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# Maximum of entropy for belief intervals under Evidence Theory

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**ABSTRACT** The Dempster-Shafer Theory (DST) or Evidence Theory has been commonly used to deal with uncertainty. It is based on the basic probability assignment concept (BPA). The upper entropy on the credal set associated with a BPA is the only uncertainty measure in DST that verifies all the necessary mathematical properties and behaviors. Nonetheless, its computation is notably complex. For this reason, many alternatives to this measure have been recently proposed, but they do not satisfy most of the mathematical requirements and present some undesirable behaviors. Belief intervals have been frequently employed to quantify uncertainty in DST in the last years, and they can represent the uncertainty-based-information better than a BPA. In this research, we develop a new uncertainty measure that consists of the maximum of entropy on the credal set corresponding to belief intervals for singletons. It verifies all the crucial mathematical requirements and presents good behavior, solving most of the shortcomings found in uncertainty measures proposed recently. Moreover, its calculation is notably easier than the upper entropy on the credal set associated with the BPA. Therefore, our proposed uncertainty measure is more suitable to be used in practical applications.

**INDEX TERMS** Dempster-Shafer Theory, belief intervals, uncertainty measures, maximum of entropy, conflict, non-specificity

## I. INTRODUCTION

The Dempster-Shafer Theory (DST), also known as Evidence Theory [1], [2], has been commonly employed to deal with uncertainty in practical applications such as *statistical classification* [3], *target identification* [4], *medical diagnosis* [5], or *face recognition* [6]. DST generalizes classical Probability Theory (PT); it is based on the concept of basic probability assignment (BPA), an extension of the probability distribution concept in PT.

An important issue in DST is to quantify the uncertainty or lack of information associated with a BPA. Many uncertainty measures have been developed in DST so far. Most of them start from the Shannon entropy [7], a well-established uncertainty measure in PT that verifies desirable properties.

Since DST extends PT, in DST, more types of uncertainty appear than in PT. According to Yager [8], in DST, we can find two types of uncertainty. The first one of them is usually called *conflict*. It corresponds to cases in which the information is focused on disjoint sets. The second type is known as *non-specificity*, which appears when the information resides

in non-singleton sets.

In [9], it was carried out a study about the set of mathematical properties that an uncertainty measure in DST that quantifies both conflict and non-specificity should satisfy. Abellán and Masegosa, in [10], extended that research. Furthermore, they analyzed the behavioral requirements that an uncertainty measure in DST has to verify. So far, the upper entropy on the credal set<sup>1</sup> associated with a BPA, proposed in [9], is the only uncertainty measure in DST that satisfies all the essential mathematical properties and behaviors [10].

Nonetheless, the algorithms proposed so far to calculate the upper entropy on the credal set corresponding to a BPA in [11]–[14] are considerably complex. For this reason, many alternatives to this measure have been proposed in the last years. For example, Shahpari and Seyedin, in [15], introduced an uncertainty measure based on the pignistic transformation of a BPA. However, in [16], it was demonstrated that this measure does not satisfy some needed properties and presents some undesirable behaviors. Afterwards, an

<sup>1</sup>A credal set is a convex and closed set of probability distributions.

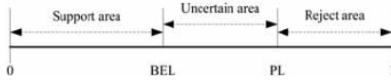


FIGURE 1. Uncertainty-based-information with belief intervals.

uncertainty measure, known as *Deng entropy*, was proposed [17]–[20]. It considers that the uncertainty degree strongly depends on the number of possible alternatives. Abellán, in [21], showed that this function does not satisfy most of the crucial mathematical properties, and its behavior in some scenarios is questionable.

Two recent measures, proposed in [22], [23], utilize a probability transformation based on the upper probabilities for singletons. These functions do not satisfy the subadditivity requirement. In consequence, they present drawbacks for BPAs defined over joint spaces. Moreover, the conflict part of these measures might be greater than 0 when all the focal elements share an element, which is not consistent.

*Belief intervals* associated with a BPA for each subset constitute a tool that has been frequently used to quantify uncertainty in DST in the last years. The belief function of a set (subset) is considered as the minimum support of information represented by that BPA on that set, and the plausibility function is considered as the maximum support of information represented by that BPA on the set [24]. As exposed in [24], belief intervals can represent the uncertainty-based-information better than a BPA. The reason is that, with belief intervals, it is possible to know the uncertain area, as shown in Figure 1. It does not happen directly using the BPA. The set of probability distributions corresponding to belief intervals extends the one associated directly with a BPA, but the first one is easier to manage than the second one.

Recently, some uncertainty measures based on the concept of belief intervals have been proposed, such as the one proposed by Yand and Dezert in [25]. Abellán and Bossé, in [26], demonstrated that this measure does not satisfy some of the required mathematical properties and behaviors for uncertainty measures in DST. Also, in [27], it was proposed an uncertainty measure that combines the Deng entropy with belief intervals. However, this measure does not verify most of the required mathematical properties [27], and it is not trivial how to extend it to more general theories than DST.

In this research, we propose a new uncertainty measure that consists of the maximum of entropy on the credal set corresponding to the belief intervals for singletons. We demonstrate that our proposed measure verifies all the required mathematical properties. It is quite remarkable that our proposal is consistent with a decrease or increase of information in a BPA. In our case, the subadditivity and additivity properties are controversial since they are based on the projections of a BPA defined over a joint set on more simple ones, and our measure is based on belief intervals. We reconsider the definition of these properties for our measure

by considering the projections of the intervals, and we show that our proposal verifies these properties with the reconsidered definitions. Also, the upper entropy on the credal set associated with belief intervals for singletons presents good behavior, solving most of the drawbacks found in recent uncertainty measures based on Deng entropy, probability transformations, and belief intervals.

Moreover, we show that the calculation of our proposal is notably simpler than the upper entropy on the credal set corresponding to the BPA, and the conflict and non-specificity parts are also easier to obtain. In consequence, our proposed measure is more suitable to be used in practical applications than the upper entropy on the credal set associated with a BPA. We also demonstrate that our proposed measure is always greater or equal than the maximum of entropy on the credal set corresponding to a BPA.

The outline of this paper is as follows: Dempster-Shafer Theory is described in Section II. Section III presents an overview of the main uncertainty measures in DST proposed in the literature, and the mathematical properties and behaviors that must be verified by an uncertainty measure in DST. In Section IV, we expose our proposal. Conclusions are given in Section V.

## II. DEMPSTER-SHAFER THEORY

Let  $X = \{x_1, x_2, \dots, x_n\}$  be a finite set considered as the set of possible alternatives. Let us denote  $\wp(X)$  the power set of  $X$ .

The basis of the Dempster-Shafer Theory, or Evidence Theory [1], [2], is the concept of *basic probability assignment*, which consists of a mapping  $m : \wp(X) \rightarrow [0, 1]$  such that  $\sum_{A \in \wp(X)} m(A) = 1$  and  $m(\emptyset) = 0$ .

If  $A \subseteq X$  satisfies that  $m(A) > 0$ , it is said that  $A$  is a *focal element* of  $m$ .

Given a BPA  $m$ , two functions are associated with it: a *belief function*  $Bel_m$ , and a *plausibility function*  $Pl_m$ . They are defined in the following way:

$$Bel_m(A) = \sum_{B|B \subseteq A} m(B), \quad Pl_m(A) = \sum_{B|B \cap A \neq \emptyset} m(B), \quad \forall A \in \wp(X). \quad (1)$$

It is easy to check that  $Bel_m(A) \leq Pl_m(A) \forall A \in \wp(X)$ . The interval  $[Bel_m(A), Pl_m(A)]$  is called the *belief interval* of  $A$ . Furthermore,

$$Pl_m(A) = 1 - Bel_m(\bar{A}), \quad \forall A \subseteq X, \quad (2)$$

where  $\bar{A}$  is the complementary of  $A$ .

For a BPA  $m$ , there exists a convex and closed set of probability distributions, also known as a credal set, corresponding to it. It is defined as follows:

$$\mathcal{P}_m = \{p \in \mathcal{P}(X) \mid Bel_m(A) \leq p(A) \quad \forall A \in \wp(X)\}, \quad (3)$$

being  $\mathcal{P}(X)$  the set of all probability distributions on  $X$ . We can observe that the condition  $Bel_m(A) \leq p(A) \forall A \subseteq X$  is equivalent to  $Bel_m(A) \leq p(A) \leq Pl_m(A) \forall A \subseteq X$ , due to the relation given in Equation (2).

Let  $X$  and  $Y$  be finite sets. Let us suppose that we have a BPA  $m$  on the product space  $X \times Y$ . The marginal BPA on  $X$ ,  $m^{\downarrow X}$ , is defined as follows:

$$m^{\downarrow X}(A) = \sum_{R|A=R_X} m(R), \quad \forall A \in \wp(X), \quad (4)$$

where  $R_X$  denotes the projection of  $R$  on  $X$ . In the same way, it is possible to define the marginal BPA on  $Y$ ,  $m^{\downarrow Y}$ .

### III. UNCERTAINTY MEASURES IN DST

It is known that, in classical possibility theory, uncertainty is measured via the Hartley measure [28], defined as follows:

$$H(A) = \log_2(|A|), \quad \forall A \subseteq X. \quad (5)$$

The type of uncertainty captured by  $H$  is often called *non-specificity*.

