Linear Maps Which are Anti-derivable at Zero

Doha Adel Abulhamil[1] · Fatmah B. Jamjoom[1] · Antonio M. Peralta[2]

Abstract

Let $T : A \rightarrow X$ be a bounded linear operator, where A is a C*-algebra, and X denotes an essential Banach A-bimodule. We prove that the following statements are equivalent:

(a) T is anti-derivable at zero (i.e., $ab = 0$ in A implies $T(b)a + bT(a) = 0$);
(b) There exist an anti-derivation $d : A \rightarrow X^{**}$ and an element $\xi \in X^{**}$ satisfying $\xi a = a\xi$, $\xi[a, b] = 0$, $T(ab) = bT(a) + T(b)a - b\xi a$, and $T(a) = d(a) + \xi a$, for all $a, b \in A$.

We also prove a similar equivalence when X is replaced with A^{**}. This provides a complete characterization of those bounded linear maps from A into X or into A^{**} which are anti-derivable at zero. We also present a complete characterization of those continuous linear operators which are *-anti-derivable at zero.

✉ Antonio M. Peralta
   aperalta@ugr.es

   Doha Adel Abulhamil
   dabulhamil0001@stu.kau.edu.sa

   Fatmah B. Jamjoom
   fjamjoom@kau.edu.sa

[1] Present Address: Mathematics Departments, College of Sciences, King Abdulaziz University, P.O. Box 9039, Jeddah 21413, Saudi Arabia

[2] Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Granada, 18071 Granada, Spain

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1 Introduction

Let us begin this note by formulating a typical problem in recent studies about pre-servers. Suppose X is a Banach A-bimodule over a complex Banach algebra A. A derivation from A to X is a linear mapping $D : A \to X$ satisfying the following algebraic identity:

$$D(ab) = D(a)b + aD(b), \quad \forall (a, b) \in A^2. \tag{1.1}$$

A derivation D is called *inner* if there exists $x_0 \in X$ such that $D(a) = \delta_{x_0}(a) = [a, x_0] = ax_0 - x_0 a$ for all $a \in A$.

A typical challenge on preservers can be posed in the following terms:

Problem 1 Suppose $T : A \to X$ is a linear map satisfying (1.1) only on a proper subset $\mathfrak{D} \subset A^2$. Is T a derivation?

There is no need to comment that the role of the set $\mathfrak{D}$ is the real core of the question. A typical example is provided by the set $\mathfrak{D}_z := \{(a, b) \in A^2 : ab = z\}$, where z is a fixed point in A. A linear map $T : A \to X$ is said to be a *derivation at a point $z \in A$* if the identity (1.1) holds for every $(a, b) \in \mathfrak{D}_z$. In the literature, a linear map which is a derivation at a point z is also called *derivable at z*.

Let us point out that there exist linear maps which are derivable at zero, but they are not derivations (for example, the identity mapping on a complex Banach algebra is a derivation at zero, but it is not a derivation).

If $T : A \to B$ is a linear mapping from A into another Banach algebra satisfying $T(ab) = T(a)T(b)$ for all $(a, b) \in \mathfrak{D}_z$, we say that T is a *homomorphism at the point z*. Linear maps which are Jordan (*-)derivations, or generalized (*-)derivations, or triple derivations, or (Jordan *)-homomorphisms at a point can be defined in similar terms. We understand that term "*-" is only employed when the involved structures are equipped with an involution.

Let us simply observe that a linear map T between Banach algebras is a homo-morphism at zero if and only if it preserves zero products (i.e., $ab = 0$ implies $T(a)T(b) = 0$). We find in this way a natural link with the results on zero products preservers (see, for example, [1,2,8,10,28,29,32,33,47–51] for additional details and results). Burgos, Cabello-Sánchez and the third author of this note explore in [6] those linear maps between C*-algebras which are *-homomorphisms at certain points of the domain, for example, at the unit element or at zero. We refer to [12,22,25,52–58] and [60] for additional related results.

According to the standard terminology (cf. [1,4,7,9,23,30]), we shall say that a linear operator G from a Banach algebra A into a Banach A-bimodule X is a *generalized derivation* if there exists $\xi \in X^{**}$ satisfying

$$G(ab) = G(a)b + aG(b) - a\xi b \ (a, b \in A).$$

Every derivation is a generalized derivation; however, there exist generalized deriva-tions which are not derivations. This notion is very useful when characterizing

(generalized) derivations in terms of annihilation of certain products of orthogonal elements (see, for example, Theorem 2.11 in [4, §2]). The just quoted reference [4] contains an illustrative survey on local, 2-local and generalized derivations.

Let us revisit some recent achievements on maps derivable at certain points. For example, every continuous linear map δ on a von Neumann algebra is a generalized derivation whenever it is derivable at zero. If we additionally assume $\delta(1) = 0$, we can conclude that δ is a derivation (see [25, Theorem 4]). Furthermore, for an infinite-dimensional Hilbert space H, a linear map $\delta : B(H) \to B(H)$ which is a generalized Jordan derivation at zero, or at 1, is a generalized derivation, even if δ is not assumed to be a priori continuous (cf. [24]). Zhu et al. prove in [59] a significant result showing that, for any Hilbert space H, a linear map $\delta : B(H) \to B(H)$ is a derivation if and only if it is a derivation at a nonzero point in $B(H)$ (see [34] for another related result).

Ghahramani and Pan [16] and Fadaee and Ghahramani [13] have recently considered certain variants of Problem 1 in their studies of continuous linear operators from a C*-algebra A into a Banach A-bimodule X behaving like derivations or anti-derivations at elements in a certain subset of A^2 determined by orthogonality conditions. Let us detail the problem.

Problem 2 Let $T : A \to X$ be a continuous linear operator which is anti-derivable at zero, i.e.,

$$T(ab) = T(b)a + bT(a) \text{ for all } (a, b) \in \mathfrak{D}_0. \tag{1.2}$$

Is T an anti-derivation or expressible in terms of an anti-derivation?

Clearly, a mapping $\mathcal{D} : A \to X$ is called an *anti-derivation* if the identity (1.2) holds for every $(a, b) \in A^2$. If A is a C*-algebra, a *-derivation (respectively, a *-anti-derivation) from A into itself, or into A^{**}, is a derivation (respectively, an anti-derivation) $d : A \to A$ satisfying $d(a^*) = d(a)^*$ for all $a \in A$.

Concerning Problem 1, Fadaee and Ghahramani prove in [13, Theorem 3.1] that for a continuous linear map $T : A \to A^{**}$, where A is a C*-algebra, the following statements hold:

(a) T is derivable at zero if and only if there is a continuous derivation $d : A \to A^{**}$ and an element $\eta \in \mathcal{Z}(A^{**})$ (the center of A^{**}) such that $T(a) = d(a) + \eta a$ for all $a \in A$;

(b) T is r-*-derivable at zero (that is, $ab^* = 0 \Rightarrow aT(b)^* + T(a)b^* = 0$) if and only if there is a continuous *-derivation $d : A \to A^{**}$ and an element $\eta \in A^{**}$ such that $T(a) = d(a) + \eta a$ for all $a \in A$ (η need not be central).

Ghahramani and Pan also considered a variant of Problem 1 in [16] in the context of (complex Banach) algebras which are zero product determined. We recall that an algebra A is called *zero product determined* if for every linear space Y and every bilinear map $V : A \times A \to Y$ satisfying $V(x, y) = 0$ for every $x, y \in A$ with $xy = 0$, there exists a linear map $T : A \to Y$ such that $V(x, y) = T(xy)$ for all $x, y \in A$. Brešar showed in [5, Theorem 4.1] that every unital algebra A (algebraically) generated by its idempotents is zero product determined. Since this is the case of $B(H)$ for any infinite-dimensional complex Hilbert space H (see [39, Theorem 1]), a property which is also enjoyed by properly infinite von Neumann algebras [39, Theorem 4], Bunce–Deddens algebras, irrotational rotation algebras, simple unital AF C*-algebras with

finitely many extremal states, UHF C*-algebras, unital simple C*-algebras of real rank zero with no tracial states [35, Corollary 4.9], [38, Theorem 4.6], properly infinite C*-algebras [31, Corollary 2.2], and von Neumann algebras of type II$_1$ [18, Theorem 2.2(a)], all these algebras are zero product determined.

