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# A Stochastic Lomax Diffusion Process: Statistical Inference and Application

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**Abstract:** In this paper, we discuss a new stochastic diffusion process in which the trend function is proportional to the Lomax density function. This distribution arises naturally in the studies of the frequency of extremely rare events. We first consider the probabilistic characteristics of the proposed model, including its analytic expression as the unique solution to a stochastic differential equation, the transition probability density function together with the conditional and unconditional trend functions. Then, we present a method to address the problem of parameter estimation using maximum likelihood with discrete sampling. This estimation requires the solution of a non-linear equation, which is achieved via the simulated annealing method. Finally, we apply the proposed model to a real-world example concerning adolescent fertility rate in Morocco.

**Keywords:** stochastic differential equation; lomax distribution; trend functions; statistical inference; simulated annealing; adolescent fertility rate



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## 1. Introduction

Stochastic diffusion models are used to analyze the evolution of phenomena in multiple fields of science, including biology, finance, energy consumption and physics. In addition to traditional applications, stochastic diffusion processes (SDPs) have attracted considerable attention as analytical tools in areas such as cell growth, population growth and environmental studies. In this respect, see for example: Lognormal [1]; Gompertz [2]; Logistic [3]; Hyperbolic [4]; Rayleigh [5]; Pearson [6]; Weibull [7] and Brennan–Schwartz [8].

However in most of these studies, the processes considered are time homogeneous, in other words, the present state of the process depend only on the previous states and not on time. In contrast observations from many fields such a as neuroscience, finance and biology, suggest otherwise. Various non-homogeneous SDPs have been proposed to reflect this time dependent behavior, see for example: Lognormal [9], Gompertz [10], Vasicek [11], Brennan–Schwartz [12], and Gamma [13] processes.

In most of the aforementioned studies, the statistical inference is based on the maximum likelihood function, which is the product of transition densities. However, in some cases the closed form of the transition density is unknown, or has complicated expression, so the maximum likelihood method remains difficult to implement. Therefore many methods based on an approximation of the maximum likelihood were developed, such as: Prakasa-Rao [14], Kloeden et al. [15], Bibby et al. [16] and among others.

The Pareto type (II) distribution or Pearson type (IV) distribution, also called Lomax distribution, was introduced and studied by Lomax [17]. This distribution is commonly used in reliability and many lifetime testing studies. It is also used to analyze business data.

The density function of a Lomax distribution on  $[0, +\infty[$  with  $\beta > 0$  (scale parameter), and  $\alpha > 0$  (shape parameter) is given by:

$$f(t) = \frac{\alpha}{\beta} \left(1 + \frac{t}{\beta}\right)^{-(\alpha+1)}. \tag{1}$$

This distribution is a special case of a more general one called the Generalized Pareto distribution, the density function of which has the following form:

$$g(t) = \frac{1}{\sigma} \left(1 + \frac{k(t - \mu)}{\sigma}\right)^{\left(-\frac{1}{k}-1\right)},$$

where  $\mu$  and  $k$  are real parameters and  $\sigma > 0$ . This distribution encompasses the Pareto distribution as a special case since if we set  $\mu = 0$  and  $k = \frac{1}{\alpha}$  we obtain the Equation (1).

In the present paper, we introduce a new Stochastic Lomax Diffusion Process (SLDP) as a non-homogeneous extension of the lognormal process, and which presents a trend function that is proportional to the Lomax density function. Moreover, the term adopted for the model we study will be improved by stochastic calculus. In this work, we will present a detailed and complete study of the Lomax Model. To this end, we will proceed as follows: In Section 2, we define the model in terms of stochastic differential equation (SDE), we then give the analytical expression of the solution of the proposed model. After which, we determine the Transition Probability Density Function (TPDF) and the trend functions. In Section 3, we deal with the problem of parameter estimation using Maximum Likelihood (ML) in the basis of discrete sampling. In this case, the system of likelihood equations does not have an explicit solution, so as a result the ML estimators cannot be given in the closed form. Then, one possible way to solve this basic problem is the use of numerical methods. In Section 4, we propose the simulated annealing method approximating the ML estimator then we show the results of the simulation of the process in Section 5. Moreover, in Section 6, we illustrate the results obtained by this method by reference to real data, namely the adolescent fertility rate in Morocco. Finally, we summarize the main conclusions drawn from this work.

