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Determining When an Algebra Is an Evolution Algebra

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Abstract: Evolution algebras are non-associative algebras that describe non-Mendelian hereditary processes and have connections with many other areas. In this paper, we obtain necessary and sufficient conditions for a given algebra A to be an evolution algebra. We prove that the problem is equivalent to the so-called SDC problem, that is, the simultaneous diagonalisation via congruence of a given set of matrices. More precisely we show that an n-dimensional algebra A is an evolution algebra if and only if a certain set of n symmetric $n \times n$ matrices $\{M_1, \ldots, M_n\}$ describing the product of A are SDC. We apply this characterisation to show that while certain classical genetic algebras (representing Mendelian and auto-tetraploid inheritance) are not themselves evolution algebras, arbitrarily small perturbations of these are evolution algebras. This is intringuing, as evolution algebras model asexual reproduction, unlike the classical ones.

Keywords: evolution algebra; multiplication structure matrices; simultaneous diagonalisation by congruence; simultaneous diagonalisation by similarity; linear pencil

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1. Introduction

Evolution algebras are non-associative algebras with a dynamic nature. They were introduced in 2008 by Tian [1] to enlighten the study of non-Mendelian genetics. Since then, a large literature has flourished on this topic (see for instance [2–17]) motivated by the fact that these algebras have connections with group theory, Markov processes, theory of knots, systems and graph theory. For instance, in [2], the theory of evolution algebras was related to that of pulse processes on weighted digraphs and applications were provided by reviewing and enlightening a report of the National Science Foundation about air pollution achieved by the Rand Corporation. A pulse process is a structural dynamic model to analyse complex networks by studying the propagation of changes, through the vertices of a weighted digraph, after introducing an initial pulse in the system at a particular vertex. It is based on a spectral analysis of the corresponding weighted digraph to face large scale decision making problems. Evolution algebras also become a proper tool to introduce useful algebraic techniques into the study of some digraphs because evolution algebras and weighted digraphs can be canonically identified.

We recall that an algebra is a linear space A provided with a product, that is, a bilinear map from $A \times A$ to A via the operation $(a, b) \to ab$. In the particular case that $(ab)c = a(bc)$, for all $a, b, c \in A$ we say that A is associative. Meanwhile, if $ab = ba$, for all $a, b \in A$, then we say that A is commutative.

An evolution algebra is defined as a commutative algebra A for which there exists a basis $B^* = \{e_i^* : i \in \Lambda\}$ such that $e_i^* e_j^* = 0$ for every $i, j \in \Lambda$ with $i \neq j$. Such a basis is called natural. Evolution algebras are, in general, non-associative. To date, most literature on evolution algebras is on finite-dimensional ones. However, in [12] it is shown that every infinite-dimensional Banach evolution algebra is the direct sum of a finite-dimensional evolution algebra and a zero-product algebra.

In this paper, we discuss necessary and sufficient conditions under which a given finite-dimensional commutative algebra is an evolution algebra, namely we determine when such a finite-dimensional algebra can be provided with a natural basis. We tackle the problem constructively by assuming an arbitrary basis B with a multiplication table given by Equation (1) below and then asking whether or not there is a change of basis from B to a natural basis B^*. In Section 2, Theorem 1, we show that this problem is equivalent to the simultaneous diagonalisation via congruence of certain $n \times n$ symmetric matrices $M_1, \ldots, M_n$, called the multiplication structure matrices obtained from the given multiplication table.

Finding concrete sufficient conditions for a given set of matrices to be simultaneously diagonalisable via congruence (SDC) is one of the 14 open problems posted in 1990 by Hiriart–Urruty [18] (see also [19,20]). It has connections with other problems such as blind-source separation in signal processing [21–24]. The SDC-problem was solved recently for complex symmetric matrices in [25].

In Theorem 2 we show that if A is a real algebra and B is a basis of A then B also is a basis of $A_{\mathbb{C}}$, the complexification of A (with the same multiplication structure matrices) and that A is an evolution algebra if and only if $A_{\mathbb{C}}$ is an evolution algebra. This reduction of the real case to the complex one allows us to apply the results in [25] to both real and complex algebras.

In Theorem 5 we determine if a given algebra A whose annihilator is zero is an evolution algebra and in Theorem 6 we do the same if its annihilator is not zero. A useful characterisation of the property of being an evolution algebra is given in the particular case that one of the multiplication structure matrices is invertible. In this case if M_{i_0} is invertible then A is an evolution algebra if and only if for each $k \neq i_0$ the matrix $M_{i_0}^{-1} M_k$ is diagonalisable by similarity and these matrices pairwise commute.

Applications of these results are provided in the final section of this paper. They also show that the conditions in the mentioned results are neither redundant nor superfluous.

We prove that some classical genetic algebras such as the gametic algebra for simple Mendelian inheritance (Example 2) or the gametic algebra for auto-tetraploid inheritance (Example 5) are not evolution algebras. Nevertheless, both of these algebras can be deformed by means of a parameter $\varepsilon > 0$ to obtain an algebra A_ε that is an evolution algebra for every value of the parameter ε, as shown in Examples 3 and 6 respectively.

2. Characterising Evolution Algebras by Means of Simultaneous Diagonalisation of Matrices by Congruence

An n-dimensional algebra A over a field $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$) is determined by means of a basis $B = \{e_1, \ldots, e_n\}$ together with a multiplication table

$$e_i e_j = \sum_{k=1}^{n} m_{ijk} \, e_k \,, \quad i, j = 1, \ldots, n \,, \tag{1}$$

where $m_{ijk} \in \mathbb{K}$, for $i, j, k = 1, \ldots, n$. In fact, if $a := \sum_{i=1}^{n} \alpha_i e_i$ and $b := \sum_{j=1}^{n} \beta_j e_j$ then, by bilinearity, the product ab is obtained from the multiplication table (1) as follows

$$ab = \left(\sum_{i=1}^{n} \alpha_i e_i \right) \left(\sum_{j=1}^{n} \beta_j e_j \right) = \sum_{k=1}^{n} \left(\sum_{i,j=1}^{n} \alpha_i \beta_j m_{ijk} \right) e_k,$$

where $m_{ijk} := \pi_k(e_ie_j)$ and $\pi_k : A \to \mathbb{K}$ is the projection over the k-th coordinate, that is $\pi_k(\sum_{i=1}^{n} \alpha_i e_i) = \alpha_k$.

These basis-dependent coefficients m_{ijk} are known as structure constants with respect to B (see [26]). For a basis B of A, the structure constants completely determine the algebra A, up to isomorphism.