A *Jordan algebra* is a non-necessarily associative algebra B over a field whose multiplication, denoted by $\circ$, is commutative and satisfies the so-called *Jordan identity*:

$$(a \circ b) \circ a^2 = a \circ (b \circ a^2) \ \ (a, b \in B).$$

Every associative algebra is a Jordan algebra when equipped with the natural Jordan product given by $a \circ b = \frac{1}{2}(ab + ba)$. A Jordan derivation from B into a Jordan B-module X is a linear mapping $D : B \to X$ satisfying $D(a \circ b) = D(a) \circ b + a \circ D(b)$ for all $a, b \in B$. For the basic background on Jordan algebras, Jordan modules and Jordan derivations, the reader is referred to [20,21] and the references therein.

In what concerns Problem 1, Ghahramani and Pan proved in [16, Theorem 3.1] that for any zero product determined unital *-algebra A, and every linear mapping $T : A \to A$, the following statements hold:

(i) T is derivable at zero if and only if there is a derivation $d : A \to A$ and an element $\eta \in \mathcal{Z}(A)$ such that $T(a) = d(a) + \eta a$ for all $a \in A$;

(ii) T is r-*-derivable at zero (that is, $ab^* = 0 \Rightarrow aT(b)^* + T(a)b^* = 0$) if and only if there is a *-derivation $d : A \to A$ and an element $\eta \in A$ such that $T(a) = d(a) + \eta a$ for all $a \in A$ (η need not be central).

When considering Problem 2 and maps which are anti-derivable at zero, the available conclusions are less determinate. Concretely, assuming that A is a C*-algebra, Theorem 3.3 in [13] proves that for any continuous linear map $T : A \to A^{**}$, the following statements hold:

(i) If T is anti-derivable at zero, there is a continuous derivation $d : A \to A^{**}$ and an element $\eta \in \mathcal{Z}(A^{**})$ such that $T(a) = d(a) + \eta a$ for all $a \in A$;

(ii) If T is r-*-anti-derivable at zero (i.e., $ab^* = 0$ in $A \Rightarrow T(b)^*a + b^*T(a) = 0$), there is a continuous *-derivation $d : A \to A^{**}$ and an element $\eta \in A^{**}$ such that $T(a) = d(a) + a\eta$ for all $a \in A$ (η need not be central).

If A is a zero product determined unital *-algebra and $T : A \to A$ is a linear mapping, Theorem 3.4 in [16] proves the following statements:

(i) If T is anti-derivable at zero, there is a Jordan derivation $d : A \to A$ and an element $\eta \in \mathcal{Z}(A)$ such that $T(a) = d(a) + \eta a$ for all $a \in A$;

(ii) If T is r-*-anti-derivable at zero, there is a Jordan *-derivation $d : A \to A$ and an element $\eta \in A$ such that $T(a) = d(a) + a\eta$ for all $a \in A$ (η need not be central).

In view of the previous result, it is natural to ask whether there exists a full characterization of those (continuous) linear maps which are (*-)anti-derivable at zero in pure algebraic terms. The main aim of this note is to complete our knowledge on these classes of continuous linear maps and to fill a natural gap which has not been fully covered. Our first main conclusion is contained in Theorem 6 where it is established that for each bounded linear operator T from a C*-algebra A into an essential Banach A-bimodule X, the following statements are equivalent:

(a) T is anti-derivable at zero;

(b) There exist an anti-derivation $d : A \to X^{**}$ and an element $\xi \in X^{**}$ satisfying $\xi a = a\xi$, $\xi[a, b] = 0$, $T(ab) = bT(a) + T(b)a - b\xi a$, and $T(a) = d(a) + \xi a$, for all $a, b \in A$.

It is further shown that if A is unital, or if X is a dual Banach A-bimodule, statement (b) above can be replaced with

(b′) There exist an anti-derivation $d : A \to X$ and an element $\xi \in X$ satisfying $\xi a = a\xi$, $\xi[a, b] = 0$, $T(ab) = bT(a) + T(b)a - b\xi a$, and $T(a) = d(a) + \xi a$, for all $a, b \in A$.

A similar conclusion holds when X is replaced with A^{**}.

In Sect. 4, we consider a C*-algebra A and an essential Banach A-bimodule equipped with an A-bimodule involution $*$ (i.e., a continuous conjugate linear mapping $x \mapsto x^*$ satisfying $(x^*)^* = x$, $(ax)^* = x^*a^*$ and $(xa)^* = a^*x^*$, for all $a \in A$, $x \in X$). We give several natural examples of bimodule involutions. A Banach A-bimodule equipped with an A-bimodule involution will be called a Banach *-A-bimodule. Suppose $T : A \to X$ is a linear mapping from a C*-algebra into a Banach *-A-bimodule. We shall say that T is r-*-anti-derivable at zero (respectively, l-*-anti-derivable at zero) if $ab^* = 0$ in A implies $T(b)^*a + b^*T(a) = 0$ in X (respectively, $a^*b = 0$ in $A \Rightarrow T(b)a^* + bT(a)^* = 0$ in X).

Let $T : A \to X$ be a bounded linear operator where A is a C*-algebra and X is an essential Banach *-A-bimodule. In Theorem 9, we prove the equivalence of the following statements:

(a) T is r-*-anti-derivable at zero (i.e., $ab^* = 0$ in $A \Rightarrow T(b)^*a + b^*T(a) = 0$ in X);
(b) There exists a *-derivation $d : A \to X^{**}$ and an element $\xi \in X^{**}$ satisfying the following properties:

(i) $d([a, b]) + [a, b]\xi + \xi^*[a, b] = 0$, for all $a, b \in A$;
(ii) $T(ab) = aT(b) + T(a)b - a\xi b$, and $T(a) = d(a) + a\xi$ for all $a, b \in A$.

The conclusion in (b) can be improved if A is unital or if X is a dual Banach A-bimodule. Finally, a complete characterization of those bounded linear operators $T : A \to X$ which are l-*-anti-derivable at zero is presented in Theorem 11 (see also Corollary 12).

In Sect. 2, we take a closer look at anti-derivations from a general C*-algebra A into a Banach A-bimodule. Theorem 4 shows that these maps are in general very scarce.

2 Are there Anti-Derivations on C*-Algebras?

If one is interested on the study of linear maps from a C*-algebra A into a Banach A-bimodule which are anti-derivable at zero, a first natural step is to explore the class of anti-derivations on C*-algebra. For this purpose, we initiate our study by paying some attention to anti-derivations. An anti-derivation from an (associative) algebra A into a Banach A bimodule X is a linear mapping $d : A \to X$ satisfying $d(ab) = d(b)a + bd(a)$ for all $(a, b) \in A^2$. An example seems to be welcome. Let

us fix an element $x_0 \in X$ satisfying

$$x_0[a, b] = [a, b]x_0. \tag{2.1}$$

for all $a, b \in A$, where $[a, b] = (ab - ba)$ denotes the Lie product or commutator of a and b. The prototype of derivation from A into X is given by $\delta_{x_0} : A \rightarrow X$, $\delta_{x_0}(a) = [a, x_0]$ ($a \in A$). The assumption (2.1) implies that $abx_0 - x_0ab = bax_0 - x_0ba$ and thus

$$\delta_{x_0}(ab) = abx_0 - x_0ab = bax_0 - x_0ba = bx_0a - x_0ba + bax_0 - bx_0a$$
$$= \delta_{x_0}(b)a + b\delta_{x_0}(a),$$

for all $a, b \in A$, witnessing that δ_{x_0} is a derivation and an anti-derivation. But, does such an element x_0 exist with the additional property that $\delta_{x_0} \neq 0$? Let us observe that δ_{x_0} also satisfies the following property:

$$\delta_{x_0}([a, b]) = [[a, b], x_0] = 0, \text{ for all } a, b \in A. \tag{2.2}$$

We shall see next that the identity in (2.2) actually characterizes anti-derivations.

Lemma 3 *Let $\delta : A \rightarrow X$ be a linear mapping from an associative algebra into an A-bimodule. Then, the following statements are equivalent:*

(a) *δ is a derivation and $\delta([a, b]) = 0$, for all $a, b \in A$;*
(b) *δ is an anti-derivation and $\delta([a, b]) = 0$, for all $a, b \in A$.*

Proof The equivalence is clear by just observing that $\delta([a, b]) = 0$, for all $a, b \in A$ if and only if $\delta(ab) = \delta(ba)$, for all $a, b \in A$. □

A central result in the theory of derivations on C*-algebras was established by Ringrose who proved that every (associative) derivation from a C*-algebra A to a Banach A-bimodule X is (automatically) continuous (compare [44]). Johnson established in [26] another result to have in mind by proving that every bounded Jordan derivation from a C*-algebra A into a Banach A-bimodule X is an associative derivation. By a result due to Russo and the third author of this note, we know that every Jordan derivation from A into X is continuous [42, Corollary 17]; consequently, every Jordan derivation from A into X is a derivation.

