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ABSTRACT

We present an interpretation of the functions appearing in the Wei–Norman factorization of the evolution operator for a Hamiltonian belonging to the SU(1,1) algebra in terms of the classical solutions of the Generalized Caldirola–Kanai (GCK) oscillator (with time-dependent mass and frequency). Choosing P^2, X^2, and the dilation operator as a basis for the Lie algebra, we obtain that, out of the six possible orderings for the Wei–Norman factorization of the evolution operator for the GCK Hamiltonian, three of them can be expressed in terms of its classical solutions and the other three involve the classical solutions associated with a *mirror* Hamiltonian obtained by inverting the mass. In addition, we generalize the Wei–Norman procedure to compute the factorization of other operators, such as a generalized Fresnel transform and the Arnold transform (and its generalizations), obtaining also in these cases a semiclassical interpretation for the functions in the exponents of the Wei–Norman factorization. The singularities of the functions appearing in the Wei–Norman factorization are related to the caustic points of Morse theory, and the expression of the evolution operator at the caustics is obtained using a limiting procedure, where the Fourier transform of the initial state appears along with the Guoy phase.

I. INTRODUCTION

The Wei–Norman (WN) factorization method[1,2] allows us to express the evolution operator (or propagator) for a system of first-order linear differential (operator) equations $y'(t) = A(t)y(t)$ as a product of a finite number of exponentials, in the case in which $A(t)$ is an element of a finite-dimensional Lie algebra. It has multiple applications, one of the most important being the factorization of the evolution operator associated with Schrödinger's equation, where $A(t)$ is proportional to the quantum Hamiltonian.

WN factorization leads to a set of nonlinear differential equations for the functions [here usually denoted by $g_i(t)$] appearing in the exponential operators. In many situations, these equations are of Riccati type, which admit a transformation, under suitable changes in the dependent variable, into second-order linear differential equations.

One of the most valuable examples in quantum mechanics are those of quadratic Hamiltonians in position Q and momentum P, which expand the $su(1,1)$ Lie algebra. The time-dependent case, whose more general expression is known as the Generalized Caldirola–Kanai (GCK) Hamiltonian,[3–5] has important applications in as diverse fields as ion trap physics,[6] photonics lattices,[7] and cosmology.[8] Thus, a method for finding the solution in this general case is crucial. The WN factorization method applies in this case, and we show in this paper that, for a basis of the quadratic Lie algebra given by P^2, Q^2, and QP, the second-order linear differential equation obtained from the Riccati equation is the Euler–Lagrange (EL) equation associated with the classical version of the GCK Hamiltonian in three of the possible orderings of the exponentials. For the other three orderings, the second-order linear differential equation is the Euler–Lagrange (EL) equation associated with a classical Hamiltonian that is a mirror version of the original GCK Hamiltonian, where the mass is inverted. This provides a geometric and

semiclassical interpretation of the WN factorization for the $SU(1,1)$ case, since all the information provided by the evolution operator is obtained from the solutions of the EL equation. Similar results can be found in the literature[9,10] (see also Refs. 11 and 12), but in those papers, just one ordering has been considered. This study could also be generalized to other Lie groups such as $SU(2)$, which will be considered elsewhere.

For the GCK case, the evolution operator can be considered as a linear integral transform[13] (see also Appendix B). Associated with the evolution operator, there are other linear integral transforms that relate different GCK systems. These are the Fresnel transform[14] and the Arnold transform,[5] along with their generalizations. The difference between the latter two is that the Arnold transform is, up to a local phase, a point (or geometric) transformation, i.e., the integral transform reduces to a local operator. The price to be paid for this simplification is an extra diffeomorphism in time in such a way that the evolution times are different in the two systems. Point transformations are preferable to non-trivial integral transforms since they preserve point (or geometric) symmetries of the system that in the case of the GCK oscillator are given by the Schrödinger group.[5] We provide in this paper a modification of the WN method that allows us to obtain also a factorization of these integral transforms. These expressions seem to be new in the literature.

One of the problems of the WN factorization method is its local character in time, in the sense that the functions $g_i(t)$ appearing in the exponentials diverge for specific finite values of t. This problem is analyzed for the $SU(1,1)$ case, and it is related to the fact that not all elements of the group can be written in a factorized way (i.e., the factorization is not onto). This problem is not specific of the WN method but appears also in other approaches such as Feynman's path integral method for computing the propagator.[15,16] The singular values of the functions $g_i(t)$ are related to the *caustics* appearing in the path integral method and are also related to the caustics appearing in Morse theory.[17] They are also related to the focal points appearing in Fourier optics.[18]

The computation of the propagator at the caustics in the path integral approach involves the use of higher order perturbations. In our case, we use a fine-tuning analysis of the limit at the caustic points to derive its expression.

In addition, the phase jumps of the propagator in the path integral method[15,16] appearing when passing through a caustic, which are ultimately related to the Maslov index and the Maslov correction, and with the Guoy phase in optics,[19] are also easily obtained in the WN method. In summary, the existence of singularities in the functions $g_i(t)$ can be easily handled within the WN method and does not prevent the computation of the evolution operator for all times.

The problem of the singularities of $g_i(t)$ and the phase jumps of the wave function when crossing a singularity seems not to be present in other works such as Refs. 9 and 10. The reason is that in those papers, the evolution operator is applied to a particular initial state (number state or coherent state of the standard harmonic oscillator), obtaining explicit expressions for the evolved wave function (this is made possible by the factorized form of the evolution operator provided by the WN method) that do not depend explicitly on $g_i(t)$ but directly on the solutions of the EL equation, which do not possess singularities. However, the phase jumps appearing in our approach when crossing a caustic can still be found in their solutions, in a form of an *arctan* function of the quotient of the two classical solutions, which experiences a jump of π when the denominator is zero (i.e., at caustic points). In a sense, our approach, focusing on the evolution operator, is more general than that of Refs. 9 and 10 where very particular initial states are considered. However, we have to tackle the problem of the singularities (inherent to the WN factorization), and this is hidden in Refs. 9 and 10 due to the nice properties of the initial states considered.

The content of this paper is as follows: In Sec. II, the WN factorization is reviewed and particularized to the Schrödinger case in Sec. II A. In Sec. III, the case of the GCK oscillator is considered and the evolution operator is factorized using two representative orderings. The functions $g_i(t)$ are related to solutions of the EL equations associated with the original classical Hamiltonian and a mirror version of it. In Sec. IV, the Fresnel transform is discussed and generalized, providing a way to compute a factorization by modification of a WN method. In Sec. V, the Arnold transform and its generalizations are reviewed, providing also a factorization by a modified WN (MWN) method. In Sec. VI, some particular versions of the Fresnel transform are recovered for particular cases of the MWN method. In Sec. VIII, the particular case of the CK oscillator, for constant frequency and damping coefficient, is discussed in detail. In particular, the problem of the caustic points [where the functions $g_i(t)$ diverge] is discussed, and the computation of the evolution operator together with the phase jumps at these points is provided within the WN method. Finally, after the Conclusion, a couple of appendices are provided: Appendix A, which discusses the groupoid property of the evolution operator in the framework of the WN method, and Appendix B, containing a review of the most important and useful results about linear integral transforms.

II. THE WEI–NORMAN FACTORIZATION METHOD

Consider the first-order linear initial value problem (IVP), where the prime indicates derivative with respect to t,

$$y'(t) = A(t)y(t), \qquad y(t_0) = y_0, \tag{1}$$

with $y(t), y_0 \in V$ and y_0 being the initial condition at t_0. Here, V is a vector space (which can be infinite-dimensional) and $A(t)$ is a family of endomorphism of V (i.e., a family of matrices or linear operators).

By Picard's existence and uniqueness theorem[20] (particularized to systems of homogeneous linear equations), if $A(t)$ is continuous and $\|A(t)\|$ is bounded on some interval $\mathcal{I} \subset \mathbb{R}$ containing t_0, then the IVP (1) has a unique solution on $\mathcal{I}$.

In most examples in physics, $A(t)$ is (up to a factor) the Hamiltonian in a classical or quantum mechanical system, and Eq. (1) corresponds to Hamilton's or Schrödinger's equation, respectively. The case of the Schrödinger equation will be treated in Sec. II A.

It is convenient to write the solution $y(t)$ as

$$y(t) = U(t, t_0)y(t_0) \qquad \forall t \in \mathcal{I}, \tag{2}$$

where $U(t, t_0)$ is known as the *matrizant*,[21] evolution operator, propagator, transfer matrix, etc., depending on the context.

Using the evolution operator, Eq. (1) is transformed into

$$\frac{\partial U(t, t_0)}{\partial t} = A(t)U(t, t_0), \qquad U(t_0, t_0) = I_V, \tag{3}$$

with I_V being the identity automorphism of V. For V finite-dimensional, it is easily checked that $U(t, t_0)$ is an automorphism of V for all $t, t_0 \in \mathcal{I}$, since we have that (see, for instance, Ref. 22)

$$\det U(t, t_0) = \det U(t_0, t_0) \exp\left(\int_{t_0}^{t} \mathrm{tr}(A(s))ds\right) = \exp\left(\int_{t_0}^{t} \mathrm{tr}(A(s))ds\right) \tag{4}$$

since $\det U(t_0, t_0) = 1$ and $A(t)$ is bounded on $\mathcal{I}$. The infinite-dimensional case will be discussed in Sec. II A.

The family of evolution operators verifies the groupoid property,

$$\begin{aligned} U(t_2, t_1)U(t_1, t_0) &= U(t_2, t_0), \\ U(t_1, t_0) &= U(t_0, t_1)^{-1} \end{aligned} \tag{5}$$

$\forall t_0, t_1, t_2 \in \mathcal{I}$. See Appendix A for a discussion of the groupoid property in the context of the Wei–Norman factorization.

Only in the time-independent case $A(t) = A, \forall t \in \mathcal{I}$, we find that $U(t, t_0) = U(t - t_0)$ and $U(t)$ defines a 1-parameter group,

$$U(t_1 + t_2) = U(t_1)U(t_2), \tag{6}$$

with A as the infinitesimal generator.

Equation (3) is more general than Eq. (1) and has the advantage that the initial condition, $U(t_0, t_0) = I_V$, is fixed. For simplicity of notation, we shall simply write $U(t) \equiv U(t, t_0)$ and take $t_0 = 0$ in most of the cases, but we should keep in mind the dependence of $U(t)$ on the initial time t_0.

