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Powers of the Stochastic Gompertz and Lognormal Diffusion Processes, Statistical Inference and Simulation

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Abstract: In this paper, we study a new family of Gompertz processes, defined by the power of the homogeneous Gompertz diffusion process, which we term the powers of the stochastic Gompertz diffusion process. First, we show that this homogenous Gompertz diffusion process is stable, by power transformation, and determine the probabilistic characteristics of the process, i.e., its analytic expression, the transition probability density function and the trend functions. We then study the statistical inference in this process. The parameters present in the model are studied by using the maximum likelihood estimation method, based on discrete sampling, thus obtaining the expression of the likelihood estimators and their ergodic properties. We then obtain the power process of the stochastic lognormal diffusion as the limit of the Gompertz process being studied and go on to obtain all the probabilistic characteristics and the statistical inference. Finally, the proposed model is applied to simulated data.

Keywords: powers of stochastic Gompertz diffusion models; powers of stochastic lognormal diffusion models; estimation in diffusion process; stationary distribution and ergodicity; trend function; application to simulated data

1. Introduction

Stochastic processes are used to model stochastic phenomena in various fields of science, engineering, economics and finance. An important category among these processes is that of Stochastic Diffusion Processes (SDP), which have received considerable attention recently, due on the one hand to their diverse applications in stochastic modelling, and on the other, to their value in addressing probabilistic statistical problems, especially those involving statistical inference. In consequence, these processes have been widely studied, and much research has been undertaken to resolve these issues of statistical inference, with particular respect to the estimation of parameters; see, among others, Bibby and Sorensen [1], Prakasa Rao [2], Chang and Cheng [3], Beskos et al. [4], Stramer and Yan [5], Shoji and Ozaki [6], Durham and Gallant [7] and Fan [8], without forgetting the works of Yenkie and Diwekar [9] and Kloeden et al. [10] and the important bibliography cited in these works.

There has been much recent interest in applying SDP, and many researchers are working on the construction of stochastic processes in order to model phenomena of interest. These processes are used in areas such as the stochastic economy, new technologies, interest rates, courses of action, insurance, finance in general, cell growth, radiotherapy, chemotherapy, emissions from energy consumption and the emissions of CO_2 and greenhouse gases. Research results have been applied to various processes, both in the homogeneous and in the non-homogeneous cases and many particular SDP have been proposed, such as Katsamaki and Skiadas [11] in the case of the exponential model, Skiadas and Giovanis [12] in the case of the Bass model, Giovanis and Skiadas [13] in the case of the logistic model, Gutiérrez et al. [14] in the case of the Rayleigh model and Román-Román et al. [15] in the case of the lognormal with exogenous factors.

Among the above-mentioned processes is the Stochastic Gompertz Diffusion Process (SGDP), which was first proposed by Ricciardi [16], who defined it in the homogeneous case by means of stochastic differential equations, for use in studies of population growth. It was subsequently used by Dennis and Patil [17] in ecology modelling. With respect to the Kolmogorov equations, it was defined by Nafidi [18], in a general way and for both the univariate and the multivariate cases.

In various papers, Gutiérrez et al. [19–21], Ferrante et al. [22], Román-Román et al. [23] and Giorno and Nobile [24], have highlighted the importance of this process, and many subsequent extensions have appeared, especially regarding the non-homogeneous case with exogenous factors (external variables) that affect the drift coefficient. In general, these extensions take one of the following two forms:

With external information (when no functional form is available): the exogenous factors are completely determined by the observed data (monthly, annual, etc.) and to obtain their functional forms interpolation methods, among others, can be used. This methodology has been applied by Gutiérrez et al. [25,26], Rupsys et al. [27] and Badurally Adam et al. [28]. In all these papers it is assumed that the coefficient drift is a linear combination of exogenous factors, obtained by linear interpolation.

Without external information: in this case there are no observed data for the exogenous factors, but they are functions of time and of certain parameters. For example, the case in which the deceleration factor is affected by exogenous factors was developed by Gutiérrez et al. [29]. Ferrante et al. [30] studied the Gompertz process in which exogenous factors are obtained as the sum of two exponential functions and Albano and Giorno [31] did so considering logarithmic exogenous factor.

The lognormal SDP and the SGDP, in turn, have been extended to the multivariate case with delay, by Frank [32], and to the bivariate case without delay by Gutiérrez et al. [33], and an application has been devised to model the emissions of CO_2 in Spain [34]. Other recent papers that have addressed questions related to SGDP include Hu [35] and Zou et al. [36].

In the present study, we define and examine a new extension of the Gompertz and lognormal diffusion processes, based on the homogeneous version of these processes, i.e., their power. Thus, we obtain two families of homogeneous diffusion processes. Firstly, we show that Gompertzian and lognormal diffusions are stable by power transformation. Them we define the proposed model as the solution to a stochastic differential equation. From this, we obtain: the explicit expression of the process, the Probability Transition Density Function (PTDF), the moments of different orders and, in particular, the conditioned and unconditioned trends of the process; the ergodicity of the process and its stationary distribution and the process parameters, estimated by maximum likelihood,with discrete sampling, determining the asymptotic properties of the likelihood estimators and the approximated confidence interval of the parameters.

