Visualization abilities and complexity of reasoning in mathematically gifted students' collaborative solutions to a visualization task: a networked analysis

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CHAPTER 14

VISUALIZATION ABILITIES AND COMPLEXITY OF REASONING IN MATHEMATICALLY GIFTED STUDENTS' COLLABORATIVE SOLUTIONS TO A VISUALIZATION TASK. A NETWORKED ANALYSIS

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Abstract. *We analyze the solutions given by secondary school mathematically gifted students to a collaborative task designed to promote the development of students' competence of visualization. Each student was provided with two different orthogonal projections of a set of buildings made of cubes and other verbal data, and they were asked to place the buildings on a squared grid. We analyze students' use of visualization abilities and the complexity of their reasoning. Results show that there is a relationship between the objective of students' actions and the kind of visualization abilities used, and, also, between students' strategies of solution and the cognitive demand necessary to fulfill them. Finally, we network both analyses to gain insight and look for global*

conclusions.

Keywords: Visualization abilities, Cognitive demand, Mathematical giftedness, Networked theories, Problem solving, Secondary school, Visualization.

Introduction

Teaching mathematics is more effective when it includes diagrams, pictures, drawings, etc. visually representing concepts and relationships. To take advantage of this teaching methodology, students should develop visualization abilities and know effective ways of using visualization as part of their mathematical reasoning. The use of visualization in mathematics classrooms is considered an important object of research in mathematics education (Battista, 2007; Presmeg, 2006; Rivera, 2011). A significant open research question on visualization is the need to identify aspects of classroom cultures which promote the use of visualization in mathematics (Presmeg, 2006). To answer this question, we investigated the promotion of visualization in cultures of collaboration between mathematically gifted students (m-gifted students hereafter). Collaborative learning has proved to be beneficial (Davis, Rimm & Siegle, 2014) for such students, but research on ways to deepen its effects is needed.

Solving mathematical tasks dealing with visualization requires the use of two kinds of elements (Gutiérrez, 1996): external data, mainly objects (e.g., pictures or real models of geometrical figures) and verbal information (e.g., written statements or oral information), and internal elements, mainly visual elements (e.g., mental images; Presmeg, 1986), visual thinking (to manage visual information) and mathematical reasoning. Two visualization processes—

interpretation of figural information (IFI) and visual processing (VP) (Bishop, 1983)—, and several visualization abilities (Del Grande, 1990) control the intrapersonal communication between external and internal data and the interpersonal communication between different subjects. We designed a workshop aimed at promoting the development of m-gifted students' use of visualization abilities and collaborative learning. To capitalize on the benefits of collaborative learning for m-gifted students, the workshop encouraged the interpersonal communication to increase the use of visualization abilities by providing each student with only a part of the data for the tasks (Figure 14.1), so they needed to share information, and they had to verbally communicate visual information efficiently to solve the problem.

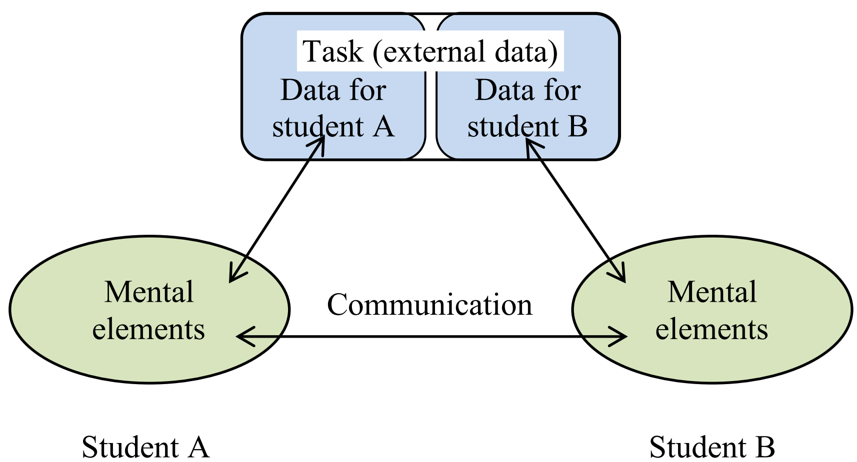


Figure 14.1. Components of the virtual workshop.

In this chapter, we present a networked analysis (Bikner-Ahsbahs & Prediger, 2010) of three pairs of m-gifted students' use of visualization abilities and the complexity of their reasoning while solving the problems. This introduction presents an overview of recent research on the constructs that conform the theoretical background of our research, describes the problems posed in the workshop, and states the research objectives.

Visualization in Mathematics Education

Researchers in psychology, mathematics and mathematics education possess diverse interpretations of terms such as visualization, visual reasoning, spatial ability, and so on. Gutiérrez (1996) presented a model integrating partial results from diverse areas, that characterize the different visualization components and that is relevant for mathematics education research (Presmeg, 2006).

Among the different definitions of visualization pertinent to mathematics education found in the literature, we highlight those comprising the types of images, processes and abilities necessary to produce, analyze, transform and communicate visual information related to objects, models and geometric concepts (Arcavi, 2003; Gutiérrez, 1996). Visualization consists of four main elements (Gutiérrez, 1996, p. 10): *Mental images* are "any kind of cognitive representation of a mathematical concept or property by means of visual or spatial elements". *External representations* are "any kind of verbal or graphical representation of concepts or properties including pictures, drawings, diagrams, etc. that helps to create or transform mental images and to do visual reasoning". A *process of visualization* is "a mental or physical action where mental images are involved". *Visualization abilities* are stable capacities of the subject which are necessary for effective learning of geometry (Bishop, 1980). In general, different visualization abilities have to be mastered "to perform the necessary processes with specific mental images for a given problem" when solving mathematical tasks (Gutiérrez, 1996, p. 10). Del Grande (1990) compiled several visualization abilities with great relevance for the development of mathematics students.

Several authors have emphasized the importance of visualization in mathematization (Arcavi, 2003; Clements & Battista, 1992) and problem solving (Ozdemir, Ayvaz-Reis & Karadag, 2012), but results of research do not show a unified position on the relationship between visualization and mathematical giftedness (Lean & Clements, 1981; Ryu, Chong & Song, 2007; Van Garderen, 2006), although other recent research has shown significant evidence of the relationship between visual perception and mathematical ability (Ramírez, 2012; Rivera, 2011).

The Complexity of Mathematical Reasoning

The tasks that teachers pose to their pupils are an important element to promote m-gifted students' learning of mathematics. There are different criteria to assess their suitability for students. A relevant criterion is the cognitive complexity of their solutions. Felmer, Pehkonen, and Kilpatrick (2016) argued that it is necessary to pose cognitively demanding tasks to make students engage in higher order thinking and improve the quality of their learning of mathematics. We agree with this criterion, since the problem solving experiment we present here was aimed to make students struggle to solve an unusual challenging task.

The *cognitive demand* of a task is "the kind and level of thinking required of students in order to successfully engage with and solve the task" (Stein, Smith, Henningsen & Silver, 2009, p. 1). Smith and Stein (1998) elaborated the *Levels of Cognitive Demand*, which organize mathematical tasks in four levels (*memorization*, *procedures without connections*, *procedures with connections*, and *doing mathematics*) depending on the cognitive effort necessary for students to solve them. This model has been acknowledged as a useful tool for teachers to promote students' higher order thinking (NCTM, 2014; Schoenfeld, 2014). We present this model in detail in Section 14.2.

The levels of cognitive demand have been used mainly to train teachers in identifying the levels of the tasks they select for their classes and maintaining their intended level during the classes (Smith & Stein, 1998). All studies we have read assigned levels of cognitive demand to tasks by analyzing their statement and the solution considered as correct by teachers. This procedure does not acknowledge that most mathematics tasks may be solved correctly in several ways, requiring from students different degrees of cognitive effort. Furthermore, there are not studies about classification of tasks that attend to the needs of m-gifted students. To overcome these issues, we have adapted the levels of cognitive demand in an innovative way to the characteristics of the visualization tasks, to analyze students' outcomes during the solution of problems (Benedicto, Gutiérrez, & Jaime, 2017). On the other hand, the characteristics of the levels, as presented in Smith and Stein (1998), are generic and a bit ambiguous, not sufficiently precise to be applied to the visualization tasks nor to the m-gifted students' answers we have analyzed, so we have also particularized the definitions of the levels of cognitive demand to the specific context of spatial visualization and the type of tasks we deal with in this chapter. This way of using the levels has proved to be a reliable framework to identify tasks adequate to students with diverse mathematical capabilities, in particular to m-gifted students (Benedicto, Acosta, Gutiérrez, Hoyos, & Jaime, 2015).

Networking Theories in Mathematics Education

In mathematics education research, several theories live together to contribute, from different approaches, to provide complementary analysis or solutions to a specific mathematics education issue. Researchers usually adopt one theoretical framework to carry out their research, but there is a growing interest in establishing links between different theories, to take advantage of the most useful components of each one by making interwoven analyses of data. Bikner-Ahsbahs and Prediger (2010) considered that:

> ... *networking strategies* are those connecting strategies that respect on the one hand the pluralism and/or modularity of autonomous theoretical approaches but are on the other hand concerned with reducing the unconnected multiplicity of theories and theoretical approaches in the scientific discipline. (p. 492, italics added)

There are different ways of networking theoretical approaches depending on the objectives aimed and the strategies used for finding connections (Bikner-Ahsbahs and Prediger, 2014). We are interested in the networking strategy of *combining*, since we have combined the theories of visualization abilities and the levels of cognitive demand to analyze the outcomes of m-gifted students solving some visualization tasks. We do not intend to merge both theories, but to use them as complementary analytical tools to gain insight into the data of the experiment.

