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Double hierarchy linguistic preference information: consistency, consensus and large-scale group decision making

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Xunjie Gou

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MEMORIA PRESENTADA POR

Xunjie Gou

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DIRECTORES

Francisco Herrera Triguero y Zeshui Xu

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Table of Contents

Chapter I

PhD dissertation

1 Introduction

Group decision making (GDM) refers to inviting a group of experts to evaluate, prioritize or select the optimal one among some available alternatives in actual decision making process. During GDM, linguistic information is more in line with the real thoughts of experts and Zadeh proposed a fuzzy linguistic approach to deal with it [Zad75a, Zad75b, Zad75c]. As well as he [Zad12] proposed a concept of Computing with words (CW) and explained it by "*Computing with words is a system of computation in which the objects of computation are words, phrases and propositions drawn from a natural language. The carriers of information are propositions. It is important to note that Computing with words is the only system of computation which offers a capability to compute with information described in a natural language.*" Motivated by CW, in recent years, lots of linguistic models were developed to represent complex linguistic information such as hesitant fuzzy linguistic term set (HFLTS) [RMH12], 2-tuple linguistic model [HM00], virtual linguistic term model [Xu05, XW17], and type-2 fuzzy sets [Men02], etc.

Additionally, for these linguistic models mentioned above, most of them can be used to express some simple linguistic information by one hierarchy linguistic label. However, because of people's cognition process and the decision making information are more and more complex, sometimes these linguistic models cannot describe some complex linguistic terms or linguistic term sets (LTSs) comprehensively and accurately. For example, some experts may tend to use complex and detailed uncertain linguistic information to represent their comprehensive opinions such that "*entirely low*", "*just right medium*", and "*a little high*". As we know, the 2-tuple linguistic model [HM00] can be used to express linguistic information by both linguistic terms and numerical values, but the numerical values may distort the meaning of original linguistic information. Then, it is necessary to consider an important issue: Does it make sense if we split each complex linguistic information into two parts with the form of "*adverb+adjective*" and express them by different kinds of linguistic terms? In fact, Zadeh has explained this idea when he dealing with a CW problem [Zad12]: "*In effect, this is the solution to the problem which I posed to you. As you can see, reduction of the original problem to the solution of a variational problem is not so simple. However, solution of the variational problem to which the original problem is reduced, is well within the capabilities of desktop computers.*" According to this idea, a concept of linguistic terms with weakened hedges was proposed [WXZ18], which regards the "*adverbs*" as a few weakened hedges expressed by other linguistic labels. However, two gaps are obvious: 1) All weakened hedges are included in a set, which will be inconvenient if different linguistic terms need different sets of weakened hedges. 2)

One weakened hedge may have different meanings when embellishing different linguistic terms.

Based on these analyses above, two requirements need to be satisfied to represent complex linguistic information with 2-tuple linguistic structure: One is that all linguistic variables should be expressed by linguistic labels without any numerical scales; The other one is that every original linguistic term in the first hierarchy LTS should have its own second hierarchy LTS that contains all modifiers. By these two motivations, this thesis proposes a concept of double hierarchy linguistic term set (DHLTS) by adding a second hierarchy LTS to each linguistic term in the first hierarchy LTS, which can be used to handle complex linguistic terms well by dividing them into two simple linguistic hierarchies where the first hierarchy LTS is the main linguistic hierarchy and the second hierarchy LTS is the linguistic feature or detailed supplementary of each linguistic term in the first hierarchy LTS. In addition, the extension of DHLTS in hesitant fuzzy environment named double hierarchy hesitant fuzzy linguistic term set (DHHFLTS) is developed to express uncertain complex linguistic information.

In this thesis, we focus on the discussions about three main aspects.

Firstly, we mainly analyze the basic concepts of DHLTS and DHHFLTS, propose some equivalent transformation functions, and then develop some operations and properties of DHHFLTSs. In addition, considering that the distance and similarity measures are fundamentally important in amounts of research fields, we define the axioms of distance and similarity measures between two DHHFLTSs, and then introduce a series of distance and similarity measures between two DHHFLTSs.

Secondly, considering that more and more experts prefer to give their preferences by making pairwise comparisons between any two alternatives, meanwhile this kind of preference reflects the relationships between different alternatives intuitively. Therefore, preference relation becomes one of the popular and effective tools. Based on the DHHFLTS and preference form, we give a concept of double hierarchy hesitant fuzzy linguistic preference relation (DHHFLPR). Then, to avoid the occurrence of some self-contradictory situations, it is very important to carry out the consistency checking and improving process for each DHHFLPR in GDM process. Therefore, we discuss some additive consistency measures for DHHFLPRs. For the purpose of judging whether a DHHFLPR is of acceptable consistency or not, we define a consistency index of DHHFLPR and develop a novel method to improve the existing methods for calculating the consistency thresholds. Then we present two convergent consistency repairing algorithms based on automatic improving method and feedback improving method respectively to improve the consistency index of a given DHHFLPR with unacceptable consistency.

Finally, with the progress of science and technology and the development of network environment, the communications between people are increasingly convenient. Large-scale group decision making (LSGDM) has become the focuses of decision-making problems. Generally, a GDM problem can be called LSGDM problem when the number of experts is more than 20 [LC06]. This thesis mainly studies LSGDM from two aspects. 1) We discuss the clustering method and the consensus reaching process in LSGDM with double hierarchy hesitant fuzzy linguistic preference information. We also propose the similarity degree-based clustering method, the double hierarchy information entropy-based weights-determining method and the consensus measures. 2) In LSGDM, sometimes some experts do not modify their preferences or even do it on the contrary way to the remaining experts, and some different opinions or minority preferences are often cited as obstacles to decision making [PMH14, XDC15]. Therefore, this thesis gives a concept of double hierarchy linguistic preference relation (DHLPR) and develops a consensus model to manage minority opinions and non-cooperative behaviors in LSGDM with DHLPRs. Additionally, to establish the consensus model, some basic tools such as the distance-based cluster method, the weight-determining method, and the comprehensive adjustment coefficient-determining method are developed.

In addition to the discussions of the core knowledge of DHLTS, DHHFLTS, DHLPR and DHHFLPR, this thesis discusses some different decision making models under different decision making contexts. We mainly discuss three different decision making contexts, i.e., multiple criteria decision making (MCDM), GDM, and LSCDM. These three decision making contexts can be expressed as follows:

(1) GDM refers to inviting a group of experts to evaluate, prioritize or select the optimal one among some available alternatives in the actual decision making process.

(2) The MCDM involves a set of feasible alternatives that are evaluated based on multiple, conflicting and non-commensurate criteria by a group of individuals.

(3) LSGDM consists of two main parts: One part is clustering. In LSGDM, the large-scale decision-making groups can be classified into several small groups for assisting and improving the efficiency of decision-making. The other one is the consensus reaching process, which aims at reaching all experts' agreements before making a decision by discussing and improving experts' preferences, guided and supervised by a moderator [PEMH14].

All in all, this thesis consists of two main parts: the first one illustrates the existing problems, the basic concepts and models, and the results obtained from the proposed models. The second part is a compilation of the main publications that are associated with this thesis.

The rest of this thesis are organized as follows: Section 2 provides some related preliminaries used throughout this contribution. In Section 3, the basic ideas and the challenges that justify the development of this thesis are discussed. Section 4 introduces the objectives of this thesis. Section 5 presents the methodologies used in this thesis. a summary of the proposals included in this thesis is presented in Section 6. Section 7 presents a discussion of the results obtained in this thesis. Section 8 discusses the conclusion of this thesis. Finally, some future works are discussed in Section 9.

Introducción

En la toma de decisiones grupales (GDM) se invita a un grupo de expertos a evaluar, priorizar o seleccionar la opción óptima entre algunas alternativas disponibles. Durante GDM, la información lingüística está más en línea con los pensamientos reales de los expertos y Zadeh [Zad75a, Zad75b, Zad75c] propuso el enfoque lingüístico difuso para enfrentarlo. Asimismo [Zad12] propuso el concepto de computación con palabras (CW) y lo explicó mediante "La computación con palabras es un sistema de computación en el cual los objetos de computación son palabras, frases y proposiciones extraídas de un lenguaje natural. Los portadores de la información son proposiciones. Es importante señalar que la computación con palabras es el único sistema de computación que ofrece la capacidad de computar con la información descrita en un lenguaje natural [Zad12]." Motivados por CW, en los últimos años, se desarrollaron muchos modelos lingüísticos para representar información lingüística compleja como el conjunto de términos lingüísticos difusos dudosos (HFLTS) [RMH12], modelo lingüístico de 2 tuplas [HM00], modelo de término lingüístico virtual [Xu05, XW17], y conjuntos borrosos de tipo 2 [Men02], etc.

En referencia a loa modelos lingüísticos previamente mencionados, la mayoría de ellos se pueden usar para expresar cierta información lingüística simple mediante una jerarquía de etiqueta lingüísticas. Sin embargo, debido a que el proceso cognitivo de las personas y la información sobre la toma de decisiones son cada vez más complejas, a veces, estos modelos lingüísticos no pueden describir algunos términos lingüísticos complejos o conjuntos de términos lingüísticos (LTS) de manera exhaustiva y precisa. Por ejemplo, algunos expertos pueden tender a utilizar información lingüística indecisa, compleja y detallada para representar sus opiniones como podría ser "enteramente bajo", "precisamente josto medio" y "un poco alto". Como sabemos, el modelo lingüístico de 2-tuplas [HM00] se puede utilizar para expresar la información lingüística mediante los términos lingüísticos y valores numéricos, pero dichos valores numéricos pueden distorsionar el significado de la información lingüística original. Por tanto, es necesario considerar la siguiente cuestión: ¿Tiene sentido si dividimos cada término de información lingüística compleja en dos partes con la forma de "adverbio+adjetivo" y los expresamos mediante diferentes tipos de términos lingüísticos? De hecho, Zadeh explicó esta idea al tratar con un problema de CW [Zad12]: "En efecto, esta es la solución al problema que le planteé. Como puede ver, la reducción del problema original a la solución de un problema variacional no es tan simple. Sin embargo, la solución del problema variacional al que se reduce el problema original está dentro de las capacidades de las computadoras de escritorio." En base a esta idea, se propuso un concepto de términos lingüísticos con coberturas debilitadas [WXZ18], que considera a los "adverbios" como unas cuantas coberturas debilitadas expresadas por otras etiquetas lingüísticas. Sin embargo, hay dos brechas obvias: 1) Todas las coberturas debilitadas se incluyen en un conjunto, lo que será inconveniente si los diferentes términos lingüísticos necesitan conjuntos diferentes de coberturas debilitadas. 2) Una cobertura debilitada puede tener diferentes significados al embellecer diferentes términos lingüísticos.

Según estos análisis anteriores, se deben cumplir dos requisitos para representar información lingüística compleja con una estructura lingüística de 2-tuplas: En primer lugar, todas las variables lingüísticas deben expresarse mediante etiquetas lingüísticas sin escalas numéricas. El segundo requisito es que cada término lingüístico original en la primera jerarquía LTS debe tener su propia segunda jerarquía LTS que contenga todos los modificadores. Por estas dos motivaciones, esta tesis propone un concepto de conjunto de términos lingüísticos de doble jerarquía (DHLTS) agregando una segunda jerarquía LTS a cada término lingüístico en la primera jerarquía LTS, que se puede usar para manejar bien los términos lingüísticos complejos al dividirlos en dos jerarquías lingüísticas simples donde la primera jerarquía LTS es la jerarquía lingüística principal y la segunda jerarquía

LTS es la característica lingüística o el complemento detallado de cada término lingüístico en la primera jerarquía LTS. Además, se ha desarrollado la extensión de DHLTS en un entorno difuso dudoso llamado doble jerarquía borrosa conjunto de términos lingüísticos difusos (DHHFLTS) para expresar información lingüística compleja incierta.

En esta tesis, nos centramos en las discusiones sobre tres aspectos principales.

En primer lugar, analizamos los conceptos básicos de DHLTS y DHHFLTS, proponemos algunas funciones de transformación equivalentes y luego desarrollamos algunas operaciones y propiedades de DHHFLTS. Adicionalmente, considerando que las medidas de distancia y similitud son fundamentalmente importantes en los campos de investigación, definimos los axiomas de las medidas de distancia y similitud entre dos DHHFLTS y luego introducimos una serie de medidas de distancia y similitud entre dos DHHFLTS.

En segundo lugar, tenenos en cuenta que cada vez más expertos prefieren dar sus preferencias haciendo comparaciones por pares entre dos alternativas. Este tipo de preferencia refleja las relaciones entre diferentes alternativas de manera intuitiva. Por lo tanto, la relación de preferencia se convierte en una de las herramientas populares y efectivas. Basándonos en el DHHFLTS y la forma de preferencia, proporcionamos un concepto de doble jerarquía dudosa relación de preferencia lingüística difusa (DHHFLPR). Para evitar la aparición de algunas situaciones autocontradictorias, es muy importante llevar a cabo el proceso de comprobación y mejora de la coherencia para cada DHHFLPR en proceso de GDM. Por lo tanto, discutimos algunas medidas de consistencia aditiva para DHHFLPRs. Con el fin de determinar si una DHHFLPR tiene una consistencia aceptable o no, definimos un índice de consistencia de DHHFLPR y desarrollamos un método novedoso para mejorar los métodos existentes para calcular los umbrales de consistencia. Posteriormente, presentamos dos algoritmos de reparación de consistencia convergentes basados en el método de mejora automática y el método de mejora de retroalimentación, respectivamente, para mejorar el índice de consistencia de un DHHFLPR determinado con una consistencia inaceptable.

Finalmente, con el progreso de la ciencia y la tecnología y el desarrollo del entorno de red, las comunicaciones entre las personas son cada vez más convenientes. La toma de decisiones en grupo a gran escala (LSGDM) se ha convertido en el foco de los problemas de toma de decisiones. En general, un problema de GDM se puede llamar problema LSGDM cuando el número de expertos es más de 20 [LC06]. Esta tesis estudia principalmente la LSGDM desde dos aspectos. 1) Discutimos el método de agrupación y el proceso de consenso en LSGDM con información jerárquica difusa de preferencia de doble jerarquía. También proponemos el método de agrupamiento basado en grados de similitud, el método de determinación de ponderaciones basado en la entropía de información de doble jerarquía y las medidas de consenso. 2) En LSGDM, a veces algunos expertos no modifican sus preferencias o incluso lo hacen de manera contraria a los expertos restantes, y algunas opiniones diferentes o preferencias minoritarias a menudo se citan como obstáculos para la toma de decisiones [PMH14, XDC15]. Por lo tanto, esta tesis da un concepto de relación de preferencia lingüística de doble jerarquía (DHLPR) y desarrolla un modelo de consenso para gestionar opiniones de minorías y comportamientos no cooperativos en LSGDM con DHLPR. Además, para establecer el modelo de consenso, se desarrollan algunas herramientas básicas como el método de clúster basado en la distancia, el método de determinación del peso y el método de determinación del coeficiente de ajuste integral.

Además de las discusiones sobre los conocimientos básicos de DHLTS, DHHFLTS, DHLPR y DHHFLPR, esta tesis analiza algunos modelos diferentes de toma de decisiones en diferentes contextos de toma de decisiones. Principalmente analizamos tres contextos diferentes de toma de decisiones, especificamente, la toma de decisiones de criterios múltiples (MCDM), la GDM y la LSCDM. Estos tres contextos de toma de decisiones se pueden expresar de la siguiente manera:

(1) MGDM invita a un grupo de expertos a evaluar, priorizar o seleccionar la opción óptima entre algunas alternativas disponibles en el proceso de toma de decisiones.

(2) MCDM involucra a un conjunto de alternativas factibles que se evalúan en función de criterios múltiples, conflictivos y no conmensurables por parte de un grupo de individuos.

(3) LSGDM consta de dos partes principales: Una parte es la agrupación en clústeres. En LSGDM, los grupos de toma de decisiones a gran escala se pueden clasificar en varios grupos pequeños para ayudar y mejorar la eficiencia de la toma de decisiones. El otro es el proceso de consenso, que apunta a alcanzar todos los acuerdos de los expertos antes de tomar una decisión al discutir y mejorar las preferencias de los expertos, guiados y supervisados por un moderador [PEMH14].

En resumen, esta tesis consta de dos partes principales: la primera ilustra los problemas abordados, los conceptos y modelos básicos y los resultados obtenidos de los modelos propuestos. La segunda parte presenta una compilación de las principales publicaciones asociadas a esta tesis.

El resto de esta tesis se organiza de la siguiente manera: la Sección 2 proporciona algunos preliminares relacionados utilizados a lo largo de esta contribución. En la Sección 3, se discuten las ideas básicas y los desafíos que justifican el desarrollo de esta tesis. La sección 4 introduce los objetivos de esta tesis. La sección 5 presenta las metodologías utilizadas en esta tesis. En la Sección 6 se presenta un resumen de las propuestas incluidas en esta tesis. La Sección 7 presenta una discusión de los resultados obtenidos en esta tesis. La sección 8 discute la conclusión de esta tesis. Finalmente, algunos trabajos futuros se discuten en la Sección 9.

2 Preliminaries

In this section, we propose the basic concepts of some linguistic representation models and several main descriptions of GDM with linguistic preference information.

2.1 Some linguistic representation models

As the fundamental of this thesis, some linguistic representation models such as the HFLTS, the 2-tuple linguistic representation model and the linguistic terms with weakened hedges (LTWHs) are introduced in this subsection.

2.1.1 Hesitant fuzzy linguistic term set

In 2012, Rodríguez et al. [RMH12] defined the concept of HFLTS as an ordered finite subset of the consecutive linguistic terms of a given LTS. Soon afterwards, Liao et al. [LXZM15] extended and formalized it mathematically as follows:

Definition 1. *(Hesitant fuzzy linguistic term set [LXZM15]). Let $x_i \in X(i = 1, 2, ..., N)$ be fixed and $S = \{s_t | t = -\tau, ..., -1, 0, 1, ..., \tau\}$ be a LTS. A HFLTS on X, H_S, is in mathematical form of $H_S = \{< x_i, h_S(x_i) > | x_i \in X\}$, where $h_S(x_i)$ is a set of some values in S and can be expressed as:*

$$h_S(x_i) = \{s_{\phi_l}(x_i) | s_{\phi_l}(x_i) \in S; l = 1, 2, ..., L; \phi_l \in \{-\tau, ..., -1, 0, 1, ..., \tau\}\} \tag{I.1}$$

with L being the number of linguistic terms in $h_S(x_i)$ and $s_{\phi_l}(x_i)(l = 1, 2, ..., L)$ in each $h_S(x_i)$ being the continuous terms in S. $h_S(x_i)$ denotes the possible degree of the linguistic variable x_i to S. For convenience, $h_S(x_i)$ is called a hesitant fuzzy linguistic element (HFLE) and Φ being the set of all HFLEs.

Besides, to make the operations of HFLTSs more reasonable, Gou and Xu [GX16] developed two equivalent transformation functions between linguistic variable and the corresponding numerical scale.

Definition 2. *(Equivalent transformation functions [GX16]). Let $S = \{s_t | t = -\tau, ..., -1, 0, 1, ..., \tau\}$ be a LTS, $h_S = \{s_{\phi_l} | s_{\phi_l} \in S; l = 1, 2, ..., L; \phi_l \in [-\tau, \tau]\}$ be a HFLE with L being the number of linguistic terms in h_S, and $h_\sigma = \{\sigma_l | \sigma_l \in [0, 1]; l = 1, 2, ..., L\}$ be a hesitant fuzzy element (HFE) [Tor10]. Then the membership degree σ_l and the subscript ϕ_l of the linguistic term s_{ϕ_l} that expresses the equivalent information to the membership degree σ_l can be transformed to each other by the following functions g and g^{-1}, respectively:*

$$g : [-\tau, \tau] \to [0, 1], g(\phi_l) = \frac{\phi_l + \tau}{2\tau} = \sigma_l \tag{I.2}$$

$$g : [0, 1] \to [-\tau, \tau], g^{-1}(\sigma_l) = (2\sigma_l - 1)\tau = \phi_l \tag{I.3}$$

Based on Definition 2, we can introduce the transformation functions between the HFLE h_S and the corresponding HFE h_σ.

Definition 3. *(Equivalent transformation functions [GX16]). The transformation functions between the HFE* $h_\sigma = \{\sigma_l|\sigma_l \in [0,1]; l = 1,2,...,L\}$ *and the HFLE* $h_S = \{s_{\phi_l}|s_{\phi_l} \in S; l = 1,2,...,L; \phi_l \in [-\tau,\tau]\}$ *are given, respectively, as follows:*

$$G : \Phi \to \Theta, G(h_S) = G(\{s_{\phi_l}|s_{\phi_l} \in S; l = 1,2,...,L; \phi_l \in [-\tau,\tau]\}) = \{\sigma_l|\sigma_l = g(\phi_l)\} = h_\sigma \quad \text{(I.4)}$$

$$G^{-1} : \Theta \to \Phi, G^{-1}(h_\sigma) = G^{-1}(\{\sigma_l|\sigma_l \in [0,1]; l = 1,2,...,L\}) = \{s_{\phi_l}|\phi_l = g^{-1}(\phi_l)\} = h_S \quad \text{(I.5)}$$

2.1.2 2-tuple linguistic representation model

Herrera and Martínez [HM00] defined the concept of 2-tuple linguistic representation model, which can be used to represent the linguistic information by a 2-tuple $(s_t, \alpha) \in \overline{S} = S \times [-0.5, 0.5)$, where $s_t \in S$ and $\alpha \in [-0.5, 0.5)$. Let $S = \{(s_t|t = -\tau, ..., -1, 0, 1, ..., \tau)\}$ be a LTS and $\beta \in [-\tau, \tau]$ be the value representing the result of a symbolic aggregation operation. Then the 2-tuple that represents the equivalent information to β is obtained as:

$$\Delta : [-\tau, \tau] \to S \times [-0.5, 0.5), \quad \text{(I.6)}$$

where

$$\Delta(\beta) = (s_t, \alpha), \quad with \begin{cases} s_t, \ t = round(\beta) \\ \alpha = \beta - t, \ \alpha \in [-0.5, 0.5) \end{cases}. \quad \text{(I.7)}$$

Function Δ is a one to one mapping function whose anti-function $\Delta^{-1} : \overline{S} \to [-\tau, \tau]$ is defined as $\Delta^{-1}(s_t, \alpha) = t + \alpha$. When $\alpha = 0$ in (s_t, α), it can be called a simple term.

2.1.3 Linguistic terms with weakened hedges

Wang et al. [WXZ18] proposed the concept of LTWHs considering that the linguistic hedges can be considered as a tool to modify the force expressed by a predefined linguistic term. As a LTWH is generated by a linguistic term of a LTS and a weakened hedge, we assume that there is a predefined LTS, associated with semantics of each term, having the form of $S = \{s_t|t = -\tau, ..., -1, 0, 1, ..., \tau\}$. Each $s_t \in S$ can be considered as an atomic term or an original term. As an ordinal linguistic computational model, the following conditions are assumed as well:

(1) The set is ordered: $s_i \geq s_j$ iff $i \geq j$;

(2) The negation operator is defined: $neg\,(s_t) = s_{2\tau - t}$.

Moreover, for qualitative decision making problem in hand, the set of all considered weakened hedges considered is denoted by $H^{(\varsigma)} = \{h_k\,|k = 0, 1, 2, \ldots, \varsigma\}$, such that hedge h_j has more weakening force than h_i if and only if $i < j$.

Then, the generation of LTWHs can be defined by

Definition 4. *(The syntactic rule [WXZ18]). Given a LTS S and a weakened hedge set (WHS)*$H^{(\varsigma)}$ *defined above, a LTWH, denoted by a 2-tuple* $l = \langle h_k, s_t\rangle$*, is generated by the following rule:*

$$\langle weakened\ hedge\rangle := h_k, \ h_k \in H^{(\varsigma)};$$

$$\langle atomic\ term\rangle := s_t, s_t \in S;$$

$$\langle LTWH\rangle := \langle weakened\ hedge\rangle\, \langle atomic\ term\rangle$$

2.2 Group decision making with linguistic preference information

In linguistic GDM process, more and more experts prefer to provide their preferences by making pairwise comparisons between any two alternatives, and this kind of preference reflects the relationships between different alternatives intuitively. Therefore, preference relation becomes one of the popular and effective tools. However, two aspects need to be considered carefully in this process:

(1) Consistency checking and improving. The above way of providing preferences may limit experts in their global perception of the alternatives, generates more information than is really necessary, and, as a consequence, the provided preferences could be inconsistent [CHVAH09]. Therefore, measuring consistency is an important step in decision making with each kind of preference relation to ensure that the preferences of experts are neither random nor illogical.

(2) Consensus reaching process. It is an essential process in GDM for enabling sufficient communications among all experts and obtaining an acceptable decision result.

In recent years, lots of preference models have been studied under linguistic environment such as linguistic preference relation (LPR) [Xu05], hesitant fuzzy linguistic preference relation (HFLPR), probabilistic linguistic preference relation (PLTS) [PWX16], etc. Under uncertain linguistic environment and as a typical linguistic preference relation, the concept of HFLPR and the basic measures of consistency and consensus are proposed in this subsection.

2.2.1 Hesitant fuzzy linguistic preference relation(HFLPR)

Given a fixed set of alternatives $A = \{A_1, A_2, \ldots, A_m\}$ and a LTS $S = \{(s_t|t = -\tau, ..., -1, 0, 1, ..., \tau)\}$, assume that experts provide pairwise comparison judgments of alternatives by linguistic representations based on S, and these linguistic representations are transformed to HFLTSs. Then the concept of HFLPR [ZX14] can be defined as follows:

Definition 5. *(Hesitant fuzzy linguistic preference relations [ZX14]). An HFLPRs B is presented by a matrix $B = (b_{ij})_{m\times m} \subset A \times A$, where $b_{ij} = \left\{ b_{ij}^{l} \,| l = 1, 2, \ldots, \#b_{ij} \right\}$ ($\#b_{ij}$ is the number of linguistic terms in b_{ij}) is an HFLTS, indicating hesitant degrees to which A_i is preferred to A_j. For all $i, j = 1, 2, ..., m$, $b_{ij}\,(i < j)$ should satisfy the following conditions:*

$$b_{ij}^{\rho(l)} \oplus b_{ji}^{\rho(l)} = s_0,\ b_{ii} = s_0,\ \#b_{ij} = \#b_{ji} \tag{I.8}$$

$$b_{ij}^{\rho(l)} < b_{ij}^{\rho(l+1)},\ b_{ji}^{\rho(l+1)} < b_{ji}^{\rho(l)} \tag{I.9}$$

where $b_{ij}^{\rho(l)}$ is the l-th linguistic term in b_{ij}.

In addition, to operate correctly between any two HFLPRs, the normalization of them is necessary by adjusting all HFLTSs and making sure that they have the same number of linguistic terms. Then, the normalization method of HFLPR is developed by Zhu and Xu [ZX14] using a parameter ς:

Definition 6. *(Normalization method [ZX14]). Assume an HFLPR $B = (b_{ij})_{m\times m}$, and an optimized parameter $\varsigma\,(0 \le \varsigma \le 1)$, using ς to add linguistic terms in $b_{ij}\,(i < j)$, and $1 - \varsigma$ to add linguistic terms in $b_{ji}\,(i < j)$; we can obtain an HFLPR, $B^N = \left(b_{ij}^{N}\right)_{m\times m}$, satisfying the condition that*

$$\#b_{ij}^{N} = \max\left\{\#b_{ij}^{N}\,|i,j=1,2,\ldots,m\right\}(i,j=1,2,\ldots,m;i\neq j) \tag{I.10}$$

where $\#b_{ij}^{N}$ *is the number of linguistic terms in* b_{ij}^{N}*. We call* $B^{N}=\left(b_{ij}^{N}\right)_{m\times m}$ *a normalized HFLPR with parameter* ς*.*

2.2.2 The consistency and consensus of HFLPR

Firstly, the consistency of HFLPR has been researched and can be defined as follows:

Definition 7. *(Additive consistency [ZX14]).Given an HFLPR* $B=(b_{ij})_{m\times m}$ *on LTS* S*.* B *can be considered consistent if* $f\left(b_{ij}^{\rho(l)N}\right)+f\left(b_{jk}^{\rho(l)N}\right)-f\left(b_{ik}^{\rho(l)N}\right)=0.5$ *for* $i,j,k=1,2,\ldots,m$*.*

Additionally, the consistency index of an HFLPR can be defined by

$$CI\left(B\right)=1-\frac{2}{3m\left(m-1\right)\left(m-2\right)}\times\frac{1}{\#b_{ij}^{N}}\sum_{i,j,k=1}^{m}\left|f\left(b_{ij}^{\rho(l)N}\right)+f\left(b_{jk}^{\rho(l)N}\right)-f\left(b_{ik}^{\rho(l)N}\right)-0.5\right| \tag{I.11}$$

By Eq. (I.11), we have $CI\left(B\right)\in[0,1]$. The bigger the value of $CI\left(B\right)$, the more consistent B will be. In addition, Dong et al. [DXL08] proposed some consistency thresholds to check whether a preference relation with linguistic preference information is of acceptable consistency. If the consistency index of HFLPR is smaller than the given consistency threshold, then some consistency repairing methods can be used to improve the HFLPR with unacceptable consistency such as the automatic method [ZX14] and the feedback-based method [ZX14, AHVFH10].

Consensus reaching process is a very important part in linguistic GDM, which makes sure that the experts and analysts have enough communications and the moderator can also assist experts in improving their preference information. Generally, consensus reaching process mainly consists of two aspects: the identification rules (IR) and the direction rules (DR). Obviously, the IR is to search the experts or the precise locations in the HFLPR that need to be improved, and the DR is a guide for the improvement.

3 Justification

As we mentioned above, the DHLTS and DHHFLTS can be used to represent some complex linguistic information accurately by 2-tuple linguistic structure. Therefore, this thesis mainly uses them to express linguistic information and research the measures of DHHFLTSs, the consistency of DHHFLPRs and the consensus reaching process in LSGDM.

Firstly, some transformations are necessary when we want to make some operations of linguistic terms. In 2004, Yager [Yag04] has provided a figure to show the process of CW:

Fig. 1. Yager's CW scheme.

Based on the Fig. 1 and Definition 2, this thesis develops two monotone functions to make the mutual transformations between the double hierarchy linguistic term (DHLT) and the numerical scale. Similarly, the monotone functions between a double hierarchy hesitant fuzzy linguistic element (DHHFLE) and a set of numerical scales can be obtained. These functions can be used to deal with decision making problems with double hierarchy linguistic information and double hierarchy hesitant fuzzy linguistic information.

The transformation process is shown in Fig. 2.

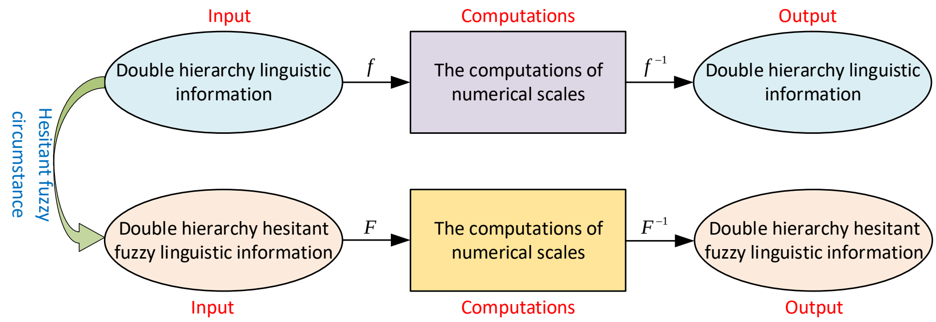

Fig. 2. The transformation process when dealing with decision-making problems

Based on Definition 9 and Fig. 2, some operational laws and measure methods among DHLTs, DHHFLTSs, DHLPRs and DHHFLPRs can be developed, which are the foundations of this work.

Additionally, the justifications of this thesis can be summarized as follows:

- To express complex linguistic information more clearly and accurately, it is necessary to provide the complete definitions of DHLTSs and DHHFLTSs, research some important properties and develop some basic operational laws of them.
- Considering that the distance and similarity measures are the basis of decision making with double hierarchy hesitant fuzzy linguistic information, so this thesis proposes some distance and similarity measures of double hierarchy hesitant fuzzy linguistic elements (DHHFLEs) and DHHFLTSs from different angles.
- To avoid occurring some self-contradictory situations, it is very important to carry out the consistency checking and repairing for each DHHFLPR. Therefore, this thesis proposes some

additive consistency measures firstly. To judge whether a DHHFLPR is of acceptable consistency or not, we introduce a consistency index, and develop some novel threshold values to judge whether a DHHFLPR is of acceptable consistency or not. Furthermore, we develop two consistency repairing algorithms based on the automatic improving method and the feedback improving method respectively to improve the DHHFLPR with unacceptable consistency.

- LSGDM or complex GDM problems are very commonly encountered in actual life, especially in the era of data. In addition, consensus reaching process is the key and focus work when dealing with LSGDM problems, which unifies all experts' opinions and ensures that the LSGDM problems can be solved smoothly. Therefore, it is necessary to develop a consensus reaching process for LSGDM with double hierarchy hesitant fuzzy linguistic preference information. To ensure the implementation of consensus reaching process, we also propose the similarity degree-based clustering method, the double hierarchy information entropy-based weights determining method and the consensus measures to make the consensus reaching process more efficient.
- In LSGDM, consensus reaching process also makes sure that the experts and analysts have enough communications and the moderator can also assist the experts in improving their preference information. However, two typical items are very common and have significant influences in consensus reaching process of LSGDM, i.e., non-cooperative behaviors [PMH14] and minority opinions [XDC15]. Although they are only the small fractions in LSGDM, it is likely to determine the direction of the decision making problem. Therefore, we focus on dealing with these preferences provided by experts or groups reasonably and accurately.

In addition, the above research results can also be applied to some practical MCDM, GDM and LSGDM problems such as evaluating the implementation status of haze controlling measures, Sichuan liquor brand assessment, and Sichuan water resource management, etc.

4 Objectives

The DHLTS enriches the vocabulary of linguistic representations by using two hierarchy LTSs where every second hierarchy LTS is a linguistic feature or detailed supplementary of the corresponding linguistic term included in the first hierarchy LTS. In addition, the DHHFLTS is the extension of DHLTS and it can represent the uncertain linguistic term information clearly with several DHLTs simultaneously. Based on DHLTS and DHHFLTS, the aim of this thesis is to analyze some basic concepts of them, to research some measure methods, and study consistency and consensus theories with double hierarchy linguistic preference information and double hierarchy hesitant fuzzy linguistic preference information and their applications in GDM, MCDM and LSGDM, etc. The specific objectives are summarized as follows:

- **To complete the concepts of DHLTSs and DHHFLTSs** Develop two equivalent transformation functions between the DHLTs (DHHFLEs) and the evaluations in [0,1] (HFE) to make the operations of double hierarchy linguistic information simpler. Then, some basic operational laws and properties of DHHFLEs are developed. In addition, A MCDM model, named double hierarchy hesitant fuzzy linguistic MULTIMOORA (DHHFL-MULTIMOORA), is proposed to deal with a practical case about selecting the best city in China by evaluating the implementation status of haze controlling measures.

- **To propose some measure methods for DHHFLTSs.** Some distance and similarity measures of DHHFLEs and DHHFLTSs are proposed from different angles. Then, a decision-making method is developed to deal with MCDM problems on the basis of these distance and similarity measures.

- **To define the concept of DHHFLPR and propose some consistency measures.** To judge whether a DHHFLPR is of acceptable consistency or not, we introduce a consistency index, and develop some novel threshold values for judging whether a DHHFLPR is of acceptable consistency or not. Furthermore, we develop two consistency repairing algorithms based on the automatic improving method and the feedback improving method respectively to improve the DHHFLPR with unacceptable consistency. Additionally, a method is set up to deal with GDM problems with double hierarchy hesitant fuzzy linguistic preference information.

- **To research the consensus reaching process for LSGDM with DHHFLPRs.** To ensure the implementation of consensus reaching process in LSGDM with double hierarchy hesitant fuzzy linguistic preference information, firstly we propose a similarity degree based clustering method cluster the experts into several small groups, it can reduce the complexity of LSGDM. Then we develop a double hierarchy information entropy-based weights-determining method considering that aggregating all experts' preference information is an important step, and some consensus measures are developed because of they are the main basis of the consensus reaching process. Finally, based on these methods and measures, a LSGDM model with DHHFLPRs is established.

- **To define the concept of DHLPR and manage minority opinions and non-cooperative behaviors in LSGMD with DHLPRs.** In LSGDM, sometimes some experts do not modify their preferences or even do it on the contrary way to the remaining experts, and some different opinions or minority preferences are often cited as obstacles to decision making. Therefore, this thesis gives the concept of DHLPR and develops a consensus model to manage minority opinions and non-cooperative behaviors in LSGDM with DHLPRs. Moreover, this thesis also establishes the consensus model, as well as develops some basic tools such as distance-based cluster method, weight-determining method, and comprehensive adjustment coefficient-determining method.

5 Methodology

Taking into account the above aims, the main idea of this work is to study the concepts of DHLTS and DHHFLTS and to research several basic measures of them, as well as to develop some consistency checking and repairing methods and consensus reaching theories of DHLPRs and DHHFLPRs in some practical linguistic decision making problems. The related methods are provided as follows:

1. **Hypothesis formulation.** When dealing with linguistic decision making problems, some reasonable and suitable hypotheses should be provided, which is an important component of linguistic decision making process. For instance, we propose a premise that the second hierarchy LTSs of all first hierarchy linguistic terms are same when we define the transformation functions and develop some operations and properties of DHLTSs and DHHFLTSs. In addition, we give a natural premise that the consistency thresholds are important indicators for judging whether a DHHFLPR is of acceptable consistency. Moreover, we assume that all experts are more willing to take the adjustment suggestions of moderators when improving the consistencies of them and building the consensus model in GDM or LSGDM process.

2. **Establishment of optimization models.** When improving the consistency of DHHFLPR, this thesis establishes two consistency repairing optimization models based on the automatic optimization method and feedback optimization method respectively to repair the DHHFLPR with unacceptable consistency. In addition, we also establish optimization models to ensure the implementation of consensus reaching process in LSGDM with DHLPRs and DHHFLPRs.

3. **Simulation analysis.** It can be used to reflect the validity and rationality of the proposed methods and models in different linguistic decision making problems. For instance, this thesis utilizes the visual method "Figure of area" to compare the inconsistent DHHFLPRs and the additive consistent DHHFLPRs more intuitively. In addition, we use the simulation analysis to show the ranking orders of all alternatives on the basis of the satisfaction degrees and to obtain that which alternative is the optimal one.

4. **Basic method research.** As the basis of this work, some measures and methods are developed in this thesis. For instance, the distance and similarity measures of DHHFLTSs are the basic tools of some decision making methods, consistency checking and consensus reaching models. Furthermore, the clustering method is a very important step when dealing with LSGDM problems.

5. **Comparative study.** This thesis makes some comparative analyses between the proposed methods and some existing methods to further analyze the characteristics and advantages (or disadvantages) of the proposed methods. For instance, we can compare the proposed decision making methods, the consistency checking and repairing models and the consensus reaching methods with some existing methods with other linguistic information.

6 Summary

In this section, we make a summary of the proposals included in this thesis, and introduce the main contents along with the obtained results associated with the journal publications. The published and submitted papers are listed as follows:

- X.J. Gou, H.C. Liao, Z.S. Xu, F. Herrera, Double hierarchy hesitant fuzzy linguistic term set and MULTIMOORA method: a case of study to evaluate the implementation status of haze controlling measures. Information Fusion, 38 (2017) 22-34.
- X.J. Gou, Z.S. Xu, H.C. Liao, F. Herrera, Multiple criteria decision making based on distance and similarity measures under double hierarchy hesitant fuzzy linguistic environment. Computers and Industrial Engineering, 126 (2018) 516-530.
- X.J. Gou, H.C. Liao, Z.S. Xu, R. Min, F. Herrera, Group decision making with double hierarchy hesitant fuzzy linguistic preference relations: consistency based measures, index and repairing algorithms and decision model. Information Sciences, 489 (2019) 93-112.
- X.J. Gou, Z.S. Xu, F. Herrera, Consensus reaching process for large-scale group decision making with double hierarchy hesitant fuzzy linguistic preference relations. Knowledge-Based Systems, 157 (2018) 20-33.
- X.J. Gou, H.C. Liao, Z.S. Xu, F. Herrera, Managing minority opinions and non-cooperative behaviors in large-scale group decision making under DHLPRs: A consensus model. Submitted to IEEE Transactions on Cybernetics,(2019).

The rest of this section is organized by five aspects mentioned in Section 4: Subsection 6.1 proposes the concepts of DHLTS and DHHFLTS, and the transformation functions between the double hierarchy linguistic model and numerical scale, as well as introduces some operational laws of DHHFLEs. Subsection 6.2 introduces a series of distance and similarity measures of DHHFLTSs. Subsection 6.3 defines the concept of DHHFLPR and proposes some consistency measures. As well as develops two consistency repairing algorithms based on the automatic improving method and the feedback improving method respectively to improve the DHHFLPR with unacceptable consistency. Section 6.4 researches the consensus reaching process for LSGDM with DHHFLPRs by developing a consensus model. Finally, Subsection 6.5 defines the concept of DHLPR and develops a consensus model to manage minority opinions and non-cooperative behaviors in LSGMD with DHLPRs.

6.1 DHLTS and DHHFLTS

The DHLTS, consists of two hierarchies fully independent LTSs, can be used to represent complex linguistic information clearly based on 2-tuple linguistic structure.

(1) The definitions of DHLTS and DHHFLTS

Definition 8. *(DHLTS). Let* $S = \{s_t \,|t = -\tau, \ldots, -1, 0, 1, \ldots, \tau\}$ *and* $O^t = \{o^t_k \,|k = -\varsigma, \ldots, -1, 0, 1, \ldots, \varsigma\}$ *be the first hierarchy LTS and the second hierarchy LTS of* s_t*, respectively. Then we call*

$$S_O = \{s_{t<o^t_k>}|t = -\tau, ..., -1, 0, 1, ..., \tau; k = -\varsigma, ..., -1, 0, 1, ..., \varsigma\} \tag{I.12}$$

the DHLTS, where $s_{t<o_k^t>}$ is called DHLT. Especially, every linguistic term in the first hierarchy LTS has its own the secondly hierarchy LTS and all the second hierarchy LTSs may be different in the actual situation. For convenient, we use a unified form $S_O = \{s_{t<o_k>} | t = -\tau, ..., -1, 0, 1, ..., \tau; k = -\varsigma, ..., -1, 0, 1, ..., \varsigma\}$ to express DHLTS.

In addition, the semantic of DHLT $s_{t<o_k>}$ is based on the linguistic terms s_t and o_k, which can be seen in Fig. 3, where we give a second hierarchy LTS $O = \{o_{-2} = far\ from, o_{-1} = a\ little, o_0 = just\ right, o_1 = much, o_2 = very\ much\}$ to the first hierarchy linguistic term s_1.

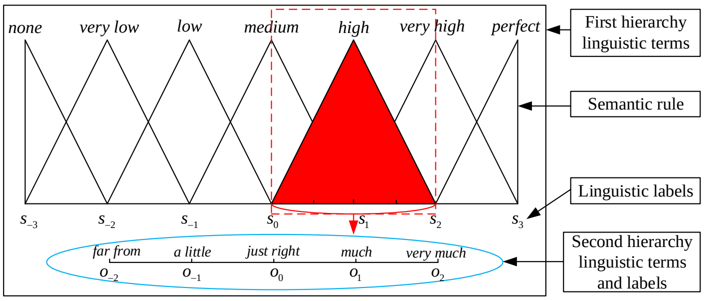

Fig. 3. The second hierarchy LTS of a linguistic term s_1 in first hierarchy LTS

To express the uncertainty of decision maker's cognitive, we extend DHLTS to hesitant linguistic environment and define the concept of DHHFLTS.

Definition 9. *(DHHFLTS). Let $x_i \in X$, $i = 1, 2, \ldots, N$ be a fixed set, S_O be a DHLTS. A DHHFLTS on X, H_{S_O}, is in mathematical term of*

$$H_{S_O} = \{< x_i, h_{S_O}(x_i) > | x_i \in X\} \tag{I.13}$$

where $h_{S_O}(x_i)$ is a set of DHLTs in H_{S_O} and can be denoted as:

$$\begin{aligned} h_{S_O}(x_i) = \Big\{ & s_{\phi_l<o_{\varphi_l}>}(x_i) \,| \\ & s_{\phi_l<o_{\varphi_l}>} \in S_O; l = 1, 2, \ldots, L;\ \phi_l = -\tau, \ldots, -1, 0, 1, \ldots, \tau;\ \varphi_l = -\varsigma, \ldots, -1, 0, 1, \ldots, \varsigma \Big\} \end{aligned} \tag{I.14}$$

with L being the number of the DHLTs in $h_{S_O}(x_i)$. $h_{S_O}(x_i)$ denotes the possible degree of the linguistic variable x_i to S_O. $s_{\phi_l<o_{\varphi_l}>}(x_i)$ $(l = 1, \ldots, L)$ in each $h_{S_O}(x_i)$ being the continuous terms in S_O. For convenience, we call $h_{S_O}(x_i)$ a DHHFLE, and all DHLTs in a DHHFLE are ranked in ascending order.

(2) Equivalent transformation functions

For the basic of some operations, measure methods and decision making models, two pairs of equivalent transformation functions for making the mutual transformations between the DHLT (or DHHFLE) and the real number (or HFE) are developed as follows:

Definition 10. *(DHHFLTS). Let S_O be a DHLTS. $h_{S_O} = \left\{ s_{\phi_l<o_{\varphi_l}>} \middle| s_{\phi_l<o_{\varphi_l}>} \in S_O; l = 1, 2, \ldots, L; \phi_l = [-\tau, \tau]; \varphi_l = [-\varsigma, \varsigma] \right\}$ be a DHHFLE with L being the number of DHLTs in h_{S_O}, and $h_\gamma = \{\gamma_l | \gamma_l \in [0, 1]; l = 1, \ldots, L\}$ be a HFE. Then the membership degree γ_l and the subscript $\phi_l < \varphi_l >$ of the DHLT $s_{\phi_l<o_{\varphi_l}>}$ that expresses the equivalent information to the membership degree γ_l can be transformed to each other by the following functions f and f^{-1}, respectively:*

$$f:[-\tau,\tau]\times[-\varsigma,\varsigma]\to[0,1]\,,f\left(\phi_l,\varphi_l\right)=\frac{1}{\tau}\times\frac{\varphi_l+\varsigma}{2\varsigma}+\frac{\tau+\phi_l-1}{2\tau}=\frac{\varphi_l+\left(\tau+\phi_l\right)\varsigma}{2\varsigma\tau}=\gamma_l \tag{I.15}$$

$$\begin{aligned}&f^{-1}:[0,1]\to[-\tau,\tau]\times[-\varsigma,\varsigma]\,,\\&f^{-1}\left(\gamma_l\right)=[2\tau\gamma_l-\tau]<o_{\varsigma(2\tau\gamma_l-\tau-[2\tau\gamma_l-\tau])}>\;=[2\tau\gamma_l-\tau]+1<o_{\varsigma((2\tau\gamma_l-\tau-[2\tau\gamma_l-\tau])-1)}>\end{aligned} \tag{I.16}$$

Then we can introduce the transformation functions F and F^{-1} between the DHHFLE h_{S_O} and the HFE h_γ:

$$\begin{aligned}&F:\Phi\times\Psi\to\Theta,\;\;F\left(h_{S_O}\right)=\\&F\left(\left\{s_{\phi_l<o_{\varphi_l}>}\middle|s_{\phi_l<o_{\varphi_l}>}\in S_O;l=1,\ldots,L;\phi_l\in[-\tau,\tau]\,;\varphi_l\in[-\varsigma,\varsigma]\right\}\right)=\left\{\gamma_l|\gamma_l=f\left(\phi_l,\varphi_l\right)\right\}=h_\gamma\end{aligned} \tag{I.17}$$

$$\begin{aligned}&F^{-1}:\Theta\to\Phi\times\Psi,\\&F^{-1}\left(h_\gamma\right)=F^{-1}\left(\left\{\gamma_l\,|\gamma_l\in[0,1]\,;l=1,\ldots,L\right\}\right)=\left\{s_{\phi_l<o_{\varphi_l}>}|\phi_l<o_{\varphi_l}>=f^{-1}\left(\gamma_l\right)\right\}=h_{S_O}\end{aligned} \tag{I.18}$$

(3) Operational laws of DHHFLEs

Based on these two equivalent transformation functions F and F^{-1}. Some basic operational laws of DHHFLEs can be developed:

Definition 11. *(DHHFLTS). Let S_O be a DHLTS. $h_{S_O}=\left\{s_{\phi_l<o_{\varphi_l}>}\middle|s_{\phi_l<o_{\varphi_l}>}\in S_O;\right.$ $\left.l=1,2,\ldots,L;\phi_l=[-\tau,\tau]\,;\varphi_l=[-\varsigma,\varsigma]\right\}$ and $h_{S_{O_i}}=\left\{s^i_{\phi_l<o_{\varphi_l}>}\middle|s^i_{\phi_l<o_{\varphi_l}>}\in S_O;l=1,2,\ldots,L;\right.$ $\left.\phi_l=[-\tau,\tau]\,;\varphi_l=[-\varsigma,\varsigma]\right\}\,(i=1,2)$ be three DHHFLEs, λ be a real number. Then*

(1) (Addiiton) $h_{S_{O_1}}\oplus h_{S_{O_2}}=F^{-1}\left(\bigcup_{\eta_1\in F\left(h_{S_{O_1}}\right),\eta_2\in F\left(h_{S_{O_2}}\right)}\{\eta_1+\eta_2-\eta_1\eta_2\}\right)$;

(2) (Multiplication) $h_{S_{O_1}}\otimes h_{S_{O_2}}=F^{-1}\left(\bigcup_{\eta_1\in F\left(h_{S_{O_1}}\right),\eta_2\in F\left(h_{S_{O_2}}\right)}\{\eta_1\eta_2\}\right)$;

(3) (Multiplication) $\lambda h_{S_O}=F^{-1}\left(\bigcup_{\eta\in F\left(h_{S_O}\right)}\left\{1-(1-\eta)^\lambda\right\}\right)$;

(4) (Power) $\left(h_{S_O}\right)^\lambda=F^{-1}\left(\bigcup_{\eta\in F\left(h_{S_O}\right)}\{\eta^\lambda\}\right)$

(5) (Complementary) $\overline{h_{S_O}}=F^{-1}\left(\bigcup_{\eta\in F\left(h_{S_O}\right)}\{1-\eta\}\right)$.

Finally, we investigate a MCDM model with double hierarchy hesitant fuzzy linguistic information, and develop a DHHFL-MULTIMOORA method to solve it by three measures including the double hierarchy hesitant fuzzy linguistic ratio system (DHHFLRS), the double hierarchy hesitant fuzzy linguistic reference point (DHHFLRP) and the double hierarchy hesitant fuzzy linguistic full multiplicative form (DHHFLFMF). Furthermore, we apply the DHHFL-MULTIMOORA method to deal with a practical case about selecting the optimal city in China by evaluating the implementation status of haze controlling measures.

The journal paper with respect to this part is:

- X.J. Gou, H.C. Liao, Z.S. Xu, F. Herrera, Double hierarchy hesitant fuzzy linguistic term set and MULTIMOORA method: a case of study to evaluate the implementation status of haze controlling measures. Information Fusion, 38 (2017) 22-34.

6.2 Distance and similarity measures of DHHFLTSs

Distance and similarity measures can be utilized to measure the deviation and closeness degrees between different arguments [LXZ14]. Therefore, in this part, we mainly propose some distance and similarity measures between the DHHFLEs and some of them between the DHHFLTSs.

(1) Distance and similarity measures between the DHHFLEs

This thesis defines the axioms of distance measure between any two DHHFLEs with four properties including Boundary, Symmetry, Complementarity and Reflexivity. Similarly, the axioms of similarity measure between any two DHHFLEs also can be obtained according to different relationships between the distance measure and the similarity measure of DHHFLEs.

Definition 12. *(DHHFLTS). Let* S_O *be a DHLTS.* $h^i_{S_O} = \left\{ s^i_{\phi_l<o_{\varphi_l}>} \middle| s^i_{\phi_l<o_{\varphi_l}>} \in S_O\,; l = 1,2, \ldots, \#h^i_{S_O} \right\}$ $(i = 1,2)$ *be two DHHFLEs. Then* $d\left(h^1_{S_O}, h^2_{S_O}\right)$ *is called the distance measure between* $h^1_{S_O}$ *and* $h^2_{S_O}$ *if it satisfies the following properties:*

(1) Boundary: $0 \leq d\left(h^1_{S_O}, h^2_{S_O}\right) \leq 1$*;*

(2) Symmetry: $d\left(h^1_{S_O}, h^2_{S_O}\right) = d\left(h^2_{S_O}, h^1_{S_O}\right)$*;*

(3) Complementarity: $d\left(h^1_{S_O}, \bar{h}^1_{S_O}\right) = 1$ *iff* $F\left(h^1_{S_O}\right) = \{0\}$ *or* $F\left(h^1_{S_O}\right) = \{1\}$*;*

(4) Reflexivity: $d\left(h^1_{S_O}, h^2_{S_O}\right) = 0$ *iff* $h^1_{S_O} = h^2_{S_O}$*.*

where $\bar{h}^1_{S_O} = \left\{ s^1_{-\phi_l<o_{-\varphi_l}>} \middle| s^1_{-\phi_l<o_{-\varphi_l}>} \in S_O\,; l = 1,2,\ldots,\#h^1_{S_O} \right\}$ *is the complement set of* $h^1_{S_O}$*, and* F *is a monotone function.*

Based on Definition 14, some distance and similarity measures between the DHHFLEs can be obtained and they consist of basic distances, the Hausdorff distance and the hesitant degrees simultaneously. In addition, Some distance and similarity measures between the DHHFLEs with preference information are developed.

(2) Distance and similarity measures between the DHHFLTSs

In some practical problems especially in MCDM problems, experts usually use a set to express their evaluation information when evaluating each alternative (or object) with respect to all attributes (or criteria). Therefore, the DHHFLTS is a perfect expression to take into account all aspects. Additionally, the weights of criteria are very important in MCDM problems, and we also need to consider them. When the evaluation information of each alternative (or object) with respect to all criteria is expressed by DHHFLTS, the distance and similarity measures between DHHFLTSs are very important to deal with MCDM problems. Similarly, this thesis develops the axioms of the distance and similarity measures between the DHHFLTSs are developed, and develops some weighted distance and similarity measures between the DHHFLTSs in discrete case, continuous case, respectively.

The journal paper with respect to this part is:

- X.J. Gou, Z.S. Xu, H.C. Liao, F. Herrera, Multiple criteria decision making based on distance and similarity measures under double hierarchy hesitant fuzzy linguistic environment. Computers and Industrial Engineering, 126 (2018) 516-530.

6.3 Consistency measures of DHHFLPRs

As we mentioned in Section 1, more and more experts prefer to give their preferences by making pairwise comparisons between any two alternatives. Meanwhile, this kind of preference reflects the relationships between different alternatives intuitively. Therefore, based on the DHHFLTS and preference form, this thesis gives the concept of DHHFLPR. In addition, to avoid the occurrence of some self-contradictory situations, it is very important to propose consistency measures for DHHFLPRs and develop consistency checking and improving method in GDM process with DHHFLPRs.

Firstly, the concept of DHHFLPR can be defined as follows:

Definition 13. *(DHHFLPR). A DHHFLPR $\tilde{H}_{S_O}$ is represented by a matrix $\tilde{H}_{S_O} = \left(h_{S_{O_{ij}}}\right)_{m\times m}$, where $h_{S_{O_{ij}}} = \left\{h^{(l)}_{S_{O_{ij}}} \middle| l = 1,2,\ldots,\#h_{S_{O_{ij}}}\right\}$ ($\#h_{S_{O_{ij}}}$ is the number of DHLTs in $h_{S_{O_{ij}}}$, $h^{(l)}_{S_{O_{ij}}}$ is the $l-th$ DHLT in $h_{S_{O_{ij}}}$) is a DHHFLE, indicating the hesitant degrees to which A_i is preferred to A_j. For all $i,j = 1,2,\ldots,m$, $h_{S_{O_{ij}}}$ $(i<j)$ satisfies the following conditions:*

$$h^{(l)}_{S_{O_{ij}}} \oplus h^{(l)}_{S_{O_{ji}}} = s_{0<o_0>}, h_{S_{O_{ii}}} = \{s_{0<O_0>}\}, \#h_{S_{O_{ij}}} = \#h_{S_{O_{ji}}} \tag{I.19}$$

$$h^{(l)}_{S_{O_{ij}}} < h^{(l+1)}_{S_{O_{ij}}}, h^{(l)}_{S_{O_{ji}}} > h^{(l+1)}_{S_{O_{ji}}} \tag{I.20}$$

Next, the normalized DHHFLPR of a DHHFLPR $\tilde{H}_{S_O} = \left(h_{S_{O_{ij}}}\right)_{m\times m}$ can be obtained, denoted by $\tilde{H}^N_{S_O} = \left(h^N_{S_{O_{ij}}}\right)_{m\times m}$, based on a linguistic expected-value of DHHFLE. Then we call $\tilde{H}_{S_O}$ an additive consistent DHHFLPR if it satisfies

$$h^N_{S_{O_{ij}}} = h^N_{S_{O_{i\rho}}} \oplus h^N_{S_{O_{\rho j}}} \; (i,j,\rho = 1,2,\ldots,m; i \neq j) \tag{I.21}$$

If $\bar{h}^N_{S_{O_{ij}}} = \frac{1}{m}\left(\overset{m}{\underset{\rho=1}{\oplus}}\left(h^N_{S_{O_{i\rho}}} \oplus h^N_{S_{O_{\rho j}}}\right)\right)$ for $i,j,\rho = 1,2,\ldots,m; i \neq j$, then $\tilde{H}_{S_O}$ is an additive consistent DHHFLPR, and $\bar{H}_{S_O}^N = \left(\bar{h}_{S_{O_{ij}}}^N\right)_{m\times m}$ is an additive consistent normalized DHHFLPR. Moreover, the consistency index (CI) of $\tilde{H}_{S_O}$ can be denoted as:

$$CI\left(\tilde{H}_{S_O}\right) = d\left(\tilde{H}_{S_O}^N, \bar{H}_{S_O}^N\right) \tag{I.22}$$

This thesis proposes some novel consistency thresholds to check whether a DHHFLPR is of acceptable consistency by making comparison with the CI of the DHHFLPR.

In some practical decision making processes with DHHFLPRs, it is common for there to be a DHHFLPR with unacceptable consistency, i.e., $CI\left(\tilde{H}_{S_O}\right) > C\bar{I}\left(\tilde{H}_{S_O}\right)$. Then, we establish two consistency repairing algorithms based on the automatic improving method and feedback improving method respectively to repair this case.

(1) Considering that the automatic improving method is time-saving, effective, and practical without the interaction of the experts, so we develop a consistency repairing algorithm based on the automatic optimization method that can repair the DHHFLPR with unacceptable consistency by automatic iterative operations. Additionally, we analyze the convergence of repair results. Finally, we establish an optimization model which can be used to obtain the DHHFLPR of acceptable consistency directly.

(2) Sometimes the experts are more likely to modify their preference relations by themselves. Meanwhile, in existing research, lots of scholars developed some feedback methods under other preference circumstances [ZX14, AHVFH10], and the feedback method can feed suggestions back to the experts and help them to improve their preferences. Therefore, this subsection also establishes a consistency repairing algorithm based on the feedback method with DHHFLPRs.

The journal article associated to this part is:

- X.J. Gou, H.C. Liao, Z.S. Xu, R. Min, F. Herrera, Group decision making with double hierarchy hesitant fuzzy linguistic preference relations: consistency based measures, index and repairing algorithms and decision model. Information Sciences, 489 (2019) 93-112.

6.4 Consensus reaching process for LSGDM with DHHFLPRs

LSGDM or complex GDM problems are very commonly encountered in actual life, especially in the era of data. This thesis develops a consensus reaching process for LSGDM with DHHFLPRs. To ensure the implementation of consensus reaching process, we also propose a similarity degree-based clustering method, a double hierarchy information entropy-based weights-determining method and some consensus measures.

An LSGDM mainly consists of two main parts:

(1) Clustering. In LSGDM, the discussions among experts are very common. However, it will surely bring forth a huge amount of work and the communications among experts also will not be smooth. To solve these problems, clustering is very necessary in the consensus reaching process because of a group with less experts is easier to discuss and improve preference information. According to some certain characteristics of experts, large-scale decision-making groups can be classified into several small groups for assisting and improving the efficiency of decision-making.

Similarity degree can be as a useful tool to reflect the relationship of any two experts. Therefore, based on the similarity measures of DHHFLTSs, this thesis develops a clustering method for LSGDM based on information entropy theory, which can be understood very clearly by a dynamic clustering figure. By this method, the experts can be divided into several small groups. Additionally, we propose a weights-determining method, which can obtain the weight of each small group, the weights of the experts included in each small group, and the weights of all experts, respectively.

(2) The other important part is the consensus reaching process, in which the experts discuss and improve their preferences, guided and supervised by a moderator. This part aims at reaching all decision makers' agreements before making decisions.

We propose some consensus measures. A model is developed, which can precisely identify the alternatives, the pairs of alternatives and the experts that do not reach the consensus threshold, and then the moderator feeds these suggestions back to each small group and experts for modifying their preference information. This consensus measures can make the consensus degree improving process more targeted.

- **Consensus measures.** In the process of clustering, the similarity matrices $SM^{ab} = \left(sm_{ij}^{ab}\right)_{m\times m}$ $(a, b = 1, 2, \ldots, n)$ associated with each pair of experts $\left(e^a, e^b\right)$ are obtained and we can establish a consensus matrix $CM = (cm_{ij})_{m\times m}$ based on these similarity matrices. Then, the consensus degree for each pair of alternatives, the consensus degree for each alternative, and the overall consensus degree for all preference relations can be obtained.

Based on the discussions above, we can make a comparison between the overall consensus degree and the given consensus threshold value ξ. If $ocd \geq \xi$, then the consensus reaching process is over; Otherwise, two steps are performed simultaneously: One is to cluster all experts into several small groups based on the given clustering methods, and the other one is to identify the alternatives, the part of alternatives, and the experts that need to improve preference relations, as well as to provide suggestions to improve them.

- **Consensus improving process.** This part mainly includes two kinds of rules: the identification rules (IR) and the direction rules (DR). The IR are mainly used to identify the alternatives, the pairs of alternatives and the experts that do not reach the given consensus threshold. The DR are utilized to send suggestions to each group and tell them how to increase the consensus level in the next round. Firstly, the moderator needs to set up a target and gives it to each group, and then each group can discuss how to change their preferences in the position (A_i, A_j). The target can be obtained by referencing the aggregation information of all experts' preferences. Finally, every group can discuss and change the corresponding preference information and the consensus reaching process is over when the overall consensus degree is bigger than the given consensus threshold value ξ.

The journal article associated to this part is:

- X.J. Gou, Z.S. Xu, F. Herrera, Consensus reaching process for large-scale group decision making with double hierarchy hesitant fuzzy linguistic preference relations. Knowledge-Based Systems, 157 (2018) 20-33.

6.5 Managing minority opinions and non-cooperative behaviors in LSGDM with DHLPRs

Similar as the concept of DHHFLPR, based on the DHLTS and preference form, this thesis also gives a concept of DHLPR, and utilizes it to express the evaluation information of all experts in LSGDM under double hierarchy linguistic preference environment.

In decision making process, let $A = \{A_1, A_2, \ldots, A_m\}$ be a fixed set of alternatives, then an additive DHLPR can be developed:

Definition 14. *(DHLPR). An additive DHLPR* $\Re$ *is represented by a matrix* $= \Re = (r_{ij})_{m\times m} \subset A \times A$*, where* $r_{ij} \in S_O$ $(i, j = 1, 2, \ldots, m)$ *is a DHLT, indicating the degree of* A_i *is preferred to* A_j*. For all* $i, j = 1, 2, \ldots, m$*,* $r_{ij}(i < j)$ *satisfies the conditions* $r_{ij} + r_{ji} = s_{0<o_0>}$ *and* $r_{ii} = s_{0<o_0>}$*.*

As we mentioned above, minority opinions and non-cooperative behaviors are very important in consensus reaching process and should be taken into consideration in LSGDM. This thesis develops a method to determine some necessary parameters in the consensus reaching process, and incorporates minority opinions and non-cooperative behaviors into the consensus model and develops an algorithm to manage them in LSGDM with DHLPRs.