In addition, we obtain the probabilistic and statistical characteristics of the lognormal process power, as a particular case of the process being studied, when the deceleration factor tends toward zero. Finally, the process and the methodology presented are applied to simulated data obtained from the explicit expression of the solution to the characteristic state equation for the process.

2. The Model and Its Basic Probabilistic Characteristics

2.1. An Overview of the Homogeneous Gompertz Stochastic Diffusion Process

Let $\{X(t); t \in [t_0, T]; t_0 \geq 0\}$ be a stochastic process taking values on $(0, \infty)$, $X(t)$ is a Gompertz diffusion process with parameters α, β and σ and which is denoted by $\text{Gomp}(\alpha; \beta; \sigma)$ if $X(t)$ satisfies Ito's Stochastic Differential Equation (SDE) as follows (see [16,18,20,37]):

$$dX(t) = [\alpha X(t) - \beta X(t) \log X(t)]\, dt + \sigma X(t) dw_t \quad ; \quad P(X(t_0) = X_{t_0}) = 1 \tag{1}$$

In the literature, the constant α ($\in \mathbb{R}$) is the intrinsic growth rate; the β ($\in \mathbb{R}$) constant is the deceleration factor, the $\sigma > 0$ constant is the diffusion coefficient, $X_{t_0} > 0$ is a fixed real number and w_t denotes the one-dimensional standard Wiener process.

The analytical expression of the unique solution to Equation (1) is given by (see, for example, [21,37])

$$X(t) = \exp\left\{ e^{-\beta(t-t_0)} \log X_{t_0} + \frac{\alpha - \sigma^2/2}{\beta} \left(1 - e^{-\beta(t-t_0)}\right) + \sigma \int_{t_0}^{t} e^{-\beta(t-\tau)} dw(\tau) \right\} \tag{2}$$

From this, we deduce that the process $X(t)$ is distributed as the following one-dimensional lognormal distribution:

$$\Lambda_1 \left(e^{-\beta(t-t_0)} \log X_{t_0} + \frac{(\alpha - \sigma^2/2)}{\beta} \left(1 - e^{-\beta(t-t_0)}\right);\quad \frac{\sigma^2}{2\beta} \left(1 - e^{-2\beta(t-t_0)}\right) \right)$$

It has been shown (see [21]), that for $\beta > 0$, $X(t)$ is ergodic and that the stationary distribution has a lognormal distribution. Hence, we have:

$$X(\infty) \sim \Lambda_1 \left(\frac{\alpha - \sigma^2/2}{\beta} \quad ; \quad \frac{\sigma^2}{2\beta} \right) \tag{3}$$

2.2. The Proposed Model

Let $\{X(t); t \in [t_0, T]; t_0 \geq 0\}$ be a $\text{Gomp}(\alpha; \beta; \sigma)$. Then, the γ-power of the Stochastic Gompertz Diffusion Process (γ-PSGDP) $X(t)$ is defined by

$$x_\gamma(t) = X^\gamma(t); \qquad \gamma \in \mathbb{R}^* \tag{4}$$

The process $\{x_\gamma(t); t \in [t_0, T]; t_0 \geq 0\}$ is also a diffusion process with values in $(0, \infty)$ and has the drift and diffusion coefficients are shown below.

By applying Ito's formula to the transform given in Equation (4), we have

$$\begin{aligned} dx_\gamma(t) &= \gamma X^{\gamma-1}(t) \left[\alpha X(t) - \beta X(t) \log X(t)\right] dt + \gamma\sigma X^\gamma(t) dW_t + \gamma(\gamma-1)\frac{\sigma^2}{2} X^\gamma(t) dt \\ &= \left[\alpha\gamma X^\gamma(t) - \beta\gamma X^\gamma(t) \log X(t)\right] dt + \gamma\sigma X^\gamma(t) dW_t \end{aligned}$$

Then, after some algebraic rearrangement, we obtain

$$dx_\gamma(t) = \left[a x_\gamma(t) - \beta x_\gamma(t) \log x_\gamma(t)\right] dt + c x_\gamma(t) dw(t)$$

This shows that the process $x_\gamma(t)$ is also a $\text{Gomp}(a; \beta; c)$ process, where:

$a = \gamma\alpha + \gamma(\gamma-1)\frac{\sigma^2}{2}$ and $c = \gamma\sigma$ and the drift and diffusion coefficients are given respectively by:

$$\begin{aligned} A_1(x) &= \left(\gamma\alpha + \frac{\gamma(\gamma-1))\sigma^2}{2}\right)x - \beta x\log(x) \\ A_2(x) &= \gamma^2\sigma^2x^2 \end{aligned}$$