A Collaborative Visualization Task

The experiment that we present was based on a set of collaborative visualization tasks that were designed to be solved by a pair of students linked by videoconference. The objective of the tasks is to place a set of colored buildings on a squared grid. Buildings are made of equal interlocking cubes, with all buildings in the same/different color having the same/different height. The data are the four side orthogonal projections (north, south, east and west views hereafter) and other data like the number of buildings of each color and some restrictions in the positions of the buildings. Each student is provided with only part of the data, which is not sufficient to solve the task. Therefore, students have to gather together their data to succeed in solving the task, with the restriction that they cannot share graphical information (pictures, drawings, etc.), although they can describe it verbally.

The buildings tasks consist of two parts. The first part is an introduction for students who do not know orthogonal projections; it presents a perspective representation of a city built on a squared grid (Figure 14.2) and students are asked to make the buildings with cubes and place them on a paper grid. Then, students are guided by the teacher to compare their view of the buildings with the orthogonal projections provided (Figure 14.2).

For the second part, each student is provided with a set of interlocking colored cubes, a 2 cm. squared grid oriented with the cardinal points and a coordinate system (Figure 14.3), two views of another set of buildings, information about the number of buildings and their colors, and some restrictions to the position of the buildings (see an example in Table 14.5). Students are asked to write the coordinate numbers in the marks near the axes based on the information provided in the views. Finally, they are asked to place the buildings (made with the cubes) on the grid (Figure 14.3 shows a solution). Several variables may modify the difficulty of the tasks, such as the number of buildings in each line of the grid, buildings hidden in some views, or the number of solutions.

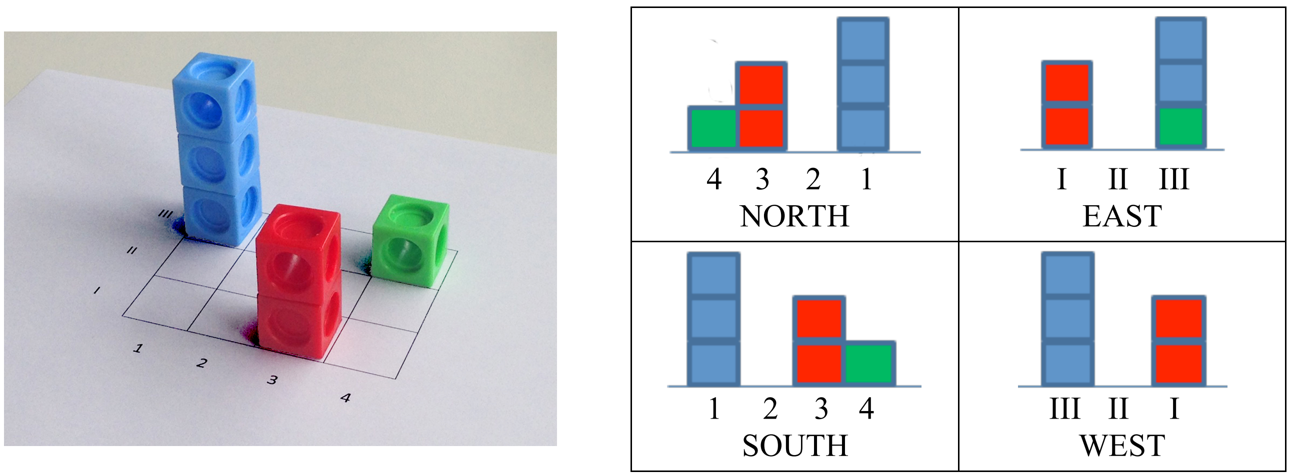


Figure 14.2. An example of the information provided in the first part of the buildings tasks.

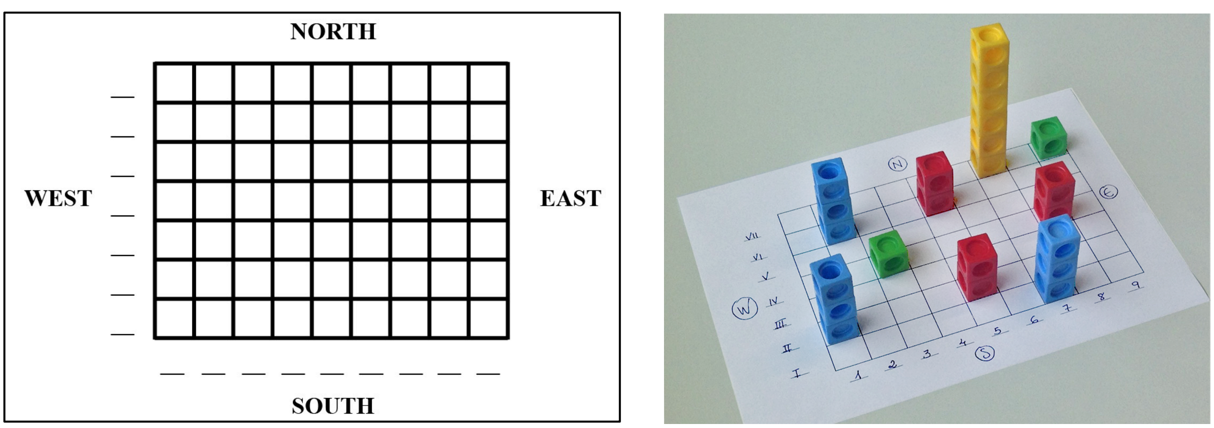


Figure 14.3. The grid provided in the second part of the buildings tasks and a solution to the task.

Research Objectives

To solve this kind of task successfully, students have to make extensive use of visualization. Furthermore, as the communication between the students is only verbal, they have to use their visualization competence to convert visual information into a meaningful verbal explanation, and vice versa. In the tasks we present, we will concentrate on looking at the visualization abilities. Our research objective is to analyze the use of visualization made by pairs of students during their interactions. Such objective is made operative by the following specific objectives:

1. Analyze the use of visualization abilities by pairs of students while solving the buildings task, looking for trends and relationships between abilities used and students' aims at that moment.
2. Analyze the variations in the cognitive effort made by pairs of students while solving the task, looking for relationships between levels of cognitive demand and kinds of actions made.
3. Relate the results from objectives 1 and 2 into a networked analysis of students' behavior, looking for relationships between use of visualization abilities and levels of cognitive demand.

Theoretical Background

We devote this section to present in detail the three theoretical components grounding the analysis of data. First, we characterize the visualization abilities as used in the context of the buildings tasks. Then, we characterize the levels of cognitive demand particularized to the specificities of the tasks. Finally, we discuss the networking of both theoretical models.

Visualization in Mathematics Education

We consider visualization as "the set of types of images, processes and skills necessary for students of geometry to produce, analyze, transform and communicate visual information related to objects, models and geometric concepts" (Gutiérrez, 1996, p. 9). We analyze the presence of visualization in students' outcomes by identifying their use of visualization abilities, which constitute one of the four main elements of visualization described in the Introduction section. We do not analyze the processes of visualization because they are ever present throughout the solution of the tasks, so they do not provide relevant information on students' behavior, and students' mental images because we did not have a reliable tool to identify them.

Del Grande (1990) characterized a set of abilities necessary for a fruitful use of visualization in mathematics. For an accurate identification of the abilities used by students in this experiment, it is necessary to make a particular characterization of each ability narrowly related to the tasks being solved. Table 14.1 presents those Del Grande's (1990) visualization abilities used by our students characterized in the specific context of the buildings tasks. Section 14.4.1 includes examples of students' answers showing the different abilities.

Table 14.1. Characterization of the visualization abilities used to solve the buildings tasks.

Abilities	Characterization of the abilities for the tasks
Figure-ground perception (FG)	- Recognize that isolated squares in the views are part of a particular building. - Discriminate one or several buildings in a view. - Recognize that different colors correspond to different buildings.
Perceptual constancy (PC)	- Recognize that the position of a building on the grid is invariant even when it is not seen in a view. - Recognize that the coordinates of a building are invariant no matter the observer's position. - Recognize that buildings that are apart on the grid continue being apart although they are seen together in a view.
Positions in space (PS)	- Identify the positions of buildings by using coordinates and/or cardinal points. - Imagine a view corresponding to another student's position. - Relate two views to determine the position of a building. - Relate several buildings on the grid by using terms like "in the same street", behind, hidden, diagonally, etc.
Spatial relationships (SR)	- Identify a relationship between the positions of two or more buildings on the grid, without depending on the observer's point of view or their coordinates, by using terms like "they touch/do not touch each other", "they are apart", etc. - Mention the heights of two buildings, e.g., to justify that one hides the other in a view.

Abilities	Characterization of the abilities for the tasks
Visual discrimination (VD)	- Compare an orthogonal projection of the buildings on the grid with the corresponding view given in the data or with another student's projection. - Compare the locations of buildings on the two students' grids. - Compare the buildings placed on a grid with verbal data.

The Model of Cognitive Demand in Visualization Tasks

The *model of Cognitive Demand* consists of four levels which allow classify tasks *and* solutions according to the cognitive effort necessary for students to solve them, and allow teachers and researchers understand the complexity of the mathematical knowledge and reasoning used by students in their solutions. Smith and Stein (1998) defined each level by a set of characteristics to be used to assign levels to tasks. We offer below a detailed characterization of the levels of cognitive demand specific for the solutions to the buildings tasks (Tables 14.2 to 14.4). This analytical framework, focused on students' use of visualization abilities, is an original contribution because, as far as we know, the model of cognitive demand has never been used to analyze visualization tasks or their answers.

We focus on the levels pertinent to our research, so we omit the level of memorization. The characteristics stated in the tables are organized in several *categories* which refer to different components of the solution to a mathematical task: the process of solution, the learning objective, the cognitive effort necessary to solve the task, the mathematical content implicit in the statement, the kind of explanations asked of students, and the systems of representation of information used by students.

