LATTICES AND MANIFOLDS OF CLASSES OF FLAT TORI

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Abstract
The topological and differentiable structures of some moduli spaces constructed from flat Riemannian tori (concretely, $H\backslash(Gl(2,\mathbb{R})^+/Sl(2,\mathbb{Z}))$, $H=O^+(2,\mathbb{R})$, $CO^+(2,\mathbb{R})$, $O(2,\mathbb{R})$, $CO(2,\mathbb{R})$) is studied by means of a cut-and-paste procedure. In the orientation preserving cases, the quotients admit a structure of orbifold with singular points corresponding to lattices in the square and hexagonal crystal systems. In the non-orientation preserving ones, the structure is a smooth manifold with piecewise smooth boundary, where the interior points corresponds to oblique lattices, and the edge of the boundary to square and hexagonal ones.

1. Introduction

It is well-known that any flat Riemannian torus can be regarded as a planar lattice, i.e., a quotient $\mathbb{R}^2/G$ where G is a group of translations generated by two independent elements. On one hand, such lattices are well-known since (at least) Bieberbach's solution to Hilbert's 18th problem, and they are classified in classical crystallographic systems. On the other, from a geometrical viewpoint, it is natural to identify some of the tori by several distinct criteria. For example, one can consider the space of all the flat Riemannian tori up to (oriented or not) isometries or conformal transformations. These spaces are topological quotients (moduli spaces) with an obvious geometrical interpretation and interest. For example, it is known that any Riemmannian torus is globally conformal to a flat one, and then, the set of all the complex structures (as Riemann surfaces) on the torus are in natural bijective correspondence with the quotient set of all the flat Riemannian tori up to oriented conformal transformations. Nevertheless, even though it is not difficult to compute their topological structure, the differentiable structure of these quotients become subtler, because they are not smooth manifolds in a standard sense. Our aim in the present article is to study carefully both, their topological and differentiable structures, identifying the crystallographic systems where the structures become singular.
The set of all the planar lattices is the quotient manifold $Gl(2,\mathbb{R})^+/Sl(2,\mathbb{Z})$. As $Sl(2,\mathbb{Z})$ is a closed subgroup, the quotient admits a natural structure of smooth 4-manifold. We reconstruct this manifold by introducing a chart in an open dense subset; this allows to consider $Gl(2,\mathbb{R})^+/Sl(2,\mathbb{Z})$ as a subset of $\mathbb{R}^4$ with some points identified. Recall that, as the structure of the quotient is known to be smooth a priori, all these identifications must be regarded also as smooth. Then, we wonder when two such lattices represent isommetric oriented flat Riemannian tori (resp. conformal oriented flat Riemannian tori; isommetric flat Riemannian tori; conformal flat Riemannian tori). The space of all these tori is naturally a further quotient $G_H=H\backslash(Gl(2,\mathbb{R})^+/Sl(2,\mathbb{Z}))$ where $H=O^+(2,\mathbb{R})$ (resp, $CO^+(2,\mathbb{R})$; $O(2,\mathbb{R})$; $CO(2,\mathbb{R})$); this is a particular case of *orbispace*. The topological structure of G_H is computed easily and, as all their identifications are carried out explicitly, one can make natural choices to fix if the identified points either preserve the differentiable structure or are singular. In fact, G_H becomes either an orbifold with two connected parts of singular points (in the orientation-preserving cases: $H=O^+(2,\mathbb{R}), CO^+(2,\mathbb{R})$) or a smooth manifold with a (connected) piecewise smooth boundary (cases $H=O(2,\mathbb{R}), CO(2,\mathbb{R})$). Even more, the explicit identifications allows to control the crystal systems of the singular, regular or boundary points. Our results can be summarized then as follows:

Theorem 1..1 *For each one of the groups H in the cases below, the quotient space $G_H = H\backslash(Gl(2,\mathbb{R})^+/Sl(2,\mathbb{Z}))$ is a topological n-manifold, eventually with boundary, with the following characteristics:*