In what follows, we shall assume that $A(t)$ can be written as a finite sum,

$$A(t) = \sum_{i=1}^{n} \alpha_i(t)X_i \qquad \forall t \in \mathcal{I}, \tag{7}$$

where $\mathcal{B} = \{X_i,\ i = 1, \ldots, n = \dim\mathcal{G}\}$ forms a basis of a Lie algebra $\mathcal{G}$, realized as endomorphisms of V, with commutation relations given by

$$[X_i, X_j] = \sum_{k=1}^{n} C_{ijk}X_k \tag{8}$$

and C_{ijk} are known as the structure constants of the Lie algebra $\mathcal{G}$ and the basis $\mathcal{B}$.

Then, the WN theorem[1,2] states that the evolution operator $U(t)$ can be factorized as

$$U(t) = e^{g_1(t)X_1} e^{g_2(t)X_2} \cdots e^{g_n(t)X_n}, \qquad g_i(0) = 0,\ i = 1, \ldots, n, \tag{9}$$

where the functions $g_i(t), i = 1, \ldots, n$ satisfy a set of non-linear first-order differential equations. This construction holds, in general, only in an open subinterval $\mathcal{J} \subset \mathcal{I}$ centered at $t_0 = 0$ (local theorem). There are some important cases where the factorization is valid for all $t \in \mathcal{I}$ (global theorem), namely, for both solvable algebras and the case of 2×2 real matrices.[2]

A. Wei–Norman factorization applied to Schrödinger's equation

The quantum evolution equation analogous to Eq. (1) is given by the Schrödinger equation [note the presence of the imaginary unit here as compared to Eq. (1)],

$$i\frac{\partial}{\partial t}|\Psi(t)\rangle = H(t)|\Psi(t)\rangle, \qquad |\Psi(t_0)\rangle = |\Psi_0\rangle, \tag{10}$$

where we have used Diracs's ket notation, i.e., $|\Psi(t)\rangle$, $|\Psi_0\rangle \in \mathcal{D}_H \subset \mathcal{H}$, and $\mathcal{H}$ is a (infinite-dimensional, in general) Hilbert space. Here, $H(t)$ is a family of essentially self-adjoint operators acting on a common dense domain $\mathcal{D}_H \subset \mathcal{H}$. Under mild conditions for $H(t)$, the Schrödinger equation has a unique solution $|\Psi(t)\rangle$ for any $|\Psi_0\rangle \in \mathcal{H}$ valid for all $t \in \mathbb{R}$. Introducing the evolution operator U as

$$|\Psi(t)\rangle = U(t, t_0)|\Psi_0\rangle \qquad \forall t \in \mathbb{R}, \tag{11}$$

this differential equation can be transformed into [note again the presence of the imaginary unit as compared to Eq. (3)]

$$i\frac{\partial U(t, t_0)}{\partial t} = H(t)U(t, t_0), \qquad U(t_0, t_0) = I_{\mathcal{H}}. \tag{12}$$

In this infinite-dimensional case, a generalization of Eq. (4) also applies, and using the fact that $H(t)$ is essentially self-adjoint (and the presence of the imaginary unit), we conclude that $U(t, t_0)$ is a family of unitary operators, satisfying the groupoid property (5). Only in the case $H(t) = H_0$, $\forall t \in \mathbb{R}$, this family constitutes a 1-parameter group with infinitesimal generator H_0. As earlier, we shall simply write $U(t)$ for $U(t, t_0)$, but we should keep in mind the dependence on t_0 if the Hamiltonian is time-dependent.

Assuming that the Hamiltonian $H(t)$ can be written as

$$H(t) = \sum_{i=1}^{n} \alpha_i(t) X_i \qquad \forall t \in \mathbb{R}, \tag{13}$$

where, now, $X_i, i = 1, \ldots, n$, are essentially self-adjoint operators having the same common dense domain $D_H \subset \mathcal{H}$, and closing the Lie algebra $\mathcal{G}$ [with commutation relations (8)], the WN factorization (9) of the evolution operator $U(t)$ can also be performed. The factorization will be valid in an interval $\mathcal{J} \subset \mathbb{R}$. The same considerations about the local or global character of the factorization also apply here since this depends (except for some particular cases such as that of 2×2 real matrices) on the structure constants of the Lie algebra $\mathcal{G}$ and not on the particular representation (either as matrices or operators) of the X_i.

III. EVOLUTION OPERATOR FOR GCK

We shall now consider the specific case of the generalized Caldirola–Kanai (GCK)[3,4] quantum Hamiltonian,

$$H(t) = \frac{1}{2m(t)}P^2 + \frac{1}{2}m(t)\omega^2(t)X^2, \tag{14}$$

where $X \equiv x$ and $P \equiv -i\frac{\partial}{\partial x}$ are the usual dimensionless position and momentum operators in one dimension. Both the mass $m(t)$ and the frequency $\omega(t)$ are taken to depend on time. This Hamiltonian appears in many physical situations, generalizing the standard harmonic oscillator [$\omega(t) = \omega_0$, $m(t) = m_0$] and the time-dependent (parametric) harmonic oscillator [$m(t) = m_0$] appearing, for instance, in ion traps.[6] The general case appears, for instance, in some cosmological models.[8]

Given that the Hamiltonian is time-dependent, the corresponding time evolution operator $U(t)$ does not take the simple textbook exponential form. Instead, we shall follow the WN factorization approach for its computation, and for that purpose, we first proceed to find the Lie algebra associated with our Hamiltonian.

The three operators

$$K = \frac{1}{2}\frac{\partial^2}{\partial x^2} = -\frac{P^2}{2}, \tag{15}$$

$$V = \frac{X^2}{2}, \tag{16}$$

and

$$S = \frac{i}{2}D, \tag{17}$$

with $D = \frac{1}{2}(XP + PX)$ as the dilation operator, form a basis of an $su(1,1)$ algebra. Note that K, V, and D are Hermitian operators, and thus, S is anti-Hermitian. The Hamiltonian in Eq. (14) is obviously an element of this algebra,

$$H(t) = -\frac{K}{m(t)} + m(t)\omega^2(t)V, \tag{18}$$

where the dilation element S is not present. However, due to the commutation relations of the Lie algebra $su(1,1)$, S will appear in the Wei–Norman factorization. We could have considered the most general element of the Lie algebra, including a term in S as well as in the Hamiltonian (see Refs. 9 and 10 for a detailed study of this case in the WN approach). However, the dilation element S of the Lie algebra is not realizable in most physical applications, and in any case, it can be easily removed by a gauge transformation.[23]

The corresponding commutation relations defining the $su(1,1)$ Lie algebra are

$$[K,V]=2S,\qquad [K,S]=K,\qquad [V,S]=-V. \tag{19}$$

Invoking now the Wei–Norman ansatz (see Refs. 1 and 2 and Sec. II), we can express the time evolution operator $U(t)$ as a product of three exponential operators,

$$U(t)=e^{g_1(t)M_1}e^{g_2(t)M_2}e^{g_3(t)M_3}, \tag{20}$$

where $B_M\equiv\{M_1,M_2,M_3\}$ is a given permutation of the basis $\{K,V,S\}$. For convenience, we have labeled the indices of the g functions according to the ordering of the exponentials.

Each exponent in $e^{g_i(t)M_i}$, $i=1,2,3$ consists of a product of an unknown time-dependent function $g_i(t)$ and the corresponding basis operator M_i. Since these operators do not commute, the explicit form of the functions g depends on the chosen permutation B_M [although the expression of $U(t)$ does not depend on it]. The advantage with this factorized form (as opposed to the exponential of a sum) is that the application of each exponential factor $e^{g_i(t)M_i}$ to a ket is straightforward, even for time-dependent functions $g_i(t)$.

Note that each exponential operator $e^{g_i(t)M_i}$ corresponds to a one-parameter transform subgroup given in Eq. (B11) of Appendix B (see also Ref. 13). This realization constitutes a twofold representation (the metaplectic representation) of the symplectic group of canonical transformations.

A. Ordering B_1

For the particular ordered basis $B_1=\{V,S,K\}$, the expression of $U(t)$ is chosen as

$$U(t)=e^{g_1(t)V}e^{g_2(t)S}e^{g_3(t)K},\qquad U(t=0)=I. \tag{21}$$

Applying the general description of the WN factorization method explained in Sec. II, we derive the three coupled differential equations satisfied by the functions $g_i(t)$,

$$\begin{aligned}
-ig_1'(t)&=\frac{W(t)}{m_0}g_1(t)^2-\frac{m_0\omega^2(t)}{W(t)}, && g_1(0)=0,\\
-ig_2'(t)&=\frac{2W(t)}{m_0}g_1(t), && g_2(0)=0,\\
-ig_3'(t)&=\frac{W(t)}{m_0}e^{g_2(t)}, && g_3(0)=0,
\end{aligned} \tag{22}$$

where $W(t)=\frac{m_0}{m(t)}$, with $m_0=m(0)$. It should be stressed that, from these equations, $g_1(t)$ and $g_3(t)$ are pure imaginary, while $g_2(t)$ is real. These, together with the hermiticity of the operators K, V, and D, guarantees that the evolution operator $U(t)$ is unitary. This turns out to be the same condition for the rest of this paper and will not be further discussed.

The first equation is an uncoupled Riccati equation for $g_1(t)$, and once solved, the second and then the third equations are solved by quadratures. In Sec. VIII, we provide the explicit solutions of these equations for the usual Caldirola–Kanai case[3,4] $m(t)=m_0e^{2\gamma t}$ and $\omega(t)=\omega_0$.

For general time-dependent functions $m(t)$ and $\omega(t)$, we shall instead transform the Riccati equation into a second-order linear differential equation to gain some physical insight. With the standard change of functions,[24]

$$g_1(t)=i\frac{m_0u_2'(t)}{W(t)u_2(t)}, \tag{23}$$

the equation for $u_2(t)$ results in

$$u_2''(t)-\frac{W'(t)}{W(t)}u_2'(t)+\omega(t)^2u_2(t)=0,\qquad u_2(0)=1,\quad u_2'(0)=0. \tag{24}$$

This is the Euler–Lagrange equation associated with the classical Hamiltonian corresponding to Eq. (14). It is the most general equation for an oscillator with time-dependent frequency and mass (or damping coefficient). See Ref. 5 for the driven case and also Ref. 9 in the framework of the WN approach.