On the other hand, in classical Probability Theory (PT), the Shannon entropy [7] is a well-established uncertainty measure. It is defined as follows:

$$S(p) = - \sum_{x \in X} p(x) \log_2(p(x)), \quad (6)$$

being  $p = (p(x))_{x \in X}$  a probability distribution on  $X$ . The type of conflict measured by the function  $S$  is known as *conflict*, which is the only one present in PT. It satisfies desirable properties [7], [9].

According to Yager [8], in DST, both types of uncertainty: non-specificity and conflict coexist. In DST, non-specificity appears when information is focused on non-singleton sets; and conflict corresponds to cases in which the information resides in sets with empty intersection.

Dubois and Prade, in [29], introduced a generalization of the Hartley measure to DST, defined as follows:

$$GH(m) = \sum_{A \in \wp(X)} m(A) \log_2(|A|). \quad (7)$$

The minimum value of  $GH$ , which is equal to 0, is attained when  $m$  is a probability distribution. When  $m(X) = 1$ ,  $GH$  obtains its maximum value, i.e  $\log_2(n)$ , where  $n = |X|$ .  $GH$  was established as a suitable non-specificity measure in DST that verifies desirable properties. In addition, it is possible to extend it to more general theories than DST [30].

In the literature, several conflict measures were proposed to extend the Shannon entropy to DST, but any of them satisfies the required properties in DST for this type of measure.

These unsuccessful attempts were replaced by a total uncertainty measure in DST that captures both conflict and non-specificity. That measure, developed by Harmanec and Klir in [12], consists of the upper entropy on the credal set associated

with a BPA  $m$ , denoted by  $S^*(\mathcal{P}_m)$ . This measure, in [9], was established as appropriated to quantify the total uncertainty in DST since it satisfied the required properties.

However, when  $S^*(\mathcal{P}_m)$  was proposed, it did not separate conflict and non-specificity. Abellan, Klir, and Moral, in [31], proposed a coherent disaggregation of  $S^*$  between conflict and non-specificity on more general theories than DST. This separation also works for DST. It can be considered:

$$S^*(\mathcal{P}_m) = S_*(\mathcal{P}_m) + (S^* - S_*)(\mathcal{P}_m), \quad (8)$$

where  $S_*(\mathcal{P}_m)$  is the minimum of entropy on  $\mathcal{P}_m$ .  $S_*(\mathcal{P}_m)$  quantifies the conflict part and  $(S^* - S_*)(\mathcal{P}_m)$  the non-specificity part. Algorithms to calculate  $S^*$  can be found in [11]–[14].

The algorithms for the calculation of  $S^*(\mathcal{P}_m)$  proposed so far are notably complex. Hence, many alternatives to this measure have been proposed in the last years.

For example, in [17], an uncertainty measure, called *Deng entropy*, was introduced. It also separates conflict and non-specificity, and it is defined as follows:

$$E_d(m) = - \sum_{A \in \wp(X)} m(A) \log_2 \left( \frac{m(A)}{2^{|A|} - 1} \right) = \sum_{A \in \wp(X)} m(A) \log_2 (2^{|A|} - 1) - \sum_{A \in \wp(X)} m(A) \log_2 (m(A)). \quad (9)$$

The first term of the previous expression quantifies the non-specificity part, whereas the second one captures conflict. According to this measure, the amount of uncertainty has to increase as there are more alternatives. Nonetheless, this function does not satisfy most of the necessary properties for an uncertainty measure in DST, and its behavior in many situations is questionable [21].

Some recent measures are based on the plausibility transformation [32], [33], which is defined as follows:

$$Pt(x) = \frac{Pl_m(\{x\})}{\sum_{x \in X} Pl_m(\{x\})}, \quad \forall x \in X. \quad (10)$$

Jirousek and Shenoy, in [22], proposed a new uncertainty measure that consists of the sum of the entropy of the plausibility transformation defined above and  $GH$ :

$$H_{JS}(m) = - \sum_{x \in X} Pt(x) \log_2 (Pt(x)) + GH(m). \quad (11)$$

The first term of the previous expression corresponds to conflict, whereas the second one captures non-specificity.

An uncertainty measure that also uses the plausibility transformation was proposed in [23]. It is defined in the following way:

$$H_{PQ}(m) = - \sum_{A \subseteq X} m(A) \log_2 (Pl_m(A)) + GH(m), \quad (12)$$

where  $Pm(A) = \sum_{x \in A} Pt(x) \forall A \subseteq X$ . The first term captures conflict, and the second one is associated with non-specificity.

As said in the Introduction, belief intervals may be more appropriate than the BPA for representing uncertainty in DST. Hence, some alternatives to  $S^*(\mathcal{P}_m)$  are based on belief intervals. A very recent example is the uncertainty measure proposed in [27], which combines belief intervals with the Deng entropy. It is defined as follows:

$$H_{inter}(m) = - \sum_{x \in X} \frac{Bel_m(\{x\}) + Pl_m(\{x\})}{2} \times \log_2 \left[ \frac{Bel_m(\{x\}) + Pl_m(\{x\})}{2} \times \exp(-(Pl_m(\{x\}) - Bel_m(\{x\}))) - \sum_{A \subseteq X || |A| \geq 2} m(A) \times \log_2 \left[ \frac{m(A)}{2^{|A|} - 1} \exp(-(Pl_m(A) - Bel_m(A))) \right] \right]. \quad (13)$$

In the previous expression, the first term corresponds to conflict, while the second one indicates non-specificity.

#### A. REQUIRED MATHEMATICAL PROPERTIES FOR UNCERTAINTY MEASURES IN DST

In [9], Klir and Wierman exposed the following set of five crucial mathematical requirements that have to be satisfied by every total uncertainty measure (TU) in DST that jointly quantifies conflict and non-specificity:

- (P1) **Probabilistic consistency:** If  $m$  is a BPA such that all its focal elements are singletons, then a TU measure must collapse to the Shannon entropy:

$$TU(m) = \sum_{x \in X} m(\{x\}) \log_2(m(\{x\})). \quad (14)$$

- (P2) **Set consistency:** If  $A \subseteq X$  verifies that  $m(A) = 1$ , then a TU measure has to coincide with the Hartley measure:

$$TU(m) = \log_2 |A|. \quad (15)$$

- (P3) **Range:** A TU measure has to take values in the interval  $[0, \log_2 |X|]$ .
- (P4) **Subadditivity:** Let  $m$  be a BPA defined on a product space  $X \times Y$ . Let us denote  $m^{\downarrow X}$  and  $m^{\downarrow Y}$  its respective marginal BPAs on  $X$  and  $Y$ . Then, the following inequality must be satisfied by every TU measure:

$$TU(m^{\downarrow X}) + TU(m^{\downarrow Y}) \geq TU(m). \quad (16)$$

- (P5) **Additivity:** Let  $m$  be a BPA defined on a product space  $X \times Y$  and let  $m^{\downarrow X}$  and  $m^{\downarrow Y}$  be its respective marginal BPAs on  $X$  and  $Y$ . Let us suppose that the marginal BPAs are not-interactive, i.e  $m(A \times B) = m^{\downarrow X}(A)m^{\downarrow Y}(B) \forall A \subseteq X, B \subseteq Y$ , and  $m(C) = 0$

if  $C \neq A \times B$ . Then, every TU measure has to satisfy that:

$$TU(m^{\downarrow X}) + TU(m^{\downarrow Y}) = TU(m). \quad (17)$$

In DST, it can appear scenarios that never occur in PT because DST is a more general theory. A probability distribution is never contained in another one. Nevertheless, in DST, the information associated with a BPA can be contained in the information corresponding to another one [10], [21]. Every uncertainty measure in DST has to take into consideration this point. For this reason, the following property is essential [10]:

- (P6) **Monotonicity:** Every TU measure in DST must take into consideration coherently an increase or decrease of information.

More formally, let us suppose that  $m_1$  and  $m_2$  are two BPAs on  $X$  such that  $\mathcal{P}_{m_1} \subseteq \mathcal{P}_{m_2}$ . Then, it must be satisfied that:

$$TU(m_1) \leq TU(m_2). \quad (18)$$

The Deng entropy only verifies the probabilistic consistency [21]. Both  $H_{JS}$  and  $H_{PQ}$  satisfy probabilistic consistency, additivity, and monotonicity, but not range, nor set consistency, nor subadditivity [22], [23]. The combination of the Deng entropy with belief intervals ( $H_{inter}$ ), only satisfies the probabilistic consistency and monotonicity properties [27].<sup>2</sup> A summary of the mathematical properties satisfied by the  $E_d$ ,  $H_{JS}$ ,  $H_{PQ}$ , and  $H_{inter}$  can be seen in Table 1. So far, the only uncertainty measure in DST that verifies the six required mathematical properties is the upper entropy on the credal set associated with a BPA  $m$ ,  $S_*(\mathcal{P}_m)$  [10].

Property	$E_d$	$H_{JS}$	$H_{PQ}$	$H_{inter}$
Probabilistic consistency	yes	yes	yes	yes
Set consistency	no	no	no	no
Range	no	no	no	no
Subadditivity	no	no	no	no
Additivity	no	yes	yes	no
Monotonicity	no	yes	yes	yes

TABLE 1. Summary of the essential mathematical properties verified by some recent uncertainty measures in DST.