Firstly, In the consensus reaching process of an LSGDM, it is common that experts may face some internal and external pressures, so there exist uncertainty and subjectivity in the opinion adjustment coefficients provided by the experts [XDC15]. Therefore, thesis develops some adjustment coefficients to improve decision credibility including subjective and objective adjustment coefficients, based on these two adjustment coefficients, the comprehensive adjustment coefficient can be also obtained.

Secondly, when dealing with minority opinions, this thesis develops a method which consists of three parts: Identifying the minority opinions, making a discussion among the experts and adjusting the corresponding weight information.

Thirdly, in the consensus reaching process of LSGDM, some experts can be regarded as the non-cooperative members if they refuse to adjust their preferences or only adjust part of preferences. Without doubt, these behaviors will lead to inaccurate result or reduce the efficiency of CRP. Therefore, we are committed to developing a method to identify and manage non-cooperative behaviors, this method also consists of three aspects: Identifying the non-cooperative group(s), measuring the non-cooperative degree, and modifying the non-cooperative behaviors.

Finally, based on the cluster method and weight-determining method discussed in Subsection 6.4, and proposed two methods for identifying and managing minority opinions and non-cooperative behaviors. An algorithm is established in this thesis to deal with LSGDM with DHLPRs. In addition, we also apply the proposed algorithm to deal with a practical LSGDM problem that is to determine the main reason of haze pollution in a city of China.

The journal article associated to this part is:

- X.J. Gou, H.C. Liao, Z.S. Xu, F. Herrera, Managing minority opinions and non-cooperative behaviors in large-scale group decision making under DHLPRs: A consensus model. Submitted to IEEE Transactions on Cybernetics.

7 Discussion of results

This section mainly makes several discussions about the results obtained in all the mentioned stages of this thesis.

7.1 DHLTS and DHHFLTS

The DHLTS is the extension of single hierarchy LTS by considering two hierarchies LTSs simultaneously with the 2-tuple linguistic structure, and it can be used to represent complex linguistic information clearly. Four important points of DHLTS are obtained: 1) All elements in DHLTS are expressed by linguistic labels without any numerical scales, which reflect the semantics of original natural languages to a greater extent; 2) The second hierarchy LTS is necessary when the set of adverbs of a first hierarchy linguistic term is large. 3) Each second hierarchy LTS can be regarded as a set of adverbs and extends the linguistic representations (richer vocabularies). 4) Each linguistic terms in the first hierarchy LTS has its own second hierarchy LTS, and usually they are different.

Based on the analyses mentioned above, DHLTS, DHHFLTS and the decision making method mainly have the following four important advantages:

a) The DHLTS consists of two hierarchy LTSs. Therefore, the basic element DHLT can be used to describe some complex linguistic more accurately and fully than the single linguistic term. Additionally, the expression of a DHLT is very intuitional and simple, and we give the linguistic labels in advance, so we can use a simple DHLT to express any complex linguistic information.

b) For the purpose of expressing some more complex uncertain linguistic information, we extend DHLTSs to hesitant fuzzy environment and develop DHHFLTSs. It is a very useful way to represent the hesitance existing in people's daily life.

c) The equivalent transformation functions can simplify the computations of original linguistic terms by transforming them into numerical scales, and do not change the essence of them by transforming the results into double hierarchy linguistic information with the anti-function.

d) The DHHFL-MULTIMOORA method is more comprehensive in dealing with MCDM problems as it utilizes the DHHFLRS, DHHFLRPs, and DHHFLFMF measures. All of them are reasonable in dealing with MCDM problems from different angles. Thus, the reliability and veracity of the decision making results would be improved greatly.

7.2 Distance and similarity measures of DHHFLTSs

This thesis develops a series of distance and similarity measures for DHHFLEs and DHHFLTSs from different angels. Obviously, each kind of distance and similarity measure owns its key point. The distance and similarity measures with preference information between DHHFLEs mainly consider that different distance measures may have different importance degrees. Additionally, we usually utilize the distance and similarity measures to deal with discrete information, but the continuous DHHFLTSs are also common and it is necessary to develop the distance and similarity measures in continuous case. Furthermore, the weight of each DHHFLE included in the DHHFLTS mainly expresses the importance degree of each DHHFLE, so giving weight information into the distance and similarity measures between DHHFLTSs is reasonable and necessary. Finally, sometimes we need to change the original information into the ordering form for practical purposes, and the ordering information can make the weights of DHHFLEs more meaningful, so we develop the ordered weighted distance and similarity measures between the DHHFLTSs.

There still exist some potential weaknesses about the DHLTS and the DHHFLTSs. Firstly, we need to introduce some more reasonable expressions for the second hierarchy LTS in the future. Secondly, these distance and similarity measures are only small parts of these fields, so it is necessary to define some other distance and similarity measures when we face some special problems.

7.3 Consistency measures of DHHFLPRs

To avoid the occurrence of some self-contradictory situations, this thesis proposes the additive consistency measures for DHHFLPRs. Additionally, to compare the inconsistent DHHFLPRs and the additive consistent DHHFLPRs more intuitively, we further utilize the visual method “Figure of area”, which is a function of MATLAB drawing toolbar. Then we obtain Fig. 3. Based on the areas of different DHHFLPRs, the area that is more regular is clearly distinguished. For example, in Fig. 4(a) and Fig. 4(b), because the changes in the areas of different colors in Fig. 4(b) are more regular than the corresponding changes in Fig. 4(a), we consider that the additive consistent DHHFLPR is more regular with respect to the areas in different colors than the inconsistent DHHFLPR.

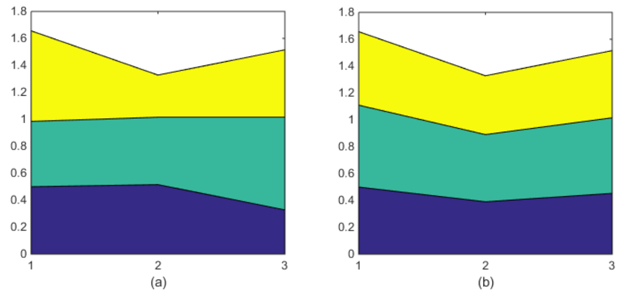

Fig. 4. The figures of area of the inconsistent DHHFLPR and the additive consistent DHHFLPR

Furthermore, the discussions about the consistency repairing methods are made as follows:

Firstly, we have discussed two different consistency repairing algorithms for the DHHFLPR with unacceptable consistency. The automatic optimization method mainly improves the DHHFLPR with unacceptable consistency by the adjusted parameter θ $(0 \leq \theta \leq 1)$. We can obtain different results if we take different values of θ. Additionally, the feedback improving method depends on the feedback mechanism, we do not change any information of the DHHFLPRs of unacceptable consistency but feed the information back to experts. They can decide whether to change the evaluation information or not, and then we can make a decision using the feedback information from the experts.

Additionally, these two methods have some advantages: For the automatic optimization method, we can obtain the decision making results very quickly because the improvement of the DHHFLPR of unacceptable consistency is automatic according to the adjusted parameter θ. Furthermore, MATLAB is utilized to do programming and it carries out the operation faster. For the feedback-based improving method, it is more in line with intelligent decision making considering that the decision makers' opinions have been given full consideration.

Finally, the application of the consistency checking and repairing methods is discussed, which is to deal with a practical group decision making problem which is to evaluate the water resources situation of some important cities in the Sichuan province.

7.4 Consensus reaching process for LSGDM with DHHFLPRs

In LSGDM problems with DHHFLPRs, we establish a consensus model to improve group consensus considering the characteristics of LSGDM. Then we apply the proposed model to a practical LSGDM problem that is to evaluate Sichuan water resource management, and we also make comparative analyses with some existing methods. Based on the consensus reaching processes and the decision making results discussed above, some analyses are summarized as follows:

Firstly, for the clustering method, we utilize the information entropy to cluster the experts. The main advantages are listed as follows: (1) By utilizing the rate of threshold change to determine the optimal classification threshold, our method can give a reasonable clustering for some experts with the high similarity degrees. (2) Our method can make the clustering process clearer by the dynamic description with a clustering figure.

Secondly, compared with other weights-determining methods, the proposed double hierarchy information entropy-based weights-determining method can be used to obtain three kinds of weight information: the weight of each group, the weights of experts included in each group, and the weights of all experts. Therefore, we have great flexibility to choose different weights when dealing with some particular problems. Additionally, this method is very simple and reasonable, so we can save lots of time in this stage.

Finally, in the consensus reaching process, we choose to do only one clustering process at the beginning of improving consensus degree. However, it is clear that the clustering may be changed when we finish every round of consensus degree improving. But our choices have an advantage: If we do not change the cluster result, the experts in each group can know each other better and then they can finish the consensus reaching process more efficiently. On the contrary, if we cluster the experts at each round, then the experts in every group need to know each other again and again, and this process will waste lots of time.

But, the consistency is not considered in the consensus reaching process in LSGDM, and we do not discuss the situation about the uncooperative experts. We will deeply discuss these issues in the future.

7.5 Managing minority opinions and non-cooperative behaviors in LSGDM with DHLPRs

This thesis develops a consensus model to manage minority opinions and non-cooperative behaviors in LSGDM with DHLPRs. The discussions about the results can be summarized from the following four parts:

Firstly, comparing with other studying of minority opinions and non-cooperative behaviors, this paper would be better to deal with non-cooperative behaviors and minority opinions simultaneously in the consensus reaching process of LSGDM with DHLPRs by proposing novel cluster method and consensus model.

Secondly, considering that giving subjective factors into the cluster process may change the accuracy of cluster results, also it is better to draw a flow chart to reflect the cluster process. This thesis proposes a distance-based cluster method, which can not only reflect the relation between any two DHLPRs, but also describe the clustering process more detailed and intuitively by a flow chart. Additionally, the proposed cluster method is only based on the original preferences and there exist no any subjective factors in the process. Furthermore, unlike similarity measure, using distance measure to establish the cluster methods can simplify some unnecessary processes.

Thirdly, the weight adjustment method is very important when dealing with the non-cooperative behaviors. This thesis develops a novel non-cooperative degree-based staircase weight adjustment function by dividing non-cooperative degrees into some more intervals, which makes the non-cooperative degree more in detail.

Fourthly, the comprehensive adjustment coefficient is vital in the consensus research process. If only utilizing the subjective adjustment coefficient and supposing that the subjective adjustment coefficient is very small in each round, then the number of iterations will be very big. If only considering the objective adjustment coefficient and neglecting the subjective adjustment coefficient, then the arbitrariness and uncertainty of subjective revision will be reduced, and the experts' own adjustment coefficients will not be brought to the forefront. Therefore, this situation will violate the original intention of LSGDM. With the comprehensive adjustment coefficient, all shortcomings can be overcome and the consensus research process will be more reasonable.

8 Concluding remarks

In this section, the results obtained from the research carried out during this PhD dissertation are presented, and they follow the goals of studying the double hierarchy linguistic preference information: consistency, consensus and large scale group decision making. This study has defined the concepts of DHLTS and DHHFLTS, and has discussed a series of distance and similarity measures of DHHFLEs and DHHFLTSs. Under double hierarchy hesitant fuzzy linguistic preference information, the consistency consistency-driven optimization-based models have been set up, as well as the consensus reaching method has been proposed to deal with LSGDM problem. Finally, a consensus model has been established to manage minority opinions and non-cooperative behaviors in LSGDM with DHLPRs.

As we mentioned above, the first objective is to introduce the concepts of DHLTS and DHHFLTS. We have proposed the concepts of DHLTS and DHHFLTS and explained them by lots of figures, as well as given the concept of the envelope of DHHFLE for understanding them better. In addition, we have set up two pairs of equivalent transformation functions, which can simplify the computations of original linguistic terms by transforming them into numerical scales, and do not change the essence of them by transforming the results into double hierarchy linguistic information with the anti-function. Then, some operations of DHHFLEs have been proposed based on the equivalent transformation functions.

Then, based on the equivalent transformation functions, we have proposed some distance and similarity measures of the DHHFLEs and the DHHFLTSs from different angles. Furthermore, we have developed a decision making method to solve MCDM problems on the basis of these distance and similarity measures. Moreover, we have applied this method to deal with a practical MCDM problem about Sichuan liquor brand assessment.

To represent some complex linguistic preference information, we have defined the concept of DHHFLPR and developed some consistency measures. Then, utilizing the linguistic expected-value of DHHFLE, we have proposed a new normalization method to transform a DHHFLPR into the corresponding normalized DHHFLPR equivalently. Additionally, for the purpose of judging whether a DHHFLPR is of acceptable consistency or not, we have defined a consistency index of the DHHFLPR and developed a novel method to improve the existing method for calculating the consistency thresholds. Two convergent consistency repairing algorithms based on the automatic improving method and the feedback improving method have been developed respectively to improve the consistency index of a given DHHFLPR of unacceptable consistency. Finally, we have proposed a weight-determining method and developed an algorithm to deal with the group decision making problem with double hierarchy hesitant fuzzy linguistic preference information. We have applied our method to deal with a practical group decision making problem involving the evaluation of the water resource situations of some important cities in Sichuan Province.

To deal with LSGDM problems, we have discussed the consensus reaching processes for LSGDM with DHHFLPRs. To ensure the implementation of consensus reaching process, we have also proposed the similarity degree-based clustering method, the double hierarchy information entropy-based weights-determining method and the consensus measures. Based on the similarity measures of DHHFLTSs, we have developed a clustering method for LSGDM based on information entropy theory, which can be understood very clearly by a dynamic clustering figure. Additionally, we have proposed a weights-determining method, which can obtain the weight of each small group, the weights of the experts included in each small group, and the weights of all experts, respectively. Furthermore, we have proposed some consensus measures and developed a model which can precisely identify the alternatives, the pairs of alternatives and the experts that do not reach the consensus threshold. And then the moderator can feed these suggestions back to each small group and experts for modifying their preference information. Finally, every group can discuss and change the corresponding preference information and the consensus reaching process is over when the overall consensus degree is bigger than the given consensus threshold value.

Similarly, in LSGDM under double hierarchy linguistic preference environment, we have established a consensus model to manage minority opinions and non-cooperative behaviors. A double hierarchy linguistic distance-based cluster method, a weights-determining method, and a consensus model for LSGDM have been developed. Additionally, this thesis has given a consensus reaching process in LSGDM which consists of the determination of comprehensive adjustment coefficient, and two methods for managing minority opinions and non-cooperative behaviors. Based on which, an algorithm for LSGDM with minority opinions and non-cooperative behaviors have been established with these proposed methods and models. Furthermore, the algorithm has been applied to a practical case study that is to determine the most main reason of haze formation in a city of China, and some comparative analyses have been made in detail.

Conclusiones

En esta sección, se presentan los resultados obtenidos de la investigación realizada durante esta tesis doctoral siguiendo los objetivos del estudio de la información de preferencia lingüística de doble jerarquía: coherencia, consenso y toma de decisiones grupales a gran escala. Este estudio ha definido los conceptos de DHLTS y DHHFLTS, y ha analizado una serie de medidas de distancia y similitud de DHHFLE y DHHFLTS. En la información de preferencia lingüística difusa dudosa de la doble jerarquía, se han establecido los modelos basados en la optimización basados en la coherencia, y se ha propuesto el método de consenso para tratar el problema LSGDM. Finalmente, se ha establecido un modelo de consenso para gestionar opiniones de minorías y comportamientos no cooperativos en LSGDM con DHLPR.

Como mencionamos anteriormente, el primer objetivo es introducir los conceptos de DHLTS y DHHFLTS. Hemos propuesto los conceptos de DHLTS y DHHFLTS y los hemos explicado apoyándonos en varios esquemas, así como el concepto de envoltura de DHHFLE para comprenderlos mejor. Además, hemos establecido dos pares de funciones de transformación equivalentes, que pueden simplificar los cálculos de los términos lingüísticos originales al transformarlos en escalas numéricas, y no cambiar su esencia al transformar los resultados en información lingüística de doble jerarquía mediante la función inversa. Luego, se han propuesto algunas operaciones de DHHFLE basadas en las funciones de transformación equivalentes.

Posteriormente, basandonos en las funciones de transformación equivalentes, hemos propuesto algunas medidas de distancia y similitud de los DHHFLE y los DHHFLTS desde diferentes perspectivas. Además, hemos desarrollado un método de toma de decisiones para resolver problemas de MCDM sobre la base de estas medidas de distancia y similitud. Asimismo, hemos aplicado este método para tratar un problema práctico de MCDM sobre la evaluación de la marca de licor de Sichuan.

Para representar alguna información compleja de preferencias lingüísticas, hemos definido el concepto de DHHFLPR y hemos desarrollado algunas medidas de coherencia. Luego, utilizando el valor lingüístico esperado de DHHFLE, hemos propuesto un nuevo método de normalización para transformar una DHHFLPR en el DHHFLPR normalizado correspondiente de manera equivalente. Además, con el fin de evaluar si una DHHFLPR tiene una consistencia aceptable o no, hemos definido un índice de consistencia de la DHHFLPR y hemos desarrollado un método novedoso para mejorar el método existente para calcular los umbrales de consistencia. Se han desarrollado dos algoritmos de reparación de consistencia convergente basados en el método de mejora automática y el método de mejora de retroalimentación, respectivamente, para mejorar el índice de consistencia de un DHHFLPR determinado de consistencia inaceptable. Finalmente, hemos mostrado un método de determinación de peso y desarrollamos un algoritmo para tratar el problema de toma de decisiones grupales con información de preferencia lingüística difusa dudosa de doble jerarquía. Hemos aplicado nuestro método para enfrentar un problema práctico de toma de decisiones grupales que involucra la evaluación de las situaciones de recursos hídricos de algunas ciudades importantes en la provincia de Sichuan.

Para lidiar con los problemas de LSGDM, hemos discutido los procesos de consenso para LSGDM con DHHFLPRs. Para garantizar la implementación del proceso de consenso, también hemos propuesto el método de agrupamiento basado en grado de similitud, el método de determinación de ponderaciones basado en la entropía de información de doble jerarquía y las medidas de consenso. Sobre la base de las medidas de similitud de los DHHFLTS, hemos desarrollado un método de agrupamiento para LSGDM basado en la teoría de la entropía de la información, que se puede entender muy claramente mediante una figura dinámica de agrupamiento. Además, hemos propuesto un método de determinación de pesos, que puede obtener el peso de cada grupo pequeño, los pesos de los expertos incluidos en cada grupo pequeño y los pesos de todos los expertos, respectivamente. Además, hemos propuesto algunas medidas de consenso y hemos desarrollado un modelo que puede identificar con precisión las alternativas, los pares de alternativas y los expertos que no alcanzan el umbral de consenso. Posteriormente el moderador puede enviar estas sugerencias a cada grupo pequeño y expertos para modificar la información de sus preferencias. Finalmente, cada grupo puede discutir y cambiar la información de preferencia correspondiente y el proceso de llegar a un consenso finaliza cuando el grado de consenso general es mayor que el valor umbral de consenso dado.

De manera similar, en LSGDM bajo un entorno de preferencia lingüística de doble jerarquía, hemos establecido un modelo de consenso para gestionar las opiniones de las minorías y los comportamientos no cooperativos. Se ha desarrollado un método de agrupamiento basado en la distancia lingüística de doble jerarquía, un método de determinación de pesos y un modelo de consenso para LSGDM. Además, este documento ha brindado un proceso de consenso en LSGDM que consiste en la determinación del coeficiente de ajuste integral y dos métodos para manejar las opiniones de las minorías y los comportamientos no cooperativos. Sobre dicha base, se ha establecido un algoritmo para LSGDM con opiniones minoritarias y comportamientos no cooperativos con estos métodos y modelos propuestos. Además, el algoritmo se ha aplicado a un caso de estudio práctico que consiste en determinar la razón principal de la formación de neblina en una ciudad de China, y se han realizado algunos análisis comparativos en detalle.

9 Future works

Along with the research of this thesis, some new and interesting topics about the double hierarchy linguistic model emerge including novel consistency-driven optimization-based models, consensus reaching methods, and linguistic representation models. In what follows, these interesting research topics are introduced which are also the future directions of our investigation.

9.1 Novel consistency-driven optimization-based models

In this thesis, we have proposed the additive consistency of DHHFLPR, and two convergent consistency repairing algorithms based on the automatic improving method and the feedback improving method have been developed respectively to improve the consistency index of DHHFLPR.

However, except the additive consistency, two novel consistency models of DHHFLPR need to be considered including the multiplicative consistency and the interval consistency.

(1) Multiplicative consistency. In existing research, scholars are more inclined to utilize multiplicative transitivity considering that it is a special case of the cycle transitivity property [BMSJ06]. In addition, amounts of scholars have proved that the multiplicative transitivity is the most appropriate property for modeling cardinal consistency of preference relations because it can avoid some gaps such as the conflict with the given range used for providing the preference values [CHVAH09]. Therefore, in the future, we will focus on investigating the multiplicative consistency of DHHFLPRs and develop a concept of acceptable multiplicative consistent DHHFLPR. Then, we are going to propose a consistency checking method to judge whether a DHHFLPR is of acceptable consistency and develop a repairing method to improve the consistency of a DHHFLPR.

(2) In general, we can only obtain partial result about the consistency index of DHHFLPR, and the result is related to the parameter used to obtain the normalized DHHFLPR. To understand the consistency degree of DHHFLPR more comprehensively, we will develop an interval consistency index (ICI) of DHHFLPR which can consist of all possible consistency indices of a DHHFLPR.

9.2 Probabilistic double hierarchy linguistic term set

In this thesis, we have proposed the concepts of DHLTS and its extension in hesitant fuzzy environment named as DHHFLTS to express complex linguistic information by combining two hierarchy LTSs with 2-tuple linguistic structure. However, in decision making process, assessment information provided by experts or aggregation results may be usually represented by some possible DHHFLEs or some DHHFLEs with probability information, and these probabilities are essential to describe the real thoughts of decision makers. So we cannot ignore them optionally when representing them directly or aggregating some decision makers' assessments.

Noticing that representing probabilities information is a new improvement and challenge for DHHFLTS, in near future we will define a novel and more general concept called probabilistic double hierarchy linguistic term set. In addition, we will also develop a method to adjust any two probabilistic double hierarchy linguistic elements and make sure that they have same probability distribution. Based on the adjusted probabilistic double hierarchy linguistic elements, some operations and a distance measure of probabilistic double hierarchy linguistic elements will be defined. Moreover, we are going to develop some novel decision making methods to deal with some MCDM problems with probabilistic double hierarchy linguistic information. Finally, we will also discuss the PDHLTSs under preference information environment and develop a probabilistic double hierarchy linguistic preference relation, and discuss the consistency checking and repairing models, the consensus reaching methods, and the applications in LSGDM.

9.3 Self-confident double hierarchy linguistic preference relations

In recent years, a novel preference relation has been developed Liu et al. [LDC+17], which considers the self-confident degrees of the basic elements of the preference relation. The self-confident degrees can be used to depict the degrees of confidences that experts have in their own evaluation information, as well as enrich the integrity of evaluation information. Additionally, the basic elements DHLTs of DHLPR are only some linguistic expressions and cannot reflect the self-confident degrees of experts. Considering that there is little research about the DHLPRs with self-confident degrees in literature, and the experts' self-confident degrees in DHLPR have to be perfected. Motivated by the research of Liu et al. [LDC+17], it is necessary to define a concept of self-confident DHLPR and develop a double hierarchy linguistic preference values and self-confident degrees Modifying (DHSM)-based consensus model to manage GDM problems with self-confident DHLPRs based on the priority ordering theory. This research consists of the following aspects:

(1) In different decision making areas, experts are various and each of them has different specialized knowledge or influence. Therefore, given each expert reasonable weight is very important in GDM. Therefore, we will fully consider all kinds of information and obtains the weight vector of experts including the subjective weights and two kinds of objective weights. Firstly, experts can evaluate themselves where the evaluation values can be regarded as the subjective weights of them; Additionally, each expert can be evaluated by the remaining experts and one kind of objective weights are obtained; Moreover, the evaluation matrix provided by each expert can be utilized to calculate the other kind of objective weights. Finally, the synthetic weights of experts can be obtained by combining all of these three weights.

(2) In the process of GDM with preference relation, the elements of priority vector reflect the importance degrees of the corresponding alternatives, and the difference between the individual priority vector and the collective priority vector represents the proximity degree of an expert's preference and group's preference. Therefore, obtaining the individual priority vector and the collective priority vector are very important to reach consensus and make decision. Based on this, in the consensus reaching process, we will develop two models to calculate the individual priority vector of each expert and the collective priority vector of all experts. These two priority vectors cannot only be used to judge whether all experts reach consensus, but also be used to obtain the ranking of all alternatives.

(3) We hope that the consensus can be reached as soon as possible, and the adjustment rounds are as small as possible. In this regard, three comparison criteria will be proposed to reflect the consensus efficiency of the proposed DHSM-based consensus model, including the number of iteration, the consensus success ratio and the distance between the original and the adjusted preference information. Motivated by the analyses above, a simulation experiment is devised to testify the proposed DHSM-based consensus model by comparing it with two other consensus reaching models: One is the DHLPR without the self-confident degrees; the other is that the self-confident degrees are not changed in the consensus reaching process.

Chapter II

Publications: Published Papers

1 Double hierarchy hesitant fuzzy linguistic term set and MULTIMOORA method: a case of study to evaluate the implementation status of haze controlling measures

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Double Hierarchy Hesitant Fuzzy Linguistic Term Set and MULTIMOORA Method: A Case of Study to Evaluate the Implementation Status of Haze Controlling Measures

Xunjie Gou[a], Huchang Liao[a], Zeshui Xu[a,*], Francisco Herrera[b,c]

[a] *Business School, Sichuan University, Chengdu 610064, China*

[b] *Department of Computer Science and Artificial Intelligence, University of Granada, E-18071 Granada, Spain*

[c] *Faculty of Computing and Information Technology, King Abdulaziz University, Jeddah, Saudi Arabia*

Abstract

In recent years, hesitant fuzzy linguistic term sets (HFLTSs) have been studied by many scholars and are becoming gradually mature. However, some shortcomings of HFLTS also emerged. To describe the complex linguistic terms or linguistic term sets more accurately and reasonably, in this paper, we introduce the novel concepts named double hierarchy linguistic term set (DHLTS) and double hierarchy hesitant fuzzy linguistic term set (DHHFLTS). The operational laws and properties of the DHHFLTSs are developed as well.

Afterwards, we investigate the multiple criteria decision making model with double hierarchy hesitant fuzzy linguistic information. We develop a double hierarchy hesitant fuzzy linguistic MULTIMOORA (DHHFL-MULTIMOORA) method to solve it. Furthermore, we apply the DHHFL-MULTIMOORA method to deal with a practical case about selecting the optimal city in China by evaluating the implementation status of haze controlling measures. Some comparisons between the DHHFL-MULTIMOORA method and the hesitant fuzzy linguistic TOPSIS method are provided to show the advantages of the proposed method.

Keywords: Multiple criteria decision making; Double hierarchy linguistic term sets, Double hierarchy hesitant fuzzy linguistic term sets; MULTIMOORA method; Haze controlling measures

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* Corresponding Author. Emails: X.J. Gou (gouxunjie@qq.com); H.C. Liao (liaohuchang@163.com); Z.S. Xu (xuzeshui@263.net); F. Herrera (herrera@decsai.ugr.es).

1. Introduction

Hesitant fuzzy linguistic term set (HFLTS), combined by hesitant fuzzy set (HFS) [1-4] and fuzzy linguistic approach [5], was developed by Rodríguez et al. [6] in 2012. It is a useful tool to deal with qualitative information given that the HFLTS can represent the linguistic information that is much more in line with people's cognitions and expressions. In recent years, amounts of scholars have researched the HFLTS theory from different research directions including information aggregation [7-9], fuzzy measures [10-14], preference relations [13,15-17], decision making [12-14,18-23], etc.

As the researches on HFLTSs have been studied in-depth and the HFLTS theory is becoming gradually mature, some shortcomings of HFLTSs, however, also emerged from two aspects:

a) In group decision making process, the aggregated hesitant fuzzy linguistic information cannot represent the important degree or the frequency of each linguistic term included in the HFLTS.
b) The HFLTS is not accurate enough to describe some more complex linguistic terms or linguistic term sets (LTSs).

For the first shortcoming, Pang et al. [24] defined a probability linguistic term set (PLTS) to generalize the HFLTSs by adding the probability information of each single linguistic term, which is a very reasonable method for saving all original linguistic information given by the experts in group decision making process. Furthermore, by utilizing the PLTSs, the experts can not only provide several linguistic evaluation values over an object (alternative or criterion), but also reflect the probability information of each element included in the LTS. Later, some scholars have studied the PLTSs from different aspects, among others: probabilistic linguistic preference relation and consistency measures [25], probabilistic linguistic vector-term sets to promote the application of multi-granular linguistic information [26], comparative procedure-based multiple criteria decision making (MCDM) problems [27], novel operational laws of PLTSs based on two equivalent transformation functions [28].

For the second shortcoming, it is obvious that sometimes the HFLTS cannot describe some complex linguistic terms or LTSs accurately. For example, let $S=\{s_{-3}=none, s_{-2}=very\ low, s_{-1}=low, s_0=medium, s_1=high, s_2=very\ high, s_3=perfect\}$ be a LTS, then we can utilize $\{s_2,s_3\}$, $\{s_{-1},s_0,s_1\}$ and $\{s_2\}$ to express the linguistic expressions "*more than very high*", "*between low and high*" and "*very high*". However, sometimes, we may need to use some more complex linguistic terms to represent our comprehensive opinions such that "*entirely high*", "*just right medium*", "*a little high*", etc. Considering that we cannot use any method or theory to solve this problem, in this paper, we introduce a novel concept: double hierarchy linguistic term set (DHLTS). Generally, the DHLTS consists of two hierarchy LTSs (denoted by the first hierarchy LTS and the second hierarchy LTS). The second hierarchy LTS is a linguistic feature or detailed supplementary of each linguistic term included in the first hierarchy LTS. Let the above LTS S be the first hierarchy LTS, and $O=$ $\{o_{-3}= far\ \ from, o_{-2}= only\ a\ little, o_{-1}= a\ little, o_0 = just\ right, o_1 = much, o_2 = very\ much, o_3 = entirely\}$ be the second hierarchy LTS. Then we can describe "*entirely high*", "*just right medium*", "*a little high*" with DHLEs (the element included in the DHLTS), which are denoted as $s_{1<o_3>}$, $s_{0<o_0>}$ and $s_{1<o_{-1}>}$, respectively. Based on the DHLTS, we can develop a double hierarchy hesitant fuzzy linguistic term set (DHHFLTS). The DHHFLTS is a novel concept, which can be used to deal with some practical MCDM problems with linguistic information.

MCDM is one of the most important branches in decision analysis theory and many fruitful results and models have been achieved related to this area. Among the widely used MCDM methodologies, the multiple multi-objective optimization by ratio analysis (MULTIMOORA) method and its extensions have been investigated by many scholars [29-40]. As an effective and comprehensive method, it combines three aspects including the ratio system, the reference point, and the full multiplicative form. The MULTIMOORA method and its extended forms have been applied to many fields such as transition economies [29], human resource management and performance management [30], EU Member States updating management [31], heating losses ranking in a building [32], supplier selection [34] and so on.

In this paper, we mainly develop a double hierarchy hesitant fuzzy linguistic MULTIMOORA (DHHFL-MULTIMOORA) method to deal with practical MCDM problems. We apply the DHHFL-MULTIMOORA method to a case of selecting the best city in China by evaluating the implementation status of haze controlling measures. Some comparisons between the DHHFL-MULTIMOORA method and the hesitant fuzzy linguistic TOPSIS method are provided to show the advantages of the proposed method.

The highlights of this paper are summarized as follows:

(1) We define the DHLTS and the DHHFLTS, both of them can be used to describe the linguistic information more accurately.

(2) The DHHFL-MULTIMOORA method with double hierarchy hesitant fuzzy linguistic information, developed in this paper, can comprehensively consider three aspects' information, which ensures the decision making result much more convincing.

(3) This paper mainly solves a practical MCDM problem, which is to select the optimal city in

China by evaluating the implementation status of haze controlling measures.

The rest of this paper are organized as follows: We review some concepts and operational laws of HFLTSs in Section 2. In Section 3, we propose the concepts of DHLTS and DHHFLTS, the basic components of which can be denoted as double hierarchy linguistic terms (DHLTs) and double hierarchy hesitant fuzzy linguistic elements (DHHFLEs), respectively. Then two equivalent transformation functions between the DHLTs (DHHFLEs) and the evaluations in [0,1] (HFE) are established. Furthermore, some basic operational laws and properties of DHHFLEs are developed in this section. In Section 4, we first propose a MCDM model with double hierarchy hesitant fuzzy linguistic information, and then develop a novel DHHFL-MULTIMOORA method. In Section 5, we apply the DHHFL-MULTIMOORA method to deal with a practical case about selecting the best city in China by evaluating the implementation status of haze controlling measures. Moreover, we make some comparisons between the DHHFL-MULTIMOORA method and the hesitant fuzzy linguistic TOPSIS method. Finally, we finish this paper with some concluding remarks and future research directions in Section 6.

2. Hesitant Fuzzy Linguistic Terms Set: Concept and Operational Laws

In 2010, Torra [1] proposed the concept of HFS on X as a function that when applied to X returns a subset of [0,1]. To be easily understood, Xia and Xu [34] expressed the HFS by a mathematical symbol $A=\left\{<x,h_A(x)>\middle|x\in X\right\}$ where $h_A(x)$ is a set of some values in [0,1], denoting the possible membership degrees of the element $x\in X$ to the set A. Additionally, $h=h_A(x)$ can be called a hesitant fuzzy element (HFE) and Θ being the set of all HFEs.

In 2012, Rodríguez et al. [6] defined the concept of HFLTS as an ordered finite subset of the consecutive linguistic terms of a given LTS. Soon afterwards, Liao et al. [13] extended and formalized it mathematically as follows:

Definition 2.1 [13]**.** Let $x_i\in X\ (i=1,2,\ldots,N)$ be fixed and $S=\left\{s_t\middle|t=-\tau,\ldots,-1,0,1,\ldots,\tau\right\}$ be a LTS. A HFLTS on X, H_S, is in mathematical form of $H_S=\left\{<x_i,h_S(x_i)>\middle|x_i\in X\right\}$, where $h_S(x_i)$ is a set of some values in S and can be expressed as:

$$h_S(x_i)=\left\{s_{\phi_l}(x_i)\middle|s_{\phi_l}(x_i)\in S;l=1,\ldots,L;\phi_l\in\{-\tau,\ldots,-1,0,1,\ldots,\tau\}\right\}$$

with L being the number of linguistic terms in $h_S(x_i)$ and $s_{\phi_l}(x_i)$ ($l=1,\ldots,L$) in each $h_S(x_i)$

being the continuous terms in S. $h_S(x_i)$ denotes the possible degree of the linguistic variable x_i to S. For convenience, $h_S(x_i)$ is called a hesitant fuzzy linguistic element (HFLE) and Φ being the set of all HFLEs.

Remark 2.1. Note that, in Definition 2.1, the linguistic terms are chosen in discrete form from S and the subscripts of $s_{\phi_l}(x_i)$, ϕ_l, belong to $\{-\tau,\ldots,-1,0,1,\ldots,\tau\}$. In order not to lose much information, there are two well known approaches to extend it to continuous form by using an interval to represent the lateral displacement between two adjacent labels, they are the 2-tuple linguistic model [41] and the linguistic alphabet [42]. In this way, we consider from now on the extension $\phi_l \in [-\tau,\tau]$, which is much general and flexible [42].

Besides, to make the operations of HFLTSs more reasonable, Gou and Xu [7] developed two equivalent transformation functions between the considered interval and the unit interval. Below we improve the definition between the transformation functions between the HFLE and the HFE.

Definition 2.2. Let $S=\{s_t \mid t=-\tau,\ldots,-1,0,1,\ldots,\tau\}$ be a LTS, $h_S=\{s_{\phi_l} \mid s_{\phi_l}\in S; l=1,\ldots,L; \phi_l\in[-\tau,\tau]\}$ be a HFLE with L being the number of linguistic terms in h_S, and $h_\sigma=\{\sigma_l \mid \sigma_l\in[0,1]; l=1,\ldots,L\}$ be a HFE. Then the membership degree σ_l and the subscript ϕ_l of the linguistic term s_{ϕ_l} that expresses the equivalent information to the membership degree σ_l can be transformed to each other by the following functions g and g^{-1}, respectively:

$$g:[-\tau,\tau]\to[0,1],\ g(\phi_l)=\frac{\phi_l+\tau}{2\tau}=\sigma_l \tag{1}$$

$$g^{-1}:[0,1]\to[-\tau,\tau],\ g^{-1}(\sigma_l)=(2\sigma_l-1)\tau=\phi_l \tag{2}$$

Based on Definition 2.2, we can introduce the transformation functions between the HFLE h_S and the HFE h_σ.

Definition 2.3. The transformation functions between the HFE $h_\sigma=\{\sigma_l \mid \sigma_l\in[0,1]; l=1,\ldots,L\}$ and the linguistic HFLE $h_S=\{s_{\phi_l} \mid s_{\phi_l}\in S; l=1,\ldots,L; \phi_l\in[-\tau,\tau]\}$ are given, respectively, as follows:

$$G:\Phi\to\Theta,\ G(h_S)=G\left(\{s_{\phi_l} \mid s_{\phi_l}\in S; l=1,\ldots,L; \phi_l\in[-\tau,\tau]\}\right)=\{\sigma_l \mid \sigma_l=g(\phi_l)\}=h_\sigma \tag{3}$$

$$G^{-1}:\Theta\to\Phi,\ G^{-1}(h_\sigma)=G^{-1}\left(\{\sigma_l \mid \sigma_l\in[0,1]; l=1,\ldots,L\}\right)=\{s_{\phi_l} \mid \phi_l=g^{-1}(\sigma_l)\}=h_S \tag{4}$$

Based on the functions G and G^{-1}, we can improve some operational laws for HFLEs.

Definition 2.4. Let $S=\left\{s_t \middle| t=-\tau,\dots,-1,0,1,\dots,\tau\right\}$ be a LTS, $h_S=\left\{s_{\phi_l} \middle| s_{\phi_l}\in S; l=1,\dots,L;\right.$ $\left.\phi_l\in[-\tau,\tau]\right\}$; $h_{S_1}=\left\{s_{\phi_l}^1 \middle| s_{\phi_l}^1\in S; l=1,\dots,L_1;\phi_l\in[-\tau,\tau]\right\}$ and $h_{S_2}=\left\{s_{\phi_l}^2 \middle| s_{\phi_l}^2\in S; l=1,2,\dots,L_2;\phi_l\in[-\tau,\tau]\right\}$ be three HFLEs (L, L_1 and L_2 are the numbers of linguistic terms included in the three HFLEs, respectively); G and G^{-1} be the equivalent transformation functions of HFLEs and HFEs, and λ be a real number. Then

(1) $h_{S_1}\oplus h_{S_2}=G^{-1}\left(\bigcup_{\sigma_1\in G(h_{S_1}),\sigma_2\in G(h_{S_2})}\left\{\sigma_1+\sigma_2-\sigma_1\sigma_2\right\}\right)$;

(2) $h_{S_1}\otimes h_{S_2}=G^{-1}\left(\bigcup_{\sigma_1\in G(h_{S_1}),\sigma_2\in G(h_{S_2})}\left\{\sigma_1\sigma_2\right\}\right)$;

(3) $h_{S_1}\ominus h_{S_2}=G^{-1}\left(\bigcup_{\sigma_1\in G(h_{S_1}),\sigma_2\in G(h_{S_2})}\left\{\theta\right\}\right)$, where $\theta=\begin{cases}\dfrac{\sigma_1-\sigma_2}{1-\sigma_2}, & \text{if } \sigma_1\geq\sigma_2 \text{ and } \sigma_2\neq 1\\ 0, & \text{otherwise}\end{cases}$;

(4) $h_{S_1}\oslash h_{S_2}=G^{-1}\left(\bigcup_{\sigma_1\in G(h_{S_1}),\sigma_2\in G(h_{S_2})}\left\{\theta\right\}\right)$, where $\theta=\begin{cases}\sigma_1/\sigma_2, & \text{if } \sigma_1\leq\sigma_2 \text{ and } \sigma_2\neq 0\\ 1, & \text{otherwise}\end{cases}$;

(5) $\overline{h_S}=G^{-1}\left(\bigcup_{\sigma\in G(h_S)}\left\{1-\sigma\right\}\right)$.

3. Double Hierarchy Linguistic Term Set and Double Hierarchy Hesitant Fuzzy Linguistic Term Set

In this section, we mainly define the concept of DHLTS, and then apply it to express hesitant fuzzy information and develop the DHHFLTS. Additionally, some operational laws and properties are proposed.

3.1. Double Hierarchy Linguistic Term Set

As we discussed in Section 1, the HFLTS can be used to express the evaluation information for an event or a decision making problem such as "*fast*", "*more*", "*between high and perfect*", etc. However, when we need to describe some more detailed sentences like "*a little fast*", "*almost 90% perfect*", and "*between much high and very high*", the HFLTS cannot describe them accurately and in detail. Therefore, we define a double hierarchy linguistic term set (DHLTS) firstly, which consists of two hierarchy fully independent LTSs. For example, let $S=\{s_t|t=-\tau,\ldots,-1,0,1,\ldots,\tau\}$ and $O=\{o_k|k=-\varsigma,\ldots,-1,0,1,\ldots,\varsigma\}$ be the first hierarchy and second hierarchy LTS, respectively.

Remark 3.1. If we let $\tau=\varsigma=3$, then these two LTSs can be denoted as:

$$S=\{s_{-3}=none, s_{-2}=very\ low, s_{-1}=low, s_0=medium, s_1=high, s_2=very\ high, s_3=perfect\}$$

$$O=\{o_{-3}=far\ \ from, o_{-2}=only\ a\ little, o_{-1}=a\ little, o_0=just\ right, o_1=much, o_2=very\ much, o_3=entirely\}$$

The LTS O indicates a linguistic feature or detailed supplementary of each linguistic term included in the LTS S. Fig. 1 shows the second hierarchy LTS O with respect to the linguistic term s_1 (*high*).

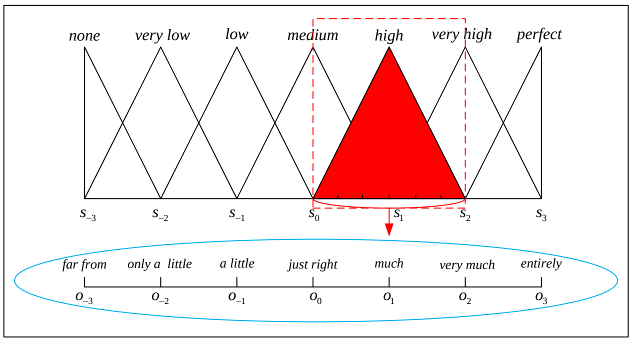

Fig. 1. The second hierarchy LTS of s_1 (*high*).

In Fig. 1, we can utilize any one linguistic term included in the second hierarchy LTS O to describe the linguistic term s_1 (*high*). For example, we can use "*only a little high*" and "*much high*" to express the different meanings of the "*high*". Obviously, the description is more correct and detailed.

Based on the analyses above, we can give the concept of DHLTS as follows:

Definition 3.1. Let $S=\{s_t|t=-\tau,\ldots,-1,0,1,\ldots,\tau\}$ and $O=\{o_k|k=-\varsigma,\ldots,-1,0,1,\ldots,\varsigma\}$ be the first hierarchy and second hierarchy LTS, respectively, and they are fully independent. A double hierarchy linguistic term set (DHLTS), S_O, is in mathematical form of

$$S_O=\{s_{t<o_k>}|t=-\tau,\ldots,-1,0,1,\ldots,\tau; k=-\varsigma,\ldots,-1,0,1,\ldots,\varsigma\} \quad (5)$$

we call $s_{t<o_k>}$ the double hierarchy linguistic term (DHLT), where o_k expresses the second hierarchy linguistic term when the first hierarchy linguistic term is s_t.

Besides, several details about the selections of the second hierarchy LTSs need to be further explained on the basis of the value of t.

(1) For the first hierarchy LTS $S=\{s_t | t=-\tau,\ldots,-1,0,1,\ldots,\tau\}$, if $t\geq 0$, then the second hierarchy LTS needs to be described in ascending order just like Fig. 1. On the contrary, if $t\leq 0$, then the second hierarchy LTS needs to be described in descending order. Moreover, we only change the orders of linguistic information, and do not change the orders of the linguistic terms o_k $(k=-\varsigma,\ldots,-1,0,1,\ldots,\varsigma)$. For example, suppose that $\varsigma=3$, then we let

$O^+=\{o_{-3}=\textit{far from}, o_{-2}=\textit{only a little}, o_{-1}=\textit{a little}, o_0=\textit{just right}, o_1=\textit{much}, o_2=\textit{very much}, o_3=\textit{entirely}\}, \textit{if } t\geq 0.$

and

$O^-=\{o_{-3}=\textit{entirely}, o_{-2}=\textit{very much}, o_{-1}=\textit{much}, o_0=\textit{just right}, o_1=\textit{a little}, o_2=\textit{only a little}, o_3=\textit{far from}\}, \textit{if } t\leq 0.$

be the second hierarchy LTSs, respectively.

(2) If $t=\tau$, then we only consider the front half of the second hierarchy LTS, i.e., $O=\{o_k | k=-\varsigma,\ldots,-1,0\}$. On the contrary, if $t=-\tau$, then we only consider the latter half of the second hierarchy LTS, i.e., $O=\{o_k | k=0,1,\ldots,\varsigma\}$.

Based on the discussions above, a figure can be drawn to show these situations, considering that we let $t=3$ and $\varsigma=2$:

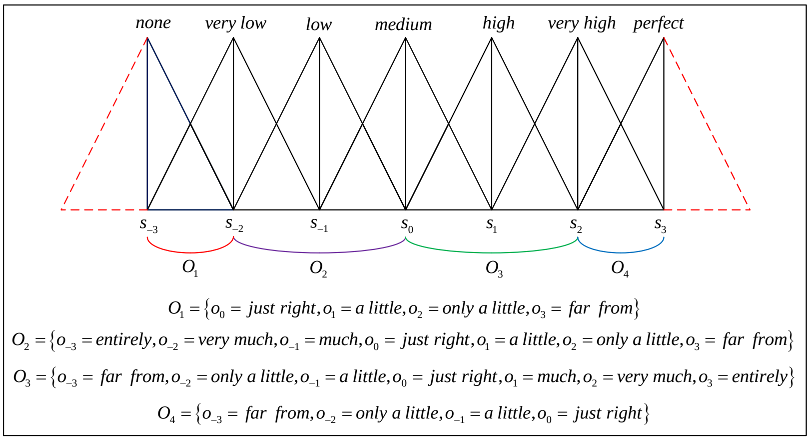

Fig. 2. The distributions of the four parts of the second hierarchy LTS.

Remark 3.2. In Fig. 2, four kinds of situations are shown on the basis of different values of t. If $t\geq 0$, then the meaning of the first hierarchy LTS $S=\{s_t | t\geq 0\}$ is positive, so the second hierarchy LTS needs to be selected with the ascending order. For example, $s_{1<o_{-2}>}$ (*only a little high*) and $s_{1<o_2>}$ (*very much high*) are two expressions of s_1, and the degree of the latter one is higher than the former. On the contrary, if $t\leq 0$, then the meaning of the first hierarchy LTS $S=\{s_t | t\leq 0\}$ is negative, so the second hierarchy LTS needs to be selected with the descending order. Specially, because both s_τ and $s_{-\tau}$ only contain a half of area compared to other linguistic terms. So we only utilize $O=\{o_k \mid k=-\varsigma,\ldots,-1,0\}$ and $O=\{o_k \mid k=0,1,\ldots,\varsigma\}$ to describe s_τ and $s_{-\tau}$, respectively.

3.2. Double Hierarchy Hesitant Fuzzy Linguistic Term Set

Obviously, we can only utilize the DHLTS S_O to express a single linguistic term. But the complex linguistic term cannot be expressed such as "*between only a little high and a little perfect*". Here we can develop S_O into hesitant fuzzy linguistic information. Then the DHHFLTS can be defined:

Definition 3.2. Let $S_O=\{s_{t<o_k>} \mid t=-\tau,\ldots,-1,0,1,\ldots,\tau;\, k=-\varsigma,\ldots,-1,0,1,\ldots,\varsigma\}$ be a DHLTS. A double hierarchy hesitant fuzzy linguistic term set (DHHFLTS) on X, H_{S_O}, is in mathematical form of

$$H_{S_O}=\{<x_i,h_{S_O}(x_i)> \mid x_i\in X\} \tag{6}$$

where $h_{S_O}(x_i)$ is a set of some values in S_O, denoted as:

$$h_{S_O}(x_i)=\{s_{\phi_l<o_{\varphi_l}>}(x_i) \mid s_{\phi_l<o_{\varphi_l}>}\in S_O;\, l=1,2,\ldots,L;\, \phi_l=-\tau,\ldots,-1,0,1,\ldots,\tau;\, \varphi_l=-\varsigma,\ldots,-1,0,1,\ldots,\varsigma\} \tag{7}$$

with L being the number of DHLTs in $h_{S_O}(x_i)$ and $s_{\phi_l<o_{\varphi_l}>}(x_i)$ ($l=1,\ldots,L$) in each $h_{S_O}(x_i)$ being the continuous terms in S_O. $h_{S_O}(x_i)$ denotes the possible degree of the linguistic variable x_i to S_O. For convenience, we call $h_{S_O}(x_i)$ the double hierarchy hesitant fuzzy linguistic element (DHHFLE), and $\Phi\times\Psi$ being the set of all DHHFLEs.

Next, we can understand the DHHFLTS much more clearly by the context-free grammar. Here we establish a context-free grammar $\aleph_{DHH}$, which generates some simple but rich linguistic expressions represented by DHHFLTSs.

Definition 3.3. Let $S_O=\{s_{t<o_k>} \mid t=-\tau,\ldots,-1,0,1,\ldots,\tau;\, k=-\varsigma,\ldots,-1,0,1,\ldots,\varsigma\}$ be a DHLTS, $\aleph_{DHH}$ be a context-free grammar. The element of $\aleph_{DHH}=\{\dot{V}_N,\dot{V}_T,\dot{I},\dot{P}\}$ can be defined as:

$\dot{V}_N=\{\langle\text{double hierarchy primary term}\rangle,\langle\text{double hierarchy composite term}\rangle,\langle\text{unary relation}\rangle,$
$\langle\text{binary relation}\rangle,\langle\text{conjunction}\rangle\}$

$\dot{V}_T=\{\text{less than; more than; between; and; } s_{-\tau},s_{1-\tau},\ldots,s_0,\ldots,s_{\tau-1},s_\tau; o_{-\varsigma},o_{1-\varsigma},\ldots,o_0,\ldots,o_{\varsigma-1},o_\varsigma\}$

$\dot{I}\in\dot{V}_N$.

For the context-free grammar $\aleph_{DHH}$, the production rules P can be defined as:

$$\dot{P} = \{ \dot{I} ::= \langle \text{double hierarchy primary term} \rangle \mid \langle \text{double hierarchy composite term} \rangle$$

$$\langle \text{double hierarchy composite term} \rangle ::= \langle \text{unary relation} \rangle \langle \text{double hierarchy primary term} \rangle \mid$$

$$\langle \text{binary relation} \rangle \langle \text{double hierarchy primary term} \rangle \langle \text{conjunction} \rangle \langle \text{double hierarchy primary term} \rangle$$

$$\langle \text{double hierarchy primary term} \rangle ::= s_{-\tau<o_{-\varsigma}>} \mid s_{-\tau<o_{-\varsigma+1}>} \mid \ldots \mid s_{\tau<o_{\varsigma-1}>} \mid s_{\tau<o_{\varsigma}>}$$

$$\langle \text{unary relation} \rangle ::= \text{less than} \mid \text{more than}$$

$$\langle \text{binary relation} \rangle ::= \text{between}$$

$$\langle \text{conjunction} \rangle ::= \text{and} \}.$$

Remark 3.3. (1) There exist some limitations about the "*unary relation*". The "*double hierarchy primary term*" cannot be $s_{-\tau<o_0>}$ if the nonterminal symbol is "*less than*". Similarly, the "*double hierarchy primary term*" cannot be $s_{\tau<o_0>}$ if the nonterminal symbol is "*more than*".

(2) For the "*binary relation*", the "*double hierarchy primary term*" on the left-hand side must be less than the "*double hierarchy primary term*" on the right-hand side.

To understand the DHHFLTS much better, here we introduce the concept of the envelope of a DHHFLE:

Definition 3.4. The envelope of a DHHFLE, $env(h_{S_O})$, is a double hierarchy linguistic interval whose limits are obtained by means of the upper bound (max) and the lower bound (min). That is

$$env(h_{S_O}) = [h_{S_O}^-, h_{S_O}^+] \tag{8}$$

which is just an uncertain linguistic variable [43].

The DHHFLE h_{S_O} contains all the elements from the lower bound $h_{S_O}^-$ to the upper bound $h_{S_O}^+$.

Example 3.1. Let $S_O = \{ s_{t<o_k>} \mid t = -\tau, \ldots, -1, 0, 1, \ldots, \tau; k = -\varsigma, \ldots, -1, 0, 1, \ldots, \varsigma \}$ be a DHLTS. Suppose that $\tau = \varsigma = 3$ and the linguistic labels are the same as those in Fig. 1. Three linguistic expressions are listed as:

(1) "*a little high*";

(2) "*between much medium and just right very high*";

(3) "*just right perfect*".

Then we can utilize the DHHFLEs $\{s_{1<o_{-1}>}\}$, $\{s_{0<o_1>}, s_1, s_{2<o_0>}\}$, and $\{s_{3<o_0>}\}$ to transform the above sentences. Besides, $env\left\{s_{0<o_1>}, s_1, s_{2<o_0>}\right\} = \left[s_{0<o_1>}, s_{2<o_0>}\right]$.

Remark 3.4. For the second linguistic expression "*between much medium and just right very high*", it contains all the linguistic terms from "*much medium*" to "*just right very high*". Therefore, we can utilize s_1 to represent the middle linguistic term without using the form of DHHFLE.

3.3. Some Operational Laws of DHHFLEs

Note that, in Definition 3.2, the DHLTs are chosen in discrete form from S_O and the value range of subscripts of $s_{\phi_l<o_{\varphi_l}>}(x_i)$ is $\{\phi_l = -\tau, \ldots, -1, 0, 1, \ldots, \tau; \varphi_l = -\varsigma, \ldots, -1, 0, 1, \ldots, \varsigma\}$. Similar to the continuous LTS, we can extend it to continuous form, i.e., $\phi_l \in [-\tau, \tau]$ and $\varphi_l \in [-\varsigma, \varsigma]$. Here we discuss two equivalent transformation functions for making the mutual transformations between the DHLT (DHHFLE) and the real number (HFE) before defining the operational laws of the DHHFLEs.

Definition 3.5. Let $S_O = \left\{s_{t<o_k>} \middle| t = -\tau, \ldots, -1, 0, 1, \ldots, \tau; k = -\varsigma, \ldots, -1, 0, 1, \ldots, \varsigma\right\}$ be a DHLTS. $h_{S_O} = \left\{s_{\phi_l<o_{\varphi_l}>} \middle| s_{\phi_l<o_{\varphi_l}>} \in S_O; l = 1, 2, \ldots, L; \phi_l = [-\tau, \tau]; \varphi_l = [-\varsigma, \varsigma]\right\}$ be a DHHFLE with L being the number of DHLTs in h_{S_O}, and $h_\gamma = \left\{\gamma_l \middle| \gamma_l \in [0,1]; l = 1, \ldots, L\right\}$ be a HFE. Then the membership degree γ_l and the subscript $\phi_l < \varphi_l >$ of the DHLT $s_{\phi_l<o_{\varphi_l}>}$ that expresses the equivalent information to the membership degree γ_l can be transformed to each other by the following functions f and f^{-1}, respectively:

$$f : [-\tau, \tau] \times [-\varsigma, \varsigma] \to [0,1],\ f(\phi_l, \varphi_l) = \begin{cases} \dfrac{1}{\tau} \times \dfrac{\varphi_l + \varsigma}{2\varsigma} + \dfrac{\tau + \phi_l - 1}{2\tau} = \dfrac{\varphi_l + (\tau + \phi_l)\varsigma}{2\varsigma\tau} = \gamma_l, & \text{if } -\tau + 1 \le \phi_l \le \tau - 1 \\ \dfrac{1}{2\tau} \times \dfrac{\varphi_l + \varsigma}{\varsigma} + \dfrac{\tau + \phi_l - 1}{2\tau} = \dfrac{\varphi_l + (\tau + \phi_l)\varsigma}{2\varsigma\tau} = \gamma_l, & \text{if } \phi_l = \tau \\ \dfrac{1}{2\tau} \times \dfrac{\varphi_l}{\varsigma} = \dfrac{\varphi_l}{2\varsigma\tau} = \gamma_l, & \text{if } \phi_l = -\tau \end{cases} \tag{9}$$

$$f^{-1} : [0,1] \to [-\tau, \tau] \times [-\varsigma, \varsigma],$$

$$f^{-1}(\gamma_l) = \begin{cases} [2\tau\gamma_l - \tau] < o_{\varsigma(2\tau\gamma_l - \tau - [2\tau\gamma_l - \tau])} >= [2\tau\gamma_l - \tau] + 1 < o_{\varsigma((2\tau\gamma_l - \tau - [2\tau\gamma_l - \tau]) - 1)} >, & \text{if } 1 - \tau \le 2\tau\gamma_l - \tau \le \tau - 1 \\ \tau - 1 < o_{\varsigma(2\tau\gamma_l - \tau - [2\tau\gamma_l - \tau])} >= \tau < o_{\varsigma \times (2\tau\gamma_l - \tau - [2\tau\gamma_l - \tau]) - \varsigma} >, & \text{if } \tau - 1 \le 2\tau\gamma_l - \tau \le \tau \\ -\tau < o_{\varsigma \times (2\tau\gamma_l - \tau - [2\tau\gamma_l - \tau])} >= 1 - \tau < o_{\varsigma((2\tau\gamma_l - \tau - [2\tau\gamma_l - \tau]) - 1)} >, & \text{if } -\tau \le 2\tau\gamma_l - \tau \le 1 - \tau \end{cases} \tag{10}$$

Based on Definition 3.5, we can introduce the transformation functions F and F^{-1} between the DHHFLE h_{S_O} and the HFE h_γ:

$$F:\Phi\times\Psi\rightarrow\Theta,\ F\left(h_{S_O}\right)=F\left(\left\{s_{\phi_l<o_{\varphi_l}>}\middle|s_{\phi_l<o_{\varphi_l}>}\in S_O;l=1,\dots,L;\phi_l\in[-\tau,\tau];\varphi_l\in[-\varsigma,\varsigma]\right\}\right)=\left\{\gamma_l\,|\,\gamma_l=f\left(\phi_l,\varphi_l\right)\right\}=h_\gamma \quad (11)$$

$$F^{-1}:\Theta\rightarrow\Phi\times\Psi,\ F^{-1}\left(h_\gamma\right)=F^{-1}\left(\left\{\gamma_l\middle|\gamma_l\in[0,1];l=1,\dots,L\right\}\right)=\left\{s_{\phi_l<o_{\varphi_l}>}\,|\,\phi_l<o_{\varphi_l}>=f^{-1}\left(\gamma_l\right)\right\}=h_{S_O} \quad (12)$$

Remark 3.5. It is noted that the second hierarchy linguistic term is a linguistic feature or detailed supplementary of each linguistic term included in the first LTS, and the second hierarchy LTSs are different when describing the upper bound, the lower bound or the median term of the first hierarchy LTS. Therefore, we divide f into three parts according to the different values of ϕ_l. Suppose that $\tau=\varsigma=3$, then we can utilize the functions F to transform the three DHLTs $s_{-3<o_1>}$, $s_{0<o_1>}$ and $s_{3<o_{-1}>}$ into $1/18$, $5/9$, $17/18$, respectively. This can be illustrated in Fig. 3.

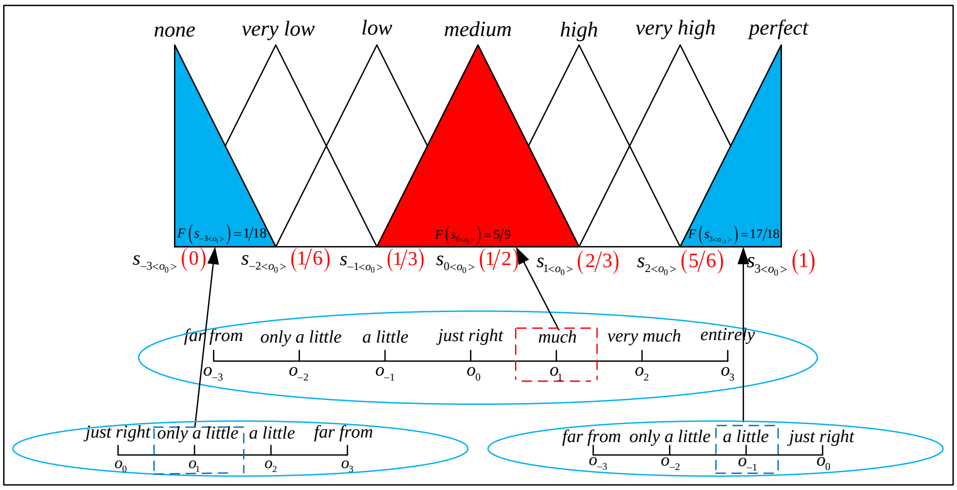

Fig. 3. Some operation results based on the equivalent transformation function F.

In Fig. 3, firstly we need to use different second hierarchy LTSs considering that the values of ϕ_l included in $s_{-3<o_1>}$, $s_{0<o_1>}$ and $s_{3<o_{-1}>}$ are different. Then we utilize the function F to calculate the equivalent real numbers: $F\left(s_{-3<o_1>}\right)=1/18$, $F\left(s_{0<o_1>}\right)=5/9$ and $F\left(s_{3<o_{-1}>}\right)=17/18$.

Remark 3.6. For the real number $\gamma\in[0,1]$, let $\tau=\varsigma=3$. The function F^{-1} can be described in three different cases:

(1) Let $\gamma=3/4$, then $-2<2\tau\gamma-\tau=1.5<2$. It follows that $s_1<s_{1.5}<s_2$. Thus, we obtain $F^{-1}(3/4)=s_{1<o_{1.5}>}=s_{2<o_{-1.5}>}$. This can be shown in Fig. 4.

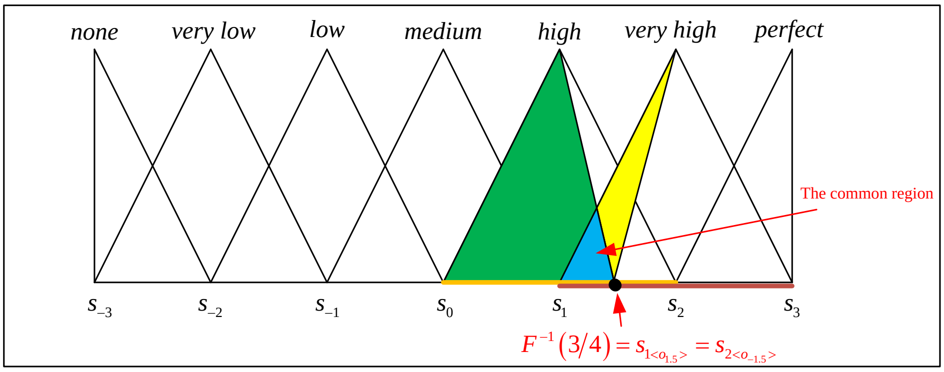

Fig 4. A special case when $1-\tau \le 2\tau\gamma-\tau \le \tau-1$**.**

(2) Let $\gamma = 11/12$, then $2 < 2\tau\gamma - \tau = 2.5 < 3$. It follows that $s_2 < s_{2.5} < s_3$. Thus, we obtain $F^{-1}(11/12) = s_{2<o_{1.5}>} = s_{3<o_{-1.5}>}$. This can be shown in Fig. 5.