In terms of the function $u_2(t)$, $g_2(t)$ can be solved as

$$g_2(t)=-\log u_2(t)^2. \tag{25}$$

Thus, the factors $e^{g_1(t)V}e^{g_2(t)S}$ appearing in the operator $U(t)$ represent a dilation by $\frac{1}{u_2(t)}$ followed by a multiplication by the phase $\exp(i\frac{m_0u_2'(t)}{2W(t)u_2(t)}x^2)$.

Once we know $g_2(t)$, $g_3(t)$ is expressed as

$$g_3(t) = \frac{i}{m_0}\int_0^t \frac{W(t')}{u_2(t')^2}\,dt' = \frac{i}{m_0}\frac{u_1(t)}{u_2(t)}, \tag{26}$$

where

$$u_1(t) = u_2(t)\int_0^t \frac{W(t')}{u_2(t')^2}\,dt', \qquad u_1(0) = 0, \quad u_1'(0) = 1, \tag{27}$$

is a new, independent solution of Eq. (24). In terms of the solutions u_1 and u_2, we have $W(t) = u_1'(t)u_2(t) - u_1(t)u_2'(t)$. Thus, $W(t)$ turns out to be the Wronskian of the two fundamental solutions of Eq. (24).

The factor $e^{g_3(t)K}$ appearing in the operator $U(t)$ represents an integral transform known as the Fresnel propagator (or Fresnel transform) [see Appendix B, the first line in Eq. (B11)].

The Fresnel propagator is a non-point (or non-geometric) transform, i.e., it is not obtained, up to a phase, as a change of variables in the argument of the function (see Appendix B). In optics, it accounts for the propagation of waves in free space, and in quantum mechanics, it is responsible for the time evolution of a free non-relativistic particle.

B. Ordering B_2

If we choose a different ordering, for instance, $B_2 = \{K, S, V\}$, the evolution operator is factorized as

$$U(t) = e^{h_1(t)K}e^{h_2(t)S}e^{h_3(t)V}. \tag{28}$$

Applying again the WN factorization method, the coupled differential equations satisfied by the functions $h_i(t)$ are

$$\begin{aligned} -ih_1'(t) &= -\frac{m_0\omega^2(t)}{W(t)}h_1(t)^2 + \frac{W(t)}{m_0}, & h_1(0) &= 0,\\ -ih_2'(t) &= \frac{2m_0\omega^2(t)}{W(t)}h_1(t), & h_2(0) &= 0,\\ -ih_3'(t) &= -\frac{m_0\omega^2(t)}{W(t)}e^{-h_2(t)}, & h_3(0) &= 0. \end{aligned} \tag{29}$$

Using again the change of functions

$$h_1(t) = -i\frac{W(t)v_2'(t)}{m_0\omega^2(t)v_2(t)}, \tag{30}$$

the equation for $v_2(t)$ takes the related form

$$v_2''(t) + \left(\frac{W'(t)}{W(t)} - 2\frac{\omega'(t)}{\omega(t)}\right)v_2'(t) + \omega^2(t)v_2(t) = 0, \qquad v_2(0) = 1, \quad v_2'(0) = 0. \tag{31}$$

The friction coefficient in this equation, $\frac{W'(t)}{W(t)} - 2\frac{\omega'(t)}{\omega(t)}$, can be interpreted as corresponding to a time-dependent mass $\tilde{m}(t) = \frac{m_0^2\omega_0^2}{m(t)\omega(t)^2}$, where $\omega_0 = \omega(0)$ has been introduced for convenience. Hence, Eq. (31) is the Euler–Lagrange equation associated with the new classical Hamiltonian

$$\tilde{H}(t) = \frac{1}{2\tilde{m}(t)}p^2 + \frac{1}{2}\tilde{m}(t)\omega^2(t)x^2. \tag{32}$$

Note that $\tilde{H}(t)$ is $H(t)$ with the functions multiplying K and V interchanged. In the particular case of constant frequency $\omega(t) = \omega_0$, the new friction coefficient has the opposite sign. Therefore, Eq. (31) corresponds to the Euler–Lagrange equation for a mirror particle in the sense of the Bateman dual system.[25]

In terms of the function $v_2(t)$, $h_2(t)$ can be solved as

$$h_2(t) = \log v_2(t)^2, \tag{33}$$

analogous to Eq. (25), but with the opposite sign.

In this case, the factors $e^{h_2(t)S}e^{h_3(t)V}$ appearing in the operator $U(t)$ represent multiplication by the phase $\exp(-i\frac{W(t)v_2'(t)}{m_0\omega^2(t)v_2(t)}x^2)$ followed by a dilation by $v_2(t)$.

Once we know $h_2(t)$, $h_3(t)$ is expressed as

$$h_3(t) = -im_0\int_0^t \frac{\omega^2(t')}{W(t')v_2(t')^2}\,dt' = -im_0\omega_0^2\frac{v_1(t)}{v_2(t)}, \tag{34}$$

where

$$v_1(t) = \frac{v_2(t)}{\omega_0^2}\int_0^t \frac{\omega(t')^2}{W(t')v_2(t')^2}\,dt', \qquad v_1(0) = 0, \quad v_1'(0) = 1, \tag{35}$$

is a new solution of Eq. (31). In terms of the solutions v_1 and v_2, we have that $v_1'(t)v_2(t) - v_1(t)v_2'(t) = \frac{\omega(t)^2}{W(t)}$ is the new Wronskian of the two fundamental solutions of Eq. (31).

Out of the other four remaining ordered basis, two of them lead to similar results to the basis B_1 with the same Euler–Lagrange equation (24), while the other two are similar to B_2 with the same Euler–Lagrange equation (31) and will not be further discussed here [see Ref. 26 for a detailed study of the Wei–Norman method for the group $SL(2,\mathbb{R})$, where the same basis is considered and the six orderings are discussed for the purpose of obtaining non-linear superposition principles].

IV. GENERAL FRESNEL TRANSFORM

In this section, we consider the possibility of relating a GCK system, with frequency $\omega(t)$ and mass $m(t)$ to the free particle. That is, we are interested in a unitary transformation $F(t)$, called the General Fresnel transform (GFT), from the Hilbert space $\mathcal{H}_t$ of solutions to the Schrödinger equation for the GCK system (14) to the corresponding Hilbert space $\mathcal{H}_t^0$ for the free particle

$$H_0 = -\frac{1}{m_0}K, \tag{36}$$

where $m_0 = m(0)$ as in Sec. III. We take the nomenclature from Ref. 14, where they denote a general linear integral transform (see Appendix B) with the integral kernel given by (B5) by a general Fresnel transform since it contains the Fresnel propagator as a particular case.

Denoting by $U(t)$ and $U_0(t)$ the evolution operators for the GCK system and the free particle, respectively, the following diagram is commutative and all operators appearing in it are unitary:

$$\begin{array}{ccc} \mathcal{H}_t & \underset{G(t)}{\overset{F(t)}{\rightleftarrows}} & \mathcal{H}_t^0 \\ & {\scriptstyle U(t)}\nwarrow \quad \nearrow{\scriptstyle U_0(t)} & \\ & \mathcal{H}_{t=0}^0 & \end{array},$$

where $G(t) = F(t)^{-1}$. From the diagram, it can be immediately seen that $G(t) = U(t)U_0(t)^{-1}$.

The corresponding differential equation for $G(t)$ is

$$i\frac{\partial G(t)}{\partial t}G(t)^{-1} = H(t) - G(t)H_0G(t)^{-1}, \qquad G(0) = I, \tag{37}$$

generalizing the Schrödinger equation for the evolution operator [Eq. (12)].

This equation can also be solved by using the WN factorization technique since both $H(t)$ and H_0 belong to the same $su(1,1)$ algebra.

Using the same basis as in Sec. III, for the specific ordering B_1, the expression of $G(t)$ is chosen as

$$G(t) = e^{f_1(t)V}e^{f_2(t)S}e^{f_3(t)K}, \tag{38}$$

where the functions $f_i(t), i = 1,2,3$ verify the new non-linear equations

$$\begin{aligned} -if_1'(t) &= \frac{W(t)}{m_0}f_1(t)^2 - \frac{m_0\omega^2(t)}{W(t)}, \\ -if_2'(t) &= \frac{2W(t)}{m_0}f_1(t), \\ -if_3'(t) &= \frac{W(t)}{m_0}e^{f_2(t)} - \frac{1}{m_0}. \end{aligned} \tag{39}$$

The equation for f_3 has an extra term $-\frac{1}{m_0}$ as compared to Eq. (22), reflecting the fact that we are now solving for the general Fresnel transform [Eq. (37)], instead of the evolution operator $U(t)$ [Eq. (12)]. As in the case of the evolution operator, we shall transform the present Riccati equation into a second-order linear differential equation to gain some physical insight. With the standard change of functions,[24]

$$f_1(t) = i\frac{m_0 u_2'(t)}{W(t)u_2(t)}, \tag{40}$$

the equation for $u_2(t)$ is

$$u_2''(t) - \frac{W'(t)}{W(t)}u_2'(t) + \omega(t)^2 u_2(t) = 0, \qquad u_2(0) = 1, \quad u_2'(0) = 0, \tag{41}$$

equivalent to Eq. (24).

As above, this is the Euler–Lagrange equation associated with the classical Hamiltonian corresponding to Eq. (14). In terms of the function $u_2(t)$, $g_2(t)$ can be solved as

$$f_2(t) = -\log u_2(t)^2. \tag{42}$$

Once we know $f_2(t)$, $f_3(t)$ is expressed as

$$f_3(t) = \frac{i}{m_0}\int_0^t \left(\frac{W(t')}{u_2(t')^2} - 1\right) dt' = \frac{i}{m_0}\left(\frac{u_1(t)}{u_2(t)} - t\right), \tag{43}$$

where

$$u_1(t) = u_2(t)\int_0^t \frac{W(t')}{u_2(t')^2}\, dt', \qquad u_1(0) = 0, \quad u_1'(0) = 1, \tag{44}$$

is a new solution of Eq. (41). In terms of the solutions u_1 and u_2, we have $W(t) = u_1'(t)u_2(t) - u_1(t)u_2'(t)$, the Wronskian of the two fundamental solutions of Eq. (41).