#### B. REQUERIMENTS OF BEHAVIOR FOR UNCERTAINTY MEASURES IN DST

In [34], it was concluded that  $S_*(\mathcal{P}_m)$  has some shortcomings. Klir and collaborators presented these drawbacks in the literature. They can be expressed as follows:

- Computing complexity.
- No separation into the two types of uncertainty coexisting in DST: conflict and non-specificity.
- Insensitivity to changes in evidence.

<sup>2</sup>In [27], it was not given a formal proof about the monotonicity of  $H_{inter}$ , but it was shown with illustrative numerical examples that  $H_{inter}$  satisfies this requirement.

Abellán and Masegosa, in [10], analyzed these considerations. They showed that, sometimes, an increase in non-specificity might produce a decrease in conflict and vice-versa. In this way, we can have similar values of TU with different values of the conflict and non-specificity parts. Hence, they exposed the set of behavioral requirements that must be satisfied by every TU measure in DST as follows:

- (RB1): The computation of a TU measure must not be too complex.
- (RB2): It must be possible to decompose a TU measure as an aggregate one that jointly quantifies conflict and non-specificity.
- (RB3): A TU measure has to be sensitive to changes of evidence, directly or via its parts of conflict and non-specificity.

Based on the "Generalized Information Theory" [9], in some situations, it is more appropriate to mathematically quantify the available information with more general theories than DST. In these cases, the *principle of uncertainty invariance* must be taken into account, which establishes that "when a representation of uncertainty in one mathematical theory is transformed into its counterpart in another theory, the amount of information must be preserved". Thus, every TU measure in DST must satisfy the following behavioral requirement:

- (RB4): It must be possible to extend a TU measure to more general theories than DST.

Both  $H_{JS}$  and  $H_{PQ}$  have an easy calculation and separate conflict and non-specificity. Furthermore, it is possible to extend them to more general theories than DST because GH is generalizable to these theories, and in all the theories that extend DST the information can be described via an upper probability function [35]. In [10], it was shown that the upper entropy satisfies RB2, RB3, and RB4, although its calculation is complex. The Deng entropy and  $H_{inter}$  are much easier to compute than  $S_*(\mathcal{P}_m)$ , and coherently separate conflict and non-specificity. However, the generalization of the Deng entropy and  $H_{inter}$  is still an open question.

#### IV. MAXIMUM OF ENTROPY ON BELIEF INTERVALS

Let  $X = \{x_1, x_2, \dots, x_n\}$  be a finite set and let  $m$  be a BPA on  $X$ . Let us denote  $Bel_m$  and  $Pl_m$  the belief and plausibility functions corresponding to  $m$ , respectively.

Let us consider the set of belief intervals for singletons:

$$\mathcal{I}_m = \{[Bel_m(\{x_i\}), Pl_m(\{x_i\})] \quad \forall i = 1, 2, \dots, n\}. \quad (19)$$

This set of belief intervals is associated with the following credal set [36]:

$$\mathcal{P}(\mathcal{I}_m) = \{p \in \mathcal{P}(X) \mid Bel_m(\{x_i\}) \leq p(\{x_i\}) \leq Pl_m(\{x_i\}), \forall i = 1, 2, \dots, n\}, \quad (20)$$

being  $\mathcal{P}(X)$  the set of all probability distributions on  $X$ .

Our proposed uncertainty measure consists of the maximum of entropy on this credal set:

$$S^*(\mathcal{P}(\mathcal{I}_m)) = \max_{p \in \mathcal{P}(\mathcal{I}_m)} S(p). \quad (21)$$

It is consistent with the *principle of minimum information* [37]. According to it, "the probability distribution that produces the maximum of entropy, compatible with the available restrictions, should be chosen".

For the calculation of  $S^*(\mathcal{P}(\mathcal{I}_m))$ , we first need to compute the belief intervals for singletons. We know that:

$$Bel_m(\{x_i\}) = m(\{x_i\}),$$

$$Pl_m(\{x_i\}) = \sum_{A \subseteq X \mid x_i \in A} m(A), \quad \forall i = 1, 2, \dots, n.$$

Thus, the computation of  $Bel_m(\{x_i\})$  is direct. For the calculation of  $Pl_m(\{x_i\})$ ,  $i = 1, 2, \dots, n$ , the following procedure can be carried out:

```

for  $i = 1$  to  $n$  do
     $Pl_m(\{x_i\}) \leftarrow Bel_m(\{x_i\})$ 
end for
for  $A \subseteq X$  do
    if  $|A| \geq 2$  then
        for  $x_i \in A$  do
             $Pl_m(\{x_i\}) \leftarrow Pl_m(\{x_i\}) + m(A)$ 
        end for
    end if
end for
    
```

We show below an example of the computation of the belief intervals for singletons.

**Example 1:** Let us consider the following BPA on the finite set  $X = \{x_1, x_2, x_3\}$ :

$$m(\{x_1\}) = 0.3, \quad m(\{x_3\}) = 0.2, \quad m(\{x_1, x_2\}) = 0.2,$$

$$m(\{x_1, x_2, x_3\}) = 0.3.$$

Taking into account that  $Bel_m(\{x_i\}) = m(\{x_i\})$ , for  $i = 1, 2, 3$ , we have the following belief values for singletons:

$$Bel_m(\{x_1\}) = 0.3, \quad Bel_m(\{x_2\}) = 0,$$

$$Bel_m(\{x_3\}) = 0.2.$$

For the plausibilities of singletons, we initially have their corresponding belief values. After considering  $\{x_1, x_2\}$  in the previous algorithm, we have the following plausibility values for singletons:

$$Pl_m(\{x_1\}) = Bel_m(\{x_1\}) + m(\{x_1, x_2\}) = 0.5,$$

$$Pl_m(\{x_2\}) = Bel_m(\{x_2\}) + m(\{x_1, x_2\}) = 0.3,$$

$$Pl_m(\{x_3\}) = Bel_m(\{x_3\}) = 0.2.$$

The final plausibility values for singletons are the following ones:

$$Pl_m(\{x_1\}) = Bel_m(\{x_1\}) +$$

$$m(\{x_1, x_2\}) + m(\{x_1, x_2, x_3\}) = 0.8,$$

$$Pl_m(\{x_2\}) = Bel_m(\{x_2\}) +$$

$$m(\{x_1, x_2\}) + m(\{x_1, x_2, x_3\}) = 0.6,$$

$$Pl_m(\{x_3\}) = Bel_m(\{x_3\}) + m(\{x_1, x_2, x_3\}) = 0.5.$$

Once we have the belief intervals for singletons, we must check that they satisfy the *reachability* condition [36]:

**Definition 1:** "A given set of probability intervals on a finite set  $X = \{x_1, x_2, \dots, x_n\}$ ,  $\mathcal{I} = \{[l(\{x_i\}), u(\{x_i\})], \forall i = 1, 2, \dots, n\}$ , is said to be reachable if, and only if, for each  $i = 1, 2, \dots, n$  and each  $v(x_i) \in [l(\{x_i\}), u(\{x_i\})]$ , there exists a probability distribution  $p \in \mathcal{P}(\mathcal{I})$  such that  $p(\{x_i\}) = v(x_i)$ , being  $\mathcal{P}(\mathcal{I})$  the credal set associated with  $\mathcal{I}$ ".

The following definition of reachability of a given set of probability intervals is equivalent to the one shown above [36]:

**Definition 2:** A given set of probability intervals  $\mathcal{I} = \{[l(\{x_i\}), u(\{x_i\})], \forall i = 1, 2, \dots, n\}$  is reachable iff it satisfies that:

$$\sum_{j=1, j \neq i}^n l(\{x_j\}) + u(\{x_i\}) \leq 1, \quad \forall i = 1, 2, \dots, n$$

$$\sum_{j=1, j \neq i}^n u(\{x_j\}) + l(\{x_i\}) \geq 1, \quad \forall i = 1, 2, \dots, n$$

The following result shows that the set of belief intervals on singletons is reachable:

**Proposition 1:** The set of belief intervals on singletons  $I_{Bel_m} = \{[Bel_m(\{x_i\}), Pl_m(\{x_i\})] \mid i = 1, \dots, n\}$  is reachable.