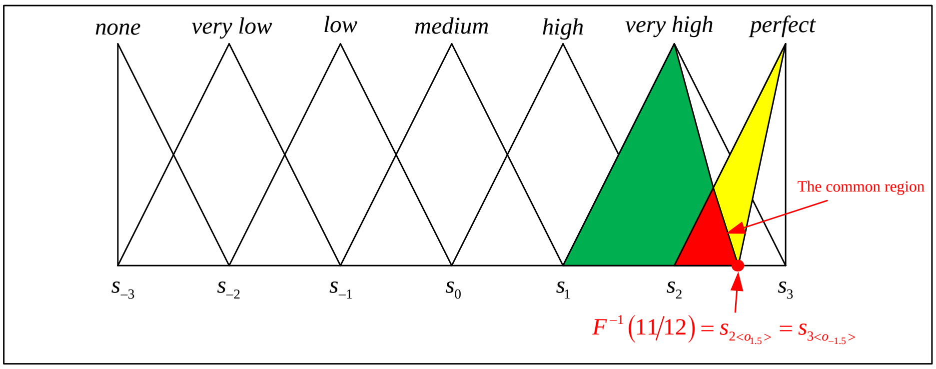

Fig. 5. A special case when $1-\tau \le 2\tau\gamma-\tau \le \tau$**.**

(3) Let $\gamma = 1/12$, then $-3 < 2\tau\gamma - \tau = -2.5 < -2$. Thus, $s_{-3} < s_{-2.5} < s_{-2}$, and so we obtain $F^{-1}(1/12) = s_{-2<o_{-1.5}>} = s_{-3<o_0>}$. This can be shown in Fig. 6.

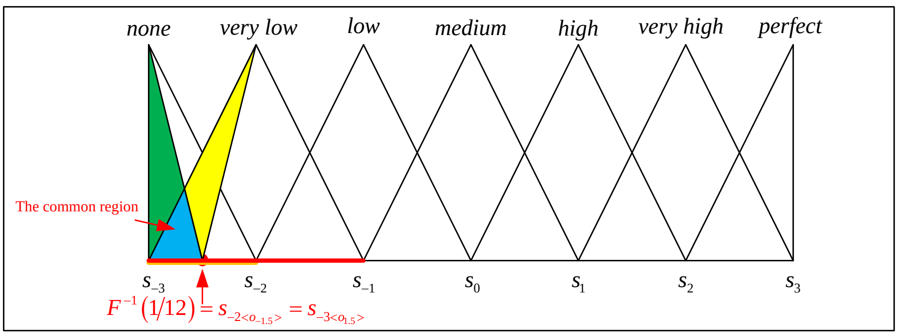

Fig. 6. A special case when $-\tau \le 2\tau\gamma-\tau \le 1-\tau$**.**

Remark 3.7. It is noted that, based on the equivalent transformation function F^{-1}, there are two equivalent DHLTs in each situation as discussed in Remark 3.6. To make the calculations more convenient, we can introduce some rules regarding to F^{-1}.

(1) If $\gamma=1$, then $F^{-1}(\gamma)=s_{\tau<o_0>}$;

(2) If $1\le 2\tau\gamma-\tau<\tau$, then $F^{-1}(\gamma)=s_{[2\tau\gamma-\tau]<o_{\varsigma(2\tau\gamma-\tau-[2\tau\gamma-\tau])}>}$;

(3) If $-1\le 2\tau\gamma-\tau\le 1$, then $F^{-1}(\gamma)=s_{0<o_{\varsigma(2\tau\gamma-\tau)}>}$;

(4) If $-\tau<2\tau\gamma-\tau\le -1$, then $F^{-1}(\gamma)=s_{[2\tau\gamma-\tau]+1<o_{\varsigma(2\tau\gamma-\tau-[2\tau\gamma-\tau]-1)}>}$;

(5) If $\gamma=-1$, then $F^{-1}(\gamma)=s_{-\tau<o_0>}$.

These five situations can be shown in Fig. 7.

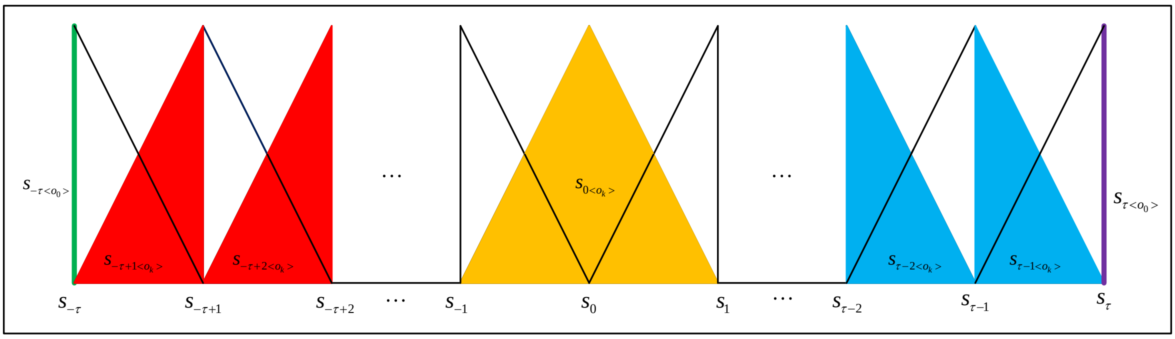

Fig. 7. The regions of each situation regarding to the function F^{-1}.

Based on Definition 2.4 and the two equivalent transformation functions F and F^{-1}. Some operational laws of DHHFLEs can be developed:

Definition 3.6. Let $S_O=\{s_{t<o_k>}|t=-\tau,\dots,-1,0,1,\dots,\tau,k=-\varsigma,\dots,-1,0,1,\dots,\varsigma\}$ be a DHLTS, $h_{S_O}=\{s_{\phi_l<o_{\varphi_l}>}|s_{\phi_l<o_{\varphi_l}>}\in S_O;l=1,2,\dots,L;\phi_l=[-\tau,\tau];\varphi_l=[-\varsigma,\varsigma]\}$, $h_{S_{O_i}}=\{s^i_{\phi_l<o_{\varphi_l}>}|s^i_{\phi_l<o_{\varphi_l}>}\in S_O;l=1,2,\dots,L;\phi_l=[-\tau,\tau];\varphi_l=[-\varsigma,\varsigma]\}(i=1,2)$ be three DHHFLEs, λ be a real number. Then

(1) (Addition) $h_{S_{O_1}}\oplus h_{S_{O_2}}=F^{-1}\left(\bigcup_{\eta_1\in F(h_{S_{O_1}}),\eta_2\in F(h_{S_{O_2}})}\{\eta_1+\eta_2-\eta_1\eta_2\}\right)$;

(2) (Multiplication) $h_{S_{O_1}}\otimes h_{S_{O_2}}=F^{-1}\left(\bigcup_{\eta_1\in F(h_{S_{O_1}}),\eta_2\in F(h_{S_{O_2}})}\{\eta_1\eta_2\}\right)$;

(3) (Multiplication) $\lambda h_{S_O}=F^{-1}\left(\bigcup_{\eta\in F(h_{S_O})}\{1-(1-\eta)^\lambda\}\right)$;

(4) (Power) $(h_{S_O})^\lambda=F^{-1}\left(\bigcup_{\eta\in F(h_{S_O})}\{\eta^\lambda\}\right)$;

(5) (Subtraction) $h_{S_{O_1}} \ominus h_{S_{O_2}} = F^{-1}\left(\bigcup_{\eta_1 \in F\left(h_{S_{O_1}}\right), \eta_2 \in F\left(h_{S_{O_2}}\right)} \{\theta\} \right)$, where $\theta = \begin{cases} \frac{\eta_1 - \eta_2}{1-\eta_2}, & if\ \eta_1 \geq \eta_2\ and\ \eta_2 \neq 1 \\ 0, & otherwise \end{cases}$;

(6) (Division) $h_{S_{O_1}} \oslash h_{S_{O_2}} = F^{-1}\left(\bigcup_{\eta_1 \in F\left(h_{S_{O_1}}\right), \eta_2 \in F\left(h_{S_{O_2}}\right)} \{\theta\} \right)$, where $\theta = \begin{cases} \frac{\eta_1}{\eta_2}, & if\ \eta_1 \leq \eta_2\ and\ \eta_2 \neq 0 \\ 0, & otherwise \end{cases}$;

(7) (Complementary) $\overline{h_{S_O}} = F^{-1}\left(\bigcup_{\eta \in F\left(h_{S_O}\right)} \{1-\eta\} \right)$;

(8) (Union) $h_{S_{O_1}} \cup h_{S_{O_2}} = \left\{ s_{t<o_{k_t}>} \middle| s_{t<o_{k_t}>} \subset h_{S_{O_1}}\ or\ s_{t<o_{k_t}>} \subset h_{S_{O_2}} \right\}$;

(9) (Intersection) $h_{S_{O_1}} \cap h_{S_{O_2}} = \left\{ s_{t<o_{k_t}>} \middle| s_{t<o_{k_t}>} \subset h_{S_{O_1}}\ and\ s_{t<o_{k_t}>} \subset h_{S_{O_2}} \right\}$.

Remark 3.8. The following points are remarkable:

(1) Based on the equivalent transformation function F, the DHHFLEs can be transformed to the HFEs. Therefore, we can develop the operational laws of DHHFLEs based on the operational laws of HFEs. Then we can obtain the results of DHHFLEs by transforming the HFEs to the DHHFLEs equivalently according to the other transformation function F^{-1}.

(2) For the formulas (1)-(2) and (5)-(6), the number of terms in the obtained results must be $\#L_1 \times \#L_2$, where $\#L_1$ and $\#L_2$ are the number of elements included in $h_{S_{O_1}}$ and $h_{S_{O_2}}$, respectively. Furthermore, the number of terms in the obtained results keeps the same as that of h_{S_O} in the formulas (3), (4) and (7).

(3) Specially, we can combine all the second hierarchy linguistic terms to one set when the DHLTs have the same first hierarchy linguistic terms. For example, if the calculation result is $\left\{ s_{0<o_1>}, s_{0<o_{1.5}>}, s_{1<o_{-0.5}>}, s_{1<o_0>} \right\}$, it can be written as $\left\{ s_{0<o_1, o_{1.5}>}, s_{1<o_{-0.5}, o_0>} \right\}$.

Example 3.2. Let $S_O = \left\{ s_{t<o_k>} \middle| t = -3, \ldots, 3, k = -3, \ldots, 3 \right\}$ be a DHLTS. Suppose that two DHHFLEs $h_{S_{O_1}} = \left\{ s_{1<o_2>}, s_{2<o_0>} \right\}$ and $h_{S_{O_2}} = \left\{ s_{-1<o_{-2}>}, s_0, s_{1<o_0>} \right\}$, as well as a real number $\lambda = \frac{1}{2}$. Then

(1) $h_{S_{O_1}} \oplus h_{S_{O_2}} = F^{-1}\left(\bigcup_{\eta_1 \in F\left(h_{S_{O_1}}\right), \eta_2 \in F\left(h_{S_{O_2}}\right)} \{\eta_1 + \eta_2 - \eta_1\eta_2\} \right)$

$$= F^{-1}\left\{ \frac{7}{9} + \frac{5}{18} - \frac{7}{9} \times \frac{5}{18}, \frac{7}{9} + \frac{5}{9} - \frac{7}{9} \times \frac{5}{9}, \frac{7}{9} + \frac{2}{3} - \frac{7}{9} \times \frac{2}{3}, \frac{5}{6} + \frac{5}{18} - \frac{5}{6} \times \frac{5}{18}, \frac{5}{6} + \frac{5}{9} - \frac{5}{6} \times \frac{5}{9}, \frac{5}{6} + \frac{2}{3} - \frac{5}{6} \times \frac{2}{3} \right\}$$

$$=\left\{s_{2<o_{0.11},o_{0.83},o_{1.22},o_{1.67},o_{2}>}\right\}$$

(2) $$h_{S_{O_1}}\otimes h_{S_{O_2}}=F^{-1}\left(\bigcup_{\eta_1\in F\left(h_{S_{O_1}}\right),\eta_2\in F\left(h_{S_{O_2}}\right)}\{\eta_1\eta_2\}\right)=F^{-1}\left\{\frac{7}{9}\times\frac{5}{18},\frac{7}{9}\times\frac{5}{9},\frac{7}{9}\times\frac{2}{3},\frac{5}{6}\times\frac{5}{18},\frac{5}{6}\times\frac{5}{9},\frac{5}{6}\times\frac{2}{3}\right\}$$

$$=\left\{s_{-1<o_{-2.11},o_{-1.83}>},s_{0<o_{-1.22},o_{-0.66},o_{0.33},o_{1}>}\right\}$$

(3) $$\lambda h_{S_{O_1}}=F^{-1}\left(\bigcup_{\eta\in F\left(h_{S_{O_1}}\right)}\left\{1-(1-\eta)^{\lambda}\right\}\right)=F^{-1}\left\{1-\left(1-\frac{7}{9}\right)^{\frac{1}{2}},1-\left(1-\frac{5}{6}\right)^{\frac{1}{2}}\right\}=\left\{s_{0<o_{0.51},o_{1.65}>}\right\}$$

(4) $$\left(h_{S_{O_1}}\right)^{\lambda}=F^{-1}\left(\bigcup_{\eta\in F\left(h_{S_O}\right)}\left\{\eta^{\lambda}\right\}\right)=F^{-1}\left\{\left(\frac{7}{9}\right)^{\frac{1}{2}},\left(\frac{5}{6}\right)^{\frac{1}{2}}\right\}=\left\{s_{2<o_{0.87},o_{1.43}>}\right\};$$

(5) $$h_{S_{O_1}}\ominus h_{S_{O_2}}=F^{-1}\left(\bigcup_{\eta_1\in F\left(h_{S_{O_1}}\right),\eta_2\in F\left(h_{S_{O_2}}\right)}\left\{\frac{\eta_1-\eta_2}{1-\eta_2}\right\}\right)=F^{-1}\left(\frac{\frac{7}{9}-\frac{5}{18}}{1-\frac{5}{18}},\frac{\frac{7}{9}-\frac{5}{9}}{1-\frac{5}{9}},\frac{\frac{7}{9}-\frac{2}{3}}{1-\frac{2}{3}},\frac{\frac{5}{6}-\frac{5}{18}}{1-\frac{5}{18}},\frac{\frac{5}{6}-\frac{5}{9}}{1-\frac{5}{9}},\frac{\frac{5}{6}-\frac{2}{3}}{1-\frac{2}{3}}\right)$$

$$=\left\{s_{0<o_{0},o_{2.25}>},s_{1<o_{0.46},o_{1.85}>},s_{3<o_{0}>}\right\}$$

(6) $$h_{S_{O_2}}\oslash h_{S_{O_1}}=F^{-1}\left(\bigcup_{\eta_1\in F\left(h_{S_{O_2}}\right),\eta_2\in F\left(h_{S_{O_1}}\right)}\left\{\frac{\eta_1}{\eta_2}\right\}\right)=F^{-1}\left\{\frac{\frac{5}{18}}{\frac{7}{9}},\frac{\frac{5}{18}}{\frac{5}{6}},\frac{\frac{5}{9}}{\frac{7}{9}},\frac{\frac{5}{9}}{\frac{5}{6}},\frac{\frac{2}{3}}{\frac{7}{9}},\frac{\frac{2}{3}}{\frac{5}{6}}\right\}=\left\{s_{0<o_{-3},o_{-2.57},o_{3}>},s_{1<o_{0.86},o_{2.4}>},s_{2<o_{0.43}>}\right\}$$

(7) $$\overline{h_{S_{O_1}}}=F^{-1}\left(\bigcup_{\eta\in F\left(h_{S_{O_1}}\right)}\{1-\eta\}\right)=F^{-1}\left\{1-\frac{7}{9},1-\frac{5}{6}\right\}=\left\{s_{-2<o_{0}>},s_{-1<o_{-2}>}\right\}.$$

Besides, we define a method to compare any two DHHFLEs:

Definition 3.7. Let $S_O=\left\{s_{t<o_k>}\middle|t=-\tau,\ldots,-1,0,1,\ldots,\tau,\,k=-\varsigma,\ldots,-1,0,1,\ldots,\varsigma\right\}$ be a DHLTS. Let $h_{S_O}=\left\{s_{\phi_l<o_{\varphi_l}>}\middle|s_{\phi_l<o_{\varphi_l}>}\in S_O;l=1,2,\ldots,L;\phi_l=[-\tau,\tau];\varphi_l=[-\varsigma,\varsigma]\right\}$ be a DHHFLE. Then we call

$$E\left(h_{S_O}\right)=\frac{1}{L}\sum_{l=1}^{L}F\left(s_{\phi_l<o_{\varphi_l}>}\right)\tag{13}$$

the expected value of h_{S_O}. Additionally, we call

$$\upsilon\left(h_{S_O}\right)=\sqrt{\frac{1}{L}\sum_{l=1}^{L}\left(F\left(s_{\phi_l<o_{\varphi_l}>}\right)-E\right)^2}\tag{14}$$

the variance of h_{S_O}.

Based on Eqs. (13) and (14), a method to compare any two DHHFLEs is developed as follows:

Definition 3.8. Let $h_{S_{O_1}}$ and $h_{S_{O_2}}$ be two DHHFLEs, then

(1) If $E\left(h_{S_{O_1}}\right) \geq E\left(h_{S_{O_2}}\right)$, then $h_{S_{O_1}}$ is bigger than $h_{S_{O_2}}$, denoted by $h_{S_{O_1}} > h_{S_{O_2}}$.

(2) If $E\left(h_{S_{O_1}}\right) = E\left(h_{S_{O_2}}\right)$, then

1) If $\upsilon\left(h_{S_{O_1}}\right) < \upsilon\left(h_{S_{O_2}}\right)$, then $h_{S_{O_1}}$ is bigger than $h_{S_{O_2}}$, denoted by $h_{S_{O_1}} > h_{S_{O_2}}$;

2) If $\upsilon\left(h_{S_{O_1}}\right) = \upsilon\left(h_{S_{O_2}}\right)$, then $h_{S_{O_1}}$ is equivalent with $h_{S_{O_2}}$, denoted by $h_{S_{O_1}} = h_{S_{O_2}}$.

3.4. Some Properties of DHHFLEs

Some properties of DHHFLEs can be concluded:

Theorem 3.1. Let $S_O = \left\{ s_{t<o_k>} \middle| t = -\tau, \ldots, -1, 0, 1, \ldots, \tau, k = -\varsigma, \ldots, -1, 0, 1, \ldots, \varsigma \right\}$ be a DHLTS. Let $h_{S_{O_1}}$, $h_{S_{O_2}}$ and $h_{S_{O_2}}$ be three DHHFLEs. Then

(1) $h_{S_{O_1}} \oplus h_{S_{O_2}} = h_{S_{O_2}} \oplus h_{S_{O_1}}$, $h_{S_{O_1}} \otimes h_{S_{O_2}} = h_{S_{O_2}} \otimes h_{S_{O_1}}$, $h_{S_{O_1}} \bigcup h_{S_{O_2}} = h_{S_{O_2}} \bigcup h_{S_{O_1}}$, $h_{S_{O_1}} \bigcap h_{S_{O_2}} = h_{S_{O_2}} \bigcap h_{S_{O_1}}$;

(2) $h_{S_{O_1}} \oplus \left(h_{S_{O_2}} \oplus h_{S_{O_3}}\right) = \left(h_{S_{O_1}} \oplus h_{S_{O_2}}\right) \oplus h_{S_{O_3}}$, $h_{S_{O_1}} \otimes \left(h_{S_{O_2}} \otimes h_{S_{O_3}}\right) = \left(h_{S_{O_1}} \otimes h_{S_{O_2}}\right) \otimes h_{S_{O_3}}$,

$h_{S_{O_1}} \bigcup \left(h_{S_{O_2}} \bigcup h_{S_{O_3}}\right) = \left(h_{S_{O_1}} \bigcup h_{S_{O_2}}\right) \bigcup h_{S_{O_3}}$, $h_{S_{O_1}} \bigcap \left(h_{S_{O_2}} \bigcap h_{S_{O_3}}\right) = \left(h_{S_{O_1}} \bigcap h_{S_{O_2}}\right) \bigcap h_{S_{O_3}}$;

(3) $h_{S_{O_1}} \bigcup \left(h_{S_{O_2}} \bigcap h_{S_{O_3}}\right) = \left(h_{S_{O_1}} \bigcup h_{S_{O_2}}\right) \bigcap \left(h_{S_{O_1}} \bigcup h_{S_{O_3}}\right)$, $h_{S_{O_1}} \bigcap \left(h_{S_{O_2}} \bigcup h_{S_{O_3}}\right) = \left(h_{S_{O_1}} \bigcap h_{S_{O_2}}\right) \bigcup \left(h_{S_{O_1}} \bigcap h_{S_{O_3}}\right)$;

Proof. (1) and (2) are the commutativity and the associativity of DHHFLEs, respectively. They are very simple to prove, so we omit the proofs of them here.

(3) Let $s_{t<o_k>} \in h_{S_{O_1}} \bigcup \left(h_{S_{O_2}} \bigcap h_{S_{O_3}}\right)$, then $s_{t<o_k>} \in h_{S_{O_1}}$ or $s_{t<o_k>} \in \left(h_{S_{O_2}} \bigcap h_{S_{O_3}}\right)$. In the first case, if $s_{t<o_k>} \in h_{S_{O_1}}$, then $s_{t<o_k>} \in h_{S_{O_1}} \bigcup h_{S_{O_2}}$ and $s_{t<o_k>} \in h_{S_{O_1}} \bigcup h_{S_{O_3}}$, then, $s_{t<o_k>} \in \left(h_{S_{O_1}} \bigcup h_{S_{O_2}}\right) \bigcap \left(h_{S_{O_1}} \bigcup h_{S_{O_3}}\right)$. In the other case, if $s_{t<o_k>} \in \left(h_{S_{O_2}} \bigcap h_{S_{O_3}}\right)$, then $s_{t<o_k>} \in h_{S_{O_2}}$ and $s_{t<o_k>} \in h_{S_{O_3}}$.

1) If $s_{t<o_k>} \in h_{S_{O_2}}$, then $s_{t<o_k>} \in h_{S_{O_1}} \bigcup h_{S_{O_2}}$;

2) If $s_{t<o_k>} \in h_{S_{O_3}}$, then $s_{t<o_k>} \in h_{S_{O_1}} \bigcup h_{S_{O_3}}$.

Therefore, $s_{t<o_k>} \in \left(h_{S_{O_1}} \bigcup h_{S_{O_2}}\right) \bigcap \left(h_{S_{O_1}} \bigcup h_{S_{O_3}}\right)$.

Let $s_{t<o_k>} \in \left(h_{S_{O_1}} \bigcup h_{S_{O_2}}\right) \bigcap \left(h_{S_{O_1}} \bigcup h_{S_{O_3}}\right)$, then $s_{t<o_k>} \in h_{S_{O_1}} \bigcup h_{S_{O_2}}$ and $s_{t<o_k>} \in h_{S_{O_1}} \bigcup h_{S_{O_3}}$. Thus,

1) $s_{t<o_k>} \in h_{S_{O_1}}$;

2) If $s_{t<o_k>} \in h_{S_{O_2}}$, there must exists $s_{t<o_k>} \in h_{S_{O_3}}$, namely, $s_{t<o_k>} \in h_{S_{O_2}} \cap h_{S_{O_3}}$; Similarly, if $s_{t<o_k>} \in h_{S_{O_3}}$, then $s_{t<o_k>} \in h_{S_{O_2}} \cap h_{S_{O_3}}$.

In conclusion, we have $s_{t<o_k>} \in h_{S_{O_1}}$ or $s_{t<o_k>} \in h_{S_{O_2}} \cap h_{S_{O_3}}$. Thus, $s_{t<o_k>} \in h_{S_{O_1}} \cup \left(h_{S_{O_2}} \cap h_{S_{O_3}}\right)$.

Similarly, we can prove the other equation $h_{S_{O_1}} \cap \left(h_{S_{O_2}} \cup h_{S_{O_3}}\right) = \left(h_{S_{O_1}} \cap h_{S_{O_2}}\right) \cup \left(h_{S_{O_1}} \cap h_{S_{O_3}}\right)$.

Theorem 3.2. Let $S_O = \left\{ s_{t<o_k>} \middle| t = -\tau, \ldots, -1, 0, 1, \ldots, \tau, k = -\varsigma, \ldots, -1, 0, 1, \ldots, \varsigma \right\}$ be a DHLTS, then

(1) If $0 \le t \le \tau - 1$, then $s_{t<o_k>} = s_{t+1<o_{k-\varsigma}>}$;

(2) If $1 - \tau \le t \le 0$, then $s_{t<o_k>} = s_{t-1<o_{k+\varsigma}>}$.

Proof. Considering that DHLT and real number (or fuzzy number) are one-to-one mapping, that is to say, one can be transformed to the other equivalently based on the function F or F^{-1}, then

(1) When $0 \le t \le \tau - 1$, there is $F\left(s_{t<o_k>}\right) = \frac{k + (\tau + t)\varsigma}{2\varsigma\tau} = \frac{k - \varsigma + (\tau + t + 1)\varsigma}{2\varsigma\tau} = F\left(s_{t+1<o_{k-\varsigma}>}\right)$.

(2) When $1 - \tau \le t \le 0$, there is $F\left(s_{t<o_k>}\right) = \frac{k + (\tau + t)\varsigma}{2\varsigma\tau} = \frac{k + \varsigma + (\tau + t - 1)\varsigma}{2\varsigma\tau} = F\left(s_{t-1<o_{k+\varsigma}>}\right)$.

Theorem 3.3. Let $S_O = \left\{ s_{t<o_k>} \middle| t = -\tau, \ldots, -1, 0, 1, \ldots, \tau, k = -\varsigma, \ldots, -1, 0, 1, \ldots, \varsigma \right\}$ be a DHLTS. Then

$$F\left(s_{t<o_k>}\right) \oplus F\left(s_{-t<o_{-k}>}\right) = 1 \tag{15}$$

Proof. If $1 - \tau \le t \le \tau - 1$, then

$$F\left(s_{t<o_k>}\right) \oplus F\left(s_{-t<o_{-k}>}\right) = \frac{k + (\tau + t)\varsigma}{2\varsigma\tau} + \frac{-k + (\tau - t)\varsigma}{2\varsigma\tau} = \frac{2\varsigma\tau}{2\varsigma\tau} = 1$$

Besides, if $t = \tau$ and $-t = -\tau$, then

$$F\left(s_{t<o_k>}\right) \oplus F\left(s_{-t<o_{-k}>}\right) = 1 + \frac{k}{2\varsigma\tau} + \frac{-k}{2\varsigma\tau} = 1$$

4. Double Hierarchy Hesitant Fuzzy Linguistic MULTIMOORA Method

This section main introduces the double hierarchy hesitant fuzzy linguistic MULTIMOORA method. We first give a brief description about the MCDM problems with the complicated linguistic information. Then, we summarize the general process of the traditional MULTIMOORA method. After that, the procedure of double hierarchy hesitant fuzzy linguistic MULTIMOORA method is developed and the algorithm is given for the convenience of application.

4.1. Multiple Criteria Decision Making Model

A MCDM problem with double hierarchy hesitant fuzzy linguistic information can be described as follows: Suppose that $A=\{A_1,A_2,\ldots,A_m\}$ is a set of alternatives, $C=\{C_1,C_2,\ldots,C_n\}$ is a set of criteria, and $w=(w_1,w_2,\ldots,w_n)^T$ is the weight vector of all criteria, where $w_j \geq 0$, $j=1,2,\ldots,n$, and $\sum_{j=1}^{n} w_j = 1$. Suppose that the two LTSs, $S=\{s_t \mid t=-\tau,\ldots,-1,0,1,\ldots,\tau\}$ and $O=\{o_k \mid k=-\varsigma,\ldots,-1,0,1,\ldots,\varsigma\}$, are the first and second hierarchy LTSs, respectively. Let $\aleph_{DHH}$ be a context-free grammar. Then the invited experts can give their original linguistic evaluation information about each alternative with respect to each criterion. We gather the evaluation information and establish an original decision making matrix $\overline{DM}=(OL)_{m\times n}$ $(i=1,2,\ldots,m;\ j=1,2,\ldots,n)$.

To find the solution, firstly, we need to transform $\overline{DM}$ into a decision making matrix $DH=\left(h_{S_{O_{ij}}}\right)_{m\times n}$ with the DHHFLEs as:

$$DH=\begin{bmatrix} h_{S_{O_{11}}} & h_{S_{O_{12}}} & \cdots & h_{S_{O_{1n}}} \\ h_{S_{O_{21}}} & h_{S_{O_{22}}} & \cdots & h_{S_{O_{2n}}} \\ \vdots & \vdots & \ddots & \vdots \\ h_{S_{O_{m1}}} & h_{S_{O_{m2}}} & \cdots & h_{S_{O_{mn}}} \end{bmatrix}$$

Then, we can utilize different MCDM methods to deal with this problem. Fig. 8 is drawn to show the process of dealing with the MCDM problem.

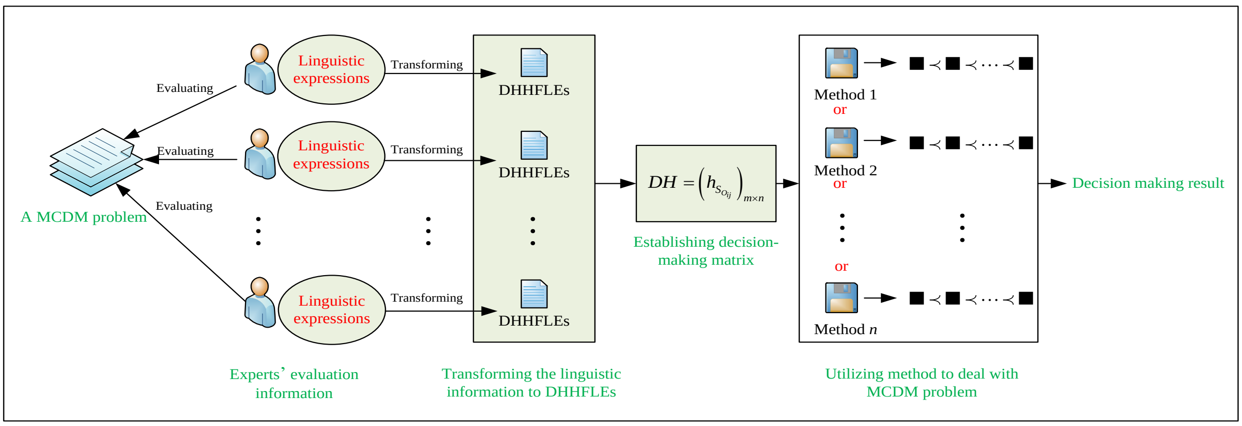

Fig. 8. The decision process of MCDM with double hierarchy hesitant fuzzy linguistic information.

4.2. The Description of the MULTIMOORA Method

Consider a general decision making problem, where the decision making matrix is denoted as $X = \left[x_{ij} \right]_{m \times n}$ with $x_{ij} \left(i = 1, 2, \ldots, m, j = 1, 2, \ldots, n \right)$ expressing the evaluation value for the i-th alternative with respect to the j-th objective (or criterion). The MULTIMOORA method [30] mainly consists of three parts: the ratio system, the reference point approach and the full multiplicative form method:

(1) The ratio system of MULTIMOORA mainly contains data normalization. Under each objective (or criterion), the data normalization of each $x_{ij} \left(i = 1, 2, \ldots, m, j = 1, 2, \ldots, n \right)$ can be calculated by

$$x_{ij}^{*} = \frac{x_{ij}}{\sum_{i=1}^{m} x_{ij}} \tag{16}$$

where $x_{ij}^{*} \left(i = 1, 2, \ldots, m, j = 1, 2, \ldots, n \right)$ reconstitute the normalized matrix. Then the summarizing index of each alternative can be derived in the following way:

$$y_i = \sum_{j=1}^{\theta} x_{ij}^{*} - \sum_{j=\theta+1}^{m} x_{ij}^{*} \tag{17}$$

where $\theta \left(\theta = 1, 2, \ldots, n \right)$ denotes the number of objectives to be maximized and $n - \theta$ denotes the number of objectives to be minimized. Then, the ranking of all alternatives can be obtained according to the summarizing indices $y_i \left(i = 1, 2, \ldots, m \right)$.

(2) The reference point for MULTIMOORA is based on the ratio system. Firstly, the reference point of each objective can be got based on $r_j = \max_i x_{ij}^{*}$. Then, every element of the normalized matrix needs to be recalculated by $\left| r_i - x_{ij}^{*} \right|$, and the final ranking is given according to the reference point and the min–max metric:

$$\min_i \left(\max_j \left| r_i - x_{ij}^{*} \right| \right) \tag{18}$$

(3) The full multiplicative form for MULTIMOORA [29] embodies maximization as well as minimization of the purely multiplicative utility function. The overall utility of the i-th alternative is represented as dimensionless numbers:

$$U_i = \frac{A_i}{B_i} \tag{19}$$

where $A_i = \prod_{j=1}^{\theta} x_{ij}$ represents the objectives of the i-th alternative to be maximized with $\theta(\theta = 1, 2, \ldots, n)$ denoting the number of objectives to be maximized. Similarly, $B_i = \prod_{j=\theta+1}^{n} x_{ij}$ represents the objectives of the i-th alternative to be minimized with $n-\theta$ denoting the number of objectives to be maximized. Then the ranking of all alternatives can be obtained based on the value of U_i.

Finally, the dominance theory [33] can be used to unite the three ranks provided by the three parts of MULTIMOORA into a single one.

Amounts of MULTIMOORA methods [29-41] have been developed with different types of decision making information and have been implemented to different fields. Baležentis and Zeng [30] extended the MULTIMOORA method with generalized interval-valued trapezoidal fuzzy numbers, which provides the means for the MCDM problems with uncertain assessments. The fuzzy MULTIMOORA method with triangular fuzzy numbers [31] was developed and applied in international comparison of the European Union Member States. Furthermore, Farzamnia and Babolghani [34] and Liu et al. [35] used the MULTIMOORA method under fuzzy environment to overcome the supplier selection problem and to evaluate the risk of failure modes, respectively. Moreover, an integrated approach of fuzzy MULTIMOORA and multi-choice conic goal programming [36] was proposed to choose the best students and define the optimum assignments among some programs. Additionally, under hesitant fuzzy environment, the MULTIMOORA-HF [37,38] is designed to facilitate group decision making with hesitant fuzzy information. Chen and Li [39] employed the MULTIMOORA method to obtain the ranking of alternatives corresponding to each ordering approach. Besides, Brauers and Zavadskas [33] discussed the concept of MULTIMOORA, and used it to decide upon a bank loan to buy property. Tian et al. [40] presented an improved MULTIMOORA approach by integrating two simplified Bonferroni mean operators and a distance measure.

4.3. The DHHFL-MULTIMOORA Method

In this paper, motivated by the classical MULTIMOORA method and its extensions [29-40], a DHHFL-MULTIMOORA method can be established by considering the three aspects comprehensively.

A. The double hierarchy hesitant fuzzy linguistic ratio system

The double hierarchy hesitant fuzzy linguistic ratio system (DHHFLRS) mainly defines the normalization of the DHHFLEs $h_{S_{O_{ij}}}$ $(i=1,2,\ldots,m;\ j=1,2,\ldots,n)$. Based on the expected values of the DHHFLEs, the normalization is performed by:

$$h^{*}_{S_{O_{ij}}}=E\left(h_{S_{O_{ij}}}\right)\Big/\sum_{i=1}^{m}E\left(h_{S_{O_{ij}}}\right),\ \text{for all } i,\ j \tag{20}$$

Additionally, we can compute the summarizing ratio Φ_i^* for each alternative:

$$\Phi_i^*=\sum_{j=1}^{\theta}h^{*}_{S_{O_{ij}}}-\sum_{j=\theta+1}^{n}h^{*}_{S_{O_{ij}}} \tag{21}$$

where θ stands for the number of profitability criteria; $m-\theta$ denotes the number of cost criteria. Therefore, Φ_i^* denotes the best performance value of the i-th alternative. Consequently, the larger the value of Φ_i^* is, the higher rank the i-th alternative would be.

B. The double hierarchy hesitant fuzzy linguistic reference point

The double hierarchy hesitant fuzzy linguistic reference point (DHHFLRP) can be established by the following steps: Firstly, we need to determine the maximal objective reference point M_j $(j=1,2,\ldots,n)$. Based on Definition 3.8, the reference point of the j-th column can be defined as:

$$\mathrm{M}_j=\begin{cases}\max\limits_{i}\left\{h_{S_{O_{ij}}}\right\},\ \text{if } j\le\theta\\ \min\limits_{i}\left\{h_{S_{O_{ij}}}\right\},\ \text{if } j>\theta\end{cases} \tag{22}$$

Then, we can calculate the distance between each DHHFLE $h_{S_{O_{ij}}}$ and M_j:

$$D\left(h_{S_{O_{ij}}},\mathrm{M}_j\right)=\sqrt{\frac{1}{L}\sum_{\substack{l=1,\\ \eta_1\in F\left(h_{S_{O_{ij}}}\right),\eta_2\in F\left(\mathrm{M}_j\right)}}^{L}\left(\eta_1^l-\eta_2^l\right)^2} \tag{23}$$

where F is the equivalent transformation function. η_1^l and η_2^l express the l-th element of $F\left(h_{S_{O_{ij}}}\right)$ and $F\left(\mathrm{M}_j\right)$, respectively. Furthermore, if two DHHFLEs have different numbers of DHLEs, then we can extend the short one with the mean value of its upper and lower bounds.

Based on Eq. (23), the final ranking of all alternatives can be obtained by the Min-Max metric:

$$\min_i\left(\max_j\left\{D\left(h_{S_{O_{ij}}},\mathrm{M}_j\right)\right\}\right) \tag{24}$$

C. The double hierarchy hesitant fuzzy linguistic full multiplicative form

The double hierarchy hesitant fuzzy linguistic full multiplicative form (DHHFLFMF) mainly considers the overall utility of the i-th alternative, which can be represented by a dimensionless number U_i yielded by:

$$U_i = \Im_i \oslash \Re_i, i = 1, 2, \ldots, m \tag{25}$$

where $\Im_i$ expresses the product of the beneficial criteria on the i-th alternative and $\Im_i = \prod_{j=1}^{\theta} E\left(h_{S_{Oij}}\right)$; $\Re_i$ denotes the product of the cost criteria on the i-th alternative and $\Re_i = \prod_{j=\theta+1}^{n} E\left(h_{S_{Oij}}\right)$. Obviously, the bigger U_i is, the higher the ranking of the alternatives $A_i \left(i = 1, 2, \ldots, m\right)$ should be.

Specially, if $\Re_i = 0$, then Eq. (25) can be denote as:

$$U_i = \Im_i, i = 1, 2, \ldots, m \tag{26}$$

Finally, we can make a decision by taking these three measures (DHHFLRS, DHHFLRP and DHHFLFMF) into consideration synthetically.

5. Case Study: The Evaluation of Air Pollution Control Measures for Treating Haze

In this section, we mainly apply the DHHFL-MULTIMOORA method to deal with a practical MCDM problem concerning the evaluation over the air pollution control measures for treating haze. Moreover, some comparisons with the hesitant fuzzy linguistic TOPSIS method are provided to show the advantages of the proposed method.

5.1. Background Description

In recent years, haze has become a huge challenge in many provinces of China. Especially in 2016, the PM2.5 concentrations in many cities, such as Shijiazhuang, Zhengzhou, Jinan, have passed 1000. Haze has taken a lot of troubles to people's daily life. More and more people go to hospital due to the diseases of lung and respiratory. Amounts of flights and expressways often need to be closed temporarily. Some primary and secondary schools can only choose to suspend classes considering the health and safety of children and so on.

In consideration of the huge harm coming from haze, China has formulated the corresponding policies as well as laws and regulations. Li Keqiang, the Premier of the State Council of China, chaired a state council executive meeting in 2014. This meeting mainly studied the deployment of further strengthening the atmospheric pollution control. Controlling the atmospheric pollution is the urgent requirement for improving people's livelihood, the key action of changing production pattern and adjusting industrial structures, and the important task for promoting the construction of ecological civilization. To improve the existing policies as soon as possible, this meeting further introduced some measures to strengthen the atmospheric pollution control (see Table 1).

Table 1. The measures to strengthen the atmospheric pollution control

Solution measures	Specific implementation projects
Speeding up the adjustment of energy structure (C_1)	C_{11}. Implementing interregional power-transmission project; controlling coal consumption; using clean coal
	C_{12}. Promoting the quality upgrading of refined oil product
	C_{13}. Carrying out the innovation of heat energy measurement; Promoting cities and towns pollution reduction
	C_{14}. Enhancing the energy conservation and environmental protection level of coal-fired boiler
Playing the incentive and guiding roles of Price, taxes and subsidies, etc. (C_2)	C_{21}. Giving tax policy support for coal bed methane power generation
	C_{22}. Establishing special fund and implementing “reward replace subsidy” for key area’s air pollution prevention and control
	C_{23}. Setting the standard for efficient involved industries and motivating enterprise who reaches the standard
	C_{24}. Improving the subsidy policy on the purchase of new energy vehicles
	C_{25}. Strongly supporting the energy conservation and environmental protection of core technology and the development of correlated industries
Clearing the responsibility of each part (C_3)	C_{31}. Implementing the evaluation of the responsibility for the control of air pollution
	C_{32}. Improving the system of environmental monitoring
	C_{33}. Completing the standard of local air pollutants emission of cement, boiler, non-ferrous industry, etc.
	C_{34}. Regulating the environment information release
Utilizing the market and law means and education (C_4)	C_{41}. Playing the effects of social forces and science technology
	C_{42}. Speeding up the formulation and revise of relevant laws and regulations
	C_{43}. Speeding up the heavy pollution weather monitoring and early warning emergency system construction
	C_{44}. Promoting to form the governance pattern about the prevention and control of atmospheric pollution

These measures are very important for controlling haze and improving the air quality. Three years have passed, the air pollution statuses of some cities have got a lot of improvements. However, most cities are even worse, especially in the mid-east region of China.

Now we investigate five cities including Nanjing, Guangzhou, Chengdu, Zhengzhou and Shijiazhuang (denoted as the set of alternatives $A=(A_1,A_2,\ldots,A_5)$), and evaluate whether these measures (denoted as the set of criteria $C=(C_1,C_2,C_3,C_4)$) were implemented effectively or not. Let two LTSs:

$$S = \{s_{-3} = none, s_{-2} = very\ bad, s_{-1} = bad, s_0 = medium, s_1 = good, s_2 = very\ good, s_3 = perfect\}$$

$$O = \begin{cases} \{o_{-3} = far\ \ from, o_{-2} = only\ a\ little, o_{-1} = a\ little, o_0 = just\ right, o_1 = much, o_2 = very\ much, o_3 = extremely\}, if\ \ s_t \geq s_0. \\ \{o_{-3} = extremely, o_{-2} = very\ much, o_{-1} = much, o_0 = just\ right, o_1 = a\ little, o_2 = only\ a\ little, o_3 = far\ \ from\}, if\ \ s_t < s_0. \end{cases}$$

be the first hierarchy LTS and the second hierarchy LTS, respectively. The invited experts gave their evaluations for each city with respect to each measure by ordinary linguistic information, and the evaluation judgments are listed in Table 2.

Table 2. The original linguistic evaluations given by the experts

	Measure C_1	Measure C_2	Measure C_3	Measure C_4
Nanjing (A_1)	*Between only a little medium and much good*	*Much very good*	*Between only a little bad and much medium*	*Between much good and very much very good*
Guangzhou (A_2)	*Between just right very good and a little perfect*	*Just right medium*	*Between a little good and a little perfect*	*Between just right very good and only a little perfect*
Chengdu (A_3)	*Just right good*	*Between just right good and just right very good*	*Between only a little very good and only a little perfect*	*Between very much bad and a little good*
Zhengzhou (A_4)	*Between much very good and just right perfect*	*Between very much very bad and much good*	*Much good*	*Just right medium*
Shijiazhuang (A_5)	*Between only a little medium and much good*	*Between very much very bad and very much medium*	*Between very much medium and very much good*	*Between very much bad and only a little medium*

5.2. Decision Making Based on the DHHFL-MULTIMOORA Method

Clearly, the problem clarified in Subsection 5.1 is a MCDM problem with double hierarchy hesitant fuzzy linguistic information. In the following, we use the DHHFL-MULTIMOORA method to deal with it.

Step 1. Transform the original linguistic evaluation information in DHHFLEs. All these DHHFLEs consist a decision making matrix shown in Table 3.

Table 3. Decision making matrix with DHHFLEs

	C_1	C_2	C_3	C_4
A_1	$\{s_{0<o_{-2}>}, s_{1<o_1>}\}$	$\{s_{2<o_1>}\}$	$\{s_{-1<o_2>}, s_{0<o_1>}\}$	$\{s_{1<o_1>}, s_{2<o_2>}\}$
A_2	$\{s_{2<o_0>}, s_{3<o_{-1}>}\}$	$\{s_{0<o_0>}\}$	$\{s_{1<o_{-1}>}, s_2, s_{3<o_{-1}>}\}$	$\{s_{2<o_0>}, s_{3<o_{-2}>}\}$
A_3	$\{s_{1<o_0>}\}$	$\{s_{1<o_0>}, s_{2<o_0>}\}$	$\{s_{2<o_{-2}>}, s_{3<o_{-2}>}\}$	$\{s_{-1<o_{-2}>}, s_0, s_{1<o_{-1}>}\}$
A_4	$\{s_{2<o_1>}, s_{3<o_0>}\}$	$\{s_{-2<o_{-2}>}, s_{-1}, s_0, s_{1<o_2>}\}$	$\{s_{1<o_1>}\}$	$\{s_{0<o_0>}\}$
A_5	$\{s_{0<o_{-2}>}, s_{1<o_1>}\}$	$\{s_{-2<o_{-2}>}, s_{-1}, s_{0<o_2>}\}$	$\{s_{0<o_2>}, s_{1<o_2>}\}$	$\{s_{-1<o_{-2}>}, s_{0<o_{-2}>}\}$

Step 2. Calculate the DHHFLRS measure

Firstly, based on Eq. (13) and Eq. (20), the expected values and the normalization results of all DHHFLEs can be obtained, shown in Table 4 and Table 5, respectively.

Table 4. The utility values of all DHHFLEs

	C_1	C_2	C_3	C_4
A_1	5/9	8/9	1/2	5/6
A_2	8/9	1/2	43/54	31/36
A_3	2/3	3/4	29/36	23/54
A_4	17/18	5/12	13/18	1/2
A_5	5/9	1/3	25/36	11/36

Table 5. The normalization results of all DHHFLEs.

	C_1	C_2	C_3	C_4
A_1	0.1538	0.3077	0.1421	0.2848
A_2	0.2462	0.1731	0.2263	0.2943
A_3	0.1846	0.2596	0.2289	0.1456
A_4	0.2616	0.1442	0.2053	0.1709
A_5	0.1538	0.1154	0.1974	0.1044

Additionally, considering that all the criteria are the profitability indicators, the summarizing ratio Φ_i^* for each alternative can be calculated according to Eq. (21). Then the ranking of alternatives and the optimal alternative can be obtained:

Table 6. The final summarizing ratios, the ranking and the optimal alternative

Φ_1^*	Φ_2^*	Φ_3^*	Φ_4^*	Φ_5^*	The rank of alternatives	The optimal alternative
0.8884	0.9399	0.8187	0.7820	0.5710	$A_2 \succ A_1 \succ A_3 \succ A_4 \succ A_5$	A_2

Step. 3. Utilize the DHHFLRP measure

Firstly, we calculate the maximal objective reference points $\mathrm{M}_j\ (j=1,2,3,4)$ based on Definition 3.8 and Eq. (22).

Table 7. The maximal objective reference point M_j of each criterion

M_1	M_2	M_3	M_4
$\{s_{2<o_1>}, s_{3<o_0>}\}$	$\{s_{2<o_1>}\}$	$\{s_{2<o_{-2}>}, s_{3<o_{-2}>}\}$	$\{s_{2<o_0>}, s_{3<o_{-2}>}\}$

Then, based on Eq. (23), we can calculate the distance between each DHHFLE $h_{S_{O_{ij}}}$ and M_j $(i=1,2,\ldots,5;\ j=1,2,3,4)$. The final ranking of all the alternatives can be obtained by Eq. (24). The calculation results are shown in Table 8.

Table 8. The distance between each DHHFLE $h_{S_{O_{ij}}}$ and M_j and the final ranking

	C_1	C_2	C_3	C_4	$\max_j\left\{D\left(h_{S_{O_{ij}}}, \mathrm{M}_j\right)\right\}$	Ranking
A_1	0.2881	0	0.2661	0.0663	0.2881	2

A_2	0.0313	0.2813	0.0571	0	0.2813	1
A_3	0.2085	0.1127	0	0.3135	0.3135	3
A_4	0	0.3891	0.1127	0.2830	0.3891	4
A_5	0.2881	0.4441	0.1250	0.4075	0.4441	5

Step 4. Utilize the DHHFLFMF measure

Based on Eq. (25) and Eq. (26), as well as Table 4, the overall expected value of the i-th alternative can be calculated. Let $i=1$, then

$$U_1 = \Im_1 \oslash \Re_1 = \prod_{j=1}^{\theta} E\left(h_{S_{O_{1j}}}\right) = \frac{5}{9} \times \frac{8}{9} \times \frac{1}{2} \times \frac{5}{6} = 0.2057$$

Similarly, we can calculate the overall expected values of the rest alternatives, and then obtain the optimal alternative. The results are shown in Table 9.

Table 9. The overall expected values of all alternatives and the optimal alternative

U_1	U_2	U_3	U_4	U_5	The ranking order	The optimal alternative
0.2057	0.3048	0.1716	0.1421	0.0393	$A_2 \succ A_1 \succ A_3 \succ A_4 \succ A_5$	A_2

Step 5. Combining the calculation results of the DHHFLRS, DHHFLRP, and DHHFLFMF measures, the final optimal alternative can be obtained, which is A_2 (Guangzhou).

Table 10. The calculation results and final optimal alternative

DHHFLRS	DHHFLRP	DHHFLFMF	The final optimal alternative
$A_2 \succ A_1 \succ A_3 \succ A_4 \succ A_5$	$A_2 \succ A_1 \succ A_3 \succ A_4 \succ A_5$	$A_2 \succ A_1 \succ A_3 \succ A_4 \succ A_5$	A_2

5.3. Further Discussions for the Case: Comparison with HFL-TOPSIS Method

To make some comparisons, in the following, we continue to discuss this case by utilizing the hesitant fuzzy linguistic TOPSIS (HFL-TOPSIS) method [18,19]. Since the weights of criteria are not considered in DHHFL-MULTIMOORA, we let the weight vector be $w = (0.25, 0.25, 0.25, 0.25)^T$ in order to eliminate its impacts.

Step 1. As we know, the traditional HFLE only considers the meaning of the first hierarchy. Therefore, we delete the second hierarchy linguistic terms and a new decision making matrix with the HFLEs $h_{S_{ij}}\ (i=1,2,\ldots,5;\ j=1,2,3,4)$ can be obtained (see Table 11).

Table 11. The decision making matrix with HFLEs

	C_1	C_2	C_3	C_4
A_1	$\{s_0, s_1\}$	$\{s_2\}$	$\{s_{-1}, s_0\}$	$\{s_1, s_2\}$
A_2	$\{s_2, s_3\}$	$\{s_0\}$	$\{s_1, s_2, s_3\}$	$\{s_2, s_3\}$
A_3	$\{s_1\}$	$\{s_1, s_2\}$	$\{s_2, s_3\}$	$\{s_{-1}, s_0, s_1\}$
A_4	$\{s_2, s_3\}$	$\{s_{-2}, s_{-1}, s_0, s_1\}$	$\{s_1\}$	$\{s_0\}$

A_5	$\{s_0, s_1\}$	$\{s_{-2}, s_{-1}, s_0\}$	$\{s_0, s_1\}$	$\{s_{-1}, s_0\}$

Step 2. Determine the positive ideal solution (PIS) h_S^+ and the negative ideal solution (NIS) h_S^- of the alternatives, respectively.

$$h_S^+ = \left(h_{S_1}^+, h_{S_2}^+, h_{S_3}^+, h_{S_4}^+\right) = \left(\{s_2, s_3\}, \{s_2\}, \{s_2, s_3\}, \{s_2, s_3\}\right)$$

$$h_S^- = \left(h_{S_1}^-, h_{S_2}^-, h_{S_3}^-, h_{S_4}^-\right) = \left(\{s_0, s_1\}, \{s_{-2}, s_{-1}, s_0\}, \{s_{-1}, s_0\}, \{s_{-1}, s_0\}\right)$$

Step 3. Calculate the deviation degrees between each alternative and the PIS as well as the deviation degrees between each alternative and the NIS based on

$$d\left(A_i, h_S^+\right) = \sum_{j=1}^{4} w_j \sqrt{\frac{1}{\# L_{ij}} \sum_{k=1}^{\# L_{ij}} \left(G\left(h_{S_{ij}}\right) - G\left(h_{S_j}^+\right)\right)^2}$$

$$d\left(A_i, h_S^-\right) = \sum_{j=1}^{4} w_j \sqrt{\frac{1}{\# L_{ij}} \sum_{k=1}^{\# L_{ij}} \left(G\left(h_{S_{ij}}\right) - G\left(h_{S_j}^-\right)\right)^2}$$

The calculation results are shown in Table 12.

Table 12. The deviation degree between each alternative and the PIS (NIS)

	A_1	A_2	A_3	A_4	A_5
$d\left(A_i, h_S^+\right)$	0.2500	0.0954	0.2009	0.2862	0.3379
$d\left(A_i, h_S^-\right)$	0.2129	0.3677	0.2869	0.2082	0.0833

Thus, $d_{\min}\left(A_i, h_S^+\right) = 0.0954$ and $d_{\max}\left(A_i, h_S^-\right) = 0.3677$ can be obtained easily.

Step 4. Calculate the closeness coefficient $C\left(A_i\right)$ for each alternative by

$$C\left(A_i\right) = \frac{d\left(A_i, h_S^-\right)}{d_{\max}\left(A_i, h_S^-\right)} - \frac{d\left(A_i, h_S^+\right)}{d_{\min}\left(A_i, h_S^+\right)}$$

Then we obtain

$$C\left(A_1\right) = -2.0244,\ C\left(A_2\right) = 0,\ C\left(A_3\right) = -1.3256,\ C\left(A_4\right) = -2.4338,\ C\left(A_5\right) = -3.3154$$

Step 5. Compare $C\left(A_i\right)\left(i = 1, 2, \ldots, 5\right)$, and the ranking of all closeness coefficients is $C\left(A_2\right) > C\left(A_3\right) > C\left(A_1\right) > C\left(A_4\right) > C\left(A_5\right)$. Therefore, the ranking of the alternatives is $A_2 \succ A_3 \succ A_1 \succ A_4 \succ A_5$. Thus, the optimal alternative is A_2.

For these two kinds of linguistic information and two MCDM methods, some analyses can be given as follows:

(1) The DHHFLTS expresses information more accurately and comprehensively than the HFLTS. For example, in the decision making matrix, the experts want to represent their opinion "*Between a*

little good and a little perfect" ($\{s_{1<o_{-1}>}, s_2, s_{3<o_{-1}>}\}$). However, we can only utilize the HFLTS $\{s_1, s_2, s_3\}$ to express its meaning. Obviously, the HFLTS is far from expressing the original meaning of experts.

(2) The DHHFL-MULTIMOORA is more comprehensive in dealing with the MCDM problems as it utilizes the DHHFLRS, DHHFLRPs, and DHHFLFMF measures. All of them are reasonable in dealing with the MCDM problems from different angles. Thus, the reliability and veracity of the decision making results would be improved greatly.

(3) The analyses on the decision making results

On the one hand, in Subsection 5.2, all the rankings of the alternatives via the DHHFLRS, DHHFLRPs, and DHHFLFMF measures are $A_2 \succ A_1 \succ A_3 \succ A_4 \succ A_5$, which further indicates that the optimal alternative is A_2. This means these three measures are suitable to deal with the MCDM problems with double hierarchy hesitant fuzzy linguistic information.

On the other hand, in Subsection 5.3, the ranking of the alternatives is $A_2 \succ A_3 \succ A_1 \succ A_4 \succ A_5$. Obviously, the orders of A_1 and A_3 are different in the DHHFL-MULTIMOORA and HFL-TOPSIS methods. The main reason for this is that the original information is changed in the HFL-TOPSIS method when we transform the DHHFLEs to the HFLEs.

6. Conclusions and Future Research Directions

In this paper, we have introduced the concept of DHLTS, and then we have utilized it to develop the DHHFLTS which is more accurate and comprehensive than the HFLTS in information representation. Some operational laws and properties of the DHHFLEs have been developed based on the equivalent transformation functions. We have drawn some figures to facilitate the understandings of the DHLTS and the DHHFLTS. Furthermore, we have proposed a DHHFL-MULTIMOORA method for dealing with the MCDM problems in which the assessments are described in double hierarchy hesitant fuzzy linguistic information. Moreover, we have applied the DHHFL-MULTIMOORA method to deal with a practical MCDM problem concerning the evaluation over the air pollution control measures for treating haze. Finally, we have made some comparisons between the DHHFL-MULTIMOORA method and the HFL-TOPSIS method.

In the future, some research directions concerning the DHHFLTSs can be developed including the double hierarchy hesitant fuzzy linguistic information aggregation, linguistic measures, preference relations and consistency analysis, personalized individual semantics [44], etc.

Furthermore, DHHFLTSs can be used to deal with new decision making model such as consensus model [45,46], large scale decision making model [47], etc. Additionally, these research results can be applied to deal with some practical problems such as medical management, water resource management, etc.

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References

[1] V. Torra. Hesitant fuzzy sets. International Journal of Intelligent Systems 25 (2010) 529-539.

[2] M.M. Xia and Z.S. Xu. Hesitant fuzzy information aggregation in decision making. International Journal of Approximate Reasoning 52 (2011) 395-407.

[3] F. Herrera, L. Martínez, V. Torra and Z.S. Xu. Hesitant fuzzy sets: An emerging tool in decision making. International Journal of Intelligent Systems 29(6) (2014) 493-494.

[4] R.M. Rodríguez, B. Bedregal, H. Bustince, Y.C. Dong, B. Farhadinia, C. Kahraman, L. Martínez, V. Torra, Y.J. Xu, Z.S. Xu and F. Herrera. A position and perspective analysis of hesitant fuzzy sets on information fusion in decision making. Towards high quality progress. InformationFusion 29 (2016) 89-97.

[5] L.A. Zadeh. The concept of a linguistic variable and its applications to approximate reasoning-Part I. Information Sciences 8 (1975) 199-249.

[6] R.M. Rodríguez, L. Martínez and F. Herrera. Hesitant fuzzy linguistic terms sets for decision making. IEEE Transactions on Fuzzy Systems 20 (2012) 109-119.

[7] X.J. Gou, Z.S. Xu and H.C. Liao. Multi-criteria decision making based on bonferroni means with hesitant fuzzy linguistic information. Soft Computing (2016) DOI: 10.1007/s00500-016-2211-1.

[8] C.P. Wei, N.Zhao and X.J. Tang. Operators and comparisons of hesitant fuzzy linguistic term sets. IEEE Transactions on Fuzzy Systems 22(3) (2014) 575-585.

[9] Y.J. Xu, A.W. Xu, J.M. Merigó and H.M. Wang. Hesitant fuzzy linguistic ordered weighted distance operators for group decision making. Journal of Applied Mathematics and Computing 49(1-2) (2015) 1-24.

[10] H.C. Liao and Z.S. Xu, Approaches to manage hesitant fuzzy linguistic information based on the cosine distance and similarity measures for HFLTSs and their application in qualitative decision making. Expert Systems with Applications 42(12) (2015) 5328-5336.

[11] G. Hesamian and M. Shams. Measuring similarity and ordering based on hesitant fuzzy linguistic term sets. Journal Intelligent of Fuzzy Systems 28(2) (2015) 983-990.

[12] H.C. Liao, Z.S. Xu and X.J. Zeng. Distance and similarity measures for hesitant fuzzy linguistic term sets and their application in multi-criteria decision making. Information Sciences 271 (2014) 125-142.

[13] H.C. Liao, Z.S. Xu, X.J. Zeng and J.M. Merigó. Qualitative decision making with correlation coefficients of hesitant fuzzy linguistic term sets. Knowledge-Based Systems 76 (2015) 127-138.

[14] J.Q. Wang, J.T. Wu, J. Wang, H.Y. Zhang and X.H. Chen. Multi-criteria decision making methods based on the Hausdorff distance of hesitant fuzzy linguistic numbers. Soft Computing 20(4) (2016) 1621-1633.

[15] H.B. Liu, J.F. Cai and L. Jiang. On improving the additive consistency of the fuzzy preference relations based on comparative linguistic expressions. Internal Journal of Intelligent Systems 29 (2014) 544-559.

[16] Z.M. Zhang and C. Wu. On the use of multiplicative consistency in hesitant fuzzy linguistic preference relations. Knowledge-Based Systems 72 (2014) 13-27.

[17] B. Zhu and Z.S. Xu. Consistency measures for hesitant fuzzy linguistic preference relations. IEEE Transactions on Fuzzy Systems 22(1) (2014) 35-45.

[18] I. Beg and T. Rashid. TOPSIS for hesitant fuzzy linguistic term sets. Internal Journal of Intelligent Systems 28 (2013) 1162- 1171.

[19] Q.Y. Tan, C.P. Wei, Q. Liu and X.Q. Feng. The hesitant fuzzy linguistic TOPSIS method based on novel information measures. Asia-Pacific Journal of Operational Research 33(5) (2016) 1-22.

[20] H.C. Liao, Z.S. Xu and X.J. Zeng. Hesitant fuzzy linguistic VIKOR method and its application in qualitative multiple criteria decision making. IEEE Transactions on Fuzzy Systems 23(5) (2015) 1343- 1355 .

[21] C.P. Wei, Z.L. Ren and R.M. Rodriguez. A hesitant fuzzy linguistic TODIM method based on a score function. International Journal of Computational Intelligence Systems 8(4) (2015) 701-712.

[22] Y. Dong, CC. Li, F. Herrera. Connecting the linguistic hierarchy and the numerical scale for the 2-tuple linguistic model and its use to deal with hesitant unbalanced linguistic information. Information Sciences 367 (2016) 259–278.