Note the presence of the term $-i\frac{t}{m_0}$ in Eq. (43), as compared to Eq. (26), whose origin is the abovementioned extra term appearing in the equation for g_3 in Eq. (39).

In addition, the general Fresnel transform can be factorized in a different basis and/or ordering. In particular, it can be factorized in the ordering given by B_2, with results similar to those of Sec. III B.

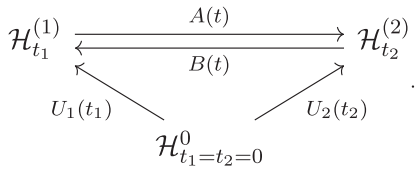
The unitary transformation $G(t)$ maps the free particle into an arbitrary GCK system. Using the explicit form of these solutions, we confirm that $G(t) = U(t)U_0(t)^{-1}$ for this particular case.

The general Fresnel transform can be further generalized to a transformation relating an arbitrary GCK system to the harmonic oscillator or even relating two arbitrary GCK systems. However, in any case, the general Fresnel transform will be a non-point transformation (see Appendix B) since it will always include an exponential factor $e^{f_3(t)K}$ (the Fresnel propagator), which is a non-trivial integral operator in this representation.

It is desirable to build a transform whose factorization does not involve the Fresnel propagator, being therefore a point transformation. This will be considered in Sec. V.

V. ARNOLD TRANSFORM

In this section, we consider the possibility of relating two different GCK systems, namely, system 1 with frequency $\omega_1(t)$ and mass $m_1(t)$ and system 2 with frequency $\omega_2(t)$ and mass $m_2(t)$, through a point transformation, with the pay-off of an additional diffeomorphism in time. That is, we are interested in a transformation $A(t)$ [and its inverse $B(t) \equiv A(t)^{-1}$] from system 1 to system 2 such that the following diagram is commutative and all operators appearing in it are unitary:



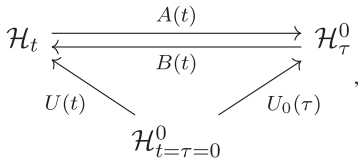
In this diagram, $\mathcal{H}^{(1)}_{t_1}$ is the Hilbert space of solutions to the Schrödinger equation for system 1 at time t_1 and $\mathcal{H}^{(2)}_{t_2}$ is the Hilbert space of solutions to the Schrödinger equation for system 2 at time t_2. $U_1(t_1)$ and $U_2(t_2)$ are the corresponding evolution operators for systems 1 and 2.

At the bottom line of this diagram, $\mathcal{H}^{0}_{t_1=t_2=0}$ represents the identical Hilbert spaces $\mathcal{H}^{(1)}_{t_1=0} \equiv \mathcal{H}^{(2)}_{t_2=0}$.

Imposing the condition that the transformation $A(t)$ is a point transformation implies that t_2 will be related to t_1 in a specific way to be determined. The corresponding unitary transformation $A(t)$ then coincides with the (generalized) quantum Arnold transform, introduced in Ref. 5 for the case in which system 1 is the free particle and in Ref. 23 for the general case.

To simplify the discussion, we shall assume that system 1 is the GCK oscillator in Eq. (14), while system 2 is the simple harmonic oscillator [$\omega_2(t) = \Omega$ and $m_2(t) = m_0 = m_1(0)$].

In this case, the commutative diagram simplifies to



where we have denoted $\tau = t_2$, $t = t_1$, $\mathcal{H}^0 \equiv \mathcal{H}^{(1)}_{\tau=0} = \mathcal{H}^{(2)}_{t=0}$, $\mathcal{H}_t \equiv \mathcal{H}^{(2)}(t)$, $U(t) \equiv U_1(t)$, and $U_0(\tau) \equiv U_2(\tau)$.

Using the commutativity of the diagram and the fact that all operators are unitary, we have that $A(t) = U_0(\tau)U(t)^{-1}$. In Refs. 5 and 23, it is shown that t and τ are related by a reparameterization in time (not depending on x or p), that is, $\tau = r(t)$,

$$A(t) = U_0(r(t))U(t)^{-1}, \qquad B(t) = A(t)^{-1} = U(t)U_0(r(t))^{-1}. \tag{45}$$

Using the expression obtained for the generalized Fresnel transform, $F(t) = U_0(t)U(t)^{-1}$, we can relate both transforms,

$$A(t) = R(t)F(t), \tag{46}$$

where

$$R(t) = U_0(r(t))U_0(t)^{-1} = U_0(r(t) - t), \tag{47}$$

where in the last equality, we have used the fact that system 2 (the simple harmonic oscillator) has a time-independent Hamiltonian.

Then, we have that $B(t)$ satisfies a generalized Schrödinger equation,

$$i\frac{\partial B(t)}{\partial t}B(t)^{-1} = H(t) - r'(t)B(t)H_0B(t)^{-1}, \qquad B(0) = I, \tag{48}$$

similar to that of the evolution operator $U(t)$ [Eq. (12)] but modified by the extra term $-r'(t)B(t)H_0B(t)^{-1}$. The WN method can be easily modified in order to obtain a factorized version of $B(t)$, as we did in Sec. IV.

Using the same basis as in Sec. III, for the particular ordered basis B_1, the expression of $B(t)$ is

$$B(t) = e^{a_1(t)V}e^{a_2(t)S}e^{a_3(t)K}, \tag{49}$$

where, now, the functions $g_i(t), i = 1, 2, 3$ verify the new non-linear equations,

$$\begin{aligned}
-ia_1'(t) &= r'(t)m_0\Omega^2\, e^{a_2(t)} + \frac{W(t)}{m_0}a_1(t)^2 - \frac{m_0\omega^2(t)}{W(t)},\\
-ia_2'(t) &= \frac{2W(t)}{m_0}a_1(t) + 2r'(t)m_0\Omega^2\, a_3(t),\\
-ia_3'(t) &= \frac{W(t)}{m_0}e^{a_2(t)} - \frac{r'(t)}{m_0} + r'(t)m_0\Omega^2\, a_3(t)^2.
\end{aligned} \tag{50}$$

Imposing the condition that $a_3(t)$ vanishes in order to cancel the Fresnel propagator in Eq. (49), we obtain the condition $r'(t) = e^{a_2(t)}W(t)$. Substituting this into the first and second equations, we obtain a coupled system of nonlinear Riccati equations that can be transformed into a second-order linear differential equation with a similar change to that in Eqs. (23) and (25),

$$\begin{aligned}
a_1(t) &= i\frac{m_0b'(t)}{W(t)b(t)},\\
a_2(t) &= -\log\, b(t)^2,
\end{aligned} \tag{51}$$

where, now, $b(t)$ satisfies the generalized Ermakov equation,[23]

$$b''(t) - \frac{W'(t)}{W(t)} b'(t) + \omega(t)^2 b(t) = \frac{\Omega^2}{b(t)^3}, \qquad b(0) = 1, \quad b'(0) = 0. \tag{52}$$

From the expression of $a_2(t)$, we derive the value of $r(t)$ solving

$$r'(t) = \frac{W(t)}{b(t)^2}, \qquad r(0) = 0. \tag{53}$$

The solutions to Eq. (52) can be obtained from the solutions to Eqs. (41)–(44) by

$$b(t)^2 = \Omega^2 u_1(t)^2 + u_2(t)^2. \tag{54}$$

Note that, since $u_1(t)$ and $u_2(t)$ are independent solutions of (24), their Wronskian never vanishes, and therefore, $u_1(t)$ and $u_2(t)$ cannot vanish simultaneously. Thus, $b(t)$ never vanishes and can be taken to be positive. This is an important feature of the Arnold transformation mapping the GCK oscillator to the harmonic oscillator.

The construction with the basis B_2 follows the same lines as in Sec. III B.

VI. PARTICULAR CASES

From the generalized Schrödinger equation (48) and the corresponding non-linear equations (50), we can recover the previous cases choosing some particular values.

For instance, if $r(t) = 0$, we obtain the Schrödinger equation (12) and Eqs. (50) reduce to Eqs. (22). When solving these equations, we can no longer impose $a_3(t) = 0$, and we are in the same situation as in Sec. III, recovering $B(t) = U(t)$ containing the Fresnel propagator. Obviously, in this case, $B(t)$ cannot be interpreted as a transformation to another system but just as unitary time evolution inside the same system.

We would now like to look at the general Fresnel transform and compare with the results given in Ref. 27 for the particular case of a simple Caldirola–Kanai oscillator. Choosing $r(t) = t$ (the time diffeomorphism is the identity), $R(t) = I$ and the generalized Schrödinger equation (48) leads to Eq. (37). The non-linear equations (50) reduce to

$$\begin{aligned} -ia_1'(t) &= m_0\Omega^2 e^{a_2(t)} + \frac{W(t)}{m_0} a_1(t)^2 - \frac{m_0\omega^2(t)}{W(t)}, \\ -ia_2'(t) &= \frac{2W(t)}{m_0} a_1(t) + 2m_0\Omega^2 a_3(t), \\ -ia_3'(t) &= \frac{W(t)}{m_0} e^{a_2(t)} - \frac{1}{m_0} + m_0\Omega^2 a_3(t)^2. \end{aligned} \tag{55}$$

For the case $\Omega = 0$, we recover the GFT $G(t)$, $a_i(t) = f_i(t), i = 1, 2, 3$, mapping the free particle to a GCK system, as in Sec. IV. This transform does contain the Fresnel propagator [since, in general, we cannot impose $a_3(t) = 0$].

If $\Omega \neq 0$, the transform maps the harmonic oscillator to a GCK system obtaining a new GFT, which also contains the Fresnel propagator. If we further demand that $a_3(t) = 0$, then $G(t)$ becomes a point transformation (see Appendix B), but this imposes a relation between the frequencies,

$$\omega(t)^2 + \frac{W''(t)}{2W(t)} - \frac{3W'(t)^2}{4W(t)^2} = \Omega^2. \tag{56}$$

For the case of the simple Caldirola–Kanai oscillator, with $m(t) = m_0 e^{2\gamma t}$ and $\omega(t) = \omega_0$, this translates into the condition $\Omega = \sqrt{\omega_0^2 - \gamma^2}$, recovering the result in Ref. 27.