If we organise the n^3 structure constants in n matrices by defining

$$M_k(B) := \begin{pmatrix} \pi_k(e_1e_1) & \pi_k(e_1e_n) \\ \vdots & \vdots \\ \pi_k(e_ne_1) & \pi_k(e_ne_n) \end{pmatrix} = \begin{pmatrix} m_{11k} & m_{1nk} \\ \vdots & \vdots \\ m_{n1k} & m_{nnk} \end{pmatrix}, \tag{2}$$

for $k = 1, ..., n$, then the product of A is given by

$$\left(\sum_{i=1}^{n} \alpha_i e_i\right)\left(\sum_{j=1}^{n} \beta_j e_j\right) = \sum_{k=1}^{n} \left(\boldsymbol{\alpha}^T M_k(B) \boldsymbol{\beta}\right) e_k\,, \tag{3}$$

where $\boldsymbol{\alpha}^T = (\alpha_1, \ldots, \alpha_n)$, $\boldsymbol{\beta}^T = (\beta_1, \ldots, \beta_n)$ and T indicates the transpose operation. This motivates the following definition.

Definition 1. *If A is an algebra, $B = \{e_1, \ldots, e_n\}$ is a basis of A and $e_ie_j = \sum_{k=1}^{n} m_{ijk}\, e_k$, for $i, j = 1, \ldots, n$, then the multiplication structure matrices (m-structure matrices for short) of A with respect to B are the $n \times n$ matrices $M_k(B) = (\pi_k(e_ie_j))$ given by Equation (2) for $k = 1, ..., n$. Note that these matrices are symmetric if and only if A is commutative. If the basis B is clear from the context then we will write $M_k := M_k(B)$ for $k = 1, ..., n$.*

We recall that an n-dimensional evolution algebra is a commutative algebra A for which there exists a basis $B^* = \{e_1^*, ..., e_n^*\}$ such that $e_i^* e_j^* = 0$ for every $i, j \in \{1, \cdots, n\}$ with $i \neq j$. Such a basis B^* is said to be a natural basis of A.

Combining the notion of evolution algebra with Definition 1 the next result is straightforward.

Proposition 1. *An evolution algebra is an algebra A provided with a basis $B^* = \{e_1^*, ..., e_n^*\}$ such that the corresponding m-structure matrices $M_1(B^*) = (\pi_1(e_i^* e_j^*)), \cdots, M_n(B^*) = (\pi_n(e_i^* e_j^*))$ are diagonal.*

Proof. $M_k(B^*)$ is diagonal for $k = 1, ..., n$, if and only if $e_i^* e_j^* = 0$, for every $i \neq j$, or equivalently if B^* is a natural basis (which means that A is an evolution algebra). □

In the next theorem we characterise when a given algebra is an evolution algebra. To this end we recall the following property.

Definition 2. *Let $M_1, \ldots, M_m$ be a set of symmetric $n \times n$ matrices. Then these matrices are (SDC) if and only if there exists a nonsingular $n \times n$ matrix P and m diagonal $n \times n$ matrices $\{D_j\}_{j=1}^m$ such that*

$$P^T M_j P = D_j, \quad j = 1, \ldots, m.$$

It is worth remarking at this point that the general problem of diagonalisation via congruence considers m symmetric matrices of dimension n, where m need not be equal to n. In reference [18], Problem 12 is stated as follows: Find sensible and palpable conditions on the symmetric matrices $\{M_1, ..., M_n\}$ ensuring they are simultaneously diagonalisable via congruence. This problem has applications in statistical signal processing and multivariate statistics [21–24] and it was solved for complex symmetric matrices in [25].

Theorem 1. *Let A be a commutative algebra over $\mathbb{K}$ with basis $B = \{e_1, \ldots, e_n\}$. Let $\{M_1, \ldots, M_n\}$ be the m-structure matrices of A with respect to B. Then A is an evolution algebra if and only if the symmetric matrices $\{M_1, \ldots, M_n\}$ are simultaneously diagonalisable via congruence.*

Proof. A is an evolution algebra if and only if A has a natural basis, say $B^* = \{e_1^*, ..., e_n^*\}$ (that is a basis such that $e_i^* e_j^* = 0$ if $i \neq j$). Let $P = (p_{ij})$ be the change of basis matrix from B to B^* (that is $e_i^* = \sum_{k=1}^n p_{ki} e_k$ for $i = 1, \ldots, n$). Then, by Equation (3),

$$e_i^* e_j^* = \left(\sum_{k=1}^n p_{ki} e_k\right)\left(\sum_{k=1}^n p_{kj} e_k\right) = \sum_{k=1}^n \left(\boldsymbol{\alpha}^T M_k \boldsymbol{\beta}\right) e_k, \tag{4}$$

where $\boldsymbol{\alpha} = P\gamma_i$ and $\boldsymbol{\beta} = P\gamma_j$ with $\gamma_i = (0, ..., 0, \overset{(i\text{-th})}{1}, 0, ...0)^T \in \mathcal{M}_{n \times 1}(\mathbb{K})$. Thus

$$e_i^* e_j^* = \sum_{k=1}^n \left(\gamma_i^T P^T M_k P \gamma_j\right) e_k = 0, \text{ for } i \neq j, \tag{5}$$

and hence $e_i^* e_j^* = 0$ if $i \neq j$ if and only if the matrix $P^T M_k P$ is diagonal for $k = 1, ..., n$. □

Since the problem of simultaneous diagonalisation of matrices via congruence was solved in [25] for complex symmetric matrices, we consider the following.

The complexification of a real algebra A is defined as the complex algebra $A_{\mathbb{C}} := A \oplus iA = \{a + ib : a, b \in A\}$, where, for $a, b, c, d \in A$ and $r, s \in \mathbb{R}$,

$$\begin{aligned}
(a + ib) + (c + id) &= (a + b) + i(b + d), \\
(r + is)(a + ib) &= ra - sb + i(rb + sa), \\
(a + ib)(c + id) &= (ac - bd) + i(ad + bc).
\end{aligned}$$

Note that every basis B of A is trivially a basis of $A_{\mathbb{C}}$ so that the real dimension of A and the complex dimension of $A_{\mathbb{C}}$ coincide.

Theorem 2. *Let A be a real algebra. Then A is an evolution algebra if and only if $A_{\mathbb{C}}$ is an evolution algebra. Moreover, if A is a real evolution algebra then every natural basis of A is a natural basis of $A_{\mathbb{C}}$.*

Proof. Suppose that as a real vector space $\dim A = n$. If A is an evolution algebra and if B is a natural basis of A then obviously B is a natural basis of $A_{\mathbb{C}}$.