[23] Y.C. Dong, X. Chen and F. Herrera. Minimizing adjusted simple terms in the consensus reaching process with hesitant linguistic assessments in group decision making. Information Sciences 297

(2015) 95-117.
[24] Q. Pang, H. Wang and Z.S. Xu. Probabilistic linguistic term sets in multi-attribute group decision making. Information Sciences 369 (2016) 128-143.
[25] Y. X. Zhang, Z. S. Xu, H. Wang and H. C. Liao. Consistency-based risk assessment with probabilistic linguistic preference relation, Applied Soft Computing 49 (2016) 817-833.
[26] Y.L. Zhai, Z.S. Xu and H.C. Liao. Probabilistic linguistic vector-term set and its application in group decision making with multi-granular linguistic information, Applied Soft Computing 49 (2016) 801-816.
[27] C.Z. Bai, R. Zhang, L.X. Qian and Y.N. Wu. Comparisons of probabilistic linguistic term sets for multi-criteria decision making. Knowledge-Based Systems (2016) DOI: 10.1016/j.knosys. 2016.12.020.
[28] X.J. Gou and Z.S. Xu. Novel basic operational laws for linguistic terms, hesitant fuzzy linguistic term sets and probabilistic linguistic term sets. Information Sciences 372 (2016) 407-427.
[29] W.K.M. Brauers and E.K. Zavadskas. Project management by MULTIMOORA as an instrument for transition economies. Technological and Economic Development of Economy 16(1) (2010) 5-24.
[30] T. Baleentis and S. Zeng. Group multi-criteria decision making based upon interval-valued fuzzy numbers: An extension of the MULTIMOORA method. Expert Systems with Applications 40(2) (2013) 543-550.
[31] W.K.M. Brauers, A. Baležentis and T. Baležentis. MULTIMOORA for the EU Member States updated with fuzzy number theory. Technological and Economic Development of Economy 17(2) (2011) 259-290.
[32] M. Kracka, W. Karel, M. Brauers and E.K. Zavadskas. Ranking heating losses in a building by applying the MULTIMOORA. Engineering Economics 21(4) (2010) 352-359.
[33] W.K.M. Brauers and E.K. Zavadskas. MULTIMOORA optimization used to decide on a bank loan to buy property. Technological and Economic Development of Economy 17(1) (2011) 174-188.
[34] E. Farzamnia and M.B. Babolghani. Group decision-making process for supplier selection using MULTIMOORA technique under fuzzy environment. Kuwait Chapter of Arabian Journal of Business and Management Review 3(11a) (2014) 203-218.
[35] H.C. Liu, X.J. Fan, P. Li and Y.Z. Chen. Evaluating the risk of failure modes with extended MULTIMOORA method under fuzzy environment. Engineering Applications of Artificial Intelligence 34 (2014) 168-177.
[36] D. Deliktas and O. Ustun. Student selection and assignment methodology based on fuzzy

MULTIMOORA and multichoice goal programming. International Transactions in operational research (2017), In press, DOI: 10.1111/itor.12185.
[37] S.Z. Zeng, A. Baležentis and W.H. Su. The multi-criteria hesitant fuzzy group decision making with MULTIMOORA method. Economic Computation & Economic Cybernetics Studies & Research 47(3) (2013) 171-184.
[38] Z.H. Li. An extension of the MULTIMOORA method for multiple criteria group decision making based upon hesitant fuzzy sets. Journal of Applied Mathematics 2014 (2014) 1-16.
[39] Z.S. Chen and Y.L. Li. Approach for group MULTIMOORA decision making based upon prospect intuitionistic trapezoidal fuzzy number Choquet integral operator. Control and Decision 29(6) (2014) 1053-1063.
[40] Z.P. Tian, J. Wang, J.Q. Wang and H.Y. Zhang. An improved MULTIMOORA approach for multi-criteria decision-making based on interdependent inputs of simplified neutrosophic linguistic information. Natural Computing and Applications. (2017), In press, DOI: 10.1007/s00521-016-2378-5.
[41] F. Herrera, L. Martinez, A 2-tuple fuzzy linguistic representation model for computing with words, IEEE Transactions on Fuzzy Systems. 8 (6) (2000) 746–752.
[42] Z.S. Xu and H. Wang. On the syntax and semantics of virtual linguistic terms for information fusion in decision making. Information Fusion 34 (2017) 43-48.
[43] Z.S. Xu. Uncertain linguistic aggregation operators based approach to multiple attribute group decision making under uncertain linguistic environment. Information Sciences 168(1-4) (2004) 171- 184.
[44] C.C Li, Y.C. Dong, F. Herrera, E. Herrera-Viedma and L. Martínez. Personalized individual semantics in computing with words for supporting linguistic group decision making. An application on consensus reaching. Information Fusion 33 (2017) 29-40.
[45] E. Herrera-Viedma, F.J. Cabrerizo, J. Kacprzyk, W. Pedrycz. A review of soft consensus models in a fuzzy environment. Information Fusion 17 (2014) 4-13.
[46] I. Palomares, FJ. Estrella, L. Martínez, F Herrera. Consensus under a fuzzy context: Taxonomy, analysis framework AFRYCA and experimental case of study. Information Fusion 20 (2014) 252-271.
[47] Y. Liu, Z.P. Fan and X. Zhang. A method for large group decision-making based on evaluation information provided by participators from multiple groups. Information Fusion 29 (2016) 132–141.

2 Multiple criteria decision making based on distance and similarity measures under double hierarchy hesitant fuzzy linguistic environment

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Multiple Criteria Decision Making based on Distance and Similarity Measures under Double Hierarchy Hesitant Fuzzy Linguistic Environment

Xunjie Gou[a,b], Zeshui Xu[a,c,*], Huchang Liao[a,b], Francisco Herrera[b,d]

[a] *Business School, Sichuan University, Chengdu 610064, China*

[b] *Department of Computer Science and Artificial Intelligence, University of Granada, E-18071 Granada, Spain*

[c] *School of Computer and Software, Nanjing University of Information Science and Technology, Nanjing, Jiangsu 210044, China*

[d] *Faculty of Computing and Information Technology, King Abdulaziz University, Jeddah, Saudi Arabia*

Abstract

The hesitant fuzzy linguistic term set (HFLTS) has been studied from different research directions. To describe the complicated linguistic information more accurately and reasonably, the double hierarchy linguistic term set (double hierarchy LTS) and double hierarchy hesitant fuzzy linguistic term set (double hierarchy HFLTS) were defined. Considering that the distance and similarity measures are the basis of decision making with double hierarchy hesitant fuzzy linguistic information, this paper proposes some distance and similarity measures of double hierarchy hesitant fuzzy linguistic elements (DHFLEs) and double hierarchy HFLTSs from different angles. We develop a decision-making method to deal with multiple criteria decision making (MCDM) problems on the basis of these distance and similarity measures. Finally, we apply this method to deal with a practical MCDM problem about Sichuan liquor brand assessment.

Keywords: Double hierarchy hesitant fuzzy linguistic term set; Distance measures; Similarity measures; Multiple criteria decision making; Sichuan liquor brand assessment

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1. Introduction

In the process of uncertain decision making, the decision makers usually utilize quantitative information to represent their evaluation results for the convenience of calculation, such as fuzzy sets

* Corresponding Author. Emails: X.J. Gou (gouxunjie@qq.com); Z.S. Xu (xuzeshui@263.net); H.C. Liao (liaohuchang@163.com); F. Herrera (herrera@decsai.ugr.es).

(FSs) (Gao, Sarlak, Parsaei, & Ferdosi, 2018; Zadeh, 1965), intuitionistic fuzzy sets (IFSs) (Atanassov, 1986) and hesitant fuzzy sets (HFSs) (Torra, 2010; Xia & Xu, 2011). Bustince et al. (2016) summarized the historical account of types of FSs and discussed their relationships. The emergence of qualitative information makes more flexible and intuitive to describe the evaluation objects (alternatives, attributes) in words or sentences. In 1975, Zadeh (1975) introduced the fuzzy linguistic approach, which has been extended into different linguistic forms in recent decades including the linguistic models based on type-2 fuzzy sets (Mende, 2002; Türkşen, 2012; Wu, 2014), hesitant fuzzy linguistic term set (HFLTS) (Rodríguez, Martínez, & Herrera, 2012), 2-tuple linguistic model (Dong, Li, & Herrera, 2016; Gao, Zhu, & Wang, 2015; Li, Zeng, & Li, 2015; Liao, Xu, & Zeng, 2014), Virtual linguistic term model (Xu & Wang, 2017) and trapezoid fuzzy linguistic variables (Liu & Su, 2012), etc.

However, considering that people's cognition process and the decision-making information are more and more complex, the linguistic information forms mentioned above cannot describe some more complex linguistic terms or linguistic term sets (LTSs) comprehensively and accurately. For example, let $S=\{s_{-3}=none, s_{-2}=very\ bad, s_{-1}=bad, s_0=medium, s_1=good, s_2=very\ good, s_3=perfect\}$ be a LTS, and we can utilize some simple linguistic terms $\{s_{-1}\}$, $\{s_0\}$ and $\{s_3\}$ to express the linguistic terms "*bad*", "*medium*" and "*perfect*". However, in some practical decision making processes, some experts may need to use some more complex and detailed uncertain linguistic information to represent their comprehensive opinions such that "*entirely good*", "*just right medium*", "*between a little bad and entirely good*", etc. Considering that the existing linguistic models are not suitable for expressing these complex linguistic information, Gou et al. (2017) defined the double hierarchy linguistic term set (double hierarchy LTS), which consists of two hierarchy LTSs (denoted by the first hierarchy LTS and the second hierarchy LTS) with the second hierarchy LTS being a linguistic feature or detailed supplementary of each linguistic term included in the first hierarchy LTS. Based on the double hierarchy LTS, they also developed double hierarchy LTS into hesitant fuzzy linguistic environment and defined a double hierarchy HFLTS, which is constituted by the double hierarchy linguistic terms (DHLTs, the elements included in the double hierarchy LTS). The double hierarchy HFLTS can be used to depict the uncertain linguistic information more specific. For example, let the above LTS S be the first hierarchy LTS, and $O=\{o_{-3}=far\ from, o_{-2}=only\ a\ little, o_{-1}=a\ little, o_0=just\ right, o_1=much, o_2=very\ much, o_3=entirely\}$ be the second hierarchy LTS. Then we can describe the "*entirely good*", "*just right medium*", "*between a little bad and entirely good*" by double hierarchy hesitant fuzzy linguistic elements (DHFLEs, the elements included in double hierarchy HFLTS), which are denoted as $\left\{s_{1<o_3>}\right\}$, $\left\{s_{0<o_0>}\right\}$ and $\left\{s_{-1<o_{-1}>}, s_0, s_{1<o_3>}\right\}$, respectively.

The double hierarchy LTSs and the double hierarchy HFLTSs are very different from some other linguistic models:

(1) Based on fuzzy sets, type-2 fuzzy sets (Dubois & Prade, 1980; Mizumoto & Tanaka, 1976) and type-n fuzzy sets (Dubois & Prade, 1980) that incorporate uncertainty about the membership function in their definitions. And the linguistic model based on type-2 fuzzy sets representation (Mende, 2002) that represents the semantics of the linguistic terms by type-2 membership functions and using interval type-2 fuzzy sets for computing with words. Similarly, double hierarchy LTSs and double hierarchy HFLTSs can be regarded as the special type-2 fuzzy sets, but they mainly utilize linguistic labels (DHLTs and DHFLEs respectively) to describe complex linguistic information directly.

(2) Compared with the 2-tuple linguistic model, double hierarchy LTSs are only established by two hierarchy LTSs, so we can understand the meaning of a linguistic information described by double hierarchy linguistic term with an enricher vocabulary. On the other hand, we can obtain the double hierarchy linguistic information without any calculation.

(3) Especially in hesitant environment, because of double hierarchy LTS consists of two hierarchy LTSs, so the linguistic information described by double hierarchy HFLTSs is more in detail than HFLTSs.

Therefore, both double hierarchy LTS and its hesitant form double hierarchy HFLTS are useful to express complex and uncertain linguistic information. Considering that the double hierarchy HFLTS is a novel concept, we shall pay more attention to the basic characteristics of the double hierarchy HFLTS to apply it to solve the MCDM problems more effectively. Specially, both the distance and similarity measures are fundamentally important in amounts of research fields including decision making (Liao, Xu, & Zeng, 2014; Xu & Wang, 2011; Xu & Xia, 2011), pattern recognition (Arevalillo-Herráez, Ferri, & Domingo, 2013; Li, Hall, & Humphreys, 1993), intelligent computing (Chen, Wang, & Juang, 2010), and recommender systems (Liao, Xu, & Zeng, 2014), distance learning techniques (Gao, Farahani, Aslam, & Hosamani, 2017), electricity markets (Gao, Sarlak, Parsaei, & Ferdosi, 2018), and ontological sparse vector learning (Gao, Zhu, & Wang, 2015), etc. In addition, these measures are also the basis of some well-known methods such as TOPSIS (Tan, Wei, Liu, & Feng, 2016), VIKOR (Liao, Xu, & Zeng, 2015), TODIM (Wei, Ren, & Rodríguez, 2015), etc. Thus, in this paper, we focus on investigating the distance and similarity measures for the double hierarchy HFLTSs, and then apply them to deal with a practical MCDM problem within the context of double hierarchy hesitant fuzzy linguistic circumstances.

To do so, the rest of this paper can be organized as follows: In Section 2, we review the concepts of the double hierarchy LTS and the double hierarchy HFLTS. In Section 3, we define the axioms of distance and similarity measures between two DHFLEs, and then introduce some basic distance and similarity measures between two DHFLEs. In Section 4, we propose some distance and similarity measures between two double hierarchy HFLTSs from three aspects including discrete case, continuous case and ordered weighted case. In Section 5, based on these distance measures, we introduce a MCDM method with double hierarchy hesitant fuzzy linguistic information, and apply this method to deal with a practical MCDM problem about Sichuan liquor brand assessment. In Section 6, we make some discussions on some advantages and limitations. Finally, we point out some concluding remarks in Section 7.

2. Preliminaries: Double hierarchy LTS and double hierarchy HFLTS

In this section, we mainly discuss the concept of double hierarchy LTS and double hierarchy HFLTS.

Definition 2.1 (Gou et al., 2017). Let $S=\left\{s_t \middle| t=-\tau,\dots,-1,0,1,\dots,\tau\right\}$ and $O=\left\{o_k \middle| k=-\varsigma,\dots,-1,0,1,\dots,\varsigma\right\}$ be the first hierarchy and the second hierarchy LTS, respectively, and they are fully independent. A double hierarchy LTS, S_O, is in mathematical form of

$$S_O=\left\{s_{t<o_k>} \middle| t=-\tau,\dots,-1,0,1,\dots,\tau;\ k=-\varsigma,\dots,-1,0,1,\dots,\varsigma\right\} \tag{1}$$

we call $s_{t<o_k>}$ the double hierarchy linguistic term (DHLT), where o_k expresses the second hierarchy linguistic term when the first hierarchy linguistic term is s_t.

For example, if we let $t=3$ and $\varsigma=2$, Fig. 1 can be drawn to show the second hierarchy LTS.

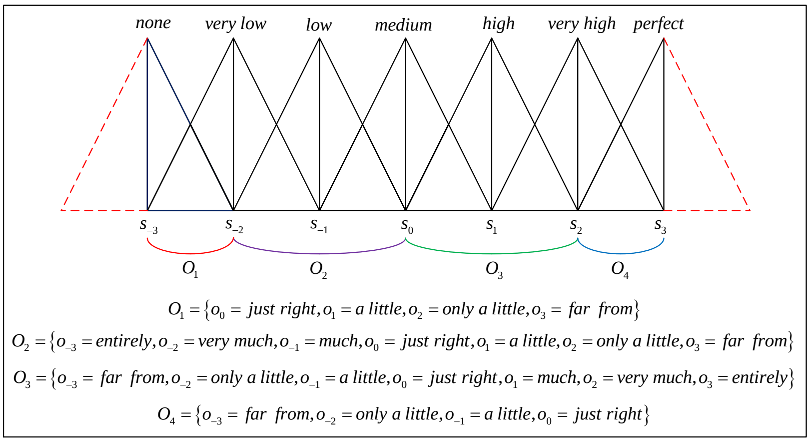

Fig. 1. The distributions of the four parts of the second hierarchy LTS

Remark 2.1. In Fig. 1, four kinds of situations are shown on the basis of different values of t. If $t \geq 0$, then the meaning of the first hierarchy LTS $S = \{s_t | t \geq 0\}$ is positive, so the second hierarchy LTS needs to be selected with the ascending order. On the contrary, if $t < 0$, then the meaning of the first hierarchy LTS $S = \{s_t | t \leq 0\}$ is negative, so the second hierarchy LTS needs to be selected with the descending order. Specially, because both s_τ and $s_{-\tau}$ only contain a half of area compared to other linguistic terms. Therefore, we only utilize $O = \{o_k | k = -\varsigma, \ldots, -1, 0\}$ and $O = \{o_k | k = 0, 1, \ldots, \varsigma\}$ to describe s_τ and $s_{-\tau}$, respectively.

Furthermore, Gou et al. (2017) developed S_O into hesitant fuzzy environment and defined the double hierarchy HFLTS.

Definition 2.2 (Gou et al., 2017**).** Let X be a fixed set, $S_O = \{s_{t<o_k>} | t = -\tau, \ldots, -1, 0, 1, \ldots, \tau; k = -\varsigma, \ldots, -1, 0, 1, \ldots, \varsigma\}$ be a double hierarchy LTS. A double hierarchy hesitant fuzzy linguistic term set (double hierarchy HFLTS) on X, H_{S_O}, is in terms of a membership function that when applied to X returns a subset of S_O, and denoted by a mathematical form:

$$H_{S_O} = \{< x_i, h_{S_O}(x_i) > | x_i \in X\} \tag{2}$$

where $h_{S_O}(x_i)$ is a set of some values in S_O, denoting the possible membership degrees of the element $x_i \in X$ to the set H_{S_O} as:

$$h_{S_O}(x_i) = \{s_{\phi_l<o_{\varphi_l}>}(x_i) | s_{\phi_l<o_{\varphi_l}>} \in S_O; l = 1, 2, \ldots, L; \phi_l = -\tau, \ldots, -1, 0, 1, \ldots, \tau; \varphi_l = -\varsigma, \ldots, -1, 0, 1, \ldots, \varsigma\} \tag{3}$$

with L being the number of the double hierarchy LTS in $h_{S_O}(x_i)$ and $s_{\phi_l<o_{\varphi_l}>}(x_i)$ ($l = 1, \ldots, L$) in each $h_{S_O}(x_i)$ being the continuous terms in S_O. $h_{S_O}(x_i)$ denotes the possible degree of the linguistic variable x_i to S_O. For convenience, we call $h_{S_O}(x_i)$ DHFLE, and double hierarchy LTS included in a DHFLE are ranked in ascending order.

Next, based on the discussion of monotonic function of Dubois (2011), we can define an monotone function for making the mutual transformations between the DHLT and the numerical scale when extending the DHLT to a continuous form, whose indexes are in the intervals $[-\tau, \tau]$ and $[-\varsigma, \varsigma]$ respectively. Like the 2-tuple linguistic terms (Herrera & Martínez, 2000) and the virtual linguistic terms (Xu & Wang, 2017), we can develop continuous function f:

Definition 2.3. Let $\bar{S}_O=\left\{s_{t<o_k>}\middle|t\in[-\tau,\tau];\ k\in[-\varsigma,\varsigma]\right\}$ be a continuous double hierarchy LTS, $h_{S_O}=\left\{s_{\phi_l<o_{\varphi_l}>}\middle|s_{\phi_l<o_{\varphi_l}>}\in\bar{S}_O;\quad l=1,2,\dots,L;\phi_l\in[-\tau,\tau];\ \varphi_l\in[-\varsigma,\varsigma]\right\}$ be a DHFLE with L being the number of linguistic terms in h_{S_O}, and $h_\gamma=\left\{\gamma_l\middle|\gamma_l\in[0,1];l=1,2,\dots,L\right\}$ be a set of numerical scales. Then the subscript (ϕ_l,φ_l) of the DHLT $s_{\phi_l<o_{\varphi_l}>}$ that expresses the equivalent information to the numerical scale γ_l can be transformed to the numerical scale γ_l by a monotone function f:

$$f:[-\tau,\tau]\times[-\varsigma,\varsigma]\to[0,1],\ f(\phi_l,\varphi_l)=\frac{\varphi_l+(\tau+\phi_l)\varsigma}{2\varsigma\tau}=\gamma_l \tag{4}$$

Additionally, let $\Phi\times\Psi$ be the set of all DHFLEs over $\bar{S}_O$, and Θ be the set of all numerical scales. Then a monotone function F between the DHFLE h_{S_O} and a set of numerical scales h_γ on the basis of f is:

$$F:\Phi\times\Psi\to\Theta,\ F\left(h_{S_O}\right)=F\left(\left\{s_{\phi_l<o_{\varphi_l}>}\middle|s_{\phi_l<o_{\varphi_l}>}\in\bar{S}_O;l=1,\dots,L;\phi_l\in[-\tau,\tau];\varphi_l\in[-\varsigma,\varsigma]\right\}\right)=\left\{\gamma_l\,|\,\gamma_l=f(\phi_l,\varphi_l);l=1,2,\dots,L\right\}=h_\gamma \tag{5}$$

Specially, if a DHFLE h_{S_O} only has a DHLT, namely, $h_{S_O}=s_{\phi<o_\varphi>}$, then F reduces to F'

$$F':\bar{S}_O\to[0,1],\ F'\left(h_{S_O}\right)=f(\phi,\varphi)=\gamma \tag{6}$$

Remark 2.2. We can discuss the monotonicity of the function f. Because $\partial f/\partial\phi_l=1/2\tau>0$ and $\partial f/\partial\varphi_l=1/2\varsigma\tau>0$, then f must be an increasing function for both ϕ_l and φ_l, namely for any $\phi_l<\phi_{l+1}$ and $\varphi_l<\varphi_{l+1}$, there is $f(\phi_l,\varphi_l)<f(\phi_{l+1},\varphi_{l+1})$. Additionally, considering that $\Phi\times\Psi$ is the set of all DHFLEs over $\bar{S}_O$, and Θ is the set of all HFEs, then the function F is also an increasing function. Therefore, the double hierarchy linguistic term can be considered as a powerful tool to represent linguistic information with labels in a double level.

3. Axioms and basic distance and similarity measures of DHFLEs

Distance and similarity measures can be utilized to measure the deviation and closeness degrees between different arguments (Liao, Xu, & Zeng, 2014). Up to now, amounts of scholars have developed a lot of distance and similarity measures including some traditional distance measures (Zavadskas et al., 2016) as the Hamming distance (Hamming, 1950), the Euclidean distance (Danielsson, 1980), and the Hausdorff metric (Hausdorff, 1957), and some ordered weighted distance measures (Grzegorzewski, 2014; Hung & Yang, 2004; Liao, Xu, & Zeng, 2014; Xu, 2005; Xu & Chen, 2008). Additionally, these distance and similarity measures have been extended into different uncertain circumstances, such as FSs (Xu, 2012), IFSs (Grzegorzewski, 2014; Hung & Yang, 2004; Xu & Chen, 2008), HFSs (Farhadinia, 2014; Xu & Xia, 2011;), LTSs (Xu & Wang, 2011; Xu, 2005) and HFLTSs (Liao, Xu, & Zeng, 2014; Liao & Xu, 2015).

In this section we shall develop some distance and similarity measures between the DHFLEs by utilizing those previous distance and similarity measures. Firstly, we discuss the axioms of distance and similarity measures between any two single DHFLEs; then some specific distance measures are defined including three basic distances, the hybrid distances, and some distances with preference information.

3.1. The axioms of distance and similarity measures between the DHFLEs

We define the axioms of distance and similarity measures between any two DHFLEs with four properties such as Boundary, Symmetry, Complementarity and Reflexivity:

Definition 3.1. Let $S_O = \left\{ s_{t<o_k>} \middle| t = -\tau, \ldots, -1, 0, 1, \ldots, \tau; k = -\varsigma, \ldots, -1, 0, 1, \ldots, \varsigma \right\}$ be a double hierarchy LTS, $h_{S_O}^i = \left\{ s_{\phi_l<o_{\varphi_l}>}^i \middle| s_{\phi_l<o_{\varphi_l}>}^i \in S_O; l = 1, 2, \ldots, \# h_{S_O}^i \right\}$ $(i = 1, 2)$ be two DHFLEs. Then $d\left(h_{S_O}^1, h_{S_O}^2\right)$ is called the distance measure between $h_{S_O}^1$ and $h_{S_O}^2$ if it satisfies the following properties:

(I) **Boundary:** $0 \le d\left(h_{S_O}^1, h_{S_O}^2\right) \le 1$;

(II) **Symmetry:** $d\left(h_{S_O}^1, h_{S_O}^2\right) = d\left(h_{S_O}^2, h_{S_O}^1\right)$;

(III) **Complementarity:** $d\left(h_{S_O}^1, \bar{h}_{S_O}^1\right) = 1$ iff $F\left(h_{S_O}^1\right) = \{0\}$ or $F\left(h_{S_O}^1\right) = \{1\}$;

(IV) **Reflexivity:** $d\left(h_{S_O}^1, h_{S_O}^2\right) = 0$ iff $h_{S_O}^1 = h_{S_O}^2$.

where $\bar{h}_{S_O}^1 = \left\{ s_{-\phi_l<o_{-\varphi_l}>}^1 \middle| s_{-\phi_l<o_{-\varphi_l}>}^1 \in S_O; l = 1, 2, \ldots, \# h_{S_O}^1 \right\}$ is the complement set of $h_{S_O}^1$, and F is a monotone function.

Definition 3.2. Let $S_O = \left\{ s_{t<o_k>} \middle| t = -\tau, \ldots, -1, 0, 1, \ldots, \tau; k = -\varsigma, \ldots, -1, 0, 1, \ldots, \varsigma \right\}$ be a double hierarchy LTS, $h_{S_O}^i =$ $\left\{ s_{\phi_l<o_{\varphi_l}>}^i \middle| s_{\phi_l<o_{\varphi_l}>}^i \in S_O; l = 1, 2, \ldots, \# h_{S_O}^i \right\}$ $(i = 1, 2)$ be two DHFLEs. Then $p\left(h_{S_O}^1, h_{S_O}^2\right)$ is called the similarity measure between $h_{S_O}^1$ and $h_{S_O}^2$ if it satisfies the following properties:

(I) **Boundary:** $0 \le p\left(h_{S_O}^1, h_{S_O}^2\right) \le 1$;

(II) **Symmetry:** $p\left(h_{S_O}^1, h_{S_O}^2\right) = p\left(h_{S_O}^2, h_{S_O}^1\right)$;

(III) **Complementarity:** $p\left(h_{S_O}^1, \bar{h}_{S_O}^1\right) = 0$ iff $F\left(h_{S_O}^1\right) = \{0\}$ or $F\left(h_{S_O}^1\right) = \{1\}$;

(IV) **Reflexivity:** $p\left(h_{S_O}^1, h_{S_O}^2\right) = 1$ iff $h_{S_O}^1 = h_{S_O}^2$,

where $\bar{h}_{S_O}^1 = \left\{ s_{-\phi_l<o_{-\varphi_l}>}^1 \middle| s_{-\phi_l<o_{-\varphi_l}>}^1 \in S_O; l = 1, 2, \ldots, \#\bar{h}_{S_O}^1 \right\}$ is the complement set of $h_{S_O}^1$, and F is a monotone function.

As we know, there usually exist some relationships between the distance measure $d\left(h_{S_O}^1, h_{S_O}^2\right)$ and the similarity measure $p\left(h_{S_O}^1, h_{S_O}^2\right)$, and the most simple and common relationship is $d\left(h_{S_O}^1, h_{S_O}^2\right) = 1 - p\left(h_{S_O}^1, h_{S_O}^2\right)$. Here we develop a more suitable and comprehensive formula to show the relationship between the distance measure and the similarity measure of the DHFLEs.

Theorem 3.1. Let $\Im : [0,1] \to [0,1]$ be a strictly monotonically decreasing real function, and $d\left(h_{S_O}^1, h_{S_O}^2\right)$ be the distance measure between any two DHFLEs $h_{S_O}^1$ and $h_{S_O}^2$. Then we call

$$p\left(h_{S_O}^1, h_{S_O}^2\right) = \frac{\Im\left(d\left(h_{S_O}^1, h_{S_O}^2\right)\right) - \Im(1)}{\Im(0) - \Im(1)} \tag{7}$$

the similarity measure between $h_{S_O}^1$ and $h_{S_O}^2$ based on the corresponding distance measure $d\left(h_{S_O}^1, h_{S_O}^2\right)$.

Obviously, Eq. (7) satisfies all conditions of similarity measures and we omit the proof of it.

Remark 3.1. For Theorem 3.1, we can establish different formulas to calculate the similarity measures between any two DHFLEs by utilizing different strictly monotonically decreasing real function such as (1) $\Im(\upsilon) = 1 - \upsilon$, (2) $\Im(\upsilon) = \dfrac{1-\upsilon}{1+\upsilon}$, (3) $\Im(\upsilon) = 1 - \upsilon e^{\upsilon - 1}$, and (4) $\Im(\upsilon) = 1 - \upsilon^2$.

As we know, different DHFLEs mainly have different numbers of double hierarchy LTS in most cases. Therefore, it is necessary to add double hierarchy LTS to the shorter DHFLE for calculating the distance and similarity measures between two DHFLEs. Let S_O be a double hierarchy LTS, $h_{S_O} = \left\{ s_{\phi_l<o_{\varphi_l}>} \middle| s_{\phi_l<o_{\varphi_l}>} \in S_O; l = 1, 2, \ldots, \# h_{S_O} \right\}$ be a DHFLE, and $\varepsilon\left(0 \le \varepsilon \le 1\right)$ be an optimized parameter. Because all double hierarchy LTS included in DHFLE are ranked in ascending order,

$s_{\phi_1 <o_{\varphi_1}>}$ and $s_{\phi_{\# h_{S_O}} <o_{\varphi_{\# h_{S_O}}}>}$ are the minimum and maximum double hierarchy LTS in h_{S_O}, respectively. Then we can add the DHLT

$$\tilde{s}_{\phi<o_\varphi>} = s_{(1-\varepsilon)\phi_1+\varepsilon\phi_{\# h_{S_O}} <o_{(1-\varepsilon)\varphi_1+\varepsilon\varphi_{\# h_{S_O}}}>} \tag{8}$$

to the shorter DHFLE. The optimized parameter ε mainly reflects the risk preferences of decision makers with $\tilde{s}_{\phi<o_\varphi>} = s_{\phi_{\# h_{S_O}} <o_{\varphi_{\# h_{S_O}}}>}$ and $\tilde{s}_{\phi<o_\varphi>} = s_{\phi_1 <o_{\varphi_1}>}$ reflect the optimism rule $\varepsilon = 1$ and the pessimism rule $\varepsilon = 0$, respectively. In this paper, we let $\varepsilon = \frac{1}{2}$ and $\tilde{s}_{\phi<o_\varphi>} = s_{\frac{(\phi_1+\phi_{\# h_{S_O}})}{2} <o_{\frac{(\varphi_1+\varphi_{\# h_{S_O}})}{2}}>}$.

3.2. Some basic distance and similarity measures between the DHFLEs

In this subsection, we mainly discuss some basic distance measures between the DHFLEs. Then the corresponding similarity measures can be obtained by Eq. (7) and thus we omit them.

Let S_O be a double hierarchy LTS, $h_{S_O}^i = \left\{ s^i_{\phi_l <o_{\varphi_l}>} \middle| s^i_{\phi_l <o_{\varphi_l}>} \in S_O ; l = 1,2,\ldots,\# h_{S_O}^i \right\}$ $(i = 1,2)$ be two DHFLEs ($\# h_{S_O}^1$ and $\# h_{S_O}^2$ being the number of double hierarchy LTS in $h_{S_O}^1$ and $h_{S_O}^2$ respectively and $\# h_{S_O}^1 = \# h_{S_O}^2 = L$. If not, we can extend the shorter one by adding double hierarchy LTS obtained by Eq. (8)). Based on the well-known Hamming distance and the Euclidean distance, we develop the Hamming distance and the Euclidean distance between $h_{S_O}^1$ and $h_{S_O}^2$, respectively:

$$d_{hd}\left(h_{S_O}^1, h_{S_O}^2\right) = \frac{1}{L}\sum_{l=1}^{L}\left|F'\left(s^1_{\phi_l <o_{\varphi_l}>}\right) - F'\left(s^2_{\phi_l <o_{\varphi_l}>}\right)\right| \tag{9}$$

$$d_{ed}\left(h_{S_O}^1, h_{S_O}^2\right) = \left(\frac{1}{L}\sum_{l=1}^{L}\left(\left|F'\left(s^1_{\phi_l <o_{\varphi_l}>}\right) - F'\left(s^2_{\phi_l <o_{\varphi_l}>}\right)\right|\right)^2\right)^{1/2} \tag{10}$$

where $s^1_{\phi_l <o_{\varphi_l}>}$ and $s^2_{\phi_l <o_{\varphi_l}>}$ are the l-th largest values in $h_{S_O}^1$ and $h_{S_O}^2$ respectively, and F' is a monotone function.

Based on the generalized idea provided by Yager (2004), let $\lambda > 0$, we can further extend the Hamming distance and the Euclidean distance into the generalized distance between $h_{S_O}^1$ and $h_{S_O}^2$:

$$d_{gd}\left(h_{S_O}^1, h_{S_O}^2\right) = \left(\frac{1}{L}\sum_{l=1}^{L}\left(\left|F'\left(s^1_{\phi_l <o_{\varphi_l}>}\right) - F'\left(s^2_{\phi_l <o_{\varphi_l}>}\right)\right|\right)^\lambda\right)^{1/\lambda} \tag{11}$$

Additionally, the generalized Hausdorff distance between $h_{S_O}^1$ and $h_{S_O}^2$ can be given as:

$$d_{ghd}\left(h_{S_O}^1, h_{S_O}^2\right) = \left(\max_{l=1,2,\ldots,L}\left(\left|F'\left(s^1_{\phi_l<o_{\varphi_l}>}\right) - F'\left(s^2_{\phi_l<o_{\varphi_l}>}\right)\right|\right)^{\lambda}\right)^{1/\lambda} \tag{12}$$

where $\lambda > 0$, and F' is a monotone function.

If $\lambda = 1$ and $\lambda = 2$, then Eq. (12) reduces to the Hamming-Hausdorff distance and Euclidean-Hausdorff distance between $h_{S_O}^1$ and $h_{S_O}^2$, respectively:

$$d_{hhd}\left(h_{S_O}^1, h_{S_O}^2\right) = \max_{l=1,2,\ldots,L}\left|F'\left(s^1_{\phi_l<o_{\varphi_l}>}\right) - F'\left(s^2_{\phi_l<o_{\varphi_l}>}\right)\right| \tag{13}$$

$$d_{ehd}\left(h_{S_O}^1, h_{S_O}^2\right) = \left(\max_{l=1,2,\ldots,L}\left(\left|F'\left(s^1_{\phi_l<o_{\varphi_l}>}\right) - F'\left(s^2_{\phi_l<o_{\varphi_l}>}\right)\right|\right)^{2}\right)^{1/2} \tag{14}$$

Furthermore, considering that the hesitance degree (Li, Zeng, & Li, 2015) is an important factor in the calculations about hesitant fuzzy environment, we can define some distance and similarity measures between the DHFLEs with hesitance degrees. Firstly, the hesitance degrees of the DHFLE and the double hierarchy HFLTS can be defined as follows:

Definition 3.3. Let $S_O = \left\{s_{t<o_k>} \middle| t = -\tau, \ldots, -1, 0, 1, \ldots, \tau,\ k = -\varsigma, \ldots, -1, 0, 1, \ldots, \varsigma\right\}$ be a double hierarchy LTS, $H_{S_O} = \left\{< x_i, h_{S_O}(x_i) > \middle| x_i \in X\right\}$ be a double hierarchy HFLTS on X, and $h_{S_O}(x_i) = \left\{s_{\phi_l<o_{\varphi_l}>} \middle| s_{\phi_l<o_{\varphi_l}>} \in S_O; l = 1, 2, \ldots, \# h_{S_O}\right\}$ ($\# h_{S_O}$ being the number of double hierarchy LTS in h_{S_O}). Then we call

$$u\left(h_{S_O}(x_i)\right) = 1 - \frac{1}{\# h_{S_O}} \tag{15}$$

and

$$u\left(H_{S_O}\right) = \frac{1}{n}\sum_{i=1}^{n} u\left(h_{S_O}(x_i)\right) \tag{16}$$

the hesitance degrees of the DHFLE $h_{S_O}(x_i)$ and the double hierarchy HFLTS H_{S_O} respectively.

Remark 3.2. The hesitance degrees $u\left(h_{S_O}(x_i)\right)$ and $u\left(H_{S_O}\right)$ reflect the degree of hesitance of a decision maker. Therefore, the larger the values are, the more hesitant the decision maker should be.

Based on the hesitance degrees of the DHFLEs, the generalized hesitance degree-based distance between two DHFLEs $h_{S_{O_1}}$ and $h_{S_{O_2}}$ can be defined as follows:

$$d_{ghdd}\left(h_{S_O}^1, h_{S_O}^2\right) = \left(\left(\left|u\left(h_{S_O}^1\right) - u\left(h_{S_O}^2\right)\right|\right)^{\lambda}\right)^{1/\lambda} \tag{17}$$

Specially, if $\lambda = 1$ and $\lambda = 2$, then Eq. (17) reduces to the Hamming-hesitance degree-based distance and the Euclidean-hesitance degree-based distance between $h_{S_{O_1}}$ and $h_{S_{O_2}}$, respectively:

$$d_{hhdd}\left(h_{S_O}^1, h_{S_O}^2\right) = \left|u\left(h_{S_O}^1\right) - u\left(h_{S_O}^2\right)\right| \tag{18}$$

$$d_{ehdd}\left(h_{S_O}^1, h_{S_O}^2\right) = \left(\left(\left|u\left(h_{S_O}^1\right) - u\left(h_{S_O}^2\right)\right|\right)^2\right)^{1/2} \tag{19}$$

Based on the three basic distance measures shown as Eqs. (11), (12) and (17), we can develop some generalized hybrid distance measures between the DHFLEs, including the generalized hybrid Hausdorff distance, the generalized hybrid hesitance degree-based distance, and the generalized hybrid Hausdorff-hesitance degree-based distance between $h_{S_O}^1$ and $h_{S_O}^2$, respectively:

$$d_{ghhd}\left(h_{S_O}^1, h_{S_O}^2\right) = \left(\frac{1}{2}\left(\frac{1}{L}\sum_{l=1}^{L}\left(\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^1\right) - F'\left(s_{\phi_l<o_{\varphi_l}>}^2\right)\right|\right)^{\lambda} + \max_{l=1,2,\ldots,L}\left(\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^1\right) - F'\left(s_{\phi_l<o_{\varphi_l}>}^2\right)\right|\right)^{\lambda}\right)\right)^{1/\lambda} \tag{20}$$

$$d_{ghhdd}\left(h_{S_O}^1, h_{S_O}^2\right) = \left(\frac{1}{2}\left(\frac{1}{L}\sum_{l=1}^{L}\left(\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^1\right) - F'\left(s_{\phi_l<o_{\varphi_l}>}^2\right)\right|\right)^{\lambda} + \left(\left|u\left(h_{S_O}^1\right) - u\left(h_{S_O}^2\right)\right|\right)^{\lambda}\right)\right)^{1/\lambda} \tag{21}$$

$$d_{ghhhdd}\left(h_{S_O}^1, h_{S_O}^2\right) = \left(\frac{1}{2}\left(\max_{l=1,2,\ldots,L}\left(\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^1\right) - F'\left(s_{\phi_l<o_{\varphi_l}>}^2\right)\right|\right)^{\lambda} + \left(\left|u\left(h_{S_O}^1\right) - u\left(h_{S_O}^1\right)\right|\right)^{\lambda}\right)\right)^{1/\lambda} \tag{22}$$

Specially, if $\lambda = 1$, then Eqs. (20-22) reduce to the hybrid Hamming-Hausdorff distance, the hybrid Hamming-hesitance degree-based distance, and the hybrid Hamming-Hausdorff-hesitance degree-based distance between $h_{S_O}^1$ and $h_{S_O}^2$, respectively:

$$d_{hhhd}\left(h_{S_O}^1, h_{S_O}^2\right) = \frac{1}{2}\left(\frac{1}{L}\sum_{l=1}^{L}\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^1\right) - F'\left(s_{\phi_l<o_{\varphi_l}>}^2\right)\right| + \max_{l=1,2,\ldots,L}\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^1\right) - F'\left(s_{\phi_l<o_{\varphi_l}>}^2\right)\right|\right) \tag{23}$$

$$d_{hhhdd}\left(h_{S_O}^1, h_{S_O}^2\right) = \frac{1}{2}\left(\frac{1}{L}\sum_{l=1}^{L}\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^1\right) - F'\left(s_{\phi_l<o_{\varphi_l}>}^2\right)\right| + \left|u\left(h_{S_O}^1\right) - u\left(h_{S_O}^2\right)\right|\right) \tag{24}$$

$$d_{hhhhdd}\left(h_{S_O}^1, h_{S_O}^2\right) = \frac{1}{2}\left(\max_{l=1,2,\ldots,L}\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^1\right) - F'\left(s_{\phi_l<o_{\varphi_l}>}^2\right)\right| + \left|u\left(h_{S_O}^1\right) - u\left(h_{S_O}^2\right)\right|\right) \tag{25}$$

If $\lambda = 2$, then Eqs. (20-22) reduce to the hybrid Euclidean-Hausdorff distance, the hybrid Euclidean-hesitance degree-based distance, and the hybrid Euclidean-Hausdorff-hesitance degree-based distance between $h_{S_O}^1$ and $h_{S_O}^2$, respectively:

$$d_{hehd}\left(h_{S_O}^1, h_{S_O}^2\right) = \left(\frac{1}{2}\left(\frac{1}{L}\sum_{l=1}^{L}\left(\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^1\right) - F'\left(s_{\phi_l<o_{\varphi_l}>}^2\right)\right|\right)^2 + \max_{l=1,2,\ldots,L}\left(\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^1\right) - F'\left(s_{\phi_l<o_{\varphi_l}>}^2\right)\right|\right)^2\right)\right)^{1/2} \tag{26}$$

$$d_{hehdd}\left(h_{S_O}^1,h_{S_O}^2\right)=\left(\frac{1}{2}\left(\frac{1}{L}\sum_{l=1}^{L}\left(\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^1\right)-F'\left(s_{\phi_l<o_{\varphi_l}>}^2\right)\right|\right)^2+\left(\left|u\left(h_{S_O}^1\right)-u\left(h_{S_O}^2\right)\right|\right)^2\right)\right)^{1/2} \tag{27}$$

$$d_{hehhdd}\left(h_{S_O}^1,h_{S_O}^2\right)=\left(\frac{1}{2}\left(\max_{l=1,2,\ldots,L}\left(\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^1\right)-F'\left(s_{\phi_l<o_{\varphi_l}>}^2\right)\right|\right)^2+\left(\left|u\left(h_{S_O}^1\right)-u\left(h_{S_O}^2\right)\right|\right)^2\right)\right)^{1/2} \tag{28}$$

Moreover, combining all these three basic distance measures together, the generalized completely hybrid Hausdorff-hesitance degree-based distance between $h_{S_O}^1$ and $h_{S_O}^2$ can be defined as:

$$d_{gchhhdd}\left(h_{S_O}^1,h_{S_O}^2\right)=\left(\frac{1}{3}\left(\frac{1}{L}\sum_{l=1}^{L}\left(\left|F\left(s_{\phi_l<o_{\varphi_l}>}^1\right)-F\left(s_{\phi_l<o_{\varphi_l}>}^2\right)\right|\right)^\lambda+\max_{l=1,2,\ldots,L}\left(\left|F\left(s_{\phi_l<o_{\varphi_l}>}^1\right)-F\left(s_{\phi_l<o_{\varphi_l}>}^2\right)\right|\right)^\lambda+\left(\left|u\left(h_{S_O}^1\right)-u\left(h_{S_O}^2\right)\right|\right)^\lambda\right)\right)^{1/\lambda} \tag{29}$$

Similarly, if $\lambda=1$ and $\lambda=2$, then Eq. (29) reduces to the completely hybrid Hamming-Hausdorff-hesitance degree-based distance and the completely hybrid Euclidean-Hausdorff-hesitance degree-based distance between $h_{S_O}^1$ and $h_{S_O}^2$, respectively:

$$d_{chhhhdd}\left(h_{S_O}^1,h_{S_O}^2\right)=\frac{1}{3}\left(\frac{1}{L}\sum_{l=1}^{L}\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^1\right)-F'\left(s_{\phi_l<o_{\varphi_l}>}^2\right)\right|+\max_{l=1,2,\ldots,L}\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^1\right)-F'\left(s_{\phi_l<o_{\varphi_l}>}^2\right)\right|+\left|u\left(h_{S_O}^1\right)-u\left(h_{S_O}^2\right)\right|\right) \tag{30}$$

$$d_{chehdd}\left(h_{S_O}^1,h_{S_O}^2\right)=\left(\frac{1}{3}\left(\frac{1}{L}\sum_{l=1}^{L}\left(\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^1\right)-F'\left(s_{\phi_l<o_{\varphi_l}>}^2\right)\right|\right)^2+\max_{l=1,2,\ldots,L}\left(\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^1\right)-F'\left(s_{\phi_l<o_{\varphi_l}>}^2\right)\right|\right)^2+\left(\left|u\left(h_{S_O}^1\right)-u\left(h_{S_O}^2\right)\right|\right)^2\right)\right)^{1/2} \tag{31}$$

Example 3.1. Let $S_O=\left\{s_{t<o_k>}\middle|t=-3,\ldots,3,k=-3,\ldots,3\right\}$ be a double hierarchy LTS, $h_{S_O}^2=\left\{s_{-1<o_{-2}>},s_0,s_{1<o_{-1}>}\right\}$ and $h_{S_O}^1=\left\{s_{-2<o_{-1}>},s_{-1},s_0,s_{1<o_2>}\right\}$ be two DHFLEs. The basic distance measures between $h_{S_O}^1$ and $h_{S_O}^2$ by different values of λ are calculated and shown in Fig. 2 and Table 1.

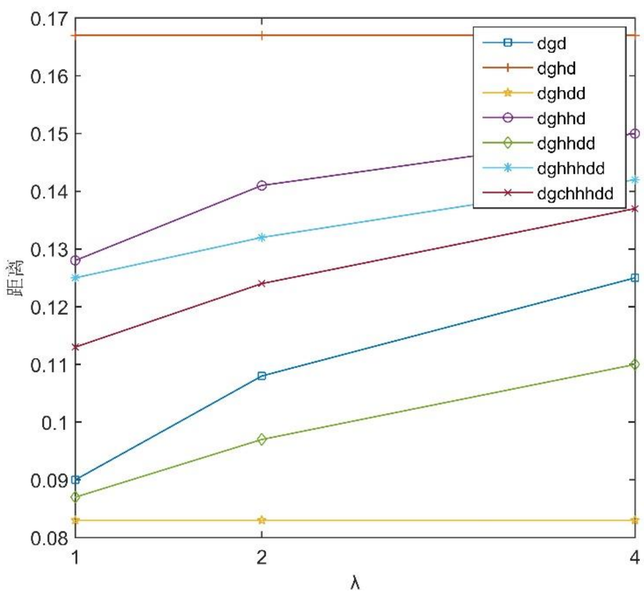

Fig. 2. The distributions of some basic distance measures based on different values of λ

Table 1. The results of some basic distance measures with different values of λ

	$d_{gd}\left(h_{S_O}^1,h_{S_O}^2\right)$	$d_{ghd}\left(h_{S_O}^1,h_{S_O}^2\right)$	$d_{ghdd}\left(h_{S_O}^1,h_{S_O}^2\right)$	$d_{ghhd}\left(h_{S_O}^1,h_{S_O}^2\right)$	$d_{ghhdd}\left(h_{S_O}^1,h_{S_O}^2\right)$	$d_{ghhhdd}\left(h_{S_O}^1,h_{S_O}^2\right)$	$d_{gchhhdd}\left(h_{S_O}^1,h_{S_O}^2\right)$
$\lambda=1$	0.090	0.167	0.083	0.129	0.087	0.125	0.113
$\lambda=2$	0.109	0.167	0.083	0.141	0.097	0.132	0.125
$\lambda=4$	0.125	0.167	0.083	0.150	0.110	0.142	0.137

Remark 3.3. In Table 1 and Fig. 2, firstly, for any distance measure, we can find that the bigger the value of λ is, the greater (or at least the same) the distance measures would be. Secondly, no matter what the value of λ is, the values of the three hybrid distance measures are between two corresponding basic distance measures. Similarly, the value of the completely hybrid distance measure is among three basic distance measures.

3.3. Some distance and similarity measures with preference information

As we discussed above, we give the same preference to membership values, Hausdorff distances and hesitance degrees. However, the decision maker usually owns different preferences for different distance measures in actual situations. Therefore, some distance measures with preference information between any two DHFLEs can be defined and the corresponding similarity measures can be omitted.

For any two DHFLEs $h_{S_O}^1$ and $h_{S_O}^2$, and combining all the three basic distance measures discussed in Subsection 3.2, the generalized completely hybrid Hausdorff-hesitance degree-preference distance between $h_{S_O}^1$ and $h_{S_O}^2$ can be defined as:

$$d_{gchhhdpd}\left(h_{S_O}^1,h_{S_O}^2\right)=\left(\frac{a}{L}\sum_{l=1}^{L}\left(\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^1\right)-F'\left(s_{\phi_l<o_{\varphi_l}>}^2\right)\right|\right)^{\lambda}+b\max_{l=1,2,\ldots,L}\left(\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^1\right)-F'\left(s_{\phi_l<o_{\varphi_l}>}^2\right)\right|\right)^{\lambda}+c\left(\left|u\left(h_{S_O}^1\right)-u\left(h_{S_O}^2\right)\right|\right)^{\lambda}\right)^{1/\lambda} \quad (32)$$

where $\lambda>0$, $0\le a,b,c\le 1$, $a+b+c=1$, and F' is a monotone function.

Next, different distance measures can be obtained by taking the values of λ, a, b, and c:

(1) If $\lambda=1$ and $\lambda=2$, then Eq. (32) reduces to the completely hybrid Hamming-Hausdorff-hesitance degree-preference distance and the completely hybrid Euclidean-Hausdorff-hesitance degree-preference distance between $h_{S_O}^1$ and $h_{S_O}^2$, respectively:

$$d_{chhhhdpd}\left(h_{S_O}^1,h_{S_O}^2\right)=\left(\frac{a}{L}\sum_{l=1}^{L}\left(\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^1\right)-F'\left(s_{\phi_l<o_{\varphi_l}>}^2\right)\right|\right)+b\max_{l=1,2,\ldots,L}\left(\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^1\right)-F'\left(s_{\phi_l<o_{\varphi_l}>}^2\right)\right|\right)+c\left(\left|u\left(h_{S_O}^1\right)-u\left(h_{S_O}^2\right)\right|\right)\right) \quad (33)$$

$$d_{chehhdpd}\left(h_{S_O}^1,h_{S_O}^2\right)=\left(\frac{a}{L}\sum_{l=1}^{L}\left(\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^1\right)-F'\left(s_{\phi_l<o_{\varphi_l}>}^2\right)\right|\right)^{2}+b\max_{l=1,2,\ldots,L}\left(\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^1\right)-F'\left(s_{\phi_l<o_{\varphi_l}>}^2\right)\right|\right)^{2}+c\left(\left|u\left(h_{S_O}^1\right)-u\left(h_{S_O}^2\right)\right|\right)^{2}\right)^{1/2} \quad (34)$$

(2) If $a+b=1$ and $c=0$, then Eq. (32) reduces to the generalized hybrid Hausdorff-

preference distance between $h_{S_O}^1$ and $h_{S_O}^2$:

$$d_{ghhpd}\left(h_{S_O}^1,h_{S_O}^2\right)=\left(\frac{a}{L}\sum_{l=1}^{L}\left(\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^1\right)-F'\left(s_{\phi_l<o_{\varphi_l}>}^2\right)\right|\right)^{\lambda}+b\max_{l=1,2,\ldots,L}\left(\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^1\right)-F'\left(s_{\phi_l<o_{\varphi_l}>}^2\right)\right|\right)^{\lambda}\right)^{1/\lambda} \quad (35)$$

If $\lambda=1$ and $\lambda=2$, then Eq. (35) reduces to the hybrid Hamming-Hausdorff-preference distance and the hybrid Euclidean-Hausdorff-preference distance between $h_{S_O}^1$ and $h_{S_O}^2$, respectively:

$$d_{hhhpd}\left(h_{S_O}^1,h_{S_O}^2\right)=\left(\frac{a}{L}\sum_{l=1}^{L}\left(\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^1\right)-F'\left(s_{\phi_l<o_{\varphi_l}>}^2\right)\right|\right)+b\max_{l=1,2,\ldots,L}\left(\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^1\right)-F'\left(s_{\phi_l<o_{\varphi_l}>}^2\right)\right|\right)\right) \quad (36)$$

$$d_{hehpd}\left(h_{S_O}^1,h_{S_O}^2\right)=\left(\frac{a}{L}\sum_{l=1}^{L}\left(\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^1\right)-F'\left(s_{\phi_l<o_{\varphi_l}>}^2\right)\right|\right)^{2}+b\max_{l=1,2,\ldots,L}\left(\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^1\right)-F'\left(s_{\phi_l<o_{\varphi_l}>}^2\right)\right|\right)^{2}\right)^{1/2} \quad (37)$$

(3) If $a+c=1$ and $b=0$, then Eq. (34) reduces to the generalized hybrid hesitance degree-preference distance between $h_{S_O}^1$ and $h_{S_O}^2$:

$$d_{ghhdpd}\left(h_{S_O}^1,h_{S_O}^2\right)=\left(\frac{a}{L}\sum_{l=1}^{L}\left(\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^1\right)-F'\left(s_{\phi_l<o_{\varphi_l}>}^2\right)\right|\right)^{\lambda}+c\left(\left|u\left(h_{S_O}^1\right)-u\left(h_{S_O}^2\right)\right|\right)^{\lambda}\right)^{1/\lambda} \quad (38)$$

Furthermore, if $\lambda=1$ and $\lambda=2$, then Eq. (38) reduces to the hybrid Hamming-hesitance degree-preference distance and the hybrid Euclidean-hesitance degree-preference distance between $h_{S_O}^1$ and $h_{S_O}^2$, respectively:

$$d_{hhhdpd}\left(h_{S_O}^1,h_{S_O}^2\right)=\left(\frac{a}{L}\sum_{l=1}^{L}\left(\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^1\right)-F'\left(s_{\phi_l<o_{\varphi_l}>}^2\right)\right|\right)+c\left(\left|u\left(h_{S_O}^1\right)-u\left(h_{S_O}^2\right)\right|\right)\right) \quad (39)$$

$$d_{hehdpd}\left(h_{S_O}^1,h_{S_O}^2\right)=\left(\frac{a}{L}\sum_{l=1}^{L}\left(\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^1\right)-F'\left(s_{\phi_l<o_{\varphi_l}>}^2\right)\right|\right)^{2}+c\left(\left|u\left(h_{S_O}^1\right)-u\left(h_{S_O}^2\right)\right|\right)^{2}\right)^{1/2} \quad (40)$$

(4) If $b+c=1$ and $a=0$, then Eq. (32) reduces to the generalized hybrid Hausdorff-hesitance degree-preference distance between $h_{S_O}^1$ and $h_{S_O}^2$:

$$d_{ghhhdpd}\left(h_{S_O}^1,h_{S_O}^2\right)=\left(b\max_{l=1,2,\ldots,L}\left(\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^1\right)-F'\left(s_{\phi_l<o_{\varphi_l}>}^2\right)\right|\right)^{\lambda}+c\left(\left|u\left(h_{S_O}^1\right)-u\left(h_{S_O}^2\right)\right|\right)^{\lambda}\right)^{1/\lambda} \quad (41)$$

Furthermore, if $\lambda=1$ and $\lambda=2$, then Eq. (41) reduces to the hybrid Hamming-Hausdorff-hesitance degree-preference distance and the hybrid Euclidean-Hausdorff-hesitance degree-preference distance between $h_{S_O}^1$ and $h_{S_O}^2$, respectively:

$$d_{hhhhdpd}\left(h_{S_O}^1,h_{S_O}^2\right)=\left(b\max_{l=1,2,\ldots,L}\left(\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^1\right)-F'\left(s_{\phi_l<o_{\varphi_l}>}^2\right)\right|\right)+c\left(\left|u\left(h_{S_O}^1\right)-u\left(h_{S_O}^2\right)\right|\right)\right) \quad (42)$$

$$d_{hehhdpd}\left(h_{S_O}^{1},h_{S_O}^{2}\right)=\left(b\max_{l=1,2,\ldots,L}\left(\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^{1}\right)-F'\left(s_{\phi_l<o_{\varphi_l}>}^{2}\right)\right|\right)^{2}+c\left(\left|u\left(h_{S_O}^{1}\right)-u\left(h_{S_O}^{2}\right)\right|\right)^{2}\right)^{1/2} \quad (43)$$

4. Some distance and similarity measures between the double hierarchy HFLTSs

In Section 3, we have developed some distance and similarity measures between two DHFLEs over only one double hierarchy linguistic variable. However, in some practical problems especially in the MCDM problems, the decision makers usually use a set to express their evaluation information when evaluating each alternative (or object) with respect to all attributes (or criteria). Therefore, the double hierarchy HFLTS is a perfect expression to take into account all aspects. Additionally, the weights of criteria are very important in the MCDM problems, and we need to consider them. When the evaluation information of each alternative (or object) with respect to all criteria is expressed by the double hierarchy HFLTS, the distance and similarity measures are very important to deal with the MCDM problems. This section mainly establishes some weighted distance and similarity measures between the double hierarchy HFLTSs.

Firstly, the axioms of the distance and similarity measures between the double hierarchy HFLTSs can be shown. Then, we develop some weighted distance and similarity measures between the double hierarchy HFLTSs in discrete case, continuous case, respectively. Finally, we propose some ordered weighted distance and similarity measures between the double hierarchy HFLTSs.

4.1. Axioms and distance and similarity measures for double hierarchy HFLTSs

Definition 4.1. Let $S_O=\left\{s_{t<o_k>}\middle|t=-\tau,\ldots,-1,0,1,\ldots,\tau;\,k=-\varsigma,\ldots,-1,0,1,\ldots,\varsigma\right\}$ be a double hierarchy LTS, $H_{S_O}^{1}=\left\{h_{S_O}^{11},h_{S_O}^{12},\ldots,h_{S_O}^{1n}\right\}$ and $H_{S_O}^{2}=\left\{h_{S_O}^{21},h_{S_O}^{22},\ldots,h_{S_O}^{2n}\right\}$ be two double hierarchy HFLTSs. Then $d\left(H_{S_O}^{1},H_{S_O}^{2}\right)$ is called the distance measure between $H_{S_O}^{1}$ and $H_{S_O}^{2}$ if it satisfies the following properties:

(I) **Boundary:** $0\le d\left(H_{S_O}^{1},H_{S_O}^{2}\right)\le 1$;

(II) **Symmetry:** $d\left(H_{S_O}^{1},H_{S_O}^{2}\right)=d\left(H_{S_O}^{2},H_{S_O}^{1}\right)$;

(III) **Complementarity:** $d\left(H_{S_O}^{1},\bar{H}_{S_O}^{1}\right)=1$ iff $H_{S_O}^{1}=\left\{s_{\tau<o_{\varsigma}>}\right\}$ or $H_{S_O}^{1}=\left\{s_{-\tau<o_{-\varsigma}>}\right\}$;

(IV) **Reflexivity:** $d\left(H_{S_O}^{1},H_{S_O}^{2}\right)=0$ iff $H_{S_O}^{1}=H_{S_O}^{2}$.

where $\bar{H}^1_{S_O} = \left\{\bar{h}^{11}_{S_O}, \bar{h}^{12}_{S_O}, \ldots, \bar{h}^{1n}_{S_O}\right\}$ is the complement set of $H^1_{S_O}$.

Definition 4.2. Let $S_O = \left\{s_{t<o_k>} \middle| t = -\tau, \ldots, -1, 0, 1, \ldots, \tau; k = -\varsigma, \ldots, -1, 0, 1, \ldots, \varsigma\right\}$ be a double hierarchy LTS, $H^1_{S_O}$ and $H^2_{S_O}$ be two DHFLEs. Then $p\left(H^1_{S_O}, H^2_{S_O}\right)$ is called the similarity measure between $H^1_{S_O}$ and $H^2_{S_O}$ if it satisfies the following properties:

(I) **Boundary:** $0 \le p\left(H^1_{S_O}, H^2_{S_O}\right) \le 1$;

(II) **Symmetry:** $p\left(H^1_{S_O}, H^2_{S_O}\right) = p\left(H^2_{S_O}, H^1_{S_O}\right)$;

(III) **Complementarity:** $p\left(H^1_{S_O}, \bar{H}^1_{S_O}\right) = 0$ iff $H^1_{S_O} = \left\{s_{\tau<o_\varsigma>}\right\}$ or $H^1_{S_O} = \left\{s_{-\tau<o_{-\varsigma}>}\right\}$;

(IV) **Reflexivity:** $p\left(H^1_{S_O}, H^2_{S_O}\right) = 1$ iff $H^1_{S_O} = H^2_{S_O}$.

where $\bar{H}^1_{S_O} = \left\{\bar{h}^{11}_{S_O}, \bar{h}^{12}_{S_O}, \ldots, \bar{h}^{1n}_{S_O}\right\}$ is the complement set of $H^1_{S_O}$.

Similar to Eq. (7), we can also establish the relationship between the distance measure and the similarity measure of the double hierarchy HFLTSs by utilizing the following formula:

$$p\left(H^1_{S_O}, H^2_{S_O}\right) = \frac{\Im\left(d\left(H^1_{S_O}, H^2_{S_O}\right)\right) - \Im(1)}{\Im(0) - \Im(1)} \tag{44}$$

Similarly, the strictly monotonically decreasing real function can be (1) $\Im(\upsilon) = 1 - \upsilon$, (2) $\Im(\upsilon) = \frac{1-\upsilon}{1+\upsilon}$, (3) $\Im(\upsilon) = 1 - \upsilon e^{\upsilon - 1}$, and (4) $\Im(\upsilon) = 1 - \upsilon^2$.

4.2. Weighted distance and similarity measures between the double hierarchy HFLTSs in discrete case

Let $S_O = \left\{s_{t<o_k>} \middle| t = -\tau, \ldots, -1, 0, 1, \ldots, \tau; k = -\varsigma, \ldots, -1, 0, 1, \ldots, \varsigma\right\}$ be a double hierarchy LTS, $H^1_{S_O} = \left\{h^{11}_{S_O}, h^{12}_{S_O}, \ldots, h^{1n}_{S_O}\right\}$ and $H^2_{S_O} = \left\{h^{21}_{S_O}, h^{22}_{S_O}, \ldots, h^{2n}_{S_O}\right\}$ be two double hierarchy HFLTSs, where $h^{1j}_{S_O} = \left\{s^{1j}_{\phi_l<o_{\varphi_l}>} \middle| s^{1j}_{\phi_l<o_{\varphi_l}>} \in S_O; l = 1, 2, \ldots, \#h^{1j}_{S_O}\right\}$ $(j = 1, 2, \ldots, n)$ ($\#h^{1j}_{S_O}$ being the number of double hierarchy LTS in $h^{1j}_{S_O}$) and $h^{2j}_{S_O} = \left\{s^{2j}_{\phi_l<o_{\varphi_l}>} \middle| s^{2j}_{\phi_l<o_{\varphi_l}>} \in S_O; l = 1, 2, \ldots, \#h^{2j}_{S_O}\right\}$ $(j = 1, 2, \ldots, n)$ ($\#h^{2j}_{S_O}$ being the number of double hierarchy LTS in $h^{2j}_{S_O}$). For $H^1_{S_O}$ and $H^2_{S_O}$ with the associated weighting vector $w = (w_1, w_2, \ldots, w_n)^T$, where $0 \le w_j \le 1$ and $\sum_{j=1}^{n} w_j = 1$, the generalized weighted distance, the generalized weighted Hausdorff distance, and the generalized weighted hesitance degree-based distance between $H_{S_O}^1$ and $H_{S_O}^2$ can be defined, respectively:

$$d_{gwd}\left(H_{S_O}^1, H_{S_O}^2\right) = \left(\sum_{j=1}^{n} \frac{w_j}{L} \sum_{l=1}^{L} \left(\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^{1j}\right) - F'\left(s_{\phi_l<o_{\varphi_l}>}^{2j}\right)\right|\right)^{\lambda}\right)^{1/\lambda} \tag{45}$$

$$d_{gwhd}\left(H_{S_O}^1, H_{S_O}^2\right) = \left(\sum_{j=1}^{n} w_j \max_{l=1,2,\ldots,L} \left(\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^{1j}\right) - F'\left(s_{\phi_l<o_{\varphi_l}>}^{2j}\right)\right|\right)^{\lambda}\right)^{1/\lambda} \tag{46}$$

$$d_{gwhdd}\left(H_{S_O}^1, H_{S_O}^2\right) = \left(\sum_{j=1}^{n} w_j \left(\left(\left|u\left(h_{S_O}^{1j}\right) - u\left(h_{S_O}^{2j}\right)\right|\right)^{\lambda}\right)\right)^{1/\lambda} \tag{47}$$

where $\lambda > 0$, and F' is a monotone function. Specially, if $\lambda = 1$ and $\lambda = 2$, then Eqs. (45)-(47) reduce to the corresponding Hamming and Euclidean distances, here we omit them.