The model proposed in this paper belongs to the family of processes γ-PSGDP $\{x_\gamma(t); t\in[t_0,T]; t_0\geq 0\}$ defined by the following SDE:

$$dx_\gamma(t) = A_1(x_\gamma(t))dt + \sqrt{A_2(x_\gamma(t))}dw(t) \quad ; \quad \mathrm{P}(x_\gamma(t_0) = x_{t_0}) = 1$$

2.3. Probabilistic Characteristics of the γ-PSGDP

Under the initial condition given, the unique solution of the SDE Equation (5) can be obtained using the relations expressed by Equations (2) and (4), from which we have

$$x_\gamma(t) = \exp\left\{e^{-\beta(t-t_0)}\log x_{t_0} + \frac{\gamma(\alpha-\sigma^2/2)}{\beta}\left(1-e^{-\beta(t-t_0)}\right) + \gamma\sigma\int_{t_0}^{t} e^{-\beta(t-\tau)}dw(\tau)\right\} \tag{5}$$

We then deduce that $x_\gamma(t)$ is distributed as a one dimensional lognormal distribution $\Lambda_1(\mu(s,t,x_{t_0}), \gamma^2\sigma^2\lambda^2(t_0,t))$, where $\mu(s,t,x_{t_0})$ and $\lambda^2(t_0,t)$ are given by

$$\begin{aligned} \mu(s,t,x_{t_0}) &= e^{-\beta(t-t_0)}\log x_{t_0} + \frac{\gamma(\alpha-\sigma^2/2)}{\beta}\left(1-e^{-\beta(t-t_0)}\right) \\ \lambda^2(t_0,t) &= \frac{1}{2\beta}\left(1-e^{-2\beta(t-t_0)}\right) \end{aligned}$$

From the homogeneity of the process, we know that $x_\gamma(t) \mid x_\gamma(s) = x_s$ has the lognormal distribution $\Lambda_1(\mu(s,t,x_s), \sigma^2\lambda^2(s,t))$, and then the PTDF of the process is

$$f(y,t\mid x,s) = \frac{1}{y}\left[2\pi\gamma^2\sigma^2\lambda^2(s,t)\right]^{-1/2}\exp\left(-\frac{[\log(y)-\mu(s,t,x)]^2}{2\gamma^2\sigma^2\lambda^2(s,t)}\right)$$

The rth conditional moment of the process is given by

$$\mathrm{E}\left(x_\gamma^r(t)\mid x_\gamma(s)=x_s\right) = \exp\left\{r\mu(s,t,x_s) + \frac{r^2\gamma^2\sigma^2}{2}\lambda^2(s,t)\right\}$$

from which the Conditional Trend Function (CTF) gives

$$\begin{aligned} \mathrm{E}\left(x_\gamma(t)\mid x_\gamma(s)=x_s\right) &= \exp\left\{e^{-\beta(t-s)}\log x_s + \frac{\gamma(\alpha-\sigma^2/2)}{\beta}\left(1-e^{-\beta(t-s)}\right)\right. \\ &\quad \left. + \frac{\gamma^2\sigma^2}{4\beta}\left(1-e^{-2\beta(t-s)}\right)\right\} \end{aligned} \tag{6}$$

Assuming the initial condition $\mathrm{P}(x_\gamma(t_0) = x_{t_0}) = 1$, the Trend Function (TF) of the process is

$$\begin{aligned} \mathrm{E}\left(x_\gamma(t)\right) &= \exp\left\{e^{-\beta(t-t_0)}\log(x_{t_0}) + \frac{\gamma(\alpha-\sigma^2/2)}{\beta}\left(1-e^{-\beta(t-t_0)}\right)\right. \\ &\quad \left. + \frac{\gamma^2\sigma^2}{4\beta}\left(1-e^{-2\beta(t-t_0)}\right)\right\} \end{aligned} \tag{7}$$

From Equation (3), we deduce that for $\beta > 0$, the stationary distribution of the process is also a lognormal distribution and thus we have:

$$x_\gamma(\infty) \sim \Lambda_1\left(\frac{\gamma(\alpha-\sigma^2/2)}{\beta}; \frac{\gamma^2\sigma^2}{2\beta}\right) \tag{8}$$

Therefore, the asymptotic trend function of the process (for $\beta > 0$) is given by

$$\mathbf{E}[x_\gamma(\infty)] = \exp\left(\frac{\gamma\left(\alpha-\sigma^2/2\right)}{\beta} + \frac{\gamma^2\sigma^2}{4\beta}\right)$$

The limit of the trend function in Equation (7) (when t tends to ∞) coincides with this asymptotic trend function.