Conversely, let $A_{\mathbb{C}}$ be an evolution algebra and $B_{\mathbb{C}} = \{e_1 + i\widetilde{e}_1, ..., e_n + i\widetilde{e}_n\}$ be a natural basis. Fix $j \in \{1, ..., n\}$. Then, for $k = 1, ..., n$, there exists complex numbers z_k and $\widetilde{z}_k$ such that $e_j = \sum_{k=1}^n z_k(e_k + i\widetilde{e}_k)$, and $\widetilde{e}_j = \sum_{k=1}^n \widetilde{z}_k(e_k + i\widetilde{e}_k)$. Since

$$e_j + i\widetilde{e}_j = \sum_{k=1}^n (z_k + i\widetilde{z}_k)(e_k + i\widetilde{e}_k),$$

it follows that $z_j + i\widetilde{z}_j = 1$ and $z_k = \widetilde{z}_k = 0$ for every $k \neq i$. Therefore, $e_j = z_j(e_j + i\widetilde{e}_j)$ and $\widetilde{e}_j = \widetilde{z}_j(e_j + i\widetilde{e}_j)$ so that, either $\widetilde{e}_j = 0$ or, otherwise, $\widetilde{z}_j \neq 0$ and $e_j = \frac{z_j}{\widetilde{z}_j}\widetilde{e}_j$. Consequently, there exist $w_k \in \mathbb{C} \setminus \{0\}$ and $\widehat{e}_k \in A$, for $k = 1, ..., n$, such that the given natural basis is $B_{\mathbb{C}} = \{w_1\widehat{e}_1, ..., w_n\widehat{e}_n\}$. Therefore we conclude that $B = \{\widehat{e}_1, ..., \widehat{e}_n\}$ is a natural basis of A, as desired.

If A has infinite dimension then a straightforward adaptation of this reasoning concludes the proof. □

Corollary 1. *Let A be a real commutative algebra, $B = \{e_1, ..., e_n\}$ a basis and $\{M_1, \ldots, M_n\}$ be the m-structure matrices of A with respect to B. Then A is an evolution algebra if and only if the matrices $M_1, \ldots, M_n$ (regarded as complex matrices) are simultaneously diagonalisable via congruence.*

2.1. Reviewing the Solution of the SDC Problem

The aim of this subsection is to review the solution of the SDC problem, that is, determining when m matrices of size $n \times n$ are simultaneously diagonalisable via congruence, which was solved in [25] for complex matrices. All matrices considered in this section are complex.

From now on, let $\mathcal{M}_n$ denote the set of all complex $n \times n$ matrices. Moreover, let $\mathcal{MS}_n$ be the set of all symmetric matrices in $\mathcal{M}_n$ and $\mathcal{GL}_n$ be the set of nonsingular matrices in $\mathcal{M}_n$.

We recall the following definition of simultaneous diagonalisation of matrices via similarity (SDS), not to be confused with Definition 2 involving simultaneous diagonalisation via congruence (SDC). Nevertheless, the solution of the problem of determining when a set of complex matrices is SDC given in [25] is related to the problem of determining the SDS of a certain set of related matrices, as we will show below.

Definition 3. *Let $N_1, ..., N_m \in \mathcal{M}_n$. These matrices are said to be simultaneously diagonalisable by similarity (SDS) if and only if there exists $P \in \mathcal{GL}_n$ such that $P^{-1}N_kP$ is diagonal for every $k = 1, ..., m$.*

The following result is well known (Theorem 1.3.12 and Theorem 1.3.21 in [27]).

Proposition 2. *Let $N_1, ..., N_m \in \mathcal{M}_n$. These matrices are SDS if and only if they are each diagonalisable by similarity and they pairwise commute.*

Remark 1. *Concerning the statement of the above theorem in [27] we point out that the fact that the symmetric matrices $\{N_1, ..., N_m\}$ commute assures that $\{N_1, ..., N_m\}$ are simultaneously diagonalisable by similarity only when $\{N_1, ..., N_m\}$ are diagonalisable matrices (and obviously not otherwise).*

In [25], to solve the SDC problem, Theorems 3 and 4 below were proved. To state them, we recall the next definition.

Definition 4. *Given $M_1, ..., M_m \in \mathcal{M}_n$, define the associated linear pencil to be the map $M : \mathbb{C}^m \to \mathcal{M}_n$ given by $M(\lambda) := \sum_{j=1}^{m} \lambda_j M_j$, for every $\lambda = (\lambda_1, ..., \lambda_m)$ in $\mathbb{C}^m$. Since, for $\lambda \neq 0$,*

$$\operatorname{rank} M(\lambda) = \operatorname{rank} M\left(\frac{\lambda}{\|\lambda\|}\right),$$

it follows that

$$\sup\{\operatorname{rank} M(\lambda) : \lambda \in \mathbb{C}^m\} = \sup\{\operatorname{rank} M(\lambda) : \lambda \in \mathbb{C}^m \text{ with } \|\lambda\| = 1\} \in \{0, 1, ..., n\}.$$

Consequently, this supremum must be achieved so that there exists $\lambda_0 \in \mathbb{C}^m$ with $\|\lambda_0\| = 1$ such that

$$r_0 := \operatorname{rank} M(\lambda_0) = \max\{\operatorname{rank} M(\lambda) : \lambda \in \mathbb{C}^m\},$$

and we say that r_0 is the maximum pencil rank of M_1, ..., M_m.

The next theorem corresponds to Theorem 7 in [25] and deals with the case when the maximum pencil rank of the matrices is n.

Theorem 3. *Let $M_1, ..., M_m \in \mathcal{MS}_n$ have maximum pencil rank n. Let $\lambda_0 \in \mathbb{C}^m$ be such that $r_0 :=$ rank $M(\lambda_0) = n$. Then $M_1, ..., M_m$ are SDC if, and only if, $M(\lambda_0)^{-1}M_1, \ ..., M(\lambda_0)^{-1}M_m$ are SDS.*

Proposition 2 gives the following result.

Corollary 2. *Let $M_1, ..., M_m \in \mathcal{MS}_n$, and $\lambda_0 \in \mathbb{C}^m$ be such that*

$$r_0 := \text{rank}M(\lambda_0) = n.$$

Then $M_1, ..., M_m$ are SDC if and only if $M(\lambda_0)^{-1}M_1, ..., M(\lambda_0)^{-1}M_m$ are all diagonalisable and pairwise commute.

Given $1 \leq r < n$, and matrices $M_r \in \mathcal{M}_r$ and $N_{n-r} \in \mathcal{M}_{n-r}$, denote by $M_r \oplus N_{n-r}$ the $n \times n$ matrix given by

$$\begin{pmatrix} M_r & 0_{r\times(n-r)} \\ 0_{(n-r)\times r} & N_{n-r} \end{pmatrix}.$$

When the pencil rank of $M_1, ..., M_m \in \mathcal{MS}_n$ is strictly less than n, then the SDC problem can be reduced to a similar one in a reduced dimension as the following result (Theorem 9 in [25]) shows.