The Level of Procedures Without Connections

Students' solutions in the level of *procedures without connections* consist of performing in a routine manner an algorithmic process already known, without the need of being aware of connections to mathematical contents underlying the tasks. These tasks are focused on getting correct answers but not on producing mathematical understanding of the underlying contents. The characteristics of this level are particularized for solutions to buildings tasks in Table 14.2.

Table 14.2. Characteristics of the level of procedures without connections.

Categories	**Characteristics of solutions**
Process of solution	Are based only on the observation and interpretation of simple explicit relationships between data available in student's part of the statement (e.g., a student places a building just by coordinating her two views of the building).
Objective	Place buildings correctly without needing to coordinate the four views or logical-deductive reasoning to understand the relationships between buildings (e.g., it is not necessary to relate the views of both students).
Cognitive effort	A successful solution requires limited cognitive effort. Little ambiguity exists about what needs to be done and how to do it, since the views available to a student clearly show how to place the building.
Implicit content	Students do not need to be aware of the implicit connections between the four views and other data and the buildings to be placed. They can be placed by using only the data of a student.
Explanations	Are focused only on describing the procedure used. It is not necessary to identify relationships between the other student's views and the building.
Representation of solution	Students use the manipulative representation to show the solution, but they might also use a graphical representation (e.g., by making some marks on the grid to indicate cells that can or cannot be locations of one or more buildings).

The Level of Procedures With Connections

Students' solutions in the level of *procedures with connections* consist of solving the task by following a solution process that is procedural but not routine, since it presents some ambiguity on how to carry it out, and students need to be aware of certain connections to mathematical contents underlying the tasks to decide on their way to the answer. These tasks are focused on discovering the underlying contents and gaining mathematical understanding of them. Table 14.3 shows the characteristics of solutions to buildings tasks in this level of cognitive demand.

Table 14.3. Characteristics of the level of procedures with connections.

Categories	Characteristics of solutions
Process of solution	Consist of following a sequence of steps based on implicit complex relationships. Students should consider different possibilities and make logical-deductive decisions about which data to combine and how to combine them (e.g., coordination of the four views to decide where to place a building).
Objective	Understand the underlying relationships between the different data, and make logical-deductive reasoning to select or reject cells in the grid for a building based on the information available (e.g., after having identified the buildings which are on a street, students analyze the data to reject or select cells to place the buildings).
Cognitive effort	Requires some degree of cognitive effort, to logically connect different elements of the task and deduce which procedure of solution should be followed.
Implicit content	To solve the tasks, students need to consider explicitly the relationships underlying their different elements, like the four views, buildings already placed, verbal data, etc.
Explanations	Requires explanations that include deductive justifications for the decisions made (e.g., about choosing or rejecting cells to place a building), based on combination of information from the views, buildings already placed and still not placed, etc.
Representation of solution	Students use the manipulative representation of the solution, but they might also use a graphical representation (e.g., by marking in different colors cells where building can or cannot be placed).

The Level of Doing Mathematics

Students' solutions in the level of *doing mathematics* require complex and non-algorithmic thinking, because there is not a predictable approach to solve them. Students have to understand the underlying mathematical contents and their relationships to make appropriate use of them while working through the tasks. Table 14.4 presents the characteristics of solutions to buildings tasks in the level of doing mathematics.

Table 14.4. Characteristics of the level of doing mathematics.

Categories	Characteristics of solutions
Process of solution	Students analyze the data in detail and coordinate the information. They identify buildings having more than one possible location, and get all possible solutions to the task (e.g., two buildings can be placed in different cells fitting the four views).
Objective	Explore and combine the information provided by the task and use logical-deductive reasoning to realize the existence of several feasible locations for some buildings, and get all possible solutions.
Cognitive effort	Solutions require a considerable cognitive effort, since there may be several solutions, so students have to be aware of this fact and take decisions, based on the data, about the possible locations of each building.
Implicit content	Students identify that there is more than one possible solution. They solve the tasks by relating the four views and other data, analyzing the different possibilities, and getting logical deductions.
Explanations	Justify the existence of several solutions, as well as rejected cells and chosen locations, based on the available information.
Representation of solution	As there are several solutions, students combine manipulative and graphical representation to mark cells that can or cannot be locations of buildings (e.g., students mark the cells around a placed building as not available for other buildings).

Networking Theories of Visualization and Cognitive Demand

As mentioned in section "Networking Theories in Mathematics Education", we consider the networking strategy of *combining* as the most interesting for our purposes in this chapter, because it is useful to make a networked analysis of empirical experiments like ours, by looking at the same data produced by the experiment from two theoretical perspectives.

Based on this analytical tool, we will analyze the pairs of students' solutions by looking at the use of visualization abilities and at their levels of cognitive demand, by means of the theoretical constructs presented in the two last sub-sections. Then, we will complete the networking by comparing and contrasting the results of both analyses to get global conclusions. Previous examples of this kind of networking may be found in the ZDM special issue in volume 40(2), 2008.

Methodology

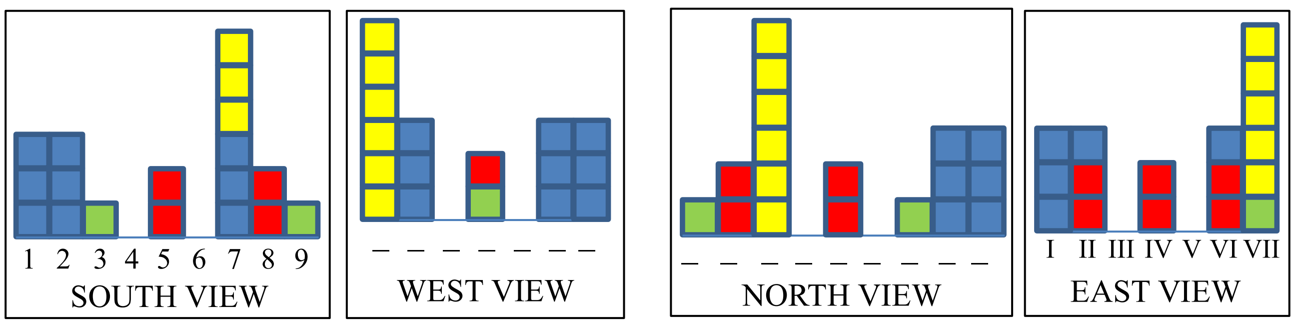
In this section, we describe the specific task posed to students, the characteristics of the m-gifted students whose solutions will be analyzed, and the two research methodologies applied to analyze the presence of visualization abilities in students' outcomes and the cognitive effort required from students to solve the task.

Description of the Experiment

The experiment consisted of posing a buildings task to several pairs of m-gifted students, the same task to all them. One student in each pair was living in Valencia (Spain) and the other one in Granada (Spain). They were linked by a group videoconference with each other and with the researchers. The researchers only intervened in students' dialog when it was evident that students had misunderstood some instruction or to answer their questions. We describe the solutions to the same task produced by three pairs of m-gifted students (we name them A1-B1, A2-B2 and A3-B3). They were aged 14-16 and studied grades 9 or 10 (lower secondary school), and were recruited from an out-of-school workshop for mathematical enrichment of m-gifted students. The pairs of students were provided with the materials mentioned in the Introduction and the information shown in Table 14.5.

Table 14.5. Information provided to the pairs of students for the second part of the buildings task.

Student A	Student B
* North-south direction: Streets are numbered with Arabic numbers 1 to 9, from west to east. * There are buildings of four colors. One is yellow, two are green and three are red. * The buildings with the same height have the same color. * The buildings are placed on the squares of the grid. The buildings cannot touch each other. * South and west orthogonal projections are:	* West-east direction: Streets are numbered with Roman numbers I to VII, from south to north. * There are 9 buildings. * Blue buildings are three floors high. * North and east orthogonal projections are:



This task has several elements of complexity: It has two solutions (Figure 14.4). There are two red buildings in street 5, which cannot be discriminated from north and south views, so it is necessary to consider other buildings and the verbal conditions (Table 14.5) to get a solution. Every view shows two blue buildings in streets 1 and 2 or I and II, which might induce students to believe that they are the same buildings. Placing the red and blue buildings is only possible by coordinating information from both students.

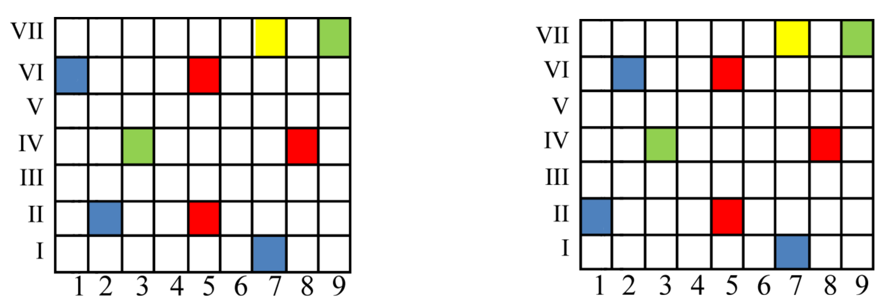


Figure 14.4. The two solutions to the buildings task analyzed.

The task may be solved by using several strategies. One is based on determining the cells in the grid where the buildings of each color may be placed by observing the views and analyzing the feasibility of the different combinations of cells. Another strategy is based on careful recursive trial and error, placing buildings on the grid and checking whether they fit or not the four views and the other data, and then making adjustments.