Let $\delta : A \rightarrow X$ be an anti-derivation from a C*-algebra into a Banach A-bimodule. It is clear that $\delta(a \circ b) = \delta(a) \circ b + a \circ \delta(b)$ ($a, b \in A$), and hence, δ is a Jordan derivation. It follows from the arguments in the previous paragraph that δ is a continuous derivation. So, every anti-derivation from a C*-algebra A into a Banach A-bimodule is continuous and a derivation; therefore, a linear mapping $T : A \rightarrow X$ is an anti-derivation if and only if it is a derivation and $T([a, b]) = 0$ for all $a, b \in A$. So, the natural question is whether there exist non-trivial derivations vanishing on all commutators.

Let A be a C*-algebra. According to the usual notation, we write $\mathfrak{c}(A) := \{[a, b] : a, b \in A\}$. The available literature contains a wide list of papers conducted to determine when an element in a C*-algebra can be expressed as a finite sum of commutators (see,

for example, [14,15,19,36–38,43] and references therein). Let us remark two concrete results Fack and de la Harpe showed in [15] that in any finite von Neumann algebra M with central trace τ, an element $x \in M$ with $\tau(x) = 0$ can be expressed as a sum of ten commutators. In another remarkable result, Halpern proved that every element a in a properly infinite von Neumann algebra M can be written as the sum of two commutators (cf. [19, Theorem 3.10]).

Let X be a Banach A-bimodule, where A is a C*-algebra. In this note, we shall deal with the bidual, X^{**}, of X, and we shall regard it as a Banach A^{**}-bimodule. For this purpose, we shall refresh our knowledge on Arens extensions and Arens regularity (cf. [3]). Let $m : X \times Y \rightarrow Z$ be a bounded bilinear map where X, Y and Z are Banach spaces. According to the construction defined by R. Arens, we define $m^*(z', x)(y) := z'(m(x, y))$ $(x \in X, y \in Y, z' \in Z^*)$. We obtain in this way a bounded bilinear mapping $m^* : Z^* \times X \rightarrow Y^*$. The same method can be applied to define $m^{**} = (m^*)^*$ and $m^{***} : X^{**} \times Y^{**} \rightarrow Z^{**}$. The mapping $x'' \mapsto m^{***}(x'', y'')$ is weak* to weak* continuous whenever we fix $y'' \in Y^{**}$, and the mapping $y'' \mapsto m^{***}(x, y'')$ is weak* to weak* continuous for every $x \in X$. The previous construction can be applied to the transposed mapping $m^t : Y \times X \rightarrow Z$, $m^t(y, x) = m(x, y)$, and we define an extension $m^{t***t} : X^{**} \times Y^{**} \rightarrow Z^{**}$. Now, the mapping $x'' \mapsto m^{t***t}(x'', y)$ is weak* to weak* continuous whenever we fix $y \in Y$, and the mapping $y'' \mapsto m^{t***t}(x'', y'')$ is weak* to weak* continuous for every $x'' \in X^{**}$. It should be remarked that the mappings m^{t***t} and m^{***} need not coincide in general (cf. [3]). The mapping m is called *Arens regular* if $m^{t***t} = m^{***}$. One of the best known examples of Arens regular maps is given by the product of any C*-algebra. That is, every C*-algebra A is Arens regular and the unique Arens extension of the product of A to $A^{**} \times A^{**}$ coincides with the product of its enveloping von Neumann algebra (cf. [11, Corollary 3.2.37]).

It is worth to recall some notions. Two projections p and q in a von Neumann algebra M are called *(Murray–von Neumann) equivalent* if there is a partial isometry $e \in M$ such that $e^*e = p$ and $ee^* = q$. We write this fact as $p \sim q$. A projection p in M is said to be *finite* if there is no projection $q < p$ that is equivalent to p. A projection p in M is *infinite* if it is not finite, and *properly infinite* if $p \neq 0$ and zp is infinite whenever z is a central projection such that $zp \neq 0$ (cf. [46, Definition V.1.15]). The von Neumann algebra M is said to be finite, infinite or properly infinite according to the property of its identity [46, Definition V.1.16].

Let M be a von Neumann algebra. A (*faithful*) *center-valued trace* on M is a linear mapping τ from M onto its center $Z(M)$ satisfying:

(a) $\tau(x^*x) = \tau(xx^*) \geq 0$ for all $x \in M$;
(b) $\tau(ax) = a\tau(x)$, for all $a \in Z(M)$, $x \in M$;
(c) $\tau(1) = 1$;
(d) $\tau(x^*x) \neq 0$ for every nonzero $x \in M$.

A von Neumann algebra M is finite if and only if it admits a faithful center-valued trace (which is further weak* continuous) [46, Theorem V.2.6].

Suppose X is a Banach A-bimodule over a C*-algebra A. Let $\pi_1 : A \times X \rightarrow X$ and $\pi_2 : X \times A \rightarrow X$ stand for the corresponding module operations given by $\pi_1(a, x) = ax$ and $\pi_2(x, a) = xa$, respectively. Given $a \in A^{**}$ and $z \in X^{**}$, we

shall write $az = \pi_1^{***}(a, z)$ and $za = \pi_2^{***}(z, a)$. It is known that X^{**} is a Banach A^{**}-bimodule (and also a Banach A-bimodule) for the just defined operations ([11, Theorem 2.6.15(iii)]). An additional property of this construction tells that

$$ax = \pi_1^{***}(a, x) = w^*\text{-}\lim_\lambda w^*\text{-}\lim_\mu a_\lambda x_\mu, \text{ and}$$
$$xa = \pi_2^{***}(x, a) = w^*\text{-}\lim_\mu w^*\text{-}\lim_\lambda x_\mu a_\lambda, \tag{2.3}$$

in the weak* topology of X^{**}, whenever (a_λ) and (x_μ) are nets in A and X, respectively, such that $a_\lambda \to a \in A^{**}$ in the weak* topology of A^{**} and $x_\mu \to x \in X^{**}$ in the weak* topology of X^{**} (cf. [11, (2.6.26)]). The reader should be warned that the module operations on X^{**} need not be separately weak* continuous. This handicap produces some difficulties in our arguments.

Let $d : A \to M$ be an (anti-)derivation from a C*-algebra into a Banach A-bimodule. We have already seen that d is continuous and hence $d^{**} : A^{**} \to X^{**}$ is weak* continuous. It follows from (2.3) that d^{**} is (anti-)derivation.

It is clear from the above that in a C*-algebra where every element coincides with a finite sum of commutators, every anti-derivation is zero. In particular, every anti-derivation from a properly infinite von Neumann algebra M into a Banach M-bimodule is zero by Halpern's theorem [19, Theorem 3.10]. We can improve this conclusion in the next result.

Theorem 4 *Let $\delta : A \to X$ be an anti-derivation from a C*-algebra to a Banach A-bimodule. Then, there exists a finite central projection p_1 in A^{**} and an element $x_0 \in X^{**}$ such that*

$$\delta^{**}(x) = \delta_{x_0}(\tau(p_1 x)) = [\tau(p_1 x), x_0], \text{ for all } x \in A^{**},$$

where $\tau : p_1 A^{**} \to Z(p_1 A^{**})$ is the (faithful) center-valued trace on $p_1 A^{**}$.

Furthermore, if we also assume that $zx = xz$ for all $z \in Z(A^{**})$ and $x \in X^{**}$ (for example, when $X = A$ or A^* or A^{**}), every anti-derivation $\delta : A \to X$ is zero.

Proof We have already commented in previous paragraphs that δ is a continuous derivation with $\delta([a, b]) = 0$ for all $a, b \in A$. Furthermore, $\delta^{**} : A^{**} \to X^{**}$ is an anti-derivation too. Therefore δ^{**} is a derivation and vanishes on every commutator of A^{**}.

By [46, Theorem V.1.19 and Lemma V.1.7], the identity of the von Neumann algebra A^{**} is uniquely written as the sum of (centrally) orthogonal projections p_1 and p_2 such that p_1 is finite and p_2 is properly infinite. It follows that A^{**} decomposes as the orthogonal direct sum of $M_1 = p_1 A^{**} p_1$ and $M_2 = p_2 A^{**} p_2$, M_1 is finite and M_2 is properly infinite. By Theorem 3.10 in [19], every element a_2 in M_2 can be written as the sum of two commutators in M_2. Furthermore, let $\tau : M_1 \to Z(M_1)$ be the faithful center-valued trace of the finite von Neumann algebra M_1. Theorem 3.2 in [15] asserts that every element b_1 in M_1 with $\tau(b_1) = 0$ coincides with the sum of ten commutators in M_1. Since for each $a_1 \in M_1$, the element $b_1 = a_1 - \tau(a_1) \in M_1$ has zero trace, we deduce that $a_1 - \tau(a_1)$ writes as the sum of ten commutators in M_1.