Theorem 4. *Let $M_1, ..., M_m \in \mathcal{MS}_n$ have maximum pencil rank r. Then the following assertions are equivalent:*

(i) *$M_1, ..., M_m$ are SDC;*
(ii) $\dim(\cap_{j=1}^m \ker M_j) = n - r$ *and there exists $P \in \mathcal{GL}_n$ satisfying $P^T M_j P = \widetilde{D}_j \oplus 0_{n-r}$ where $\widetilde{D}_j \in \mathcal{MS}_r$ is diagonal for $1 \leq j \leq m$.*

Moreover, if either of the above conditions is satisfied, then the pencil $\widetilde{D}$ associated with the matrices $\widetilde{D}_1, ..., \widetilde{D}_m$ is non-singular. Indeed, if $\lambda_0 \in \mathbb{C}^m$ with $\|\lambda_0\| = 1$ is such that $r =$ rank $M(\lambda_0)$ then $\widetilde{D}(\lambda_0) \in \mathcal{GL}_r$.

2.2. Checking When an Algebra Is an Evolution Algebra

We apply the above results to the m-structure matrices $M_1, \ldots, M_n$ of an algebra A with respect to a basis $B = \{e_1, ..., e_n\}$ as in (2). For a real algebra A we consider the complexification $A_{\mathbb{C}}$ provided with the same basis B.

We recall that the annihilator of an algebra A is the set

$$\text{Ann}(A) = \{b \in A : ab = ba = 0, \text{ for every } a \in A\}.$$

This set is an ideal of A.

Lemma 1. *Let A be a commutative algebra and $B = \{e_1, ..., e_n\}$ be a basis of A. Let $\{M_1, \ldots, M_n\}$ be the m-structure matrices of A with respect to B. Then*

$$\text{Ann}(A) = \{\sum_{i=1}^n \beta_i e_i : (\beta_1, ..., \beta_n)^T \in \cap_{j=1}^n \ker M_j\}.$$

Proof. Since $\left(\sum_{i=1}^n \alpha_i e_i\right)\left(\sum_{j=1}^n \beta_j e_j\right) = \sum_{k=1}^n \left(\boldsymbol{\alpha}^T M_k \boldsymbol{\beta}\right) e_k$, as shown in (3) we have that if $(\beta_1, ..., \beta_n)^T \in \cap_{j=1}^n \ker M_j$ then $b := \sum_{j=1}^n \beta_j e_j \in \text{Ann}(A)$ as $ab = ba = 0$ for every $a \in A$ (because $M_k\boldsymbol{\beta} = \mathbf{0}$).

Conversely, if $b := \sum_{j=1}^{n} \beta_j e_j \in \mathrm{Ann}(A)$ then $e_i b = 0$ for every $i = 1, ..., n$. It follows that,

$$(0, ..., 0, \overset{(i-\mathrm{th})}{1}, 0, ..., 0) M_k (\beta_1, ..., \beta_n)^T = 0,$$

for $i, k \in \{1, ..., n\}$. Fixing k and running i we deduce that, for each $k = 1, ..., n$,

$$(\beta_1, ..., \beta_n)^T \in \ker M_k,$$

Consequently, $(\beta_1, ..., \beta_n)^T \in \cap_{j=1}^{n} \ker M_j$, as desired. □

Theorem 5. *Let A be a complex commutative algebra with $\mathrm{Ann}(A) = \{0\}$. Let $B = \{e_1, \ldots, e_n\}$ be a basis of A, and let $M_1, \ldots, M_n$ be the m-structure matrices of A with respect to B .*

(i) If $M_1, ..., M_n$ have maximum pencil rank n, and $\lambda_0 \in \mathbb{C}^n$ with $\|\lambda_0\| = 1$ is such that $\mathrm{rank}\, M(\lambda_0) = n$ then A is an evolution algebra if and only if each of the matrices $M(\lambda_0)^{-1} M_1, ..., M(\lambda_0)^{-1} M_n$ are diagonalisable and they pairwise commute.

(ii) If $M_1, \ldots, M_n$ have maximum pencil rank $r < n$ then A is not an evolution algebra.

Proof. (i) If $\lambda_0 \in \mathbb{C}^n$ with $\|\lambda_0\| = 1$ is such that $\mathrm{rank}\, M(\lambda_0) = n$ then, by Corollary 2, we conclude that A is an evolution algebra if and only if the matrices $M(\lambda_0)^{-1} M_1, ..., M(\lambda_0)^{-1} M_n$ are diagonalisable and they pairwise commute. (ii) Otherwise the maximum pencil rank of $\{M_1, \ldots, M_n\}$ is $r < n$ and, by the above lemma, $\dim \mathrm{Ann}(A) = \cap_{j=1}^{n} \ker M_j = 0 \neq n - r$. Consequently, by Theorem 4, we conclude that A is not an evolution algebra. □

Corollary 3. *Let A be a complex commutative algebra and let $B = \{e_1, \ldots, e_n\}$ be a basis of A. Let $M_1, \ldots, M_n$ be the m-structure matrices of A with respect to B. If M_{i_0} is invertible for some $1 \leq i_0 \leq n$ then $\mathrm{Ann}(A) = 0$, and A is an evolution algebra if and only if the matrices $M_{i_0}^{-1} M_1, ..., M_{i_0}^{-1} M_n$ are diagonalisable for $j = 1, ...n$ and they pairwise commute.*

Proof. Since $\mathrm{Ann}(A) \subseteq \ker M_{i_0}$ by Lemma 1, we obtain that if M_{i_0} is invertible then $\mathrm{Ann}(A) = 0$. Moreover, for $\lambda_0 = (0, ..., 0, \overset{(i_0-\mathrm{th})}{1}, 0, ..., 0)$ we have

$$\mathrm{rank}(M(\lambda_0)) = \mathrm{rank}(M_{i_0}) = n$$

and the result follows from Theorem 5. □

If A is an algebra with $\mathrm{Ann}(A) \neq \{0\}$ (suppose that $\dim \mathrm{Ann}(A) = r > 0$) then we can fix a basis of $\mathrm{Ann}(A)$ which can be extended to a basis of A. Therefore we obtain a basis $\widetilde{B} = \{e_1, ..., e_r, e_{r+1}, ..., e_n\}$ of A such that $\{e_{r+1}, ..., e_n\}$ is a basis of $\mathrm{Ann}(A)$ and the m-structure matrices $M_1(\widetilde{B}), \ldots, M_n(\widetilde{B})$ of A with respect to $\widetilde{B}$ satisfy $M_k(\widetilde{B}) = \widetilde{M}_k \oplus 0_{n-r}$, for certain $r \times r$ matrices $\widetilde{M}_k \in \mathcal{MS}_r$.