However, we should be careful when imposing the condition $a_3(t) = 0$. For this case, Eqs. (55) lead to a solution for $a_1(t)$ given by a constant,

$$a_1(t) = im_0\gamma, \qquad a_2(t) = 2\gamma t, \tag{57}$$

and thus, it cannot satisfy the initial condition $a_1(0) = 0$. Therefore, the general Fresnel transform $G(t)$ is no longer the identity operator at $t = 0$ as was assumed in Eq. (37). This can be accounted for by modifying the commutative diagram in Sec. IV with a new one of the form

$$\begin{array}{ccc} \mathcal{H}_t & \xleftarrow{G(t)} & \mathcal{H}_t^0 \\ {\scriptstyle U(t)}\uparrow & & \uparrow{\scriptstyle U_0(t)}, \\ \mathcal{H}_{t=0} & \xleftarrow[T]{} & \mathcal{H}_{t=0}^0 \end{array} \tag{58}$$

where, now, $G(t) = U(t)TU_0(t)^{-1}$ satisfies $G(0) = T$, with T being a non-trivial unitary transformation relating $\mathcal{H}^0_{t=0}$ and $\mathcal{H}_{t=0}$. For the case of the CK oscillator, $T = e^{\frac{i}{2}m_0\gamma X^2}$, and this reproduces the results in Ref. 27 for the case of a point general Fresnel transform.

VII. INTEGRAL EXPRESSIONS OF THE EVOLUTION, GENERALIZED FRESNEL, AND ARNOLD TRANSFORMS

Using the results of Appendix B, we provide the explicit expression, as integral transforms, of the operators discussed in Secs. III, IV and V. In particular, using Eqs. (B11)–(B13), the factorized version (for the ordering B_1) of the evolution operator, generalized Fresnel transform, and Arnold transform can be obtained.

A. Integral expressions of operators

For the evolution operator,

$$\begin{aligned}\Psi_U(x,t) &= U(t)\Psi_0(x) = e^{g_1(t)V}e^{g_2(t)S}e^{g_3(t)K}\Psi_0(x)\\ &= e^{\frac{1}{2}g_1(t)x^2}e^{\frac{g_2(t)}{4}}\frac{1}{\sqrt{2\pi g_3(t)}}\int_{-\infty}^{\infty}dx'e^{-\frac{1}{2g_3(t)}\left(e^{\frac{g_2(t)}{2}}x-x'\right)^2}\Psi_0(x')\\ &= \sqrt{\frac{m_0 s(u_2(t))}{2\pi i u_1(t)}}e^{\frac{im_0u_2'(t)}{2W(t)u_2(t)}x^2}\int_{-\infty}^{\infty}dx'e^{\frac{im_0u_2(t)}{2u_1(t)}\left(\frac{x}{|u_2(t)|}-x'\right)^2}\Psi_0(x'),\end{aligned} \tag{59}$$

where $\Psi_0(x)$ stands for any initial state (at $t_0 = 0$) for the GCK oscillator and $\Psi_U(x,t)$ stands for its evolution under $U(t)$. In the last equation, s stands for the *sign* function.

For the GFT transform,

$$\begin{aligned}\Psi_G(x,t) &= G(t)\psi(x,t) = e^{f_1(t)V}e^{f_2(t)S}e^{f_3(t)K}\psi(x,t)\\ &= e^{\frac{1}{2}f_1(t)x^2}e^{\frac{f_2(t)}{4}}\frac{1}{\sqrt{2\pi f_3(t)}}\int_{-\infty}^{\infty}dx'e^{-\frac{1}{2f_3(t)}\left(e^{\frac{f_2(t)}{2}}x-x'\right)^2}\psi(x',t)\\ &= \sqrt{\frac{m_0 s(u_2(t))}{2\pi i(u_1(t)-tu_2(t))}}e^{\frac{im_0u_2'(t)}{2W(t)u_2(t)}x^2}\int_{-\infty}^{\infty}dx'e^{\frac{im_0u_2(t)}{2(u_1(t)-tu_2(t))}\left(\frac{x}{|u_2(t)|}-x'\right)^2}\psi(x',t),\end{aligned} \tag{60}$$

where $\psi(x,t)$ represents an arbitrary state for the free particle and $\Psi_G(x,t)$ represents its transformed state for the GCK oscillator. Note that both states are given at the same time t.

For the Arnold transform,

$$\begin{aligned}\Psi_B(x,t) &= B(t)\psi(x,\tau) = e^{a_1(t)V}e^{a_2(t)S}\psi(x,r(t))\\ &= e^{\frac{1}{2}a_1(t)x^2}e^{\frac{a_2(t)}{4}}\psi(e^{\frac{a_2(t)}{2}}x,r(t))\\ &= \frac{1}{\sqrt{b(t)}}e^{\frac{im_0b'(t)}{2W(t)b(t)}x^2}\psi\left(\frac{x}{b(t)},\int_0^t\frac{W(t')}{b(t')^2}dt'\right),\end{aligned} \tag{61}$$

where $\psi(x,\tau)$, now, represents an arbitrary state for the standard harmonic oscillator at time τ and $\Psi_B(x,t)$ represents its transformed state for the GCK oscillator at time t. Both times are related by $\tau = r(t) = \int_0^t\frac{W(t')}{b(t')^2}dt' = \arctan\left(\frac{\Omega u_1(t)}{u_2(t)}\right)$.

It should be stressed that $\Psi_U(x,t) = \Psi_G(x,t) = \Psi_B(x,t)$ if $\psi(x,0) = \Psi_0(x)$, as a consequence of the commutative diagrams in Secs. IV and V. Therefore, we shall focus on the evolution operator in the rest of this section.

B. Caustic points

Note that the integral expressions of $U(t)$ and $G(t)$ are undefined when $u_2(t) = 0$ [although $u_1(t)$ appears in the denominator inside the Fresnel propagator, when $u_1(t) = 0$, the Fresnel propagator is the identity; see Appendix B]. In addition, when $u_2(t) = 0$, all functions $g_i(t)$ [or $f_i(t)$] diverge. More precisely, both g_1 and g_3 have a simple pole, while g_2 has a logarithmic singularity [the same applies for $f_i(t)$].

This singular behavior is spurious in the sense that the evolution operator is well defined for all times, but the factorization (for the chosen basis and order) is not possible for certain elements of the SU(1,1) group (see Refs. 13 and 28). It should be stressed that even if the evolution operator in its factorized form cannot be computed at the singularities, its action on particularly well-behaved initial states (such as number states or coherent states of the standard harmonic oscillator) is defined for all times; see the discussion below.

The time values t_s when $u_2(t_s) = 0$ have an interesting feature, and they play an important role in the semiclassical theory of calculus of variations, where they are referred to as *caustic* or *focal* points, as there are infinite solutions of the classical equations verifying the boundary conditions. For these values, the phase of the evolution operator experiences a *jump*, known as Guoy phase[19] (see also Ref. 29), which can

be observed experimentally, and it is related to the Maslov index and Morse theory.[17] For an account of this phenomenon, see Ref. 15 and references therein, where the computation of the propagator in the path integral approach for the standard harmonic oscillator is discussed.

As a consequence of this singular behavior, if analytic expressions for the solutions u_1 and u_2 are not available, then $g_i(t)$ have to be computed numerically from Eqs. (22). Then, it will not be possible to obtain their values outside of the interval defined by two consecutive zeros of u_2 containing t_0, and therefore, it will not be possible to compute the evolution operator (or the GFT) in its factorized form outside this interval (unless a different t_0 in a different interval is chosen).

However, when analytic expressions for u_1 and u_2 are available, even if the $g_i(t)$ are singular at the zeros of u_2, the evolution operator in its factorized form can be computed directly for all values of t, except at the zeros of u_2. A relevant question is thus if there is any way of computing the expression of $U(t)$ at the zeros of $u_2(t)$ using its factorized form.

Our approach consists in performing a limiting procedure with a careful analysis of the behavior of the solutions $u_1(t)$ and $u_2(t)$ near a zero of $u_2(t)$. For that purpose, if, as before, t_s is a zero of $u_2(t)$ (there can be none, a finite, or an infinite number of them),

$$u_2(t_s) = 0, \qquad u_2'(t_s) \neq 0, \qquad u_1(t_s) = -\frac{W(t_s)}{u_2'(t_s)} \neq 0 \tag{62}$$

since u_2 is not a trivial solution and the Wronskian $W(t_s)$ cannot be zero. The limit is then

$$\lim_{t\to t_s^{\pm}} U(t)\Psi_0(x) = e^{\pm i\frac{\pi}{4}} e^{-m_0 \frac{u_1'(t_s)u_2'(t_s)}{W(t_s)^2} V} e^{m_0 \frac{|u_2'(t_s)|}{W(t_s)} S} \hat{\Psi}_0(\mp x), \tag{63}$$

where $\hat{\Psi}_0$ is the Fourier transform of Ψ_0. The $\pm$ sign at the right-hand side appears since $s(u_2(t_s^{\pm})u_2'(t_s^{\pm})) = \pm 1$, $s(.)$ being the sign function.

Observe that at the singular value t_s, the initial function Ψ_0 is Fourier transformed and rescaled by $m_0\frac{|u_2'(t_s)|}{W(t_s)}$ (in addition to a local phase). When crossing the singularity, the wave function experiences a *parity reversal* and a *phase jump* by $e^{\pm i\frac{\pi}{2}}$. This is the abovementioned Guoy phase or Maslov correction. In addition, since the phase varies continuously between two consecutive singular values of t, the effect of the phase jump is accumulative, i.e., if starting from $t_0 = 0$ we need to pass through k singularities to reach t, then the total phase shift is $e^{ik\frac{\pi}{2}}$.

C. Action of the evolution operator on particular initial states

Let us illustrate the action of the evolution operator on a couple of initial states. We shall consider two diametrically opposite states: a plane wave and a Gaussian state. The plane wave is an eigenstate of the free particle Hamiltonian, and it is not normalizable. The Gaussian state is an eigenstate of the standard harmonic oscillator, and it is a very well-behaved state (Gaussian functions are usually used as regularizing functions).