*Proof:*

For each  $A \subseteq X$ , let us define  $\alpha_i(A) = |A| - 1$  if  $x_i \in A$  and  $\alpha_i(A) = |A|$  else,  $\forall i = 1, 2, \dots, n$ . Thus,

$$Bel_m(\{x_i\}) + \sum_{j=1, j \neq i}^n Pl_m(\{x_j\}) =$$

$$m(\{x_i\}) + \sum_{j=1, j \neq i}^n \sum_{A \subseteq X | x_j \in A} m(A) =$$

$$\sum_{i=1}^n m(x_i) + \sum_{A \subseteq X, |A| \geq 2} \alpha_i(A) m(A) \geq$$

$$\sum_{A \subseteq X} m(A) = 1, \quad \forall i = 1, 2, \dots, n$$

since  $m(A) \geq 0 \forall A \subseteq X$  and  $\alpha_i(A) \geq 1 \forall A$  such that  $|A| \geq 2$ .

$$Pl_m(\{x_i\}) + \sum_{j=1, j \neq i}^n Bel_m(\{x_j\}) =$$

$$\sum_{A \subseteq X | x_i \in A} m(A) + \sum_{j=1, j \neq i}^n m(\{x_j\}) =$$

$$\sum_{j=1}^n m(\{x_j\}) + \sum_{A \subseteq X | x_i \in A \wedge |A| \geq 2} m(A) \leq$$

$$\sum_{A \subseteq X} m(A) = 1, \quad \forall i = 1, 2, \dots, n.$$

□

Now, we can utilize the algorithm presented in [38] to obtain the maximum of entropy on the credal set corresponding to a reachable set of probability intervals. We introduce the following notation:

- $Min(p, In_x)$  indicates the minimum value of the distribution  $p$  among the components whose index belongs to the index set  $In_x$ .
- $Sig(p, In_x)$  denotes the second minor value of the distribution  $p$  among the components whose index belongs to the index set  $In_x$ . If that second minor value does not exist, then  $Sig(p, In_x) = -1$ .
- $N_{min}(p, In_x)$  indicates the number of indices in the index set  $In_x$  that attain the minimum value of the distribution  $p$  among the components whose index belongs to  $In_x$ .
- $Min(a, b, c)$  denotes the minimum value of  $\{a, b, c\}$ , real numbers.

Then, the following procedure can be used to obtain the probability distribution that attains the maximum of entropy on  $\mathcal{P}(\mathcal{I}_m)$ , which we denote  $\hat{p}$ :

```

index_set ← {1, 2, ..., n}
for i = 1 to n do
     $\hat{p}(\{x_i\}) \leftarrow Bel_m(\{x_i\})$ 
end for
mass ← 1 -  $\sum_{i=1}^n Bel_m(\{x_i\})$ 
while mass > 0 do
    for i ∈ index_set do
        if  $\hat{p}(\{x_i\}) = Pl_m(\{x_i\})$  then
            index_set ← index_set \ {i}
        end if
    end for
    min ← Min( $\hat{p}$ , index_set)
    second ← Sig( $\hat{p}$ , index_set)
    m ← Nmin( $\hat{p}$ , index_set)
    for i ∈ index_set do
        if  $\hat{p}(\{x_i\}) = min$  then
            if second = -1 then
                 $\hat{p}(\{x_i\}) \leftarrow \hat{p}(\{x_i\}) +$ 
                    Min(Pl( $\{x_i\}) -  $\hat{p}(\{x_i\})$ ,  $\frac{mass}{m}$ , 1)
                mass ← mass -
                    Min(Pl( $\{x_i\}) -  $\hat{p}(\{x_i\})$ ,  $\frac{mass}{m}$ , 1)
            else
                 $\hat{p}(\{x_i\}) \leftarrow \hat{p}(\{x_i\}) +$ 
                    Min(Pl( $\{x_i\}) -  $\hat{p}(\{x_i\})$ ,
                        second -  $\hat{p}(\{x_i\})$ ,  $\frac{mass}{m}$ )
                mass ← mass -
                    Min(Pl( $\{x_i\}) -  $\hat{p}(\{x_i\})$ ,
                        second -  $\hat{p}(\{x_i\})$ ,  $\frac{mass}{m}$ )
            end if
        end if
    end for$$$$ 
```

**end if**  
**end for**  
**end while**

We separate  $S^*(\mathcal{P}(\mathcal{I}_m))$  into two measures that capture conflict and non-specificity in a similar way as the maximum of entropy on the credal set associated with the BPA:

$$S^*(\mathcal{P}(\mathcal{I}_m)) = S_*(\mathcal{P}(\mathcal{I}_m)) + (S^* - S_*)(\mathcal{P}(\mathcal{I}_m)), \quad (22)$$

being  $S_*(\mathcal{P}(\mathcal{I}_m))$  the minimum of entropy on  $\mathcal{P}(\mathcal{I}_m)$ . The first term indicates conflict, whereas the second one quantifies the non-specificity part.

The term of conflict,  $S_*(\mathcal{P}(\mathcal{I}_m))$ , is equal to 0 if, and only if,  $\mathcal{P}(\mathcal{I}_m)$  contains a degenerate probability distribution. This only happens if  $\exists \bar{p} \in \mathcal{P}(\mathcal{I}_m)$  such that  $\bar{p}(\{x_j\}) = 1$  and  $\bar{p}(\{x_i\}) = 0 \forall i \neq j$ . There exists such probability distribution iff  $Pl_m(\{x_j\}) = 1$ , i.e. iff  $x_j$  belongs to all the focal sets. Therefore, there is no conflict only when all the focal sets have a non-empty intersection. The maximum value of  $S_*(\mathcal{P}(\mathcal{I}_m))$ ,  $\log_2(n)$ , is obtained when  $\mathcal{P}(\mathcal{I}_m)$  only contains the uniform probability distribution, which is quite logical. For these reasons, we can say that our conflict measure makes sense, and it does not present problems when all the focal sets are not disjunct, unlike the Deng entropy [21]. In such situations, the conflict value of  $H_{JS}$  and  $H_{PD}$  might be greater than 0. Indeed, in these cases, the plausibility transformation for many elements may be lower than 1, as well as the sum of the plausibility transformations of the elements belonging to many subsets. Consequently, both  $H_{JS}$  and  $H_{PD}$  have an undesirable behavior when all the focal sets share an element.

For calculating  $S_*(\mathcal{P}(\mathcal{I}_m))$ , we utilize the following Lemma, proved by Wasserman and Kadane in [39]:

**Lemma 1:**

“Let  $X$  be a discrete variable that takes values in  $\{x_1, x_2, \dots, x_n\}$ . Let  $p$  and  $q$  be two probability distributions on  $X$ . We denote  $p(\{x_i\})$  as  $p_i$  and  $q(\{x_i\})$  as  $q_i \forall i = 1, 2, \dots, n$ , in such a way that  $p = (p_1, p_2, \dots, p_n)$  and  $q = (q_1, q_2, \dots, q_n)$ . Let  $p^*$  (respectively,  $q^*$ ) be the array  $p$ , (respectively,  $q$ ) ordered decreasingly. If  $\sum_{i=1}^j p_i^* \leq \sum_{i=1}^j q_i^*, \forall j = 1, 2, \dots, n$  then  $S(p) \geq S(q)$ ”.

Let us denote  $\bar{p}$  the distribution of minimum of entropy in  $\mathcal{P}(\mathcal{I}_m)$  and  $((Bel_m)_1, (Bel_m)_2, \dots, (Bel_m)_n)$  (respectively,  $((Pl_m)_1, (Pl_m)_2, \dots, (Pl_m)_n)$ ) the array of beliefs (respectively, plausibilities) for singletons. Let  $Pl_m^*$  be the array of plausibilities for singletons ordered decreasingly. Let  $Bel'_m$  be the array of beliefs for singletons ordered in the same way as  $Pl_m^*$ . Let  $\bar{p}^*$  be the array of  $\bar{p}$  ordered decreasingly. Then,  $\bar{p}$  can be obtained via the following procedure:

**for**  $i = 1$  to  $n$  **do**  
     $\bar{p}_i \leftarrow (Bel_m)_i$   
**end for**  
 $mass \leftarrow 1 - \sum_{i=1}^n (Bel_m)_i$   
 $r \leftarrow 1$

$first\_step \leftarrow false$   
**while**  $first\_step = false$  **do**  
    **if**  $(Pl_m^*)_r - (Bel'_m)_r < mass$  **then**  
         $\bar{p}_r^* \leftarrow (Pl_m^*)_r$   
         $mass \leftarrow mass - (Pl_m^*)_r + (Bel'_m)_r$   
         $r \leftarrow r + 1$   
    **else**  
         $first\_step \leftarrow true$   
    **end if**  
**end while**  
 $k \leftarrow \arg \max_{l \geq r} \{(Bel'_m)_l + mass\}$   
 $\bar{p}_r^* \leftarrow (Bel'_m)_k + mass$

We may note that, at the end of the while loop of the algorithm,  $r \leq n$  because the belief intervals of singletons are reachable, as shown in Proposition 1. The following result shows that the probability distribution obtained in the previous procedure attains the minimum of entropy in  $\mathcal{P}(\mathcal{I}_m)$ .

**Theorem 1:** The probability distribution  $\bar{p}$  obtained in the previous algorithm verifies that  $S_*(\mathcal{P}(\mathcal{I}_m)) = S(\bar{p})$ .

*Proof:* With the same notation as in the previous algorithm, we have that

$$\bar{p}^* = ((Pl_m^*)_1, \dots, (Pl_m^*)_{r-1}, \alpha_k, (Bel''_m)_{r+1}, \dots, (Bel''_m)_n), \text{ where}$$

$$(Bel''_m)_i \in \{(Bel_m)_1, \dots, (Bel_m)_n\} \forall i = r+1, \dots, n,$$

$$(Bel''_m)_i \geq (Bel''_m)_j \forall i, j \in \{r+1, \dots, n\} \text{ with } j \leq i,$$

$$\text{and } \alpha_k \in [(Bel'_m)_k, (Pl_m^*)_k].$$

Let us suppose now that  $q \in \mathcal{P}(\mathcal{I}_m)$  and let  $q^*$  be its corresponding array ordered decreasingly.