- *Case $H=O^+(2,\mathbb{R})$ (set of all the flat Riemannian tori up to oriented isometries). G_H is homeomorphic to $\mathbb{R}^3$ an admits a natural structure of 3-orbifold whose singular points are distributed in two lines: one corresponds with the lattices in the square crystal system, and the other with lattices in the hexagonal one.*
- *Case $H=CO^+(2,\mathbb{R})$ (set of all the flat Riemannian tori up to oriented conformal diffeomorphisms –or, equivalently, set of all the Riemann surface structures on a torus). G_H is homeomorphic to $\mathbb{R}^2$ an admits a natural structure of 2-orbifold with two singular points: one is the class of all the lattices in the square crystal system, and the other the class of all the lattices in the hexagonal one.*
- *Case $H=O(2,\mathbb{R})$ (set of all the flat Riemannian tori up to isometries). G_H is homeomorphic to a closed semi-space of $\mathbb{R}^3$ (i.e., $\mathbb{R}^+_0\times\mathbb{R}^2$, with $\mathbb{R}^0_+=[0,\infty[$), where the non-boundary points are all the classes of lattices in the oblique crystal system. It also admits a natural structure of smooth 3-manifold with piecewise smooth boundary ∂G_H such that ∂G_H contains:*
  - *Two disjoint singular lines: one with classes of lattices in the square crystal system, and the other with classes in the hexagonal one.*
  - *Three disjoint regular planes (separated by the singular lines): one of them contain the classes of lattices in the rectangular crystal system, and the other two classes in the centered rectangular one.*
- *Case $H=CO(2,\mathbb{R})$ (set of all the flat Riemannian tori upt to conformal transformations): G_H is homeomorphic to a closed semi-plane $\mathbb{R}^+_0\times\mathbb{R}$, where the non-boundary points are all the classes of lattices in the oblique crystal system. It also admits a natural structure of smooth 2-manifold with piecewise smooth boundary ∂G_H such that ∂G_H contains:*
  - *Two singular points: one is the class of the lattices in the square crystal system, and the other in the hexagonal one.*
  - *Three disjoint regular lines (separated by the singular points): one of them contain the classes of lattices in the rectangular crystal system, and the other two classes in the centered rectangular one.*

2. A cut-and-paste construction for flat quotient tori

Let Gl$(2,\mathbb{R})$ be the group of regular matrixes 2×2. Given a basis $B=\{w_1,w_2\}$ of $\mathbb{R}^2$, the lattice generated by B is the conmutative subgroup $G(=G(w_1,w_2))=\{mw_1+nw_2 : m,n\in\mathbb{Z}\}$ of $\mathbb{R}^2$; this lattice is also generated by any two independent $v_1,v_2\in G$ which span a parallelogram of minimum area. The associated torus is the quotient set $T_G=\mathbb{R}^2/G$, which inherits the canonical flat connection and orientation of $\mathbb{R}^2$, as well as the usual Riemannian metric and its associated conformal structure. Such planar lattices are classically clasified into five crystallographic groups: oblique (S_2), rectangular (D_2^p), centered rectangular (D_2^c), square (D_4) and hexagonal (D_6), see fig.1.

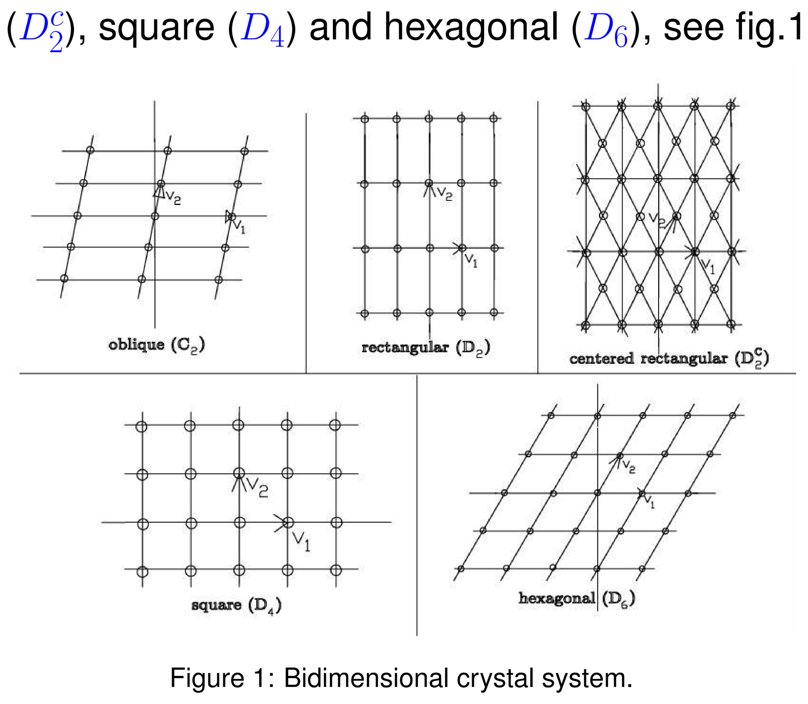

Figure 1: Bidimensional crystal system.