For the plane wave,

$$\Psi_0(x) = \frac{1}{\sqrt{2\pi}} e^{iPx}, \qquad \hat{\Psi}_0(x) = \delta(x - P), \tag{64}$$

$$\Psi_U(x,t) = \frac{1}{\sqrt{2\pi|u_2(t)|}} e^{-i\frac{u_1'(t)}{2m_0u_2'(t)}P^2} e^{i\frac{m_0u_2'(t)}{2W(t)u_2(t)}\left(x+\frac{W(t)s(u_2(t))}{m_0u_2'(t)}P\right)^2}, \tag{65}$$

$$\lim_{t\to t_s^{\pm}} \Psi_U(x,t) = e^{\pm i\frac{\pi}{4}} \sqrt{\frac{iW(t_s^{\pm})}{m_0|u_2'(t_s^{\pm})|}} e^{-i\frac{u_1'(t_s^{\pm})}{2m_0u_2'(t_s^{\pm})}P^2} \delta\left(x \pm \frac{W(t_s^{\pm})}{m_0|u_2'(t_s^{\pm})|}P\right). \tag{66}$$

In the last equation, we have made use of the limit of the non-relativistic free propagator,

$$\lim_{a\to 0} \sqrt{\frac{m_0}{2\pi i a}} e^{i\frac{m_0}{2a}(x-x')^2} = \delta(x - x'). \tag{67}$$

It can be easily checked that Eq. (66) coincides with the general case of Eq. (63), once P in the exponential is written in terms of x as a consequence of the Dirac delta.

For the Gaussian state,

$$\Psi_0(x) = \left(\frac{m_0\Omega}{\pi}\right)^{1/4} e^{-\frac{1}{2}m_0\Omega x^2}, \qquad \hat{\Psi}_0(x) = \left(\frac{1}{m_0\Omega\pi}\right)^{1/4} e^{-\frac{1}{2m_0\Omega}x^2}, \tag{68}$$

$$\Psi_U(x,t) = \left(\frac{m_0\Omega}{\pi}\right)^{1/4} \frac{1}{\sqrt{b(t)}} e^{-\frac{i}{2}\arctan\left(\frac{\Omega u_1(t)}{u_2(t)}\right)} e^{\frac{im_0}{2}\frac{b'(t)}{W(t)b(t)}x^2} e^{-\frac{1}{2}m_0\Omega\frac{x^2}{b(t)^2}}, \tag{69}$$

$$\lim_{t\to t_s^{\pm}} \Psi_U(x,t) = \left(\frac{m_0\Omega}{\pi}\right)^{1/4} e^{\pm i\frac{\pi}{4}} \sqrt{\frac{|u_2'(t_s^{\pm})|}{\Omega W(t_s^{\pm})}}\, e^{-\frac{im_0}{2}\frac{u_1'(t_s^{\pm})u_2'(t_s^{\pm})}{W(t_s^{\pm})^2}x^2} e^{-\frac{1}{2}m_0\frac{u_2'(t_s^{\pm})^2}{\Omega W(t_s^{\pm})^2}x^2}, \tag{70}$$

where we have taken into account that $\lim_{t\to t_s^{\pm}} \arctan\left(\frac{\Omega u_1(t)}{u_2(t)}\right) = \mp\frac{\pi}{2}$. It can be easily checked that Eq. (70) coincides with the general case of Eq. (63).

It is worth to note that the evolution operator acting on a Gaussian state [Eq. (69)] is well defined for all values of time, except for the phase jump at t_s (compare with Refs. 9 and 10, where the Wei–Norman factorization and the same basis and ordering B_1 are used). Thus, it could seem that no singularities appear in this case, but this is due to the regularizing properties of the Gaussian function. For the plane wave case in Eq. (65), it is evident that the evolution operator is not defined at the singular times since the limit is a delta function and must be computed with extreme care [cf. (67)].

The similarities with Fourier optics should also be remarked,[18] where a thin lens, in the paraxial approximation, behaves as an *optical Fourier transform*. There, a plane wave with momentum P arriving to it is transformed, at the focal point, into a light spot (Dirac delta) at position (proportional to) $-P$, a situation completely analogous to Eqs. (64)–(66). The regularized version of this situation [Eqs. (68)–(70)] has an optical analog on a Gaussian beam *focused* into another Gaussian beam (its Fourier transform) at the focal point of the lens.

VIII. CALDIROLA–KANAI EXAMPLE

In the original Caldirola–Kanai example, with constant frequency $\omega(t) = \omega_0$ and constant damping coefficient 2γ, or, equivalently, mass function $m(t) = m_0 e^{2\gamma t}$, the EL equation (24) can be solved analytically, with solutions (following the notation of Sec. III A)

$$\begin{aligned} u_1(t) &= \frac{1}{\omega_d} e^{-\gamma t} \sin(\omega_d t), \\ u_2(t) &= \frac{1}{\omega_d} e^{-\gamma t}(\omega_d \cos(\omega_d t) + \gamma \sin(\omega_d t)), \end{aligned} \tag{71}$$

where $\omega_d = \sqrt{\omega_0^2 - \gamma^2}$ is the displaced frequency that can be positive, zero, or pure imaginary, leading to the underdamped, critical, or overdamped regimes, respectively. In the following, we shall discuss the underdamped regime for $\gamma \in [0, \omega_0)$. In Fig. 1, a plot of the two functions is shown for this case.

We shall consider in Subsections VIII A–VIII D, the factorized form of the evolution operator and the related integral transforms, particularized to the Caldirola–Kanai example.

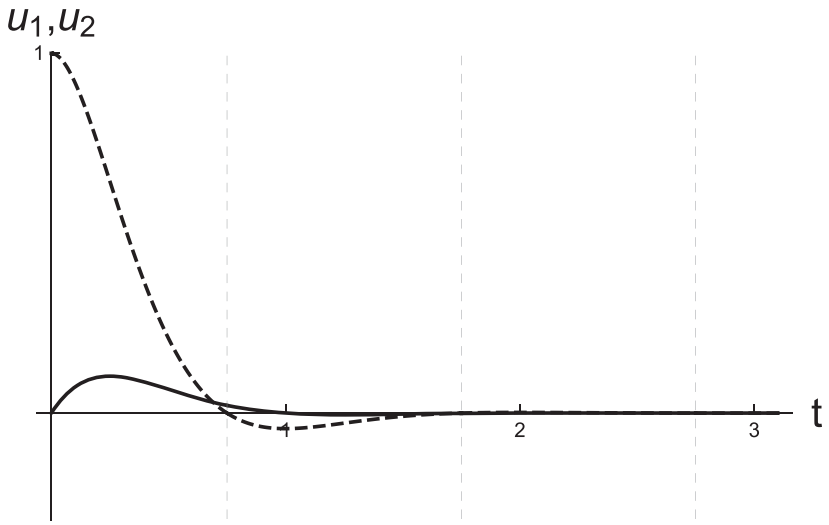


FIG. 1. Graphs of u_1 and u_2 (dashed) for $\gamma = \omega_d = \pi$ and $m_0 = 1$. The dashed vertical lines indicate the zeros of u_2.

A. Evolution operator

The evolution operator with the ordering B_1 [Eq. (21)] is obtained once Eqs. (22) are solved. Using the solutions in terms of u_1 and u_2 given by Eqs. (23), (25), and (26), we find

$$g_1(t) = -\frac{im_0\omega_0^2 e^{2\gamma t}\,\sin(\omega_d t)}{\gamma\,\sin(\omega_d t) + \omega_d\,\cos(\omega_d t)}, \tag{72}$$

$$g_2(t) = -\log\left(\frac{e^{-\gamma t}}{\omega_d}(\gamma\,\sin(\omega_d t) + \omega_d\,\cos(\omega_d t)\right)^2, \tag{73}$$

$$g_3(t) = \frac{i}{m_0}\frac{\sin(\omega_d t)}{\gamma\,\sin(\omega_d t) + \omega_d\,\cos(\omega_d t)}. \tag{74}$$

It should be stressed that in the particular case of the CK Hamiltonian, $g_1(t) = -\frac{m_0^2\omega_0^2}{W(t)}g_3(t)$. Note also that for $t = t_k = \frac{1}{\omega_d}(k\pi - \arctan\frac{\omega_d}{\gamma})$, $k \in \mathbb{Z}$, g_1 and g_3 have a simple pole, while g_2 has a logarithmic singularity. See Fig. 2 for a graph of these functions showing the singularities.

Thus, for the CK case, the evolution operator in its factorized form (and for the ordering B_1) with the initial condition $t_0 = 0$ can only be computed inside each interval $\mathcal{J}_k = (t_k, t_{k+1})$, $k \in \mathbb{Z}$, of width $\frac{\pi}{\omega_d}$, and not at the singular values t_k. In particular, the interval $\mathcal{J}_k$ containing the initial condition $t_0 = 0$ is obtained for $k = 0$. In this section, we shall use the same notation t_k for $k = 0$ and the initial time t_0 of Eq. (1), the meaning being obvious from the context.

The times t_k are the concrete realization of the singular times t_s discussed in Sec. VII, and all results obtained there applies here particularized to the CK case.

B. Fresnel transform to the free particle

In the case of the Fresnel transform to the free particle, the expressions are identical to those of the evolution operator,

$$\begin{aligned} g_1(t) &= -\frac{im_0\omega_0^2 e^{2\gamma t}\,\sin(\omega_d t)}{\gamma\,\sin(\omega_d t) + \omega_d\,\cos(\omega_d t)}, \\ g_2(t) &= -\log\left(\frac{e^{-\gamma t}}{\omega_d}(\gamma\,\sin(\omega_d t) + \omega_d\,\cos(\omega_d t)\right)^2, \\ g_3(t) &= \frac{i}{m_0}\left(\frac{\sin(\omega_d t)}{\gamma\,\sin(\omega_d t) + \omega_d\,\cos(\omega_d t)} - t\right), \end{aligned} \tag{75}$$

except that the last function includes a shift by $-\frac{i}{m_0}t$. The Fresnel transform to the free particle has the same local character as the evolution operator; see Sec. VII.