For  $1 \leq j \leq r-1$  clearly:

$$\sum_{i=1}^j \bar{p}_i^* = \sum_{i=1}^j (Pl_m^*)_i \geq \sum_{i=1}^j q_i^*.$$

For  $j \in [r, n]$ :

$$\sum_{i=1}^j \bar{p}_i^* = 1 - \sum_{i=j+1}^n (Bel''_m)_i \geq 1 - \sum_{i=j+1}^n q_i^* = \sum_{i=1}^j q_i^*.$$

In consequence,  $\sum_{i=1}^j \bar{p}_i^* \geq \sum_{i=1}^j q_i^*, \forall j = 1, \dots, n$ .

Due to Lemma 1,  $S(\bar{p}) \leq S(q)$ . Thus, we can conclude that  $\bar{p}$  is the distribution with the minimum entropy on  $\mathcal{P}(\mathcal{I}_m)$ .  $\square$

We show below an example about the procedure to obtain  $S_*(\mathcal{P}(\mathcal{I}_m))$ .

**Example 2:**

Let be the finite set  $X = \{x_1, x_2, x_3, x_4\}$  and let us consider the following BPA on  $X$ :

$$m(\{x_1\}) = 0.1, \quad m(\{x_2, x_3\}) = 0.6,$$

$$m(\{x_1, x_4\}) = 0.3.$$

We have the following belief intervals for singletons:

$$\begin{aligned} x_1 &\rightarrow [0.1, 0.4], \\ x_2 &\rightarrow [0, 0.6], \\ x_3 &\rightarrow [0, 0.6], \\ x_4 &\rightarrow [0, 0.3]. \end{aligned}$$

If we carry out the steps of the previous algorithm, we obtain the following values for  $\bar{p} = (\bar{p}(\{x_1\}), \bar{p}(\{x_2\}), \bar{p}(\{x_3\}), \bar{p}(\{x_4\}))$ , the distribution of minimum entropy on the credal set corresponding to these belief intervals:

$$\begin{aligned} p &= (0.1, 0, 0, 0) \\ p &= (0.1, 0, 0.6, 0) \\ p &= (0.4, 0, 0.6, 0) \end{aligned}$$

Regarding the non-specificity part of  $S^*(\mathcal{P}(\mathcal{I}_m))$ ,  $(S^* - S_*)(\mathcal{P}(\mathcal{I}_m))$ , it is equal to 0 (its minimum value) when  $\mathcal{P}(\mathcal{I}_m)$  contains a single probability distribution, i.e when  $Bel_m(\{x_i\}) = Pl_m(\{x_i\}) \forall i = 1, 2, \dots, n$ , and it happens if, and only if,  $m$  is a probability distribution. The maximum value of  $(S^* - S_*)(\mathcal{P}(\mathcal{I}_m))$ , which is equal to  $\log_2(n)$ , is obtained when  $\mathcal{P}(\mathcal{I}_m)$  contains the uniform probability distribution and a degenerate one. Hence, we can say that  $(S^* - S_*)(\mathcal{P}(\mathcal{I}_m))$  makes a lot of sense as a non-specificity measure.

### A. MATHEMATICAL PROPERTIES OF OUR NEW UNCERTAINTY MEASURE

We analyze which of the necessary mathematical properties for an uncertainty measure in DST, described in Section III-A, are satisfied by our proposed measure  $S^*(\mathcal{P}(\mathcal{I}_m))$ .

- **Probabilistic consistency:** If all the focal elements of  $m$  are singletons, then  $Bel_m(\{x_i\}) = m(\{x_i\}) = Pl_m(\{x_i\}) \forall i = 1, \dots, n$ . In this situation,  $\mathcal{P}(\mathcal{I}_m)$  only contains a probability distribution  $p$ , given by  $p(\{x_i\}) = m(\{x_i\})$ ,  $\forall i = 1, \dots, n$ , and it is obvious that  $S^*(\mathcal{P}(\mathcal{I}_m))$  coincides with the Shannon entropy.
- **Set consistency:** If  $\exists A \subseteq X$  such that  $m(A) = 1$ , then  $Bel_m(\{x_i\}) = Pl_m(\{x_i\}) = 0 \forall x_i \notin A$ . In addition,  $Bel_m(\{x_i\}) = 0$ , and  $Pl_m(\{x_i\}) = 1 \forall x_i \in A$ . It can be easily deduced that, in this case, the probability distribution that attains that maximum of entropy on  $\mathcal{P}(\mathcal{I}_m)$  is the one given by:

$$\hat{p}(\{x_i\}) = \begin{cases} \frac{1}{|A|} & \text{if } x_i \in A \\ 0 & \text{if } x_i \notin A \end{cases}$$

Thus,

$$\begin{aligned} S^*(\mathcal{P}(\mathcal{I}_m)) &= S(\hat{p}) = - \sum_{x_i \in A} \frac{1}{|A|} \log_2 \left( \frac{1}{|A|} \right) = \\ &= -|A| \times \left[ \frac{1}{|A|} \log_2 \left( \frac{1}{|A|} \right) \right] = \log_2(|A|), \end{aligned}$$

and  $S^*(\mathcal{P}(\mathcal{I}_m))$  collapse to the Hartley measure.

- **Range:** The minimum value of  $S^*(\mathcal{P}(\mathcal{I}_m))$  is equal to 0. It is obtained when  $\mathcal{P}(\mathcal{I}_m)$  just contains a degenerate probability distribution  $p$ , i.e  $p(\{x_i\}) = 1$  for some  $i \in \{1, \dots, n\}$  and  $p(\{x_j\}) = 0, \forall j \neq i$ .  $S^*(\mathcal{P}(\mathcal{I}_m))$  attains its maximum value when the uniform probability distribution belongs to  $\mathcal{P}(\mathcal{I}_m)$ . That maximum value is equal to  $\log_2(n)$ . Therefore,  $S^*(\mathcal{P}(\mathcal{I}_m))$  verifies the range property.

- **Subadditivity:** This property is based on the projections of a BPA defined over a joint space  $X \times Y$  on its respective marginal sets  $X$  and  $Y$ , and our presented uncertainty measure considers the belief intervals for singletons and its corresponding credal sets. Hence, the subadditivity property for our uncertainty measure is controversial because it might make more sense to consider the projections of the belief intervals than the projections of the BPA. For this reason, we reconsider the definition of this property for our proposed uncertainty measure. We need some concepts related to probability intervals and credal sets.

Firstly, we define the projections of the belief intervals for singletons of a BPA defined on a product space on the marginal sets. This definition is based on the one given in [36] for the marginalization of probability intervals.

**Definition 3:** Let  $X = \{x_1, \dots, x_n\}$  and  $Y = \{y_1, \dots, y_{n'}\}$  be finite sets. Let  $m$  be a BPA defined on  $X \times Y$ , and  $Bel_m$  and  $Pl_m$  the belief and plausibility functions corresponding to  $m$ , respectively. Let us suppose that  $\mathcal{I}_m = \{[l_{ij}, u_{ij}] \mid l_{ij} = Bel_m(\{x_i, y_j\}), u_{ij} = Pl_m(\{x_i, y_j\})\}$  is the set of belief intervals for singletons. Let us denote  $P_*(\mathcal{I}_m)$  and  $P^*(\mathcal{I}_m)$  the lower and upper probabilities associated with  $\mathcal{I}_m$ , respectively. The marginal set of belief intervals on  $X$  is defined as follows:

$$\begin{aligned} \mathcal{I}_m^{\downarrow X} &= \{[l_i, u_i] \mid l_i = P_*(\mathcal{I}_m)(x_i \times Y), \\ u_i &= P^*(\mathcal{I}_m)(x_i \times Y), \quad \forall i = 1, 2, \dots, n\}. \end{aligned}$$

The definition of the marginal set of belief intervals on  $Y$ ,  $\mathcal{I}_m^{\downarrow Y}$ , is analogous.

Now, we define the projections of the credal set corresponding to belief intervals for singletons associated with a BPA defined on a product space on the corresponding marginal sets.

**Definition 4:** Let  $X = \{x_1, \dots, x_n\}$  and  $Y = \{y_1, \dots, y_{n'}\}$  be finite sets. Let  $m$  be a BPA defined on  $X \times Y$  and  $\mathcal{P}(\mathcal{I}_m)$  the credal set associated with belief intervals for singletons. The marginal credal set of  $\mathcal{P}(\mathcal{I}_m)$  on  $X$  is defined as follows:

$$\begin{aligned} \mathcal{P}(\mathcal{I}_m)^{\downarrow X} &= \{p_X \mid \exists p \in \mathcal{P}(\mathcal{I}_m) : p_X(x_i) = \\ &= \sum_{j=1}^{n'} p(x_i, y_j), \quad \forall i = 1, \dots, n\}. \end{aligned}$$



Analogously, it is possible to define the marginal credal set of  $\mathcal{P}(\mathcal{I}_m)$  on  $Y$ , which we denote  $\mathcal{P}(\mathcal{I}_m)^{\downarrow Y}$ .