3. Statistical Inference on the Model

3.1. Likelihood Parameter Estimation

In the present study, with discrete sampling, we estimate the parameters α, σ^2 and β of the model by applying Maximum Likelihood (ML) methodology, following the same scheme as in Gutiérrez et al. [21]. To do so, we consider a discrete sampling of the process $x_\gamma(t_1) = x_1, x_\gamma(t_2) = x_2, \ldots, x_\gamma(t_n) = x_n$ for times $t_1, t_2, \ldots, t_n$ and assume, moreover, that the length of the time intervals $[t_{i-1}, t_i]$ $(i = 2, ..., n)$ is equal to constant h i.e., $t_i - t_{i-1} = h$ and an initial distribution $\mathrm{P}\left[x_\gamma(t_1) = x_1\right] = 1$. Then the associated likelihood function can be obtained by the following expression:

$$\mathbb{L}(x_1, \ldots, x_n, \alpha, \beta, \sigma^2) = \prod_{j=2}^{n} f\left(x_j, t_j \mid x_{j-1}, t_{j-1}\right)$$

The variable change can be used to work with a known probability function and to calculate the maximum probability estimators in a simpler way, considering the following transformation: $v_1 = x_1, v_{i,\beta} = \lambda_\beta^{-1}(\log(x_i) - e^{-\beta h}\log(x_{i-1}))$, for $i = 2, \ldots, n$ and denoting $\mathbf{V}_\beta = (v_{2,\beta}, \ldots, v_{n,\beta})'$. Thus, in terms of $\mathbf{V}_\beta$, the likelihood function is expressed as follows:

$$\mathbb{L}_{\mathbf{V}_\beta}(\mathbf{a}_\gamma, \beta, c_\gamma^2) = \left[2\pi c_\gamma^2\lambda_\beta^2\right]^{-(n-1)/2} \exp\left(-\frac{1}{2c_\gamma^2}(\mathbf{V}_\beta - \nu_\beta\mathbf{a}_\gamma\mathbf{U})'(\mathbf{V}_\beta - \nu_\beta\mathbf{a}_\gamma\mathbf{U})\right)$$

where $\mathbf{a}_\gamma = \gamma\left(\alpha - \frac{\sigma^2}{2}\right)$, $c_\gamma = \gamma\sigma$, $\nu_\beta = \lambda_\beta^{-1}(1-e^{-\beta h})/\beta$, $\lambda_\beta^2 = \frac{1}{2\beta}(1-e^{-2h\beta})$ and $\mathbf{U} = (1, \ldots, 1)'$ is a vector of the order $(n-1)$.

By differentiating the log-likelihood function with respect to $\mathbf{a}_\gamma$ and c_γ^2, we obtain the following equations:

$$\begin{aligned} \mathbf{U}'\mathbf{V}_\beta &= \hat{\mathbf{a}}_\gamma\nu_\beta\mathbf{U}'\mathbf{U} \\ (n-1)\hat{c}_\gamma^2 &= (\mathbf{V}_\beta - \hat{\mathbf{a}}_\gamma\nu_\beta\mathbf{U})'(\mathbf{V}_\beta - \hat{\mathbf{a}}_\gamma\nu_\beta\mathbf{U}) \end{aligned}$$

The third likelihood equation is obtained by differentiating the log-likelihood function with respect to β and by using the effect that $\mathbf{V}_\beta = \lambda_\beta^{-1}(\mathrm{J}_x - e^{-\beta h}\mathrm{I}_x)$ with $\mathrm{J}_x = (\log(x_2), \ldots, \log(x_n))'$ and $\mathrm{I}_x = (\log(x_1), \ldots, \log(x_{n-1}))'$. After various operations, we have

$$\mathrm{I}_x'\left(\mathbf{V}_\beta - \hat{\mathbf{a}}_\gamma\nu_\beta\mathbf{U}\right) = 0$$

Taking into account that $\mathbf{U}'\mathbf{U} = n-1$ and after algebraic rearrangement (not shown), the ML estimators of $\mathbf{a}_\gamma$ and c_γ^2 are

$$(n-1)\hat{\mathbf{a}}_\gamma = \nu_\beta^{-1}\mathbf{U}'\mathbf{V}_{\mathbf{fi}} \tag{9}$$

$$(n-1)\hat{c}_\gamma^2 = \mathbf{V}'_\beta \mathrm{H}_{\mathbf{U}}\mathbf{V}_\beta \tag{10}$$

The ML estimator of β is given by

$$\hat{\beta} = \frac{1}{h}\log\left(\frac{\mathrm{I}'_x\mathrm{H}_{\mathbf{U}}\mathrm{I}_x}{\mathrm{I}'_x\mathrm{H}_{\mathbf{U}}\mathrm{J}_x}\right) \tag{11}$$

where $\mathrm{H}_{\mathbf{U}} = \mathrm{I}_{n-1} - \frac{1}{n-1}\mathbf{U}\mathbf{U}'$is idempotent and a symmetric matrix and I_{n-1} denotes the identity matrix.

