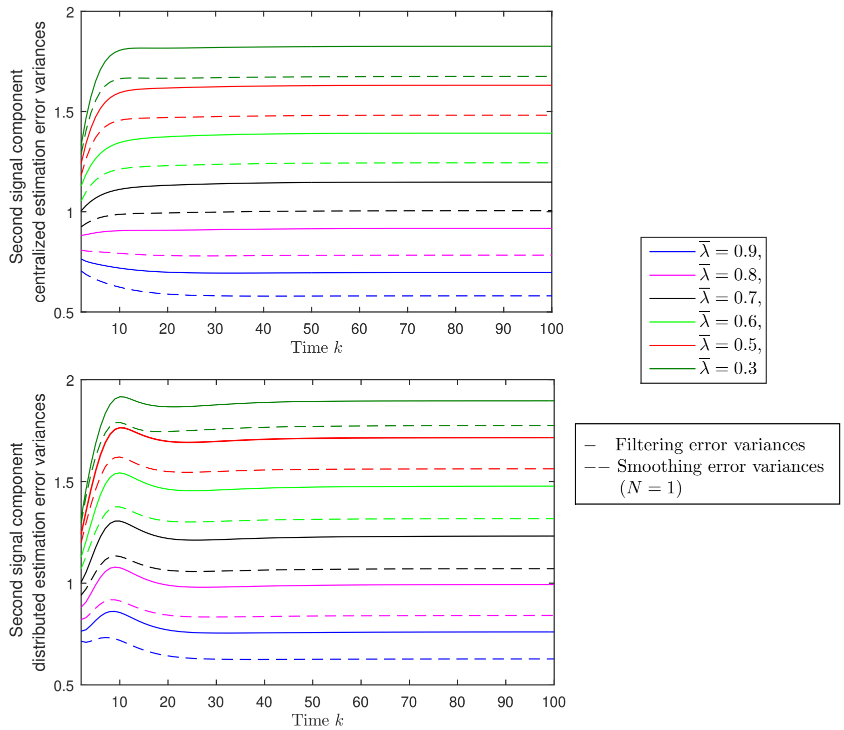

Figure 3: Second signal component error variances for the distributed and centralized estimators when $\overline{\beta} = 0.9$ and $\overline{\lambda} = 0.3,\ 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9$.

4, the distributed and centralized filtering and smoothing error variances of the second signal component are displayed in Figure 3 for different values of the probability $\overline{\lambda}$. As in Figure 2, it is observed that the distributed and centralized error variances become smaller as the probability $\overline{\lambda}$ increases, which means that the performance of both estimators improves when the missing probability $1 - \overline{\lambda}$ decreases. This fact was expected, since more information is available when the probability $\overline{\lambda}$ is greater and, therefore, the accuracy of the estimators is improved for higher values of such probability $\overline{\lambda}$. Similar results and, consequently, the same conclusions are obtained for the first signal component.

6. Conclusions

In this paper, the LS linear fusion filtering and fixed-point smoothing problems have been considered under the distributed and centralized fusion schemes, for a class of discrete-time networked systems with random parameter matrices and time-correlated channel noise. More precisely, random measurement matrices have been considered in the sensor output model and also in the data available after the transmission from the sensors to the processing center where the estimation is carried out. Hence, the proposed model provides a general framework to deal with network-induced uncertainties in both the sensor measurements and the transmission channels. The additive channel noise is assumed to obey a dynamic linear equation perturbed by white noise and, as is usual in such cases, the time-differencing methodology has been applied to linearly transform the available measurements with time-correlated noise into new ones that do not depend on the time-correlated noise. Taking into account that the LS linear estimator of the signal based on the original measurements is equal to the LS linear estimator based on the new ones, the estimation problem has been reformulated as that of obtaining recursive algorithms for the distributed and centralized fusion estimators of the signal based on the transformed measurements, for which an innovation-based methodology has been adopted. Finally, a computer simulation example has shown that the proposed estimation algorithms are suitable for practical implementation and cover a great variety of engineering problems that fit in the system model under consideration, with multiple uncertainties in both the sensor measurements and the transmission, such as continuous and discrete gain degradation, missing measurements or multiplicative noise.

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Appendix A. Proof of Lemma 1

By substituting (2) with (3) into (6), $y_k^{(i)}$ is rewritten as follows:

$$\begin{array}{l} y_k^{(i)} = H_k^{(i)} z_k^{(i)} - D_{k-1}^{(i)} H_{k-1}^{(i)} z_{k-1}^{(i)} + \xi_{k-1}^{(i)}, \quad k \geq 2; \\ y_1^{(i)} = H_1^{(i)} z_1^{(i)} + D_0^{(i)} \eta_0^{(i)} + \xi_0^{(i)}. \end{array} \tag{26}$$

From this expression, the following statements are deduced:

(a) Clearly, $\{y_k^{(i)}\}_{k\geq 1}$, $i=1,\dots,m$, are zero-mean processes and from (26) it is evident that $\Sigma_k^{y^{(ij)}}$ satisfy (9).