We have the following proposition:

**Proposition 2:** If  $m$  is a BPA defined over a product space  $X \times Y$ ,  $\mathcal{I}_m$  is the set of belief intervals for singletons, and  $\mathcal{P}(\mathcal{I}_m)$  is its corresponding credal set, it is satisfied that:

$$\mathcal{P}(\mathcal{I}_m)^{\downarrow X} = \mathcal{P}(\mathcal{I}_m^{\downarrow X}), \quad \mathcal{P}(\mathcal{I}_m)^{\downarrow Y} = \mathcal{P}(\mathcal{I}_m^{\downarrow Y}).$$

*Proof:*

We use the same notation as in the previous definitions. Let  $p_X \in \mathcal{P}(\mathcal{I}_m)^{\downarrow X}$ . Then,  $\exists p \in \mathcal{P}(\mathcal{I}_m)$  such that  $p_X(\{x_i\}) = \sum_{j=1}^{n'} p(\{x_i, y_j\}) \forall i = 1, 2, \dots, n$ . Since  $p \in \mathcal{P}(\mathcal{I}_m)$  we have that

$$\begin{aligned} l_{ij} &\leq p(\{x_i, y_j\}) \leq u_{ij}, \\ \forall i &= 1, \dots, n, j = 1, \dots, n' \Rightarrow \\ \sum_{j=1}^{n'} l_{ij} &\leq \sum_{j=1}^{n'} p(\{x_i, y_j\}) = p_X(\{x_i\}) \leq \\ &\sum_{j=1}^{n'} u_{ij} \quad \forall i = 1, \dots, n, \end{aligned}$$

which implies that  $p_X \in \mathcal{P}(\mathcal{I}_m^{\downarrow X})$ .

Let us suppose that  $p_X \in \mathcal{P}(\mathcal{I}_m^{\downarrow X})$ . Then:

$$\sum_{j=1}^{n'} l_{ij} \leq p_X(\{x_i\}) \leq \sum_{j=1}^{n'} u_{ij}, \quad \forall i = 1, \dots, n.$$

For each  $i = 1, \dots, n$ , there are 3 possibilities:

- 1)  $p_X(\{x_i\}) = \sum_{j=1}^{n'} l_{ij}$
- 2)  $p_X(\{x_i\}) = \sum_{j=1}^{n'} u_{ij}$
- 3)  $p_X(\{x_i\}) = \lambda_i$ , where  $\sum_{j=1}^{n'} l_{ij} < \lambda_i < \sum_{j=1}^{n'} u_{ij}$ .

We consider:

$$p(\{x_i, y_j\}) = \begin{cases} l_{ij} & \text{if } p_X(\{x_i\}) = \sum_{j=1}^{n'} l_{ij}, \\ u_{ij} & \text{if } p_X(\{x_i\}) = \sum_{j=1}^{n'} u_{ij}, \\ \alpha_{ij} & \text{if } p_X(\{x_i\}) = \lambda_i \end{cases}$$

with  $\sum_{j=1}^{n'} l_{ij} < \lambda_i < \sum_{j=1}^{n'} u_{ij}$ ,  $l_{ij} \leq \alpha_{ij} \leq u_{ij}$ , in such a way that  $\sum_{j=1}^{n'} \alpha_{ij} = \lambda_i \forall i \in \{1, \dots, n\}$  such that  $\sum_{j=1}^{n'} l_{ij} < p_X(\{x_i\}) < \sum_{j=1}^{n'} u_{ij}$ .

Clearly,  $p \in \mathcal{P}(\mathcal{I}_m)$  and  $p_X(\{x_i\}) = \sum_{j=1}^{n'} p(\{x_i, y_j\}) \forall i = 1, \dots, n$ . Consequently,  $p_X \in \mathcal{P}(\mathcal{I}_m)^{\downarrow X}$ .

The proof of  $\mathcal{P}(\mathcal{I}_m)^{\downarrow Y} = \mathcal{P}(\mathcal{I}_m^{\downarrow Y})$  is analogous.  $\square$

Thus, when a BPA is defined on a joint set, marginalizing over the belief intervals for singletons is equivalent to marginalizing over the corresponding credal set for our proposed uncertainty measure.

Concerning subadditivity, we have the following proposition:

**Proposition 3:** Let  $X = \{x_1, \dots, x_n\}$  and  $Y = \{y_1, \dots, y_{n'}\}$  be two finite sets. Let  $m$  be a BPA defined on  $X \times Y$ ,  $\mathcal{I}_m$  the set of belief intervals for singletons, and  $\mathcal{P}(\mathcal{I}_m)$  its corresponding credal set. It is satisfied that:

$$S^*(\mathcal{P}(\mathcal{I}_m)) \leq S^*(\mathcal{P}(\mathcal{I}_m^{\downarrow X})) + S^*(\mathcal{P}(\mathcal{I}_m^{\downarrow Y})).$$

Taking into account Proposition 2, the proof of this result is identical to the one given in [38] for the subadditivity property for the extension of  $S^*(\mathcal{P}_m)$  to general credal sets. Therefore, when the belief intervals corresponding to a BPA defined over a product space are projected on the marginal sets, the amount of information according to our proposed uncertainty measure is not increased. In that sense, we could say that  $S^*(\mathcal{P}(\mathcal{I}_m))$  is subadditive.

The following example shows that the belief intervals of the projected BPAs does not coincide with the projections of the belief intervals.

**Example 3:** Let us consider the finite sets  $X = \{x_1, x_2, x_3\}$ , and  $Y = \{y_1, y_2\}$ . Let  $m$  be the BPA on  $X \times Y$  given by:

$$\begin{aligned} m(\{z_{11}, z_{12}, z_{21}\}) &= 0.7, \quad m(\{z_{31}, z_{32}\}) = 0.1, \\ m(\{z_{11}, z_{12}, z_{21}, z_{22}, z_{31}, z_{32}\}) &= 0.2, \end{aligned}$$

where we denote  $z_{ij} = (x_i, y_j)$ , for  $i = 1, 2, 3, j = 1, 2$ . For singletons, we have the following belief intervals:

$$\begin{aligned} z_{11} &\rightarrow [Bel_m(\{z_{11}\}), Pl_m(\{z_{11}\})] = [0, 0.9] \\ z_{12} &\rightarrow [Bel_m(\{z_{12}\}), Pl_m(\{z_{12}\})] = [0, 0.9] \\ z_{21} &\rightarrow [Bel_m(\{z_{21}\}), Pl_m(\{z_{21}\})] = [0, 0.9] \\ z_{22} &\rightarrow [Bel_m(\{z_{22}\}), Pl_m(\{z_{22}\})] = [0, 0.2] \\ z_{31} &\rightarrow [Bel_m(\{z_{31}\}), Pl_m(\{z_{31}\})] = [0, 0.3] \\ z_{32} &\rightarrow [Bel_m(\{z_{32}\}), Pl_m(\{z_{32}\})] = [0, 0.3] \end{aligned}$$

Let  $m_X$  be the marginal BPA of  $m$  on  $X$ . We have that:

$$m_X(\{x_1, x_2\}) = 0.7,$$

$$m_X(\{x_3\}) = 0.1, \quad m_X(X) = 0.2.$$

The belief intervals for singletons associated with  $m_X$  are the following ones:

$$\begin{aligned} x_1 &\rightarrow [0, 0.9] \\ x_2 &\rightarrow [0, 0.9] \\ x_3 &\rightarrow [0.1, 0.3] \end{aligned}$$

Nevertheless, the result of the projection of the belief intervals for singletons associated with  $m$  on  $X$  is the following one:

$$x_1 \rightarrow [0, 1]$$

$$x_2 \rightarrow [0, 1]$$

$$x_3 \rightarrow [0, 0.6]$$

For these reasons, we consider that, with the maximum of entropy on the credal set associated with the belief intervals for singletons, the subadditivity property should be utilized with the projections of the belief intervals, instead of the projections of the BPA, as the definition of subadditivity exposed in Section III-A. Our proposed measure is subadditive with this adapted definition of the property.

- **Additivity:** With this property, as with subadditivity, for our proposed measure, it makes more sense to consider the projections of the intervals instead of the marginals BPAs. Thus, we need a definition of independence that is applied to credal sets instead of BPAs. For this purpose, we use the concept of strong independence [40]. **Definition 5:** “Let  $X$  and  $Y$  be finite sets. Let  $m$  be a BPA on the product space  $X \times Y$ . Let us consider  $\mathcal{P}(\mathcal{I}_m)$  the credal set associated with the belief intervals for singletons. Let  $\mathcal{P}(\mathcal{I}_m)^{\downarrow X}$  and  $\mathcal{P}(\mathcal{I}_m)^{\downarrow Y}$  be the corresponding marginal credal sets on  $X$  and  $Y$ , respectively. It is said that there is *strong independence* under  $\mathcal{P}(\mathcal{I}_m)$  iff  $\mathcal{P}(\mathcal{I}_m) = CH(\mathcal{P}(\mathcal{I}_m)^{\downarrow X} \times \mathcal{P}(\mathcal{I}_m)^{\downarrow Y})$ .”<sup>3</sup>

Now, we have the following result:

**Proposition 4:** Let  $X$  and  $Y$  be finite sets. Let us consider a BPA  $m$  on  $X \times Y$ . Let  $\mathcal{P}(\mathcal{I}_m)$  be the credal set associated with the belief intervals for singletons. Let us suppose that  $\mathcal{P}(\mathcal{I}_m)^{\downarrow X}$  and  $\mathcal{P}(\mathcal{I}_m)^{\downarrow Y}$  are the corresponding marginal credal sets on  $X$  and  $Y$ , respectively. If there is strong independence under  $\mathcal{P}(\mathcal{I}_m)$ , then

$$S^*(\mathcal{P}(\mathcal{I}_m)) = S^*(\mathcal{P}(\mathcal{I}_m^{\downarrow X})) + S^*(\mathcal{P}(\mathcal{I}_m^{\downarrow Y})).$$

The proof of this proposition is identical to the one provided in [38] for the additivity requirement for the maximum of entropy on credal sets if we take into consideration Proposition 2.