3.2. Asymptotic Properties of the Parameter Drift Estimators

Let X be a random variable with a distribution function given by Equation (8); then $\log(X)$ is distributed as a normal distribution $N_1\left(\frac{\gamma(\alpha-\sigma^2/2)}{\beta}; \frac{\gamma^2\sigma^2}{2\beta}\right)$. If $\beta > 0$, the process under consideration has ergodic properties, and for $\theta^* = (a_\gamma, \beta) \in (a_{\gamma,1}, a_{\gamma,2}) \times (\beta_1, \beta_2)$, with $\beta_1 > 0$, we have

$$\mathcal{L}_\theta\left(\sqrt{T}(\hat{\theta}-\theta)\right) \to \mathcal{N}_2\left(0, \mathbb{I}^{-1}(\theta)\right) \quad ; \quad \text{when} \quad T \to \infty \tag{12}$$

$\mathbb{I}(\theta)$ is the information matrix and is given by $\mathbb{I}(\theta) = \mathbb{E}_\theta\left(\frac{\dot{A}_1(X)\dot{A}_1^*(X)}{A_2(X)}\right)$

where $\dot{A}_1(x)$ is the following vector: $\dot{A}_1(x) = \left(\frac{\partial A_1(x)}{\partial\alpha}; \frac{\partial A_1(x)}{\partial\beta}\right)^*$

Then, we have

$$\mathbb{I}(\theta) = \frac{1}{\gamma^2\sigma^2}\mathbb{E}_\theta\begin{pmatrix} \gamma^2 & -\gamma\log(X) \\ -\gamma\log(X) & \log^2(X) \end{pmatrix} = \frac{1}{\sigma^2}\begin{pmatrix} 1 & -\frac{\alpha-\sigma^2/2}{\beta} \\ -\frac{\alpha-\sigma^2/2}{\beta} & \frac{\sigma^2}{2\beta} + \frac{(\alpha-\sigma^2/2)^2}{\beta^2} \end{pmatrix}$$

and the inverse is

$$\mathbb{I}^{-1}(\theta) = \begin{pmatrix} \sigma^2 + \frac{2}{\beta}(\alpha - \frac{\sigma^2}{2})^2 & 2\alpha - \sigma^2 \\ 2\alpha - \sigma^2 & 2\beta \end{pmatrix} \tag{13}$$

An approximated, asymptotic confidence region of θ and an approximated, asymptotic marginal confidence interval of α and β can be obtained from Equations (12) and (13). The above-mentioned region is given, for a large T, by

$$\mathrm{P}\left[T\left(\theta-\hat{\theta}\right)^* \hat{\mathbb{I}}(\theta)\left(\theta-\hat{\theta}\right) \le \chi^2_{2,\xi}\right] = 1-\xi$$

obtaining $\widehat{\mathbb{I}}(\theta)$ by replacing the parameters by their estimators and where $\chi^2_{2,\xi}$ represents the upper 100ξ per cent points of the chi squared distribution with two degrees of freedom.

The $\xi\%$ confidence (marginal) intervals for parameters α and β are given, for a large T, by

$$\mathrm{P}\left(\alpha \in \left[\hat{\alpha} \pm \frac{1}{\gamma}\lambda_{\xi}\left(\frac{\hat{\beta}\hat{\sigma}^2 + 2(\hat{\alpha} - \hat{\sigma}^2/2)^2}{\hat{\beta}T}\right)^{1/2}\right]\right) = 1 - \xi \tag{14}$$

$$\mathrm{P}\left(\beta \in \left[\hat{\beta} \pm \lambda_{\xi}(2\hat{\beta}/T)^{1/2}\right]\right) = 1 - \xi \tag{15}$$

where λ_{ξ} represents the 100ξ per cent points of the normal standard distribution.

Note that in Equations (14) and (15) we have assumed that σ is known with a value $\sigma = \hat{\sigma}$.

4. Powers of the Lognormal Diffusion Process

The Stochastic Lognormal Diffusion Process (SLDP) is known to be a particular case of the Gompertz diffusion process when the deceleration factor $\beta = 0$ (see, for example [21]). Then, the power of the SLDP can be obtained from that of the SGDP by tending β to zero.

Then, if the SLDP $Y(t)$ is given by the following SDE:

$$dY(t) = \alpha Y(t)dt + \sigma Y(t)dw_t$$

The resulting γ-PSLDP ($y_{\gamma}(t) = Y^{\gamma}(t)$) is governed by the following SDE:

$$dy_{\gamma}(t) = \left(\gamma\alpha + \frac{\gamma(\gamma-1))\sigma^2}{2}\right)y_{\gamma}dt + \gamma\sigma y_{\gamma}dw(t) \tag{16}$$

The same approach can be used to derive all the probabilistic properties and statistics for the γ-PSLDP process, taking $\beta = 0$ on the perspective equations established for the properties of γ-PSGDP in the previous sections, except as regards the symptotic properties of the drift parameter estimators (we already know that there is no asymptotic distribution in the case of the SLDP). For the latter case, we can obtain the exact distributions of the estimators, together with the confidence intervals for the process parameters (see [21]).