Theorem 6. *Let A be a commutative complex algebra with $\mathrm{Ann}(A) \neq \{0\}$. Let $\widetilde{B} = \{e_1, ..., e_r, e_{r+1}, ..., e_n\}$ be a basis of A such that $\{e_{r+1}, ..., e_n\}$ is a basis of $\mathrm{Ann}(A)$. Let $M_1(\widetilde{B}), ..., M_n(\widetilde{B})$ be the m-structure matrices of A with respect to $\widetilde{B}$ with $M_k(\widetilde{B}) = \widetilde{M}_k \oplus 0_{n-r}$, where $\widetilde{M}_k \in \mathcal{MS}_r$. Then A is an evolution algebra if and only if there exists $\|\lambda_0\| = 1$ such that the pencil $\widetilde{M}(\lambda_0)$ is invertible, each of the matrices $\widetilde{M}(\lambda_0)^{-1} \widetilde{M}_1, ..., \widetilde{M}(\lambda_0)^{-1} \widetilde{M}_n$, is diagonalisable by similarity and they pairwise commute.*

Proof. From Equation (2) it is clear that $M_k(\widetilde{B}) = \widetilde{M}_k \oplus 0_{n-r}$, for certain $r \times r$ matrices $\widetilde{M}_k$. On the other hand, there exists $\|\lambda_0\| = 1$ such that the pencil $\widetilde{M}(\lambda_0)$ is invertible if and only if the maximum pencil rank of $M_k(\widetilde{B})$ is r. If this happens then $\dim(\cap_{j=1}^{n} \ker M_j(\widetilde{B})) = n - r$, as $\dim \mathrm{Ann}(A) = \dim(\cap_{j=1}^{n} \ker M_j(\widetilde{B}))$ by Lemma 1. If $\widetilde{M}(\lambda_0)$ is invertible then, by Corollary 2, we have that $\widetilde{M}_1, ..., \widetilde{M}_n$

are SDC if and only if each of the matrices $\widetilde{M}(\lambda_0)^{-1}\widetilde{M}_1, ..., \widetilde{M}(\lambda_0)^{-1}\widetilde{M}_n$ is diagonalisable by similarity and they pairwise commute. Since the matrices $\widetilde{M}_1, ..., \widetilde{M}_n$ are SDC (by $P_r \in \mathcal{GL}_r$) if and only if the matrices $M_1(\widetilde{B}), ..., M_n(\widetilde{B})$ are SDC (by $P_n := P_r \oplus I_{n-r}$), the result follows from Theorem 1. □

Remark 2. *The above result shows that the condition that $A/\text{Ann}(A)$ be an evolution algebra is a necessary condition for A to be an evolution algebra. This is known because it was proved in [3] that the quotient of an evolution algebra by an ideal is an evolution algebra. However Theorem 6 proves that this condition is not sufficient (which is new). In fact, if $\dim \text{Ann}(A) := r < n$, and we consider a basis $\widetilde{B}$, as in Theorem 6 above, with m-structure matrices given by $M_k(\widetilde{B}) = \widetilde{M}_k \oplus 0_{n-r}$ for $k = 1, ..., n$, then A is an evolution algebra if, and only if, $\widetilde{M}_1, ..., \widetilde{M}_n$ are SDC. Suppose then that $\widetilde{M}_1, ..., \widetilde{M}_r$ are SDC but that $\widetilde{M}_1, ..., \widetilde{M}_n$, are not SDC. It turns out that $A/\text{Ann}(A)$ is an evolution algebra but A is not (because the m-structure matrices of $A/Ann(A)$ with respect to the basis $\widetilde{B}_{A/Ann(A)} = \{e_1 + Ann(A), ..., e_r + Ann(A)\}$ are precisely $\widetilde{M}_1, ..., \widetilde{M}_r$). It is easy to come up with particular examples of this situation (see Remark 3 below).*

We conclude this section by providing a procedure, obtained from Theorems 1, 5, 3 and 6 above, to determine in a finite number of steps whether or not a given commutative algebra A with fixed basis $B = \{e_1, ..., e_n\}$ is an evolution algebra. Let $M_1, ..., M_n$ be the m-structure matrices of A with respect to B.

While one can try to check directly, see Example 1 below, if the matrices $M_1, ..., M_n$ are SDC this is generally not easy to do. Alternatively, to determine if A is an evolution algebra we can proceed as follows.

Check if any one of the matrices $M_1, ..., M_n$ is invertible.

(a) Suppose that M_{i_0} is invertible, for some $1 \le i_0 \le n$. If $M_{i_0}^{-1}M_1, ..., M_{i_0}^{-1}M_n$ are all diagonalisable (by similarity) and they pairwise commute then we can conclude that A is an evolution algebra, and otherwise we conclude that A is not an evolution algebra.

(b) If none of the matrices $M_1, ..., M_n$ is invertible then we determine $Ann(A)$, that is, by means of (3), we describe those elements $a \in A$ such that $ae_i = 0$ for every $i = 1, ..., n$.

(b.1) If $Ann(A) = \{0\}$ then we check if there exists some $\lambda_0 = (\lambda_1, ..., \lambda_n) \in \mathbb{C}^n$ with $\|\lambda_0\| = 1$ such that $M(\lambda_0) := \sum_{i=1}^{n} \lambda_i M_i$ is invertible. If such a λ_0 does not exist then we conclude that A is not an evolution algebra. Otherwise we have that A is an evolution algebra if, and only if, the matrices $M(\lambda_0)^{-1}M_1, ..., M(\lambda_0)^{-1}M_n$ are all diagonalisable (by similarity) and they pairwise commute.

(b.2) If $Ann(A) \neq \{0\}$ then we construct a basis $\widetilde{B} = \{\widetilde{e}_1, ..., \widetilde{e}_r, \widetilde{e}_{r+1}, ..., \widetilde{e}_n\}$, such that $\{\widetilde{e}_{r+1}, ..., \widetilde{e}_n\}$ is a basis of $Ann(A) \neq \{0\}$. We then have $M_k(\widetilde{B}) = \widetilde{M}_k \oplus 0_{n-r}$ for $k = 1, ..., n$ and $r \times r$ matrices $\widetilde{M}_1, ..., \widetilde{M}_n$. Next, we check if there exists $\lambda_0 = (\lambda_1, ..., \lambda_n) \in \mathbb{C}^n$ with $\|\lambda_0\| = 1$ such that $\widetilde{M}(\lambda_0) := \sum_{i=1}^{n} \lambda_i \widetilde{M}_i$ is invertible as an $r \times r$ matrix. In particular, this is the case whenever $\widetilde{M}_{i_0}$ is invertible for some $1 \le i_0 \le n$ (in which case we can choose $\widetilde{M}(\lambda_0) = \widetilde{M}_{i_0}$). If such a λ_0 does not exist then we conclude that A is not an evolution algebra. Otherwise, we have that A is an evolution algebra if, and only if, the matrices $\widetilde{M}(\lambda_0)^{-1}\widetilde{M}_1, ..., \widetilde{M}(\lambda_0)^{-1}\widetilde{M}_n$ are all diagonalisable (by similarity) and they pairwise commute.