Additionally, some generalized hybrid weighted distance measures can be defined as the generalized hybrid weighted Hausdorff distance, the generalized hybrid weighted hesitance degree-based distance, the generalized hybrid weighted Hausdorff-hesitance degree-based distance, and the generalized completely hybrid weighted Hausdorff-hesitance degree-based distance between $H_{S_O}^1$ and $H_{S_O}^2$, respectively:

$$d_{ghwhd}\left(H_{S_O}^1, H_{S_O}^2\right) = \left(\sum_{j=1}^{n} \frac{w_j}{2}\left(\frac{1}{L}\sum_{l=1}^{L}\left(\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^{1j}\right) - F'\left(s_{\phi_l<o_{\varphi_l}>}^{2j}\right)\right|\right)^{\lambda} + \max_{l=1,2,\ldots,L}\left(\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^{1j}\right) - F'\left(s_{\phi_l<o_{\varphi_l}>}^{2j}\right)\right|\right)^{\lambda}\right)\right)^{1/\lambda} \tag{48}$$

$$d_{ghwhdd}\left(H_{S_O}^1, H_{S_O}^2\right) = \left(\sum_{j=1}^{n} \frac{w_j}{2}\left(\frac{1}{L}\sum_{l=1}^{L}\left(\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^{1j}\right) - F'\left(s_{\phi_l<o_{\varphi_l}>}^{2j}\right)\right|\right)^{\lambda} + \left(\left|u\left(h_{S_O}^{1j}\right) - u\left(h_{S_O}^{2j}\right)\right|\right)^{\lambda}\right)\right)^{1/\lambda} \tag{49}$$

$$d_{ghwhhdd}\left(H_{S_O}^1, H_{S_O}^2\right) = \left(\sum_{j=1}^{n} \frac{w_j}{2}\left(\max_{l=1,2,\ldots,L}\left(\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^{1j}\right) - F'\left(s_{\phi_l<o_{\varphi_l}>}^{2j}\right)\right|\right)^{\lambda} + \left(\left|u\left(h_{S_O}^{1j}\right) - u\left(h_{S_O}^{2j}\right)\right|\right)^{\lambda}\right)\right)^{1/\lambda} \tag{50}$$

$$d_{gchwhhdd}\left(H_{S_O}^1, H_{S_O}^2\right) = \left(\sum_{j=1}^{n} \frac{w_j}{3}\left(\frac{1}{L}\sum_{l=1}^{L}\left(\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^{1j}\right) - F'\left(s_{\phi_l<o_{\varphi_l}>}^{2j}\right)\right|\right)^{\lambda} + \max_{l=1,2,\ldots,L}\left(\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^{1j}\right) - F'\left(s_{\phi_l<o_{\varphi_l}>}^{2j}\right)\right|\right)^{\lambda} + \left(\left|u\left(h_{S_O}^{1j}\right) - u\left(h_{S_O}^{2j}\right)\right|\right)^{\lambda}\right)\right)^{1/\lambda} \tag{51}$$

where $\lambda > 0$, and F' is a monotone function. Similarly, if $\lambda = 1$ and $\lambda = 2$, then Eqs. (48)-(51) reduce to the corresponding Hamming and Euclidean distances, we also omit them.

In addition, if we consider the preference information about the Hausdorff distances, the hesitance degrees and the membership values, then the generalized completely hybrid weighted Hausdorff-hesitance degree-preference distance between $H_{S_O}^1$ and $H_{S_O}^2$ can be defined as:

$$d_{gchwhhdpd}\left(H_{S_O}^1, H_{S_O}^2\right) = \left(\sum_{j=1}^{n} w_j\left(\frac{a}{L}\sum_{l=1}^{L}\left(\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^{1j}\right) - F'\left(s_{\phi_l<o_{\varphi_l}>}^{2j}\right)\right|\right)^{\lambda} + b\max_{l=1,2,\ldots,L}\left(\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^{1j}\right) - F'\left(s_{\phi_l<o_{\varphi_l}>}^{2j}\right)\right|\right)^{\lambda} + c\left(\left|u\left(h_{S_O}^{1j}\right) - u\left(h_{S_O}^{2j}\right)\right|\right)^{\lambda}\right)\right)^{1/\lambda} \tag{52}$$

where $\lambda > 0$, $0 \le a,b,c \le 1$, $a+b+c=1$, and F' is a monotone function.

Next, we can obtain different distance measures based on the values of λ, a, b, and c:

(1) If $a+b=1$ and $c=0$, then Eq. (52) reduces to the generalized hybrid weighted Hausdorff-preference distance $H_{S_O}^1$ and $H_{S_O}^2$:

$$d_{ghwhpd}\left(H_{S_O}^1, H_{S_O}^2\right)=\left(\sum_{j=1}^{n} w_j\left(\frac{a}{L}\sum_{l=1}^{L}\left(\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^{1j}\right)-F'\left(s_{\phi_l<o_{\varphi_l}>}^{2j}\right)\right|\right)^{\lambda}+b\max_{l=1,2,\ldots,L}\left(\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^{1j}\right)-F'\left(s_{\phi_l<o_{\varphi_l}>}^{2j}\right)\right|\right)^{\lambda}\right)\right)^{1/\lambda} \quad (53)$$

(2) If $a+c=1$ and $b=0$, then Eq. (52) reduces to the generalized hybrid weighted hesitance degree-preference distance between $H_{S_O}^1$ and $H_{S_O}^2$:

$$d_{ghwhdpd}\left(H_{S_O}^1, H_{S_O}^2\right)=\left(\sum_{j=1}^{n} w_j\left(\frac{a}{L}\sum_{l=1}^{L}\left(\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^{1j}\right)-F'\left(s_{\phi_l<o_{\varphi_l}>}^{2j}\right)\right|\right)^{\lambda}+c\left(\left|u\left(h_{S_O}^{1j}\right)-u\left(h_{S_O}^{2j}\right)\right|\right)^{\lambda}\right)\right)^{1/\lambda} \quad (54)$$

(3) If $b+c=1$ and $a=0$, then Eq. (52) reduces to the generalized hybrid weighted Hausdorff-hesitance degree-preference distance between $H_{S_O}^1$ and $H_{S_O}^2$:

$$d_{ghwhhdpd}\left(H_{S_O}^1, H_{S_O}^2\right)=\left(\sum_{j=1}^{n} w_j\left(b\max_{l=1,2,\ldots,L}\left(\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^{1j}\right)-F'\left(s_{\phi_l<o_{\varphi_l}>}^{2j}\right)\right|\right)^{\lambda}+c\left(\left|u\left(h_{S_O}^{1j}\right)-u\left(h_{S_O}^{2j}\right)\right|\right)^{\lambda}\right)\right)^{1/\lambda} \quad (55)$$

Similarly, if $\lambda=1$ and $\lambda=2$, then Eqs. (52)-(55) reduce to the corresponding Hamming and Euclidean distances, here we omit them.

4.3. Weighted distance and similarity measures between the double hierarchy HFLTSs in continuous case

Obviously, all the distance and similarity measures discussed above are in discrete case. If both the universe of discourse and the weights of elements are continuous, we can define some distance and similarity measures between the double hierarchy HFLTSs in continuous case.

Let $x\in[\alpha,\beta]$, and $w(x)$ be the weight of x, where $0\le w(x)\le 1$ and $\int_{\alpha}^{\beta} w(x)dx=1$. Let $H_{S_O}^1$ and $H_{S_O}^2$ be two double hierarchy HFLTSs over the element x. Then we can define the generalized continuous weighted distance, the generalized continuous weighted Hausdorff distance, and the generalized continuous weighted hesitance degree-based distance between $H_{S_O}^1$ and $H_{S_O}^2$, respectively:

$$d_{gcwd}\left(H_{S_O}^1, H_{S_O}^2\right)=\left(\int_{\alpha}^{\beta} w(x)\frac{1}{L}\sum_{l=1}^{L}\left(\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^{1}(x)\right)-F'\left(s_{\phi_l<o_{\varphi_l}>}^{2}(x)\right)\right|\right)^{\lambda}dx\right)^{1/\lambda} \quad (56)$$

$$d_{gcwhd}\left(H_{S_O}^1, H_{S_O}^2\right)=\left(\int_{\alpha}^{\beta} w(x)\max_{l=1,2,\ldots,L}\left(\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^{1}(x)\right)-F'\left(s_{\phi_l<o_{\varphi_l}>}^{2}(x)\right)\right|\right)^{\lambda}dx\right)^{1/\lambda} \quad (57)$$

$$d_{gcwhdd}\left(H_{S_O}^1, H_{S_O}^2\right)=\left(\int_\alpha^\beta w(x)\left(\left|u\left(h_{S_O}^1(x)\right)-u\left(h_{S_O}^2(x)\right)\right|\right)^\lambda dx\right)^{1/\lambda} \tag{58}$$

Specially, if $\lambda=1$, then Eqs. (56)-(58) reduce to the continuous weighted Hamming distance, the continuous weighted Hamming-Hausdorff distance, and the continuous weighted Hamming-hesitance degree-based distance between $H_{S_O}^1$ and $H_{S_O}^2$, respectively:

$$d_{cwhd}\left(H_{S_O}^1, H_{S_O}^2\right)=\left(\int_\alpha^\beta w(x)\frac{1}{L}\sum_{l=1}^{L}\left|F'\left(s^1_{\phi_l<o_{\varphi_l}>}(x)\right)-F'\left(s^2_{\phi_l<o_{\varphi_l}>}(x)\right)\right|dx\right) \tag{59}$$

$$d_{cwhhd}\left(H_{S_O}^1, H_{S_O}^2\right)=\left(\int_\alpha^\beta w(x)\max_{l=1,2,\ldots,L}\left|F'\left(s^1_{\phi_l<o_{\varphi_l}>}(x)\right)-F'\left(s^2_{\phi_l<o_{\varphi_l}>}(x)\right)\right|dx\right) \tag{60}$$

$$d_{cwhhdd}\left(H_{S_O}^1, H_{S_O}^2\right)=\left(\int_\alpha^\beta w(x)\left|u\left(h_{S_O}^1(x)\right)-u\left(h_{S_O}^2(x)\right)\right|dx\right) \tag{61}$$

If $\lambda=2$, then Eqs. (56)-(58) reduce to the continuous weighted Euclidean distance, the continuous weighted Euclidean-Hausdorff distance, and the continuous weighted Euclidean-hesitance degree-based distance between $H_{S_O}^1$ and $H_{S_O}^2$, respectively:

$$d_{cwed}\left(H_{S_O}^1, H_{S_O}^2\right)=\left(\int_\alpha^\beta w(x)\frac{1}{L}\sum_{l=1}^{L}\left(\left|F'\left(s^1_{\phi_l<o_{\varphi_l}>}(x)\right)-F'\left(s^2_{\phi_l<o_{\varphi_l}>}(x)\right)\right|\right)^2dx\right)^{1/2} \tag{62}$$

$$d_{cwehd}\left(H_{S_O}^1, H_{S_O}^2\right)=\left(\int_\alpha^\beta w(x)\max_{l=1,2,\ldots,L}\left(\left|F'\left(s^1_{\phi_l<o_{\varphi_l}>}(x)\right)-F'\left(s^2_{\phi_l<o_{\varphi_l}>}(x)\right)\right|\right)^2dx\right)^{1/2} \tag{63}$$

$$d_{cwehdd}\left(H_{S_O}^1, H_{S_O}^2\right)=\left(\int_\alpha^\beta w(x)\left(\left|u\left(h_{S_O}^1(x)\right)-u\left(h_{S_O}^2(x)\right)\right|\right)^2 dx\right)^{1/2} \tag{64}$$

Additionally, we can define some hybrid continuous weighted distance measures, such as the generalized hybrid continuous weighted Hausdorff distance, the generalized hybrid continuous weighted hesitance degree-based distance, the generalized hybrid continuous weighted Hausdorff-hesitance degree-based distance, the generalized completely hybrid continuous weighted distance, and the generalized completely hybrid continuous weighted distance between $H_{S_O}^1$ and $H_{S_O}^2$, respectively:

$$d_{ghcwhd}\left(H_{S_O}^1, H_{S_O}^2\right)=\left(\int_\alpha^\beta \frac{w(x)}{2}\left(\frac{1}{L}\sum_{l=1}^{L}\left(\left|F'\left(s^1_{\phi_l<o_{\varphi_l}>}(x)\right)-F'\left(s^2_{\phi_l<o_{\varphi_l}>}(x)\right)\right|\right)^\lambda+\max_{l=1,2,\ldots,L}\left(\left|F'\left(s^1_{\phi_l<o_{\varphi_l}>}(x)\right)-F'\left(s^2_{\phi_l<o_{\varphi_l}>}(x)\right)\right|\right)^\lambda\right)dx\right)^{1/\lambda} \tag{65}$$

$$d_{ghcwhdd}\left(H_{S_O}^1, H_{S_O}^2\right)=\left(\int_\alpha^\beta \frac{w(x)}{2}\left(\frac{1}{L}\sum_{l=1}^{L}\left(\left|F'\left(s^1_{\phi_l<o_{\varphi_l}>}(x)\right)-F'\left(s^2_{\phi_l<o_{\varphi_l}>}(x)\right)\right|\right)^\lambda+\left(\left|u\left(h_{S_O}^1(x)\right)-u\left(h_{S_O}^2(x)\right)\right|\right)^\lambda\right)dx\right)^{1/\lambda} \tag{66}$$

$$d_{ghcwhhdd}\left(H_{S_O}^1, H_{S_O}^2\right)=\left(\int_\alpha^\beta \frac{w(x)}{2}\left(\max_{l=1,2,\ldots,L}\left(\left|F'\left(s^1_{\phi_l<o_{\varphi_l}>}(x)\right)-F'\left(s^2_{\phi_l<o_{\varphi_l}>}(x)\right)\right|\right)^\lambda+\left(\left|u\left(h_{S_O}^1(x)\right)-u\left(h_{S_O}^2(x)\right)\right|\right)^\lambda\right)dx\right)^{1/\lambda} \tag{67}$$

$$d_{gchcwd}\left(H_{S_O}^1, H_{S_O}^2\right)=\left(\int_{\alpha}^{\beta}\frac{w(x)}{3}\left(\frac{1}{L}\sum_{l=1}^{L}\left(\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^1(x)\right)-F'\left(s_{\phi_l<o_{\varphi_l}>}^2(x)\right)\right|\right)^{\lambda}+\max_{l=1,2,\ldots,L}\left(\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^1(x)\right)-F'\left(s_{\phi_l<o_{\varphi_l}>}^2(x)\right)\right|\right)^{\lambda}+\left(\left|u\left(h_{S_O}^1(x)\right)-u\left(h_{S_O}^2(x)\right)\right|\right)^{\lambda}\right)dx\right)^{1/\lambda} \quad (68)$$

Specially, if $\lambda=1$ and $\lambda=2$, then Eqs. (65)-(68) reduce to the corresponding Hamming and Euclidean distances, here we omit them.

4.4. Ordered weighted distance and similarity measures between the double hierarchy HFLTSs

In recent years, lots of scholars have researched the ordered weighted distance and similarity measures under different uncertain environments. Xu and Chen (2012) defined several ordered weighted distance measures, which are suitable to be used in many actual fields, including group decision making, medical diagnosis, data mining, and pattern recognition. Based on Xu and Chen' distance measures, Yager (2010) generalized and provided a variety of ordered weighted averaging norms and similarity measures. Merigó and Gil-Lafuente (2010) introduced an ordered weighted averaging distance operator. Furthermore, on the basis of hesitant fuzzy information, Xu and Xia (2011) developed a variety of distance measures and the corresponding similarity measures for HFSs. Liao, Xu and Zeng (2014) and Liao and Xu (2015) proposed a family of distance and similarity measures between two HFLTSs. In what follows, we develop some ordered weighted distance measures between the double hierarchy HFLTSs.

Firstly, the generalized ordered weighted distance between $H_{S_O}^1$ and $H_{S_O}^2$ is defined as:

$$d_{gowd}\left(H_{S_O}^1, H_{S_O}^2\right)=\left(\sum_{j=1}^{n}w_j\left(\frac{1}{L}\sum_{l=1}^{L}\left(\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^{1\sigma(j)}\right)-F'\left(s_{\phi_l<o_{\varphi_l}>}^{2\sigma(j)}\right)\right|\right)^{\lambda}\right)\right)^{1/\lambda} \quad (69)$$

where $\lambda>0$ and $\sigma(j):(1,2,\ldots,n)\to(1,2,\ldots,n)$ is a permutation satisfying

$$\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^{1\sigma(j+1)}\right)-F'\left(s_{\phi_l<o_{\varphi_l}>}^{2\sigma(j+1)}\right)\right|\geq\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^{1\sigma(j)}\right)-F'\left(s_{\phi_l<o_{\varphi_l}>}^{2\sigma(j)}\right)\right|,\quad j=1,2,\ldots,n-1.$$

Similarly, the generalized ordered weighted Hausdorff distance between $H_{S_O}^1$ and $H_{S_O}^2$ is defined as:

$$d_{gowhd}\left(H_{S_O}^1, H_{S_O}^2\right)=\left(\sum_{j=1}^{n}w_j\left(\max_{l=1,2,\ldots,L}\left(\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^{1\sigma'(j)}\right)-F'\left(s_{\phi_l<o_{\varphi_l}>}^{2\sigma'(j)}\right)\right|\right)^{\lambda}\right)\right)^{1/\lambda} \quad (70)$$

where $\lambda>0$ and $\sigma'(j):(1,2,\ldots,n)\to(1,2,\ldots,n)$ is a permutation satisfying

$$\max_{l=1,2,\ldots,L}\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^{1\sigma'(j+1)}\right)-F'\left(s_{\phi_l<o_{\varphi_l}>}^{2\sigma'(j+1)}\right)\right|\geq\max_{l=1,2,\ldots,L}\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^{1\sigma'(j)}\right)-F'\left(s_{\phi_l<o_{\varphi_l}>}^{2\sigma'(j)}\right)\right|,\quad j=1,2,\ldots,n-1.$$

and the generalized ordered weighted hesitance degree-based distance between $H_{S_O}^1$ and $H_{S_O}^2$ is defined as:

$$d_{gowhdd}\left(H_{S_O}^1, H_{S_O}^2\right) = \left(\sum_{j=1}^{n} w_j \left(\left(\left|u\left(h_{S_O}^{1\sigma''(j)}\right) - u\left(h_{S_O}^{2\sigma''(j)}\right)\right|\right)^{\lambda}\right)\right)^{1/\lambda} \tag{71}$$

where $\lambda > 0$ and $\sigma''(j):(1,2,\ldots,n)\to(1,2,\ldots,n)$ is a permutation satisfying

$$\left|u\left(h_{S_O}^{1\sigma''(j+1)}\right) - u\left(h_{S_O}^{2\sigma''(j+1)}\right)\right| \geq \left|u\left(h_{S_O}^{1\sigma''(j)}\right) - u\left(h_{S_O}^{2\sigma''(j)}\right)\right|, \quad j = 1,2,\ldots,n-1.$$

Specially, if $\lambda = 1$, then Eqs. (69)-(71) reduce to the ordered weighted Hamming distance, the ordered weighted Hamming-Hausdorff distance, and the ordered weighted Hamming-hesitance degree-based distance between $H_{S_O}^1$ and $H_{S_O}^2$, respectively:

$$d_{owhd}\left(H_{S_O}^1, H_{S_O}^2\right) = \left(\sum_{j=1}^{n} w_j \left(\frac{1}{L}\sum_{l=1}^{L}\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^{1\sigma(j)}\right) - F'\left(s_{\phi_l<o_{\varphi_l}>}^{2\sigma(j)}\right)\right|\right)\right) \tag{72}$$

$$d_{owhhd}\left(H_{S_O}^1, H_{S_O}^2\right) = \left(\sum_{j=1}^{n} w_j \left(\max_{l=1,2,\ldots,L}\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^{1\sigma'(j)}\right) - F'\left(s_{\phi_l<o_{\varphi_l}>}^{2\sigma'(j)}\right)\right|\right)\right) \tag{73}$$

$$d_{owhhdd}\left(H_{S_O}^1, H_{S_O}^2\right) = \left(\sum_{j=1}^{n} w_j \left(\left|u\left(h_{S_O}^{1\sigma''(j)}\right) - u\left(h_{S_O}^{2\sigma''(j)}\right)\right|\right)\right) \tag{74}$$

If $\lambda = 2$, then Eqs. (69)-(71) reduce to the ordered weighted Euclidean distance, the ordered weighted Euclidean-Hausdorff distance, and the ordered weighted Euclidean-hesitance degree-based distance between $H_{S_O}^1$ and $H_{S_O}^2$, respectively:

$$d_{owed}\left(H_{S_O}^1, H_{S_O}^2\right) = \left(\sum_{j=1}^{n} w_j \left(\frac{1}{L}\sum_{l=1}^{L}\left(\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^{1\sigma(j)}\right) - F'\left(s_{\phi_l<o_{\varphi_l}>}^{2\sigma(j)}\right)\right|\right)^2\right)\right)^{1/2} \tag{75}$$

$$d_{owehd}\left(H_{S_O}^1, H_{S_O}^2\right) = \left(\sum_{j=1}^{n} w_j \left(\max_{l=1,2,\ldots,L}\left(\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^{1\sigma'(j)}\right) - F'\left(s_{\phi_l<o_{\varphi_l}>}^{2\sigma'(j)}\right)\right|\right)^2\right)\right)^{1/2} \tag{76}$$

$$d_{owehdd}\left(H_{S_O}^1, H_{S_O}^2\right) = \left(\sum_{j=1}^{n} w_j \left(\left(\left|u\left(h_{S_O}^{1\sigma''(j)}\right) - u\left(h_{S_O}^{2\sigma''(j)}\right)\right|\right)^2\right)\right)^{1/2} \tag{77}$$

Additionally, we can define three generalized hybrid distance measures such as:

(1) The generalized hybrid ordered weighted Hausdorff distance between $H_{S_O}^1$ and $H_{S_O}^2$:

$$d_{ghowhd}\left(H_{S_O}^1, H_{S_O}^2\right) = \left(\sum_{j=1}^{n} \frac{w_j}{2}\left(\frac{1}{L}\sum_{l=1}^{L}\left(\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^{1\dot{\sigma}(j)}\right) - F'\left(s_{\phi_l<o_{\varphi_l}>}^{2\dot{\sigma}(j)}\right)\right|\right)^{\lambda} + \max_{l=1,2,\ldots,L}\left(\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^{1\dot{\sigma}(j)}\right) - F'\left(s_{\phi_l<o_{\varphi_l}>}^{2\dot{\sigma}(j)}\right)\right|\right)^{\lambda}\right)\right)^{1/\lambda} \tag{78}$$

where $\lambda > 0$ and $\dot{\sigma}:(1,2,\ldots,n)\to(1,2,\ldots,n)$ is a permutation satisfying

$$\frac{1}{L}\sum_{l=1}^{L}\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^{1\dot{\sigma}(j+1)}\right) - F'\left(s_{\phi_l<o_{\varphi_l}>}^{2\dot{\sigma}(j+1)}\right)\right| + \max_{l=1,2,\ldots,L}\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^{1\dot{\sigma}(j)}\right) - F'\left(s_{\phi_l<o_{\varphi_l}>}^{2\dot{\sigma}(j+1)}\right)\right| \geq \frac{1}{L}\sum_{l=1}^{L}\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^{1\dot{\sigma}(j)}\right) - F'\left(s_{\phi_l<o_{\varphi_l}>}^{2\dot{\sigma}(j)}\right)\right| + \max_{l=1,2,\ldots,L}\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^{1\dot{\sigma}(j)}\right) - F'\left(s_{\phi_l<o_{\varphi_l}>}^{2\dot{\sigma}(j)}\right)\right|$$

(2) The generalized hybrid ordered weighted hesitance degrees distance between $H_{S_O}^1$ and

$H_{S_O}^2$:

$$d_{ghowhdd}\left(H_{S_O}^1,H_{S_O}^2\right)=\left(\sum_{j=1}^{n}\frac{w_j}{2}\left(\frac{1}{L}\sum_{l=1}^{L}\left(\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^{1\ddot{\sigma}(j)}\right)-F'\left(s_{\phi_l<o_{\varphi_l}>}^{2\ddot{\sigma}(j)}\right)\right|\right)^{\lambda}+\left(\left|u\left(h_{S_O}^{1\ddot{\sigma}(j)}\right)-u\left(h_{S_O}^{2\ddot{\sigma}(j)}\right)\right|\right)^{\lambda}\right)\right)^{1/\lambda} \quad (79)$$

where $\lambda>0$ and $\ddot{\sigma}:(1,2,\ldots,n)\rightarrow(1,2,\ldots,n)$ is a permutation satisfying

$$\frac{1}{L}\sum_{l=1}^{L}\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^{1\ddot{\sigma}(j+1)}\right)-F'\left(s_{\phi_l<o_{\varphi_l}>}^{2\ddot{\sigma}(j+1)}\right)\right|+\left|u\left(h_{S_O}^{1\ddot{\sigma}(j+1)}\right)-u\left(h_{S_O}^{2\ddot{\sigma}(j+1)}\right)\right|\geq\frac{1}{L}\sum_{l=1}^{L}\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^{1\ddot{\sigma}(j)}\right)-F'\left(s_{\phi_l<o_{\varphi_l}>}^{2\ddot{\sigma}(j)}\right)\right|+\left|u\left(h_{S_O}^{1\ddot{\sigma}(j)}\right)-u\left(h_{S_O}^{2\ddot{\sigma}(j)}\right)\right|$$

(3) The generalized hybrid ordered weighted Hausdorff-hesitance degree-based distance between $H_{S_O}^1$ and $H_{S_O}^2$:

$$d_{ghowhhdd}\left(H_{S_O}^1,H_{S_O}^2\right)=\left(\sum_{j=1}^{n}\frac{w_j}{2}\left(\max_{l=1,2,\ldots,L}\left(\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^{1\ddot{\sigma}(j)}\right)-F'\left(s_{\phi_l<o_{\varphi_l}>}^{2\ddot{\sigma}(j)}\right)\right|\right)^{\lambda}+\left(\left|u\left(h_{S_O}^{1\ddot{\sigma}(j)}\right)-u\left(h_{S_O}^{2\ddot{\sigma}(j)}\right)\right|\right)^{\lambda}\right)\right)^{1/\lambda} \quad (80)$$

where $\lambda>0$ and $\ddot{\sigma}:(1,2,\ldots,n)\rightarrow(1,2,\ldots,n)$ is a permutation satisfying

$$\max_{l=1,2,\ldots,L}\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^{1\ddot{\sigma}(j+1)}\right)-F'\left(s_{\phi_l<o_{\varphi_l}>}^{2\ddot{\sigma}(j+1)}\right)\right|+\left|u\left(h_{S_O}^{1\ddot{\sigma}(j+1)}\right)-u\left(h_{S_O}^{2\ddot{\sigma}(j+1)}\right)\right|\geq\max_{l=1,2,\ldots,L}\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^{1\ddot{\sigma}(j)}\right)-F'\left(s_{\phi_l<o_{\varphi_l}>}^{2\ddot{\sigma}(j)}\right)\right|+\left|u\left(h_{S_O}^{1\ddot{\sigma}(j)}\right)-u\left(h_{S_O}^{2\ddot{\sigma}(j)}\right)\right|$$

Specially, if $\lambda=1$ and $\lambda=2$, then it is obvious that Eqs. (78)-(80) reduce to their Hamming and Euclidean distance measures respectively. Here we omit them.

Furthermore, by combining all these three distance measures together, the generalized completely hybrid ordered weighted distance can be defined as:

$$d_{gchowd}\left(H_{S_O}^1,H_{S_O}^2\right)=\left(\sum_{j=1}^{n}\frac{w_j}{3}\left(\frac{1}{L}\sum_{l=1}^{L}\left(\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^{1\tilde{\sigma}(j)}\right)-F'\left(s_{\phi_l<o_{\varphi_l}>}^{2\tilde{\sigma}(j)}\right)\right|\right)^{\lambda}+\max_{l=1,2,\ldots,L}\left(\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^{1\tilde{\sigma}(j)}\right)-F'\left(s_{\phi_l<o_{\varphi_l}>}^{2\tilde{\sigma}(j)}\right)\right|\right)^{\lambda}+\left(\left|u\left(h_{S_O}^{1\tilde{\sigma}(j)}\right)-u\left(h_{S_O}^{2\tilde{\sigma}(j)}\right)\right|\right)^{\lambda}\right)\right)^{1/\lambda} \quad (81)$$

where $\lambda>0$ and $\tilde{\sigma}:(1,2,\ldots,n)\rightarrow(1,2,\ldots,n)$ is a permutation satisfying

$$\frac{1}{L}\sum_{l=1}^{L}\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^{1\tilde{\sigma}(j+1)}\right)-F'\left(s_{\phi_l<o_{\varphi_l}>}^{2\tilde{\sigma}(j+1)}\right)\right|+\max_{l=1,2,\ldots,L}\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^{1\tilde{\sigma}(j+1)}\right)-F'\left(s_{\phi_l<o_{\varphi_l}>}^{2\tilde{\sigma}(j+1)}\right)\right|+\left|u\left(h_{S_O}^{1\tilde{\sigma}(j+1)}\right)-u\left(h_{S_O}^{2\tilde{\sigma}(j+1)}\right)\right|$$

$$\geq\frac{1}{L}\sum_{l=1}^{L}\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^{1\tilde{\sigma}(j)}\right)-F'\left(s_{\phi_l<o_{\varphi_l}>}^{2\tilde{\sigma}(j)}\right)\right|+\max_{l=1,2,\ldots,L}\left|F'\left(s_{\phi_l<o_{\varphi_l}>}^{1\tilde{\sigma}(j)}\right)-F'\left(s_{\phi_l<o_{\varphi_l}>}^{2\tilde{\sigma}(j)}\right)\right|+\left|u\left(h_{S_O}^{1\tilde{\sigma}(j)}\right)-u\left(h_{S_O}^{2\tilde{\sigma}(j)}\right)\right|$$

Similarly, we also omit the corresponding Hamming and Euclidean distances when $\lambda=1$ and $\lambda=2$.

5. A multiple criteria decision making method and application

In recent years, lots of MCDM methods are developed such as TOPSIS (Tan, Wei, Liu, & Feng, 2016), TODIM (Wei, Ren, & Rodríguez, 2015), VIKOR (Liao, Xu, & Zeng, 2015) and MULTIMOORA (Gou et al., 2017). TOPSIS is attractive as limited subjective input is needed from decision-makers. Many authors argue that TOPSIS is an easy and useful method helping a decision-maker select the best choice according to both the minimal distance from the positive-ideal solution and the maximal distance from the negative-ideal solution (Zavadskas et al., 2013). Therefore, this paper proposes a MCDM method with double hierarchy hesitant fuzzy linguistic information based on TOPSIS model, and then applies this method to a practical MCDM problem about Sichuan liquor brand assessment.

5.1. A MCDM method

A MCDM problem with double hierarchy hesitant fuzzy linguistic information can be described as follows: Let $A=\{A_1,A_2,\ldots,A_m\}$ be a set of alternatives, $C=\{C_1,C_2,\ldots,C_n\}$ be a set of criteria, and $w=(w_1,w_2,\ldots,w_n)^T$ be the weight vector of all criteria with $w_j\geq 0$, $j=1,2,\ldots,n$, and $\sum_{j=1}^{n}w_j=1$. Let $S_O=\{s_{t<o_k>}\,|t=-\tau,\ldots,-1,0,1,\ldots,\tau;\,k=-\varsigma,\ldots,-1,0,1,\ldots,\varsigma\}$ be a double hierarchy LTS. The invited experts can give their linguistic evaluation information about each alternative with respect to each criterion. We gather the evaluation information and establish a decision making matrix $DM=\left(h_{S_O}^{ij}\right)_{m\times n}$ $(i=1,2,\ldots,m;\ j=1,2,\ldots,n)$ shown as:

$$DM=\left(h_{S_O}^{ij}\right)_{m\times n}=\begin{pmatrix} h_{S_O}^{11} & h_{S_O}^{12} & \cdots & h_{S_O}^{1n} \\ h_{S_O}^{21} & h_{S_O}^{22} & \cdots & h_{S_O}^{2n} \\ \vdots & \vdots & \ddots & \vdots \\ h_{S_O}^{m1} & h_{S_O}^{m2} & \cdots & h_{S_O}^{mn} \end{pmatrix}=\begin{pmatrix} H_{S_O}^{1} \\ H_{S_O}^{2} \\ \vdots \\ H_{S_O}^{m} \end{pmatrix} \tag{82}$$

Obviously, the double hierarchy HFLTSs $H_{S_O}^{i}=\left\{h_{S_O}^{i1},h_{S_O}^{i2},\ldots,h_{S_O}^{in}\right\}(i=1,2,\ldots,m)$ can be used to express all evaluation information on the alternatives $A_i\ (i=1,2,\ldots,m)$. Then, a MCDM method can be shown as follows:

Step 1. For each criterion C_j, we can obtain the smallest $h_{S_O}^{j-}$ and largest DHFLE $h_{S_O}^{j+}$, respectively:

$$h_{S_O}^{j-}=\begin{cases} \min\limits_{i=1,2,\ldots,m}\left\{h_{S_O}^{ij}\right\}, & \text{for benefit criterion } C_j \\ \max\limits_{i=1,2,\ldots,m}\left\{h_{S_O}^{ij}\right\}, & \text{for cost criterion } C_j \end{cases} \tag{83}$$

$$h_{S_O}^{j+}=\begin{cases} \max\limits_{i=1,2,\ldots,m}\left\{h_{S_O}^{ij}\right\}, & \text{for benefit criterion } C_j \\ \min\limits_{i=1,2,\ldots,m}\left\{h_{S_O}^{ij}\right\}, & \text{for cost criterion } C_j \end{cases} \tag{84}$$

Combining all the smallest DHFLEs and largest DHFLEs, respectively, we can obtain the double hierarchy hesitant fuzzy linguistic negative ideal solution $H_{S_O}^{-}=\left\{h_{S_O}^{1-},h_{S_O}^{2-},\ldots,h_{S_O}^{n-}\right\}$ and the double hierarchy hesitant fuzzy linguistic positive ideal solution $H_{S_O}^{+}=\left\{h_{S_O}^{1+},h_{S_O}^{2+},\ldots,h_{S_O}^{n+}\right\}$.

Step 2. Calculate the distance $d\left(H_{S_O}^{i},H_{S_O}^{-}\right)$ between each alternative $H_{S_O}^{i}$ and the double hierarchy hesitant fuzzy linguistic negative ideal solution $H_{S_O}^{-}$, and the distance $d\left(H_{S_O}^{i},H_{S_O}^{+}\right)$ between each alternative $H_{S_O}^{i}$ and the double hierarchy hesitant fuzzy linguistic positive ideal solution $H_{S_O}^{+}$, respectively. Clearly, the larger the distance $d\left(H_{S_O}^{i},H_{S_O}^{-}\right)$ is, the better the alternative would be, while the smaller the value of $d\left(H_{S_O}^{i},H_{S_O}^{+}\right)$ is, the better the alternative would be.

Step 3. Calculate the satisfaction degree of each given alternative A_i based on the following formula:

$$\phi(A_i)=\frac{(1-\theta)d\left(H_{S_O}^{i},H_{S_O}^{-}\right)}{\theta d\left(H_{S_O}^{i},H_{S_O}^{+}\right)+(1-\theta)d\left(H_{S_O}^{i},H_{S_O}^{-}\right)} \tag{85}$$

where the parameter θ expresses the risk preferences of the decision maker and $0\le\theta\le1$. If $\theta>0.5$, then the decision maker is pessimist; if $\theta<0.5$, then the decision maker is optimist.

Step 4. Obviously, the bigger the satisfaction degree is, the better the alternative should be. Therefore, we can obtain the final ranking order of all alternatives.

Step 5. End.

The flowchart of this MCDM method can be drawn in Fig. 3.

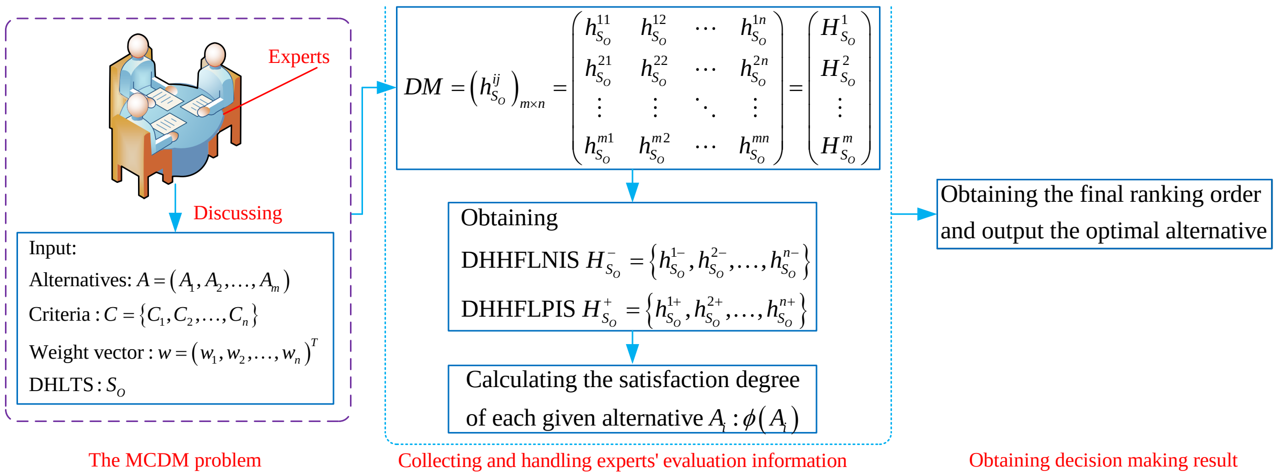

Fig. 3. The flowchart of the MCDM method

5.2. Case study: Sichuan liquor brand assessment

Chinese liquor has a thousand years of history, which also carries Chinese culture. Meanwhile, the liquor industry has very high rates of return and profitability. In China, both Sichuan and Guizhou provinces are the largest scale and the optimal production quality white liquor producing regions, and support the development of the entire Chinese liquor industry. At present, the whole liquor market has the following characteristics:

(1) The brand competition will be the main theme of the next stage liquor competition because of young consumers' rational consumption.

(2) The work of government will further affect the development direction of the whole liquor industry, such as forbidding driving after drinking, tax adjustment, etc.

(3) The living spaces of middle and small-sized and low side competition enterprises are more and more small.

Nowadays, according to the development of economy and the constantly improvement of consuming stratums, liquors of middle and top grades will be the theme of Chinese liquor industry development in the future, as well as the main battlefield of Chinese liquor competition. However, Sichuan liquor lacks the hard core in the true sense. Therefore, according to the awkward situation of Sichuan liquor industry, it is necessary to analyze and research the development strategy of Sichuan liquor industry, and then analyze the preference relations and consuming behaviors of consumers from their cognitive perspectives about each Sichuan liquor brand. Thus, the above work can provide a series of adjustment strategy to Sichuan liquor enterprises and promote the development of Sichuan liquor enterprises much better.

In order to investigate the consumers' cognitions about Sichuan liquor, we choose five Sichuan liquor brands, namely, Wuliangye Yibin (A_1), Luzhou Old Cellar (A_2), Ichiro liquor (A_3), Tuopai liquor (A_4) and Jian Nan Chun (A_5). Then we investigate the cognitions of consumers based on four criteria such as product price (C_1), product classification (C_2), consumer group (C_3) and distribution channel (C_4). Based on the following two LTSs:

$$S=\{s_{-3}=none, s_{-2}=very\ bad, s_{-1}=bad, s_0=medium, s_1=good, s_2=very\ good, s_3=perfect\}$$

$$O=\begin{cases}\{o_{-3}=far\ \ from, o_{-2}=only\ a\ little, o_{-1}=a\ little, o_0=just\ right, o_1=much, o_2=very\ much, o_3=extremely\},\ if\ \ s_t\geq s_0.\\ \{o_{-3}=extremely, o_{-2}=very\ much, o_{-1}=much, o_0=just\ right, o_1=a\ little, o_2=only\ a\ little, o_3=far\ \ from\},\ if\ \ s_t< s_0.\end{cases},$$

we summarize the survey results and the evaluation information for each alternative with respect to each criterion and express these information by the DHFLEs. All evaluation information establishes the decision making matrix (Table 2). Furthermore, the weight vector of these criteria is

$w = (0.1, 0.3, 0.2, 0.4)^T$.

Table 2. Decision making matrix with DHFLEs

	C_1	C_2	C_3	C_4
A_1	$\{s_{0<o_{-2}>}, s_1, s_{2<o_1>}\}$	$\{s_{2<o_1>}\}$	$\{s_{-1<o_2>}, s_{0<o_1>}\}$	$\{s_{1<o_1>}, s_{2<o_2>}\}$
A_2	$\{s_{2<o_0>}, s_{3<o_{-1}>}\}$	$\{s_{-2<o_{-2}>}, s_{-1}, s_0, s_{1<o_2>}\}$	$\{s_{-2<o_{-2}>}, s_{-1}, s_{0<o_2>}\}$	$\{s_{1<o_{-2}>}, s_2, s_{3<o_{-2}>}\}$
A_3	$\{s_{1<o_0>}\}$	$\{s_{1<o_0>}, s_{2<o_0>}\}$	$\{s_{-1<o_{-2}>}, s_{0<o_{-2}>}\}$	$\{s_{0<o_2>}, s_{1<o_2>}\}$
A_4	$\{s_{2<o_1>}, s_{3<o_0>}\}$	$\{s_{0<o_0>}\}$	$\{s_{1<o_1>}\}$	$\{s_{0<o_0>}\}$
A_5	$\{s_{0<o_{-2}>}, s_{1<o_1>}\}$	$\{s_{1<o_{-1}>}, s_2, s_{3<o_{-1}>}\}$	$\{s_{-1<o_{-2}>}, s_0, s_{1<o_{-1}>}\}$	$\{s_{2<o_{-2}>}, s_{3<o_{-2}>}\}$

In what follows, we utilize our method to deal with this MCDM problem:

Firstly, we need to obtain the double hierarchy hesitant fuzzy linguistic negative ideal solution:

$$H_{S_O}^{-} = \left\{\left\{s_{0<o_{-2}>}, s_{1<o_1>}\right\}, \left\{s_{-2<o_{-2}>}, s_{-1}, s_0, s_{1<o_2>}\right\}, \left\{s_{-1<o_{-2}>}, s_{0<o_{-2}>}\right\}, \left\{s_{0<o_0>}\right\}\right\}$$

and the double hierarchy hesitant fuzzy linguistic positive ideal solution:

$$H_{S_O}^{+} = \left\{\left\{s_{2<o_1>}, s_{3<o_0>}\right\}, \left\{s_{2<o_1>}\right\}, \left\{s_{1<o_1>}\right\}, \left\{s_{1<o_1>}, s_{2<o_2>}\right\}\right\}$$

Additionally, we utilize the generalized completely hybrid weighted Hausdorff-hesitance degree-based distance to calculate the distance between each alternative and the double hierarchy hesitant fuzzy linguistic negative ideal solution $d_{gchwhhdd}\left(H_{S_O}^{1}, H_{S_O}^{-}\right)$, and the distance between each alternative and the double hierarchy hesitant fuzzy linguistic positive ideal solution $d_{gchwhhdd}\left(H_{S_O}^{1}, H_{S_O}^{+}\right)$, respectively. In this process, we let λ be 1, 2 and 5, respectively.

Furthermore, based on Eq. (87), we can calculate the satisfaction degree of each alternative. And then, the ranking orders of all alternatives can be obtained and shown in Fig. 4 and Table 3.

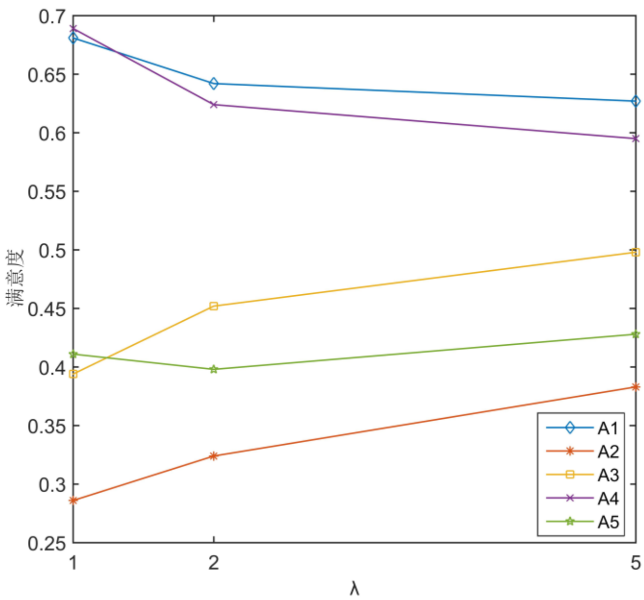

Fig 4. The satisfaction degrees and ranking orders based on the generalized completely hybrid weighted

Hausdorff-hesitance degree-based distance

Table 3. The satisfaction degrees and the ranking orders based on the generalized completely hybrid weighted Hausdorff-hesitance degree-based distance

	A_1	A_2	A_3	A_4	A_5	Ranking order
$\lambda=1$	0.681	0.286	0.394	0.689	0.411	$A_4 \succ A_1 \succ A_5 \succ A_3 \succ A_2$
$\lambda=2$	0.642	0.324	0.452	0.624	0.398	$A_1 \succ A_4 \succ A_3 \succ A_5 \succ A_2$
$\lambda=5$	0.627	0.383	0.498	0.595	0.428	$A_1 \succ A_4 \succ A_3 \succ A_5 \succ A_2$

On the other hand, by utilizing the generalized completely hybrid ordered weighted distance, we can also obtain the ranking orders of all alternatives in Fig. 5 and Table 4.

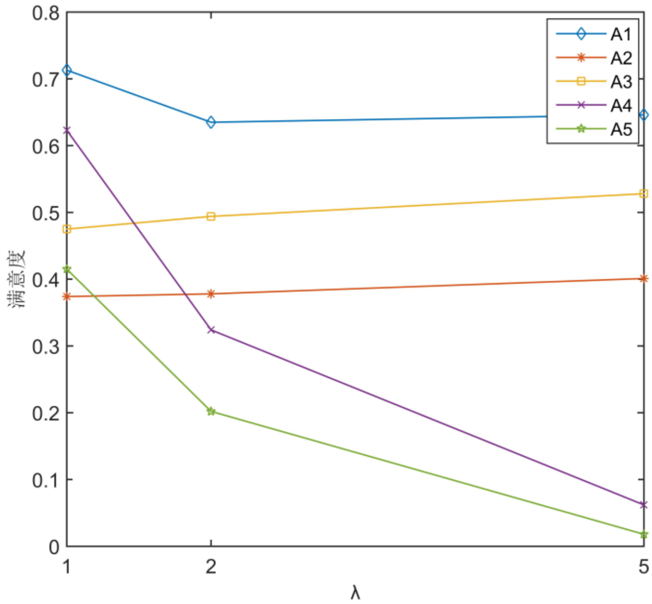

Fig 5. The satisfaction degrees and ranking orders based on the generalized completely hybrid ordered weighted distance

Table 4. The satisfaction degrees and ranking orders based on the generalized completely hybrid ordered weighted distance

	A_1	A_2	A_3	A_4	A_5	Ranking order
$\lambda=1$	0.713	0.374	0.475	0.623	0.415	$A_1 \succ A_4 \succ A_3 \succ A_5 \succ A_2$
$\lambda=2$	0.635	0.378	0.494	0.324	0.202	$A_1 \succ A_3 \succ A_2 \succ A_4 \succ A_5$
$\lambda=5$	0.646	0.401	0.528	0.062	0.018	$A_1 \succ A_3 \succ A_2 \succ A_4 \succ A_5$

For the generalized completely hybrid weighted Hausdorff-hesitance degree-based distance, as we have seen in Table 3 and Fig. 4, the changes of the ranking orders are very small when we utilize the different values of λ. Specifically, for the alternatives A_1 and A_4, the satisfaction degrees of them are gradually decreased with the increase of the value of λ; For the alternatives A_2 and A_3, the satisfaction degrees of them are gradually increased with the increase of the value of λ; For the alternative A_5, its satisfaction degrees have three different stages of change. Additionally, for the generalized completely hybrid ordered weighted distance, as we have seen in Table 4 and Fig. 5, the changes of the ranking orders of A_1 and A_2 are small and the changes of the ranking orders of the rest alternatives are very apparent when we utilize different values of λ. Finally, by considering that we change the orders of all DHFLEs included in each double hierarchy HFLTS when we utilize the generalized completely hybrid weighted Hausdorff-hesitance degree-based distance to calculate the satisfaction degrees of these alternatives, it is reasonable that the changes of the ranking orders of some alternatives are very apparent.

5.3. Comparison analyses

We can transform the DHFLEs into hesitant fuzzy linguistic elements (HFLEs) (the basic elements of HFLTS) by deleting the second hierarchy linguistic information. Then the Table 2 can be changed to Table 5:

Table 5. Decision making matrix with HFLEs

	C_1	C_2	C_3	C_4
A_1	$\{s_0, s_1, s_2\}$	$\{s_2\}$	$\{s_{-1}, s_0\}$	$\{s_1, s_2\}$
A_2	$\{s_2, s_3\}$	$\{s_{-2}, s_{-1}, s_0, s_1\}$	$\{s_{-2}, s_{-1}, s_0\}$	$\{s_1, s_2, s_3\}$
A_3	$\{s_1\}$	$\{s_1, s_2\}$	$\{s_{-1}, s_0\}$	$\{s_0, s_1\}$
A_4	$\{s_2, s_3\}$	$\{s_0\}$	$\{s_1\}$	$\{s_0\}$
A_5	$\{s_0, s_1\}$	$\{s_1, s_2, s_3\}$	$\{s_{-1}, s_0, s_1\}$	$\{s_2, s_3\}$

Then we utilize two other methods to deal with this MCDM problem including HFL-TOPSIS method (Tan, Wei, Liu, & Feng, 2016) and HFL-VIKOR method (Liao, Xu, & Zeng, 2015). The decision-making results are shown in Table 6.

Table 6. Decision making matrix with HFLEs

Methods	Ranking orders
HFL-TOPSIS	$A_5 \succ A_1 \succ A_4 \succ A_2 \succ A_3$
HFL-VIKOR	$A_4 \succ A_3 \succ A_2 \succ A_1 \succ A_5$

Obviously, the results among these three methods are very different. The reasons can be shown as:

a) By transforming the DHFLEs into HFLEs, we lose lots of original linguistic information.
b) In both our method and the HFL-VIKOR methods, the weights of criteria are considered, but the HFL-TOPSIS does not utilize this parameter.

Therefore, we can obtain that the double hierarchy hesitant fuzzy linguistic information can express the original linguistic information more accurately. Additionally, it is necessary to consider some important parameters in some specific MCDM problems.

6. Discussions on the advantages and limitations

In the following, we analyze the double hierarchy LTS and the double hierarchy HFLTS, the current results on distance and similarity measures and the potential weakness.

(1) For the double hierarchy LTS and the double hierarchy HFLTS, which mainly have some advantages:

a) The double hierarchy LTS consists of two hierarchy LTSs, therefore, the basic element DHLT can be used to describe some complex linguistic more accurately and fully than the single LTS. Additionally, the expression of a DHLT is very intuitional and simple, and we give the linguistic labels in advance, so we can use a very simple DHLT to express any complex linguistic information.

b) For the purpose of expressing some more complex uncertain linguistic information, we develop double hierarchy LTSs into hesitant fuzzy environment and obtain double hierarchy HFLTSs. It is a very useful way to represent the hesitance existing in people's daily life. Therefore, it is different from the use of Type-2 fuzzy sets, which were developed from fuzzy sets.

c) We define these monotone functions for making the mutual transformations between the DHLT (or DHFLE) and the numerical scale (or the set of the numerical scales) when extending the DHLT and the DHFLE to the continuous forms, which is similar with some existing researches. For example, Yager (2004) proposed that an ordered scale often arises from the use of linguistic values to describe membership. Li et al. (2017) developed a personalized individual semantics for CW based on the numerical scale which can be used to transform linguistic terms into real numbers equivalently. García-Lapresta and Pérez-Román (2018) introduced the ordered qualitative scales using proximity measures between consecutive labels and metrizable distances. Therefore, we can fix the double hierarchy LTS and the double hierarchy HFLTS as another tool together with the above three mentioned references for including more knowledge for the experts to represent the linguistic evaluations.

(2) In this paper, we mainly develop a series of distance and similarity measures for DHFLEs and double hierarchy HFLTSs from different angels. Obviously, each kind of distance and similarity measures owns its key point. The distance and similarity measures with preference information between DHFLEs mainly consider that different distance measures may have different importance degrees. Additionally, we usually utilize the distance and similarity measures to deal with discrete information, but the continuous double hierarchy HFLTSs are also common and it is necessary to develop the distance and similarity measures in continuous case. Furthermore, the weight of each DHFLE included in the double hierarchy HFLTS mainly expresses the importance degree of each DHFLE, so giving weight information into the distance and similarity measures between double

hierarchy HFLTSs is reasonable and necessary. Finally, sometimes we need to change the original information into the ordering form for practical purposes, and the ordering information can make the weights of DHFLEs more meaningful, so we develop the ordered weighted distance and similarity measures between the double hierarchy HFLTSs.

(3) There still exist some potential weaknesses about the double hierarchy LTS and the double hierarchy HFLTSs:

a) In order to fully analyze the second hierarchy LTS, four kinds of conditions are given in Fig. 1, and we have given them some corresponding explanations. However, it is very complex when we deal with practical decision making problems. Therefore, we need to introduce some more reasonable expressions for the second hierarchy LTS in the future.
b) These distance and similarity measures are only small parts of these fields, so it is necessary to define some other distance and similarity measures when we face some special problems.

7. Conclusions

In this paper, we have proposed some distance and similarity measures of the DHFLEs and the double hierarchy HFLTSs from different angles including the axioms of distance and similarity measures of the DHFLEs and the double hierarchy HFLTSs, the basic distance and similarity measures of the DHFLEs, the distance and similarity measures with preference information, the weighted distance and similarity measures of the double hierarchy HFLTSs in discrete case and continuous case, and the ordered weighted distance and similarity measures of the double hierarchy HFLTSs. Furthermore, we have developed a decision making method to solve the MCDM problems on the basis of these distance and similarity measures. Finally, we have applied this method to deal with a practical MCDM problem about Sichuan liquor brand assessment.

In the future, these distance and similarity measures can be used as the basic tools to make the corresponding calculations for double hierarchy hesitant fuzzy linguistic preference relations, consistency analysis, personalized individual semantics, etc. Additionally, they can also be applied to deal with some practical problems such as medical management, water resource management, etc.

References

Arevalillo-Herráez, M., Ferri, F.J., & Domingo J. (2013). A naive relevance feedback model for content-based image retrieval using multiple similarity measures. Pattern Recognition, 43(3), 9-

629.

Atanassov, K.T. (1986). Intuitionistic fuzzy set. Fuzzy Sets and Systems, 20, 87-96.

Bustince, H., Barrenechea, E., Pagola, M., Fernandez, J., Xu, Z.S., Bedregal, B., Montero, J., Hagras, H., Herrera, F., & Baets, B.D. (2016). A historical account of types of fuzzy sets and their relationships. IEEE Transactions on Fuzzy Systems, 24(1), 179-194.

Chen, Y.S., Wang, W.H., & Juang, J.G. (2010). Application of intelligent computing to autonomous vehicle control. IEEE Congress on Evolutionary Computation, 1-8.

Danielsson, P.E. (1980). Euclidean distance Mapping. Computer Graphics and Image Processing, 14, 227-248.

Dong, Y.C., Li, C.C., & Herrera, F. (2016). Connecting the linguistic hierarchy and the numerical scale for the 2-tuple linguistic model and its use to deal with hesitant unbalanced linguistic information. Information Sciences, 367, 259-278.

Dubois, D., & Prade, H. (1980). Fuzzy Sets and Systems: Theory and Applications. New York: Kluwer.

Dubois, D. (2011). The role of fuzzy sets indecision sciences: Old techniques and new directions. Fuzzy Sets and Systems, 184, 3-28.

Farhadinia, B. (2014). Distance and similarity measures for higher order hesitant fuzzy sets. Knowledge-Based Systems, 55, 43-48.

Gao, W., Farahani, M. R., Aslam, A., & Hosamani, S. (2017). Distance learning techniques for ontology similarity measuring and ontology mapping. Cluster Computing, 20(2), 959-968.

Gao, W., Sarlak, V., Parsaei, M. R., & Ferdosi, M. (2018). Combination of fuzzy based on a meta-heuristic algorithm to predict electricity price in an electricity markets. Chemical Engineering Research and Design, 131, 333-345.

Gao, W., Zhu, L.L., & Wang, K.Y. (2015). Ontology sparse vector learning algorithm for ontology similarity measuring and ontology mapping via ADAL technology. International Journal of Bifurcation and Chaos, 25(14), 1540034.

García-Laprestaa, J.L., & Pérez-Román, D. (2018). Aggregating opinions in non-uniform ordered qualitative scales. Applied Soft Computing, 67, 652-657.

Gou, X.J., Liao, H.C., Xu, Z.S., & Herrera, F. (2017). Double hierarchy hesitant fuzzy linguistic term set and MULTIMOORA method: A case of study to evaluate the implementation status of haze controlling measures. Information Fusion, 38, 22-34.

Grzegorzewski, P. (2004). Distances between intuitionistic fuzzy sets and/or interval-valued fuzzy sets based on the Hausdorff metric. Fuzzy Sets and Systems, 148, 319-328.

Hamming, R.W. (1950). Error detecting and error correcting codes. The Bell System Technical

Journal, 29(2), 147-160.

Hausdorff, F. (1957). Set theory, Chelsea, New York, 1957.

Herrera, F., & Martínez, L. (2000). A 2-tuple fuzzy linguistic representation model for computing with words. IEEE Transactions on Fuzzy Systems, 8, 746-752.

Hung, W.L., & Yang, M.S. (2004). Similarity measures of intuitionistic fuzzy sets based on Hausdorff distance. Pattern Recognition Letters, 25, 1603-1611.

Li, C.C., Dong, Y.C., Herrera, F., Herrera-Viedma, E., & Martínez, L. (2017). Personalized individual semantics in computing with words for supporting linguistic group decision making: An application on consensus reaching. Information Fusion, 33, 29-40.

Li, D.Q., Zeng, W.Y., & Li, J.H. (2015). New distance and similarity measures on hesitant fuzzy sets and their applications in multiple criteria decision making. Engineering Applications of Artificial Intelligence, 40, 11-16.

Li, X., Hall, N.S., & Humphreys, G.W. (1993). Discrete distance and similarity measures for pattern candidate selection. Pattern Recognition, 26(6), 843-851.

Liao, H.C., Xu, Z.S., & Zeng, X.J. (2014). Distance and similarity measures for hesitant fuzzy linguistic term sets and their application in multi-criteria decision making. Information Sciences, 271, 125-142.

Liao, H.C., & Xu, Z.S. (2015). Approaches to manage hesitant fuzzy linguistic information based on the cosine distance and similarity measures for HFLTSs and their application in qualitative decision making. Expert Systems with Applications, 42(12), 5328-5336.

Liao, H.C., Xu, Z.S., & Zeng, X.J. (2015). Hesitant fuzzy linguistic VIKOR method and its application in qualitative multiple criteria decision making. IEEE Transactions on Fuzzy Systems, 23(5), 1343-1355 .

Liu, P.D., & Su, Y. (2012). Multiple attribute decision making method based on the trapezoid fuzzy linguistic hybrid harmonic averaging operator. Informatica, 36(1), 83-90.

Martínez, L., & Herrera, F. (2012). An overview on the 2-tuple linguistic model for computing with words in decision making: extensions, applications and challenges. Information Sciences, 207, 1-18.

Mende, J.M. (2002). An architecture for making judgement using computing with words. International Journal of Applied Mathematics and Computer Science, 12(3), 325-335.

Merigó, J.M., & Gil-Lafuente, A.M. (2013). Induced 2-tuple linguistic generalized aggregation operators and their application in decision-making. Information Sciences, 236, 1-16.

Merigó, J.M., & Gil-Lafuente, A.M. (2010). New decision-making techniques and their application in the selection of financial products. Information Sciences, 180, 2085-2094.

Mizumoto, M. & Tanaka, K. (1976). Some properties of fuzzy sets of type 2. Information Control, 31, 312-340.

Rodríguez, R.M., Martínez, L., & Herrera, F. (2012). Hesitant fuzzy linguistic terms sets for decision making. IEEE Transactions on Fuzzy Systems, 20, 109-119.

Torra, V. (2010). Hesitant fuzzy sets. International Journal of Intelligent Systems, 25, 529-539.

Tan, Q.Y., Wei, C.P., Liu, Q., & Feng, X.Q. (2016). The hesitant fuzzy linguistic TOPSIS method based on novel information measures. Asia-Pacific Journal of Operational Research, 33(5), 1-22.

Türkşen, I.B. (2012). Type 2 representation and reasoning for CWW. Fuzzy Sets and Systems, 127, 17-36.

Wei, C.P., Ren, Z.L., & Rodríguez, R.M. (2015). A hesitant fuzzy linguistic TODIM method based on a score function. International Journal of Computational Intelligence Systems, 8(4), 701-712.

Wu, D.R. (2014). A reconstruction decoder for computing with words. Information Sciences, 255, 1-15.

Xia, M.M., & Xu, Z.S. (2011). Hesitant fuzzy information aggregation in decision making. International Journal of Approximate Reasoning, 52, 395-407.

Xu, Y.J., & Wang, H.M. (2011) Distance measure for linguistic decision making. Systems Engineering Procedia, 1, 450-456.

Xu, Z.S. (2005). An approach based on similarity measure to multiple attribute decision making with trapezoid fuzzy linguistic variables. Fuzzy Systems and Knowledge Discovery: Lecture Notes in Computer Science, 3613, 110-117.

Xu, Z.S. (2012).Fuzzy ordered distance measures. Fuzzy Optimization and Decision Making, 11, 73-97.

Xu, Z.S., & Chen, J. (2008). An overview of distance and similarity measures of intuitionistic fuzzy sets. International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, 16, 529-555.

Xu, Z.S. & Chen, J. (2008). Ordered weighted distance measure. Journal of Systems Science and Systems Engineering, 16, 529-555.

Xu, Z.S., & Wang, H. (2017). On the syntax and semantics of virtual linguistic terms for information fusion in decision making. Information Fusion, 34, 43-48.

Xu, Z.S., & Xia, M.M. (2011). Distance and similarity measures for hesitant fuzzy sets. Information Sciences, 181, 2128-2138.

Yager, R.R. (2004). Generalized OWA aggregation operators. Fuzzy Optimization and Decision Making, 3, 93-107.

Yager, R.R. (2010). Norms induced from OWA operators. IEEE Transactions on Fuzzy Systems, 18, 57-66.

Yager, R.R. (2004). On the retranslation process in Zadeh's paradigm of computing with words. IEEE Transactions on Systems, Man, and Cybernetics—Part B: Cybernetics, 34(2), 1184-1195.

Zadeh, L.A. (1975). The concept of a linguistic variable and its applications to approximate reasoning-Part I. Information Sciences, 8, 199-249.

Zadeh, L.A. (1965). Fuzzy sets. Information and Control, 8, 338-356.

Zavadskas, E.K., Mardani, A., Turskis, Z., Jusoh, A., & Nor, K.M. (2016). Development of TOPSIS method to solve complicated decision-making problems—An overview on developments from 2000 to 2015. International Journal of Information Technology & Decision Making, 15(3), 645-682.

Zavadskas, E.K., Turskis, Z., Volvaciovas, R., & Kildiene, S. (2013). Multi-criteria assessment model of technologies. Studies in Informatics and Control, 22(4), 249-258.

3 Group decision making with double hierarchy hesitant fuzzy linguistic preference relations: consistency based measures, index and repairing algorithms and decision model

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Group Decision Making with Double Hierarchy Hesitant Fuzzy Linguistic Preference Relations: Consistency based Measures, Index and Repairing Algorithms and Decision Model

Xunjie Gou[a,b], Huchang Liao[a,b], Zeshui Xu[a,c,*], Rui Min[b], Francisco Herrera[b,d]

[a] *Business School, Sichuan University, Chengdu 610064, China*

[b] *Andalusian Research Institute in Data Science and Computational Intelligence (DaSCI), University of Granada, Granada 18071, Spain*

[c] *School of Computer and Software, Nanjing University of Information Science and Technology, Nanjing, Jiangsu 210044, China*

[d] *Faculty of Computing and Information Technology, King Abdulaziz University, Jeddah, Saudi Arabia*

Abstract

Group decision making, refers to inviting a group of decision makers to evaluate, prioritize or select the optimal one among some available alternatives in the actual decision making process. Considering that the double hierarchy hesitant fuzzy linguistic term set can describe natural languages clearly, in this paper, we define the concept of double hierarchy hesitant fuzzy linguistic preference relation (DHHFLPR) and propose some additive consistency measures. To judge whether a DHHFLPR is of acceptable consistency or not, we introduce a consistency index, and develop some novel threshold values for judging whether a DHHFLPR is of acceptable consistency or not. Furthermore, we develop two consistency repairing algorithms based on the automatic improving method and the feedback improving method respectively, to improve the DHHFLPR with unacceptable consistency. Additionally, a method is set up to deal with group decision making problems with double hierarchy hesitant fuzzy linguistic preference information. Finally, the proposed method is validated by a case study that is used to evaluate the water resource situations of some important cities in Sichuan Province, and some comparative analyses are given to show the efficiency of the proposed method.