Since commutators in A^{**} are sums of commutators in M_1 and M_2, we deduce from the properties of δ^{**} that

$$\delta^{**}(a_1 + a_2) = \delta^{**}(\tau(a_1)), \text{ for all } a_1 \in M_1, a_2 \in M_2.$$

On the other hand, X^{**} is a dual $Z(M_1)$-bimodule with respect to the restricted bimodule operations, and $\delta^{**}|_{Z(M_1)} : Z(M_1) \to X^{**}$ is a derivation and an anti-derivation. Since every commutative C*-algebra $\mathcal{C}$ is amenable (i.e., for every Banach $\mathcal{C}$-bimodule Y, every derivation from $\mathcal{C}$ into Y^* is inner (see, for example, [11, Theorem 5.6.2(i)]), there exists $x_0 \in X^{**}$ such that $\delta^{**}(z) = [z, x_0]$ for all $z \in Z(M_1)$. This finishes the proof of the first statement.

Let us deal with the last statement. Suppose that $zx = xz$ for all $z \in Z(A^{**})$ and $x \in X^{**}$. In this case, $\delta^{**}(z) = [z, x_0] = 0$ for all $z \in Z(M_1)$, and thus, $\delta = 0$. $\qquad\square$

Let $A = C[0, 1]$ and $X = \mathbb{C}$ equipped with the bimodule operations defined by $f \cdot \lambda := f(0)\lambda$ and $\lambda \cdot f := f(1)\lambda$. According to this structure, there exist elements $f \in C[0, 1]$ such that $f \cdot 1 \neq 1 \cdot f$. The mapping $\delta_1 : C[0, 1] \to X$ is a nonzero anti-derivation.

3 Linear Maps Anti-Derivable at Zero

Let A and B be C*-algebras. It is known that every bounded bilinear form $V : A \times B \to \mathbb{C}$ admits a unique norm preserving separately weak* continuous extension to $A^{**} \times B^{**}$ (cf. [27, Lemma 2.1]). Actually, the same conclusion also holds when A and B are JB*-triples (see [41, Lemma 1]).

Along this note, the self-adjoint part of a C*-algebra A will be denoted by A_{sa}.

Let A be a Banach algebra. If instead of requiring A to be zero product determined we only request that for every Banach space Y and every continuous bilinear form $V : A \times A \to Y$ satisfying $V(a, b) = 0$ for every $a, b \in A_{sa}$ with $ab = 0$, there exist continuous functionals $\phi, \varphi \in A^*$ such that $V(a, b) = \phi(ab) + \varphi(ba)$ for all $a, b \in A$, a celebrated theorem due to Goldstein (see [17, Theorem 1.10]) affirms that every C*-algebra satisfies this latter property. This is one of the advantages in the study of derivations on C*-algebras.

Let X be a Banach A-bimodule over a Banach algebra A. According to the usual terminology, we shall say that X is *essential* if the linear span of the set $\{axb : a, b \in A, x \in X\}$ is dense in X.

If A is a non-unital C*-algebra, 1 denotes the unit in A^{**}, and (u_λ) is a bounded approximate unit in A (cf. [40, Theorem 1.4.2]), it is known that $(u_\lambda) \to 1$ in the weak* topology of A^{**}. Furthermore, if we regard X^{**} as a Banach A^{**}-bimodule, it follows from the basic properties commented in the first section that

$$(\eta a)1 = w^*\text{-}\lim_\lambda (\eta a)u_\lambda = w^*\text{-}\lim_\lambda \eta(au_\lambda) = \|.\|\text{-}\lim_\lambda \eta(au_\lambda) = \eta a,$$

for all $a \in A$ and $\eta \in X$. Assuming that X is essential, we get $\eta 1 = \pi_2^{***}(\eta, 1) = \eta$ (and similarly $1\eta = \pi_1^{***}(1, \eta) = \eta$) for all $\eta \in X$. Actually, $\lim_\lambda \|\eta u_\lambda - \eta\| = 0 =$

$\lim_\lambda \|u_\lambda \eta - \eta\|$ for all $\eta \in X$. Let us take $\eta \in X^{**}$, and pick via Goldstine's theorem a bounded net (η_μ) in X converging to η in the weak* topology of X^{**}. Since $\pi_2^{***}(\cdot, 1)$ is weak* continuous, we have

$$\eta 1 = \pi_2^{***}(\eta, 1) = w^*\text{-}\lim_\mu \pi_2^{***}(\eta_\mu, 1) = w^*\text{-}\lim_\mu \eta_\mu 1 = w^*\text{-}\lim_\mu \eta_\mu = \eta. \quad (3.1)$$

Our next result is a modular version of [13, Lemma 2.2].

Lemma 5 *Let X be a Banach A-bimodule over a C^*-algebra A. Let ξ be an element in X satisfying the following property: $h\xi k = 0$ for every $h, k \in A_{sa}$ with $hk = 0$. Let 1 denote the unit of A^{**}. Then, the element $\eta = 1\xi 1 \in X^{**}$ satisfies $a\eta b = a\xi b$, for all $a, b \in A$, and commutes with every element in A, that is, $\eta a = a\eta$, for all $a \in A$.*

Proof Since A may be non-unital, we shall consider A^{**} and the space X^{**} as a Banach A^{**}-bimodule. Let $\eta = 1\xi 1 \in X^{**}$. By the basic properties of the A^{**}-bimodule X^{**}, we have $a\eta b = a\xi b$ for all $a, b \in A$ and $\eta 1 = 1\eta = \eta$.

Let us fix an arbitrary $\phi \in X^*$ and define the bounded bilinear form given by $V_\phi : A \times A \to \mathbb{C}$, $V_\phi(a, b) = \phi(a\eta b) = \phi(a\xi b)$. It follows from the hypothesis that $V_\phi(h, k) = 0$, for every $h, k \in A_{sa}$ with $hk = 0$, witnessing that V_ϕ is an orthogonal form in the sense of Goldstein [17]. Theorem 1.9 in [17] implies the existence of two functionals $\varphi_1, \varphi_2 \in A^*$ satisfying $V_\phi(a, b) = \varphi_1(ab) + \varphi_2(ba)$ for all $a, b \in A$. We denote by the same symbol V_ϕ the (unique) separate weak* continuous extension of V_ϕ to $A^{**} \times A^{**}$. We can therefore conclude that

$$\phi(a\eta) = V_\phi(a, 1) = \varphi_1(a) + \varphi_2(a) = V_\phi(1, a) = \phi(\eta a),$$

for all $a \in A$. The arbitrariness of $\phi \in X^*$ combined with the Hahn–Banach theorem implies that $\eta a = a\eta$, for all $a \in A$. $\qquad\square$

We can now present our characterization of those continuous linear maps on a C^*-algebra which are anti-derivable at zero.

Theorem 6 *Let $T : A \to X$ be a bounded linear operator where A is a C^*-algebra and X is an essential Banach A-bimodule. Then, the following are equivalent:*

(a) *T is anti-derivable at zero;*
(b) *There exists an anti-derivation $d : A \to X^{**}$ and an element $\xi \in X^{**}$ satisfying $\xi a = a\xi$, $\xi[a, b] = 0$, $T(ab) = bT(a) + T(b)a - b\xi a$, and $T(a) = d(a) + \xi a$, for all $a, b \in A$;*

Proof $(a) \Rightarrow (b)$ Suppose T is anti-derivable at zero. Let us pick $h_1, k, h_2 \in A_{sa}$ with $h_j k = 0$ (and thus $kh_j = 0$) for $j = 1, 2$, it follows from the hypothesis that $T(k)h_2 + kT(h_2) = 0$ and therefore

$$h_1 T(k)h_2 = h_1(T(k)h_2 + kT(h_2)) = 0. \quad (3.2)$$

This shows that the mapping $T : A \to X$ satisfies the hypotheses of [4, Theorem 2.11]; we therefore conclude from the just quoted result that $T : A \to X$ is a generalized

derivation, that is, there exists $\xi \in X^{**}$ such that

$$T(ab) = T(a)b + aT(b) - a\xi b, \quad \forall a, b \in A. \tag{3.3}$$

By replacing ξ with $1\xi 1$, we can assume that $1\xi = \xi 1 = \xi$ and (3.3) holds.