## 2. The Model and Its Characteristics

### 2.1. The Model

The proposed model is the one-dimensional non-homogeneous SDP  $\{x(t), t \in [t_1, T], t_1 \geq 0\}$  taking values on  $[0, \infty]$  and with drift and diffusion coefficients:

$$\begin{cases} A_1(x) = -\frac{\alpha}{t+\beta} x & , \\ A_2(x) = \sigma^2 x^2 & , \end{cases} \tag{2}$$

where  $\sigma > 0, \beta > -t_1$  and  $\alpha$  are real parameters.

Alternatively, the process defined above can be considered as the unique solution to the following SDE:

$$dx(t) = -\frac{\alpha}{t+\beta} x(t)dt + \sigma x(t)dw(t), \quad x(t_1) = x_{t_1}, \tag{3}$$

where  $w(t)$  is the one-dimensional standard Wiener process and  $x_{t_1}$  is fixed in  $\mathbb{R}_+^*$ .

### 2.2. Distribution of the Process

The SDE in Equation (3) has a unique solution (see Kloeden et al [15]). In order to obtain this solution, we consider the appropriate transformation  $y(t) = \log(x(t))$ , then, by means of Itô formula, the Equation (3) becomes:

$$dy(t) = \left(-\frac{\alpha}{t + \beta} - \frac{\sigma^2}{2}\right)dt + \sigma dw(t),$$

$$y(t_1) = \ln(x_{t_1}).$$

The solution to which is:

$$y(t) = y(s) + \int_s^t \left(-\frac{\alpha}{\theta + \beta} - \frac{\sigma^2}{2}\right)d\theta + \sigma(w(t) - w(s)).$$

For  $s \in [t_1, t]$ . Hence, we deduce the expression of the solution to SDE in Equation (3):

$$x(t) = x_s \left(\frac{s + \beta}{t + \beta}\right)^\alpha \exp\left[-\frac{\sigma^2}{2}(t - s) + \sigma(w(t) - w(s))\right], \tag{4}$$

then  $x(t)|x(s) = x_s$  follows a Lognormal distribution:

$$x(t)|x(s) = x_s \sim \Lambda\left(\log(x_s) + \alpha \log\left(\frac{s + \beta}{t + \beta}\right) - \frac{\sigma^2}{2}(t - s), \sigma^2(t - s)\right).$$

where  $\Lambda$  is the Lognormal distribution. As a result, the TPDF of this process is found to be:

$$f(x, t|x_s, s) = \frac{1}{x\sqrt{2\pi(t - s)\sigma^2}} \exp\left(-\frac{\left[\log\left(\frac{x}{x_s}\right) + \alpha \log\left(\frac{t + \beta}{s + \beta}\right) + \frac{\sigma^2}{2}(t - s)\right]^2}{2\sigma^2(t - s)}\right).$$

### 2.3. Trend Functions

From the properties of the Lognormal distribution, the main characteristics of the process can be determined, in particular the  $r$ -th conditional moment of the process is given by:

$$E(x^r|x(s) = x_s) = \exp\left[r\left(\log(x_s) + \alpha \log\left(\frac{s + \beta}{t + \beta}\right) - \frac{\sigma^2}{2}(t - s)\right) + \frac{r^2}{2}\sigma^2(t - s)\right].$$

Then, by considering the case where  $r = 1$  in the previous expression, the conditional trend function of the process is:

$$E(x(t)|x(s) = x_s) = x_s \left(\frac{s + \beta}{t + \beta}\right)^\alpha. \tag{5}$$