Analysis of Students' solutions to the Buildings Task

The main sources of information were the video recordings of computer screens and students' dialogs, which were transcribed. To analyze a pair of students' solution, we first divided the protocol into fragments corresponding to the different actions. Then, each students' outcome was analyzed twice, to identify the visualization abilities exhibited during their actions and reasoning and to characterize the levels of cognitive demand associated to their reasoning. Our objective in the networked analysis is to identify trends in and relationships between the use of abilities and levels of cognitive demand.

A global observation of students' solutions to the buildings task evidenced several *phases* in the solutions, devoted to different types of actions performed by the students characterized by their operational aims:

- *Placement* (of buildings): Students try to place buildings on the grid. These were the most frequent and time consuming actions.
- *Checking*: Students compare the buildings placed on the grid with the views and verbal data to check whether the buildings' positions and colors are correct or not.
- *Correction* (of errors): Students realize that some buildings are misplaced on the grid, and they try to identify their correct positions.
- *Request of information*: A student asks the other student to provide him with data from his views,

positions of buildings, verbal data in the statement, etc. in a quite systematic way.

- *Recapitulation*: A pair of students share the positions of the buildings placed on their grids to verify whether both grids match or not.

To identify the visualization abilities put to work by students, we looked carefully into each student's outcome, since most of them showed one or more abilities. To complete this analysis, we counted the number of appearances of each ability in every phase of the solution.

To identify the levels of cognitive demand exhibited in students' reasoning while solving the buildings task, we looked globally at each phase of solution, since the cognitive effort made by students cannot be reliably identified in a single outcome, but instead it is necessary to consider the whole students' dialogue along each phase. To complete this analysis, we put together the levels of cognitive demand of the consecutive phases of the solution.

As an example, we analyze below a short fragment of the dialog between A1 and B1. They had already placed the yellow building in (7,VII) and a green building in (9,VII) (refer to Figure 14.4 and Table 14.5). Now they began trying to place the red buildings:

B1: (12:41) *But the red* [building] *is not hidden by the yellow one. The red, in fact, is in street VI in my east view.*

A1: *In street VI?*

B1: *In the north-south street, street 6* [B1 really meant east-west street VI].

A1: *Ok. I don't see the red in street VI, I have a blue building.*

B1: *You don't see the red! Ah!*

A1: *I don't see the red.*

B1: *Of course, I only see a blue square* [in street VI from east view]*, which should be... the one behind it. Therefore, it should be in street 7 and street VI.*

We identify occurrences of the ability of positions in space when the students used coordinates to talk about possible positions of the red building (e.g., *The red, in fact, is in street VI in my east view*). We also identify the ability of figure-ground perception when students identified the blue building behind the red one (*I only see a blue square, which should be... the one behind it*).

Students showed reasoning in the level of procedures without connections, since B1 worked in identifying the position of a red building by using only his own data, taking and applying simple explicit relationships found in his two views. Even though B1 got information about A1's data and views, he did not use it to place a red building in (7,VI). B1's objective was not to gain a global understanding of the set of red buildings, but only to place one of them. B1's explanations were only descriptions of what he observed in his views.

Analysis of Students' Use of Visualization Abilities

We present the analysis of students' use of visualization abilities during the solution of the buildings task. We first present examples of use of the different visualization abilities. To complete this section, we provide an overview of the three pairs of students by comparing their ways of solution.

Visualization Abilities in Students' Answers

We present examples of students' use of different Del Grande's (1990) visualization abilities. Refer to Table 14.5 for the orthogonal views and other data provided to each student, and to Figure 14.4 for the solutions to the task.

Ability of figure-ground perception: Students put to work this ability to isolate buildings or parts of buildings from their context with different aims:

- To recognize that isolated squares in the views are part of buildings partially hidden:

B2: (16:55) *I believe that there is a three-floor building... I see it in the east view, in* [street] *VI there is a three-floor blue building.*

- To recognize that squares in different colors represent buildings of different height:

B1: (20:28) *I have green, red and yellow squares. I believe that a building cannot have a part in a color and the other part in another color.*

Ability of perceptual constancy: It is necessary when students have to recognize that:

- The position of a building on the grid does not change when it is hidden behind another building:

B1: (11:30) *In the south view, then, in street 7 north-south, there is some blue building that I cannot see* [from the north view].

A1: *Ok. Do you see the yellow building complete* [from north view]*, or is there another building that I cannot see* [from the south view]*?*

- Two buildings may be apart on the grid even when they are seen together in a view:

A3: (40:18) [referring to the blue buildings in streets I and II] *One in (II,7), for instance, and another one, for instance, in (3,I), and they would look like an entire building. Do you understand me?*

Ability of positions in space: This is the ability most frequently used by students, because the context of the problem is a grid with coordinate axes. This ability is present when:

- Students identify positions of buildings by mentioning the coordinates and/or cardinal points of cells in the grid:

A1: (37:22) *Wait. The blues* [blue buildings] *in (1,II) and (2,VI) can be placed in (1,VI) and (2,II)*.

- A student imagines buildings in a view from the other student's data. B1 described what he thought should be seen in street II from A1's west view.:

B1: (27:31) *... Furthermore,* [in my east view,] *the one* [blue building] *in street II is hidden by the red* [building] *in* [street] *II, which you see, in your west view, hidden by the blue one. In your west view, you only see one blue building in the* [street] *II, right?*

- Students coordinate two views to determine the position of a building on the grid. The two views may be from the same student or one view from each student:

B1: (30:05) *In my north view, I see two blue buildings in* [streets] *2 and 1, so this street, the north-south, must have a blue* [building]. *And, if you tell me that in your west view it hides the red* [building] *in VI,... Do you see the red building in your west view?*

- Students relate one building to others by using terms like "in the same street", behind, etc.:

A3: (33:10) [A red building] *Could be behind the tall building in (7,VII)* [the yellow building].

Ability of spatial relationships: This ability is used by students when they have to relate several buildings:

- To identify an internal relationship between buildings. A characteristic of this use of the ability is to verbalize terms like touch, diagonally, they are apart, etc.:

A3: (29:17) [A3 and B3 had placed the buildings shown in Figure 14.5] *Then, the* [blue] *one in... (7,VI) cannot be there, because it touches the yellow one. It should be in another place.*

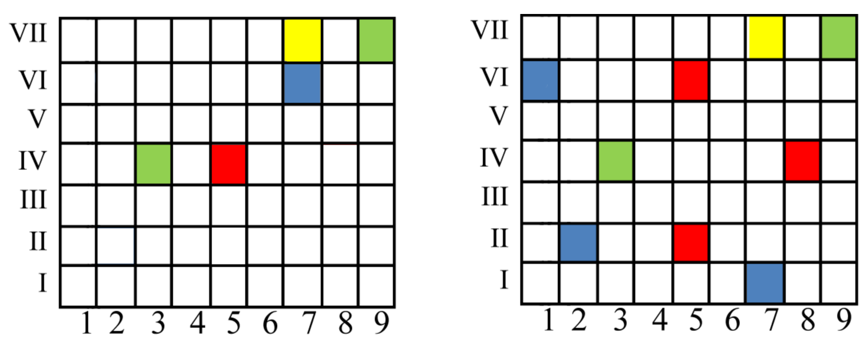


Figure 14.5. Buildings placed on the grid by A3 and B3 (29:17) (left) and a solution (right).

- To compare or relate their heights:

B1: (39:00) *... I see that the red building hides the green one* [from the east view, in street IV] *but it does not hide the blue* [buildings] *because they are taller...*

Ability of visual discrimination: This ability is necessary to compare visual pieces of information, like buildings on the grid and in some view(s), looking for similarities or differences:

- To compare a view of the buildings already placed on the grid to the same view in the data:

B2: (16:26) [B2 is inclined comparing his views with the buildings already placed on the grid] *The building in (V,3) does not fit* [the east view].

- To compare the positions of buildings on both students' grids, to check whether they match:

B1: (33:27) *I tell you, from right and top. Yellow (7,VII), green (9,VII) and (3,IV), red (5,II),...*

A1: *Yes. It's ok. Everything fits my views.*

- To check the positions of buildings and verbal data:

A1: (20:50) *Look, my clue says that there is one yellow, two greens and three reds. Three reds!?*

B1: *Yes, three reds. Because, in my east view, there are three reds in* [streets] *VI, IV and II...*

A1's surprise was because his views showed one and two red buildings (Table 14.5).

Global Analysis of Students' Answers

This section offers a synthesis of the visualization abilities showed by the students in their solutions. The charts in Figures 14.6 to 14.11 show the relationship between the phases of solution of the task and the abilities used. The ability most used by all pairs of students was positions in space, since it was necessary to share or transmit information about placement of buildings or to refer to cells in the grid, mainly by means of coordinates or cardinal points.

The pair A1-B1 used 144 times the visualization abilities along their solution of the buildings task. Figures 14.6 and 14.7 show that A1 and B1, apart from the ability of positions in space (56.9% of all appearances of visualization abilities), also used quite frequently the abilities of figure-ground perception (18.8%) and visual discrimination (15.3%). They used the other abilities too, but sporadically and without any apparent pattern of use.