3. Some Examples and Applications

We discuss some examples where our approach is useful to determine whether or not certain classical genetic algebras are evolution algebras. Mostly these algebras are defined in the literature as real algebras but, in our case, they can be regarded as complex algebras (with the same basis, and hence with the same structure m-structure matrices) as shown in Theorem 2 and Corollary 1.

We will consider the class of gametic algebras discussed by Etherington [28]. Gametic algebras, widely used in genetics, are simply baric algebras: they are endowed with a weight function. To decide if these algebras are evolution algebras or not we do not need further background about them. Nevertheless the reader is referred to [29,30] for a review of these algebras.

Example 1. *Let A be the algebra with basis $B = \{e_1, e_2\}$ and $e_1^2 = e_1$, $e_1e_2 = e_2 = e_2e_1$, $e_2^2 = e_1$. Define $\xi : A \to \mathbb{K}$ by $\xi(\alpha e_1 + \beta e_2) = \alpha + \beta$. Obviously ξ is linear and if $a = \alpha e_1 + \beta e_2$ and if $b = \gamma e_1 + \delta e_2$ then*

$$ab = (\alpha\gamma + \beta\delta)e_1 + (\alpha\delta + \beta\gamma)e_2,$$

so that $\xi(ab) = (\alpha\gamma + \beta\delta) + (\alpha\delta + \beta\gamma) = (\alpha + \beta)(\gamma + \delta) = \xi(a)\xi(b)$, and hence ξ is a non-zero algebra homomorphism. Consequently A is a baric algebra [28].

The corresponding m-structure matrices with respect to B are $M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $M_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Since for $P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ we have that $P^T M_1 P = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $P^T M_2 P = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$, by Theorem 1, we obtain that A is an evolution algebra. In fact $\widetilde{B} = \{\widetilde{e}_1, \widetilde{e}_2\}$ with $\widetilde{e}_1 = e_1 - e_2$ and $\widetilde{e}_2 = e_1 + e_2$ is a natural basis of A, as $\widetilde{e}_1\widetilde{e}_2 = 0$.

Remark 3. *Let M_1 and M_2 be as above and consider a matrix M_3 that does not commute with M_2, say for instance $M_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then we have that $M_1^{-1}M_2$ and $M_1^{-1}M_3$ do not commute so that, by the proof of Theorem 6 (or alternatively using Section 3.3 in [25]), the 3×3 matrices $M_1 \oplus 0_{1\times1}$, $M_2 \oplus 0_{1\times1}$ and $M_3 \oplus 0_{1\times1}$ are not SDC, while M_1 and M_2 are SDC. Therefore, the algebra $\widetilde{A}$ with basis $\widetilde{B} = \{e_1, e_2, e_3\}$ and product $e_1^2 = e_1 + e_3$, $e_2^2 = e_1 - e_3$, $e_3^2 = 0$, $e_1e_2 = e_2 = e_2e_1$, $e_1e_3 = e_3e_1 = e_2e_3 = e_3e_2 = 0$ is an algebra such that $\mathrm{Ann}(\widetilde{A}) = \mathbb{K}e_3$. By Theorem 6 (see also Remark 2) we have that $\widetilde{A}$ is therefore not an evolution algebra whereas $\widetilde{A}/\mathrm{Ann}(\widetilde{A})$ is an evolution algebra isomorphic to the evolution algebra A in Example 1.*

Example 2 (Gametic algebra for simple Mendelian inheritance)**.** *Let A_0 denote a commutative 2-dimensional algebra over $\mathbb{R}$, corresponding to the gametic algebra describing simple Mendelian inheritance (see [30]). In terms of the basis $B = \{e_1, e_2\}$ the multiplication table is*

$$e_1^2 = e_1, \quad e_1e_2 = e_2e_1 = \frac{1}{2}(e_1 + e_2), \quad e_2^2 = e_2\,.$$

The associated m-structure matrices M_1, M_2 can be read off easily:

$$M_1 = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}.$$

It is easy to check that A_0 is a baric algebra, with weight function defined by $\xi(e_1) = \xi(e_2) = 1$. Note that $M_1^{-1} = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 2 \\ 2 & -4 \end{pmatrix}$ while

$$M_1^{-1}M_2 = \begin{pmatrix} 0 & 2 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -2 & -3 \end{pmatrix}$$

is not diagonalisable by similarity, as $\lambda = -1$ is the unique eigenvalue and the associated eigenspace has dimension 1. Therefore, by Corollary 3, we obtain that A_0 is not an evolution algebra. (This last assertion can also be deduced from Theorem 1, with more tedious calculations, by directly checking that M_1 and M_2 are not SDC).

We will now deform this algebra in order to construct an evolution algebra.

Example 3 (Evolution algebra for deformed Mendelian inheritance). *Consider a deformation of the algebra A_0 of the previous example. We denote these deformed algebras by A_ε, which depend on the free parameter $\varepsilon \in \mathbb{R}$. In terms of the basis $B = \{e_1, e_2\}$, the multiplication table for A_ε is given by*

$$e_1^2 = (1-\varepsilon)e_1 + \varepsilon\, e_2, \quad e_1 e_2 = e_2 e_1 = \frac{1}{2}(e_1 + e_2), \quad e_2^2 = e_2\,.$$

The associated m-structure matrices M_1, M_2 are now:

$$M_1 = \begin{pmatrix} 1-\varepsilon & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} \varepsilon & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}.$$

For genetic applications, we restrict $0 < \varepsilon \leq 1$ so that all coefficients in these matrices are non-negative. Moreover, A_ε is baric with weight function defined by $\xi(e_1) = \xi(e_2) = 1$, for any ε. In fact $\xi(e_i e_j) = \xi(e_i)\xi(e_j) = 1$, for $i, j = 1, 2$. Obviously, the undeformed case corresponds to $\varepsilon = 0$.