2..1 Canonical representatives for each lattice

Among the possible generators of the lattice G, a pair of representatives (v_1,v_2) of each lattice will be chosen in the following concrete way.
First, take v_1' such that $\|v_1'\|$=Min$\{\|x\|: x\in G\}$, v_1' lies in the closed upper semiplane and its angle θ_1' with the x-axis is the smallest possible one. Then, choose v_2' analogously, such that $\|v_2'\|$=Min$\{\|x\|: x\in G\setminus\{mv_1' : m\in\mathbb{Z}\}\}$, v_2' lies in the closed upper semiplane with smallest angle θ_2' with the x-axis.
Finally, let v_1 be the one of the v_i''s with smaller angle with the x-axis and v_2 the one with bigger angle.
Let θ be the angle between v_1 and v_2: $\theta\in]0,\pi$-$\theta_1[$.
For each $v_1\equiv(r_1,\theta_1)$, vector v_2 is characterizied by and lie in one of the following regions R_{r_1,θ_1}:

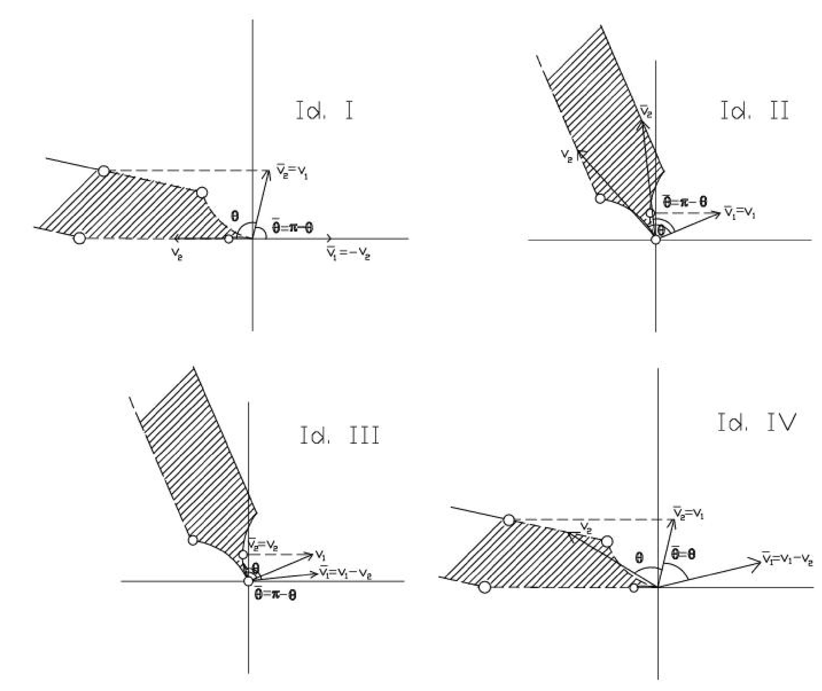

Figure 2: Identifications in R_{r_1,θ_1}.

Proposition 2..2 *The map between the set of all the flat torus $Gl^+(2,\mathbb{R})/Sl(2,\mathbb{Z})$ and the set $X=\{(r_1,\theta_1,r_2,\theta_2) : r_1\in\mathbb{R}^+$, $\theta_1\in[0,\frac{2\pi}{3}[\quad and \quad (r_2,\theta_2)\in R_{r_1,\theta_1}\}$ which assigns to each torus T_G the value of $(r_1,\theta_1,r_2,\theta_2)$ corresponding to the canonical representatives of the lattice G, is a point bijection.*

2..2 Manifold structure: identifications.

Up to now, the bijection between the quotient set $Gl^+(2,\mathbb{R})/Sl(2,\mathbb{Z})$ and X has been defined at a point-set level. In order to obtain a homeomorphism, a topology will be defined in X by enlarging it with the points in the boundary (endowed with the natural topology), and identifying suitable boundary points. The explicit identifications are:

I If $\theta_2=\pi$,: $(r_1,\theta_1,r_2,\pi)\sim(r_2,0,r_1,\theta_1)$.

II If $r_2=\frac{-r_1}{2cos\theta}$: $(r_1,\theta_1,\frac{-r_1}{2cos\theta},\theta_1+\theta)\sim(r_1,\theta_1,\frac{-r_1}{2cos\theta},\theta_1+\pi-\theta)$.