In addition, taking into account the initial condition  $P(x(t_1) = x_{t_1}) = 1$ , the trend function of the process is:

$$E(x(t)) = x_{t_1} \left(\frac{t_1 + \beta}{t + \beta}\right)^\alpha. \tag{6}$$

We note here that:

- The trend function as defined in Equation (6) is proportional to the Lomax density function Equation (1).
- Otherwise, in the absence of white noise (i.e.,  $\sigma = 0$ ) the solution to equation Equation (3) is  $x(t) = x_s \left(\frac{s + \beta}{t + \beta}\right)^\alpha$  which is proportional in this case to the Lomax density function [18], with shape parameter  $\alpha$  and scale parameter  $\beta$ , which can be denoted  $P(II)(\beta, \alpha, \mu = 0)$ .

### 3. Maximum Likelihood Estimation

We consider a discrete sample of  $n$  observations of the process  $x(t)$  which we denote here  $(x_i)_{i=1,\dots,n}$ , let  $(t_0 < t_1 < \dots < t_n)$  denotes the moments when the process was observed with  $x_i = x(t_i)$  moreover we set  $t_i - t_{i-1} = h$ , and finally  $\theta = (\alpha, \sigma)$  is the parameters vector.

We know that the likelihood function  $l(x, \theta)$  is the product of the densities functions:

$$l(x, \theta) = \prod_{i=2}^n f(x_i, t_i | x_{i-1}, t_{i-1})$$

$$= \prod_{i=2}^n \frac{1}{x\sqrt{2\pi h\sigma^2}} \exp\left(-\frac{\left[\log\left(\frac{x_i}{x_{i-1}}\right) + \alpha\log\left(\frac{t_i+\beta}{t_{i-1}+\beta}\right) + \frac{\sigma^2}{2}h\right]^2}{2\sigma^2h}\right).$$

The Log-likelihood is given by:

$$L(x, \theta) = \log(l(x, \theta))$$

$$= \sum_{i=2}^n -\log(x_i) - \frac{1}{2}\log(2\pi h) - \frac{1}{2}\log(\sigma^2)$$

$$- \frac{1}{2\sigma^2h} \left[ \log\left(\frac{x_i}{x_{i-1}}\right) + \alpha\log\left(\frac{t_i+\beta}{t_{i-1}+\beta}\right) + \frac{\sigma^2}{2}h \right]^2$$

$$= -\frac{n-1}{2}\log(\sigma^2) - \frac{n-1}{2}\log(2\pi h) - \sum_{i=2}^n \left[ \log(x_i) + \frac{1}{2\sigma^2h} \left( C_{i,\alpha,\beta} + \frac{\sigma^2}{2}h \right)^2 \right],$$

where  $C_{i,\alpha,\beta} = \log\left(\frac{x_i}{x_{i-1}}\right) + \alpha\log\left(\frac{t_i+\beta}{t_{i-1}+\beta}\right)$ .

We differentiate this function with respect to the elements of vector  $\theta$  to obtain the following equations:

$$\sum_{i=2}^n \left( C_{i,\alpha,\beta} + \frac{\sigma^2h}{2} \right) \log\left(\frac{t_i+\beta}{t_{i-1}+\beta}\right) = 0, \tag{7}$$

$$\left( \sum_{i=2}^n C_{i,\alpha,\beta}^2 \right) - (n-1)\sigma^2h - \frac{n-1}{4}\sigma^4h^2 = 0, \tag{8}$$

$$\sum_{i=2}^n \frac{C_{i,\alpha,\beta} + \frac{\sigma^2h}{2}}{(t_i+\beta)(t_{i-1}+\beta)} = 0. \tag{9}$$

Equation (8) is a second-degree equation in  $\sigma^2$ , which admits two solutions (since the discriminant is  $\delta = (n-1)^2h^2 - \left(\sum_{i=2}^n C_{i,\alpha,\beta}^2\right)(n-1)h^2 > 0$ ). Therefore, from the non-negative solution corresponding to  $\sigma^2$ , the estimator  $\hat{\sigma}^2$  is given by:

$$\hat{\sigma}^2 = \frac{2}{h} \left( \left( 1 + \frac{1}{n-1} \sum_{i=2}^n C_{i,\alpha,\beta}^2 \right)^{1/2} - 1 \right). \tag{10}$$