4.1. Estimated Trend Functions

In the same way as in Gutiérrez et al. [21], by Zehna's theorem [38], the Estimated Conditional Trend (ECT) and the Estimated Trend (ET) functions can be obtained from Equations (6) and (7) by replacing the parameters by their estimators. Furthermore, we can obtain an approximated and asymptotic confidence interval of the ETF and ECTF by means of the approximated and asymptotic confidence interval of the parameters given by Equations (14) and (15).

5. Simulation and Application

The trajectory of the model can be obtained by simulating the exact solution of SDE Equation (4) obtained in Equation (5). From this explicit solution, the simulated trajectories of the process are obtained from the following discretising time interval $[t_0, T]$: $t_i = t_0 + ih$, for $i = 1, \ldots, N$ (N is an integer and h is the discretization step), taking into account that the random variable in the latter expression $\sigma(w_t) - w(t_1)$ is distributed as a one-dimensional normal distribution $\mathcal{N}(0, \sigma^2(t - t_1))$ ([39]).

Table 1 shows the simulated data and the ETF for different powers, considering $h = 1$, $N = 30$, and the initial value $x_1 = 0.99$. We estimate the parameters by maximum likelihood, reserving the values observed for the time $t = 30$ for comparison with the corresponding prediction by the model. The results are shown in Table 2.

Table 1. Simulated data and estimated trend function.

Time	$x_1(t)$	ETF-x_1	$x_{1.5}(t)$	ETF-$x_{1.5}$	$x_2(t)$	ETF-x_2
1	0.99	0.99	0.99	0.99	0.99	0.99
2	2.1831	2.1832	3.2364	3.2369	4.7957	4.7960
3	3.5272	3.5271	6.6380	6.6385	12.4861	12.4876
4	4.7180	4.7181	10.2628	10.2613	22.3149	22.3122
5	5.6288	5.6286	13.3648	13.3620	31.7343	31.7261
6	6.2651	6.2645	15.6878	15.6818	39.2796	39.2767
7	6.6848	6.6846	17.2845	17.2802	44.7154	44.7063
8	6.9539	6.9531	18.3316	18.3276	48.3607	48.3586
9	7.1220	7.1211	18.9998	18.9933	50.7075	50.7176
10	7.2251	7.2250	19.4136	19.4087	52.1922	52.2041
11	7.2894	7.2887	19.6703	19.6649	53.1189	53.1268
12	7.3285	7.3277	19.8262	19.8219	53.7088	53.6943
13	7.3520	7.3514	19.9247	19.9177	54.0539	54.0414
14	7.3663	7.3658	19.9776	19.9761	54.2598	54.2531
15	7.3742	7.3746	20.0117	20.0115	54.3836	54.3818
16	7.3792	7.3799	20.0323	20.0330	54.4489	54.4601
17	7.3820	7.3831	20.0497	20.0461	54.4903	54.5076
18	7.3841	7.3851	20.0598	20.0540	54.5492	54.5364
19	7.3849	7.3863	20.0641	20.0588	54.5629	54.5539
20	7.3862	7.3870	20.0648	20.0617	54.5623	54.5645
21	7.3875	7.3874	20.0633	20.0635	54.5783	54.5710
22	7.3877	7.3877	20.0654	20.0645	54.5922	54.5749
23	7.3885	7.3879	20.0662	20.0652	54.5997	54.5773
24	7.3882	7.3880	20.0587	20.0656	54.6148	54.5787
25	7.3881	7.3880	20.0626	20.0658	54.6020	54.5796
26	7.3883	7.3881	20.0638	20.0660	54.5914	54.5801
27	7.3890	7.3881	20.0599	20.0661	54.6196	54.5804
28	7.3878	7.3881	20.0549	20.0661	54.6297	54.5806
29	7.3873	7.3881	20.0507	20.0661	54.6110	54.5807
Prediction						
30	7.3872	7.3881	20.0473	20.0662	54.6221	54.5808

Table 2. Starting values used in the simulation and estimation of the parameters.

	σ	α	β
Starting Values	0.0001	1	0.5
γ	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$
1	0.0000852	0.999952	0.500008
1.5	0.0001498	1.00043	0.500377
2	0.0001606	1.00003	0.500052

Figure 1 shows the fit and the prediction obtained for $x_\gamma(t)$ using the ETF ($\gamma = 1$ $\gamma = 1.5$ and $\gamma = 2$) (see Table 1).

Figure 2 shows 10 simulated trajectories for $x_\gamma(t)$ ($\gamma = 1$ $\gamma = 1.5$ and $\gamma = 2$), taking as the values for α, β and σ those obtained by maximum likelihood estimation (see Table 2). For each trajectory, 2901 data are generated by considering $h = 0.01$, and initial value $x_1 = 0.99$.