(b) Substituting (1) into (26) and using the independence hypotheses we have

$$
\begin{aligned}
E\big[x_k y_h^{(i)T}\big] &= E[x_k x_h^T]\overline{C}_h^{(i)T}\overline{H}_h^{(i)T} - E[x_k x_{h-1}^T]\overline{C}_{h-1}^{(i)T}\overline{H}_{h-1}^{(i)T}D_{h-1}^{(i)T}, \quad h\geq 2;\\
E\big[x_k y_1^{(i)T}\big] &= E[x_k x_1^T]\overline{C}_1^{(i)T}\overline{H}_1^{(i)T},
\end{aligned}
$$

and the separable form of the signal covariance *(H1)* leads to (10).

(c) Using again (26) for $y_h^{(j)}$, the model hypotheses easily lead to $E\big[V_k^{(i)}y_h^{(j)T}\big]=0$, for $h<k$, and $\mathcal{V}_k^{(ij)}=E\big[V_k^{(i)}y_k^{(j)T}\big]=E[V_k^{(i)}z_k^{(j)T}H_k^{(j)T}]$. Then, just replacing $V_k^{(i)}$ by (8), we obtain

$$
\mathcal{V}_k^{(ij)}=-D_k^{(i)}E\big[\big(H_k^{(i)}z_k^{(i)}-\overline{H}_k^{(i)}\overline{C}_k^{(i)}x_k\big)z_k^{(j)T}H_k^{(j)T}\big],
$$

thus proving expression (12), since $E\big[x_k z_k^{(j)T}H_k^{(j)T}\big]=A_kB_k^T\overline{C}_k^{(j)}\overline{H}_k^{(j)}$.

□

Appendix B. Proof of Theorem 2

From the expression of the filters and the recursive expression of the vectors $O_k^{(i)}$ in Theorem 1, (16)-(18) are easily obtained.

- To obtain (19) for $J_{k-1,k}^{(ij)}=E\big[O_{k-1}^{(i)}y_k^{(j)T}\big]-E\big[O_{k-1}^{(i)}\widehat{y}_{k/k-1}^{(j)T}\big]$, firstly we use (15) for $\widehat{y}_{k/k-1}^{(j)}$, which leads to

$$
J_{k-1,k}^{(ij)}=E\big[O_{k-1}^{(i)}y_k^{(j)T}\big]-r_{k-1}^{(ij)}\Psi_{A_k}^{(j)T}-J_{k-1}^{(ij)}\Pi_{k-1}^{(j)-1}\mathcal{V}_{k-1}^{(j)T}.
$$

  Next, using (7) for $y_k^{(j)}$, expressing $E\big[O_{k-1}^{(i)}x_h^T\big]=E\big[O_{k-1}^{(i)}\widehat{x}_{h/k-1}^{(i)T}\big]$ by the OPL and writing $\widehat{x}_{h/k-1}^{(i)}=A_hO_{k-1}^{(i)}$, $h=k,k-1$, we obtain

$$
E\big[O_{k-1}^{(i)}y_k^{(j)T}\big]=r_{k-1}^{(i)}\Psi_{A_k}^{(j)T}+E\big[O_{k-1}^{(i)}V_{k-1}^{(j)T}\big].
$$

  Finally, from definition (14) for $O_{k-1}^{(i)}$ and (c) in Lemma 1, $E\big[O_{k-1}^{(i)}V_{k-1}^{(j)T}\big]=J_{k-1}^{(i)}\Pi_{k-1}^{(i)-1}\mathcal{V}_{k-1}^{(ji)T}$ and (19) is proven.

- To obtain (20), starting from $\Pi_k^{(ij)} = \Sigma_k^{y^{(ij)}} - E\big[y_k^{(i)}\widehat{y}_{k/k-1}^{(j)T}\big] - E\big[\widehat{y}_{k/k-1}^{(i)}\mu_k^{(j)T}\big]$, firstly we use (7) for $y_k^{(i)}$ and we reason as in the previous item to obtain

$$E\big[y_k^{(i)}\widehat{y}_{k/k-1}^{(j)T}\big] = \Psi_{A_k}^{(i)}E[O_{k-1}^{(j)}\widehat{y}_{k/k-1}^{(j)T}] + \mathcal{V}_{k-1}^{(ij)}\Pi_{k-1}^{(j)-1}E[\mu_{k-1}^{(j)}\widehat{y}_{k/k-1}^{(j)T}].$$

Then, from (15) for $\widehat{y}_{k/k-1}^{(j)}$ and $\widehat{y}_{k/k-1}^{(i)}$:

$$\begin{aligned} E\big[y_k^{(i)}\widehat{y}_{k/k-1}^{(j)T}\big] &= \Psi_{A_k}^{(i)}\Big(r_{k-1}^{(j)}\Psi_{A_k}^{(j)T} + J_{k-1}^{(j)}\Pi_{k-1}^{(j)-1}\mathcal{V}_{k-1}^{(j)T}\Big)\\ &\quad+\mathcal{V}_{k-1}^{(ij)}\Pi_{k-1}^{(j)-1}\big(\Psi_{A_k}^{(j)}J_{k-1}^{(j)} + \mathcal{V}_{k-1}^{(j)}\big)^T,\\ E[\widehat{y}_{k/k-1}^{(i)}\mu_k^{(j)}] &= \Psi_{A_k}^{(i)}J_{k-1,k}^{(ij)} + \mathcal{V}_{k-1}^{(i)}\Pi_{k-1}^{(i)-1}\Pi_{k-1,k}^{(ij)}, \end{aligned}$$

and expression (20) for $\Pi_k^{(ij)}$ is obtained.