***Keywords*:** Double hierarchy hesitant fuzzy linguistic preference relation; additive consistency measures; consistency repairing algorithms; group decision making; water resource management

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* Corresponding Author. Emails: X.J. Gou (gou_xunjie@163.com); H.C. Liao (liaohuchang@163.com); Z.S. Xu (xuzeshui@263.net); R. Min (minminmail@hotmail.com); F. Herrera (herrera@decsai.ugr.es).

1. Introduction

Group decision making refers to inviting a group of decision makers to evaluate, prioritize or select the optimal one among some available alternatives in the actual decision making process. During group decision making, linguistic information is more in line with the real thoughts of decision makers and Zadeh [34-36] proposed a fuzzy linguistic approach to deal with it. As well as he proposed a concept of Computing with words (CW). Zadeh [33] also explained CW by "*Computing with words is a system of computation in which the objects of computation are words, phrases and propositions drawn from a natural language. The carriers of information are propositions. It is important to note that Computing with words is the only system of computation which offers a capability to compute with information described in a natural language.*" And he divided CW into two levels. In Level 1 CW (CW1), the objects of computation are some simple linguistic terms such as words, phrases and simple propositions. In Level 2 CW (CW2), the objects of computation include possibly complex propositions, and semantics of natural languages play an important role. Motivated by the CW2, in recent years, lots of linguistic models based on fuzzy set theory were developed to represent complex linguistic information such as hesitant fuzzy linguistic term set (HFLTS) [7, 13, 16, 18], 2-tuple linguistic model [11, 14], virtual linguistic term model [31, 32], and type-2 fuzzy sets [2, 15, 28].

Complex linguistic information can be found around us in our daily lives. For example, a teacher is hesitant when he/she gives the mark of a student, and he/she may utilize a HFLTS {good, very good, perfect} to express his/her opinion. However, all the linguistic terms included in this HFLTS have the same important degrees, which is not always adequate when representing the real thoughts of people. Therefore, one question is raised: How should we represent natural languages more accurately? With this in mind, four novel proposals have been developed to solve this problem: Firstly, Pang et al. [17] proposed a probabilistic linguistic term set (PLTS), which mainly consists of two parts: one is to utilize weights to represent the important degrees of natural languages given by people directly; the other one is to show the frequencies of linguistic terms. However, considering that sometimes the weights of linguistic terms included in complex linguistic information cannot be expressed clearly by PLTS such as "*more than fast*", Durand and Truck [5] developed a mapping function to compute weights and assigned them to corresponding linguistic terms. Additionally, Zhang et al. [37] introduced a probabilistic distribution of several linguistic terms, and developed the concept of distribution linguistic preference relations. Obviously, all of the above linguistic models consist of linguistic terms and numerical values simultaneously. To only utilize linguistic labels to represent complex linguistic information, Gou et al. [9] proposed a double hierarchy linguistic term set, and its hesitant extension named double hierarchy hesitant fuzzy linguistic term set. Double hierarchy linguistic term set adds a second hierarchy linguistic term set and uses linguistic labels to represent the important degrees of complex linguistic terms rather than numerical values, but the second hierarchy linguistic terms of different first hierarchy linguistic terms have no inevitable relation. In fact, all these four linguistic models belong to the CW2. Considering that the double hierarchy hesitant fuzzy linguistic term set can be used to reflect complex linguistic information intuitively, it will serve as the basis for this study and its basic element is called a double hierarchy hesitant fuzzy linguistic element (DHHFLE).

In group decision making, preference relations are popular and powerful techniques for decision maker preference modeling [24]. A large number of preference relations have been proposed in the literature such as the fuzzy preference relations [23], the linguistic preference relations [37], the multiplicative preference relations [19], and the hesitant fuzzy linguistic preference relation (HFLPR) [8, 25, 30, 38, 39]. Consistency measures of preference relations are the vital basis of group decision making and have been studied extensively, which show that the supplied preferences satisfy some transitive properties [30]. Consistency measures include two parts: (1) judging whether each preference relation is of acceptable consistency; (2) improving the preference relation with unacceptable consistency.

Up to now, two critical defects of existing consistency measures are being more and more apparent:

1) It is common that the normalization procedure is very necessary for making calculations expediently. But almost all methods complete it by adding or deleting some linguistic terms [39]. Obviously, these methods may cause the original information loss and make calculations complex.

2) Considering that there are some unreasonable places in the calculations of consistency thresholds under linguistic preference information environment, it is necessary to improve the existing consistency thresholds as the novel references for consistency improving processes.

To solve these two defects successfully, whilst considering that the double hierarchy hesitant fuzzy linguistic term set can describe linguistic evaluation information comprehensively and correctly, as well as there exists no any research available regarding its preference information. In this paper, the decision makers' linguistic evaluation information can establish some preference matrices with double hierarchy hesitant fuzzy linguistic information, denoted as double hierarchy hesitant fuzzy linguistic preference relation (DHHFLPR). In addition, to avoid the occurrence of some self-contradictory situations, it is very important to carry out the consistency checking and improving process for each DHHFLPR in a group decision making process. In this paper, we discuss some additive consistency measures for DHHFLPRs and the main contributions of this paper are summarized as follows:

a) For the first defect above, we develop a new normalization method by utilizing the linguistic expected-value of each DHHFLE to transform the DHHFLPR into the normalized DHHFLPR equivalently. The linguistic expected-value of the DHHFLE can be obtained by aggregating all elements of a DHHFLE into a double hierarchy linguistic term. With this method, we will not lose any linguistic terms and can make the calculations simpler.

b) For the purpose of judging whether a DHHFLPR is of acceptable consistency or not, we define a consistency index of the DHHFLPR and develop a novel method to improve the existing methods for calculating the consistency thresholds. Then we present two convergent consistency repairing algorithms based on automatic improving method and feedback improving method respectively to improve the consistency index of a given DHHFLPR with unacceptable consistency.

c) We propose a weight-determining method for obtaining the weight information of each decision maker, and then develop an algorithm to deal with the group decision making problem with double hierarchy hesitant fuzzy linguistic preference information.

Nowadays, the Sichuan water resource is an important water system in China. The protection of water quality of Sichuan water resources has become a crucial issue for the economic and social stability and rapid development of China. Therefore, the evaluation of water resource situations is a very important study carried out every year. In this paper, a case study is set up to apply our method to deal with a practical group decision making problem which is to evaluate the water resource situations of some important cities in Sichuan province.

To do so, the rest of this paper is organized as follows: Section 2 mainly discusses some basic concepts. Section 3 defines DHHFLPR, the additive consistent DHHFLPR, and the consistency index of DHHFLPR. Section 4 develops two convergent consistency repairing algorithms. Section 5 develops an algorithm to deal with the group decision making problem with DHHFLPRs. Section 6 sets up a case study to handle the Sichuan water resource management problem, and makes some comparative analyses with existing methods. Section 7 gives some discussions for highlighting the advantages of the proposed methods. Finally, we make some conclusions and propose some future research directions in Section 8.

2. Introducing the double hierarchy linguistic term set and the hesitant extension

In this section, we discuss three essential issues regarding double hierarchy linguistic term set and its hesitant extension with the aim of understanding them better.

2.1. What is double hierarchy linguistic term set and the hesitant extension

As we discussed in the Introduction, we can only utilize linguistic terms to represent complex linguistic information directly based on the double hierarchy linguistic term set and the hesitant extension. Suppose that $S=\{s_t|t=-\tau,\ldots,-1,0,1,\ldots,\tau\}$ and $O=\{o_k|k=-\varsigma,\ldots,-1,0,1,\ldots,\varsigma\}$ are the first hierarchy and the second hierarchy linguistic term set, respectively. A double hierarchy linguistic term set, S_O, is in mathematical form of

$$S_O=\{s_{t<o_k>}|t=-\tau,\ldots,-1,0,1,\ldots,\tau;\ k=-\varsigma,\ldots,-1,0,1,\ldots,\varsigma\} \quad (1)$$

we call $s_{t<o_k>}$ the double hierarchy linguistic term, where o_k expresses the second hierarchy linguistic term when the first hierarchy linguistic term is s_t. The second hierarchy linguistic term set of different first hierarchy linguistic terms may be different.

Then the distributions of four parts of the second hierarchy linguistic term set can be shown in Fig. 1:

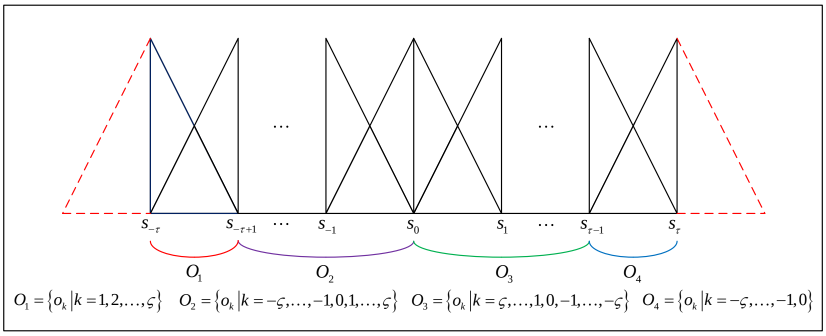

Fig. 1. The distributions of four parts of the second hierarchy linguistic term set.

Remark 1. In Fig. 1, four kinds of situations are shown on the basis of different values of t. If $t\geq 0$, then the meaning of the first hierarchy linguistic term set $S=\{s_t|t\geq 0\}$ is positive, so the second hierarchy linguistic term set needs to be selected with an ascending order. On the contrary, if $t<0$, then the meaning of the first hierarchy linguistic term set $S=\{s_t|t\leq 0\}$ is negative, so the second hierarchy linguistic term set needs to be selected with a descending order. Specially, both s_τ and $s_{-\tau}$ only contain a half of the area compared to other linguistic terms. Therefore, we only utilize $O=\{o_k|k=-\varsigma,\ldots,-1,0\}$ and $O=\{o_k|k=0,1,\ldots,\varsigma\}$ to describe s_τ and $s_{-\tau}$, respectively. In particular, the second hierarchy linguistic term sets with respect to different first hierarchy linguistic terms may be different. For convenience, we only utilize a uniform linguistic term set

$O=\left\{o_k \middle| k=-\varsigma,\dots,-1,0,1,\dots,\varsigma\right\}$ to express the second hierarchy linguistic term set.

Then, Gou et al. [9] extended S_O into hesitant fuzzy environment and developed a new concept: A double hierarchy hesitant fuzzy linguistic term set on X, H_{S_O}, is in mathematical form of

$$H_{S_O}=\left\{< x_i, h_{S_O}(x_i) > \middle| x_i \in X\right\}$$

where $h_{S_O}(x_i)$ is a set of some values in S_O, denoted as

$$h_{S_O}(x_i)=\left\{s_{\phi_l<o_{\varphi_l}>}(x_i) \middle| s_{\phi_l<o_{\varphi_l}>} \in S_O; l=1,2,\dots,L; \phi_l \in \{-\tau,\dots,-1,0,1,\dots,\tau\}; \varphi_l \in \{-\varsigma,\dots,-1,0,1,\dots,\varsigma\}\right\}$$

with L being the number of the double hierarchy linguistic terms in $h_{S_O}(x_i)$ and $s_{\phi_l<o_{\varphi_l}>}(x_i)$ ($l=1,2,\dots,L$) in each $h_{S_O}(x_i)$ being the terms in S_O. $h_{S_O}(x_i)$ denotes the possible degree of the linguistic variable x_i to S_O. For convenience, we call $h_{S_O}(x_i)$ the DHHFLE.

2.2. Why propose the double hierarchy linguistic term set and the hesitant extension?

We have discussed that the semantics of natural languages play an important role in CW2 in the Introduction. Therefore, how to represent complex linguistic information with correct semantics is the most important area of study. In recent years, lots of complex linguistic models based on fuzzy set theory have been developed to represent natural languages such as HFLTS [18], 2-tuple linguistic model [11, 14], virtual linguistic term model [31, 32], and type-2 fuzzy sets [2, 28], etc. In the semantic representation aspect, each linguistic model has its unique method:

- A hesitant fuzzy linguistic term can be used to express complex linguistic information by taking more than one linguistic terms;
- A 2-tuple linguistic term takes use of a linguistic term and a real number to represent its information;
- The semantic of a virtual linguistic term can be obtained by means of a proper linguistic modifier;
- The linguistic model based on type-2 fuzzy set representation that represents the semantics of the linguistic terms by type-2 membership functions.

However, if we only want to represent a complex linguistic term as "*only a little high*" or "*far from perfect*", there will always be more or less defects in the existing linguistic models. To solve this problem, Gou et al. [9] added a second hierarchy linguistic term set to the first linguistic term set as

S and defined the double hierarchy linguistic term set, which consists of two hierarchy linguistic term sets. They are denoted by a first hierarchy linguistic term set with classical feature linguistic labels and a second hierarchy linguistic term set as a linguistic feature or detailed supplementary of each linguistic term included in the first hierarchy linguistic term set. We can utilize linguistic labels to represent the modifiers (important degrees or weights) instead of numerical values. Let the above S be the first hierarchy linguistic term set, $O=\{o_{-3}= \textit{farfrom}, o_{-2}=\textit{only a little}, o_{-1}=\textit{a little}, o_0=\textit{just right}, o_1=\textit{much}, o_2=\textit{very much}, o_3=\textit{entirely}\}$ be the second hierarchy linguistic term set, then we can use the double hierarchy linguistic term $\{s_{1<o_{-2}>}\}$ to represent the "*only a little high*". Obviously, based on the double hierarchy linguistic term set, the real meaning of any one complex linguistic term can be obtained directly.

All in all, there are two important advantages to using the double hierarchy linguistic term set:

a) It is very intuitive and can be understood by making one to one correspondence with the given two linguistic term sets;

b) By introducing the second hierarchy linguistic term set, the auxiliary linguistic hierarchy can be expressed more accurately.

Furthermore, Gou et al. [9] extended double hierarchy linguistic term set to hesitant fuzzy environment and defined the double hierarchy hesitant fuzzy linguistic term set, which is a more reasonable linguistic model to represent natural languages in CW2.

2.3. How is a double hierarchy linguistic term set and the hesitant extension simpler applied?

Considering that the double hierarchy linguistic term set consists of two linguistic term sets, one question may arise: Are the computations among double hierarchy linguistic terms or DHHFLEs very complex? According to this question, some methods are developed to reduce the difficulty of computations.

Firstly, based on the discussion of monotonic function of Dubois [4] and virtual linguistic terms [32], Gou et al. [10] defined a monotonic function for making the mutual transformations between the double hierarchy linguistic term and the numerical scale when extending the double hierarchy linguistic term to a continuous form. The monotonic function provides convenience for using the mathematical expressions to make the operations among double hierarchy linguistic terms, as well as reducing the difficulty of computation. The monotonic function can be shown as follows:

Definition 2.1 [10]. Let $\bar{S}_O=\{s_{t<o_k>} \mid t\in[-\tau,\tau];\ k\in[-\varsigma,\varsigma]\}$ be a continuous double hierarchy linguistic term set, $h_{S_O}=\{s_{\phi_l<o_{\varphi_l}>} \mid s_{\phi_l<o_{\varphi_l}>}\in\bar{S}_O; l=1,2,\dots,L; \phi_l\in[-\tau,\tau];\ \varphi_l\in[-\varsigma,\varsigma]\}$ be a DHHFLE with

L being the number of linguistic terms in h_{S_O}, and $h_\gamma = \{\gamma_l | \gamma_l \in [0,1]; l = 1,2,\ldots,L\}$ be a hesitant fuzzy set. Then the subscript (ϕ_l, φ_l) of the double hierarchy linguistic term $s_{\phi_l < o_{\varphi_l} >}$ that expresses the equivalent information to the membership degree γ_l can be transformed to the membership degree γ_l by using a function f:

$$f : [-\tau, \tau] \times [-\varsigma, \varsigma] \to [0,1],\ f(\phi_l, \varphi_l) = \frac{\varphi_l + (\tau + \phi_l)\varsigma}{2\varsigma\tau} = \gamma_l$$

When we extend a double hierarchy linguistic term set into hesitant fuzzy environment, let $\Phi \times \Psi$ be the set of all DHHFLEs over $\bar{S}_O$, and Θ be the set of all hesitant fuzzy sets. Then a transformation function F between the DHHFLE h_{S_O} and hesitant fuzzy set h_γ on the basis of f is:

$$F : \Phi \times \Psi \to \Theta,\ F(h_{S_O}) = F\left(\left\{ s_{\phi_l < o_{\varphi_l} >} \middle| s_{\phi_l < o_{\varphi_l} >} \in \bar{S}_O; l = 1,\ldots,L; \phi_l \in [-\tau, \tau]; \varphi_l \in [-\varsigma, \varsigma] \right\}\right) = \{\gamma_l \,|\, \gamma_l = f(\phi_l, \varphi_l); l = 1,2,\ldots,L\} = h_\gamma$$

Secondly, the topic of this paper is to deal with DHHFLPRs, so some operations of the DHHFLEs with some conditions need to be developed. Suppose $h_{S_O} = \left\{ s_{\phi_l < o_{\varphi_l} >} \middle| s_{\phi_l < o_{\varphi_l} >} \in S_O; l = 1,2,\ldots,\# h_{S_O} \right\}$, $h_{S_{O_i}} = \left\{ s_{\phi_l^i < o_{\varphi_l^i} >} \middle| s_{\phi_l^i < o_{\varphi_l^i} >} \in S_O; l = 1,2,\ldots,\# h_{S_O}^i \right\}$ $(i = 1,2; \# h_{S_O}^1 = \# h_{S_O}^2)$ are three DHHFLEs, $\lambda (0 \le \lambda \le 1)$ is a real number. Based on some operations of linguistic term sets, such as $s_\alpha \oplus s_\beta = s_{\alpha+\beta}$, $\lambda s_\alpha = s_{\lambda\alpha}$. Then

(1) Addition: $h_{S_{O_1}} \oplus h_{S_{O_2}} = \bigcup_{s_{\phi_l^1 < o_{\varphi_l^1} >} \in h_{S_{O_1}}, s_{\phi_l^2 < o_{\varphi_l^2} >} \in h_{S_{O_2}}} \left\{ s_{\phi_l^1 + \phi_l^2 < o_{\varphi_l^1 + \varphi_l^2} >} \right\};\ if\ \phi_l^1 + \phi_l^2 \le \tau, \varphi_l^1 + \varphi_l^2 \le \varsigma$;

(2) Multiplication: $\lambda h_{S_O} = \bigcup_{s_{\phi_l < o_{\varphi_l} >} \in h_{S_O}} \left\{ s_{\lambda\phi_l < o_{\lambda\varphi_l} >} \right\};\ 0 \le \lambda \le 1$;

(3) Complementary operation: $\overline{h_{S_O}} = \bigcup_{s_{\phi_l < o_{\varphi_l} >} \in h_{S_O}} \left\{ s_{-\phi_l < o_{-\varphi_l} >} \right\}$.

Remark 2. These operations are made simpler by only calculating the subscripts of DHHFLEs. Specially, if each of these three DHHFLEs h_{S_O}, $h_{S_{O_1}}$ and $h_{S_{O_2}}$ only has one double hierarchy linguistic term, respectively. Then the above three operations can be reduced to the operations of double hierarchy linguistic terms: $\oplus_{i=1}^{2} s_{\phi^i < o_{\varphi^i} >} = s_{\phi^1 + \phi^2 < o_{\varphi^1 + \varphi^2} >}$, $\lambda s_{\phi < o_\varphi >} = s_{\lambda\phi < o_{\lambda\varphi} >}$, and $\overline{s_{\phi < o_\varphi >}} = s_{-\phi < o_{-\varphi} >}$.

3. DHHFLPR: Additive consistency and Index

As we mentioned in the Introduction, there are some shortcomings such as the normalization methods, consistency index and consistency thresholds in existing consistency measures. In this section, a novel concept of DHHFLPR is defined first, then we develop an additive consistency measure method and a consistency index of DHHFLPR on the basis of the distance measure of DHHFLEs to judge whether a DHHFLPR is of acceptable consistency or not.

3.1. Double hierarchy hesitant fuzzy linguistic preference relation

Suppose that we are dealing with a group decision making problem in a double hierarchy hesitant fuzzy linguistic environment. Let $A=\{A_1,A_2,\ldots,A_m\}$ be a set of alternatives. A group of decision makers $E=(e^1,e^2,\ldots,e^R)$ are invited to utilize linguistic expressions based on S_O to provide their pairwise comparison judgments of all alternatives. These linguistic expressions can be transformed into the DHHFLEs, then the concept of DHHFLPR can be defined as follows:

Definition 3.1. A DHHFLPR $\tilde{H}_{S_O}$ is represented by a matrix $\tilde{H}_{S_O}=\left(h_{S_{O_{ij}}}\right)_{m\times m}$, where $h_{S_{O_{ij}}}=\left\{h^{(l)}_{S_{O_{ij}}}\middle| l=1,2,\ldots,\#h_{S_{O_{ij}}}\right\}$ ($\#h_{S_{O_{ij}}}$ is the number of double hierarchy linguistic terms in $h_{S_{O_{ij}}}$, $h^{(l)}_{S_{O_{ij}}}$ is the l-th double hierarchy linguistic term in $h_{S_{O_{ij}}}$) is a DHHFLE, indicating the hesitant degrees to which A_i is preferred to A_j. For all $i,j=1,2,\ldots,m$, $h_{S_{O_{ij}}}$ $(i<j)$ satisfies the following conditions:

$$h^{(l)}_{S_{O_{ij}}}\oplus h^{(l)}_{S_{O_{ji}}}=s_{0<o_0>},\quad h_{S_{O_{ii}}}=\left\{s_{0<O_0>}\right\},\quad \#h_{S_{O_{ij}}}=\#h_{S_{O_{ji}}} \tag{2}$$

and

$$h^{(l)}_{S_{O_{ij}}}<h^{(l+1)}_{S_{O_{ij}}},\quad h^{(l)}_{S_{O_{ji}}}>h^{(l+1)}_{S_{O_{ji}}} \tag{3}$$

Remark 3. Based on the operations of DHHFLEs, we can utilize $h^{\sigma(l)}_{S_{O_{ij}}}\oplus h^{\sigma(l)}_{S_{O_{ji}}}=s_{0<o_0>}$ to check the first condition of the DHHFLPR. Furthermore, considering that a DHHFLE is an ordered finite subset of the consecutive linguistic terms of a double hierarchy linguistic term set, then we can also define that the double hierarchy linguistic terms in the upper triangle are arranged in an ascending order, while in the lower triangle are arranged in a descending order. That is to say, $h^{(l)}_{S_{O_{ij}}}<h^{(l+1)}_{S_{O_{ij}}}$ and $h^{(l)}_{S_{O_{ji}}}>h^{(l+1)}_{S_{O_{ji}}}$. For example, one DHHFLPR can be established as:

$$\tilde{H}_{S_O}=\begin{pmatrix} \{s_{0<o_0>}\} & \{s_{-1<o_1>}, s_0, s_{1<o_2>}\} & \{s_{-1<o_{-2}>}, s_{0<o_1>}\} \\ \{s_{1<o_{-1}>}, s_0, s_{-1<o_{-2}>}\} & \{s_{0<o_0>}\} & \{s_{2<o_1>}\} \\ \{s_{1<o_2>}, s_{0<o_{-1}>}\} & \{s_{-2<o_{-1}>}\} & \{s_{0<o_0>}\} \end{pmatrix}$$

3.2. Additive consistency measure method of DHHFLPRs

For the purpose of judging whether a DHHFLPR is with acceptable consistency or not, we define an additive consistency measure method for DHHFLPR. To do so, the normalization of DHHFLPR is the first and very important step considering it is very common that some DHHFLEs have different numbers of double hierarchy linguistic terms. To carry out the normalization process in a more reasonable manner and keep all linguistic information intact, we develop a linguistic expected-value for DHHFLE based on Remark 2. Suppose that $h_{S_O}=\left\{s_{\phi_l<o_{\varphi_l}>}\middle|s_{\phi_l<o_{\varphi_l}>}\in\bar{S}_O; l=1,2,\ldots,\#h_{S_O}\right\}$ is a DHHFLE, $\Phi\times\Psi$ is the set of all DHHFLEs over $\bar{S}_O$. Then a linguistic expected-value of h_{S_O}, denoted as $le\left(h_{S_O}\right)$, is obtained by

$$le:\Phi\times\Psi\to\bar{S}_O,\ le\left(h_{S_O}\right)=\frac{1}{\#h_{S_O}}\overset{\#h_{S_O}}{\underset{l=1}{\oplus}}s_{\phi_l<o_{\varphi_l}>}=s_{\frac{1}{\#h_{S_O}}\sum_{l=1}^{\#h_{S_O}}\phi_l<o_{\frac{1}{\#h_{S_O}}\sum_{l=1}^{\#h_{S_O}}\varphi_l}>} \tag{4}$$

Additionally, the normalized DHHFLPR of a DHHFLPR $\tilde{H}_{S_O}=\left(h_{S_{O_{ij}}}\right)_{m\times m}$ can be obtained, denoted by $\tilde{H}_{S_O}^N=\left(h_{S_{O_{ij}}}^N\right)_{m\times m}$, satisfying

$$h_{S_{O_{ij}}}^N=le\left(h_{S_{O_{ij}}}\right),\quad i,j=1,2,\ldots,m \tag{5}$$

From Eq. (4), it is obvious that $le\left(h_{S_O}\right)$ is a double hierarchy linguistic term. Thus, every basic element included in the normalized DHHFLPR of one DHHFLPR is also a double hierarchy linguistic term. Then the DHHFLPR of Remark 3 can be normalized by Eq. (5) and can be shown as:

$$\tilde{H}_{S_O}^N=\begin{pmatrix} \{s_{0<o_0>}\} & \{s_{0<o_1>}\} & \{s_{-1/2<o_{-1/2}>}\} \\ \{s_{0<o_{-1}>}\} & \{s_{0<o_0>}\} & \{s_{2<o_1>}\} \\ \{s_{1/2<o_{1/2}>}\} & \{s_{-2<o_{-1}>}\} & \{s_{0<o_0>}\} \end{pmatrix}$$

Next, the definition of an additive consistency for DHHFLPR can be given:

Definition 3.2. Let $\tilde{H}_{S_O}=\left(h_{S_{O_{ij}}}\right)_{m\times m}$ be a DHHFLPR and $\tilde{H}_{S_O}^N=\left(h_{S_{O_{ij}}}^N\right)_{m\times m}$ be its normalized

DHHFLPR, then we call $\tilde{H}_{S_O}$ an additive consistent DHHFLPR if it satisfies

$$h_{S_{O_{ij}}}^N = h_{S_{O_{i\rho}}}^N \oplus h_{S_{O_{\rho j}}}^N \left(i, j, \rho = 1, 2, \ldots, m; i \neq j\right) \tag{6}$$

Theorem 3.1. Let $\tilde{H}_{S_O} = \left(h_{S_{O_{ij}}}\right)_{m\times m}$ be a DHHFLPR and $\tilde{H}_{S_O}^N = \left(h_{S_{O_{ij}}}^N\right)_{m\times m}$ be its normalized DHHFLPR. If $\bar{h}_{S_{O_{ij}}}^N = \frac{1}{m}\left(\overset{m}{\underset{\rho=1}{\oplus}}\left(h_{S_{O_{i\rho}}}^N \oplus h_{S_{O_{\rho j}}}^N\right)\right)$ for $i, j, \rho = 1, 2, \ldots, m; i \neq j$, then $\tilde{H}_{S_O}$ is an additive consistent DHHFLPR, and $\bar{H}_{S_O}^N = \left(\bar{h}_{S_{O_{ij}}}^N\right)_{m\times m}$ is an additive consistent normalized DHHFLPR.

Proof. Since $\bar{h}_{S_{O_{i\rho}}}^N \oplus \bar{h}_{S_{O_{\rho j}}}^N = \left(\frac{1}{m}\left(\sum_{b=1}^{m}\left(h_{S_{O_{ib}}}^N \oplus h_{S_{O_{b\rho}}}^N\right)\right) \oplus \frac{1}{m}\left(\overset{m}{\underset{b=1}{\oplus}}\left(h_{S_{O_{\rho b}}}^N \oplus h_{S_{O_{bj}}}^N\right)\right)\right) = \left(\frac{1}{m}\left(\overset{m}{\underset{b=1}{\oplus}}\left(h_{S_{O_{ib}}}^N \oplus h_{S_{O_{bj}}}^N\right) \oplus h_{S_{O_{b\rho}}}^N \oplus h_{S_{O_{\rho b}}}^N\right)\right)$ and considering that $\tilde{H}_{S_O}^N$ is a normalized DHHFLPR, which satisfies $h_{S_{O_{b\rho}}}^N \oplus h_{S_{O_{\rho b}}}^N = \left\{s_{0<o_0>}\right\}$. Therefore,

$$\bar{h}_{S_{O_{i\rho}}}^N \oplus \bar{h}_{S_{O_{\rho j}}}^N = \left(\frac{1}{m}\left(\overset{m}{\underset{b=1}{\oplus}}\left(h_{S_{O_{ib}}}^N \oplus h_{S_{O_{bj}}}^N\right) \oplus s_{0<o_0>}\right)\right) = \left(\frac{1}{m}\overset{m}{\underset{b=1}{\oplus}}\left(h_{S_{O_{ib}}}^N \oplus h_{S_{O_{bj}}}^N\right)\right) = \bar{h}_{S_{O_{ij}}}^N$$

Based on Definition 3.2, $\bar{H}_{S_O}^N = \left(\bar{h}_{S_{O_{ij}}}^N\right)_{m\times m}$ is an additive consistent normalized DHHFLPR, which completes the proof of Theorem 3.1.■

Remark 4. Theorem 3.1 mainly provides the method that obtains the additive consistent normalized DHHFLPR. Meanwhile, it also gives a necessary condition which can be used to judge whether a normalized DHHFLPR is the additive consistent normalized DHHFLPR. Considering that checking the consistency of a DHHFLPR is the first and important step when dealing with double hierarchy linguistic preference information, so Theorem 3.1 is the most critical foundation of this paper.

Example 3.1. Let $S_O = \left\{s_{t<o_k>} \middle| t = -4, \ldots, 0, \ldots, 4; k = -4, \ldots, 0, \ldots, 4\right\}$ be a double hierarchy linguistic term set. For two DHHFLPRs

$$\tilde{H}_{S_O}^1 = \begin{pmatrix} \left\{s_{0<o_0>}\right\} & \left\{s_{-1<o_1>}, s_{0<o_2>}\right\} & \left\{s_{1<o_{-2}>}, s_{2<o_1>}\right\} \\ \left\{s_{1<o_{-1}>}, s_{0<o_{-2}>}\right\} & \left\{s_{0<o_0>}\right\} & \left\{s_{-1<o_{-2}>}\right\} \\ \left\{s_{-1<o_2>}, s_{-2<o_{-1}>}\right\} & \left\{s_{-1<o_2>}\right\} & \left\{s_{0<o_0>}\right\} \end{pmatrix}$$

$$\tilde{H}_{S_O}^2 = \begin{pmatrix} \left\{s_{0<o_0>}\right\} & \left\{s_{-1<o_1>}, s_{0<o_2>}\right\} & \left\{s_{-1<o_{-2}>}, s_0, s_{1<o_1>}\right\} & \left\{s_{1<o_{-2}>}\right\} \\ \left\{s_{1<o_{-1}>}, s_{0<o_{-2}>}\right\} & \left\{s_{0<o_0>}\right\} & \left\{s_{0<o_{-1}>}, s_{1<o_1>}\right\} & \left\{s_{1<o_1>}, s_{2<o_2>}\right\} \\ \left\{s_{1<o_2>}, s_0, s_{-1<o_{-1}>}\right\} & \left\{s_{0<o_1>}, s_{-1<o_{-1}>}\right\} & \left\{s_{0<o_0>}\right\} & \left\{s_{0<o_{-3}>}, s_1, s_{2<o_1>}\right\} \\ \left\{s_{-1<o_2>}\right\} & \left\{s_{-1<o_{-1}>}, s_{-2<o_{-2}>}\right\} & \left\{s_{0<o_3>}, s_{-1}, s_{-2<o_{-1}>}\right\} & \left\{s_{0<o_0>}\right\} \end{pmatrix}$$

The normalized DHHFLPRs $\tilde{H}_{S_O}^{1N} = \left(h_{S_{O_{ij}}}^{1N}\right)_{3\times 3}$ and $\tilde{H}_{S_O}^{2N} = \left(h_{S_{O_{ij}}}^{2N}\right)_{4\times 4}$ can be obtained:

$$\tilde{H}_{S_O}^{1N} = \begin{pmatrix} \{s_{0<o_0>}\} & \{s_{-1/2<o_{3/2}>}\} & \{s_{3/2<o_{-1/2}>}\} \\ \{s_{1/2<o_{-3/2}>}\} & \{s_{0<o_0>}\} & \{s_{-1<o_{-2}>}\} \\ \{s_{-3/2<o_{1/2}>}\} & \{s_{1<o_2>}\} & \{s_{0<o_0>}\} \end{pmatrix}, \quad \tilde{H}_{S_O}^{2N} = \begin{pmatrix} \{s_{0<o_0>}\} & \{s_{-1/2<o_{3/2}>}\} & \{s_{0<o_{-1/3}>}\} & \{s_{1<o_{-2}>}\} \\ \{s_{1/2<o_{-3/2}>}\} & \{s_{0<o_0>}\} & \{s_{1/2<o_0>}\} & \{s_{3/2<o_{3/2}>}\} \\ \{s_{0<o_{1/3}>}\} & \{s_{-1/2<o_0>}\} & \{s_{0<o_0>}\} & \{s_{1<o_{-2/3}>}\} \\ \{s_{-1<o_2>}\} & \{s_{-3/2<o_{-3/2}>}\} & \{s_{-1<o_{2/3}>}\} & \{s_{0<o_0>}\} \end{pmatrix}$$

We utilize Theorem 3.1 to obtain the additive consistent normalized DHHFLPRs $\bar{H}_{S_O}^{1N} = \left(\bar{h}_{S_{O_{ij}}}^{1N}\right)_{3\times 3}$ and $\bar{H}_{S_O}^{2N} = \left(\bar{h}_{S_{O_{ij}}}^{2N}\right)_{4\times 4}$, respectively:

$$\bar{H}_{S_O}^{1N} = \begin{pmatrix} \{s_{0<o_0>}\} & \{s_{1/2<o_{3/2}>}\} & \{s_{1/2<o_{-1/2}>}\} \\ \{s_{-1/2<o_{-3/2}>}\} & \{s_{0<o_0>}\} & \{s_{0<o_{-2}>}\} \\ \{s_{-1/2<o_{1/2}>}\} & \{s_{0<o_2>}\} & \{s_{0<o_0>}\} \end{pmatrix}, \quad \bar{H}_{S_O}^{2N} = \begin{pmatrix} \{s_{0<o_0>}\} & \{s_{-1/2<o_{-5/24}>}\} & \{s_{0<o_{-1/8}>}\} & \{s_{1<o_{-1/2}>}\} \\ \{s_{1/2<o_{5/24}>}\} & \{s_{0<o_0>}\} & \{s_{1/2<o_{1/12}>}\} & \{s_{3/2<o_{-7/24}>}\} \\ \{s_{0<o_{1/8}>}\} & \{s_{-1/2<o_{-1/12}>}\} & \{s_{0<o_0>}\} & \{s_{1<o_{-3/8}>}\} \\ \{s_{-1<o_{1/2}>}\} & \{s_{-3/2<o_{7/24}>}\} & \{s_{-1<o_{3/8}>}\} & \{s_{0<o_0>}\} \end{pmatrix}$$

Remark 5. To compare the inconsistent DHHFLPRs and the additive consistent DHHFLPRs more intuitively, we can further utilize the visual method "Figure of area", which is a function of MATLAB drawing toolbar. Then we obtain Fig. 2. Based on the areas of different DHHFLPRs, the area that is more regular is clearly distinguished. For example, in Fig. 2(a) and Fig. 2(b), because the changes in the areas of different colors in Fig. 2(b) are more regular than the corresponding changes in Fig. 2(a), we consider that the additive consistent DHHFLPR $\bar{H}_{S_O}^{1N}$ is more regular with respect to the areas in different colors than the inconsistent DHHFLPR $\tilde{H}_{S_O}^{1N}$. Similarly, the additive consistent DHHFLPR $\bar{H}_{S_O}^{2N}$ is more regular with respect to the areas in different colors than the inconsistent DHHFLPR $\tilde{H}_{S_O}^{2N}$ based on Fig. 2(c) and Fig. 2(d).

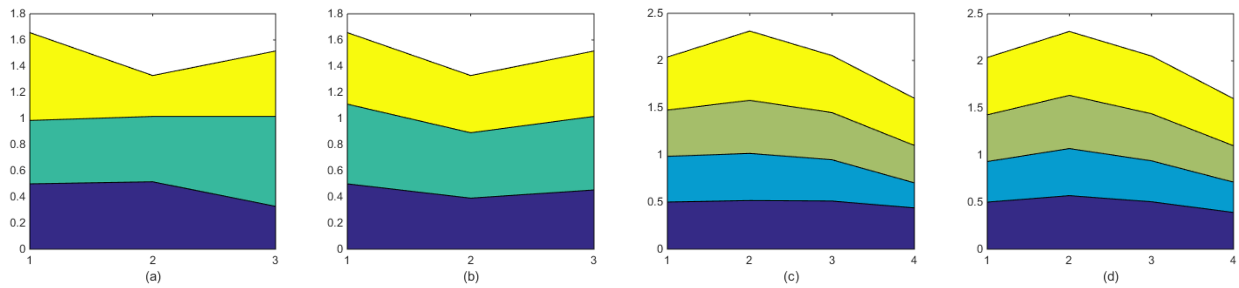

Fig. 2. The figures of area of $\tilde{H}_{S_O}^{1N}$, $\bar{H}_{S_O}^{1N}$, $\tilde{H}_{S_O}^{2N}$ and $\bar{H}_{S_O}^{2N}$

3.3. Consistency index of DHHFLPRs

When dealing with DHHFLPRs, judging whether a DHHFLPR is of acceptable consistency or not is of great importance. Therefore, how to calculate the consistency and judge whether it can be accepted is the focus of this subsection. Here, we introduce a consistency index for DHHFLPRs on the basic of distance measure [10]. Meanwhile, we develop a novel method to improve the existing method for calculating the consistency thresholds.

Firstly, one kind of distance measure of DHHFLEs can be shown as follows:

Definition 3.3. Let $\bar{S}_O = \left\{ s_{t<o_k>} \middle| t \in [-\tau, \tau];\ k \in [-\varsigma, \varsigma] \right\}$ be a continuous double hierarchy linguistic term set, $h_{S_O}^i = \left\{ s_{\phi_l<o_{\varphi_l}>}^i \middle| s_{\phi_l<o_{\varphi_l}>}^i \in S_O; l = 1, 2, \ldots, \# h_{S_O}^i \right\} (i = 1, 2)$ be two DHHFLEs, $le\left(h_{S_O}^i\right) = \left\{ s_{\phi^{le}<o_{\varphi^{le}}>}^i \right\} (i = 1, 2)$ be the linguistic expected-value of $h_{S_O}^1$ and $h_{S_O}^2$, respectively. Then

$$d\left(h_{S_O}^1, h_{S_O}^2\right) = \left| F'\left(le\left(h_{S_O}^1\right)\right) - F'\left(le\left(h_{S_O}^2\right)\right) \right| = \left| \gamma^1 - \gamma^2 \right| \quad (7)$$

where F' is an equivalent transformation function and

$$F' : \bar{S}_O \to [0,1],\ F'\left(le\left(h_{S_O}\right)\right) = f\left(\frac{1}{\# h_{S_O}} \sum_{l=1}^{\# h_{S_O}} \phi_l, \frac{1}{\# h_{S_O}} \sum_{l=1}^{\# h_{S_O}} \varphi_l \right) = \gamma$$

Given two DHHFLPRs $\tilde{H}_{S_O}^1 = \left(h_{S_{O_{ij}}}^1\right)_{m \times m}$ and $\tilde{H}_{S_O}^2 = \left(h_{S_{O_{ij}}}^2\right)_{m \times m}$, $\tilde{H}_{S_O}^{1N} = \left(h_{S_{O_{ij}}}^{1N}\right)_{m \times m}$ and $\tilde{H}_{S_O}^{2N} = \left(h_{S_{O_{ij}}}^{2N}\right)_{m \times m}$ are their corresponding normalized DHHFLPRs, then the distance measure between $\tilde{H}_{S_O}^1$ and $\tilde{H}_{S_O}^2$ is:

$$d\left(\tilde{H}_{S_O}^1, \tilde{H}_{S_O}^2\right) = \left(\frac{2}{m(m-1)} \sum_{i<j}^{m} \left(d\left(h_{S_{O_{ij}}}^{1N}, h_{S_{O_{ij}}}^{2N} \right) \right)^2 \right)^{1/2} = \left(\frac{2}{m(m-1)} \sum_{i<j}^{m} \left(\gamma_{ij}^1 - \gamma_{ij}^2 \right)^2 \right)^{1/2} \quad (8)$$

Obviously, the distance measure $d\left(\tilde{H}_{S_O}^1, \tilde{H}_{S_O}^2\right)$ satisfies properties: 1) $0 \le d\left(\tilde{H}_{S_O}^1, \tilde{H}_{S_O}^2\right) \le 1$; 2) $d\left(\tilde{H}_{S_O}^1, \tilde{H}_{S_O}^2\right) = 0$ if and only if $\tilde{H}_{S_O}^1 = \tilde{H}_{S_O}^2$; 3) $d\left(\tilde{H}_{S_O}^1, \tilde{H}_{S_O}^2\right) = d\left(\tilde{H}_{S_O}^2, \tilde{H}_{S_O}^1\right)$.

Example 3.2 (Continued with Example 3.1). Based on Eq. (8), we obtain

$$d\left(\tilde{H}_{S_O}^{1N}, \bar{H}_{S_O}^{1N}\right) = \left(\frac{2}{2 \times 3} \sum_{i<j}^{3} \left(d\left(h_{S_{O_{ij}}}^{1N}, \bar{h}_{S_{O_{ij}}}^{1N} \right) \right)^2 \right)^{1/2} = 0.1250$$

$$d\left(\tilde{H}_{S_O}^{2N}, \bar{H}_{S_O}^{2N}\right) = \left(\frac{2}{3 \times 4} \sum_{i<j}^{4} \left(d\left(h_{S_{O_{ij}}}^{2N}, \bar{h}_{S_{O_{ij}}}^{2N} \right) \right)^2 \right)^{1/2} = 0.0371$$

As we know, no matter what kind of preference relation, consistency index is a necessary tool to check whether a preference relation is of acceptable consistency or not. Similarly, it is necessary to develop a consistency index for DHHFLPR:

Definition 3.4. Let $\tilde{H}_{S_O}=\left(h_{S_{O_{ij}}}\right)_{m\times m}$ be a DHHFLPR. $\tilde{H}^N_{S_O}=\left(h^N_{S_{O_{ij}}}\right)_{m\times m}$ and $\bar{H}^N_{S_O}=\left(\bar{h}^N_{S_{O_{ij}}}\right)_{m\times m}$ are its normalized DHHFLPR and additive consistent normalized DHHFLPR, respectively. A consistency index (CI) of $\tilde{H}_{S_O}$ can be denoted as:

$$CI\left(\tilde{H}_{S_O}\right)=d\left(\tilde{H}^N_{S_O},\bar{H}^N_{S_O}\right) \tag{9}$$

The consistency index $CI\left(\tilde{H}_{S_O}\right)$ satisfies $0\le CI\left(\tilde{H}_{S_O}\right)\le 1$. Additionally, the smaller the consistency index $CI\left(\tilde{H}_{S_O}\right)$ is, the more consistent the DHHFLPR $\tilde{H}_{S_O}$ should be.

Dong et al. [3] proposed some consistency thresholds to check whether a preference relation with linguistic preference information is of acceptable consistency. Here we introduce a novel and reasonable method to improve these consistency thresholds. Firstly, in order to make the method more clear, it is necessary to transform the function f into a new form. Let T be the number of linguistic terms in the first hierarchy linguistic term set S. Obviously, we get $2\tau=T-1$. Then the function f is equal to

$$f:[-\tau,\tau]\times[-\varsigma,\varsigma]\to[0,1],\ f\left(\phi_l,\varphi_l\right)=\frac{\varphi_l+\left(\tau+\phi_l\right)\varsigma}{\left(T-1\right)\varsigma}$$

Let $\frac{\varphi_l+\left(\tau+\phi_l\right)\varsigma}{\varsigma}=\Delta^l$, then $f\left(\phi_l,\varphi_l\right)=\frac{\Delta^l}{T-1}$. Considering that $le\left(h_{S_O}\right)$ is a double hierarchy linguistic term, then Eq. (7) can be rewritten as

$$d\left(h^1_{S_O},h^2_{S_O}\right)=\left|F'\left(le\left(h^1_{S_O}\right)\right)-F'\left(le\left(h^2_{S_O}\right)\right)\right|=\left|\frac{\Delta^1}{T-1}-\frac{\Delta^2}{T-1}\right|=\left|\frac{\Delta^1-\Delta^2}{T-1}\right| \tag{10}$$

Therefore, Eq. (9) can be developed into

$$CI\left(\tilde{H}_{S_O}\right)=d\left(\tilde{H}^N_{S_O},\bar{H}^N_{S_O}\right)=\left(\frac{2}{m\left(m-1\right)}\sum_{i<j}^{m}\left(\left|\frac{\Delta_{ij}-\bar{\Delta}_{ij}}{T-1}\right|\right)^2\right)^{1/2}=\frac{1}{\left(T-1\right)}\left(\frac{2}{m\left(m-1\right)}\sum_{i<j}^{m}\left(\left|\Delta_{ij}-\bar{\Delta}_{ij}\right|\right)^2\right)^{1/2} \tag{11}$$

Let $\mathbb{R}_{ij}=\left|\Delta_{ij}-\bar{\Delta}_{ij}\right|$. Then $CI\left(\tilde{H}_{S_O}\right)=\frac{1}{\left(T-1\right)}\left(\frac{2}{m\left(m-1\right)}\sum_{i<j}^{m}\mathbb{R}_{ij}^{\ 2}\right)^{1/2}$. Considering that the value of $\mathbb{R}_{ij}\left(i<j\right)$ is independent normally distributed with a mean of 0 and standard deviation of ϑ, similar to the analyses of Dong et al. [3], we can obtain that $\frac{m\left(m-1\right)}{2}\left(\left(T-1\right)\times\frac{1}{\vartheta}\times CI\left(\tilde{H}_{S_O}\right)\right)^2$ is a chi-

square distribution with freedom degree $\frac{m(m-1)}{2}$, i.e., $\frac{m(m-1)}{2}\left((T-1)\times\frac{1}{\vartheta}\times CI\left(\tilde{H}_{S_O}\right)\right)^2 \sim \chi^2\left(\frac{m(m-1)}{2}\right)$, on the condition that $\mathbb{R}_{ij}\left(i<j\right)$ is independent normally distributed with a mean of 0 and standard deviation of ϑ, namely, $\mathbb{R}_{ij} \sim N\left(0,\vartheta^2\right)$. As we know, the freedom degree of $\chi^2 = \sum_{i<j}^{m}\left(\frac{\mathbb{R}_{ij}}{\vartheta}\right)^2$ is $\frac{m(m-1)}{2}$. This is a one-sided right-tailed test. At a significance level α, the critical value of χ^2 is λ_α. Let

$$\overline{CI}\left(\tilde{H}_{S_O}\right)=\frac{\vartheta}{(T-1)}\left(\frac{2}{m(m-1)}\lambda_\alpha\right)^{1/2} \tag{12}$$

be the consistency threshold. Therefore, if $CI\left(\tilde{H}_{S_O}\right)\le \overline{CI}\left(\tilde{H}_{S_O}\right)$, then $\tilde{H}_{S_O}$ is a DHHFLPR of acceptable consistency; Otherwise, if $CI\left(\tilde{H}_{S_O}\right)> \overline{CI}\left(\tilde{H}_{S_O}\right)$, then $\tilde{H}_{S_O}$ is a DHHFLPR of unacceptable consistency.

As we discussed above, the parameters α and ϑ are decided by the decision makers or according to practical situations. We let $\alpha=0.1$ and $\vartheta=2$, and then calculate the values of consistency thresholds $\overline{CI}\left(\tilde{H}_{S_O}\right)$ for different m and T, which can be shown in Table 3.1.

Table 3.1. The values of consistency thresholds based on different m and T

	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$	$m=8$
$T=5$	0.2207	0.3030	0.3488	0.3774	0.3970	0.4112
$T=9$	0.1103	0.1515	0.1744	0.1887	0.1985	0.2056
$T=17$	0.0552	0.0758	0.0872	0.0944	0.0993	0.1028

In Example 3.2, we obtain $CI\left(\tilde{H}^1_{S_O}\right)=0.1250$ and $CI\left(\tilde{H}^2_{S_O}\right)=0.0371$. In Table 3.1, $\overline{CI}\left(\tilde{H}^1_{S_O}\right)=0.1103$ and $\overline{CI}\left(\tilde{H}^2_{S_O}\right)=0.1515$. We obtain $CI\left(\tilde{H}^1_{S_O}\right)>\overline{CI}\left(\tilde{H}^1_{S_O}\right)$ and $CI\left(\tilde{H}^2_{S_O}\right)<\overline{CI}\left(\tilde{H}^2_{S_O}\right)$. So $\tilde{H}^2_{S_O}$ is a DHHFLPR of acceptable consistency, and $\tilde{H}^1_{S_O}$ is a DHHFLPR of unacceptable consistency.

Remark 6. In this subsection, the consistency index for DHHFLPR is proposed. And then a novel specific calculation process of consistency thresholds is given, which is a novel method and more reasonable than the existing method based on the more correct parameter found in Eq. (14). Therefore, the results of Table 3.1 can be used as important references when judging whether a preference relation with linguistic information is of acceptable consistency. Additionally, according to the consistency index and consistency thresholds, if a DHHFLPR is of acceptable consistency, then there is no need for it to be optimized. Otherwise, it is necessary to develop some methods to improve it, which will be discussed in the next section.

4. Consistency Repairing Algorithms

In some practical decision making processes with DHHFLPRs, it is common for there to be a DHHFLPR $\tilde{H}_{S_O} = \left(h_{S_{O_{ij}}}\right)_{m\times m}$ of unacceptable consistency, namely, $CI\left(\tilde{H}_{S_O}\right) > \overline{CI}\left(\tilde{H}_{S_O}\right)$. In this case, we need to repair the DHHFLPR $\tilde{H}_{S_O}$ until it reaches the consistency threshold. To improve the consistency, two existing methods have been developed: the automatic method [39] and the feedback-based method [1, 6, 12, 39]. Similarly, we establish two consistency repairing algorithms based on the automatic improving method and feedback improving method respectively to repair the DHHFLPR of unacceptable consistency.

4.1. Consistency repairing algorithm based on automatic optimization method

Considering that the automatic improving method is time-saving, effective, and practical without the interaction of the decision makers, so we develop a consistency repairing algorithm based on the automatic optimization method that can repair the DHHFLPR with unacceptable consistency by automatic iterative operations. Additionally, we analyze the convergence of repair results. Finally, we establish an optimization model which can be used to obtain the DHHFLPR of acceptable consistency directly.

Algorithm 4.1. The consistency repairing algorithm based on the automatic optimization method

Step 1. Let $\left(\tilde{H}_{S_O}\right)^{(\mathbb{Z})} = \left(\left(h_{S_{O_{ij}}}\right)_{m\times m}\right)^{(\mathbb{Z})}$ ($\mathbb{Z}=0$, $\left(\tilde{H}_{S_O}\right)^{(\mathbb{Z})}$ expresses the $\mathbb{Z}$-th power of $\tilde{H}_{S_O}$, indicating the number of iterations). Based on Eqs. (4)-(5) and Theorem 3.1, we can calculate the normalized DHHFLPR $\left(\tilde{H}^N_{S_O}\right)^{(\mathbb{Z})} = \left(\left(h^N_{S_{O_{ij}}}\right)_{m\times m}\right)^{(\mathbb{Z})}$ and the consistent normalized DHHFLPR $\left(\bar{H}^N_{S_O}\right)^{(\mathbb{Z})} = \left(\left(\bar{h}^N_{S_{O_{ij}}}\right)_{m\times m}\right)^{(\mathbb{Z})}$, respectively.

Step 2. Calculate $\overline{CI}\left(\tilde{H}_{S_O}\right)$ based on Eq. (12) or Table 3.1.

Step 3. Calculate $CI\left(\left(\tilde{H}_{S_O}\right)^{(\mathbb{Z})}\right)=d\left(\left(\tilde{H}_{S_O}^N\right)^{(\mathbb{Z})},\left(\bar{H}_{S_O}^N\right)^{(\mathbb{Z})}\right)$ based on Eq. (9). If $CI\left(\left(\tilde{H}_{S_O}\right)^{(\mathbb{Z})}\right)\le C\bar{I}\left(\tilde{H}_{S_O}\right)$, then go to step 5; If $CI\left(\left(\tilde{H}_{S_O}\right)^{(\mathbb{Z})}\right)> C\bar{I}\left(\tilde{H}_{S_O}\right)$, then go to Step 4.

Step 4. Let $\theta\left(0\le\theta\le1\right)$ be an adjusted parameter. Utilize the formula

$$\left(h_{S_{O_{ij}}}^N\right)^{(\mathbb{Z}+1)}=(1-\theta)\left(h_{S_{O_{ij}}}^N\right)^{(\mathbb{Z})}\oplus\theta\left(\bar{h}_{S_{O_{ij}}}^N\right)^{(\mathbb{Z})}\quad\left(i,j=1,2,\ldots,m;i\neq j\right)\tag{13}$$

to obtain the modified normalized DHHFLPR $\left(\tilde{H}_{S_O}^N\right)^{(\mathbb{Z}+1)}=\left(\left(h_{S_{O_{ij}}}^N\right)^{(\mathbb{Z}+1)}\right)_{m\times m}$. Let $\mathbb{Z}=\mathbb{Z}+1$ and go back to Step 3.

Step 5. Let $*H_{S_O}=\left(\tilde{H}_{S_O}^N\right)^{(\mathbb{Z})}$ and output the modified normalized DHHFLPR $*H_{S_O}$.

Based on Eq. (13), it is obvious that all the consistent normalized DHHFLPRs are the same no matter what the value of $\mathbb{Z}$ is. Namely, $\left(\bar{H}_{S_O}^N\right)^{(0)}=\left(\bar{H}_{S_O}^N\right)^{(1)}=\left(\bar{H}_{S_O}^N\right)^{(2)}=\cdots$.

Considering that the presented algorithm is convergent, so we can get a more consistent DHHFLPR after the consistency repairing process. The following theorem shows the convergence.

Theorem 4.1. Let $\tilde{H}_{S_O}=\left(h_{S_{O_{ij}}}\right)_{m\times m}$ be a DHHFLPR, $\theta\left(0\le\theta\le1\right)$ be the adjusted parameter, and $*\tilde{H}_{S_O}$ be the modified normalized DHHFLPR obtained by Algorithm 4.1. Then $CI\left(*\tilde{H}_{S_O}\right)<CI\left(\tilde{H}_{S_O}\right)$.

Proof. From Algorithm 4.1, $\left(\tilde{H}_{S_O}^N\right)^{(\mathbb{Z})}=\left(\left(h_{S_{O_{ij}}}^N\right)^{(\mathbb{Z})}\right)_{m\times m}$ is the modified normalized DHHFLPR in the $\mathbb{Z}$-th power of $\tilde{H}_{S_O}$. Suppose that the modified normalized DHHFLPR in the $\mathbb{Z}+1$-th power of $\tilde{H}_{S_O}$ is $\left(\tilde{H}_{S_O}^N\right)^{(\mathbb{Z}+1)}=\left(\left(h_{S_{O_{ij}}}^N\right)^{(\mathbb{Z}+1)}\right)_{m\times m}$. Based on Eq. (13) and Remark 6, we obtain

$$\left(h_{S_{O_{ij}}}^N\right)^{(\mathbb{Z}+1)}=(1-\theta)\left(h_{S_{O_{ij}}}^N\right)^{(\mathbb{Z})}\oplus\theta\bar{h}_{S_{O_{ij}}}^N\tag{14}$$

Then,

$$d\left(\left(h_{S_{O_{ij}}}^N\right)^{(\mathbb{Z}+1)},\bar{h}_{S_{O_{ij}}}^N\right)=\left|\left(\frac{\Delta_{ij}}{T-1}\right)^{(\mathbb{Z}+1)}-\frac{\bar{\Delta}_{ij}}{T-1}\right|=\left|(1-\theta)\left(\frac{\Delta_{ij}}{T-1}\right)^{(\mathbb{Z})}+\theta\left(\frac{\bar{\Delta}_{ij}}{T-1}\right)-\frac{\bar{\Delta}_{ij}}{T-1}\right|=\left|(1-\theta)\left(\frac{\Delta_{ij}}{T-1}\right)^{(\mathbb{Z})}-(1-\theta)\left(\frac{\bar{\Delta}_{ij}}{T-1}\right)\right|=(1-\theta)\left|\left(\frac{\Delta_{ij}}{T-1}\right)^{(\mathbb{Z})}-\left(\frac{\bar{\Delta}_{ij}}{T-1}\right)\right|\tag{15}$$

Considering $0\le\left(1-\theta\right)\le1$, we can obtain

$$\left(1-\theta\right)\left|\left(\frac{\Delta_{ij}}{T-1}\right)^{(\mathbb{Z})}-\left(\frac{\bar{\Delta}_{ij}}{T-1}\right)\right|\leq\left|\left(\frac{\Delta_{ij}}{T-1}\right)^{(\mathbb{Z})}-\left(\frac{\bar{\Delta}_{ij}}{T-1}\right)\right| \tag{16}$$

Combining Eq. (15) and Eq. (16), we have

$$\left|\left(\frac{\Delta_{ij}}{T-1}\right)^{(\mathbb{Z}+1)}-\left(\frac{\bar{\Delta}_{ij}}{T-1}\right)\right|\leq\left|\left(\frac{\Delta_{ij}}{T-1}\right)^{(\mathbb{Z})}-\left(\frac{\bar{\Delta}_{ij}}{T-1}\right)\right| \tag{17}$$

Then we have $\frac{1}{T-1}\left(\frac{2}{m(m-1)}\sum_{i<j}^{m}\left(\left(\Delta_{ij}\right)^{(\mathbb{Z}+1)}-\left(\bar{\Delta}_{ij}\right)\right)^{2}\right)^{1/2}\leq\frac{1}{T-1}\left(\frac{2}{m(m-1)}\sum_{i<j}^{m}\left(\left(\Delta_{ij}\right)^{(\mathbb{Z})}-\left(\bar{\Delta}_{ij}\right)\right)^{2}\right)^{1/2}$. Therefore, we obtain $CI\left(\left(\tilde{H}_{S_O}\right)^{(\mathbb{Z}+1)}\right)\leq CI\left(\left(\tilde{H}_{S_O}\right)^{(\mathbb{Z})}\right)$.

In a similar way, we can also get $CI\left(\left(\tilde{H}_{S_O}\right)^{(\mathbb{Z})}\right)\leq CI\left(\tilde{H}_{S_O}\right)$, i.e., $CI\left(*\tilde{H}_{S_O}\right)< CI\left(\tilde{H}_{S_O}\right)$, which completes the proof of Theorem 4.1.■

Remark 7. Firstly, Theorem 4.1 mainly shows that utilizing the consistency repairing algorithm based on the automatic optimization method, the consistency index of the repaired DHHFLPR is always smaller than the original DHHFLPR. Secondly, for Algorithm 4.1, the adjusted parameter $\theta\left(0\leq\theta\leq1\right)$ is very important. It determines the number of iterations and the accuracy of modification to the original HFLPR. Therefore, it is very important to choose a proper value of θ to reduce the number of iterations and simultaneously let the modified normalized DHHFLPR be close to its original normalized DHHFLPR as much as possible. Zhu and Xu [39] calculated the average iterations of θ when $T=9$, $\alpha=0.1$ and $\vartheta=2$. The results and the corresponding values of $C\bar{I}\left(\tilde{H}_{S_O}\right)$ are listed in Table 4.1:

Table 4.1. The averaged values of iterations in Algorithm 4.1 ($T=9$, $\alpha=0.1$, $\vartheta=2$)

θ	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$	$m=8$
0.20	0.88	0.98	1.00	1.05	1.04	1.06
0.10	1.65	1.63	1.69	1.74	1.75	1.76
0.08	1.96	1.88	2.12	2.14	2.09	2.13
0.05	2.95	3.22	3.03	3.08	3.05	3.16
0.01	13.78	13.81	13.02	14.14	13.66	14.129
$C\bar{I}$	0.1103	0.1515	0.1744	0.1887	0.1985	0.2056

Next, we can set up an example to show the working process of Algorithm 4.1.

Example 4.1 (Continued with Example 3.2). The DHHFLPR $\tilde{H}_{S_O}^{1}$ needs to be improved.

Firstly, based on Table 3.1, there is $C\bar{I}\left(\tilde{H}_{S_O}^{1}\right)=0.1103$. From Example 3.2 and Definition 3.4, we know $CI\left(\tilde{H}_{S_O}^{1}\right)=d\left(\tilde{H}_{S_O}^{1N},\bar{H}_{S_O}^{1N}\right)=0.1250>C\bar{I}\left(\tilde{H}_{S_O}^{1}\right)$. Suppose $\theta=0.2$, and based on Eq. (13), we can get the modified normalized DHHFLPR $\left(\tilde{H}_{S_O}^{1N}\right)^{(1)}$:

$$\left(\tilde{H}_{S_O}^{1N}\right)^{(1)} = \begin{pmatrix} \left\{s_{0<o_0>}\right\} & \left\{s_{-3/10<o_{3/2}>}\right\} & \left\{s_{13/10<o_{-1/2}>}\right\} \\ \left\{s_{3/10<o_{-3/2}>}\right\} & \left\{s_{0<o_0>}\right\} & \left\{s_{-4/5<o_{-2}>}\right\} \\ \left\{s_{-13/10<o_{1/2}>}\right\} & \left\{s_{4/5<o_2>}\right\} & \left\{s_{0<o_0>}\right\} \end{pmatrix}$$

Go back and calculate $CI\left(\left(\tilde{H}_{S_O}^{1N}\right)^{(1)}\right) = d\left(\left(\tilde{H}_{S_O}^{1N}\right)^{(1)}, \left(\bar{H}_{S_O}^{1N}\right)^{(1)}\right) = 0.1000 < \overline{CI}\left(\tilde{H}_{S_O}^{1}\right)$, so the normalized DHHFLPR $\left(\tilde{H}_{S_O}^{1N}\right)^{(1)}$ is of acceptable consistency. Let $*\tilde{H}_{S_O}^{1} = \left(\tilde{H}_{S_O}^{1N}\right)^{(1)}$ and output $*\tilde{H}_{S_O}^{1}$.

Additionally, based on the modeling method proposed by Dong et al. [3] and an optimization model of HFLPR introduced by Zhu and Xu [39], we can develop an optimization model of the DHHFLPR to improve its consistency. Suppose that $\tilde{H}_{S_O} = \left(h_{S_{O_{ij}}}\right)_{m\times m}$ is a DHHFLPR with unacceptable consistency, $\tilde{H}_{S_O}^{N} = \left(h_{S_{O_{ij}}}^{N}\right)_{m\times m}$ being its normalized DHHFLPR. To obtain the modified normalized DHHFLPR $*\tilde{H}_{S_O} = \left(*h_{S_{O_{ij}}}\right)_{m\times m}$ with acceptable consistency and reduce the loss of original information, we set up $*h_{S_{O_{ij}}} = h_{S_{O_{ij}}}^{N} \oplus y_{ij}$, in which $y_{ij}\left(i, j = 1, 2, \ldots, m; i < j\right)$ are the adjusted DHHFLEs. An optimization model can be established as follows:

$$\begin{aligned} &\min_{y}\left(\frac{2}{m(m-1)}\sum_{i<j}^{m}\left|F'\left(y_{ij}\right)\right|\right) \\ &s.t.\begin{cases} F'\left(y_{ij}\right) + F'\left(y_{ji}\right) = 0 \\ CI\left(*\tilde{H}_{S_O}\right) \le \overline{CI}\left(\tilde{H}_{S_O}\right) \end{cases} \end{aligned} \tag{18}$$

where $CI\left(*\tilde{H}_{S_O}\right) = \left(\frac{2}{m(m-1)}\sum_{i<j}^{m}\left(d\left(*h_{S_{O_{ij}}}^{N}, *\bar{h}_{S_{O_{ij}}}^{N}\right)\right)^{2}\right)^{1/2}$.