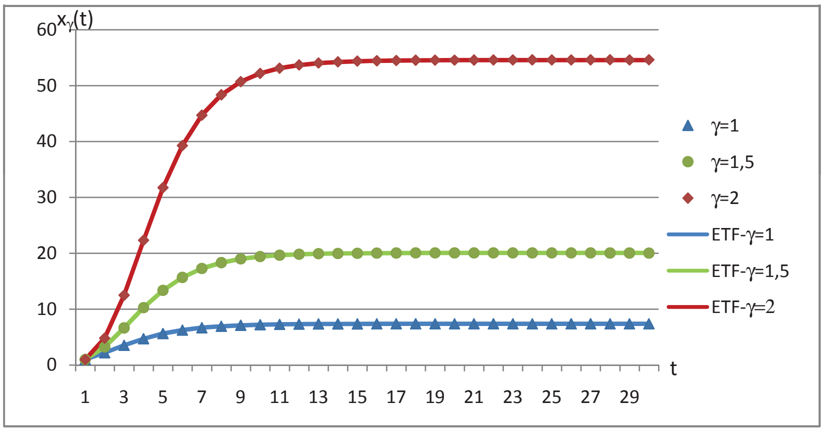

Figure 1. Fit and prediction based on ETF.

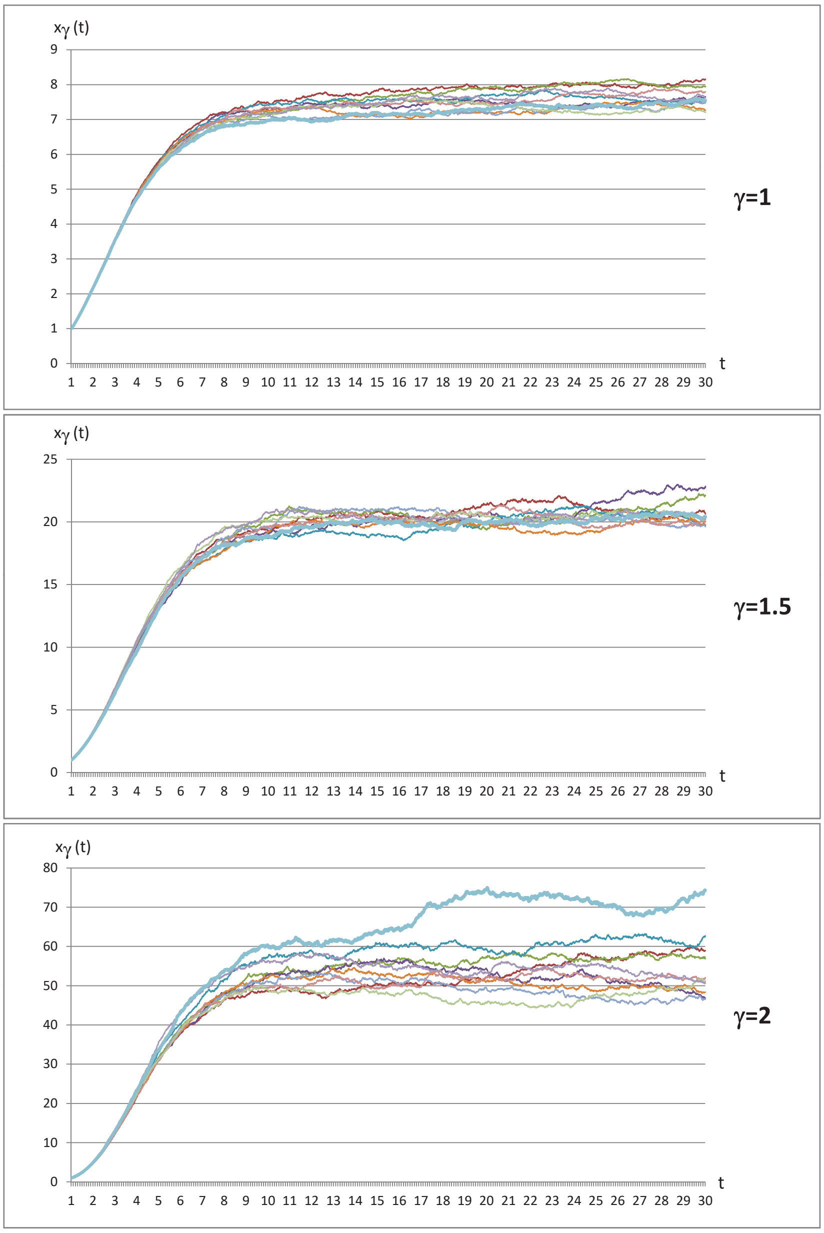

Figure 2. Fit and prediction based on ETF.

Figure 3 shows a trajectory whose values are the average of those obtained in the simulation of 100 trajectories, with the ETF. The values used in the simulation and the results obtained by estimating the parameters are shown in Table 3.