By replacing  $\sigma^2$  by  $\hat{\sigma}^2$  in Equation (7), the estimator of  $\alpha$  is satisfying the following non-linear equation:

$$\sum_{i=2}^n \left( C_{i,\alpha,\beta} + \left( 1 + \frac{1}{n-1} \sum_{i=2}^n C_{i,\alpha,\beta}^2 \right)^{1/2} - 1 \right) \log\left(\frac{t_i+\beta}{t_{i-1}+\beta}\right) = 0, \tag{11}$$

On the other hand, substituting  $\sigma^2$  by  $\hat{\sigma}^2$  in Equation (10),

$$\sum_{i=2}^n \frac{C_{i,\alpha,\beta} + \left(1 + \frac{1}{n-1} \sum_{i=2}^n C_{i,\alpha,\beta}^2\right)^{1/2} - 1}{(t_i + \beta)(t_{i-1} + \beta)} = 0. \tag{12}$$

Obviously, this is a set of non-linear equations whose solutions may be difficult to find. To address this problem we use numerical resolution methods.

#### 4. Computational Aspects

In this paper we suggest the simulated annealing (SA) method for solving the equations Equations (11) and (12). Hereafter is the description of the method.

Simulated annealing is a stochastic optimisation algorithm, developed in 1983 by [19], which approaches the global optimum of a given cost function by means of a random search. The fundamental idea of the algorithm is inspired by the process of annealing of metals in metallurgy. At each step of the simulated annealing algorithm a new point is randomly generated, if the new point improves the cost function it is accepted, otherwise, it is accepted with a probability  $\exp(-\Delta f/T)$ , where  $f$  is the cost function and  $T$  is the temperature. Accepting points that don't improve the cost function allows the algorithm to escape local optima. The main disadvantage of this method is that the adjustment of the parameters (initial temperature, minimum temperature, cooling process and stopping conditions ...) considerably affects the time required to reach the extremum.

#### 5. Simulation

To illustrate the process described by Equation (3), let us consider an equidistant discretisation of the interval  $[s, T]$  with  $t_i = t_{i-1} + (i - 1)h$  for  $i = 2, \dots, N$ . Let  $(t_1 = s)$  and assume a discretisation step  $h = \frac{T-s}{N}$  where  $N$  denotes the size of the sample. A total of 25 trajectories of the process were simulated, with  $s = 0, T = 1000$  and  $N = 2000$  and  $x_s = 100$ .

The results of the simulation, together with the Estimated Trend Function (ETF) of the process, are illustrated in Figure 1.

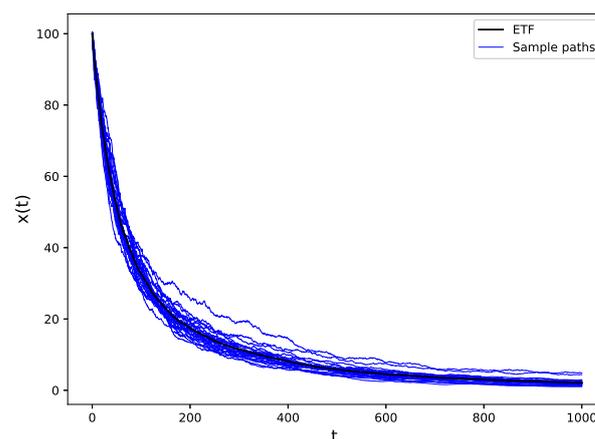


Figure 1. Simulated sample paths vs. the Estimated Trend Function (ETF) for  $\alpha = 1.5, \beta = 90$   $\sigma = 0.01$ .