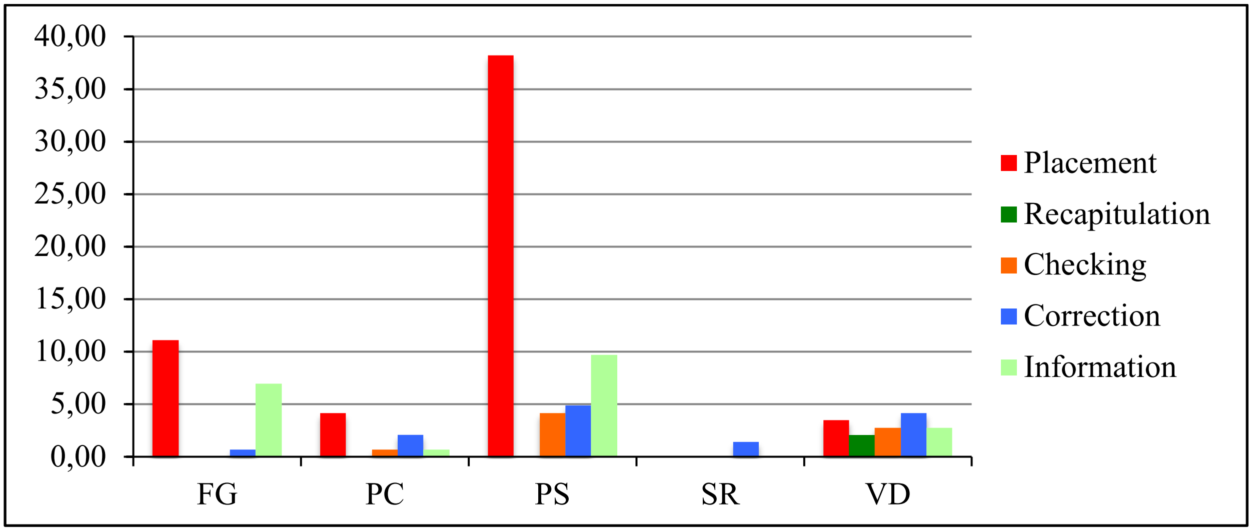


Figure 14.6. Distribution (%) of A1 and B1's use of visualization abilities between the phases of solution.

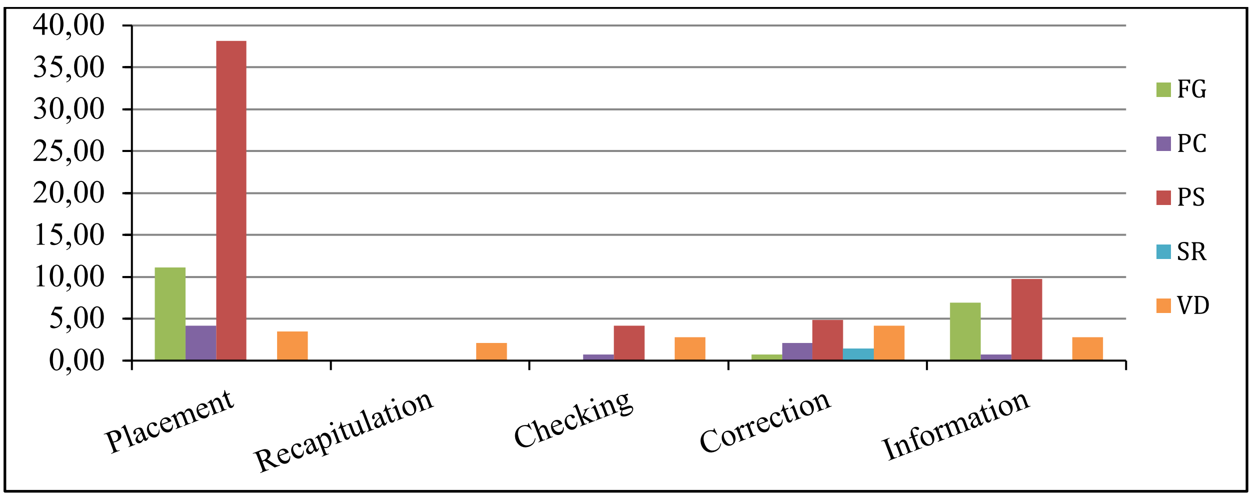


Figure 14.7. Distribution (%) of A1 and B1's use of visualization abilities over each phase of solution.

The pair A2-B2 used 145 times the visualization abilities along their solution. A2 and B2 took much more time than the other pairs to solve the buildings task, mainly because they used the abilities of recapitulation and checking many more times than the other pairs. According to Figures 14.8 and 14.9, the most used ability was positions in space (51%), but its difference of use with respect to the other abilities was smaller than for the other pairs of students. The ability of visual discrimination (31%) was also used often by A2 and B2, mainly in the phases of recapitulation and checking.

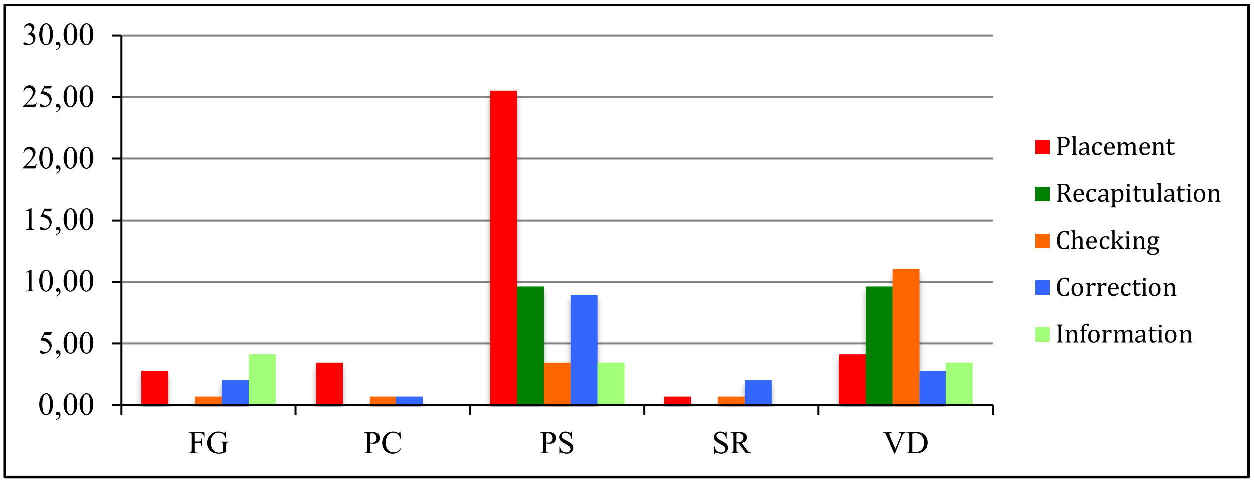


Figure 14.8. Distribution (%) of A2 and B2's use of visualization abilities between the phases of solution.

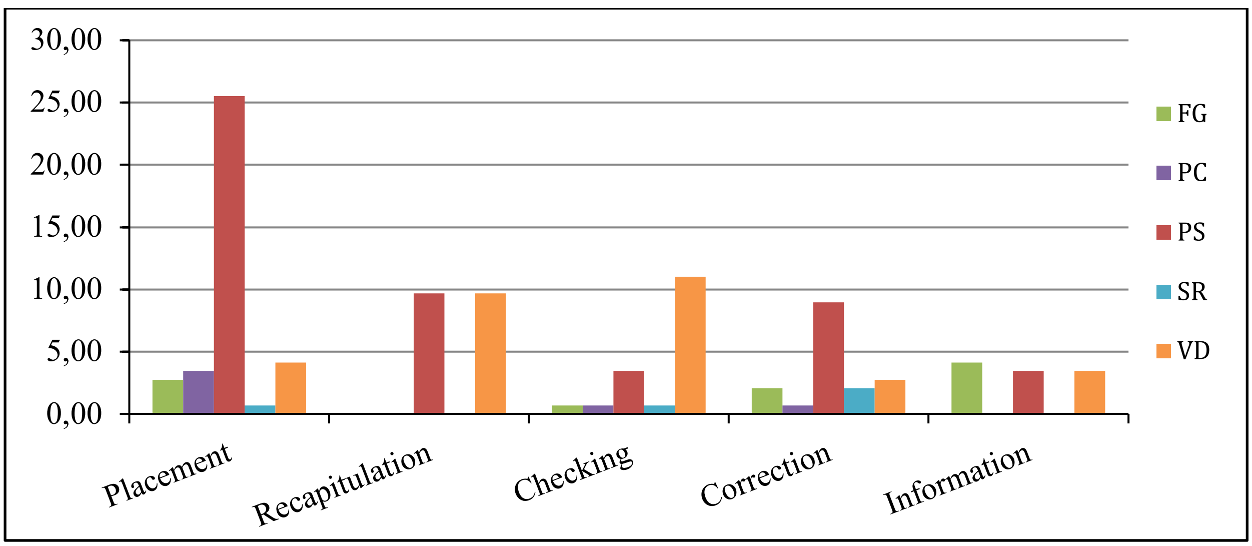


Figure 14.9. Distribution (%) of A2 and B2's use of visualization abilities over each phase of solution.

The pair A3-B3 only used 79 times the visualization abilities along their solution. A3 and B3 made a very efficient solution, devoting most time to actions of placement of buildings, and they did not use the phases of correction and information (Figures 14.10 and 14.11). As a consequence, the ability most used by A3 and B3 was positions in space (55.7%). They only used significantly figure-ground perception (19%) and spatial relationships (12.7%).