Let us study whether A_ε is an evolution algebra by using Theorem 5. First of all, the maximal rank of the linear pencil $M(\boldsymbol{\lambda}) = \lambda_1 M_1 + \lambda_2 M_2$ is $r = 2$ because M_1 is nonsingular for all ε, so we can take $\boldsymbol{\lambda}_0 = (1, 0)$. Thus $M(\boldsymbol{\lambda}_0) = M_1$. To see that A_ε is an evolution algebra we prove that $M_1^{-1} M_2$ is diagonalisable by similarity. It is easy to check that

$$M_1^{-1} M_2 = \begin{pmatrix} 1 & 2 \\ 4\varepsilon - 2 & 4\varepsilon - 3 \end{pmatrix}$$

and that if

$$P = \begin{pmatrix} 1 & 1 \\ -1 & 2\varepsilon - 1 \end{pmatrix}$$

then

$$\begin{aligned} P^{-1} M_1^{-1} M_2 P &= \\ &= \begin{pmatrix} \frac{1}{2\varepsilon}(2\varepsilon - 1) & -\frac{1}{2\varepsilon} \\ \frac{1}{2\varepsilon} & \frac{1}{2\varepsilon} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 4\varepsilon - 2 & 4\varepsilon - 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 2\varepsilon - 1 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 \\ 0 & 4\varepsilon - 1 \end{pmatrix}. \end{aligned}$$

Since

$$\begin{aligned} P^T M(\boldsymbol{\lambda}_0) P = P^T M_1 P &= \begin{pmatrix} 1 & -1 \\ 1 & 2\varepsilon - 1 \end{pmatrix} \begin{pmatrix} 1-\varepsilon & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 2\varepsilon - 1 \end{pmatrix} \\ &= \begin{pmatrix} -\varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix}, \end{aligned}$$

and $\det P = 2\varepsilon$, *we conclude by Theorem 5 that the algebra A_ε is an evolution algebra if and only if $\varepsilon \neq 0$. For completeness, we show the diagonalisation of the original matrices:*

$$P^T M_1 P = \begin{pmatrix} -\varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix}, \qquad P^T M_2 P = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon(4\varepsilon - 1) \end{pmatrix},$$

which shows by Theorem 1, that A_ε is an evolution algebra for every $\varepsilon > 0$, having $B = \{e_1 - e_2, e_1 + (2\varepsilon - 1)e_2\}$ as a natural basis.

Example 4. *The annihilator of every algebra A_ε in the above example is zero as one of its m-structure matrices is invertible. To get a similar example with algebras having non-zero annihilator consider for instance the algebra A_ε with natural basis $\widehat{B} = \{e_1, e_2, e_3\}$ and product given by*

$$e_1^2 = (1-\varepsilon)e_1 + \varepsilon\, e_2 - \varepsilon\, e_3, \;\; e_2^2 = e_2 - e_3;\; e_3^2 = 0,$$
$$e_1e_2 = e_2e_1 = \frac{1}{2}(e_1 + e_2 - e_3), \quad e_1e_3 = e_3e_1 = e_2e_3 = e_3e_2 = 0.$$

Here, the m-structure matrices are $M_k(\widehat{B}) = M_k \oplus 0$ (for $i = 1,2,3$) where 0 denotes the 1×1 zero matrix, M_1 and M_2 are given in the above example and $M_3 = -M_2$. Hence if

$$P = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 2\varepsilon - 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

we obtain, from the calculations in the above example, that $P^T M_k(\widehat{B}) P$ is diagonal for every $k = 1,2,3$ and hence A_ε is an evolution algebra. Nevertheless, for $\varepsilon = 0$ we do not obtain an evolution algebra. Indeed, if we denote this algebra by A then the quotient algebra $A/\mathrm{Ann}(A)$ is exactly the algebra A_0 in Example 2 which is not an evolution algebra and, consequently, A is not an evolution algebra (see Remark 2).

Example 5 (Gametic algebra for auto-tetraploid inheritance)**.** *Let T_0 denote a 3-dimensional commutative algebra over $\mathbb{R}$, considered the simplest case of special train algebras in polyploidy Chapter 15 in [28] (see also [29,30]). In terms of the basis $\{e_1, e_2, e_3\}$ the multiplication table is given by*

$$e_1^2 = e_1, \quad e_2^2 = e_1e_3 = \frac{1}{6}(e_1 + 4e_2 + e_3),$$
$$e_3^2 = e_3, \quad e_2e_3 = \frac{1}{2}(e_2 + e_3), \quad e_1e_2 = \frac{1}{2}(e_1 + e_2)\,.$$

The corresponding m-structure matrices M_1, M_2, M_3 are

$$M_1 = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & 0 \\ \frac{1}{6} & 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & \frac{1}{2} & \frac{2}{3} \\ \frac{1}{2} & \frac{2}{3} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{2} & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & 0 & \frac{1}{6} \\ 0 & \frac{1}{6} & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{2} & 1 \end{pmatrix}.$$

The algebra T_0 is baric, with a weight function defined by $\xi(e_j) = 1,\;\; j = 1,2,3$.
To see that this algebra is not an evolution algebra note that

$$M_1^{-1} = \begin{pmatrix} 0 & 0 & 6 \\ 0 & 6 & -18 \\ 6 & -18 & 18 \end{pmatrix},$$

and that

$$M_1^{-1}M_2 = \begin{pmatrix} 0 & 0 & 6 \\ 0 & 6 & -18 \\ 6 & -18 & 18 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{2} & \frac{2}{3} \\ \frac{1}{2} & \frac{2}{3} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{2} & 0 \end{pmatrix} = \begin{pmatrix} 4 & 3 & 0 \\ -9 & -5 & 3 \\ 3 & 0 & -5 \end{pmatrix}$$

is not diagonalisable by similarity because it has a single eigenvalue ($\lambda = -2$) and the dimension of the associated eigenspace is 1 (indeed, $(1, -2, 1)^T$ generates it). Consequently, A is not an evolution algebra by Corollary 3.

On the other hand,

$$M_1^{-1}M_3 = \begin{pmatrix} 0 & 0 & 6 \\ 0 & 6 & -18 \\ 6 & -18 & 18 \end{pmatrix} \begin{pmatrix} 0 & 0 & \frac{1}{6} \\ 0 & \frac{1}{6} & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{2} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 6 \\ -3 & -8 & -15 \\ 3 & 6 & 10 \end{pmatrix}$$

so that

$$M_1^{-1}M_2M_1^{-1}M_3 = M_1^{-1}M_3M_1^{-1}M_2 = \begin{pmatrix} -5 & -12 & -21 \\ 15 & 31 & 51 \\ -12 & -21 & -32 \end{pmatrix}.$$

This proves that, in Theorem 5, the condition that the matrices $M(\lambda_0)^{-1}M_1, ..., M(\lambda_0)^{-1}M_n$ pairwise commute is not sufficient to ensure that the given algebra is an evolution algebra (see also Proposition 2).