III For $r_2=2r_1\cos\theta$: $(r_1,\theta_1,2r_1\cos\theta,\theta_1+\theta)\sim(r_1,\theta_1+2\theta-\pi,2r_1\cos\theta,\theta_1+\theta)$.

IV For $r_2=\frac{r_1}{2\cos\theta}$: $(r_1,\theta_1,\frac{r_1}{2\cos\theta},\theta_1+\theta)\sim(\frac{r_1}{2\cos\theta},\theta_1-\theta,r_1,\theta_1)$

Let $\hat{X}$ be the quotient set of $\bar{X}$ by the relation of equivalence $\sim$, endowed with the quotient topology. Any class has one and only one element of X.
Then, the map $F : Gl^+(2,\mathbb{R})/Sl(2,\mathbb{Z}) \longrightarrow \hat{X}$ which maps each lattice to the class of its canonical representatives, is a homeomorphism, and allows to find the not smooth points of Th. 1.1.

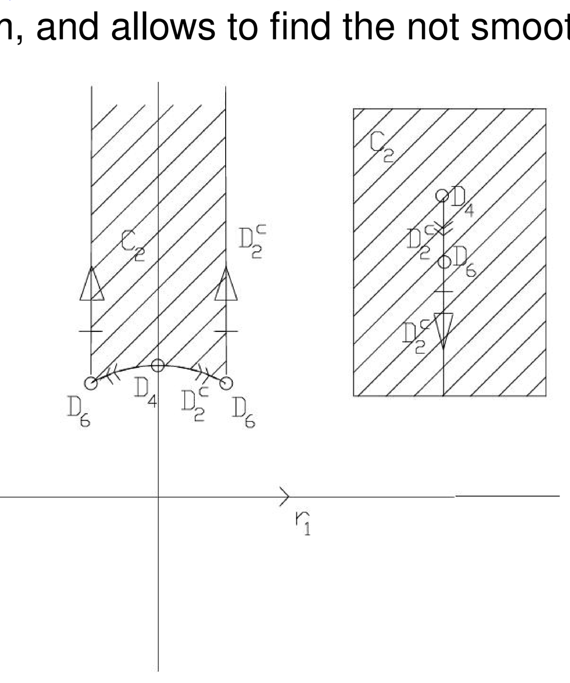
Figure 3: Case $H=CO^+(2,\mathbb{R})$ (if $r_1=1$ the set is $CO^+(2,\mathbb{R})$).

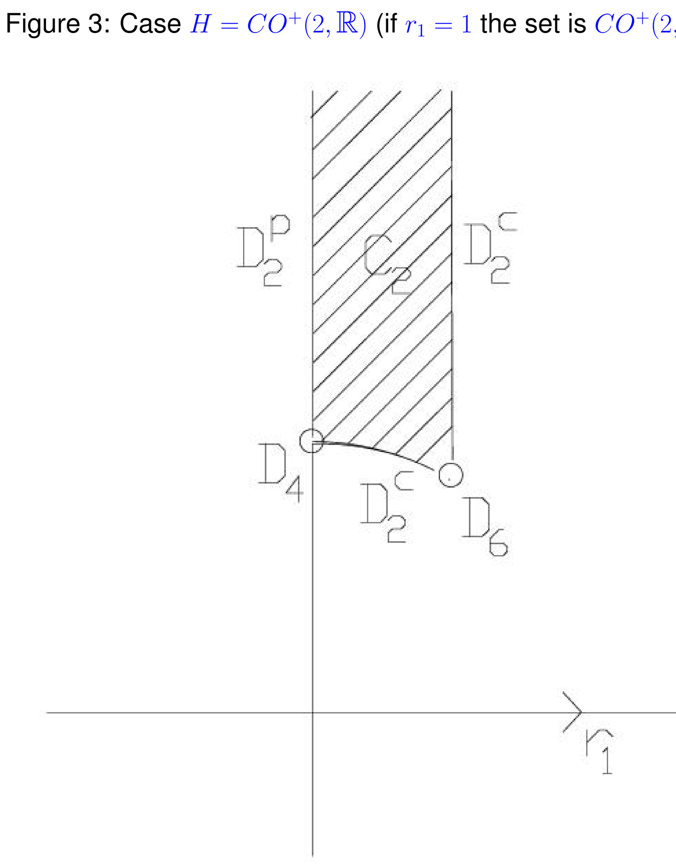
Figure 4: Cases $O(2,\mathbb{R})$ (if $r_1=1$ the set is $CO(2,\mathbb{R})$).

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