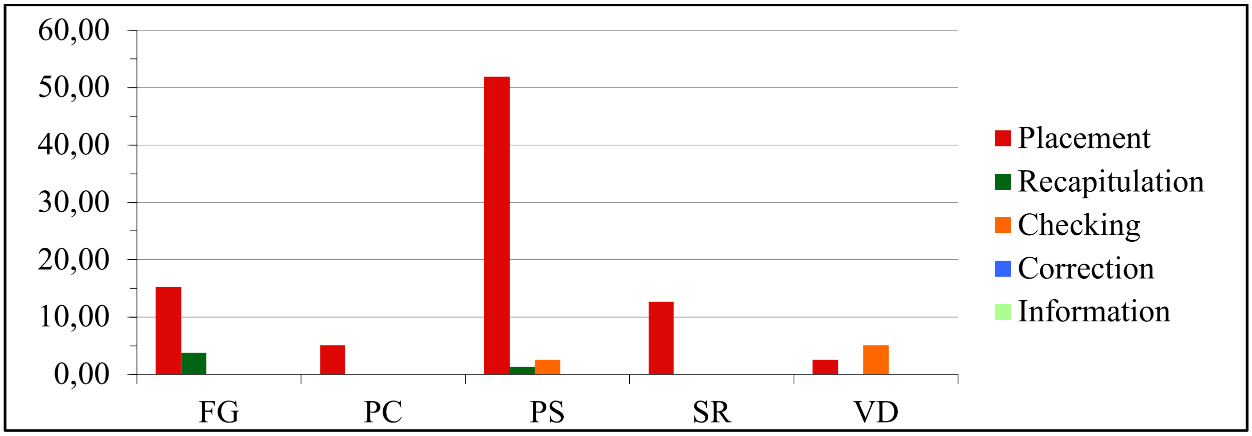


Figure 14.10. Distribution (%) of A3 and B3's use of visualization abilities between the phases of solution.

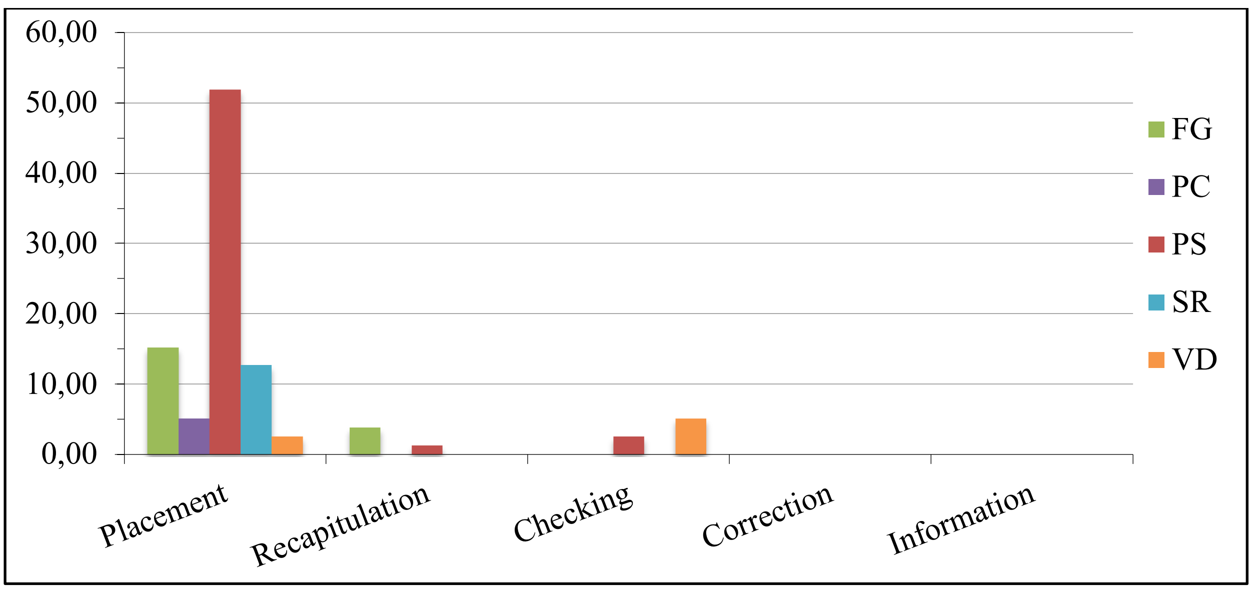


Figure 14.11. Distribution (%) of A3 and B3's use of visualization abilities over each phase of solution.

A global overview of the three pairs of students' solutions shows some patterns of behavior in their use of visualization abilities. All of them made scarce use of the abilities of perceptual constancy and spatial relationships, mainly because the task required little use of them. The students in the sample were aware of the perceptual constancy of the buildings on the grid, so they usually did not make explicit use of this ability. The spatial relationships ability was necessary mainly to identify whether two buildings touch each other or to locate several buildings in the same street (buildings that, in some views, are seen superposed or are hidden by a taller building).

Other patterns of behavior are specific to different levels of efficiency in solving the buildings task. Students A1-B1 and A3-B3 were the most efficient solving it. Some characteristics of their behavior were:

- The phase of placement of buildings occupied most time of solution (70-80% of the time).
- The ability of figure-ground perception was mostly used in the placement phase (80% of the times A3-B3 used this ability, 59.3% for A1-B1, and only 28.6% for A2-B2).

Students A2-B2 were the less efficient in solving the task. Some characteristics of these students that explain their difficulties, were:

- They used much time in the phases of solution different from placement: 36% of the time devoted to placement of buildings, 20% to checking, 18% to recapitulation, 16% to correction of errors, and 10% to requests of information.
- The ability of visual discrimination was quite used by A2-B2, mainly in the phases of recapitulation (31.1% of the times they used this ability) and checking (35.6%).

Analysis of the Cognitive Complexity of Students' Visual Reasoning

In this section, we analyze the cognitive effort required by the visual reasoning made by the m-gifted students when solving the buildings task. We have assigned levels of cognitive demand, as characterized in section "The Model of Cognitive Demand in Visualization Tasks", to students' outcomes. We first present a classification of the strategies which may be used by students to solve this kind of task, analyzing their levels of cognitive demand and presenting examples taken from our experiments. Then, we analyze the trajectory of each pair of students' levels of cognitive demand while solving the buildings task.

Classification of Strategies of Solutions According to Their Level of Cognitive Demand

The strategies used by m-gifted students to communicate information to each other and get the positions of buildings on the grid are the main source of information to understand why they expended more or less cognitive effort to solve the buildings tasks. We have identified six types of strategies that are present throughout their solutions of the buildings task and analyzed the cognitive demand required by each of these strategies.

Strategies requiring the level of Procedures Without Connections

Students do not connect the contents underlying the tasks (views, verbal data, buildings already placed, and relationships between them), since each student can manage his own information but they are not able to combine the information shared. Sharing pieces of information without combining them, or providing directions to help the other student to place a building are strategies typical of this level of cognitive demand. These strategies require only a limited cognitive effort, but they cannot be used to get the correct position of all buildings, since only the yellow and green ones can be placed by using just the data available to one student. We have identified two strategies in this level of cognitive demand:

Share and Build: Students exchange pieces of information about a building, but they do not combine them operatively so, finally, a student gets a (maybe correct) location for the building by using only his own data. In the following excerpt, A3 and B3 did not make sense of the data they shared, so they were not able to combine their views to find a correct cell for a blue building. Then, A3, considering only his own views, located a blue building in (7,VI), which is a wrong position.

A3: (24:50) *In the south view, in* [street] *7, I have a three-floor blue* [building] *hiding half yellow building. Do you see it?*

...

B3: *I see three blues in I, II and VI* [east view].

A3: *I have them in I, II and VI too* [west view]. *Do you have them together in I and II?*

B3: *Yes.*

A3: *Then, it has to be in VI, hasn't it? Because it is detached. (VI,7), right?*

B3: *Yes, maybe.*

Build and Direct: A student places a building on the grid by using only his views and then he guides the other student to place the building in the same cell. We see below that B2, after having (wrongly) deduced from his views that there are two blue buildings in (1,I) and (2,II), gave directions to A2 to help him place two blue and a red buildings.

B2: (10:53) *I've found the place of the blues* [buildings]. *It is in the south-west corner, the one with 3 floors, the first one. The second, diagonally... towards the north-east corner. I better use coordinates... The south-west corner is (1,I). There is a three-floor blue in the corner (1,I).* [A] *Three* [floor blue building] *in (2,II) and* [a] *two* [-floor red building] *in (8,II).*

Strategies requiring the level of Procedures With Connections

Students need to be aware of certain connections between contents underlying the task and be able to use them to decide on how to proceed to the answer, which requires some degree of cognitive effort. These strategies do not help students realize that there may be several correct positions for some buildings. We have identified three different strategies:

Study Positions: Students discard cells where a building cannot be located based on the observation and application of implicit relationships between views and buildings already placed on the grid. Students might not be able to get the position of a building but they identify the possible positions.

The dialog below shows an example of this strategy. A3 and B3 combined their views to get the correct conclusion that there is a red building in street 8. Then, A3 identified as possible locations for the red building all non empty streets in his west view, and B3 discarded the cells not having a red building in his east view.

A3: (21:59) [Figure 14.12 shows the buildings already placed] *Let's see, in the* [street] *8, in both north view and south, we see the red building without hiding anything.*

B3: *Yes, it is the same.*

...

A3: *Then, I do not have it in the west view. It is in* [street] *I or II or IV, or VI or VII. Do you have something in...?*

B3: *I have that it may be in II, IV or VI.*

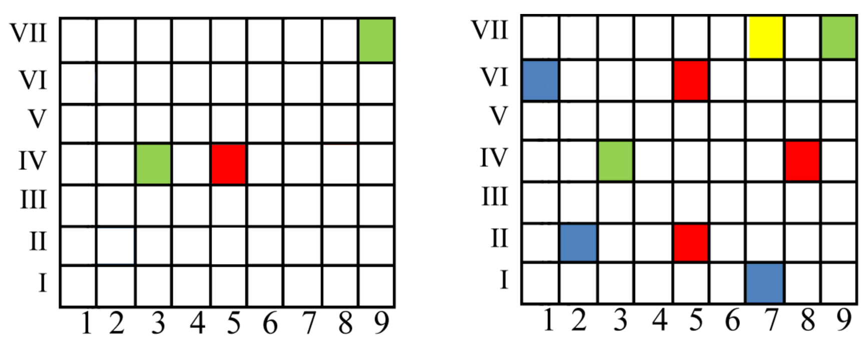


Figure 14.12. Buildings placed by A3 and B3 (21:59) (left) and a solution (right).