Example 6 (Evolution algebra for deformed auto-tetraploid inheritance)**.** *Consider now a deformation of the algebra T_0 of the previous example. We denote this deformed algebra by T_ε, which depends on the free parameter $\varepsilon \in \mathbb{R}$. In terms of the basis $\{e_1, e_2, e_3\}$ the multiplication table for T_ε is:*

$$e_1^2 = e_1 + 2\,\varepsilon(e_1 + 4e_2), \quad e_2^2 = \frac{1}{6}(e_1 + 4e_2 + e_3) - \varepsilon(3e_2 - 13e_3), \quad e_3^2 = e_3 + 10\,\varepsilon e_3\,,$$

$$e_1e_3 = \frac{1}{6}(e_1 + 4e_2 + e_3) + 10\,\varepsilon e_3,\ e_2e_3 = \frac{1}{2}(e_2 + e_3) + 10\,\varepsilon e_3,\ e_1e_2 = \frac{1}{2}(e_1 + e_2) + 10\,\varepsilon e_3\,.$$

The corresponding m-strucuture matrices M_1, M_2, M_3 are

$$M_1 = \begin{pmatrix} 1+2\,\varepsilon & \frac{1}{2} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & 0 \\ \frac{1}{6} & 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 8\,\varepsilon & \frac{1}{2} & \frac{2}{3} \\ \frac{1}{2} & \frac{2}{3} - 3\,\varepsilon & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{2} & 0 \end{pmatrix},$$

$$M_3 = \begin{pmatrix} 0 & 10\,\varepsilon & \frac{1}{6} + 10\,\varepsilon \\ 10\,\varepsilon & \frac{1}{6} + 13\,\varepsilon & \frac{1}{2} + 10\,\varepsilon \\ \frac{1}{6} + 10\,\varepsilon & \frac{1}{2} + 10\,\varepsilon & 1 + 10\,\varepsilon \end{pmatrix}.$$

For genetic applications, we restrict $0 < \varepsilon \leq 2/9$, so all coefficients in the above matrices are non-negative. The algebra T_ε is baric, with weight function defined by $\xi(e_j) = 1 + 10\,\varepsilon,\ \ j = 1, 2, 3$.

Let us consider whether T_ε is an evolution algebra. First of all, the maximal rank of the linear pencil $M(\boldsymbol{\lambda}) = \lambda_1 M_1 + \lambda_2 M_2 + \lambda_3 M_3$ is $r = 3$ because M_1 is nonsingular for all ε, so we can take $\boldsymbol{\lambda}_0 = (1, 0, 0)$. Thus $M(\boldsymbol{\lambda}_0) = M_1$. By Theorem 5, a necessary condition is that the matrices $M_1^{-1}M_2$ and $M_1^{-1}M_3$ are simultaneously diagonalisable by similarity: in particular, they must commute. Let us write these matrices explicitly:

$$M_1^{-1}M_2 = \begin{pmatrix} 4 & 3 & 0 \\ -9 & -5 - 18\,\varepsilon & 3 \\ 3 & 18\,\varepsilon & -5 \end{pmatrix},$$

$$M_1^{-1}M_3 = \begin{pmatrix} 1 + 60\,\varepsilon & 3 + 60\,\varepsilon & 6 + 60\,\varepsilon \\ -3(1 + 40\,\varepsilon) & -2(4 + 51\,\varepsilon) & -15(1 + 8\,\varepsilon) \\ 3(1 - 4\,\varepsilon - 240\,\varepsilon^2) & 6(1 - 5\,\varepsilon - 120\,\varepsilon^2) & 2(5 - 6\,\varepsilon - 360\,\varepsilon^2) \end{pmatrix}.$$

It is straightforward to show that these matrices commute for all ε (even for $\varepsilon = 0$). Regarding the Jordan decomposition for $M_1^{-1}M_2$ and $M_1^{-1}M_3$ we find that if $\varepsilon > 0$ then these matrices are simultaneously diagonalisable: in fact, there is a nonsingular matrix P such that $P^{-1}M_1^{-1}M_2P$ is diagonal:

$$P^{-1}M_1^{-1}M_2P = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2-9\varepsilon-3S_\varepsilon & 0 \\ 0 & 0 & 2-9\varepsilon+3S_\varepsilon \end{pmatrix}, \qquad S_\varepsilon = \sqrt{3\varepsilon(3\varepsilon+4)}\,.$$

Explicitly, in terms of the radical S_ε,

$$P = \begin{pmatrix} 1 & 1 & 1 \\ -2 & -2-3\varepsilon-S_\varepsilon & -2-3\varepsilon+S_\varepsilon \\ 1-12\varepsilon & 1+3\varepsilon+S_\varepsilon & 1+3\varepsilon-S_\varepsilon \end{pmatrix}.$$

We find $\det P = -24\varepsilon S_\varepsilon$ *which shows there is a problem at $\varepsilon = 0$. It is easy to show that at $\varepsilon = 0$ the Jordan form of $M_1^{-1}M_2$ is not diagonal. For $\varepsilon > 0$ the Jordan form of $M_1^{-1}M_2$ is diagonal and so is the Jordan form of $M_1^{-1}M_3$:*

$$P^{-1}M_1^{-1}M_3P = \begin{pmatrix} 1-72\varepsilon-720\varepsilon^2 & 0 & 0 \\ 0 & 1+9\varepsilon+3S_\varepsilon & 0 \\ 0 & 0 & 1+9\varepsilon-3S_\varepsilon \end{pmatrix}.$$

For completeness we show the diagonalisation of the original matrices:

$$P^TM_1P = \varepsilon\begin{pmatrix} -2 & 0 & 0 \\ 0 & 4+3\varepsilon+S_\varepsilon & 0 \\ 0 & 0 & 4+3\varepsilon-S_\varepsilon \end{pmatrix},$$

$$P^TM_2P = -2\varepsilon\begin{pmatrix} -2 & 0 & 0 \\ 0 & 4+39\varepsilon+27\varepsilon^2+(9\varepsilon+7)S_\varepsilon & 0 \\ 0 & 0 & 4+39\varepsilon+27\varepsilon^2-(9\varepsilon+7)S_\varepsilon \end{pmatrix},$$

$$P^TM_3P = \varepsilon\begin{pmatrix} \alpha & 0 & 0 \\ 0 & 4+75\varepsilon+54\varepsilon^2+(18\varepsilon+13)S_\varepsilon & 0 \\ 0 & 0 & 4+75\varepsilon+54\varepsilon^2-(18\varepsilon+13)S_\varepsilon \end{pmatrix},$$

where $\alpha = -2+144\varepsilon+1440\varepsilon^2$.

4. Conclusions and Discussion

In this paper we determine completely whether a given algebra A is an evolution algebra, by translating the question to a recently solved problem, namely the problem of simultaneous diagonalisation via congruence of the m-structure matrices of A. This is relevant because evolution algebras have strong connections with areas such as group theory, Markov processes, theory of knots, and graph theory, among others. In fact, every evolution algebra can be canonically regarded as a weighted digraph when a natural basis is fixed, and because of this evolution algebras may introduce useful algebraic techniques into the study of some digraphs.

We also consider applications of our results to classical genetic algebras. Strikingly, the classical cases of Mendelian and auto-tetraploid inheritance are not evolution algebras, while slight deformations of them produce evolution algebras. This is interesting because evolution algebras are supposed to describe asexual reproduction, unlike these classical cases. In future work we will more closely study the relation between baric algebras and evolution algebras, in order to better understand this phenomenon.

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