C. Arnold transform to the harmonic oscillator

In the case of the Arnold transform to the harmonic oscillator, the solution to the generalized Ermakov equation (52) is given by

$$b(t) = \frac{e^{-\gamma t}}{\omega_d}\sqrt{(\gamma^2 + \Omega^2)\sin^2(\omega_d t) + \gamma\omega_d\,\sin(2\omega_d t) + \omega_d^2\cos^2(\omega_d t)}. \tag{76}$$

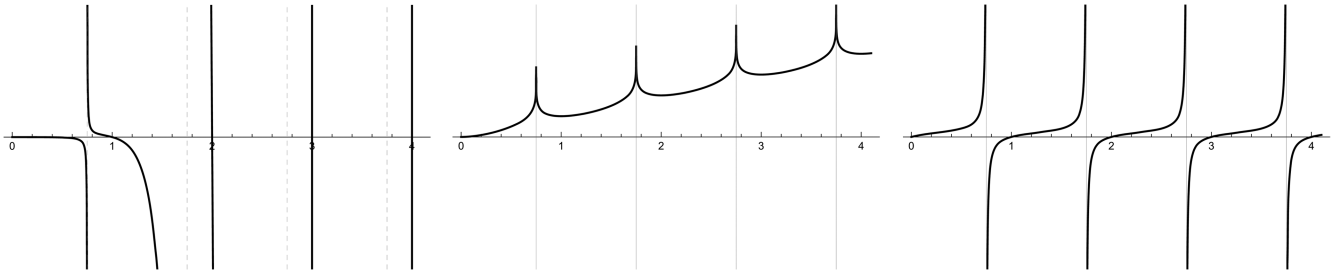


FIG. 2. Graphs of $-ig_1$, g_2, and $-ig_3$, respectively, for $\gamma = \omega_d = \pi$ and $m_0 = 1$. The dashed lines indicate the asymptotes (zeros of u_2) at the singular values.

With this solution, the expression of $g_i(t)$ are easily computed using (51),

$$\begin{aligned} g_1(t) &= -\frac{ie^{2\gamma t}\,\sin(\omega_d t)\left(\gamma\left(\omega_0^2+\Omega^2\right)\sin(\omega_d t)+\omega_d\left(\omega_0^2-\Omega^2\right)\cos(\omega_d t)\right)}{\left(\gamma^2+\Omega^2\right)\sin^2(\omega_d t)+\gamma\omega_d\,\sin(2\omega_d t)+\omega_d^2\cos^2(\omega_d t)}, \\ g_2(t) &= -\log\left(\frac{e^{-2\gamma t}\left(\left(\gamma^2+\Omega^2\right)\sin^2(\omega_d t)+\gamma\omega_d\,\sin(2\omega_d t)+\omega_d^2\cos^2(\omega_d t)\right)}{\omega_d^2}\right), \\ g_3(t) &= 0, \end{aligned} \tag{77}$$

where the last function is zero by construction (point transformation). In this case, the information is encoded in the diffeomorphism in time appearing in Eq. (45).

It should be stressed that the solution $b(t)$ to the generalized Ermakov equation never vanishes; therefore, the functions g_i have no singularities (except at infinity). Thus, the Arnold transform to the harmonic oscillator has a global character; see Sec. VII. See also Ref. 30 for a detailed discussion of this case.

D. Arnold transform to the free particle

In the case of the Arnold transform to the free particle, the expression of the $g_i(t)$ is

$$\begin{aligned} g_1(t) &= -\frac{im_0\omega_0^2 e^{2\gamma t}\,\sin(\omega_d t)}{\gamma\,\sin(\omega_d t)+\omega_d\,\cos(\omega_d t)}, \\ g_2(t) &= -\log\left(\frac{e^{-\gamma t}}{\omega_d}\left(\gamma\,\sin(\omega_d t)+\omega_d\,\cos(\omega_d t)\right)^2\right), \\ g_3(t) &= 0, \end{aligned} \tag{78}$$

where, again, the last function is zero by construction. As in the case of the evolution operator or the Fresnel transform, the g_i diverges for the same finite values t_k. In fact the Arnold transform to the free particle, the GFT, and the evolution operator coincide when $\Psi_0(x)=\psi(x,0)$; see Sec. VII. See also Refs. 5 and 31 for a discussion of this case.

IX. CONCLUSIONS

In this paper, we have applied the Wei–Norman method, with $SU(1,1)$ algebra, to the GCK oscillator, obtaining a factorization for the evolution operator in terms of functions $g_i(t)$ appearing in the exponential factors that can be expressed in terms of the solutions of the Euler–Lagrange equations associated with the classical version of the GCK Hamiltonian or a mirror version of it (with inverted mass), depending on the chosen ordered basis of the Lie algebra. In this way, we provide a semiclassical interpretation to the WN method, building the evolution operator in terms of classical solutions.

We also provide factorizations, by means of a modified WN method, for other integral transforms such as the Fresnel transform and the Arnold transforms and their generalizations, which relate different GCK systems in a unitary way. The Arnold transform is characterized by the condition of being a point transformation (i.e., a local transform), implying that its factorization involves only two factors. The pay-off for this simplicity is the need for an extra reparameterization in time, implying that the Arnold transform maps solutions of one system into the other but with different, although related, evolution times.

One of the problems with factorization techniques such as the WN method is the impossibility of obtaining the factorization for certain time values t_s, where all the functions g_i diverge. These correspond to zeros of $u_2(t)$ and are usually denoted in the literature focal points or caustics. We have been able to obtain the expression of the evolution operators at the caustics using a fine-tuning analysis of the limiting process when t approaches t_s, obtaining that the evolution operator involves the Fourier transform of the initial state, followed by a rescaling and a local phase. In addition to this, there is a parity transformation and a phase jump when crossing the caustic.

The example of the CK oscillator (with constant frequency and damping coefficient), as the simplest example of time-dependent Hamiltonian with $SU(1,1)$ symmetry, is thoroughly discussed.

The techniques developed in this paper can also be applied to other groups, such as $SU(2)$, with applications in the case of a spin in the presence of a time-dependent magnetic field[32] or in photonic systems such as waveguide arrays with a z-dependent refraction index and couplings.[33]

It would also be interesting to generalize our construction to other cases where the WN method is not directly applicable but where we can still use the interaction picture,[34] apply a mean field approximation,[35] or a perturbative treatment.

A further interesting study would be the construction of Wigner functions using the WN method, thus making a connection between the semiclassical description provided by the Wigner functions and the one provided in this paper.

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APPENDIX A: GROUPOID PROPERTY OF THE EVOLUTION OPERATOR IN THE WEI–NORMAN FORMULATION

In this appendix, we shall delve into the groupoid property satisfied by the evolution operator given in Eq. (5).

The main difference of the groupoid property (5) with respect to the group property (6) [apart from the obvious dependence in two parameters (t, t_0) in the case of groupoid] is that the composition of two evolution operators $U(t_1, t_0)$ and $U(t_2, t_1')$, $U(t_2, t_1')U(t_1, t_0)$, requires $t_1' = t_1$, i.e., not all elements of the groupoid can be composed.

In addition, the main difference of the groupoid with respect to the semigroup is that in the semigroup, the existence of the inverse is not guaranteed for all elements, not even the existence of the identity element.

The structure of a groupoid is more similar to a group than to a semigroup. Elements of a groupoid can be unitary, for instance (as in the case of the Schrödinger equation), but elements of a semigroup cannot be unitary, as it happens in the case of the evolution under the heat equation or in open systems.

Let us study the implications of the groupoid property in the Wei–Norman context, i.e., what are the conditions required on the functions g_i in the WN factorization in order for the evolution operator to satisfy the groupoid property (5). To simplify the notation, we shall consider the case of the GCK oscillator studied in Sec. III, where the Lie algebra is $SU(1,1)$. We shall also restrict to the ordering B_1 of Sec. III A for concreteness. Consider the factorized evolution operator given in Eq. (21), which, restoring the dependence on the initial time t_0, can be written as

$$U(t, t_0) = e^{g_1(t,t_0)V} e^{g_2(t,t_0)S} e^{g_3(t,t_0)K}, \qquad U(t_0, t_0) = I. \tag{A1}$$

In this case, the groupoid property (5) for the Wei–Norman factorized evolution operator is written as

$$\begin{aligned} U(t_2, t_0) &= e^{g_1(t_2,t_0)V} e^{g_2(t_2,t_0)S} e^{g_3(t_2,t_0)K} = U(t_2, t_1)U(t_1, t_0) \\ &= e^{g_1(t_2,t_1)V} e^{g_2(t_2,t_1)S} e^{g_3(t_2,t_1)K} e^{g_1(t_1,t_0)V} e^{g_2(t_1,t_0)S} e^{g_3(t_1,t_0)K}. \end{aligned} \tag{A2}$$

It can be shown that this equation implies the following restriction for the functions g_i:

$$\begin{aligned} g_1(t_2, t_0) &= g_1(t_2, t_1) + \frac{e^{g_2(t_2,t_1)} g_1(t_1, t_0)}{1 - g_1(t_1, t_0) g_3(t_2, t_1)}, \\ g_2(t_2, t_0) &= g_2(t_2, t_1) + g_2(t_1, t_0) - \log\left[1 - g_1(t_1, t_0) g_3(t_2, t_1)\right]^2, \\ g_3(t_2, t_0) &= g_3(t_1, t_0) + \frac{e^{g_2(t_1,t_0)} g_3(t_2, t_1)}{1 - g_1(t_1, t_0) g_3(t_2, t_1)}. \end{aligned} \tag{A3}$$

It should be stressed that these equations are compatible with the unitarity conditions mentioned in Sec. III A, namely, that g_1 and g_3 are pure imaginary, while g_2 is real.