It is not hard to check from (3.3) that the mapping $d : A \to X^{**}, d(a) = T(a) - \xi a$ is a derivation satisfying $T(a) = d(a) + \xi a$ for all $a \in A$.

If we pick $h, k \in A_{sa}$ with $hk = 0$ (and thus $kh = 0$). We deduce from the hypothesis that $T(h)k + hT(k) = 0$, and by (3.3):

$$0 = T(hk) = T(h)k + hT(k) - h\xi k,$$

identities which combined give $h\xi k = 0$ (for any $h, k \in A_{sa}$ with $hk = 0$). Lemma 5 guarantees that $\xi a = a\xi$ for all $a \in A$.

We shall next show that d is an anti-derivation. Let $(Y, \odot)$ denote the opposite Banach A-bimodule X^{op}, that is, $y \odot a = ay$ and $a \odot y = ya$ for all $a \in A$, $y \in Y$. Let us pick $h_1, k, h_2 \in A_{sa}$ with $h_j k = 0$ for $j = 1, 2$. We have seen in (3.3) that $h_1 \odot T(k) \odot h_2 = h_2 T(k) h_1 = 0$. Then, the mapping $\tilde{T} : A \to Y$, $\tilde{T}(a) = T(a)$ $(a \in A)$ satisfies that $h_1 \odot \tilde{T}(k) \odot h_2 = 0$ for every $h_1, k, h_2 \in A_{sa}$ with $h_j k = 0$. We deduce from [4, Theorem 2.11] the existence of $\eta \in X^{**}$ such that $T(ab) = T(a) \odot b + a \odot T(b) - a \odot \eta \odot b$, $\forall a, b \in A$, equivalently,

$$T(ab) = bT(a) + T(b)a - b\eta a, \quad \forall a, b \in A. \tag{3.4}$$

Replacing η with $1\eta 1$, we can always assume that $\eta = \eta 1 = 1\eta$.

By mimicking the arguments above, fix $h, k \in A_{sa}$ with $hk = 0$. We deduce from the hypothesis (with $kh = 0$) that $T(h)k + hT(k) = 0$, and by (3.4)

$$0 = T(kh) = T(h)k + hT(k) - h\eta k.$$

By combining the previous two identities, we get $h\eta k = 0$ for all $h, k \in A_{sa}$ with $hk = 0$. A new application of Lemma 5 guarantees that $\eta a = a\eta$ for all $a \in A$.

Now, combining the fact that ξ and η commute with any element in A, and (3.3) and (3.4) with $a = b$, we have $\eta a^2 = \xi a^2$ for all $a \in A$. Since A is a C*-algebra, it follows that $a\eta = \eta a = \xi a = a\xi$ for all $a \in A$. Therefore, there is no loss of generality in assuming $\xi = \eta$ in (3.3) and (3.4).

Now, let us apply (3.4) and (3.3) to deduce the following identities:

$$T(ab) = T(b)a + bT(a) - b\xi a, \text{ and } T(ba) = T(b)a + bT(a) - b\xi a,$$

for all $a, b \in A$. Therefore,

$$T([a, b]) = T(ab - ba) = 0 \text{ for all } a, b \in A. \tag{3.5}$$

Let us analyze the identity (3.3). Let (u_λ) be an approximate unit in A. Since the identity

$$T(u_\lambda b) = T(u_\lambda)b + u_\lambda T(b) - u_\lambda \xi b$$

holds for every λ, T^{**} is weak* continuous (and hence $T^{**}(u_\lambda) \to T^{**}(1)$ in the weak* topology), $u_\lambda T(b) \to T(b)$ in norm because X is essential, the product of A^{**} is separately weak* continuous [45, Theorem 1.7.8], $\pi_1^{***}(\cdot, \xi b)$ is weak* continuous (and thus $u_\lambda \xi b = \pi_1^{***}(u_\lambda, \xi b) \to \pi_1^{***}(1, \xi b) = 1\xi b = \xi b$ in the weak* topology), we conclude that

$$T(b) = T^{**}(1)b + T(b) - 1\xi b, \text{ or equivalently, } T^{**}(1)b = 1\xi b = \xi b, \quad (3.6)$$

for all $b \in A$.

Let us recall that a continuous bilinear mapping $V : A \times A \to X$ *preserves zero products* if

$$ab = 0 \text{ in } A \Rightarrow V(a, b) = 0.$$

By [1, Example 1.3(2.), Theorem 2.11 and Definition 2.2], every continuous bilinear mapping V preserving zero products satisfies $V(ab, c) = V(a, bc)$ for all $a, b, c \in A$. By hypothesis, the mapping $V(a, b) := T(b)a + bT(a)$ is continuous and preserves zero products and therefore

$$T(c)ab + cT(ab) = V(ab, c) = V(a, bc) = T(bc)a + bcT(a),$$

for all $a, b, c \in A$. If in the above equality we replace c with u_λ, where (u_λ) is an approximate unit in A, and we take weak* limits, we get

$$T^{**}(1)ab + T(ab) = T(b)a + bT(a), \text{ for all } a, b \in A.$$

Since $\xi ab = T^{**}(1)ab$ (cf. (3.6)) and $T(ab) = T(ba) = d(ba) + \xi ba$, it follows that

$$\xi ab + d(ba) + \xi ba = d(b)a + \xi ba + bd(a) + \xi ba = d(ba) + 2\xi ba,$$

witnessing that $\xi[a, b] = 0$.

Therefore, by (3.5)

$$d([a, b]) = T([a, b]) - \xi[a, b] = 0 \text{ for all } a, b \in A.$$

Lemma 3 proves that d is an anti-derivation.

$(b) \Rightarrow (a)$ Suppose there exist an anti-derivation $d : A \to X^{**}$ and an element $\xi \in X^{**}$ satisfying the stated properties. Let us take $a, b \in A$ with $ab = 0$. It follows from the assumptions that

$$T(b)a + bT(a) = (d(b) + \xi b)a + b(d(a) + \xi a) = d(ab) + 2\xi ba = 0 + 2\xi ab = 0.$$

$\square$

Remark 7 If in Theorem 6 the C*-algebra A is unital or if X is a dual Banach A-bimodule, statement (b) can be replaced with the following:

(b′) There exist an anti-derivation $d : A \rightarrow X$ and an element $\xi \in X$ satisfying $\xi a = a\xi$, $\xi[a, b] = 0$, $T(ab) = bT(a) + T(b)a - b\xi a$, and $T(a) = d(a) + \xi a$, for all $a, b \in A$.

A closer look at the proof of Theorem 6 shows that the desired statement will follow as soon as we prove that the element ξ lies in X. If A is unital, this is clear because $T(1) = d(1) + \xi = \xi \in X$. If X is a dual Banach space, we can repeat the arguments in the proof of [1, Proposition 4.3 or Theorem 4.6].

Every C*-algebra A is an essential A-bimodule because it admits a bounded approximate unit (see [40, Theorem 1.4.2]). The second dual, A^{**}, of A is an A-bimodule with respect to the natural product. In general, A^{**} need not be an essential A-bimodule, consider, for example, $A = c_0$ and $A^{**} = \ell_\infty$. However, if A is unital, A^{**} is an essential A-bimodule. Despite that A^{**} is not in general an essential A-bimodule, A is weak* dense in A^{**} by Goldstine's theorem, and it is known that A admits a bounded approximate unit (see [40, Theorem 1.4.2]) which converges to the unit of A^{**} in the weak* topology. Applying these special properties, the proofs of [4, Lemma 2.10, Theorem 2.11] remain valid to characterize when a bounded linear operator $T : A \rightarrow A^{**}$ is a generalized derivation. Therefore, the proof of Theorem 6 can be combined with Theorem 4 to get the following result:

Theorem 8 *Let $T : A \rightarrow A^{**}$ be a bounded linear operator where A is a C*-algebra. Then, the following are equivalent:*

(a) *T is anti-derivable at zero;*
(b) *There exists $\xi \in A^{**}$ satisfying $\xi a = a\xi$, $\xi[a, b] = 0$, and $T(a) = \xi a$, for all $a, b \in A$.*

The preceding theorem can be regarded as a generalization of [16, Corollary 3.8(i)].