Therefore, our proposed measure verifies the additivity requirement with this adapted definition of the property.

- **Monotonicity:** Let  $m_1$  and  $m_2$  be two BPAs on a finite set  $X = \{x_1, x_2, \dots, x_n\}$ . Let  $Bel_{m_1}$  and  $Bel_{m_2}$  be the belief functions associated with  $m_1$  and  $m_2$ , respectively. Let  $\mathcal{P}(\mathcal{I}_{m_1})$  and  $\mathcal{P}(\mathcal{I}_{m_2})$  be the credal sets corresponding to the belief intervals for singletons associated with  $m_1$  and  $m_2$ , respectively. Let us consider the credal sets associated with  $m_j$ , for  $j = 1, 2$ :

$$\mathcal{P}_{m_j} = \{p \in \mathcal{P}(X) \mid p(A) \geq Bel_{m_j}(A) \quad \forall A \subseteq X\}.$$

Let us suppose that  $\mathcal{P}_{m_1} \subseteq \mathcal{P}_{m_2}$ . In such case, it is easy to check that  $Bel_{m_1}(A) \geq Bel_{m_2}(A) \quad \forall A \subseteq X$ .

<sup>3</sup>CH indicates the convex hull of a set.

In particular,  $Bel_{m_1}(\{x_i\}) \geq Bel_{m_2}(\{x_i\}) \quad \forall i = 1, 2, \dots, n$ .

Let  $Pl_{m_j}$  be the plausibility function corresponding to  $Bel_{m_j}$ , for  $j = 1, 2$ . We have that:

$$Bel_{m_1}(X \setminus \{x_i\}) \geq Bel_{m_2}(X \setminus \{x_i\})$$

$$\Rightarrow 1 - Pl_{m_1}(\{x_i\}) \geq 1 - Pl_{m_2}(\{x_i\})$$

$$\Rightarrow Pl_{m_2}(\{x_i\}) \geq Pl_{m_1}(\{x_i\}), \quad \forall i = 1, \dots, n.$$

Consequently, if  $p \in \mathcal{P}(\mathcal{I}_{m_1})$ , then:

$$Bel_{m_2}(\{x_i\}) \leq Bel_{m_1}(\{x_i\}) \leq p(x_i) \leq$$

$$Pl_{m_1}(\{x_i\}) \leq Pl_{m_2}(\{x_i\}) \quad \forall i = 1, \dots, n,$$

which implies that  $p \in \mathcal{P}(\mathcal{I}_{m_2})$ .

Hence, if  $\mathcal{P}_{m_1} \subseteq \mathcal{P}_{m_2}$ , then  $\mathcal{P}(\mathcal{I}_{m_1}) \subseteq \mathcal{P}(\mathcal{I}_{m_2})$ . Therefore,  $S^*(\mathcal{P}(\mathcal{I}_{m_1})) \leq S^*(\mathcal{P}(\mathcal{I}_{m_2}))$ , and we conclude that  $S^*(\mathcal{P}(\mathcal{I}_m))$  satisfies the monotonicity property.

For uncertainty measures based on belief intervals, it might make more sense to apply an adapted definition of the monotonicity property that considers the inclusion among the belief intervals [41], instead of the inclusion among the credal sets corresponding to the BPAs. However, as shown above,  $S^*(\mathcal{P}(\mathcal{I}_m))$  verifies the monotonicity requirement under both inclusions.

Summarizing, unlike other alternatives to  $S^*(\mathcal{P}_m)$  such as the Deng entropy,  $H_{JS}$ ,  $H_{PQ}$ , or  $H_{inter}$  (see Table 1),  $S^*(\mathcal{P}(\mathcal{I}_m))$  verifies the six essential mathematical properties for a TU measure in DST, although the subadditivity and additivity properties for our proposed measure are controversial. Remark that the monotonicity requirement is crucial since a total uncertainty measure in DST must be consistent with an increase or decrease of information.

## B. BEHAVIORS OF OUR PROPOSAL

Now, we analyze the behavioral requirements described in Section III-B for our proposed measure  $S^*(\mathcal{P}(\mathcal{I}_m))$ .

- (RB1):  $S^*(\mathcal{P}(\mathcal{I}_m))$  is not as direct to obtain as some recent measures such as the Deng entropy. Nonetheless, its calculation is considerably easier than the calculation of  $S^*(\mathcal{P}_m)$  because the algorithms proposed in Section IV to compute the belief intervals and, after, calculate the maximum of entropy on the corresponding credal set, are much simpler than the algorithms to calculate  $S^*(\mathcal{P}_m)$  proposed in [11]–[14]. It is since, for the calculation of  $S^*(\mathcal{P}(\mathcal{I}_m))$ , only the extreme values of the belief intervals for singletons are taken into account, while it is necessary to consider the power set to obtain  $S^*(\mathcal{P}_m)$ .
- (RB2): As we have shown,  $S^*(\mathcal{P}(\mathcal{I}_m))$  can be coherently disaggregated in two measures that quantify conflict and non-specificity. This separation is done similarly as the maximum of entropy on the credal set associated with  $m$ ,  $S^*(\mathcal{P}(\mathcal{I}_m)) = S_*(\mathcal{P}(\mathcal{I}_m)) +$

$(S^* - S_*)(\mathcal{P}(\mathcal{I}_m))$ . We have provided an algorithm to calculate  $S_*(\mathcal{P}(\mathcal{I}_m))$ . Nevertheless, the only algorithm developed in the literature for the computation of  $S_*(\mathcal{P}_m)$  is the one proposed in [42], whose complexity is notably high.

- (RB3): For analyzing the sensitivity to changes in the evidence of  $m$ , consistently with the studies carried out in [34] and [10], we use the following example:

**Example 4:** Let be the finite  $X = \{x_1, x_2\}$  and let us suppose that we have the BPA on  $X$  given by:

$$m(\{x_1\}) = m_1, \quad m(\{x_2\}) = m_2,$$

$$m(\{x_1, x_2\}) = m_{12} = 1 - m_1 - m_2.$$

where  $0 \leq m_i \leq 1$  for  $i = 1, 2$ , and  $m_1 + m_2 \leq 1$ . For singletons, we have the following belief intervals:

$$x_1 \rightarrow [m_1, 1 - m_2],$$

$$x_2 \rightarrow [m_2, 1 - m_1].$$

It is known that the conflict part of  $m$  depends on the interaction of  $m_1$  and  $m_2$ , whereas the non-specificity part is quantified by  $m(\{x_1, x_2\}) = m_{12} = 1 - m_1 - m_2$ .

Without losing generality, we assume that the value of  $m_1$  is known. We distinguish two cases:

- **Case 1:**  $m_1 \geq 0.5$ . We have that  $m_2 \leq 0.5 \leq m_1 \Rightarrow$   
 $S^*(\mathcal{P}(\mathcal{I}_m)) = S(m_1, 1 - m_1)$ ,  $S_*(\mathcal{P}(\mathcal{I}_m)) = S(m_2, 1 - m_2)$ ,  
 $(S^* - S_*)(\mathcal{P}(\mathcal{I}_m)) = S(m_1, 1 - m_1) - S(m_2, 1 - m_2)$ .<sup>4</sup>

The amount of total uncertainty keeps constant. The conflict part increases as  $m_2$  increases, which is logical if we take into consideration that  $m_2 \leq 0.5 \leq m_1$ . The non-specificity part increases when  $m_2$  decreases, i.e when  $m_{12}$  increases. Remark that the non-specificity part of  $m$  is indicated by  $m_{12}$ . Thus, we can say that the variations of the conflict and non-specificity parts of  $S^*(\mathcal{P}(\mathcal{I}_m))$  as  $m_2$  changes are pretty coherent.

- **Case 2:**  $m_1 < 0.5$ . In this case:  
 $S^*(\mathcal{P}(\mathcal{I}_m)) = S(\alpha_2, 1 - \alpha_2)$ , where  $\alpha_2 = \max\{m_2, 0.5\}$ , and  
 $S_*(\mathcal{P}(\mathcal{I}_m)) = S(\alpha, 1 - \alpha)$ , being  $\alpha = \min\{m_1, m_2\}$ .