Combine and Build: Students exchange information and combine it operatively to correctly place a building. In the excerpt below, A1 and B1 looked for the position of a green building. Figure 14.13 shows the buildings already placed. They combined operatively data from both students and succeeded in finding the correct position of that green building.

B1: (20:51) *Ok. Have you placed a green* [building] *in 3 north-south, IV east-west?*

A1: *Yes, I have a green there or in (IV,9). I have as* [possible] *greens (IV,9) and (IV,3).*

B1: *I believe that it is in (IV,3) because, in my east view, I see the second green building you say in street (9,VII). You do not see it from your west view...*

A1: *Because the yellow hides it. Ok. So there is a green there. Ok... So, as you said, the other green is in (3,IV).*

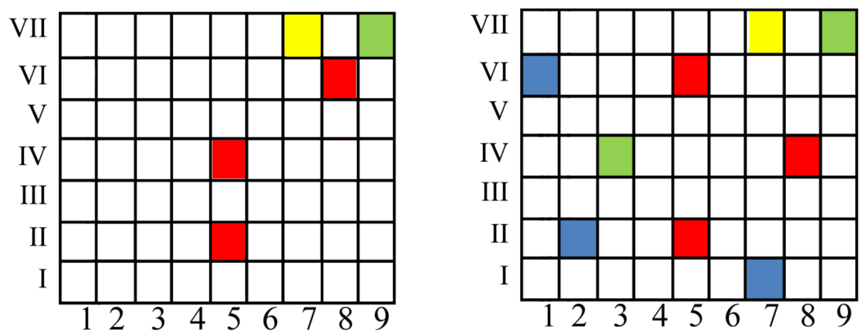


Figure 14.13. Buildings placed by A1 and B1 (20:51) (left) and a solution (right).

Not All Possible: Students do not note that two blue buildings may be placed correctly in another cell. In the following excerpt, B1 had already placed some buildings (Figure 14.14). To try placing the blue buildings, the students shared information from all the views and decided to place a blue building in (2,VI) without realizing that the blue buildings in streets 1 and 2 could be correctly placed also in another position.

B1: (30:10) *As, in my north view, I see two blue buildings in* [streets] *2 and 1,* [then] *in street 2 north-south there must be a blue* [building]. *If you tell me that, in your west view,* [the blue building] *hides the red* [building] *in* [street] *VI,... Then, it* [the blue one] *would be in 2 north-south, VI west. Or in 1... In your west view, do you see the red building in* [street] *VI?*

A1: *No.*

B1: *Ok. I also have a red building in* [street] *VI in my west view and behind it* [I see] *a blue* *square. Then, there is a blue building hiding the red one* [from the east view].

A1: *But, in the north or south views, the* [red building] *which is in* [street] *VI east-west is in* [street] *1 or 2 north-south?*

B1: *It is in* [street] *2, because in my north view I see two blue buildings in* [streets] *2 and 1. Then, it* [the blue building] *has to be in* [street] *2, because in the 1 we had already placed one* [blue building] *which is the one hiding the red* [building] *in your west view.*

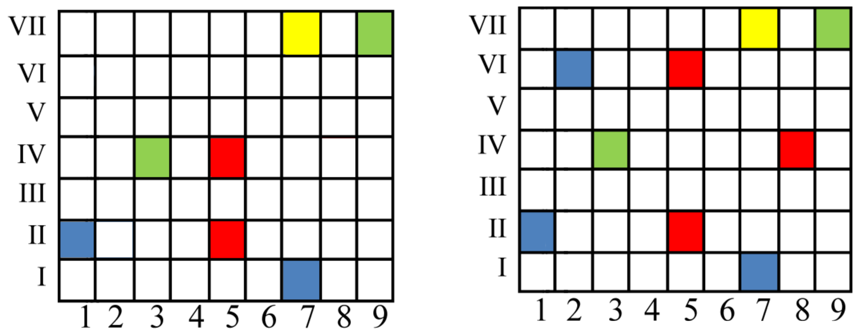


Figure 14.14. Buildings placed by B1 (30:10) (left) and a solution (right).

Strategies requiring the level of Doing Mathematics

These strategies require a complex non-algorithmic thinking, and considerable cognitive effort, to explore the implicit relationships between the data available and make appropriate operational use of them, allowing students deduce that some buildings may be correctly placed in several cells and find all possible solutions (two blue buildings in the task we are analyzing). We have identified one strategy in this level:

All Possible: Students A3 and B3 had already correctly placed all buildings except the blue ones in streets 1 and 2 (Figure 14.15). They explored all the possible positions of those blue buildings to find out the two solutions (Figure 14.4).

A3: (44:27) *I don't know whether it* [a blue building] *is in (VI,1) or (VI,2).*

B3: *I see.*

A3: *If one is in (VI,2), then the other has to be in (II,1). And, if one is in (VI,1), then the other has to be in (II,2). It wouldn't matter, I think. Because there are not more buildings, are they?*

B3: *No, the nine are there* [placed in the grid].

A3: *Then, they could be in both cells.*

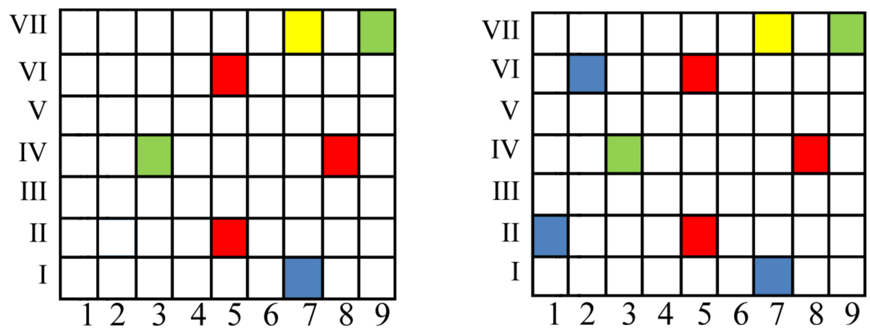


Figure 14.15. Buildings placed by A3 and B3 (44:27) (left) and a solution (right).

Trajectories of Cognitive Demand of Students' Solutions

In previous pages we have presented, exemplified and analyzed the cognitive demand required by the different strategies used by the three pairs of students to solve the buildings task. To summarize our analysis of the complexity of those students' visualization reasoning, we present three graphs showing the trajectory of each pair of students' cognitive demand during the solution of the task (Figures 14.16 to 14.18); the horizontal axis represents the strategies used in the consecutive phases of the solution and the color of the marks corresponds to the buildings students were dealing with. We have analyzed only the phases of placement and correction of errors, since these are the only phases where students' actions might end up placing buildings on the grid.

A1 and B1 started (Figure 14.16) placing the yellow building without needing to combine their information, so requiring a cognitive effort in the level of procedures without connections. Next, they tried independently (each student using only his views) to place the green buildings, in the same level of cognitive demand. They did not succeed, so they shifted to a combine-and-build strategy, requiring from them a higher level of cognitive effort to correctly place the green buildings. A1 and B1 made several partially successful attempts to place the red buildings without combining their pieces of information operatively, which required from them a reduced cognitive demand. When they used the strategy of combine-and-build, they were able to get the correct locations of all red buildings. Finally, A1 and B1 worked the same way to place the blue buildings, but they did not realize the existence of more than one possible solution (a not-all strategy).

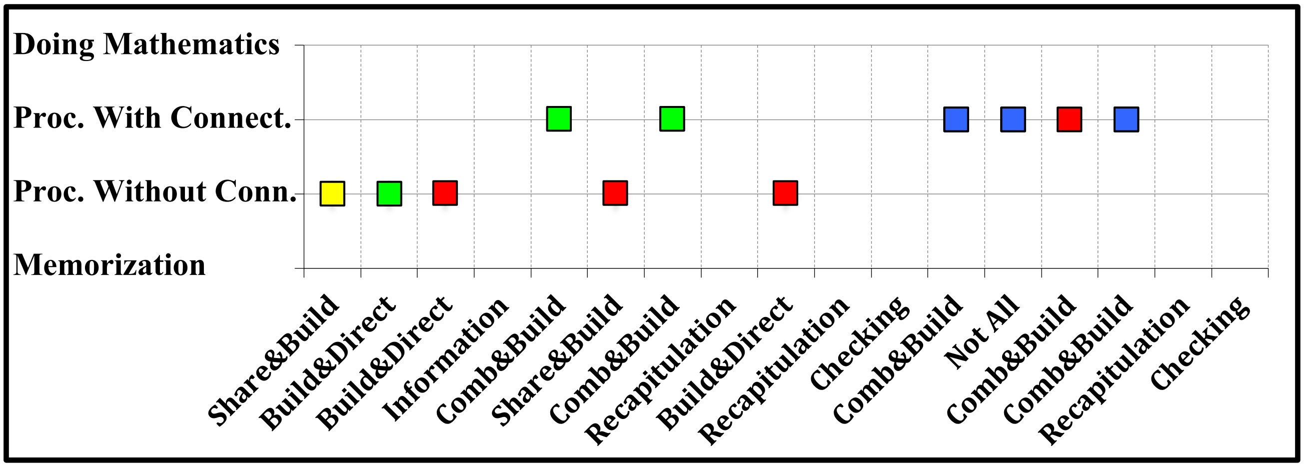


Figure 14.16. Levels of cognitive demand of A1 and B1's strategies over the phases of solution.

The graph in Figure 14.17 shows that A2 and B2 solved the task in 23 phases and only 13 of them included location of some building. During the first part of their solution, students' level of cognitive demand was procedures without connections, which allowed them to correctly place the green buildings but not the red and blue ones. When A2 and B2 began doing real collaborative work, moving to the strategy combine-and-build, they were able to place correctly the red and blue buildings, although they did not realize the two possible solutions of blue buildings.