Based on this model, we can optimize $*\tilde{H}_{S_O}$ as discussed in Example 4.1. All adjusted DHHFLEs $y_{ij}\left(i, j = 1, 2, 3\right)$ are obtained and the modified normalized DHHFLPR $*\tilde{H}_{S_O}^{1'} = \left(*h_{S_{O_{ij}}}^{1'}\right)_{3\times 3}$ is established:

$$*\tilde{H}_{S_O}^{1'} = \begin{pmatrix} \{s_{0<o_0>}\} & \{s_{-173/518<o_{1424/783}>}\} & \{s_{3/2<o_{-1/2}>}\} \\ \{s_{173/518<o_{-1424/783}>}\} & \{s_{0<o_0>}\} & \{s_{-216/259<o_{-1254/265}>}\} \\ \{s_{-3/2<o_{1/2}>}\} & \{s_{216/259<o_{1254/265}>}\} & \{s_{0<o_0>}\} \end{pmatrix}$$

Then, the consistency index $CI\left(*\tilde{H}_{S_O}^{1'}\right) = 0.1103 = C\overline{I}\left(\tilde{H}_{S_O}^{1}\right)$.

Remark 8. From Algorithm 4.1, we can determine that the modified normalized DHHFLPR $*H_{S_O}^{1}$ is of acceptable consistency from several iterations. Simultaneously, the number of iterations can be controlled by using different values of the adjusted parameter θ. Additionally, using the above optimization model, we only need to calculate it one time to obtain the modified normalized DHHFLPR $*\tilde{H}_{S_O}^{1'}$ with acceptable consistency, but the calculation is complex. Thus, if the DHHFLPR is simple, the optimization model is suitable; otherwise, we can use Algorithm 4.1.

4.2. Consistency repairing algorithm based on the feedback method

Considering that sometimes the decision makers are more likely to modify their preference relations by themselves, then Algorithm 4.1 is not suitable any more. In existing research, lots of scholars developed some feedback methods under other preference circumstances [1, 6, 12, 39], and the feedback method can feed suggestions back to the decision makers and help them to improve their preferences. Therefore, this subsection establishes a consistency repairing algorithm based on the feedback method under DHHFLPR. Firstly, a novel concept of interval-valued double hierarchy hesitant fuzzy linguistic term set is defined.

Definition 4.1. An interval-valued double hierarchy hesitant fuzzy linguistic term set on X, $H_{\bar{S}_{\bar{O}}}$, is given in the mathematical form of $H_{\bar{S}_{\bar{O}}} = \left\{< x_i, h_{\bar{S}_{\bar{O}}}(x_i) > \middle| x_i \in X\right\}$, where $h_{\bar{S}_{\bar{O}}}$ is a set of some values in S_O, denoted by $h_{\bar{S}_{\bar{O}}} = \left\{h_{\bar{S}_{\bar{O}_{ij}}}^{(l)} \middle| l = 1, 2, \ldots, h_{\bar{S}_{\bar{O}}}\right\}$. We call $h_{\bar{S}_{\bar{O}}}$ interval-valued DHHFLE, and call $h_{\bar{S}_{\bar{O}_{ij}}}^{(l)}$ interval-valued double hierarchy linguistic term. $h_{\bar{S}_{\bar{O}_{ij}}}^{(l)} = \left[\left(h_{\bar{S}_{\bar{O}_{ij}}}^{(l)}\right)^{\mu}, \left(h_{\bar{S}_{\bar{O}_{ij}}}^{(l)}\right)^{\nu}\right]$ satisfies $\left(h_{\bar{S}_{\bar{O}_{ij}}}^{(l)}\right)^{\mu}, \left(h_{\bar{S}_{\bar{O}_{ij}}}^{(l)}\right)^{\nu} \in S_O$ and $\left(h_{\bar{S}_{\bar{O}_{ij}}}^{(l)}\right)^{\mu} \le \left(h_{\bar{S}_{\bar{O}_{ij}}}^{(l)}\right)^{\nu}$.

Then an interval-valued DHHFLPR can be defined as follows:

Definition 4.2. We call $\tilde{H}_{\bar{S}_{\bar{O}}} = \left(h_{\bar{S}_{\bar{O}_{ij}}}\right)_{m\times m} \subset A \times A$ an interval-valued DHHFLPR, where

$h_{\bar{S}_{\bar{O}_{ij}}} = \left\{ h_{\bar{S}_{\bar{O}_{ij}}}^{(l)} \middle| l = 1,2,\ldots,\# h_{\bar{S}_{\bar{O}_{ij}}} \right\}$ is an interval-valued DHHFLE indicating the preferences in an interval to which A_i over A_j, and $h_{\bar{S}_{\bar{O}_{ij}}}^{(l)} = \left[\left(h_{\bar{S}_{\bar{O}_{ij}}}^{(l)} \right)^{\mu}, \left(h_{\bar{S}_{\bar{O}_{ij}}}^{(l)} \right)^{\nu} \right]$. For all $i, j = 1,2,\ldots,m$, $h_{\bar{S}_{\bar{O}_{ij}}} \left(i \le j \right)$ should satisfy that:

$$\left(h_{\bar{S}_{\bar{O}_{ij}}}^{(l)} \right)^{\mu} \oplus \left(h_{\bar{S}_{\bar{O}_{ji}}}^{(l)} \right)^{\nu} = \left(h_{\bar{S}_{\bar{O}_{ij}}}^{(l)} \right)^{\nu} \oplus \left(h_{\bar{S}_{\bar{O}_{ji}}}^{(l)} \right)^{\mu} = s_{0<o_0>}, \quad h_{\bar{S}_{\bar{O}_{ii}}} = \left\{ s_{0<o_0>} \right\} \text{ and } \# h_{\bar{S}_{\bar{O}_{ij}}} = \# h_{\bar{S}_{\bar{O}_{ji}}} \tag{19}$$

and

$$h_{\bar{S}_{\bar{O}_{ij}}}^{(l)} \le h_{\bar{S}_{\bar{O}_{ij}}}^{(l+1)}, h_{\bar{S}_{\bar{O}_{ji}}}^{(l+1)} \le h_{\bar{S}_{\bar{O}_{ji}}}^{(l)} \tag{20}$$

where $h_{\bar{S}_{\bar{O}_{ij}}}^{(l)}$ is the l-th interval-valued double hierarchy linguistic term in $h_{\bar{S}_{\bar{O}_{ij}}}$.

Then, an algorithm is established to show the feedback-based improving method:

Algorithm 4.2. The consistency repairing algorithm based on the feedback method

Step 1. Let $\left(\tilde{H}_{S_O} \right)^{(\mathbb{Z})} = \left(\left(h_{S_{O_{ij}}} \right)_{m \times m} \right)^{(\mathbb{Z})}$ be a DHHFLPR. $\left(\tilde{H}_{S_O}^{N} \right)^{(\mathbb{Z})} = \left(\left(h_{S_{O_{ij}}}^{N} \right)_{m \times m} \right)^{(\mathbb{Z})}$ and $\left(\bar{H}_{S_O}^{N} \right)^{(\mathbb{Z})} = \left(\left(\bar{h}_{S_{O_{ij}}}^{N} \right)_{m \times m} \right)^{(\mathbb{Z})}$ be the normalized DHHFLPR and the consistent normalized DHHFLPR, respectively.

Step 2. Calculate $\overline{CI} \left(\tilde{H}_{S_O} \right)$ based on Eq. (12) or Table 3.1.

Step 3. Calculate $CI \left(\left(\tilde{H}_{S_O} \right)^{(\mathbb{Z})} \right) = d \left(\left(\tilde{H}_{S_O}^{N} \right)^{(\mathbb{Z})}, \left(\bar{H}_{S_O}^{N} \right)^{(\mathbb{Z})} \right)$ based on Eq. (9). If $CI \left(\left(\tilde{H}_{S_O} \right)^{(\mathbb{Z})} \right) \le \overline{CI} \left(\tilde{H}_{S_O} \right)$, then go to Step 6; If $CI \left(\left(\tilde{H}_{S_O} \right)^{(\mathbb{Z})} \right) > \overline{CI} \left(\tilde{H}_{S_O} \right)$, then go to Step 4.

Step 4. Construct an interval-valued DHHFLPR $\tilde{H}_{\bar{S}_{\bar{O}}} = \left(\left(h_{\bar{S}_{\bar{O}_{ij}}} \right)_{m \times m} \right)^{(\mathbb{Z})} = \left(\left(\left\{ h_{\bar{S}_{\bar{O}_{ij}}}^{(l)} \middle| l = 1,2,\ldots,\# h_{\bar{S}_{\bar{O}_{ij}}} \right\} \right)_{m \times m} \right)^{(\mathbb{Z})}$, where $\left(h_{\bar{S}_{\bar{O}_{ij}}}^{(l)} \right)^{(\mathbb{Z})} = \left[\min \left\{ \left(h_{S_{O_{ij}}}^{(l)} \right)^{(\mathbb{Z})}, \left(\bar{h}_{S_{O_{ij}}}^{(l)} \right)^{(\mathbb{Z})} \right\}, \max \left\{ \left(h_{S_{O_{ij}}}^{(l)} \right)^{(\mathbb{Z})}, \left(\bar{h}_{S_{O_{ij}}}^{(l)} \right)^{(\mathbb{Z})} \right\} \right]$. Then we return $\tilde{H}_{\bar{S}_{\bar{O}}}$ to the decision maker and ask him to provide new preference information.

Step 5. Receive all the preference information of the decision makers and establish the modified normalized DHHFLPR $\left(\tilde{H}_{S_O}^{N} \right)^{(\mathbb{Z}+1)} = \left(\left(h_{S_{O_{ij}}}^{N} \right)_{m \times m} \right)^{(\mathbb{Z}+1)}$. Let $\mathbb{Z} = \mathbb{Z} + 1$. Go back to Step 3.

Step 6. Let $* \tilde{H}_{S_O} = \left(\tilde{H}_{S_O}^{N} \right)^{(\mathbb{Z})}$, and output the modified normalized DHHFLPR $* \tilde{H}_{S_O}$.

Fig. 3 shows the consistency improving process of Algorithm 4.2.

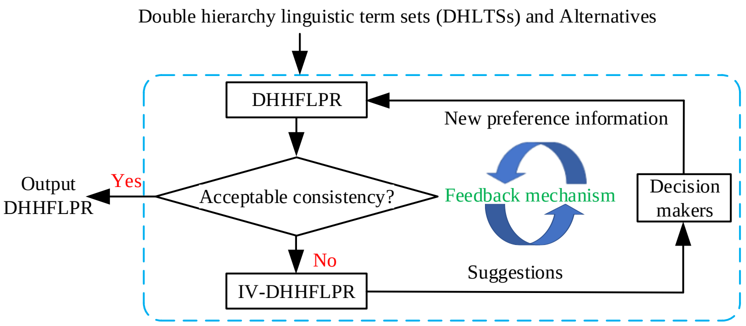

Fig 3. The feedback-based improving method

Similar to Theorem 4.1, we can give the following theorem:

Theorem 4.2. Let $\tilde{H}_{S_O} = \left(h_{S_{O_{ij}}}\right)_{m\times m}$ be a DHHFLPR, $\theta\left(0\le\theta\le 1\right)$ be the adjusted parameter, and $*H_{S_O}^{N}$ be the modified normalized DHHFLPR obtained by Algorithm 4.2. Then $CI\left(*H_{S_O}^{N}\right) < CI\left(\tilde{H}_{S_O}\right)$.

As we discussed above, the Theorem 4.2 mainly shows that utilizing the consistency repairing algorithm based on the feedback method, the consistency index of the repaired DHHFLPR is also smaller than the original DHHFLPR. Considering that Theorem 4.2 is similar as Theorem 4.1, its proof is omitted.

Example 4.3 (Continued with Example 3.2). $\tilde{H}_{S_O}^{1}$ is a DHHFLPR of unacceptable consistency and thus it needs to be improved. Let $\tilde{H}_{S_O}^{1} = \left(\tilde{H}_{S_O}^{1}\right)^{(0)}$, and we can get the normalized DHHFLPR $\left(\tilde{H}_{S_O}^{1N}\right)^{(0)} = \left(\left(h_{S_{O_{ij}}}^{1N}\right)_{3\times 3}\right)^{(0)}$ and the consistent normalized DHHFLPR $\left(\bar{H}_{S_O}^{1N}\right)^{(0)} = \left(\left(\bar{h}_{S_{O_{ij}}}^{1N}\right)_{3\times 3}\right)^{(0)}$, respectively. Then we get $CI\left(\left(\tilde{H}_{S_O}^{1}\right)^{(0)}\right) = 0.1250 > \overline{CI}\left(\tilde{H}_{S_O}^{1}\right)$. So we construct an interval-valued DHHFLPR $\tilde{H}_{\bar{S}_{\bar{O}}} = \left(\left(h_{\bar{S}_{\bar{O}_{ij}}}\right)_{3\times 3}\right)^{(0)}$ based on $\tilde{H}_{S_O}^{1N}$ and $\bar{H}_{S_O}^{1N}$:

$$\tilde{H}_{\bar{S}_{\bar{O}}} = \begin{pmatrix} \left\{s_{0<o_0>}\right\} & \left\{\left[s_{-1/2<o_{3/2}>}, s_{1/2<o_{3/2}>}\right]\right\} & \left\{\left[s_{1/2<o_{-1/2}>}, s_{3/2<o_{-1/2}>}\right]\right\} \\ \left\{\left[s_{-1/2<o_{-3/2}>}, s_{1/2<o_{-3/2}>}\right]\right\} & \left\{s_{0<o_0>}\right\} & \left\{\left[s_{-1<o_{-2}>}, s_{0<o_{-2}>}\right]\right\} \\ \left\{\left[s_{-3/2<o_{1/2}>}, s_{-1/2<o_{1/2}>}\right]\right\} & \left\{\left[s_{0<o_2>}, s_{1<o_2>}\right]\right\} & \left\{s_{0<o_0>}\right\} \end{pmatrix}$$

Then, return $\tilde{H}_{\bar{S}_{\bar{O}}}$ to the decision maker and ask him to propose new preference information. Collecting all the preferences to establish the modified normalized DHHFLPR

$\left(\tilde{H}_{S_O}^{1N}\right)^{(1)} = \left(\left(h_{S_{O_{ij}}}^{1N}\right)_{3\times3}\right)^{(1)}$. Suppose that the decision makers give a modified normalized DHHFLPR $\left(\tilde{H}_{S_O}^{1N}\right)^{(1)}$ as:

$$\left(\tilde{H}_{S_O}^{1N}\right)^{(1)} = \begin{pmatrix} \{s_{0<o_0>}\} & \{s_{0<o_1>}\} & \{s_{1<o_{-1}>}\} \\ \{s_{0<o_{-1}>}\} & \{s_{0<o_0>}\} & \{s_{1<o_1>}\} \\ \{s_{-1<o_1>}\} & \{s_{-1<o_{-1}>}\} & \{s_{0<o_0>}\} \end{pmatrix}$$

Then we obtain $CI\left(\left(\tilde{H}_{S_O}^{1N}\right)^{(1)}\right) = d\left(\left(\tilde{H}_{S_O}^{1N}\right)^{(1)}, \left(\bar{H}_{S_O}^{1N}\right)^{(1)}\right) = 0.0313 \le \overline{CI}\left(\tilde{H}_{S_O}^{1}\right) = 0.1103$. Let $*\tilde{H}_{S_O}^{1''} = \left(\tilde{H}_{S_O}^{1N}\right)^{(1)}$, and output the modified normalized DHHFLPR $*\tilde{H}_{S_O}^{1''}$.

5. Group decision making with DHHFLPRs

In this section, we first describe the group decision making problem with DHHFLPRs. Then a decision maker weight-determining method is developed on the basis of information entropy theory. Finally, an algorithm is proposed to deal with the group decision making problem with DHHFLPRs.

5.1. Group decision making problem with DHHFLPR

For a group decision making problem with DHHFLPRs, let $A = \{A_1, A_2, \ldots, A_m\}$ be a set of alternatives, $E = \{e^1, e^2, \ldots, e^R\}$ be a set of decision makers invited to provide their linguistic preference information by making pairwise comparisons among alternatives, and $w = (w_1, w_2, \ldots, w_R)^T$ be the weight vector of the decision makers with $0 \le w_r \le 1$ and $\sum_{r=1}^{R} w_r = 1$. Each decision maker's linguistic preference information can be established by DHHFLPR and denoted as $\tilde{H}_{S_O}^{r} = \left(h_{S_{O_{ij}}}^{r}\right)_{m\times m}$ $(r = 1, 2, \ldots, R)$.

5.2. Group decision making model

When developing the group decision making method with DHHFLPRs, determining the decision makers' weights becomes an important step. Thus, a weight-determining method is developed to obtain the weights of decision makers at first, and then an algorithm is set up.

We mainly utilize the information entropy theory to determine the weights of the decision makers. The first step is to obtain each decision maker's ordering vector $U^r = (u_1^r, u_2^r, \ldots, u_m^r)^T$ $(r = 1, 2, \ldots, R)$

for all alternatives, which can be calculated by

$$u_i^r = \left(\sum_{j=1}^{m} F'\left(le\left(h_{S_{O_{ij}}}^r \right) \right) \right) \Big/ \sum_{i=1}^{m} \sum_{j=1}^{m} F'\left(le\left(h_{S_{O_{ij}}}^r \right) \right), \quad i = 1, 2, \ldots, m \tag{21}$$

And then the information entropy of each decision maker e^r $\left(r = 1, 2, \ldots, R\right)$ can be obtained by

$$IE\left(U^r\right) = -\frac{1}{\log_2 m} \bullet \sum_{i=1}^{m} u_i^r \log_2 u_i^r \tag{22}$$

Information entropy indicates the uncertainty degree and randomness of evaluation information. Therefore, the smaller the information entropy, the bigger the certainty degree of the evaluation information, which means that this decision maker plays an significant role and then we need to give him a bigger weight. So let w_r be weight of the r-th decision maker, then

$$w_r = \left(IE\left(U^r\right) \right)^{-1} \Big/ \sum_{r=1}^{R} \left(IE\left(U^r\right) \right)^{-1} \tag{23}$$

Furthermore, a double hierarchy hesitant fuzzy linguistic weighted averaging operator (DHFLWA) can be defined. Suppose that $h_{S_O}^i = \left\{ s_{\phi_l^i < o_{\varphi_l^i} >} \middle| s_{\phi_l^i < o_{\varphi_l^i} >} \in S_O; l = 1, 2, \ldots, \# h_{S_O}^i \right\} \left(i = 1, 2, \ldots, n\right)$ is a collection of DHHFLEs. A DHFLWA operator is a mapping $M^R \to M$, such that

$$\text{DHFLWA}\left(h_{S_O}^1, h_{S_O}^2, \ldots, h_{S_O}^n\right) = \sum_{i=1}^{n} \left(w_i \bullet le\left(h_{S_O}^i\right) \right) \tag{24}$$

where $w = \left(w_1, w_2, \ldots, w_n\right)^T$ is the weight vector of $h_{S_O}^i$ with $0 \le w_i \le 1$ and $\sum_{i=1}^{n} w_i = 1$ $\left(i = 1, 2, \ldots, n\right)$.

Theorem 5.1. Let $h_{S_O}^i = \left\{ s_{\phi_l^i < o_{\varphi_l^i} >} \middle| s_{\phi_l^i < o_{\varphi_l^i} >} \in S_O; l = 1, 2, \ldots, \# h_{S_O}^i \right\}$ $\left(i = 1, 2, \ldots, n\right)$ be a collection of DHHFLEs, and $le\left(h_{S_O}^i\right) = \frac{1}{\# h_{S_O}^i} \sum_{l=1}^{\# h_{S_O}^i} s_{\phi_l^i < o_{\varphi_l^i} >} = s_{\phi^i < o_{\varphi^i} >}$, $w = \left(w_1, w_2, \ldots, w_n\right)^T$ be the weight vector of $h_{S_O}^i$ $\left(i = 1, 2, \ldots, n\right)$ with $0 \le w_i \le 1$ and $\sum_{i=1}^{n} w_i = 1$. Then $\text{DHFLWA}\left(h_{S_O}^1, h_{S_O}^2, \ldots, h_{S_O}^n\right) = s_{\sum_{i=1}^{n} w_i \phi^i < o_{\sum_{i=1}^{n} w_i \varphi^i} >}$.

Remark 9. Based on the operational laws of DHHFLEs and considering that every DHHFLE only contains a double hierarchy linguistic term, we can sum all linguistic terms included in the first hierarchy and the second hierarchy respectively and obtain the aggregation result. In group decision making problem with DHHFLPRs, this aggregation method is very suitable in the decision making process. Meanwhile, this aggregation method can be used as the most basic tool for the following group decision making model with DHHFLPRs.

Additionally, Gou et al. [9] developed a method to calculate the excepted value of a DHHFLE.

Definition 5.1 [9]**.** Let $h_{S_O} = \left\{ s_{\phi_l < o_{\varphi_l} >} \middle| s_{\phi_l < o_{\varphi_l} >} \in \bar{S}_O; l = 1, 2, \ldots, L; \phi_l \in [-\tau, \tau]; \varphi_l \in [-\varsigma, \varsigma] \right\}$ be a DHHFLE. We call

$$E : \bar{S}_O \to [0,1], \; E\left(h_{S_O}\right) = \frac{1}{L} \sum_{l=1}^{L} F\left(s_{\phi_l < o_{\varphi_l} >}\right) \tag{25}$$

the expected value of h_{S_O}.

Then, a group decision making model with DHHFLPRs can be established as follows:

Algorithm 5.1. A group decision making model with DHHFLPRs

Step 1. Let $A = \{A_1, A_2, \ldots, A_m\}$ be a set of alternatives, $E = \{e^1, e^2, \ldots, e^R\}$ be a set of decision makers and their preference information can establish some DHHFLPRs $\tilde{H}_{S_O}^r = \left(h_{S_{O_{ij}}}^r\right)_{m \times m} (r = 1, 2, \ldots, R)$.

Step 2. Calculate the normalized DHHFLPRs $\tilde{H}_{S_O}^{rN} = \left(h_{S_{O_{ij}}}^{rN}\right)_{m \times m} (r = 1, 2, \ldots, R)$ and the consistent normalized DHHFLPR $\bar{H}_{S_O}^{rN} = \left(\bar{h}_{S_{O_{ij}}}^{rN}\right)_{m \times m} (r = 1, 2, \ldots, R)$, respectively.

Step 3. Utilize Algorithm 4.1 or Algorithm 4.2 to ensure that each normalized DHHFLPR is of acceptable consistency.

Step 4. Calculate the decision makers' weight vector $w = (w_1, w_2, \ldots, w_R)^T$ based on Eqs. (21)-(23).

Step 5. Aggregate all of the normalized DHHFLPRs into a synthetical normalized DHHFLPR using the DHFLWA operator, denoted as $\hat{H}_{S_O}^N = \left(\hat{h}_{S_{O_{ij}}}^N\right)_{m \times m}$.

Step 6. Calculate the synthetical value of each alternative by formula $SV(A_i) = \sum_{j=1}^{m} E\left(\hat{h}_{S_{O_{ij}}}^N\right)$.

Step 7. Rank all the alternatives based on the values of $SV(A_i)\,(i = 1, 2, \ldots, m)$.

Remark 10. At the end of the Algorithm 5.1, it is necessary to develop a rank-reversal experiment to check the effectiveness of this algorithm by adding some other alternatives based on the Ref. [20, 21]. In this experiment, if the ranking order of the original alternative is not changed, then this algorithm is effective. Otherwise, the algorithm should be improved.

6. Case study: Sichuan water resource management

In this section, the proposed method is validated by a case study of evaluating the water resource situations of some cities in Sichuan Province, and some comparative studies with others methods are made.

6.1. Problem description

Water resources require indispensable solutions to sustain human life and that of all living things. In China, the average volume of renewable water is estimated to be about 2.812 trillion cubic meters per year, ranked the fifth in the word. Meanwhile, Sichuan water resources are very abundant and prominent in China. As one of the upper reaches of the Yangtze River system, Sichuan water resources are important water systems in China. The protection of water quality of Sichuan water resources has become a crucial issue of economic and social stability and the rapid development of China. However, in recent years, with the development of the society's productivity and industrialization, urbanization in China has accelerated. The problems of Sichuan water resource development, protection and management are facing an increasingly severe test, and the grim reality of global climate change has made these problems more urgent. At present, the main problems that are being faced include sustainable utilization of regional water resources, rational development of water resources, water condition detection, rational exploration and utilization of water resources, integrated management of water resources, the harmonious development between economy and environment, etc. To solve the problems of water resource development, protection and management, a lot of experts and scholars carried out research and some achievements have been made [22, 26, 27]. In some ways, these studies have solved some problems, but the reality has been unsatisfactory. For example, in 2016, the amount of water was once again insufficient in the irrigation period of Dujiangyan, Sichuan province; the water quality in Liangshan state still cannot reach the national average. Therefore, these realities have prompted the authorities to think about other ways to solve the problems of water resource development, protection and management in Sichuan province.

Because of this, the water resources of each city in the Sichuan province will be assessed annually. Additionally, amounts of studies have utilized the definite data to make analyses and calculations when dealing with water resource development, protection and management problems. However, in reality, it is very difficult to measure the key indicators of water resource management such as maintaining the quantity of water resources, the water quality and so on, and complex uncertainties often arise. Therefore, we can utilize double hierarchy hesitant fuzzy linguistic information to express some immeasurable phenomenons. Based on these water resource comprehensive assessment indices (criteria) and the double hierarchy linguistic term set $S_O = \left\{ s_{t<o_k>} \middle| t = -4,\dots,-1,0,1,\dots,4;\ k = -4,\dots,-1,0,1,\dots,4 \right\}$ with

$S = \{s_{-4} = extremely\ bad,\ s_{-3} = very\ bad,\ s_{-2} = bad,\ s_{-1} = slightly\ bad,\ s_0 = medium,\ s_1 = slightly\ good,\ s_2 = good,\ s_3 = very\ good,\ s_4 = extremely\ good\}$

$O = \{o_{-4} = far\ \ from,\ o_{-3} = scarcely,\ o_{-2} = only\ a\ little,\ o_{-1} = a\ little,\ o_0 = just\ right,\ o_1 = much,\ o_2 = very\ much, o_3 = extremely\ much, o_4 = entirely\}$

, we invite five experts $E = \{e^1, e^2, \dots, e^5\}$ to evaluate the water resource situations of four typical and important cities in the Sichuan province, including Chengdu (A_1), Nanchong (A_2), Panzhihua (A_3) and Dazhou (A_4). Collecting the linguistic preference information of each expert, five DHHFLPRs can be established and shown in Tables 6.1-6.5.

Table 6.1. The evaluation preference information of the expert e^1

	A_1	A_2	A_3	A_4
A_1	$\{s_{0<o_0>}\}$	$\{s_{-1<o_1>}, s_{0<o_2>}\}$	$\{s_{1<o_{-2}>}, s_{2<o_3>}\}$	$\{s_{-1<o_{-2}>}\}$
A_2	$\{s_{1<o_{-1}>}, s_{0<o_{-2}>}\}$	$\{s_{0<o_0>}\}$	$\{s_{0<o_{-1}>}, s_{1<o_1>}\}$	$\{s_{-2<o_1>}, s_{-1<o_2>}\}$
A_3	$\{s_{-1<o_2>}, s_{-2<o_{-3}>}\}$	$\{s_{0<o_1>}, s_{-1<o_{-1}>}\}$	$\{s_{0<o_0>}\}$	$\{s_{2<o_{-3}>}, s_{3<o_3>}\}$
A_4	$\{s_{1<o_2>}\}$	$\{s_{2<o_{-1}>}, s_{1<o_{-2}>}\}$	$\{s_{-2<o_3>}, s_{-3<o_{-3}>}\}$	$\{s_{0<o_0>}\}$

Table 6.2. The evaluation preference information of the expert e^2

	A_1	A_2	A_3	A_4
A_1	$\{s_{0<o_0>}\}$	$\{s_{1<o_1>}\}$	$\{s_{2<o_2>}, s_{3<o_{-1}>}\}$	$\{s_{0<o_3>}\}$
A_2	$\{s_{-1<o_{-1}>}\}$	$\{s_{0<o_0>}\}$	$\{s_{1<o_{-1}>}, s_{2<o_1>}\}$	$\{s_{-1<o_3>}, s_{0<o_2>}\}$
A_3	$\{s_{-2<o_{-2}>}, s_{-3<o_1>}\}$	$\{s_{-1<o_1>}, s_{-2<o_{-1}>}\}$	$\{s_{0<o_0>}\}$	$\{s_{-2<o_1>}, s_{-1<o_3>}\}$
A_4	$\{s_{0<o_{-3}>}\}$	$\{s_{1<o_{-3}>}, s_{0<o_{-2}>}\}$	$\{s_{2<o_{-1}>}, s_{1<o_{-3}>}\}$	$\{s_{0<o_0>}\}$

Table 6.3. The evaluation preference information of the expert e^3

	A_1	A_2	A_3	A_4
A_1	$\{s_{0<o_0>}\}$	$\{s_{2<o_2>}\}$	$\{s_{-1<o_2>}, s_{-1<o_3>}\}$	$\{s_{-2<o_{-2}>}, s_{-1<o_{-2}>}\}$
A_2	$\{s_{-2<o_{-2}>}\}$	$\{s_{0<o_0>}\}$	$\{s_{0<o_{-1}>}, s_{0<o_1>}\}$	$\{s_{-1<o_1>}, s_{-1<o_2>}\}$
A_3	$\{s_{1<o_{-2}>}, s_{1<o_{-3}>}\}$	$\{s_{0<o_1>}, s_{0<o_{-1}>}\}$	$\{s_{0<o_0>}\}$	$\{s_{1<o_{-3}>}, s_{2<o_3>}\}$
A_4	$\{s_{2<o_2>}, s_{1<o_2>}\}$	$\{s_{1<o_{-1}>}, s_{1<o_{-2}>}\}$	$\{s_{-1<o_3>}, s_{-2<o_{-3}>}\}$	$\{s_{0<o_0>}\}$

Table 6.4. The evaluation preference information of the expert e^4

	A_1	A_2	A_3	A_4
A_1	$\{s_{0<o_0>}\}$	$\{s_{-1<o_{-1}>}\}$	$\{s_{1<o_{-2}>}, s_{2<o_1>}\}$	$\{s_{2<o_{-3}>}\}$
A_2	$\{s_{1<o_1>}\}$	$\{s_{0<o_0>}\}$	$\{s_{2<o_1>}, s_{3<o_{-1}>}\}$	$\{s_{3<o_3>}\}$
A_3	$\{s_{-1<o_2>}, s_{-2<o_{-1}>}\}$	$\{s_{-2<o_{-1}>}, s_{-3<o_1>}\}$	$\{s_{0<o_0>}\}$	$\{s_{-1<o_3>}\}$
A_4	$\{s_{-2<o_3>}\}$	$\{s_{-3<o_{-3}>}\}$	$\{s_{1<o_{-3}>}\}$	$\{s_{0<o_0>}\}$

Table 6.5. The evaluation preference information of the expert e^5

	A_1	A_2	A_3	A_4
A_1	$\{s_{0<o_0>}\}$	$\{s_{2<o_1>}, s_{3<o_1>}\}$	$\{s_{1<o_{-2}>}, s_{2<o_1>}\}$	$\{s_{-1<o_2>}\}$

A_2	$\{s_{-2<o_1>}, s_{-3<o_1>}\}$	$\{s_{0<o_0>}\}$	$\{s_{1<o_{-2}>}\}$	$\{s_{1<o_1>}, s_{2<o_{-1}>}\}$
A_3	$\{s_{-1<o_2>}, s_{-2<o_{-1}>}\}$	$\{s_{-1<o_2>}\}$	$\{s_{0<o_0>}\}$	$\{s_{2<o_3>}, s_{3<o_1>}\}$
A_4	$\{s_{1<o_{-2}>}\}$	$\{s_{-1<o_{-1}>}, s_{-2<o_1>}\}$	$\{s_{-2<o_{-3}>}, s_{-3<o_{-1}>}\}$	$\{s_{0<o_0>}\}$

6.2. The application of the group decision making model with DHHFLPRs

We can utilize Algorithm 5.1 to solve this group decision making problem. Considering that the first step has been discussed above, so we start the decision making process from Step 2.

Step 2. Calculate the normalized DHHFLPR $\tilde{H}_{S_O}^{rN} = \left(h_{S_{O_{ij}}}^{rN}\right)_{4\times4}$ $(r = 1, 2, \ldots, 5)$ and the consistent normalized DHHFLPR $\bar{H}_{S_O}^{rN} = \left(\bar{h}_{S_{O_{ij}}}^{rN}\right)_{4\times4}$ $(r = 1, 2, \ldots, 5)$, respectively.

Step 3. Based on Eq. (9), the consistency indexes of all decision makers' DHHFLPRs $CI\left(\tilde{H}_{S_O}^{r}\right)(r = 1, 2, \ldots, 5)$ can be obtained and shown in Table 6.6.

Table 6.6. The consistency index of each decision maker's DHHFLPR $CI\left(\tilde{H}_{S_O}^{r}\right)(r = 1, 2, \ldots, 5)$

	e^1	e^2	e^3	e^4	e^5
$CI\left(\tilde{H}_{S_O}^{r}\right)$	0.1809	0.0292	0.1564	0.0500	0.1872

Clearly, $CI\left(\tilde{H}_{S_O}^{1}\right), CI\left(\tilde{H}_{S_O}^{3}\right), CI\left(\tilde{H}_{S_O}^{5}\right) > 0.1515$, which means the DHHFLPRs $\tilde{H}_{S_O}^{1}$, $\tilde{H}_{S_O}^{3}$, $\tilde{H}_{S_O}^{5}$ are of unacceptable consistency and the other DHHFLPRs are of acceptable consistency.

A. Utilizing the consistency repairing algorithm based on the automatic optimization method

Utilizing Algorithm 4.1, we can improve these three DHHFLPRs. The improved normalized DHHFLPRs can be obtained with the adjusted parameter $\theta = 0.2$ and the corresponding consistency indices are $CI\left(*\tilde{H}_{S_O}^{1}\right) = 0.1447$, $CI\left(*\tilde{H}_{S_O}^{3}\right) = 0.1251$, and $CI\left(*\tilde{H}_{S_O}^{5}\right) = 0.1498$.

B. Utilizing the consistency repairing algorithm based on the feedback improving method

Utilizing Algorithm 4.2, we can establish three interval-valued DHHFLPRs $\tilde{H}_{\bar{S}_{\bar{O}}}^{r} = \left(h_{\bar{S}_{\bar{O}_{ij}}}^{r}\right)_{4\times4}$ $(r = 1, 3, 5)$, and return them to the experts and ask them to provide their new preferences. Collecting the feedback information and obtaining three improved normalized DHHFLPRs shown in Tables 6.7-6.9, and the corresponding consistency indices are $CI\left(*\tilde{H}_{S_O}^{1'}\right) = 0.0959$, $CI\left(*\tilde{H}_{S_O}^{5'}\right) = 0.0749$, and $CI\left(*\tilde{H}_{S_O}^{1'}\right) = 0.0832$.

Table 6.7. The improved DHHFLPR $\tilde{H}_{S_O}^{1'N}$

	A_1	A_2	A_3	A_4

A_1	$\{s_{0<o_0>}\}$	$\{s_{0<o_0>}\}$	$\{s_{1<o_{-1}>}\}$	$\{s_{0<o_{-2}>}\}$
A_2	$\{s_{0<o_0>}\}$	$\{s_{0<o_0>}\}$	$\{s_{0<o_1>}\}$	$\{s_{-1<o_1>}\}$
A_3	$\{s_{-1<o_1>}\}$	$\{s_{0<o_{-1}>}\}$	$\{s_{0<o_0>}\}$	$\{s_{1<o_2>}\}$
A_4	$\{s_{0<o_2>}\}$	$\{s_{1<o_{-1}>}\}$	$\{s_{-1<o_{-2}>}\}$	$\{s_{0<o_0>}\}$

Table 6.8. The improved DHHFLPR $\tilde{H}_{S_O}^{3'N}$

	A_1	A_2	A_3	A_4
A_1	$\{s_{0<o_0>}\}$	$\{s_{1<o_2>}\}$	$\{s_{-1<o_3>}\}$	$\{s_{-1<o_{-1}>}\}$
A_2	$\{s_{-1<o_{-2}>}\}$	$\{s_{0<o_0>}\}$	$\{s_{-1<o_1>}\}$	$\{s_{-1<o_1>}\}$
A_3	$\{s_{1<o_{-3}>}\}$	$\{s_{1<o_{-1}>}\}$	$\{s_{0<o_0>}\}$	$\{s_{1<o_{-1}>}\}$
A_4	$\{s_{1<o_1>}\}$	$\{s_{1<o_{-1}>}\}$	$\{s_{-1<o_1>}\}$	$\{s_{0<o_0>}\}$

Table 6.9. The improved DHHFLPR $\tilde{H}_{S_O}^{5'N}$

	A_1	A_2	A_3	A_4
A_1	$\{s_{0<o_0>}\}$	$\{s_{1<o_2>}\}$	$\{s_{1<o_{-1}>}\}$	$\{s_{1<o_{-2}>}\}$
A_2	$\{s_{-1<o_{-2}>}\}$	$\{s_{0<o_0>}\}$	$\{s_{0<o_1>}\}$	$\{s_{1<o_1>}\}$
A_3	$\{s_{-1<o_1>}\}$	$\{s_{0<o_{-1}>}\}$	$\{s_{0<o_0>}\}$	$\{s_{2<o_{-1}>}\}$
A_4	$\{s_{-1<o_2>}\}$	$\{s_{-1<o_{-1}>}\}$	$\{s_{-2<o_1>}\}$	$\{s_{0<o_0>}\}$

Step 4. Based on Eqs. (21)-(23), the weight vector of the decision makers can be calculated as:

(1) Under the automatic optimization method: $w=(0.1973,0.2009,0.1983,0.2041,0.1994)^T$.

(2) Under the feedback improving method: $w=(0.1974,0.2011,0.1984,0.2042,0.1989)^T$.

Step 5. Aggregate all the normalized DHHFLPRs into the synthetical normalized DHHFLPRs $\hat{H}_{S_O}^{N}=\left(\hat{h}_{S_{O_{ij}}}^{N}\right)_{4\times4}$ and $\hat{H}_{S_O}^{N'}=\left(\hat{h}_{S_{O_{ij}}}^{N'}\right)_{4\times4}$ by the DHFLWA operator. The aggregated results based on these two methods can be shown in Table 6.10 and Table 6.11, respectively.

Table 6.10. The synthetical normalized DHHFLPR $\hat{H}_{S_O}^{N}=\left(\hat{h}_{S_{O_{ij}}}^{N}\right)_{4\times4}$ based on the automatic optimization method

	A_1	A_2	A_3	A_4
A_1	$\{s_{0<o_0>}\}$	$\{s_{0.69<o_{0.8}>}\}$	$\{s_{1.12<o_{0.53}>}\}$	$\{s_{-0.1<o_{-0.24}>}\}$
A_2	$\{s_{-0.69<o_{-0.8}>}\}$	$\{s_{0<o_0>}\}$	$\{s_{0.99<o_{-0.37}>}\}$	$\{s_{0.34<o_{1.58}>}\}$
A_3	$\{s_{-1.12<o_{-0.53}>}\}$	$\{s_{-0.99<o_{0.37}>}\}$	$\{s_{0<o_0>}\}$	$\{s_{0.58<o_{1.38}>}\}$
A_4	$\{s_{0.1<o_{0.24}>}\}$	$\{s_{-0.34<o_{-1.58}>}\}$	$\{s_{-0.58<o_{-1.38}>}\}$	$\{s_{0<o_0>}\}$

Table 6.11. The synthetical normalized DHHFLPR $\hat{H}_{S_O}^{N'}=\left(\hat{h}_{S_{O_{ij}}}^{N'}\right)_{4\times4}$ based on the feedback-based improving method

	A_1	A_2	A_3	A_4
A_1	$\{s_{0<o_0>}\}$	$\{s_{0.39<o_{0.79}>}\}$	$\{s_{1.01<o_{0.3}>}\}$	$\{s_{0.41<o_{-1}>}\}$
A_2	$\{s_{-0.39<o_{-0.79}>}\}$	$\{s_{0<o_0>}\}$	$\{s_{0.61<o_{0.6}>}\}$	$\{s_{0.32<o_{1.71}>}\}$
A_3	$\{s_{-1.01<o_{-0.3}>}\}$	$\{s_{-0.61<o_{-0.6}>}\}$	$\{s_{0<o_0>}\}$	$\{s_{0.29<o_{1.01}>}\}$
A_4	$\{s_{-0.41<o_1>}\}$	$\{s_{-0.32<o_{-1.71}>}\}$	$\{s_{-0.29<o_{-1.01}>}\}$	$\{s_{0<o_0>}\}$

Step 6. Calculate the synthetical value of each alternative

(1) Using the automatic improving method: $SV = \{2.2478, 2.0922, 1.8467, 1.8132\}$;

(2) Using the feedback-based improving method: $SV = \{2.2291, 2.1140, 1.8373, 1.8195\}$.

Step 7. Rank all the alternatives based on $SV(A_i)$ $(i = 1,2,3,4)$: $A_1 \succ A_2 \succ A_3 \succ A_4$. Therefore, the rank of the water resource situations of these four cities is Chengdu $\succ$ Nanchong $\succ$ Panzhihua $\succ$ Dazhou .

We have also set up some discussions when we finish the case study with the proposed method in Section 6.2:

Firstly, the proposed normalization method mainly has two advantages: 1) The calculation becomes simple by transforming all DHHFLEs into the corresponding double hierarchy linguistic terms. 2) The obtained double hierarchy linguistic terms consist of all original linguistic information, so the proposed normalization method can avoid the original information loss.

Secondly, considering that we can utilize the adjusted parameter $\theta\,(0 \le \theta \le 1)$ to adjust the number of iterations and to improve the accuracy of modification to the original DHHFLPR, the reasonable value of θ can be chosen based on Table 4.1. Therefore, if we choose the reasonable value of θ, based on MATLAB software, we can quickly obtain the modified normalized DHHFLPR of acceptable consistency on the basis of the automatic improving method. Additionally, the feedback-based improving method can feed the suggestions back to the decision makers and help them improve their preferences, so the consistency repairing algorithm based on the feedback method to satisfies the decision maker's willingness.

Thirdly, after the consistency checking and repairing processes, we can calculate the weight vector of all decision makers and aggregate all DHHFLPRs, then we obtain the final decision result. Obviously, the consistency indexes of all repaired DHHFLPRs when using different methods are different, and the repaired places in each DHHFLPR are also different. But both of two methods obtain the same ranking of alternatives: $A_1 \succ A_2 \succ A_3 \succ A_4$. Therefore, there is no significant impact on the outcome based on these two different consistency checking and repairing methods.

6.3. Comparative analysis

Considering that there is not any consistency research about double hierarchy linguistic preference information, so it is more suitable to make comparative analyses between the double hierarchy linguistic information and single hierarchy linguistic information. Therefore, we need to delete the second hierarchy linguistic and transform all DHHFLPRs into HFLPRs. Then, we can utilize some existing methods [25, 30, 38, 39] to deal with this group decision making problem with HFLPRs.

Firstly, based on Zhu and Xu's method [39], the consistency index of each HFLPR $CI\left(\tilde{H}_S^r\right)$ $\left(r=1,2,\ldots,5\right)$ is obtained:

Table 6.12. The consistency index of each HFLPR $CI\left(\tilde{H}_S^r\right)$ under each criterion

	e^1	e^2	e^3	e^4	e^5
$CI\left(\tilde{H}_S^r\right)$	0.1614	0.0260	0.1146	0.0469	0.1770

Clearly, $CI\left(\tilde{H}_S^1\right), CI\left(\tilde{H}_S^5\right) > 0.1515$. Based on the automatic optimization method with the adjusted parameter $\theta = 0.4$ and the feedback-based improving method, we obtain $CI\left(*\tilde{H}_S^1\right) = 0.1066$, $CI\left(*\tilde{H}_S^5\right) = 0.1166$, $CI\left(*\tilde{H}_S^{1'}\right) = 0.0920$ and $CI\left(*\tilde{H}_S^{5'}\right) = 0.0920$. Thus, all of them are of acceptable consistency. Then the final ranking of the cities is $A_1 \succ A_2 \succ A_4 \succ A_3$.

Secondly, based on Wu and Xu's method [30], the decision making process is as follows:

We establish the expected 2-tuple linguistic preference relation for each HFLPRs obtained above, then the consistency checking and repairing process is shown in Table 6.13:

Table 6.13. The consistency checking and repairing process for $\tilde{H}_S^1$ and $\tilde{H}_S^5$

Iteration	$CI\left(\tilde{H}_S^1\right)$	$\left(i_1, j_1\right)$	Modified preference	$CI\left(\tilde{H}_S^r\right) \le \overline{CI}$	Iteration	$CI\left(\tilde{H}_S^5\right)$	$\left(i_1, j_1\right)$	Modified preference	$CI\left(\tilde{H}_S^r\right) \le \overline{CI}$
0	0.1614	(3,4)	$h''^1_{S_{34}} = \{s_0, s_1\}$	No	0	0.1770	(1,4)	$h''^5_{S_{34}} = \{s_2\}$	No
1	0.1198	—	—	Yes	1	0.0938	—	—	Yes

Using the method of Wu and Xu [30], the final ranking of all alternatives is $A_1 \succ A_2 \succ A_4 \succ A_3$.

Based on the consistency checking and repairing methods of Zhu and Xu [39] and Wu and Xu [26], we can obtain the same ranking result $A_1 \succ A_2 \succ A_4 \succ A_3$. Zhu and Xu [39] utilizes two consistency repairing methods to deal with HFLPRs including the automatic optimization method and the feedback-based improving method. The automatic optimization method is mainly based on the adjusted parameter and the feedback-based improving method is provided by establishing the interval-valued HFLPR to guide the consistency repairing direction. In contrast, we can use Wu and Xu's [30] method to obtain the precise location where the preference information needs to be repaired. Clearly, the feedback-based improving methods of Zhu and Xu [39] and the proposed method need to establish the corresponding interval-valued HFLPR and interval-valued DHHFLPR, and sometimes more than one linguistic term needs to be changed, but each linguistic term change is very small and smooth.

Additionally, we can set two small experiments to verify whether the proposed method has rank reversal. Firstly, we add one alternative (denoted as A_6) and give some evaluation information to the Tables 6.1-6.5 randomly and obtain five 5×5 matrices, and then check whether the new alternative changes the rank of original 5 ones. By calculation, we obtain that $A_1 \succ A_2 \succ A_3 \succ A_5 \succ A_4$. Secondly, we add two alternatives (denoted as A_6 and A_7) and give some evaluation information to the Tables 6.1-6.5 randomly and obtain five 6×6 matrices, and then check whether the new alternative changes the rank of original 5 ones. By calculation, we obtain that $A_1 \succ A_2 \succ A_3 \succ A_4 \succ A_6 \succ A_5$. In these two experiments, the ranking order of the original alternatives is not changed. Therefore, the proposed method is a rank reversal free method.

7. Discussion

Based on the decision making processes above and the basic characteristics of different consistency checking and repairing models [25, 30, 39], the details of these models are summarized as follows:

(1) Firstly, we have discussed two different consistency repairing algorithms for the DHHFLPR of unacceptable consistency. The automatic optimization method mainly improves the DHHFLPR of unacceptable consistency by utilizing the adjusted parameter $\theta\left(0\le\theta\le1\right)$. We can obtain different results if we take different values of θ. However, we can also take a suitable adjusted parameter based on Table 4.1 which shows the average values of iterations in Algorithm 4.1. Additionally, the feedback improving method depends on the feedback mechanism, we do not change any information of the DHHFLPRs of unacceptable consistency but feed the information back to the experts. They can decide whether to change the evaluation information or not, and then we can make a decision using the feedback information from the experts.

These two methods have some advantages: For the automatic optimization method, we can obtain the decision making results very quickly because the improvement of the DHHFLPR of unacceptable consistency is automatic according to the adjusted parameter θ. Furthermore, MATLAB is utilized to do programming and it carries out the operation faster. For the feedback-based improving method, it is more in line with intelligent decision making considering that the experts' opinions have been given full consideration.

(2) Compared with Ref. [30, 39], the advantages of the DHHFLPR can be summarized as follows: Firstly, considering that the DHHFLPR is established by DHHFLEs, it contains more detailed information. Because of this, the evaluation information included in the DHHFLPR can be enlarged or minified compared with the HFLPR. Thus, we can find that the consistency index of each

DHHFLPR is different from the corresponding HFLPR. Additionally, to obtain the normalized HFLPRs, Zhu and Xu [39] added some linguistic terms to make sure that each element included in HFLPR have same length, and therefore their method consists of two shortcomings: 1) the original information is changed; 2) the calculations are more complex. For the method of Wu and Xu [30], they proposed the expected 2-tuple linguistic preference relation and possibility distribution method, which can ensure the integrity of the original information. In fact, the expected 2-tuple linguistic preference relation is equivalent to the virtual linguistic preference relation [32], and the form of the latter one is simpler.

(3) Compared with Ref. [25]. Wang and Xu [25] discussed the additive consistency measure and the weak consistency measure of extended hesitant fuzzy linguistic preference relations based on the symmetric hesitant preference relation graph. The process is intuitive but the calculation is complex and the calculation of the length of the paths will lose the original information. Therefore, if we extend this method into DHHFLPR, the calculation will be more complex and we cannot avoid the information loss. Obviously, the proposed method is more suitable when we deal with double hierarchy hesitant fuzzy linguistic information.

(4) Considering that the HFLTS lacks the second hierarchy linguistic terms of DHHFLTS, and the normalization method of Zhu and Xu [39] has some shortcomings. Therefore, compared with Table 6.6 and Table 6.13 and from the final ranking orders, we can find that the incomplete linguistic information and the normalization method of Zhu and Xu [39] have changed the consistency degrees to different extents, and have made some deviations in decision making results.

8. Concluding remark

In this paper, we have defined the concept of DHHFLPR and developed some additive consistency measures. Then, utilizing the linguistic expected-value of DHHFLE, we have proposed a new normalization method to transform the DHHFLPR into the normalized DHHFLPR equivalently. Additionally, for the purpose of judging whether a DHHFLPR is of acceptable consistency or not, we have defined a consistency index of the DHHFLPR and develop a novel method to improve the existing method for calculating the consistency thresholds. We have developed two convergent consistency repairing algorithms based on the automatic improving method and the feedback improving method respectively to improve the consistency index of a given DHHFLPR of unacceptable consistency. Finally, we have proposed a weight-determining method and developed an algorithm to deal with the group decision making problem with double hierarchy hesitant fuzzy linguistic preference information. We have applied our method to deal with a practical group decision making problem involving the evaluation of the water resource situations of some important cities in the Sichuan province and made some comparative analyses with the existing method.

General, the additive consistency measures and convergent consistency repairing algorithms discussed in this paper have the following advantages:

(1) The proposed normalization method simplifies the calculations and does not lose any information.

(2) We analyze the shortcomings of the calculation method of the existing consistency thresholds, and give some new consistency thresholds as the novel references for consistency improving processes.

(3) We present two convergent consistency repairing algorithms based on the automatic improving method and the feedback improving method respectively from different angles.

(4) A case study is set up to apply the proposed method to deal with a practical group decision making problem which is to evaluate the water resources situation of some important cities in the Sichuan province.

In the future, some research directions concerning the DHHFLPR can be developed including the consensus model, large-scale group decision making and incomplete DHHFLPR, among others. Additionally, motivated by the Characteristic Objects Method [20, 21], we will be applied to a double hierarchy hesitant fuzzy environment and a rank reversal free method will be developed.

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References

[1] S. Alonso, E. Herrera-Viedma, F. Chiclana and F. Herrera. A web based consensus support system for group decision making problems and incomplete preferences. Information Sciences 180 (2010) 4477-4495.

[2] H. Bustince, E. Barrenechea, M. Pagola, J. Fernandez, Z.S. Xu, B. Bedregal, J. Montero, H. Hagras, F. Herrera and B.D. Baets. A historical account of types of fuzzy sets and their relationships. IEEE Transactions on Fuzzy Systems 24(1) (2016) 179-194.

[3] Y.C. Dong, Y.F. Xu and H.Y. Li. On consistency measures of linguistic preference relations.

European Journal of Operational Research 189(2) (2008) 430-444.

[4] D. Dubois and H. Prade. Fuzzy Sets and Systems: Theory and Applications. New York: Kluwer, 1980.

[5] M. Durand, I. Truck. A new proposal to deal with hesitant linguistic expressions on preference assessments. Information Fusion 41 (2018) 176-181.

[6] C. Fu and S. L. Yang. The group consensus based evidential reasoning approach for multiple attributive group decision analysis. European Journal of Operational Research 206(3) (2010) 601-608.

[7] X.J. Gou, Z.S. Xu and H.C. Liao. Hesitant fuzzy linguistic entropy and cross-entropy measures and alternative queuing method for multiple criteria decision making. Information Sciences 388-389 (2017) 225-246.

[8] X.J. Gou, Z.S. Xu and H.C. Liao. Group decision making with compatibility measures of hesitant fuzzy linguistic preference relations. Soft Computing (2017) DOI: 10.1007/s00500-017-2871-5.

[9] X.J. Gou, Z.S. Xu, H.C. Liao and F. Herrera. Double hierarchy hesitant fuzzy linguistic MULTIMOORA method for evaluating the implementation status of haze controlling measures. Information Fusion 38 (2017) 22-34.

[10] X.J. Gou, Z.S. Xu, H.C. Liao and F. Herrera. Multiple criteria decision making based on distance and similarity measures under double hierarchy hesitant fuzzy linguistic environment. Computers & Industrial Engineering 126 (2018) 516-530.

[11] F. Herrera and L. Martínez. A 2-tuple fuzzy linguistic representation model for computing with words. IEEE Transactions on Fuzzy Systems 8(6) (2000) 746-752.

[12] E. Herrera-Viedma, S. Alonso, F. Chiclana and F. Herrera. A consensus model for group decision making with incomplete fuzzy preference relations. IEEE Transactions on Fuzzy Systems 15(5) (2007) 863-877.

[13] H.C. Liao, Z.S. Xu, E. Herrera-Viedma and F. Herrera. Hesitant Fuzzy Linguistic Term Set and Its Application in Decision Making: A State-of-the-Art Survey. International Journal of Fuzzy Systems (2018) DOI: 10.1007/s40815-017-0432-9.

[14] L. Martínez and F. Herrera. An overview on the 2-tuple linguistic model for computing with words in decision making: extensions, applications and challenges Information Sciences 207 (2012) 1-18.

[15] J.M. Mendel. An architecture for making judgement using computing with words. International Journal of Applied Mathematics and Computer Science 12(3) (2002) 325-335.

[16] R. Montes, A.M. Sánchez, P. Villar and F. Herrera. A web tool to support decision making in the housing market using hesitant fuzzy linguistic term sets. Applied Soft Computing 35 (2015) 949-

957.

[17] Q. Pang, H. Wang, Z.S. Xu, Probabilistic linguistic term sets in multi-attribute group decision making. Information Sciences 369 (2016) 128-143.

[18] R.M. Rodríguez, L. Martínez and F. Herrera. Hesitant fuzzy linguistic term sets for decision making. IEEE Transactions on Fuzzy Systems 20 (2012) 109-119.

[19] T.L. Saaty. The Analytic Hierarchy Process, McGraw-Hill, New York, 1980.

[20] W. Sałabun. The Characteristic Objects Method: A new distance-based approach to multicriteria decision-making problems. Journal of Multi-Criteria Decision Analysis 22(1-2) (2015) 37-50.

[21] W. Sałabun. The Characteristic Objects Method: a new approach to identify a multi-criteria group decision-making model. International Journal of Computer Technology & Applications 5(5) (2014) 1597-1602.

[22] Y.Z. Shang, S.B. Lu, X.F. Li, P.F. Hei, X.H. Lei, J.G. Gong, J.H. Liu, J.Q. Zhai and H. Wang. Balancing development of major coal bases with available water resources in China through 2020. Applied Energy 194 (2017) 735-750.

[23] T. Tanino. Fuzzy preference orderings in group decision making. Fuzzy Sets and Systems 12(2) (1984) 117-131.

[24] R. Ureña, F. Chiclana, J.A. Morente-Molinera and E. Herrera-Viedma. Managing incomplete preference relations in decision making: are view and future trends. Information Sciences 302 (2015) 14-32.

[25] H. Wang and Z.S. Xu. Some consistency measures of extended hesitant fuzzy linguistic preference relations. Information Sciences 297 (2015) 316-331.

[26] B.L. Wang, Y.P. Cai, X.A. Yin, Q. Tan and Y. Hao. An integrated approach of system dynamics, orthogonal experimental design and inexact optimization for supporting water resources management under uncertainty. Water Resources Management 31(5) (2017) 1665-1694.

[27] I. Winz, G. Brierley and S. Trowsdale. The use of system dynamics simulation in water resources management. Water Resources Management 23(7) (2009) 1301-1323.

[28] D.R. Wu. A reconstruction decoder for computing with words. Information Sciences 255 (2014) 1-15.

[29] J. Wu, F. Chiclana and E. Herrera-Viedma. Trust based consensus model for social network in an incomplete linguistic information context. Applied Soft Computing 35 (2015) 827-839.

[30] Z.B. Wu and J.P. Xu. Managing consistency and consensus in group decision making with hesitant fuzzy linguistic preference relations. Omega 65(3) (2016) 28-40.

[31] Z. S. Xu. A method based on linguistic aggregation operators for group decision making with

linguistic preference relations. Information Sciences 166(1-4) (2004) 19-30.

[32] Z.S. Xu and H. Wang. On the syntax and semantics of virtual linguistic terms for information fusion in decision making. Information Fusion 34 (2017) 43-48.

[33] L.A. Zadeh. Computing with Words: What is computing with words (CWW)? Springer, Berlin Heidelberg (2012) 1-40.

[34] L.A. Zadeh. The concept of a linguistic variable and its applications to approximate reasoning-Part I. Information Sciences 8 (1975) 199-249.

[35] L.A. Zadeh. The concept of a linguistic variable and its applications to approximate reasoning-Part II. Information Sciences 8 (1975) 301-357.

[36] L.A. Zadeh, The concept of a linguistic variable and its applications to approximate reasoning-Part III. Information Sciences 9 (1975) 43-80.

[37] G.Q. Zhang, Y.C. Dong and Y.F. Xu. Consistency and consensus measures for linguistic preference relations based on distribution assessments. Information Fusion 17 (2014) 46-55.

[38] Z.M. Zhang and C. Wu. On the use of multiplicative consistency in hesitant fuzzy linguistic preference relations. Knowledge-Based Systems 72 (2014) 13-27.

[39] B. Zhu and Z.S. Xu. Consistency measures for hesitant fuzzy linguistic preference relations. IEEE Transactions on Fuzzy System 22(1) (2014) 35-45.

4 Consensus reaching process for large-scale group decision making with double hierarchy hesitant fuzzy linguistic preference relations

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Consensus Reaching Process for Large-scale Group Decision Making with Double Hierarchy Hesitant Fuzzy Linguistic Preference Relations

Xunjie Gou[a,b], Zeshui Xu[a,c*], Francisco Herrera[b,d]

[a] *Business School, Sichuan University, Chengdu 610064, China*
[b] *Department of Computer Science and Artificial Intelligence, University of Granada, Granada 18071, Spain*
[c] *School of Computer and Software, Nanjing University of Information Science and Technology, Nanjing, Jiangsu 210044, China*
[d] *Faculty of Computing and Information Technology, King Abdulaziz University, Jeddah, Saudi Arabia*

Abstract

Large-scale group decision making (LSGDM) or complex group decision making (GDM) problems are very commonly encountered in actual life, especially in the era of data. At present, double hierarchy hesitant fuzzy linguistic term set is a reasonable linguistic expression when describing some complex linguistic preference information. In this paper, we develop a consensus reaching process for LSGDM with double hierarchy hesitant fuzzy linguistic preference relations. To ensure the implementation of consensus reaching process, we also propose the similarity degree-based clustering method, the double hierarchy information entropy-based weights-determining method and the consensus measures. Finally, we apply our model to deal with a practical problem that is to evaluate Sichuan water resource management and make some comparisons with the existing approaches.

***Keywords*:** Double hierarchy hesitant fuzzy linguistic preference relations; Large-scale group decision making; Consensus reaching process; Clustering; Weights-determining method; Water resource management

1. Introduction

Group decision making (GDM) is considered as a decision situation in which a group of decision makers or experts are invited to provide their preference information for achieving a common solution to a problem consisting of more than two objects or alternatives. In recent years, GDM has been

* Corresponding Author. Emails: X.J. Gou (gouxunjie@qq.com); Z.S. Xu (xuzeshui@263.net); F. Herrera (herrera@decsai.ugr.es)

widely studied [1,4,9,11,12,14,38]. However, with the rapid development of society and the increasingly complex economic environment, management and decision-making tasks are becoming more and more difficult. Meanwhile, with the progress of science and technology and the development of network environment, the communications between people are increasingly convenient. Therefore, large-scale group decision making (LSGDM) has become the focus of decision-making problems. Generally, a GDM problem can be called LSGDM problem when the number of decision makers is more than 20 [19]. Now LSGDM are very commonly encountered in actual life, especially in the era of data [10,14-18,20,22,23,25,29,30,32,40,41].

An LSGDM consists of two main parts:

(1) One part is clustering. Because of the number of decision makers is numerous, and the decision makers exist differences in cognitive ability, judgment level, special emphasis, etc. Therefore, some scholars applied clustering methods into the process of LSGDM [22,29,30]. According to some certain characteristics of decision makers, large-scale decision-making groups can be classified into several small groups for assisting and improving the efficiency of decision-making.

(2) The other important part is the consensus reaching process, in which the decision makers discuss and improve their preferences, guided and supervised by a *moderator* [8,21,27]. This part aims at reaching all decision makers' agreements before making decisions.

When dealing with the LSGDM problems, the first and most important step is to collect the evaluation information of the decision makers. Additionally, considering that the qualitative information is more in line with the real thoughts of the decision makers, especially linguistic information is the most real response of people's cognitive process. Therefore, it is very reasonable to collect the linguistic information as the original evaluations of the decision makers in LSGDM. However, a practical and critical issue arises: How to express linguistic information more exactly and intuitively? For example, to express their true ideas more concretely, some decision makers usually give their evaluation information by several uncertain linguistic terms as "*only a little low*" and "*far from very high*". In fact, we can analyze the above linguistic terms by dividing them into two hierarchies, one is the basic linguistic hierarchy and the other one is the auxiliary linguistic hierarchy. Then "*only a little low*" can be divided into "*low*" and "*only a little*", and "*far from very high*" can be divided into "*very high*" and "*far from*". Clearly, the basic linguistic hierarchy consists of some simple linguistic terms, so it is more important to express the auxiliary linguistic hierarchy. A double hierarchy linguistic term set (DHLTS) [5] was developed, which consists of two hierarchy linguistic term sets (LTSs) denoted by a first hierarchy LTS with classical feature linguistic labels and a second hierarchy LTS as a linguistic feature or detailed supplementary of each linguistic term included in the first hierarchy LTS. Each basic element included in the DHLTS called double hierarchy linguistic term (DHLT). Suppose that $S=\{s_{-3}=none, s_{-2}=very\ low, s_{-1}=low, s_0=medium, s_1=high, s_2=very\ high, s_3=perfect\}$ and $O=\{o_{-3}=far\ from, o_{-2}=only\ a\ little, o_{-1}=a\ little, o_0=just\ right, o_1=much, o_2=very\ much, o_3=entirely\}$ are the first hierarchy LTS and second hierarchy LTS, respectively, then we can utilize DHLTs $s_{-1<o_{-2}>}$ and $s_{2<o_{-3}>}$ to describe linguistic terms "*only a little low*" and "*far from very high*", respectively.

There are two very important advantages: a) The DHLT is very intuitive and can be understood by making one to one correspondence with the two given LTSs; b) by introducing the second hierarchy LTS, the auxiliary linguistic hierarchy can be expressed more accurately.