In addition, the condition for the inverse in (5) implies

$$\begin{aligned} g_1(t_0, t_1) &= -\frac{e^{-g_2(t_1,t_0)} g_1(t_1, t_0)}{1 - e^{-g_2(t_1,t_0)} g_1(t_1, t_0) g_3(t_1, t_0)}, \\ g_2(t_0, t_1) &= -g_2(t_1, t_0) - \log\left[1 - e^{-g_2(t_1,t_0)} g_1(t_1, t_0) g_3(t_1, t_0)\right]^2, \\ g_3(t_0, t_1) &= -\frac{e^{-g_2(t_1,t_0)} g_3(t_1, t_0)}{1 - e^{-g_2(t_1,t_0)} g_1(t_1, t_0) g_3(t_1, t_0)}. \end{aligned} \tag{A4}$$

Equations (A3) are of functional type; therefore, the question arises if these functional equations are equivalent to the WN equations (22) (under the assumption of differentiability or at least continuity of the g_i). The answer is affirmative, but some minor modifications should be made to Eq. (22) to make explicit the dependence on t_0,

$$
\begin{aligned}
-ig_1'(t,t_0) &= \frac{1}{m(t)}g_1(t,t_0)^2 - m(t)\omega^2(t), \quad & g_1(t_0,t_0) &= 0,\\
-ig_2'(t,t_0) &= \frac{2}{m(t)}g_1(t,t_0), & g_2(t_0,t_0) &= 0,\\
-ig_3'(t,t_0) &= \frac{1}{m(t)}e^{g_2(t,t_0)}, & g_3(t_0,t_0) &= 0,
\end{aligned}
\tag{A5}
$$

where the prime indicates the derivative with respect to the first argument. We restore the use of $m(t)$ instead of $W(t)$ in the equations since now $m_0 = m(0)$ does not make sense in this setting. Now, we are going a step forward, and we calculate the *initial velocities* [using the initial conditions in Eq. (A5)],

$$
\begin{aligned}
g_1'(t_0,t_0) &= -im(t_0)\omega^2(t_0),\\
g_2'(t_0,t_0) &= 0,\\
g_3'(t_0,t_0) &= \frac{i}{m(t_0)}.
\end{aligned}
\tag{A6}
$$

These conditions are clearly valid for any value of t_0; therefore, we can rewrite Eqs. (A5) as

$$
\begin{aligned}
g_1'(t,t_0) &= g_3'(t,t)g_1(t,t_0)^2 + g_1'(t,t), \quad & g_1(t_0,t_0) &= 0,\\
g_2'(t,t_0) &= 2g_3'(t,t)g_1(t,t_0), & g_2(t_0,t_0) &= 0,\\
g_3'(t,t_0) &= g_3'(t,t)e^{g_2(t,t_0)}, & g_3(t_0,t_0) &= 0.
\end{aligned}
\tag{A7}
$$

If now we compute the derivatives $g_i'(t,t_0)$ as $g_i'(t,t_0) = \lim_{h\to 0}\frac{g_i(t+h,t_0)-g_i(t,t_0)}{h}$ and express $g_i(t+h,t_0)$ in terms of $g_j(t+h,t)$ and $g_j(t,t_0)$ using Eq. (A3), we arrive to Eqs. (A7) [using also that $g_2'(t,t) = 0$ from Eq. (A6)].

For completeness, we provide the expression of the WN functions $g_i(t,t_0)$ in terms of the solutions of the classical equations of motion, generalizing Eqs. (23), (25), and (26). First of all, we should obtain the solutions of Eqs. (24) and (27) but by satisfying the same initial conditions at arbitrary t_0 instead of $t_0 = 0$. Denoting by $u_1(t,t_0)$ and $u_2(t,t_0)$ those solutions, they are related to $u_1(t)$ and $u_2(t)$ by a canonical transformation,

$$
\begin{pmatrix} u_1(t,t_0)\\ u_2(t,t_0)\end{pmatrix} = \frac{1}{W(t_0)}\begin{pmatrix} u_2(t_0) & -u_1(t_0)\\ -u_2'(t_0) & u_1'(t_0)\end{pmatrix}\begin{pmatrix} u_1(t)\\ u_2(t)\end{pmatrix}.
\tag{A8}
$$

Now, it can be checked that the following functions

$$
\begin{aligned}
g_1(t,t_0) &= im(t)\frac{u_2'(t,t_0)}{u_2(t,t_0)},\\
g_2(t,t_0) &= -\log u_2(t,t_0)^2,\\
g_3(t,t_0) &= \frac{i}{m(t_0)}\frac{u_1(t,t_0)}{u_2(t,t_0)},
\end{aligned}
\tag{A9}
$$

satisfy Eq. (A5).

APPENDIX B: LINEAR INTEGRAL TRANSFORMS AND ITS RELATION TO LINEAR CANONICAL TRANSFORMATIONS IN PHASE SPACE

Linear integral transforms[13,14] are operators characterized by an integral kernel,

$$
\hat{C}\Psi(x) = \int_{-\infty}^{\infty} C(x,x')\Psi(x')dx',
\tag{B1}
$$

where $C(x,x')$ is the integral *kernel* of the transform $\hat{C}$.

A well-known example of this kind of transform is Fourier Transform (FT), where the integral kernel is $C_{\text{Fourier}}(x,x') = \frac{1}{\sqrt{2\pi}}e^{-ix'x}$, with its generalizations to the Fractional Fourier Transform (FrFT), with kernel

$$C_{\mathrm{FrFT}}(x,x') = \frac{1}{\sqrt{2\pi i \sin\theta}} e^{i\frac{(x^2+x'^2)\cos\theta - 2xx'}{2\sin\theta}}. \tag{B2}$$

In the previous equation, choosing $\theta = \omega t$, we recover the propagator describing time evolution in a harmonic oscillator,

$$C_{\mathrm{osc}}(x,x',t) = \sqrt{\frac{m\omega}{2\pi i\hbar \sin(\omega t)}} e^{i\frac{m\omega}{\hbar}\frac{(x^2+x'^2)\cos(\omega t) - 2xx'}{2\sin(\omega t)}}, \tag{B3}$$

where the dimensional constants (m, ω, and $\hbar$) have been restored.

Another important integral transform, with applications in optics, is the Fresnel Transform, with kernel

$$C_{\mathrm{Fresnel}}(x,x',z) = \frac{1}{i\lambda z} e^{i\frac{\pi}{\lambda z}(x-x')^2}, \tag{B4}$$

describing propagation in free space along the z direction of waves of wavelength λ.

A natural generalization of these transforms consists in considering the most general quadratic polynomial in x and x' in the exponent of the integral kernel, thus leading to what is known as linear canonical transform[13] or Generalized Fresnel Transform (GFT),[14] with kernel

$$C_M(x,x') = \frac{1}{\sqrt{2\pi i b}} e^{i\frac{ax'^2 - 2xx' + dx^2}{2b}}, \tag{B5}$$

which is the exponential of a quadratic form related to the matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{B6}$$

satisfying $\det(M) = ad - bc = 1$. Thus, $M \in SL(2,\mathbb{R}) \approx Sp(1,\mathbb{R})$. Note that c does not appear in the expression of the kernel, given that $c = \frac{ad-1}{b}$ from the condition on the determinant.

Equation (B5) is not well defined in the case $b = 0$, i.e., for lower triangular matrices. In this case, $d = \frac{1}{a} \neq 0$, and the integral kernel can be written as[13]

$$C_{M(b=0)}(x,x') = \frac{1}{\sqrt{a}} e^{i\frac{c}{2a}x'^2} \delta(x - x'/a). \tag{B7}$$

The resulting integral transformation in this case is

$$\hat{C}_{M(b=0)}\Psi(x) = \frac{1}{\sqrt{a}} e^{i\frac{c}{2a}x^2} \Psi(x/a) \tag{B8}$$

and is called *geometric* or *point* transformation. A point transformation in phase space is a canonical transformation $(\vec{q},\vec{p}) \to (\vec{Q},\vec{P})$ induced by a coordinate transformation, i.e., if $\vec{Q} = \vec{Q}(\vec{q},t)$, then $\vec{P} = J(\vec{q},t)^{-1}\vec{p}$, where $J(\vec{q},t)$ is the Jacobian of the coordinate transformation. A point transformation does not involve an integral (and it is therefore local), corresponding to a dilation by $1/a$ followed by a multiplication by a phase quadratic in x. The factor $\frac{1}{\sqrt{a}}$ multiplying the function ensures the unitarity of the dilation.

Note that the GFT transform is a unitary transformation with the usual $L^2(\mathbb{R})$ scalar product. The action on operators X and P by this unitary transformation is

$$\begin{aligned} \hat{C}_M X \hat{C}_M^\dagger &= dX - bP \equiv X', \\ \hat{C}_M P \hat{C}_M^\dagger &= -cX + aP \equiv P'. \end{aligned} \tag{B9}$$

Thus, $\begin{pmatrix} X' \\ P' \end{pmatrix} = M^{-1}\begin{pmatrix} X \\ P \end{pmatrix}$, and therefore, $\hat{C}_M$ constitutes a unitary twofold representation of the group $SL(2,\mathbb{R}) \approx Sp(1,\mathbb{R})$, known as the metaplectic representation.

1. One-parameter transform subgroups

Since the GFT transform constitutes a unitary representation of the $SL(2,\mathbb{R})$ group of linear canonical transformations in phase space, we can use well-known factorization formulas in terms of uniparametric subgroups. Let us define the subgroups

$$M^{\mathrm{f}} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \qquad M^{\mathrm{g}} = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \qquad M^{\mathrm{d}} = \begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix}. \tag{B10}$$

Explicitly,

$$\hat{C}_{M^f}\Psi(x) = \frac{1}{\sqrt{2\pi i b}}\int_{-\infty}^{\infty} dx' e^{\frac{i}{2b}(x-x')^2}\Psi(x'),$$
$$\hat{C}_{M^g}\Psi(x) = e^{\frac{ic}{2}x^2}\Psi(x), \tag{B11}$$
$$\hat{C}_{M^d}\Psi(x) = \frac{1}{\sqrt{a}}\Psi(x/a).$$

These transformations are well known in the optics literature. The transformation generated by M^f is known as the Fresnel propagator in free space (in quantum mechanics, it corresponds to the evolution operator of a free Galilean particle), the transformation generated by M^g is known as the quadrature phase operator (or Gauss–Weierstrass operator), and the transformation generated by M^d is known as the dilation operator (in quantum mechanics, it corresponds to a squeezing operator).

2. Infinitesimal generators of integral transforms

It is useful to find the differential operators that generate the integral transform for each one of the uniparametric subgroups, in the sense that

$$\hat{C}_{M(t)}\Psi(x) = e^{tN}\Psi(x). \tag{B12}$$

The differential operator N can be computed by differentiation with respect to the parameter t at $t = 0$ (assuming that $t = 0$ corresponds to the identity transform). For each one of the uniparametric subgroups discussed in Appendix B 1, the corresponding infinitesimal generators are[13]

$$N^f = \frac{i}{2}\frac{\partial^2}{\partial x^2} = iK,$$
$$N^g = \frac{i}{2}x^2 = iV, \tag{B13}$$
$$N^d = -\left(x\frac{\partial}{\partial x} + \frac{1}{2}\right) = -2S,$$

where we have taken $b = t, c = t$, and $a = e^t$ in Eq. (B10). Thus, up to constants, they coincide with the operators K, V, and S defined in Eqs. (15)–(17), showing that operators such as the evolution operator, the general Fresnel transform, or the Arnold transform are, in fact, linear integral transforms that can be written as the product of the uniparametric subgroups of integral transforms discussed in Appendix B 1.

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