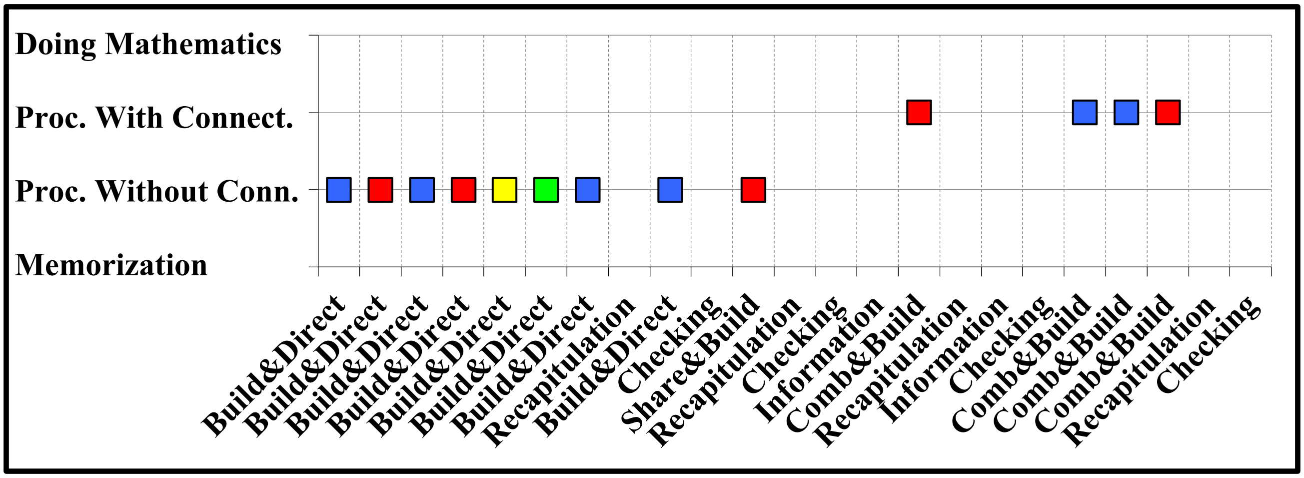


Figure 14.17. Levels of cognitive demand of A2 and B2's strategies over the phases of solution.

A3 and B3 were the most efficient students. Figure 14.18 shows that 12 out of the 15 phases of their solution were devoted to place buildings. The students worked mostly collaboratively, except during the first two phases of the solution: A3 started placing green and red buildings based only on his views. Next, A3 and B3 combined information from their views, by using the strategy of combine-and-build, and they succeeded in placing the second green building and the red buildings. When they first tried to place the blue buildings, they had some difficulties because they could not combine their pieces of information operatively. When A3 and B3 succeeded in combining operatively their information, by means of strategies combine-and-build, they correctly placed all the buildings and even identified the two possible solutions, showing a cognitive effort in the level of doing mathematics.

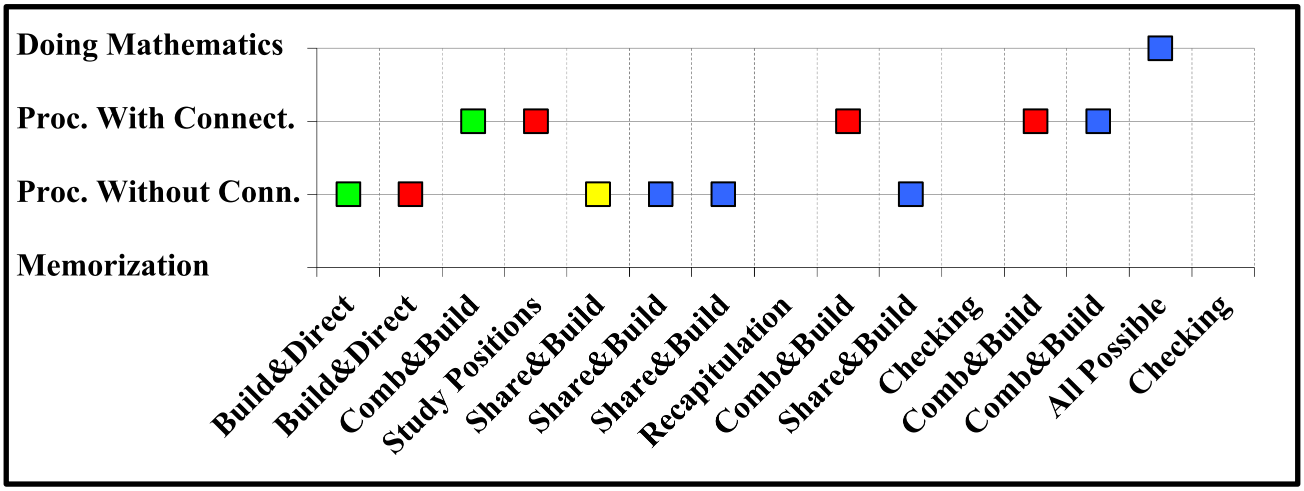


Figure 14.18. Levels of cognitive demand of A3 and B3's strategies over the phases of solution.

Networked Analysis of Students' Visualization Behavior

We have presented in sections "Analysis of Students' Use of Visualization Abilities" and "Analysis of the Cognitive Complexity of Students' Visual Reasoning" two parallel analyses of three pairs of students' solutions to a visualization task, taking into consideration the use of visualization abilities and the levels of cognitive demand posed to students by the task and their strategies of solution. Those analyses focus on different aspects of students' solutions, so some relationships and concordance between them should be expected, although it has never been explored. In this section, we make an interwoven analysis trying to relate both points of view. Our sample is only three pairs of students, so we do not pretend to get any generalizable conclusion, and we have found that students' behaviors to be quite different. A3-B3 were the most efficient and successful solvers of the buildings task, since they needed the least number of phases (12) to solve it and found the two solutions. A1-B1 were also very efficient solving the task, needing a few more phases (16) than A3-B3 but they only found one solution. In contrast, A2-B2 had more difficulties, needed almost twice as many phases (23) as A3-B3 and they required more help.

The aim of this networked analysis is to explore a possible relationship between the use of the visualization abilities and students' levels of higher cognitive demand (procedures with connections and doing mathematics). To do it, we have focused on the phases of placement of buildings and correction of errors, since these are the only phases in which the levels of cognitive demand can be evaluated. Table 14.6 presents a synthesis of the quantitative data (absolute and percentage) describing the solutions. For instance, the pair A1-B1 used 55 times the ability of positions in space in the phases of placement, which represents 38.2% of the 144 times they used any visualization ability throughout their solution. And A1-B1 showed high levels of cognitive demand in 28 out of the 55 times they used the ability of positions in space in the phases of placement (50.9% of them).

Table 14.6. Use of visualization abilities related to levels of cognitive demand.

		Number of occurrences of each ability			
		Placement Phases		Correction Phases	
Abilities	Students	Total[1]	With high cognitive demand[2]	Total[1]	With high cognitive demand[2]
Positions in space	A1-B1	55 (38.2%)	28 (50.9%)	7 (4.9%)	0 (0.0%)
	A2-B2	37 (25.5%)	21 (56.8%)	13 (9.0%)	12 (92.3%)
	A3-B3	41 (51.9%)	29 (70.7%)	0 (0.0%)	0 (0.0%)
Visual discrimination	A1-B1	5 (3.5%)	2 (40.0%)	6 (4.2%)	6 (100%)
	A2-B2	6 (4.1%)	4 (66.7%)	4 (2.8%)	2 (50.0%)
	A3-B3	2 (2.5%)	0 (0.0%)	0 (0.0%)	0 (0.0%)

[1] Percentages with respect to the total number of uses of visualization abilities along the solution.
[2] Percentages with respect to the number of occurrences of the ability in the phases.

The ability of positions in space is the only one that has been consistently and extensively used by students throughout all phases during their solutions; this result is reasonable given the characteristics of the buildings tasks. The ability of visual discrimination is the other ability having a significant presence in students' outcomes, but less consistently than the ability of positions in space. Respect to the use of the higher levels of cognitive demand, the data from our experiments do not show any clear trend or relationship between the visualization abilities used by students and the higher levels of cognitive demand required from them to solve the buildings task. We could only raise a relationship between the use of higher levels of cognitive demand and the ability of positions in space.

Conclusions

Mathematically gifted students need to be posed challenging problems and tasks that help them progress in the learning of mathematical content and the development of their mathematical capabilities, in particular their competence with mathematical visualization. In this chapter, we have presented a kind of challenging task, the buildings task, that is useful to improve students' visualization abilities while demanding from them a high level of cognitive activity.

The objective of this research was to analyze students' solutions to a buildings task i) to identify their use of visualization abilities and ii) to evaluate the level of cognitive demand used by them to solve the task successfully. We have adopted a networking position to combine both analyses to gain a deeper knowledge of students' activity. Each pair of students solved the task in a different way, which allowed us to get some conclusions that, due to the small sample, we do not claim are generalizable.

A buildings task may be designed to have several solutions. To find all them, students have to work collaboratively, communicate efficiently, use visualization abilities, and reach the highest level of cognitive demand. When our students did not succeed in sharing and combining operatively information, they were unable to correctly place some buildings. The use of the visualization abilities was more necessary when the solution to the task required from students higher levels of cognitive demand.

All pairs of students progressed in learning to work collaboratively and using more demanding reasoning, to manage their visualization abilities, and to improve their communication with each other. Our analysis shows that m-gifted students can understand and learn quickly new, more efficient strategies of solution.

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