Using the simulated annealing method to solve Equations (11) and (12), we obtained the estimators  $\hat{\alpha}_i, \hat{\beta}_i$  and  $\hat{\sigma}_i^2$  of each trajectory, and then considered the mean values of the estimators given by  $\bar{\alpha} = \frac{1}{25} \sum_{i=1}^{25} \hat{\alpha}_i, \bar{\beta} = \frac{1}{25} \sum_{i=1}^{25} \hat{\beta}_i$  and  $\bar{\sigma}^2 = \frac{1}{25} \sum_{i=1}^{25} \hat{\sigma}_i^2$ .

The values obtained using this method are  $\bar{\alpha} = 1.511872058, \bar{\beta} = 90.002739556$  and  $\bar{\sigma}^2 = 0.000099480$ . We then calculated the mean value of the simulated paths at each time

step, namely,  $\bar{x}(t_i) = \frac{1}{m} \sum_{j=1}^m x^j(t_i)$ , where  $x^j$  is the sample path  $j$ ,  $t_i$  is the time step  $i$  and  $m$  is the number of simulated trajectories. We also obtained the estimated trend functions of the process using each method, the result are plotted in Figure 2.

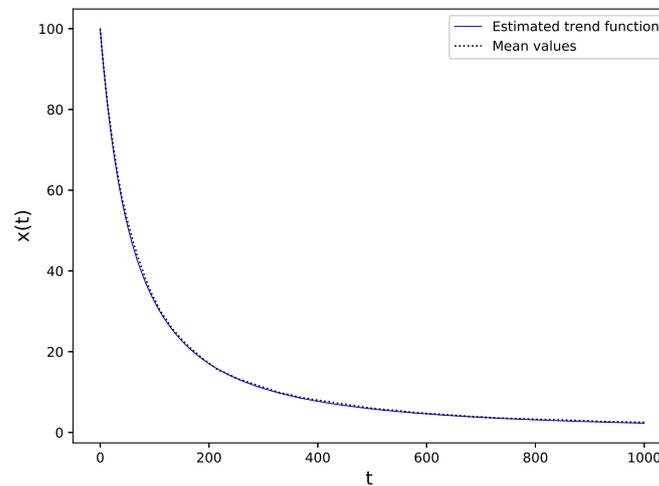


Figure 2. ETFs vs. the mean values of simulated data.

## 6. Application

### 6.1. Data Description

In this application, we examined the variable  $x(t)$  defined by the adolescent fertility rate, which is the number of births per 1000 women aged 15 to 19, these data are annual and are available on the site: <https://data.worldbank.org/>. The average value for Morocco during the period from 1979 to 2018 is 42.395655 with a minimum value of 30.6810 in 2018 and a maximum value of 92.9376 in 1979. Table 1 illustrates the observed values, as well as the ETF and Estimated Conditional Trend Function (ECTF) during this period.

In Figure 3, real data are plotted against trend functions (conditional and unconditional). The unconditional trend function provides a good estimates for the real values, the accuracy of those estimates can be more accurate if we consider the conditional trend function.

Table 2 shows the estimated data for the years from 2016 to 2018 that were not used in the modeling and the actual data.

### 6.2. Goodness of Fit of the Model

The absolute mean error in percentage (MAPE) is the average of the deviations in absolute value compared to the observed values. It is a practical indicator of comparison, it makes it possible to evaluate the forecasts obtained from the models. We denote by  $y_i$ ,  $\hat{y}_i$  and  $n$  respectively the real values, the values predicted by the model and the number of predictions, so we have:

$$MAPE = \frac{1}{n} \sum_{i=1}^n \frac{|\hat{y}_i - y_i|}{y_i} \times 100.$$

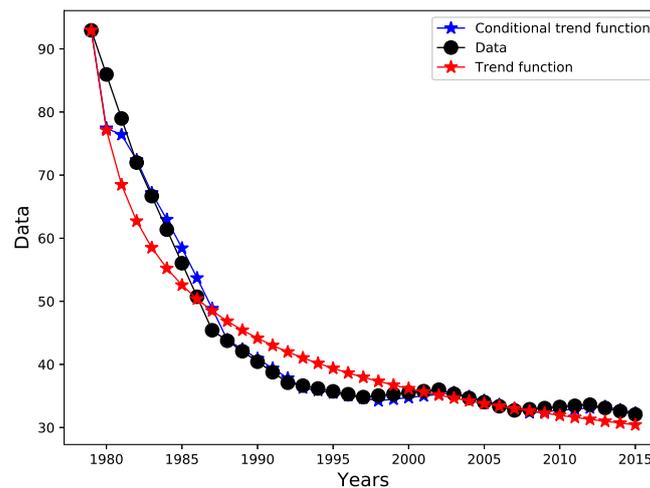
The symmetric mean percentage absolute error (SMAPE) is a measure of precision based on relative errors and is defined as follows:

$$SMAPE = \frac{100}{n} \sum_{i=1}^n \frac{|\hat{y}_i - y_i|}{(|\hat{y}_i| + |y_i|)/2}.$$

The values obtained for the MAPE and SMAPE are: 1.756265 and 1.759680, respectively. The MAPE value is less than 10, so according to Lewis [20] the values obtained by this model are “very precise”.

**Table 1.** Table of adolescent fertility rate data by year.

Year	Data	TF	CTF	Year	Data	TF	CTF
1979	92.9376	92.9376	92.9376	1998	35.0552	37.3231	34.2368
1980	85.9524	77.139	77.4169	1999	35.2894	36.7257	34.495
1981	78.9672	68.4703	76.4049	2000	35.5236	36.1646	34.751
1982	71.982	62.7125	72.3839	2001	35.7578	35.636	35.0052
1983	66.6672	58.4969	67.1766	2002	35.992	35.1368	35.2577
1984	61.3524	55.2209	62.9552	2003	35.3362	34.6643	35.5087
1985	56.0376	52.5705	58.4222	2004	34.6804	34.2161	34.8799
1986	50.7228	50.3627	53.6944	2005	34.0246	33.7901	34.2491
1987	45.408	48.4826	48.8365	2006	33.3688	33.3844	33.6166
1988	43.7456	46.8537	43.8876	2007	32.713	32.9973	32.9824
1989	42.0832	45.4225	42.4135	2008	32.894	32.6275	32.3468
1990	40.4208	44.1505	40.9082	2009	33.075	32.2737	32.5376
1991	38.7584	43.0093	39.3787	2010	33.256	31.9346	32.7279
1992	37.096	41.9769	37.8303	2011	33.437	31.6092	32.9175
1993	36.641	41.0364	36.2668	2012	33.618	31.2966	33.1067
1994	36.186	40.1744	35.873	2013	33.1012	30.996	33.2954
1995	35.731	39.3801	35.472	2014	32.5844	30.7065	32.7923
1996	35.276	38.6448	35.0651	2015	32.0676	30.4274	32.2886
1997	34.821	37.9611	34.6531				



**Figure 3.** Real data vs. trend function and conditional trend function.

**Table 2.** Forecasted values by year.

Year	Data	TF	CTF
2016	31.5508	30.1582	31.7841
2017	31.034	29.8982	31.279
2018	30.681	29.6468	30.7733

### 7. Conclusions

In this study of the stochastic Lomax diffusion process, from a theoretical point of view, we conclude that we can determine the basic probabilistic characteristics of the model and we obtain its parameter estimators. Using the maximum likelihood method in the basis of discrete sampling, we obtained a series of non linear equations which were solved by computational methods. We used the simulated annealing method to estimate the parameters of the model. Hence, a set of statistical results are obtained and show that the proposed process is enable to be applied to real data.

The Lomax model is applied to fit data for adolescent fertility rate in Morocco. The ETF presented a good description of the changing levels of the fertility rate. Furthermore, the pe-

riod from 2016 to 2018 improved good forecasts. Then, the resulting values obtained by the MAPE and SMAPE were calculated and showed good results. Taking into account these points, we deduced that the methodology applied in the study of this new model was efficient and present a high degree of accuracy.

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