Consequently, the conflict part depends on the minimum value of  $m_1$  and  $m_2$ , which is very logical. For the non-specificity part, we distinguish three cases:

- 1)  $m_2 \leq m_1 < 0.5$ . In such case:  
 $S^*(\mathcal{P}(\mathcal{I}_m)) = S(0.5, 0.5)$ ,  $S_*(\mathcal{P}(\mathcal{I}_m)) = S(m_2, 1 - m_2)$ , and  $(S^* - S_*)(\mathcal{P}(\mathcal{I}_m)) =$

<sup>4</sup>Within this example,  $S(a, 1 - a)$  with  $a \in [0, 1]$  denotes  $S(p_a)$ , where  $p_a(\{x_1\}) = a$  and  $p_a(\{x_2\}) = 1 - a$ .

$S(0.5, 0.5) - S(m_2, 1 - m_2)$ . The total uncertainty keeps constant, and the conflict part increases as  $m_2$  increases. Hence, the non-specificity part decreases as  $m_2$  increases ( $m(\{x_1, x_2\})$  decreases), which makes a lot of sense.

- 2)  $m_1 \leq m_2 < 0.5$ . Then,  $S^*(\mathcal{P}(\mathcal{I}_m)) = S(m_1, 1 - m_1)$ , which implies that the conflict part does not vary. In addition,  $S^*(\mathcal{P}(\mathcal{I}_m)) = S(0.5, 0.5)$ . Therefore, the total uncertainty, conflict and non-specificity values keep constant. It could be considered such an undesirable behavior. Nevertheless, in this case, since  $1 - m_i > 0.5$  for  $i = 1, 2$ , it might make sense to consider a total uncertainty value because the plausibility of each singleton is greater than 0.5.
- 3)  $m_1 < 0.5 \leq m_2$ . In this case,  $S^*(\mathcal{P}(\mathcal{I}_m)) = S(m_2, 1 - m_2)$ ,  $S_*(\mathcal{P}(\mathcal{I}_m)) = S(m_1, 1 - m_1)$ , and  $(S^* - S_*)(\mathcal{P}(\mathcal{I}_m)) = S(m_2, 1 - m_2) - S(m_1, 1 - m_1)$ . The conflict part keeps constant, and the non-specificity value decreases as  $m_2$  increases, i.e, as  $m(\{x_1, x_2\})$  decreases, which is quite coherent.

From the previous example, we can conclude that, as happens with  $S^*(\mathcal{P}_m)$  (See [10] for more details),  $S^*(\mathcal{P}(\mathcal{I}_m))$  is sensitive to changes of evidence, directly or via its parts of conflict and non-specificity. Using the same example, it is easy to check the sensitivity to changes in the evidence of that the Deng entropy,  $H_{JS}$ ,  $H_{PQ}$ , and  $H_{inter}$ .

- (RB4): In all the generalizations of the Probability Theory, the information can be expressed by a lower probability function, which always has associated an upper probability function [35]. Therefore, it is immediate to conclude that  $S^*(\mathcal{P}(\mathcal{I}_m))$  can be easily extended to more general theories than DST because, in them, it is possible to consider the lower and upper probabilities for singletons.

Hence, our proposed measure verifies all the requirements of behavior for uncertainty measures in DST. Table 2 shows a summary of the behavioral requirements satisfied by Deng entropy,  $H_{inter}$ ,  $S^*(\mathcal{P}_m)$ , and  $S^*(\mathcal{P}(\mathcal{I}_m))$ .

Requirement	$E_d$	$H_{inter}$	$S^*(\mathcal{P}_m)$	$S^*(\mathcal{P}(\mathcal{I}_m))$
Complexity	low	low	high	medium
RB2	yes	yes	yes	yes
RB3	yes	yes	yes	yes
RB4	no	no	yes	yes

**TABLE 2.** Summary of the crucial behavioral requirements verified by some recent uncertainty measures in DST.

Indeed,  $H_{JS}$  and  $H_{PD}$  satisfy all the behavioral requirements for uncertainty measures in DST. Nonetheless, we must remark that, as said before, these measures present an

undesirable behavior when all the focal sets share an element, unlike our proposal.

Let us analyze the relation between  $S^*(\mathcal{P}(\mathcal{I}_m))$  and  $S^*(\mathcal{P}_m)$ . As the following result shows, the credal set corresponding to a BPA is always contained in the credal set associated with the belief intervals for singletons corresponding to that BPA:

**Proposition 5:** If  $m$  is a BPA on a finite set  $X = \{x_1, x_2, \dots, x_n\}$ , then

$$\mathcal{P}_m \subseteq \mathcal{P}(\mathcal{I}_m).$$

*Proof:*

Let  $p \in \mathcal{P}_m$ . Then, for each  $A \subseteq X$ , we have that  $Bel(A) \leq p(A)$ . In particular,  $Bel_m(\{x_i\}) \leq p(\{x_i\}) \forall i = 1, 2, \dots, n$ . Besides,

$$Bel_m(X \setminus \{x_i\}) \leq p(X \setminus \{x_i\}) \Rightarrow$$

$$1 - Pl_m(\{x_i\}) \leq 1 - p(\{x_i\}) \Rightarrow$$

$$p(\{x_i\}) \leq Pl_m(\{x_i\}), \quad \forall i = 1, 2, \dots, n.$$

Consequently,

$$Bel_m(\{x_i\}) \leq p(\{x_i\}) \leq Pl_m(\{x_i\}), \quad \forall i = 1, 2, \dots, n,$$

which implies that  $p \in \mathcal{P}(\mathcal{I}_m)$ .  $\square$

From the previous result, it is immediate that  $S^*(\mathcal{P}(\mathcal{I}_m)) \geq S^*(\mathcal{P}_m)$ . Thus, our proposed measure provides an upper bound of the maximum of entropy on the credal set associated with a BPA. Moreover,  $S_*(\mathcal{P}(\mathcal{I}_m)) \leq S_*(\mathcal{P}_m)$ . In this way, the conflict value provided by our uncertainty measure is always lower or equal than the conflict value captured by  $S^*(\mathcal{P}_m)$ . Nonetheless, the non-specificity value of  $S^*(\mathcal{P}(\mathcal{I}_m))$  is always greater or equal than the non-specificity value corresponding to  $S^*(\mathcal{P}_m)$ . It makes sense since the main difference between uncertainty in DST and PT resides in the non-specificity part, and our proposed measure enhances this idea.

## V. CONCLUSIONS AND FUTURE WORK

In this work, we have considered the belief intervals to quantify the uncertainty-based information in Evidence Theory. As said in the introduction, belief intervals are more suitable than a basic probability assignment to represent the uncertainty since they allow knowing the uncertain area for each subset. More specifically, we have proposed a new uncertainty measure that consists of the maximum of entropy on the credal set corresponding to belief intervals for singletons. It has been presented as an alternative to the maximum of entropy on the credal set associated with a BPA, which is the only uncertainty measure proposed so far that verifies all the mathematical properties and requirements of behavior for uncertainty measures in DST.

Our proposed measure satisfies the probabilistic consistency, set consistency, range, and monotonicity properties. Remark that the last one is essential because an uncertainty

measure must be consistent with an increase or decrease of information. The additivity and subadditivity properties are controversial for the maximum of entropy on the credal set corresponding to belief intervals for singletons since these properties consider the projections of a BPA defined on a product space on the corresponding marginal sets, which might not be very coherent for our proposal. Thus, we have reconsidered the concepts of subadditivity and additivity for our proposed measure by considering the projections of the belief intervals, and we have shown that our proposal verifies these properties with the reconsidered definitions.

Furthermore, we have shown that our new uncertainty measure satisfies the behavioral requirements for an uncertainty measure in DST: it coherently separates conflict and non-specificity, it is sensitive to changes in evidence (directly or via its parts of conflict and nonspecificity), and it is possible to extend it to more general theories than DST. In this way, our proposed measure overcomes most of the drawbacks of some measures recently proposed.

We have provided an algorithm for the calculation of the maximum of entropy on the credal set corresponding to belief intervals for singletons, which is considerably simpler than the ones proposed in the literature for the computation of the maximum of entropy on the credal set corresponding to a BPA. The reason is that, for the calculation of our proposed measure, it is just necessary to consider the extreme values of the belief intervals and not the power set, unlike the upper entropy on the credal set associated with a BPA. Also, for the same reason, the conflict and non-specificity parts are easier to obtain with the maximum of entropy on the credal set corresponding to belief intervals for singletons. Therefore, our proposed uncertainty measure is more suitable to be employed in practical applications than the upper entropy on the credal set corresponding to a BPA.

In addition, we have demonstrated that the maximum of entropy on the credal set associated with belief intervals for singletons provides an upper bound of the maximum of entropy on the credal set associated with the BPA. More specifically, the conflict part of our proposed measure is always lower or equal than conflict part of the upper entropy on the credal set corresponding to the BPA, whereas the non-specificity part is always greater or equal. In consequence, our proposed uncertainty measure enhances the idea that the difference between uncertainty in DST and Probability Theory resides in the non-specificity part.

As future work, firstly, the algorithm to calculate the uncertainty measure presented here could be simplified. We consider that reducing the complexity of the algorithm to calculate the maximum entropy for BPAs, in general, is also possible and can be another interesting task for us. Secondly, we want to apply our new measure to specific cases where the theory of evidence is used to represent the available information. The most immediate applications may be in the case of information fusion in the field of the sensors, as applied in [4].

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