4 Linear Maps *-Anti-Derivable at Zero

In this section, we shall deal with continuous linear maps which are r-*-anti-derivable at zero. We shall first recall the basic theory on bimodules equipped with an involution. Let X be a Banach A-bimodule over a C*-algebra A. By an *A-bimodule involution* on X, we mean a continuous conjugate linear mapping $X \rightarrow X$, $x \mapsto x^*$, satisfying $(x^*)^* = x$, $(ax)^* = x^*a^*$, and $(xa)^* = a^*x^*$, for all $a \in A$, $x \in X$. The natural involutions on A and on A^{**} are A-bimodule involutions when A and A^{**} are regarded as Banach A-bimodules. Another typical example can be given in the following way: For each functional $\varphi \in A^*$ and $a \in A$, the functionals $a\varphi$, $\varphi a \in A^*$ are defined by $(a\varphi)(b) = \varphi(ba)$, and $(\varphi a)(b) = \varphi(ab)$, for all $b \in A$, respectively. These operations

define a structure of Banach A-bimodule on A^*. Furthermore, for each $\varphi \in A^*$ we define $\varphi^* \in A^*$ by $\varphi^*(b) := \overline{\varphi(b^*)}$ ($\forall b \in A$). It is easy to check that $(a\varphi)^* = \varphi^*a^*$ and $(\varphi a)^* = a^*\varphi^*$ for all $a \in A$, $\varphi \in A^*$. Therefore, $\varphi \mapsto \varphi^*$ defines an A-bimodule involution on A^*.

Suppose $x \mapsto x^*$ is an A-bimodule involution on a Banach A-bimodule X. We shall regard X^* as a Banach A-bimodule with module operations given by $(a\phi)(x) = \phi(xa)$, and $(\phi a)(x) = \phi(ax)$, for all $a \in A$, $x \in X$ and $\phi \in X^*$. We shall consider the natural involutions on X and X^{**} naturally induced by the A-bimodule involution of X, defined by $\phi^*(x) := \overline{\phi(x^*)}$ ($\forall \phi \in X^*, x \in X$) and $z^*(\phi) := \overline{z(\phi^*)}$ ($\forall \phi \in X^*, z \in X^{**}$). Clearly, the involution $z \mapsto z^*$ is weak* continuous on X^{**}. Let $a \in A$, $z \in X^{**}$, and let $(x_\mu) \subset X$ a bounded net converging to z in the weak* topology of X^{**}. By the properties of the module operation on X^{**} (see page 7), we have

$$(az)^* = \pi_1^{***}(a, z)^* = w^* - \lim_\mu \pi_1(a, x_\mu)^* = w^* - \lim_\mu \pi_2(x_\mu^*, a^*)$$
$$= \pi_2^{***}(z^*, a^*) = z^*a^*,$$

and

$$(za)^* = \pi_2^{***}(z, a)^* = w^* - \lim_\mu \pi_2(x_\mu, a)^* = w^* - \lim_\mu \pi_1(a^*, x_\mu^*)$$
$$= \pi_1^{***}(a^*, z^*) = a^*z^*.$$

A Banach A-bimodule equipped with an A-bimodule involution will be called a *Banach *-A-bimodule*. Along this section, X will stand for a Banach *-A-bimodule over a C*-algebra A. A linear mapping $T : A \to X$ will be called *r-*-anti-derivable at zero* (respectively, *l-*-anti-derivable at zero*) if $ab^* = 0$ in A implies $T(b)^*a + b^*T(a) = 0$ in X (respectively, $a^*b = 0$ in $A \Rightarrow T(b)a^* + bT(a)^* = 0$ in X). It is easy to see that T is r-*-anti-derivable at zero if and only if the mapping $S : A \to X$, $S(a) := T(a^*)^*$ ($\forall a \in A$) is l-*-anti-derivable at zero.

We can now state our main conclusion for continuous linear maps which are r-*-anti-derivable at zero.

Theorem 9 *Let $T : A \to X$ be a bounded linear operator where A is a C*-algebra and X is an essential Banach *-A-bimodule. Then, the following statements are equivalent:*

(a) *T is r-*-anti-derivable at zero (i.e., $ab^* = 0$ in $A \Rightarrow T(b)^*a + b^*T(a) = 0$ in X);*

(b) *There exists a *-derivation $d : A \to X^{**}$ and an element $\xi \in X^{**}$ satisfying the following properties:*

    (i) *$d([a, b]) + [a, b]\xi + \xi^*[a, b] = 0$, for all $a, b \in A$;*

    (ii) *$T(ab) = aT(b) + T(a)b - a\xi b$, and $T(a) = d(a) + a\xi$ for all $a, b \in A$.*

If we further assume that A is unital or X is a dual Banach A-bimodule, we can replace (b) with the following:

(b′) *There exists a *-derivation $d : A \to X$ and an element $\xi \in X$ satisfying properties (i)–(ii) above.*

Proof $(a) \Rightarrow (b)$ Suppose T is r-*-anti-derivable at zero. As in the proof of Theorem 6 we observe that given $h_1, h_2, k \in A_{sa}$ with $h_j k = 0$ for $j = 1, 2$, we have $T(h_1)^* k + h_1 T(k) = 0$, and consequently,

$$0 = (T(h_1)^* k + h_1 T(k)) h_2 = h_1 T(k) h_2. \tag{4.1}$$

Theorem 2.11 in [4] assures that T is a generalized derivation, that is, there exists an element $\xi \in X^{**}$ satisfying

$$T(ab) = T(a)b + aT(b) - a\xi b, \quad \forall a, b \in A. \tag{4.2}$$

By replacing ξ with $1\xi 1$, we can assume that $\xi = 1\xi = \xi 1$. It is routine to check that the mapping $d : A \to X^{**}$, $d(a) = T(a) - a\xi$ is a derivation and $T(a) = d(a) + a\xi$ for all $a \in A$.

The same arguments employed in the proof of Theorem 6 (3.6) prove that

$$T^{**}(1)b = 1\xi b = \xi b, \text{ for all } b \in A. \tag{4.3}$$

We consider the continuous bilinear mapping $V : A \times A \to X$ defined by $V(a, b) := T(b^*)^* a + bT(a)$. If $a(b^*)^* = ab = 0$, the hypothesis implies that $V(a, b) = 0$. Therefore, V preserves zero products. We conclude from [1, Example 1.3(2.), Theorem 2.11 and Definition 2.2] that $V(ab, c) = V(a, bc)$, or equivalently,

$$T(c^*)^* ab + cT(ab) = T(c^* b^*)^* a + bcT(a)$$

for all $a, b, c \in A$. So, if (u_λ) is an approximate unit in A, we have

$$T(u_\lambda)^* ab + u_\lambda T(ab) = T(u_\lambda b^*)^* a + bu_\lambda T(a),$$

for all $\lambda, a, b \in A$. We can take norm limits on the right-hand side. For the left-hand side, we observe that the bimodule operations $\pi_2^{***}(\cdot, ab)$ and $\pi_1^{***}(\cdot, T(ab))$ are weak* continuous. Taking weak* limits in λ in the previous equality, we derive

$$T^{**}(1)^* ab + T(ab) = T^{**}(1)^* ab + 1T(ab) = T(b^*)^* a + bT(a), \tag{4.4}$$

and

$$b^* a^* T^{**}(1) + T(ab)^* = a^* T(b^*) + T(a)^* b^*, \tag{4.5}$$

for all $a, b \in A$. Replacing a with u_λ, we get

$$b^* u_\lambda T^{**}(1) + T(u_\lambda b)^* = u_\lambda T(b^*) + T(u_\lambda)^* b^*,$$

for all $\lambda, b \in A$. Now, taking weak* limits in λ we obtain

$$b^* T^{**}(1) + T(b)^* = T(b^*) + T^{**}(1)^* b^*,$$

equivalently,

$$T(b)^* - T^{**}(1)^*b^* = T(b^*) - b^*T^{**}(1), \text{ for all } b \in A.$$

Consequently,

$$T(h) + T^{**}(1)^*h = T(h)^* + hT^{**}(1), \quad \forall h \in A_{sa}, \tag{4.6}$$

that is,

$$d(h) + h\xi + T^{**}(1)^*h = d(h)^* + \xi^*h + hT^{**}(1), \quad \forall h \in A_{sa}.$$