- Finally, (21) is proven just expressing $\Pi_{k-1,k}^{(ij)} = E\big[\mu_{k-1}^{(i)}y_k^{(j)T}\big] - E\big[\mu_{k-1}^{(i)}\widehat{y}_{k/k-1}^{(j)T}\big]$ and adopting a similar reasoning using (7) for $y_k^{(j)}$ and (15) for $\widehat{y}_{k/k-1}^{(j)}$, which leads to

$$E\big[\mu_{k-1}^{(i)}y_k^{(j)T}\big] = J_{k-1}^{(i)T}\Psi_{A_k}^{(j)T} + \mathcal{V}_{k-1}^{(ji)T},$$

$$E\big[\mu_{k-1}^{(i)}\widehat{y}_{k/k-1}^{(j)T}\big] = J_{k-1}^{(ji)T}\Psi_{A_k}^{(j)T} + \Pi_{k-1}^{(ij)}\Pi_{k-1}^{(j)-1}\mathcal{V}_{k-1}^{(j)T}.$$

From these expectations, expression (21) is straightforward.

$\square$

Appendix C. Proof of Theorem 3

Using the expressions of the local smoothers, $\widehat{x}_{k/k+N}^{(i)}$, $i = 1\ldots, m$, given in Theorem 1, the recursive expression (22) for the cross-covariance matrices between any two local smoothers, is immediately deduced. To derive (23) for $\Phi_{k,k+N}^{(ij)} = E\big[\widehat{x}_{k/k+N-1}^{(i)}y_{k+N}^{(j)T}\big] - E\big[\widehat{x}_{k/k+N-1}^{(i)}\widehat{y}_{k+N/k+N-1}^{(j)T}\big]$ we use (15) for $\widehat{y}_{k+N/k+N-1}^{(j)}$ and definition of $E_{k,k+N}^{(ij)}$ to obtain:

$$\begin{aligned}\Phi_{k,k+N}^{(ij)} = E\big[\widehat{x}_{k/k+N-1}^{(i)}y_{k+N}^{(j)T}\big] &- E_{k,k+N-1}^{(ij)}\Psi_{A_{k+N}}^{(j)T}\\ &- E\big[\widehat{x}_{k/k+N-1}^{(i)}\mu_{k+N-1}^{(j)T}\big]\Pi_{k+N-1}^{(j)-1}\mathcal{V}_{k+N-1}^{(j)T}.\end{aligned}$$

- The first expectation in the above expression is obtained as in Appendix B, using (7) for $y^{(j)}_{k+N}$, the OPL and the expression of the predictor and the filter, $\widehat{x}^{(i)}_{h/k+N-1} = A_h O^{(i)}_{k+N-1},\ h = k+N,\ k+N-1$; namely:

$$E\big[\widehat{x}^{(i)}_{k/k+N-1} y^{(j)T}_{k+N}\big] = E^{(i)}_{k,k+N-1}\Psi^{(j)T}_{A_{k+N}} + \mathcal{X}^{(i)}_{k,k+N-1}\Pi^{(i)-1}_{k+N-1}\mathcal{V}^{(ji)T}_{k+N-1}.$$

- For the second expectation, we use again the recursive expression of the smoothers, $\widehat{x}^{(i)}_{k/k+N-1} = \widehat{x}^{(i)}_{k/k+N-2} + \mathcal{X}^{(i)}_{k,k+N-1}\Pi^{(i)-1}_{k+N-1}\mu^{(i)}_{k+N-1}$, and then:

$$E\big[\widehat{x}^{(i)}_{k/k+N-1}\mu^{(j)T}_{k+N-1}\big] = \Phi^{(ij)}_{k,k+N-1} + \mathcal{X}^{(i)}_{k,k+N-1}\Pi^{(i)-1}_{k+N-1}\Pi^{(ij)}_{k+N-1}.$$

Substituting these expectations, expression (23) is immediately derived, and the initial condition, $\Phi^{(ij)}_{k} = A_k J^{(ij)}_{k-1,k}$, is clear from $\widehat{x}^{(i)}_{k/k-1} = A_k O_{k-1}$ and definition of $J^{(ij)}_{k-1,k}$.

Finally, expression (24) for $E^{(ij)}_{k,k+N}$ is straightforward from the recursive expressions of the smoothers, $\widehat{x}^{(i)}_{k/k+N}$, and vectors $O^{(j)}_{k+N}$, given in Theorem 1. Its initial condition, $E^{(ij)}_{k} = A_k r^{(ij)}_{k}$, is also clear from $\widehat{x}^{(i)}_{k/k} = A_k O_k$ and the definition of $r^{(ij)}_{k}$.

□

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