Table 3. Starting values used in the simulation and estimation of the parameters.

	σ	α	β
Starting values	0.0001	1	0.5
γ	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$
1.5	0.0000106801	1.00006	0.50003

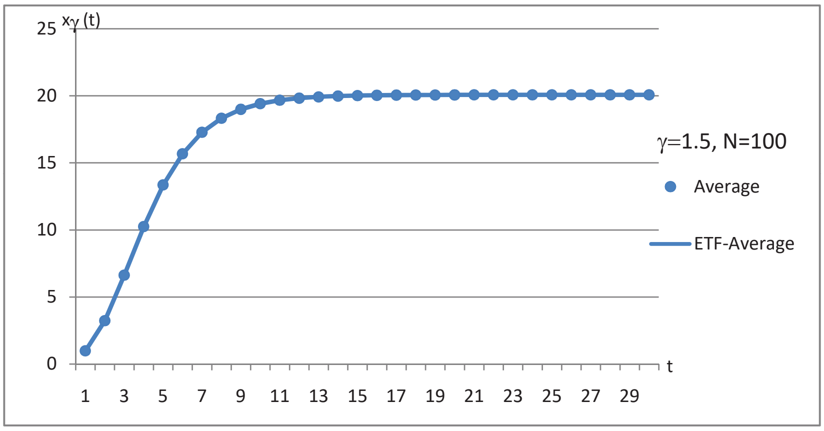

Figure 3. Fit and prediction based on ETF.

The variation of the mean and standard error of the estimators is studied, taking into account how N and h change. The results are shown in Table 4.

20 process paths are simulated with N observations each. The parameters are estimated using the equations (ref Eq11), (ref Eq12) and (ref Eq13), obtaining a vector of 20 components corresponding to the different estimators. For these, the sample mean is calculated and the Standard Error (SE).

The next step is to study the evolution of the mean and the standard error of the estimators with respect to the variation in the number N and in h. The results of this study are shown in Table 4.

The true parameter values considered in this simulation are $\alpha = 1$, $\beta = 0.5$, $\sigma = 0.0001$ and the start point is $x_1 = 0.99$, and $t_1 = 0$ and $\gamma = 1.5$.

The calculations have been made using the Mathematica program, in which a program has been implemented.

Table 4. Mean and standard error of the estimators.

h	N	Mean ($\hat{\sigma}$)	SE ($\hat{\sigma}$)	Mean ($\hat{\alpha}$)	SE ($\hat{\alpha}$)	Mean ($\hat{\beta}$)	SE ($\hat{\beta}$)
0.05	100	0.025108	0.114736	1.000132	0.000503	0.500144	0.000439
0.05	500	0.000112	0.000005	0.999637	0.000839	0.499770	0.000809
0.05	1000	0.000116	0.000005	1.000181	0.000953	0.500090	0.000915
0.1	100	0.000106	0.000008	1.000027	0.000350	0.500007	0.000262
0.1	500	0.000123	0.000010	1.000020	0.000654	0.500044	0.000647
0.1	1000	0.000141	0.000016	0.999081	0.000736	0.499156	0.000672
0.5	100	0.000143	0.000030	1.000046	0.000253	0.500002	0.000274
0.5	500	0.000329	0.000069	0.999171	0.000616	0.499202	0.000570
0.5	1000	0.000491	0.000141	0.998581	0.000730	0.498779	0.000584
1	100	0.000230	0.000074	0.999610	0.000381	0.499638	0.000359
1	500	0.000584	0.000217	0.999034	0.000597	0.499092	0.000541
1	1000	0.000908	0.000318	0.997592	0.001211	0.498045	0.000923

6. Conclusions

This article presents a study of the Gamma Power Stochastic Gompertz Diffusion Process (γ-PSGDP), including all its probabilistic properties and the corresponding statistical inference. As a particular case in the limit comparison test, we also study the Gamma Power Stochastic Lognormal Diffusion Process (γ-PSLDP).

A simulation study was conducted, analysing different process trajectories.

In the future, it will be possible to apply these models to fit real data and to obtain goodness of fit results between the processes and the data. We will also study the possibility of defining all these processes in their non-homogeneous form, by introducing exogenous factors, and considering the use of numerical methods to obtain the estimates.

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Abbreviations

The following abbreviations are used in this manuscript:

SDP	Stochastic Diffusion Processes
SGDP	Stochastic Gompertz Diffusion Process
PTDF	Probability Transition Density Function
SDE	Stochastic Differential Equation
γ-PSGDP	γ-Power of the Stochastic Gompertz Diffusion Process
γ-PSLDP	γ- Power of the Stochastic Lognormal Diffusion Process
CTF	Conditional Trend Function
TF	Trend Function (TF)
ML	Maximum Likelihood
SLDP	Stochastic Lognormal Diffusion Process
ECT	Estimated Conditional Trend
ET	Estimated Trend
SE	Standard Error

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