Multiplying by $k_1 \in A_{sa}$ on the left and by $k_2 \in A_{sa}$ on the right and applying (4.3), we get

$$
\begin{aligned}
k_1 d(h)k_2 + k_1 h\xi k_2 + k_1 \xi^* h k_2 &= k_1 d(h)k_2 + k_1 h\xi k_2 + k_1 T^{**}(1)^* h k_2 \\
&= k_1 d(h)^* k_2 + k_1 \xi^* h k_2 + k_1 h T^{**}(1) k_2 \\
&= k_1 d(h)^* k_2 + k_1 \xi^* h k_2 + k_1 h\xi k_2,
\end{aligned}
$$

for all $h, k_1, k_2 \in A_{sa}$. It then follows that

$$ad(h)b = ad(h)^*b, \text{ for all } h \in A_{sa}, a, b \in A.$$

Since the mappings $\pi_1^{***}(\cdot, d(h)b)$ and $\pi_1^{***}(\cdot, d(h)^*b)$ are weak* continuous, we can replace a with u_λ and take weak* limits in λ to deduce that

$$d(h)b = d(h)^*b, \text{ and } b^*d(h)^* = b^*d(h), \text{ for all } h \in A_{sa}, b \in A. \tag{4.7}$$

Now, by the local Gelfand theory, for each $h \in A_{sa}$, there exist $h_1, h_2 \in A_{sa}$ with $h_1 h_2 = h_2 h_1 = h$. If we apply (4.7) and the fact that d is a derivation, we arrive at

$$
\begin{aligned}
d(h)^* &= d(h_1 h_2)^* = (d(h_1)h_2 + h_1 d(h_2))^* = h_2 d(h_1)^* + d(h_2)^* h_1 \\
&= h_2 d(h_1) + d(h_2)h_1 = d(h_2 h_1) = d(h).
\end{aligned}
$$

The arbitrariness of $h \in A_{sa}$ proves that $d(a)^* = d(a^*)$ for all $a \in A$, witnessing that d is a *-derivation.

We claim that

$$d([a, b]) + \xi^*[a, b] + [a, b]\xi = 0 \text{ for all } a, b \in A. \tag{4.8}$$

Namely, by (4.3) and (4.4) for all $a, b, c \in A$ we have

$$c\xi^*ab + cT(ab) = cT^{**}(1)^*ab + cT(ab) = cT(b^*)^*a + cbT(a),$$

equivalently

$$c\xi^*ab + cd(ab) + cab\xi = cd(b^*)^*a + c\xi^*ba + cbd(a) + cba\xi$$
$$= cd(ba) + c\xi^*ba + cba\xi,$$

where in the last equality we applied that d is *-derivation. Therefore,

$$cd([a, b]) + c\xi^*[a, b] + c[a, b]\xi = 0, \text{ for all } a, b, c \in A.$$

Replacing c with u_λ, where (u_λ) is an approximate unit in A, and having in mind that the maps $\pi_1^{***}(\cdot, d([a, b]))$ and $\pi_1^{***}(\cdot, \xi^*[a, b])$ are weak* continuous, by taking weak* limits in λ we obtain the identity claimed in (4.8).

$(b) \Rightarrow (a)$ Suppose there exists a *-derivation $d : A \to X^{**}$ and an element $\xi \in X^{**}$ satisfying properties (i)–(ii) in the statement. Let us fix $a, b \in A$ with $ab^* = 0$. It follows from the assumptions that

$$T(b)^*a + b^*T(a) = d(b)^*a + \xi^*b^*a + b^*d(a) + b^*a\xi$$
$$= d(b^*)a + b^*d(a) + \xi^*b^*a + b^*a\xi$$
$$= d(b^*a) + \xi^*b^*a + b^*a\xi = d([b^*, a]) + \xi^*[b^*, a] + [b^*, a]\xi$$
$$= (\text{by } (i)) = 0.$$

The proof of the last statement can be obtained with the arguments we gave in Remark 7. $\qquad\square$

As we commented before, for a C*-algebra A, the proofs of [4, Lemma 2.10, Theorem 2.11] remain valid to characterize when a bounded linear operator $T : A \to A^{**}$ is a generalized derivation. So, the proofs of Theorem 9 remains valid to get the following result:

Theorem 10 *Let $T : A \to A^{**}$ be a bounded linear operator, where A is a C*-algebra. Then, the following statements are equivalent:*

(a) *T is r-*-anti-derivable at zero (i.e., $ab^* = 0$ in $A \Rightarrow T(b)^*a + b^*T(a) = 0$ in A^{**});*
(b) *There exists a *-derivation $d : A \to A^{**}$ and an element $\xi \in A^{**}$ satisfying the following properties:*

   (i) *$d([a, b]) + [a, b]\xi + \xi^*[a, b] = 0$, for all $a, b \in A$;*
   (ii) *$T(ab) = aT(b) + T(a)b - a\xi b$, and $T(a) = d(a) + a\xi$ for all $a, b \in A$.*

The description of those continuous linear operators which are l-*-anti-derivable at zero is a straight consequence of Theorem 9.

Theorem 11 *Let $T : A \to X$ be a bounded linear operator where A is a C*-algebra and X is an essential Banach *-A-bimodule. Then, the following statements are equivalent:*

(a) *T is l-*-anti-derivable at zero (i.e., $a^*b = 0$ in $A \Rightarrow T(b)a^* + bT(a)^* = 0$ in X);*

(b) *There exists a *-derivation $d : A \to X^{**}$ and an element $\eta \in X^{**}$ satisfying the following properties:*

(i) $d([a, b]) + [a, b]\eta^* + \eta[a, b] = 0$, *for all $a, b \in A$;*

(ii) $T(ab) = aT(b) + T(a)b - a\eta b$, *and $T(a) = d(a) + \eta a$ for all $a, b \in A$.*

If we further assume that A is unital or X is a dual Banach A-bimodule, we can replace (b) with the following:

(b$'$) *There exists a *-derivation $d : A \to X$ and an element $\eta \in X$ satisfying the properties (i)–(ii) above.*

Proof By observing that T is l-*-anti-derivable at zero if and only if the mapping $S : A \to X$, $S(a) := T(a^*)^*$ ($\forall a \in A$) is r-*-anti-derivable at zero, an application of Theorem 9 tells that this is the case if and only if there exists a *-derivation $d : A \to X^{**}$ and an element $\xi \in X^{**}$ satisfying the following properties:

(i) $d([a, b]) + [a, b]\xi + \xi^*[a, b] = 0$, for all $a, b \in A$;

(ii) $S(ab) = aS(b) + S(a)b - a\xi b$, and $S(a) = d(a) + a\xi$ for all $a, b \in A$.

Taking $\eta = \xi^*$, the rest can be straightforwardly checked by the reader. □

Corollary 12 *Let $T : A \to A^{**}$ be a bounded linear operator where A is a C*-algebra. Then, the following statements are equivalent:*

(a) *T is l-*-anti-derivable at zero;*

(b) *There exists a *-derivation $d : A \to A^{**}$ and an element $\eta \in A^{**}$ satisfying the following properties:*

(i) $d([a, b]) + [a, b]\eta^* + \eta[a, b] = 0$, *for all $a, b \in A$;*

(ii) $T(ab) = aT(b) + T(a)b - a\eta b$, *and $T(a) = d(a) + \eta a$ for all $a, b \in A$.*

References

1. Alaminos, J., Bresar, M., Extremera, J., Villena, A.: Maps preserving zero products. Stud. Math. **193**(2), 131–159 (2009)
2. Araujo, J., Jarosz, K.: Biseparating maps between operator algebras. J. Math. Anal. Appl. **282**, 48–55 (2003)
3. Arens, R.: The adjoint of a bilinear operation. Proc. Am. Math. Soc. **2**, 839–848 (1951)
4. Ayupov, S., Kudaybergenov, K., Peralta, A.M.: A survey on local and 2-local derivations on C*- and von Neumann algebras, Contemporary Mathematics. Am. Math. Soc. **672**, 73–126 (2016)
5. Brešar, M.: Multiplication algebra and maps determined by zero products. Linear Multilinear Algebra **60**, 763–768 (2012)
6. Burgos, M., Cabello-Sánchez, J., Peralta, A.M.: Linear maps between C*-algebras that are *-homomorphisms at a fixed point. Quaest. Math. **42**(2), 151–164 (2019)
7. Burgos, M., Fernández-Polo, F.J., Garcés, J., Peralta, A.M.: Local triple derivations on C*-algebras. Commun. Algebra **42**(3), 1276–1286 (2014)
8. Burgos, M., Fernández-Polo, F.J., Garcés, J.J., Martínez Moreno, J., Peralta, A.M.: Orthogonality preservers in C*-algebras, JB*-algebras and JB*-triples. J. Math. Anal. Appl. **348**, 220–233 (2008)
9. Burgos, M., Fernández-Polo, F.J., Peralta, A.M.: Local triple derivations on C*-algebras and JB*-triples. Bull. Lond. Math. Soc. **46**(4), 709–724 (2014)
10. Chebotar, M.A., Ke, W.-F., Lee, P.-H., Wong, N.-C.: Mappings preserving zero products. Stud. Math. **155**, 77–94 (2003)
11. Dales, H.G.: Banach Algebras and Automatic Continuity. London Mathematical Society Monographs (New Series), vol. 24. Oxford Science Publications, Oxford (2000)
12. Essaleh, A.B.A., Peralta, A.M.: Linear maps on C*-algebras which are derivations or triple derivations at a point. Linear Algebra Appl. **538**, 1–21 (2018)
13. Fadaee, B., Ghahramani, H.: Linear maps behaving like derivations or anti-derivations at orthogonal elements on C*-algebras. Malays. Math. Sci. Soc, Bull (2019). https://doi.org/10.1007/s40840-019-00841-6
14. Fack, Th: Finite sums of commutators in C*-algebras. Ann. Inst. Fourier Grenoble **32**, 129–137 (1982)
15. Fack, Th, de la Harpe, P.: Sommes de commutateurs dans les alg'ebres de von Neumann finies continues. Ann. Inst. Fourier Grenoble **30**, 49–73 (1980)
16. Ghahramani, H., Pan, Z.: Linear maps on *-algebras acting on orthogonal elements like derivations or anti-derivations. Filomat **32**(13), 4543–4554 (2018)
17. Goldstein, S.: Stationarity of operator algebras. J. Funct. Anal. **118**(2), 275–308 (1993)
18. Goldstein, S., Paszkiewicz, A.: Linear combinations of projections in von Neumann algebras. Proc. Am. Math. Soc. **116**, 175–183 (1992)
19. Halpern, H.: Commutators in properly infinite von Neumann algebras. Trans. Am. Math. Soc. **139**, 55–73 (1969)
20. Hanche-Olsen, H., Størmer, E.: Jordan Operator Algebras. Pitman, London (1984)
21. Hejazian, S., Niknam, A.: Modules Annihilators and module derivations of JB*-algebras. Indian J. Pure Appl. Math. **27**, 129–140 (1996)
22. Hou, J.C., Qi, X.F.: Additive maps derivable at some points on J-subspace lattice algebras. Linear Algebra Appl. **429**, 1851–1863 (2008)
23. Jamjoom, F.B., Peralta, A.M., Siddiqui, A.A.: Jordan weak amenability and orthogonal forms on JB*-algebras. Banach J. Math. Anal. **9**(4), 126–145 (2015)
24. Jing, W.: On Jordan all-derivable points of $B(H)$. Linear Algebra Appl. **430**(4), 941–946 (2009)
25. Jing, W., Lu, S.J., Li, P.T.: Characterization of derivations on some operator algebras. Bull. Aust. Math. Soc. **66**, 227–232 (2002)
26. Johnson, B.E.: Symmetric amenability and the nonexistence of Lie and Jordan derivations. Math. Proc. Camb. Philos. Soc. **120**(3), 455–473 (1996)
27. Johnson, B.E., Kadison, R.V., Ringrose, J.R.: Cohomology of operator algebras III. Reducing to normal cohomology. Bull. Sot. Math. France **100**, 73–96 (1972)
28. Leung, C.-W., Tsai, C.-W., Wong, N.-C.: Linear disjointness preservers of W*-algebras. Math. Z. **270**, 699–708 (2012)
29. Leung, C.-W., Wong, N.-C.: Zero product preserving linear maps of CCR C*-algebras with Hausdorff spectrum. J. Math. Anal. Appl. **361**, 187–194 (2010)

30. Li, J., Pan, Z.: Annihilator-preserving maps, multipliers, and derivations. Linear Algebra Appl. **423**, 5–13 (2010)
31. Lin, Y.-F., Mathieu, M.: Jordan isomorphism of purely infinite C*-algebras. Quart. J. Math. **58**, 249–253 (2007)
32. Liu, J.-H., Chou, C.-Y., Liao, C.-J., Wong, N.-C.: Disjointness preservers of AW*-algebras. Linear Algebra Appl. **552**, 71–84 (2018)
33. Liu, J.-H., Chou, C.-Y., Liao, C.-J., Wong, N.-C.: Linear disjointness preservers of operator algebras and related structures. Acta Sci. Math. (Szeged) **84**(1–2), 277–307 (2018)
34. Lu, F.: Characterizations of derivations and Jordan derivations on Banach algebras. Linear Algebra Appl. **430**(8–9), 2233–2239 (2009)
35. Marcoux, L.W.: On the Linear Span of the Projections in Certain Simple C*-algebras. Indiana Univ. Math. J. **51**, 753–771 (2002)
36. Marcoux, L.W.: Sums of small numbers of commutators. J. Oper. Theory **56**, 111–142 (2006)
37. Marcoux, L.W.: Projections, commutators and Lie ideals in C*-algebras. Math. Proc. R. Ir. Acad. **110A**, 31–55 (2010)
38. Marcoux, L.W., Murphy, G.J.: Unitarily-invariant linear spaces in C*-algebras. Proc. Am. Math. Soc. **126**(12), 3597–3605 (1998)
39. Pearcy, C., Topping, D.: Sum of small numbers of idempotent. Michigan Math. J. **14**, 453–465 (1967)
40. Pedersen, G.K.: C*-algebras and their automorphism groups. Academic Press, London (1979)
41. Peralta, A.M., Rodríguez Palacios, A.: Grothendieck's inequalities for real and complex JBW*-triples. Proc. Lond. Math. Soc. (3) **83**(3), 605–625 (2001)
42. Peralta, A.M., Russo, B.: Automatic continuity of triple derivations on C*-algebras and JB*-triples. J. Algebra **399**, 960–977 (2014)
43. Pop, C.: Finite sums of commutators. Proc. Am. Math. Soc. **130**, 3039–3041 (2002)
44. Ringrose, J.R.: Automatic continuity of derivations of operator algebras. J. Lond. Math. Soc. **2**(5), 432–438 (1972)
45. Sakai, S.: C*- and W*-Algebras. Springer, Berlin (1971)
46. Takesaki, M.: Theory of Operator Algebras I. Springer, New York (2003)
47. Tsai, C.-W.: The orthogonality structure determines a C*-algebra with continuous trace. Oper. Matrices **5**, 529–540 (2011)
48. Tsai, C.-W., Wong, N.-C.: Linear orthogonality preservers of standard operator algebras. Taiwan. J. Math. **14**, 1047–1053 (2010)
49. Wolff, M.: Disjointness preserving operators in C*-algebras. Arch. Math. **62**, 248–253 (1994)
50. Wong, N.-C.: Triple homomorphisms of C*-algebras. Southeast Asian Bull. Math. **29**, 401–407 (2005)
51. Wong, N.-C.: Zero product preservers of C*-algebras. Contemp. Math. **435**, 377–380 (2007)
52. Zhang, Y.F., Hou, J.C., Qi, X.F.: Characterizing derivations for any nest algebras on Banach space by their behaviors at an injective operator. Linear Algebra Appl. **449**(15), 312–333 (2014)
53. Zhang, Y.F., Hou, J.C., Qi, X.F.: All-derivable subsets for nest algebras on Banach spaces. Int. Math. Forum **9**(1–4), 1–11 (2014)
54. Zhu, J.: All-derivable points of operator algebras. Linear Algebra Appl. **427**, 1–5 (2007)
55. Zhu, J., Xiong, C.P.: Generalized derivable mappings at zero point on nest algebras. Acta Math. Sin. **45**, 783–788 (2002)
56. Zhu, J., Xiong, C.P.: Generalized derivable mappings at zero point on some reflexive operator algebras. Linear Algebra Appl. **397**, 367–379 (2005)
57. Zhu, J., Xiong, C.P.: Derivable mappings at unit operator on nest algebras. Linear Algebra Appl. **422**, 721–735 (2007)
58. Zhu, J., Xiong, C.P.: All-derivable points in continuous nest algebras. J. Math. Anal. Appl. **340**, 845–853 (2008)
59. Zhu, J., Xiong, Ch., Li, P.: Characterizations of all-derivable points in $B(H)$. Linear Multilinear Algebra **64**(8), 1461–1473 (2016)
60. Zhu, J., Zhao, S.: Characterizations all-derivable points in nest algebras. Proc. Am. Math. Soc. **141**(7), 2343–2350 (2013)