Additionally, considering that sometimes some experts may be hesitant about their judgments, Gou et al. [5] developed DHLTS into hesitant fuzzy environment and proposed a double hierarchy hesitant fuzzy linguistic term set (DHHFLTS) and its basic element is called double hierarchy hesitant fuzzy linguistic element (DHHFLE). Then the linguistic terms "*between just right high and a little very high*" and "*more than only a little perfect*" can be expressed by the DHHFLEs $\{s_{1<o_0>}, s_{2<o_{-1}>}\}$ and $\{s_{3<o_{-2}>}, s_{3<o_{-1}>}, s_{3<o_0>}\}$, respectively.

Considering that the DHHFLTS can describe some complex linguistic terms more truly and completely, as well as some decision makers are more likely to express their evaluation information by making pairwise between any two alternatives, then Gou et al. [4] defined the concept of double hierarchy hesitant fuzzy linguistic preference relation (DHHFLPR), and developed some consistency checking and repairing methods. It is necessary to describe the decision makers' evaluation information more accurately and take full account of the case of hesitance. Meanwhile, the preference information obtained by making pairwise between any two alternatives is more clearly to reflect the relationship between two alternatives. Therefore, it is very suitable and significant to apply DHHFLPRs into LSGDM.

Because the clustering and the consensus reaching process are two important constituent parts, the main work in this paper is to discuss the clustering method and the consensus reaching process in LSGDM under double hierarchy hesitant fuzzy linguistic preference information. The main contributions of this paper are summarized as follows:

(1) Based on the similarity measures of DHHFLTSs, we develop a clustering method for LSGDM based on information entropy theory, which can be understood very clearly by a dynamic clustering figure. By this method, the decision makers can be divided into several small groups. Additionally, we propose a weights-determining method, which can obtain the weight of each small group, the weights of the decision makers included in each small group, and the weights of all decision

makers, respectively.

(2) We propose some consensus measures. A model is developed, which can precisely identify the alternatives, the pairs of alternatives and the decision makers that do not reach the consensus threshold, and then the moderator feeds these suggestions back to each small group and decision makers for modifying their preference information. This consensus measures can make the consensus degree improving process more targeted.

(3) Collecting the results obtained above, we establish a LSGDM model. It can be used to deal with LSGDM step by step. Moreover, a case study is set up to apply our model to deal with a practical LSGDM problem that is to evaluate Sichuan water resource management.

To do so, the rest of this paper is organized as follows: Section 2 introduces some basic concepts and the reviews of LSGDM. Section 3 discusses a similarity degree-based clustering method, proposes a double hierarchy information entropy-based weights-determining method, develops some consensus measures and establishes a LSGDM model. Section 4 applies our model to deal with a practical LSGDM problem that is to evaluate the implementation status of some policies in Sichuan water resource management. In addition, we make some comparison analyses with some existing methods from different angles. Finally, we conclude the paper in Section 5.

2. Preliminaries

In this section, we mainly discuss two parts: the basic concepts of DHLTS, DHHFLTS and DHHFLPR, and the descriptions of the LSGDM.

2.1. Double hierarchy hesitant fuzzy linguistic information

Suppose that $S=\{s_t|t=-\tau,\ldots,-1,0,1,\ldots,\tau\}$ and $O=\{o_k|k=-\varsigma,\ldots,-1,0,1,\ldots,\varsigma\}$ are the first and the second hierarchy LTS, respectively. Then a DHLTS is denoted by

$$S_O=\left\{s_{t<o_k>}\middle|t=-\tau,\ldots,-1,0,1,\ldots,\tau;\,k=-\varsigma,\ldots,-1,0,1,\ldots,\varsigma\right\} \tag{1}$$

where $s_{t<o_k>}$ is called double hierarchy linguistic term (DHLT), and o_k expresses the second hierarchy linguistic term when the first hierarchy linguistic term is s_t.

Let $t=\varsigma=3$, Fig.1 can show the distributions of four parts of the second hierarchy LTS [4].

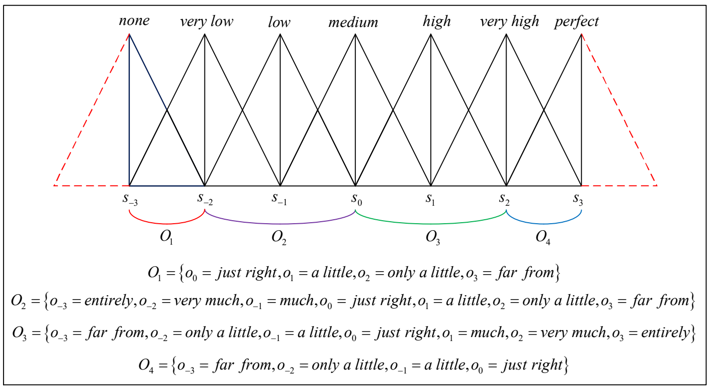

Fig. 1. The distributions of the four parts of the second hierarchy LTS.

Remark 2.1. In Fig. 1, four kinds of situations are shown on the basis of different values of t. If $t \ge 0$, then the meaning of the first hierarchy LTS $S=\{s_t \mid t \ge 0\}$ is positive, so the second hierarchy LTS needs to be selected with the ascending order. On the contrary, if $t<0$, then the meaning of the first hierarchy LTS $S=\{s_t \mid t \le 0\}$ is negative, so the second hierarchy LTS needs to be selected with the descending order. Specially, because both s_τ and $s_{-\tau}$ only contain a half of area compared to other linguistic terms. Therefore, we only utilize $O=\{o_k \mid k=-\varsigma,\dots,-1,0\}$ and $O=\{o_k \mid k=0,1,\dots,\varsigma\}$ to describe s_τ and $s_{-\tau}$, respectively.

Furthermore, Gou et al. [4] developed S_O into hesitant fuzzy environment and defined the DHHFLTS.

Definition 2.1 [4]. Let X be a fixed set, $S_O=\{s_{t<o_k>} \mid t=-\tau,\dots,-1,0,1,\dots,\tau; k=-\varsigma,\dots,-1,0,1,\dots,\varsigma\}$ be a DHLTS. A double hierarchy hesitant fuzzy linguistic term set (DHHFLTS) on X, H_{S_O}, is in terms of a membership function that when applied to X returns a subset of S_O, and denoted by a mathematical form:

$$H_{S_O}=\{<x_i, h_{S_O}(x_i)> \mid x_i \in X\} \tag{2}$$

where $h_{S_O}(x_i)$ is a set of some values in S_O, denoting the possible membership degrees of the element $x_i \in X$ to the set H_{S_O} as:

$$h_{S_O}(x_i)=\left\{s_{\phi_l<o_{\varphi_l}>}(x_i) \mid s_{\phi_l<o_{\varphi_l}>} \in S_O; l=1,2,\dots,L; \phi_l=-\tau,\dots,-1,0,1,\dots,\tau; \varphi_l=-\varsigma,\dots,-1,0,1,\dots,\varsigma\right\} \tag{3}$$

with L being the number of the DHLTs in $h_{S_O}(x_i)$ and $s_{\phi_l<o_{\varphi_l}>}(x_i)$ ($l=1,\dots,L$) in each $h_{S_O}(x_i)$ being the continuous terms in S_O. $h_{S_O}(x_i)$ denotes the possible degree of the linguistic variable x_i

to S_O. For convenience, we call $h_{S_O}(x_i)$ DHHFLE, and DHLTs included in a DHHFLE are ranked in ascending order.

Next, based on the discussion of monotonic function [3], we define an monotone function for making the mutual transformations between the DHLT and the numerical scale when extending the DHLT to a continuous form, whose indices are in the intervals $[-\tau,\tau]$ and $[-\varsigma,\varsigma]$ respectively. Like the 2-tuple linguistic terms [2,7,12] and the virtual linguistic terms [34-37], we can develop a continuous function f:

Definition 2.2. Let $\bar{S}_O=\{s_{t<o_k>} | t\in[-\tau,\tau];\ k\in[-\varsigma,\varsigma]\}$ be a continuous DHLTS, $h_{S_O}=\{s_{\phi_l<o_{\varphi_l}>} | s_{\phi_l<o_{\varphi_l}>}\in\bar{S}_O; l=1,2,\ldots,L; \phi_l\in[-\tau,\tau];\ \varphi_l\in[-\varsigma,\varsigma]\}$ be a DHHFLE with L being the number of linguistic terms in h_{S_O}, and $h_\gamma=\{\gamma_l | \gamma_l\in[0,1]; l=1,2,\ldots,L\}$ be a set of numerical scales. Then the subscript (ϕ_l,φ_l) of the DHLT $s_{\phi_l<o_{\varphi_l}>}$ that expresses the equivalent information to the numerical scale γ_l can be transformed to the numerical scale γ_l by a monotone function f:

$$f:[-\tau,\tau]\times[-\varsigma,\varsigma]\rightarrow[0,1], f(\phi_l,\varphi_l)=\frac{\varphi_l+(\tau+\phi_l)\varsigma}{2\varsigma\tau}=\gamma_l \tag{4}$$

Additionally, let $\Phi\times\Psi$ be the set of all DHHFLEs over $\bar{S}_O$, Θ be the set of all numerical scales. Then a monotone function F between a DHHFLE h_{S_O} and a set of numerical scales h_γ based on f is:

$$F:\Phi\times\Psi\rightarrow\Theta,\ F(h_{S_O})=F\left(\{s_{\phi_l<o_{\varphi_l}>} | s_{\phi_l<o_{\varphi_l}>}\in\bar{S}_O; l=1,\ldots,L; \phi_l\in[-\tau,\tau]; \varphi_l\in[-\varsigma,\varsigma]\}\right)=\{\gamma_l \mid \gamma_l=f(\phi_l,\varphi_l); l=1,2,\ldots,L\}=h_\gamma \tag{5}$$

Specially, if a DHHFLE h_{S_O} only has a DHLT, namely, $h_{S_O}=s_{\phi<o_\varphi>}$, then F reduces to F'

$$F':\bar{S}_O\rightarrow[0,1],\ F'(h_{S_O})=f(\phi,\varphi)=\gamma \tag{6}$$

Before giving the definition of additive DHHFLPR, it is necessary to develop the addition and multiplication operations for DHHFLEs under some conditions:

Definition 2.3. Let $S_O=\{s_{t<o_k>} | t=-\tau,\ldots,-1,0,1,\ldots,\tau; k=-\varsigma,\ldots,-1,0,1,\ldots,\varsigma\}$ be a DHLTS, $h_{S_O}=\{s_{\phi_l<o_{\varphi_l}>} | s_{\phi_l<o_{\varphi_l}>}\in S_O; l=1,2,\ldots,\#h_{S_O}\}$, $h_{S_{O_i}}=\left\{s_{\phi_l^i<o_{\varphi_l^i}>} \middle| s_{\phi_l^i<o_{\varphi_l^i}>}\in S_O; l=1,2,\ldots,\#h_{S_O}^i\right\}(i=1,2)$ ($\#h_{S_O}^1=\#h_{S_O}^2$) be three DHHFLEs, λ be a real number and $0<\lambda<1$. Then

(1) $h_{S_{O_1}} \oplus h_{S_{O_2}} = \bigcup_{s_{\phi^1_{\sigma(l)}<o_{\varphi^1_{\sigma(l)}}>} \in h_{S_{O_1}}, s_{\phi^2_{\sigma(l)}<o_{\varphi^2_{\sigma(l)}}>} \in h_{S_{O_2}}} \left\{ s_{\phi^1_{\sigma(l)}+\phi^2_{\sigma(l)}<o_{\varphi^1_{\sigma(l)}+\varphi^2_{\sigma(l)}}>} \right\}$; *if* $\phi^1_{\sigma(l)} + \phi^2_{\sigma(l)} \le \tau$ *and* $\varphi^1_{\sigma(l)} + \varphi^1_{\sigma(l)} \le \varsigma$;

(2) $\lambda h_{S_O} = \bigcup_{s_{\phi_{\sigma(l)}<o_{\varphi_{\sigma(l)}}>} \in h_{S_O}} \left\{ s_{\lambda\phi_{\sigma(l)}<o_{\lambda\varphi_{\sigma(l)}}>} \right\}$; $0 < \lambda < 1$;

Specially, if all these three DHHFLEs h_{S_O}, $h_{S_{O_1}}$ and $h_{S_{O_2}}$ only have one DHLT, respectively. Then Definition 2.3 is changed to the operational laws of DHLTs: $\overset{2}{\underset{i=1}{\oplus}} s_{\phi^i<o_{\varphi^i}>} = s_{\phi^1+\phi^2<o_{\varphi^1+\varphi^2}>}$ and $\lambda s_{\phi<o_\varphi>} = s_{\lambda\phi<o_{\lambda\varphi}>}$.

Suppose that $A = \{A_1, A_2, \ldots, A_m\}$ is a fixed set of alternatives. Then a DHHFLPR can be developed:

Definition 2.4 [4]**.** A DHHFLPR $\tilde{H}_{S_O}$ is presented by a matrix $\tilde{H}_{S_O} = \left(h_{S_{O_{ij}}}\right)_{m\times m} \subset A \times A$, where $h_{S_{O_{ij}}} = \left\{ h^{\sigma(l)}_{S_{O_{ij}}} \middle| l = 1, 2, \ldots, \# h_{S_{O_{ij}}} \right\}$ ($\# h_{S_{O_{ij}}}$ is the number of DHLT in $h_{S_{O_{ij}}}$, $h^{\sigma(l)}_{S_{O_{ij}}}$ is the l-th DHLT in $h_{S_{O_{ij}}}$) is a DHHFLE, indicating hesitant degrees to which A_i is preferred to A_j. For all $i, j = 1, 2, \ldots, m$, $h_{S_{O_{ij}}}$ $(i < j)$ satisfies the conditions:

(1) $h^{\sigma(l)}_{S_{O_{ij}}} + h^{\sigma(l)}_{S_{O_{ji}}} = s_{0<o_0>}$, $h_{S_{O_{ii}}} = s_{0<O_0>}$, and $\# h_{S_{O_{ij}}} = \# h_{S_{O_{ji}}}$;

(2) $h^{\sigma(l)}_{S_{O_{ij}}} < h^{\sigma(l+1)}_{S_{O_{ij}}}$ and $h^{\sigma(l)}_{S_{O_{ji}}} > h^{\sigma(l+1)}_{S_{O_{ji}}}$.

2.2. LSGDM

LSGDM has been studied in some different fields and mainly includes two parts: consensus models and clustering methods. Firstly, some of the consensus models are based on self-organizing maps [22], graphical monitoring tool (MENTOR) [23], expert weighting methodology [25], and minimum adjustment cost feedback mechanism-based consensus model [33], etc. Additionally, two consensus models were built to deal with some LSGDM problems with non-cooperative behaviors and minority opinions [32], and individual concerns and satisfactions [40], respectively. Furthermore, with the hesitant fuzzy information [31], a consensus model for LSGDM was introduced, which is distinguished from previous studies about the obtained clusters and the feedback mechanism [30]. Zhang [41] proposed a consistency-and consensus-based model based on probabilistic linguistic term sets (PLTSs) [24] under LSGDM. Secondly, amounts of clustering methods were developed including k-means clustering method [30], fuzzy c-mean clustering method [22], interval type-2 fuzzy equivalence clustering analysis [29], the partial binary tree DEA-DA cyclic classification model [15], and the hierarchical clustering approach [43], etc. In this paper, these two parts are also the contents which we need to focus on discussing.

Next, an LSGDM problem under double hierarchy hesitant fuzzy linguistic preference information can be described as: Let $A=\{A_1,A_2,\dots,A_m\}$ be a set of alternatives, $E=\{e^1,e^2,\dots,e^n\}$ be a set of decision makers, and $w=(w_1,w_2,\dots,w_n)^T$ be the weight vector of decision makers with $0\le w_i\le 1$ and $\sum_{i=1}^{n}w_i=1$. Suppose that $\tilde{H}_{S_O}^a=\left(h_{S_{O_{ij}}}^a\right)(i,j=1,2,\dots,m;\ a=1,2,\dots,n)$ be a DHHFLPR which indicates that the decision maker e^a gives his/her evaluations for all alternatives by making pairwise comparisons. Without loss of generality, we let $n\ge 20$ and $m\ge 3$.

Before we start work, the normalization of DHHFLPR is necessary. In order to not lose the original information, we normalize DHHFLPR based on the linguistic expected value of DHHFLE [4]:

Definition 2.4 [4]**.** Let $\bar{S}_O=\left\{s_{t<o_k>}\middle|t\in[-\tau,\tau];\ k\in[-\varsigma,\varsigma]\right\}$ be a continuous DHLTS, $h_{S_O}=\left\{s_{\phi_l<o_{\varphi_l}>}\middle|s_{\phi_l<o_{\varphi_l}>}\in\bar{S}_O;\ l=1,2,\dots,\#h_{S_O}\right\}$ be a DHHFLE, $\Phi\times\Psi$ be the set of all DHHFLEs over $\bar{S}_O$. Then

$$le:\Phi\times\Psi\to\bar{S}_O,\ le\left(h_{S_O}\right)=\frac{1}{\#h_{S_O}}\mathop{\oplus}_{l=1}^{\#h_{S_O}}s_{\phi_l<o_{\varphi_l}>}=s_{\phi^*<o_{\varphi^*}>}\tag{7}$$

can be called the linguistic expected value of the DHHFLE h_{S_O}, where $\phi^*=\frac{1}{\#h_{S_O}}\sum_{l=1}^{\#h_{S_O}}\phi_l$ and $\varphi^*=\frac{1}{\#h_{S_O}}\sum_{l=1}^{\#h_{S_O}}\varphi_l$.

Suppose that $\tilde{H}_{S_O}=\left(h_{S_{O_{ij}}}\right)_{m\times m}\subset A\times A$ is a DHHFLPR, then we call

$$\tilde{H}_{S_O}^N=\left(le\left(h_{S_{O_{ij}}}\right)\right)_{m\times m}\subset A\times A\tag{8}$$

a normalized DHHFLPR (NDHHFLPR), which satisfies $le\left(h_{S_{O_{ij}}}\right)\oplus le\left(h_{S_{O_{ji}}}\right)=s_{0<o_0>}$, $le\left(h_{S_{O_{ii}}}\right)=s_{0<O_0>}$, and $\#le\left(h_{S_{O_{ij}}}\right)=\#le\left(h_{S_{O_{ji}}}\right)$.

3. A consensus reaching process in LSGDM with DHHFLPRs

In general, consensus reaching process is a very important part in LSGDM, which makes sure that the decision makers and analysts have enough communications and the moderator can also assist the decision makers in improving their preference information. In this section, we research an consensus reaching process for dealing with the LSGDM problems with DHHFLPRs. And this process mainly consists of four parts:

a) The similarity degree-based clustering algorithm. Similarity degree can be as a useful tool to reflect the relation of any two decision makers. Therefore, we can use it to develop a clustering algorithm to cluster the decision makers into several small groups.

b) Double hierarchy information entropy-based weights-determining method. During the consensus reaching process, aggregating all decision makers' preference information is an important step. Meanwhile, information entropy can be used as a useful method to reflect the important degrees of each group and every decision maker. Therefore, based on the clustering result, an information entropy-based weights-determining method under double hierarchy linguistic information is established.

c) The consensus measures. Based on the similarity degree discussed in the clustering algorithm, some consensus measures can be developed, which are the main basis of the consensus reaching process.

d) The LSGDM model. This model can be established based on all the results discussed above and consists two parts, one is the consensus reaching process, and the other one is to make decisions.

3.1. Similarity degree-based clustering algorithm

In LSGDM, the discussions among the decision makers is very common. However, it will surely bring forth a huge amount of work and the communications among the decision makers also will not be smooth. To solve these problems, clustering is very necessary in the consensus reaching process because of a group with less decision makers is easier to discuss and improve preference information. Therefore, in this subsection, we introduce how to cluster the decision makers in LSGDM on the basis of similarity measure. Firstly, the concept of similarity degree between two DHHFLEs can be defined as follows:

Definition 3.1. Let $h_{S_O}^1$ and $h_{S_O}^2$ be two DHHFLEs, then the similarity degree between $h_{S_O}^1$ and

$h_{S_O}^2$ is

$$sd\left(h_{S_O}^1, h_{S_O}^2\right) = 1 - d\left(h_{S_O}^1, h_{S_O}^2\right) = 1 - \left|F'\left(le\left(h_{S_O}^1\right)\right) - F'\left(le\left(h_{S_O}^2\right)\right)\right| \tag{9}$$

where F' is the membership function as Eq. (6). Clearly, $0 \le sd\left(h_{S_O}^1, h_{S_O}^2\right) \le 1$, and the $sd\left(h_{S_O}^1, h_{S_O}^2\right)$ is closer to 1, the more similar between $h_{S_O}^1$ and $h_{S_O}^2$ will be.

Then a similarity matrix $SM^{ab} = \left(sm_{ij}^{ab}\right)_{m\times m}$ $\left(i, j = 1, 2, \dots, m; a, b = 1, 2, \dots, n\right)$ for each pair of decision makers $\left(e^a, e^b\right)$ can be established:

$$SM^{ab} = \begin{pmatrix} sm_{11}^{ab} & sm_{12}^{ab} & \cdots & sm_{1m}^{ab} \\ sm_{21}^{ab} & sm_{22}^{ab} & \cdots & sm_{2m}^{ab} \\ \vdots & \vdots & \ddots & \vdots \\ sm_{m1}^{ab} & sm_{m2}^{ab} & \cdots & sm_{mm}^{ab} \end{pmatrix} \tag{10}$$

where sm_{ij}^{ab} expresses the similarity degree between e^a and e^b in the position $\left(i, j\right)$ and

$$sm_{ij}^{ab} = sd\left(h_{S_{O_{ij}}}^a, h_{S_{O_{ij}}}^b\right) \tag{11}$$

In general, the higher similarity degree two decision makers have, the greater possibility they belong to the same group. Therefore, a similarity degree-based clustering method can be developed as follows:

Algorithm 3.1. Similarity degree-based clustering algorithm

Step 1. Establish the overall similarity matrix. Based on Eq. (9) and Eq. (10), we can obtain a similarity matrix $SM = \left(sm_{ij}^{ab}\right)_{m\times m}$ associated with each pair of decision makers $\left(e^a, e^b\right)$ $\left(a, b = 1, 2, \dots, n\right)$, and then aggregate all the similarity matrices and obtain the overall similarity matrix $OSM = \left(osm^{ab}\right)_{n\times n}$, where

$$osm^{ab} = \frac{2}{m\left(m-1\right)} \sum_{i=1}^{m} \sum_{i<j}^{m} sm_{ij}^{ab} \tag{12}$$

Step 2. Choose the classification threshold. Ranking all the different elements of the upper triangular matrix of OSM (except the diagonal elements) following the order from big to small, denoted by $\eta_1 > \eta_2 > \cdots > \eta_p > \cdots > \eta_q$, where $q \le \frac{n\left(n-1\right)}{2}$. Let $\eta = \eta_p$, obviously, $\eta \in \left[0, 1\right]$.

Step 3. Determine the optimal classification threshold η^*. Let C_p be the rate of threshold change, obtained by

$$C_p = \frac{\eta_{p-1} - \eta_p}{n_p - n_{p-1}} \tag{13}$$

where η_{p-1} and η_p are the $p-1$-th and p-th classification threshold, respectively; n_p and n_{p-1} are the number of the p-th and $p-1$-th classification, respectively. If $n_p = n$, then the operation is over. If

$$C_\mu = \max_p \left\{ C_p \right\} \tag{14}$$

then we call the μ-th classification threshold the optimal classification threshold, namely, $\eta^* = \eta_\mu$.

Step 4. Determine the classification result. Firstly, we collect all pairs of decision makers $\left(e^a, e^b\right)$ into an overall group where $osm^{ab} \geq \eta^*$, denoted by $B_1, B_2, \ldots, B_\zeta$, and then combine the elements of overall group into a group if they satisfy $B_{\zeta_i} \cap B_{\zeta_j} \neq \varnothing \quad \left(\zeta_i \neq \zeta_j; \zeta_i, \zeta_j = 1, 2, \ldots, \zeta\right)$. When $B_{\zeta_i} \cap B_{\zeta_j} = \varnothing$, then we can obtain the classification result of the large-scale group members, denoted as $B_\iota \left(\iota = 1, 2, \ldots, \Upsilon\right)$.

Remark 3.1. This clustering is mainly based on the similarity degree between any two decision makers, which means that two decision makers can be deemed as a cluster if they have a high enough similarity degree. From Steps 1-3, we can obtain all pairs of decision makers $\left(e^a, e^b\right)$, which can be collected into an overall group, denoted by $B_1, B_2, \ldots, B_\zeta$. For example, if $B_{\zeta_i} \cap B_{\zeta_j} \neq \varnothing$, then it is obvious that these two pairs of decision makers have same decision maker. Therefore, all decision makers are included in B_{ζ_i} and B_{ζ_j}. Similarly, we can obtain the final clustering result.

The similarity degree-based clustering process in LSGDM can be described in Fig. 2:

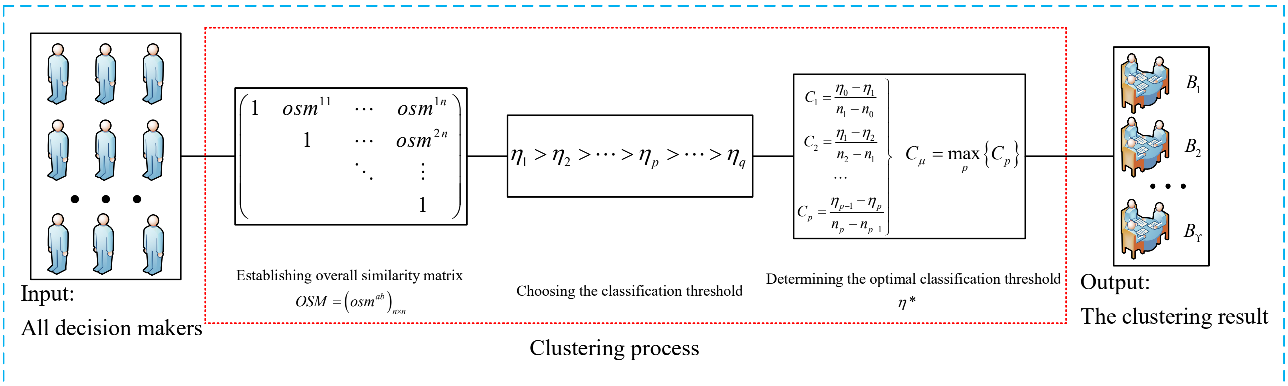

Fig. 2. The similarity degree-based clustering process in LSGDM.

3.2. Double hierarchy information entropy-based weights-determining method

At present, there exist a lot of weights-determining methods in decision making, such as the dynamic weights-determining approach based on the intuitionistic fuzzy Bayesian network [6], the two-layer weights-determining method [17], the AHP method [26], the Delphi [13] methods, the entropy-based method [33], the TOPSIS-based methods [38], the projection method [39], and the combined weighting methods [9,11], etc. In this paper, we also need to develop a weight-determining method for LSGDM. Based on the clustering result discussed in Subsection 3.1, a double hierarchy information entropy-based weights-determining method can be developed. This method can obtain three kinds of weights information including the weight of each group, the weights of the decision makers included in each group, and the weights of all decision makers. The process of this method can be shown as follows:

Step 1. Determine the weight of each group mostly based on the number of decision makers. Suppose that the decision makers $e^1, e^2, \ldots, e^n$ are divided into T groups, and the t-th group contains φ_t decision makers, then the weight of each group ω_t can be obtained by

$$\omega_t = \frac{\varphi_t^2}{\sum_{i=1}^{T} \varphi_i^2}, \quad i = 1, 2, \ldots, T \tag{15}$$

Step 2. Utilize information entropy theory to determine the weights of decision makers included in each group. The first step is to obtain every decision maker's ordering vector $U^a = \left(u_1^a, u_2^a, \ldots, u_m^a\right)$ $\left(a = 1, 2, \ldots, n\right)$ for all alternatives, which can be calculated by

$$u_i^a = \frac{\sum_{j=1}^{m} F'\left(le\left(h_{S_{O_{ij}}}^a\right)\right)}{\sum_{i=1}^{m} \sum_{j=1}^{m} F'\left(le\left(h_{S_{O_{ij}}}^a\right)\right)}, \quad i = 1, 2, \ldots, m \tag{16}$$

Then the information entropy of the decision maker e^a can be obtained by

$$IE\left(U^a\right) = -\frac{1}{\log_2 m} \bullet \sum_{i=1}^{m} u_i^a \log_2 u_i^a \tag{17}$$

Information entropy indicates the uncertainty degree and the randomness of evaluation information. Therefore, the smaller the information entropy is, the bigger the certainty degree will be, which means that the corresponding decision maker plays a significant role and it is necessary to give him/her a bigger weight. Therefore, let $\overline{\omega}_t^a$ be the weight of the a-th decision maker included in the t-th group, then

$$\overline{\omega}_t^a = \frac{\left(IE\left(U^a\right)\right)^{-1}}{\sum_{a=1}^{\varphi_t}\left(IE\left(U^a\right)\right)^{-1}} \tag{18}$$

Step 3. Obtain the weight of every decision maker by combining these two weight information:

$$w_a = \omega_t \bullet \overline{\omega}_t^a \tag{19}$$

3.3. Consensus measures

Firstly, the fundamental of consensus reaching process in LSGDM can be shown in Fig. 3:

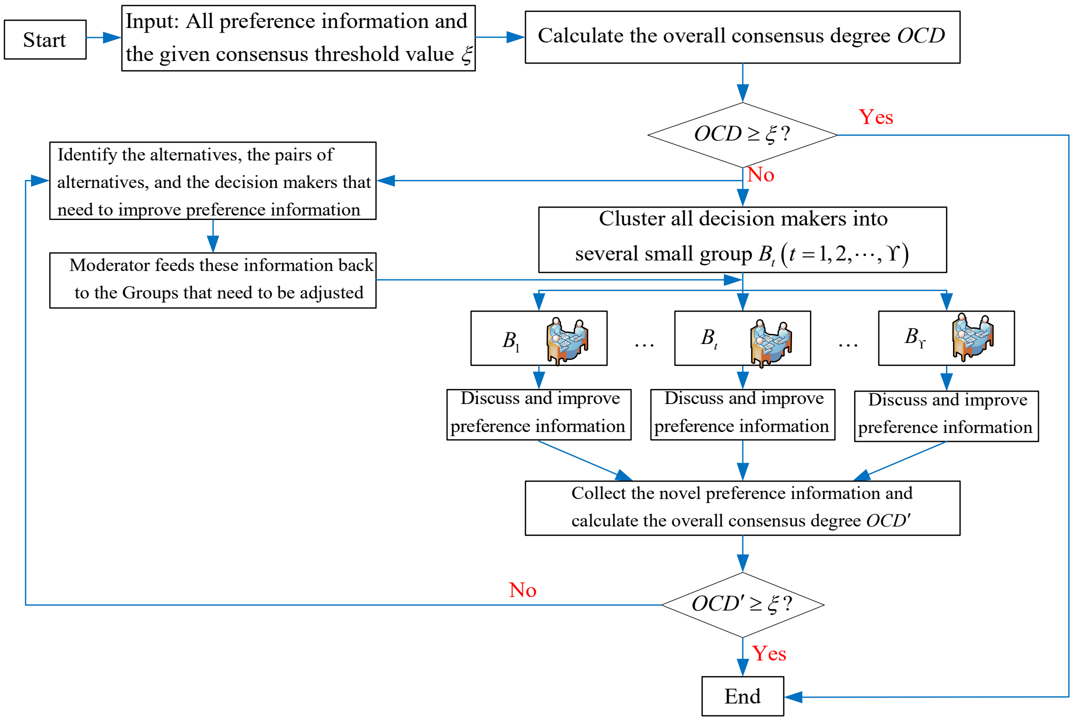

Fig 3. the fundamental of consensus reaching process in LSGDM.

From Fig. 3, there exist four main issues in consensus reaching process:

(1) How to calculate the overall consensus degree.

(2) How to identify the alternatives, the part of alternatives, and the decision makers that need to improve preference relations.

(3) How to discuss and improve the preference relation in each group.

(4) How to determine some necessary parameters.

For the first issue, some consensus degrees can be developed to solve it. At the beginning, we aggregate all similarity matrices $SM^{ab} = \left(sm_{ij}^{ab}\right)_{m\times m}$ $(a, b = 1, 2, \ldots, n)$ associated with each pair of decision makers $\left(e^a, e^b\right)$ and establish a consensus matrix $CM = \left(cm_{ij}\right)_{m\times m}$ based on the similarity degrees, where

$$cm_{ij} = \frac{2}{n(n-1)} \sum_{a=1}^{n-1} \sum_{b=a+1}^{n} sm_{ij}^{ab}, \quad i, j = 1, 2, \ldots, m \tag{20}$$

Next, we determine the consensus degrees of all decision makers based on the following three parts:

(1) Consensus degree for each pair of alternatives. Considering each element cm_{ij} included in the consensus matrix $CM = (cm_{ij})_{m\times m}$ means the consensus level among all decision makers for the pair of alternatives (A_i, A_j), so we can use it to express the consensus degree for (A_i, A_j), denoted as $cdpa_{ij}$ and

$$cdpa_{ij} = cm_{ij} \; (i, j = 1, 2, \ldots, m) \tag{21}$$

Obviously, the bigger the value of $cdpa_{ij}$ is, the greater agreement among all decision makers on the pair of alternatives (A_i, A_j) will be. Therefore, we can utilize this measure to obtain which position has a poor consensus level.

(2) Consensus degree for each alternative. By aggregating all elements included in each row of consensus matrix $CM = (cm_{ij})_{m\times m}$, the consensus degree for every alternative A_i, denoted by cda_i, can be developed to measure the consensus level among all decision makers for this alternative:

$$cda_i = \frac{1}{m-1} \sum_{j=1, i\neq j}^{m} cdpa_{ij} \; (j = 1, 2, \ldots, m) \tag{22}$$

(3) Overall consensus degree for all preference relations. the overall consensus degree for all preference relations, denoted by ocd, can be used to measure the total consensus level among all decision makers and control the progress of the consensus researching process. It can be obtained by

$$ocd = \min_i \{cda_i\} \; (i = 1, 2, \ldots, m) \tag{23}$$

Based on the discussions above, we propose three parts to determine different consensus degrees. And then we can make a comparison between overall consensus degree ocd and the given consensus threshold value ξ. If $ocd \geq \xi$, then the consensus reaching process is over; Otherwise, two steps are performed simultaneously: One is to cluster all decision makers into several small groups based on Subsection 3.1, and the other one is to identify the alternatives, the part of alternatives, and the decision makers that need to improve preference relations and how to improve them. Next we only need to solve the second issues. Our method includes two kinds of rules: the identification rules (IR) and the direction rules (DR).

(1) The identification rules (IR)

The identification rules are mainly used to identify the alternatives, the pairs of alternatives and the decision makers that do not reach the given consensus threshold.

(I) Identify the alternatives (IR-1): Let AL be the set of alternatives in which the consensus degree cda_i is lower than the given consensus threshold value ξ. Then we can identify the alternatives based on

$$AL = \left\{ A_i \middle| cda_i < \xi,\ i = 1,2,\ldots,m \right\} \tag{24}$$

Obviously, AL is a set and it may contain many alternatives. However, if we only want to change one alternative in each consensus reaching process, then the set AL can be developed as:

$$AL = \left\{ A_i \middle| \min_i \left\{ cda_i < \xi,\ i = 1,2,\ldots,m \right\} \right\} \tag{25}$$

(II) Identify the pairs of alternatives (IR-2): For any alternative $A_i \in AL$, this rule is utilized to identify which pair of alternatives $\left(A_i, A_j\right)$ needs to be improved. These pairs of alternatives are named as a set PAL_i and can be obtained by

$$PAL_i = \left\{ (i,j) \middle| A_i \in AL \wedge cdpa_{ij} < \xi \right\} \tag{26}$$

Obviously, combining IR-1 and IR-2, we can determine which position needs to be changed.

(III) Identify the decision makers (IR-3): The decision makers who need to improve their preference relations can be decided by making some discussions among all decision makers in each group. Additionally, the next method can also be used as a reference for each group:

Let DM_{ij} be a set of decision makers who should change their preference information. Then we can calculate the distance between any of the decision makers $e^a \left(a = 1,2,\ldots,n\right)$ and all the others $e^b \left(a \neq b\right)$ at the position $\left(A_i, A_j\right)$ based on the formula below:

$$d_{ij}^a = \sum_{b=1,b\neq a}^{n} \left(1 - sd\left(h_{S_{O_{ij}}}^a, h_{S_{O_{ij}}}^b\right)\right) = n - 1 - \sum_{b=1,b\neq a}^{n} sm_{ij}^{ab} \tag{27}$$

The decision maker DM_{ij} who should change preference at $\left(A_i, A_j\right)$ can be determined based on

$$DM_{ij} = \left\{ e^* \middle| \left(A_i, A_j\right) \in PAL_i \wedge d_{ij}^* = \max_a \left\{ d_{ij}^a \right\} \right\} \tag{28}$$

Combining IR-1, IR-2 and IR-3, it is very easy to determine which decision maker and his/her position needs to be changed. Suppose that a decision maker $e^* \in DM_{ij}$, and he/she needs to change preference information $h^{*}_{S_{O_{ij}}}$, then a set can be set up to express these elements:

$$\Delta = \left\{ \left(*,(i,j)\right) \middle| e^{*} \in DM_{ij} \wedge \left(A_i, A_j\right) \in PAL_i \right\} \tag{29}$$

(2) The direction rules (DR)

These rules are utilized to send suggestions to each group and tell them how to increase the consensus level in the next round. Firstly, the moderator needs to set up a target and gives it to each group, and then each group can discuss how to change their preferences in the position $\left(A_i, A_j\right)$. The target can be obtained by referencing the aggregation information of all decision makers' preferences.

Definition 3.2 [4]**.** Let $h^{a}_{S_{O_{ij}}}$ $\left(a = 1,2,\ldots,n\right)$ be a set of DHHFLEs, then we call

$$DHHFLWA\left(h^{1}_{S_{O_{ij}}}, h^{2}_{S_{O_{ij}}}, \ldots, h^{n}_{S_{O_{ij}}}\right) = \overset{n}{\underset{a=1}{\oplus}} w_a le\left(h^{a}_{S_{O_{ij}}}\right) \tag{30}$$

the double hierarchy hesitant fuzzy linguistic weighted average (DHHFLWA) operator, where $w = \left(w_1, w_2, \ldots, w_n\right)^T$ is the weight vector of DHHFLEs and satisfies $0 \le w_a \le 1$ and $\sum_{a=1}^{n} w_a = 1$ $\left(a = 1,2,\ldots,n\right)$.

Based on Eq. (30), the group DHHFLPRs $\tilde{H}^{c}_{S_O} = \left(h^{c}_{S_{O_{ij}}}\right)_{m \times m}$ can be established, and we call $h^{c}_{S_{O_{ij}}}$ the group preference element. Then the direction rules can be designed as follows:

(I) $DR\text{-}1$: If $h^{*}_{S_{O_{ij}}} < h^{c}_{S_{O_{ij}}}$, then the decision maker e^{*} should increase his/her evaluation associated with the pair of alternatives $\left(A_i, A_j\right)$.

(II) $DR\text{-}2$: If $h^{*}_{S_{O_{ij}}} > h^{c}_{S_{O_{ij}}}$, then the decision maker e^{*} should decrease his/her evaluation associated with the pair of alternatives $\left(A_i, A_j\right)$.

When the decision makers know how to change their evaluations associated with the pair of alternatives $\left(A_i, A_j\right)$, the next problem is to decide the extent of the change. Suppose that $\left(h^{*}_{S_{O_{ij}}}\right)^{(\mathbb{C}+1)}$ and $\left(h^{*}_{S_{O_{ij}}}\right)^{(\mathbb{C})}$ are the $\mathbb{C}+1$-th and $\mathbb{C}$-th round preferences of the decision maker e^{*}, respectively. Then the general range is

$$\left(h^{*}_{S_{O_{ij}}}\right)^{(\mathbb{C}+1)} \in \left[\min\left\{ \left(h^{*}_{S_{O_{ij}}}\right)^{(\mathbb{C})}, \left(h^{c}_{S_{O_{ij}}}\right)^{(\mathbb{C})} \right\}, \max\left\{ \left(h^{*}_{S_{O_{ij}}}\right)^{(\mathbb{C})}, \left(h^{c}_{S_{O_{ij}}}\right)^{(\mathbb{C})} \right\} \right] \tag{31}$$

In fact, we can always find a parameter $\lambda \in (0,1)$, Eq. (26) is equivalent with

$$\left(h_{S_{O_{ij}}}^{*}\right)^{(\mathbb{C}+1)} = \lambda\left(h_{S_{O_{ij}}}^{*}\right)^{(\mathbb{C})} \oplus (1-\lambda)\left(h_{S_{O_{ij}}}^{c}\right)^{(\mathbb{C})} \tag{32}$$

Remark 3.2. For the consensus reaching process of this paper, the solutions of these two issues only are the references for each group. For the third issue, each group can be free to discuss and decide how to improve the preference information. Therefore, each decision maker who needs to improve their preference information has two choices: Change or no to change. For the first one, this group can discuss how to improve the preference information based on Eq. (31). But for the second one, we also have two choices: Delete this decision maker or change his/her preference information based on Eq. (32) randomly.

Theorem 3.1. For any alternative A_i, if its related preference information needs to be changed, and the identification rules and the direction rules have been applied, then

$$(cda_i)^{(\mathbb{C}+1)} > (cda_i)^{(\mathbb{C})} \tag{33}$$

Proof. To prove $(cda_i)^{(\mathbb{C}+1)} > (cda_i)^{(\mathbb{C})}$, it is equivalent to prove $(cm_{ij})^{(\mathbb{Z}+1)} > (cm_{ij})^{(\mathbb{Z})}$ and

$$\frac{2}{n(n-1)}\sum_{a=1}^{n-1}\sum_{b=a+1}^{n}\left(sm_{ij}^{ab}\right)^{(\mathbb{C}+1)} > \frac{2}{n(n-1)}\sum_{a=1}^{n-1}\sum_{b=a+1}^{n}\left(sm_{ij}^{ab}\right)^{(\mathbb{C})} \tag{34}$$

Based on Eqs. (9) and (11), Eq. (34) can be rewritten as:

$$\frac{2}{n(n-1)}\sum_{a=1}^{n-1}\sum_{b=a+1}^{n} sd\left(le\left(\left(h_{S_{O_{ij}}}^{a}\right)^{(\mathbb{C}+1)}\right), le\left(\left(h_{S_{O_{ij}}}^{b}\right)^{(\mathbb{C}+1)}\right)\right) > \frac{2}{n(n-1)}\sum_{a=1}^{n-1}\sum_{b=a+1}^{n} sd\left(le\left(\left(h_{S_{O_{ij}}}^{a}\right)^{(\mathbb{C})}\right), le\left(\left(h_{S_{O_{ij}}}^{b}\right)^{(\mathbb{C})}\right)\right) \tag{35}$$

which is equal to

$$\frac{2}{n(n-1)}\sum_{a=1}^{n-1}\sum_{b=a+1}^{n} d\left(le\left(\left(h_{S_{O_{ij}}}^{a}\right)^{(\mathbb{C}+1)}\right), le\left(\left(h_{S_{O_{ij}}}^{b}\right)^{(\mathbb{C}+1)}\right)\right) < \frac{2}{n(n-1)}\sum_{a=1}^{n-1}\sum_{b=a+1}^{n} d\left(le\left(\left(h_{S_{O_{ij}}}^{a}\right)^{(\mathbb{C})}\right), le\left(\left(h_{S_{O_{ij}}}^{b}\right)^{(\mathbb{C})}\right)\right) \tag{36}$$

Without a loss of generality, let e^1 be the decision maker who needs to change his /her preference for the part (A_i, A_j), then Eq. (31) can be developed into

$$d\left(le\left(\left(h_{S_{O_{ij}}}^{1}\right)^{(\mathbb{C}+1)}\right), le\left(\left(h_{S_{O_{ij}}}^{2}\right)^{(\mathbb{C}+1)}\right)\right) + d\left(le\left(\left(h_{S_{O_{ij}}}^{1}\right)^{(\mathbb{C}+1)}\right), le\left(\left(h_{S_{O_{ij}}}^{3}\right)^{(\mathbb{C}+1)}\right)\right) + \cdots + d\left(le\left(\left(h_{S_{O_{ij}}}^{1}\right)^{(\mathbb{C}+1)}\right), le\left(\left(h_{S_{O_{ij}}}^{n}\right)^{(\mathbb{C}+1)}\right)\right)$$

$$< d\left(le\left(\left(h_{S_{O_{ij}}}^{1}\right)^{(\mathbb{C})}\right), le\left(\left(h_{S_{O_{ij}}}^{2}\right)^{(\mathbb{C})}\right)\right) + d\left(le\left(\left(h_{S_{O_{ij}}}^{1}\right)^{(\mathbb{C})}\right), le\left(\left(h_{S_{O_{ij}}}^{3}\right)^{(\mathbb{C})}\right)\right) + \cdots + d\left(le\left(\left(h_{S_{O_{ij}}}^{1}\right)^{(\mathbb{C})}\right), le\left(\left(h_{S_{O_{ij}}}^{n}\right)^{(\mathbb{C})}\right)\right) \tag{37}$$

Based on Eq. (32), we have

$$d\left(le\left(\left(h_{S_{O_{ij}}}^{1}\right)^{(\mathbb{C}+1)}\right), le\left(\left(h_{S_{O_{ij}}}^{2}\right)^{(\mathbb{C}+1)}\right)\right)$$

$$=\left|F\left(\lambda le\left(\left(h^{1}_{S_{O_{ij}}}\right)^{(\mathbb{C})}\right)+(1-\lambda)le\left(\left(h^{c}_{S_{O_{ij}}}\right)^{(\mathbb{C})}\right)\right)-F\left(\lambda le\left(\left(h^{2}_{S_{O_{ij}}}\right)^{(\mathbb{C})}\right)+(1-\lambda)le\left(\left(h^{2}_{S_{O_{ij}}}\right)^{(\mathbb{C})}\right)\right)\right|$$

$$=\left|\lambda F\left(le\left(\left(h^{1}_{S_{O_{ij}}}\right)^{(\mathbb{C})}\right)-le\left(\left(h^{2}_{S_{O_{ij}}}\right)^{(\mathbb{C})}\right)\right)\right.$$

$$\left.+\frac{(1-\lambda)}{n}\left(F\left(le\left(\left(h^{1}_{S_{O_{ij}}}\right)^{(\mathbb{C})}\right)-le\left(\left(h^{2}_{S_{O_{ij}}}\right)^{(\mathbb{C})}\right)\right)+F\left(le\left(\left(h^{2}_{S_{O_{ij}}}\right)^{(\mathbb{C})}\right)-le\left(\left(h^{2}_{S_{O_{ij}}}\right)^{(\mathbb{C})}\right)\right)+\cdots+F\left(le\left(\left(h^{c}_{S_{O_{ij}}}\right)^{(\mathbb{C})}\right)-le\left(\left(h^{2}_{S_{O_{ij}}}\right)^{(\mathbb{C})}\right)\right)\right)\right|$$

$$<\lambda d\left(le\left(\left(h^{1}_{S_{O_{ij}}}\right)^{(\mathbb{C})}\right),le\left(\left(h^{2}_{S_{O_{ij}}}\right)^{(\mathbb{C})}\right)\right)+\frac{(1-\lambda)}{n}\left(d\left(le\left(\left(h^{1}_{S_{O_{ij}}}\right)^{(\mathbb{C})}\right),le\left(\left(h^{2}_{S_{O_{ij}}}\right)^{(\mathbb{C})}\right)\right)+\cdots+d\left(le\left(\left(h^{m}_{S_{O_{ij}}}\right)^{(\mathbb{C})}\right),le\left(\left(h^{2}_{S_{O_{ij}}}\right)^{(\mathbb{C})}\right)\right)\right)$$

Because the consensus degree between e^1 and e^2 are smallest, then we obtain

$$d\left(le\left(\left(h^{1}_{S_{O_{ij}}}\right)^{(\mathbb{C}+1)}\right),le\left(\left(h^{2}_{S_{O_{ij}}}\right)^{(\mathbb{C}+1)}\right)\right)<\left(\lambda+\frac{(1-\lambda)(n-1)}{n}\right)d\left(le\left(\left(h^{1}_{S_{O_{ij}}}\right)^{(\mathbb{C})}\right),le\left(\left(h^{2}_{S_{O_{ij}}}\right)^{(\mathbb{C})}\right)\right)$$

Therefore, we have

$$d\left(le\left(\left(h^{1}_{S_{O_{ij}}}\right)^{(\mathbb{C}+1)}\right),le\left(\left(h^{2}_{S_{O_{ij}}}\right)^{(\mathbb{C}+1)}\right)\right)+\cdots+d\left(le\left(\left(h^{1}_{S_{O_{ij}}}\right)^{(\mathbb{C}+1)}\right),le\left(\left(h^{n}_{S_{O_{ij}}}\right)^{(\mathbb{C}+1)}\right)\right)$$

$$<\left(\lambda d\left(le\left(\left(h^{1}_{S_{O_{ij}}}\right)^{(\mathbb{C})}\right),le\left(\left(h^{2}_{S_{O_{ij}}}\right)^{(\mathbb{C})}\right)\right)+\frac{(1-\lambda)}{n}\left(d\left(le\left(\left(h^{1}_{S_{O_{ij}}}\right)^{(\mathbb{C})}\right),le\left(\left(h^{2}_{S_{O_{ij}}}\right)^{(\mathbb{C})}\right)\right)+\cdots+d\left(le\left(\left(h^{m}_{S_{O_{ij}}}\right)^{(\mathbb{C})}\right),le\left(\left(h^{2}_{S_{O_{ij}}}\right)^{(\mathbb{C})}\right)\right)\right)\right)$$

$$+\cdots$$

$$+\left(\lambda d\left(le\left(\left(h^{1}_{S_{O_{ij}}}\right)^{(\mathbb{C})}\right),le\left(\left(h^{n}_{S_{O_{ij}}}\right)^{(\mathbb{C})}\right)\right)+\frac{(1-\lambda)}{n}\left(d\left(le\left(\left(h^{1}_{S_{O_{ij}}}\right)^{(\mathbb{C})}\right),le\left(\left(h^{n}_{S_{O_{ij}}}\right)^{(\mathbb{C})}\right)\right)+\cdots+d\left(le\left(\left(h^{m}_{S_{O_{ij}}}\right)^{(\mathbb{C})}\right),le\left(\left(h^{n}_{S_{O_{ij}}}\right)^{(\mathbb{C})}\right)\right)\right)\right)$$

$$<\left(\left(\lambda+\frac{(1-\lambda)(n-1)}{n}\right)\left(d\left(le\left(\left(h^{1}_{S_{O_{ij}}}\right)^{(\mathbb{C})}\right),le\left(\left(h^{2}_{S_{O_{ij}}}\right)^{(\mathbb{C})}\right)\right)+\cdots+d\left(le\left(\left(h^{1}_{S_{O_{ij}}}\right)^{(\mathbb{C})}\right),le\left(\left(h^{n}_{S_{O_{ij}}}\right)^{(\mathbb{C})}\right)\right)\right)\right)$$

$$<d\left(le\left(\left(h^{1}_{S_{O_{ij}}}\right)^{(\mathbb{C})}\right),le\left(\left(h^{2}_{S_{O_{ij}}}\right)^{(\mathbb{C})}\right)\right)+\cdots+d\left(le\left(\left(h^{1}_{S_{O_{ij}}}\right)^{(\mathbb{C})}\right),le\left(\left(h^{n}_{S_{O_{ij}}}\right)^{(\mathbb{C})}\right)\right).$$

which completes the proof of Theorem 3.1.

For the final issue, some parameters need to be determined such as the given consensus threshold value ξ, and the number of iteration, denoted by CT. Xu et al. [32] analyzed these two parameters and obtained that it is reasonable to set ξ to fall within the interval [0.7386, 0.85], and the maximum number of iterations may belong to [0, 6]. In this paper, we can also determine two kinds of parameters in these two intervals respectively. However, both of them only are the references and the final values of them must be combined with the practical decision-making problem.

3.4. An LSGDM model with DHHFLPRs

In an LSGDM, a moderator is invited to give the revision suggestions to the decision makers and guide them to modify their preference information. Then an LSGDM model with DHHFLPRs can be shown as follows:

Step 1. Check whether all decision makers' preference information reaches the given consensus threshold based on Eqs. (21-23). If so, go to Step 5, else go to Step 2.

Step 2. Cluster all decision makers into several categories based on Subsection 3.1 (This step only happens in the first time of consensus reaching process). Then go to Step 3.

Step 3. Identify the alternatives, the pairs of alternatives and the decision makers that need to improve their consensus degrees on the basis of Eqs. (24-29). The moderator feeds the above two kinds of information to all groups, then every group conducts a discussion. Every group can discuss and change the corresponding preference information based on Remark 3.2. Then go to Step 4.

Step 4. Collect all modified evaluation information of each group and go back to Step 1.

Step 5. Calculate all decision makers' weights and obtain the final group DHHFLPR. Then we obtain the synthetical value of each alternative and the ranking order.

Step 6. End.

This LSGDM model with DHHFLPRs can be shown in Fig. 4.

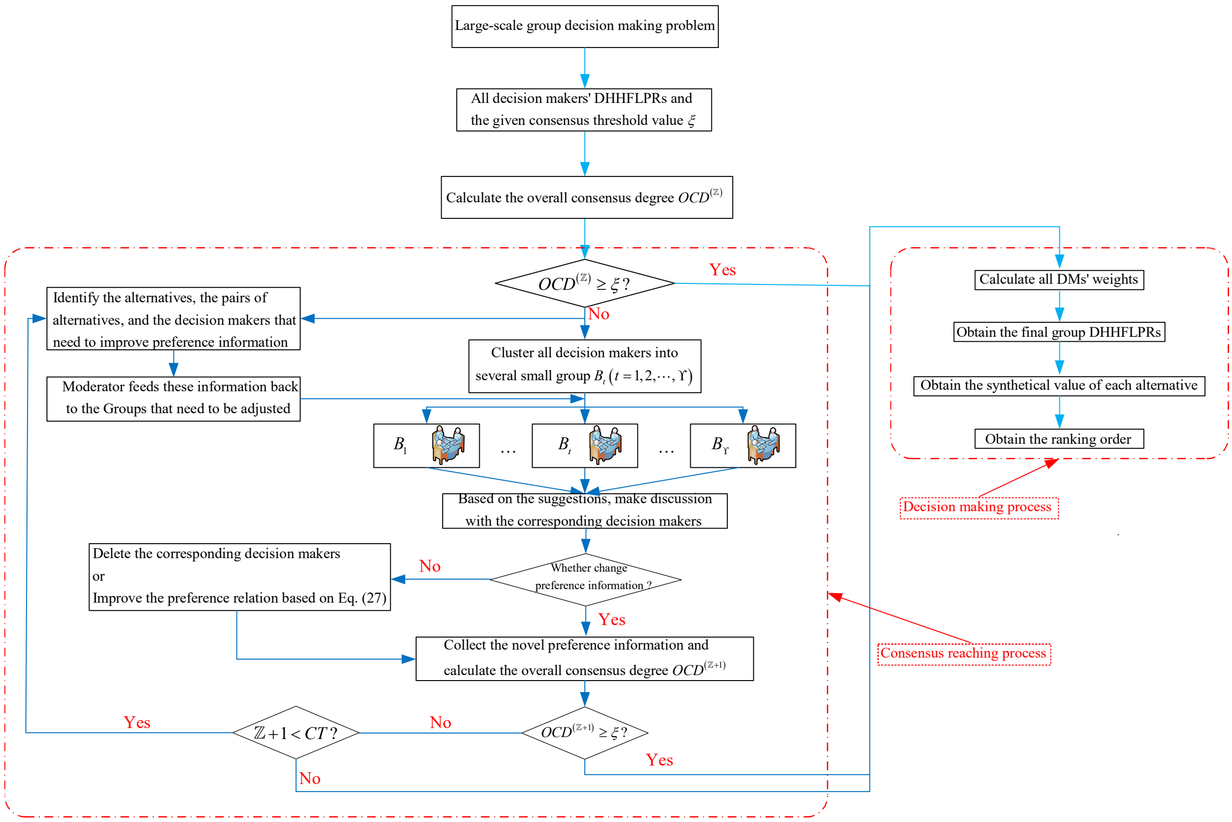

Fig. 4. LSGDM model with DHHFLPRs.

4. The case study: Water resource management

In this section, we make an overview about the current situation of water resource management, and summarize some implementation opinions introduced by Sichuan province to take the strict water resources management. Then we apply our method to deal with a practical LSGDM problem about water resource management.

4.1. Water resource management

In China, the state council's opinions on the implementation of the strictest water resources management system was promulgated. According to the practical situation, Sichuan province introduced the following implementation opinions to take the strict water resources management as a strategic move for accelerating the transformation of economic development mode:

(1) Establish a total water control system. This measure mainly contains implementing the total amount of water control, strengthening water resources development and utilization management, strict water intaking permits, strengthening the unified deployment of water resources, strict groundwater management and protection, and strengthening the collection and use of water resources expenditure, etc.

(2) Establish water efficiency control system. This measure mainly contains accelerating the development of water-saving society, enhancing water management, and strengthening the oversight and management of water saving, etc.

(3) Establish water functional area to restrict the pollution system. This measure mainly contains strict water function area management, strengthening the pollution discharge outlets of rivers management, strengthening water conservation, strengthening the protection of drinking water, and carrying out pilot and creation of water ecological civilization.

(4) Promote the comprehensive implementation of the most stringent water resources management system. This measure mainly contains strengthening the leadership of water resources management, establishing water resources management responsibility and examination system, improving the investment mechanism of water resources management, enhancing the team construction, improving the system and strengthening supervision.

Obviously, each policy discussed above is an important measure and all of them can be used to take the strict water resources management more efficiently. Therefore, in order to evaluate the implementation status of the above policies, a review meeting is hold and 20 decision makers $E=\left\{e^1,e^2,\ldots,e^{20}\right\}$ are invited to provide their preference information about the evaluations of four important cities: Chengdu (A_1), Panzhihua (A_2), Liangshan (A_3), and Nanchong (A_4). Let $S_O = \left\{ s_{t<o_k>} \middle| t = -4, \ldots, -1, 0, 1, \ldots, 4; k = -4, \ldots, -1, 0, 1, \ldots, 4 \right\}$ be a DHLTS with

$$S = \{ s_{-4} = extremely\ bad, s_{-3} = very\ bad, s_{-2} = bad, s_{-1} = slightly\ bad, s_0 = medium, s_1 = slightly\ good, s_2 = good, s_3 = very\ good, s_4 = extremely\ good \}$$

$$O = \{ o_{-4} = far\ from, o_{-3} = scarcely, o_{-2} = only\ a\ little, o_{-1} = a\ little, o_0 = just\ right, o_1 = much, o_2 = very\ much, o_3 = extremely\ much, o_4 = entirely \}$$

Then the decision makers provide their evaluations with linguistic information, we collect these linguistic information and transform them into DHHFLEs, which can be contained in the following DHHFLPRs $\tilde{H}^r_{S_O}\left(r = 1, 2, \ldots, 20\right)$:

$$\tilde{H}^{1}_{S_O} = \begin{pmatrix} \{s_{0<o_0>}\} & \{s_{0<o_1>}\} & \{s_{1<o_{-2}>}, s_{1<o_2>}\} & \{s_{-1<o_{-1}>}\} \\ \{s_{0<o_{-1}>}\} & \{s_{0<o_0>}\} & \{s_{0<o_0>}\} & \{s_{-1<o_1>}\} \\ \{s_{-1<o_2>}, s_{-1<o_{-2}>}\} & \{s_{0<o_0>}\} & \{s_{0<o_0>}\} & \{s_{2<o_{-3}>}\} \\ \{s_{1<o_1>}\} & \{s_{1<o_{-1}>}\} & \{s_{-2<o_3>}\} & \{s_{0<o_0>}\} \end{pmatrix} \quad \tilde{H}^{2}_{S_O} = \begin{pmatrix} \{s_{0<o_0>}\} & \{s_{1<o_{-2}>}, s_{2<o_{-2}>}\} & \{s_{1<o_{-2}>}\} & \{s_{0<o_2>}, s_{1<o_{-1}>}\} \\ \{s_{-1<o_2>}, s_{-2<o_2>}\} & \{s_{0<o_0>}\} & \{s_{-1<o_1>}, s_{0<o_{-1}>}\} & \{s_{-2<o_{-1}>}, s_{-1<o_2>}\} \\ \{s_{-1<o_2>}\} & \{s_{1<o_{-1}>}, s_{0<o_1>}\} & \{s_{0<o_0>}\} & \{s_{0<o_{-3}>}\} \\ \{s_{0<o_{-2}>}, s_{-1<o_1>}\} & \{s_{2<o_1>}, s_{1<o_{-2}>}\} & \{s_{0<o_3>}\} & \{s_{0<o_0>}\} \end{pmatrix}$$

$$\tilde{H}^{3}_{S_O} = \begin{pmatrix} \{s_{0<o_0>}\} & \{s_{0<o_{-1}>}\} & \{s_{1<o_{-1}>}\} & \{s_{-1<o_3>}\} \\ \{s_{0<o_1>}\} & \{s_{0<o_0>}\} & \{s_{0<o_2>}\} & \{s_{-2<o_1>}, s_{-1<o_1>}\} \\ \{s_{-1<o_1>}\} & \{s_{0<o_{-2}>}\} & \{s_{0<o_0>}\} & \{s_{2<o_{-1}>}, s_{3<o_{-1}>}\} \\ \{s_{1<o_{-3}>}\} & \{s_{2<o_{-1}>}, s_{1<o_{-1}>}\} & \{s_{-2<o_1>}, s_{-3<o_1>}\} & \{s_{0<o_0>}\} \end{pmatrix} \quad \tilde{H}^{4}_{S_O} = \begin{pmatrix} \{s_{0<o_0>}\} & \{s_{1<o_1>}\} & \{s_{2<o_2>}, s_{3<o_{-1}>}\} & \{s_{0<o_3>}\} \\ \{s_{-1<o_{-1}>}\} & \{s_{0<o_0>}\} & \{s_{1<o_{-1}>}, s_{2<o_1>}\} & \{s_{-1<o_2>}, s_{-1<o_3>}\} \\ \{s_{-2<o_{-2}>}, s_{-3<o_1>}\} & \{s_{-1<o_1>}, s_{-2<o_{-1}>}\} & \{s_{0<o_0>}\} & \{s_{-2<o_1>}, s_{-1<o_3>}\} \\ \{s_{0<o_{-3}>}\} & \{s_{1<o_{-2}>}, s_{1<o_{-3}>}\} & \{s_{2<o_{-1}>}, s_{1<o_{-3}>}\} & \{s_{0<o_0>}\} \end{pmatrix}$$

$$\tilde{H}^{5}_{S_O} = \begin{pmatrix} \{s_{0<o_0>}\} & \{s_{1<o_2>}, s_{2<o_2>}\} & \{s_{1<o_{-1}>}\} & \{s_{0<o_1>}, s_{1<o_1>}\} \\ \{s_{-1<o_{-2}>}, s_{-2<o_{-2}>}\} & \{s_{0<o_0>}\} & \{s_{-1<o_{-1}>}, s_{0<o_{-1}>}\} & \{s_{-2<o_{-1}>}, s_{-1<o_2>}\} \\ \{s_{-1<o_1>}\} & \{s_{1<o_1>}, s_{0<o_1>}\} & \{s_{0<o_0>}\} & \{s_{0<o_{-3}>}\} \\ \{s_{0<o_{-1}>}, s_{-1<o_{-1}>}\} & \{s_{2<o_1>}, s_{1<o_{-2}>}\} & \{s_{0<o_3>}\} & \{s_{0<o_0>}\} \end{pmatrix} \quad \tilde{H}^{6}_{S_O} = \begin{pmatrix} \{s_{0<o_0>}\} & \{s_{1<o_{-2}>}, s_{2<o_{-2}>}\} & \{s_{0<o_2>}\} & \{s_{1<o_{-1}>}, s_{2<o_{-3}>}\} \\ \{s_{-1<o_2>}, s_{-2<o_2>}\} & \{s_{0<o_0>}\} & \{s_{-1<o_1>}, s_{0<o_{-1}>}\} & \{s_{-2<o_{-1}>}, s_{-1<o_2>}\} \\ \{s_{0<o_{-2}>}\} & \{s_{1<o_{-1}>}, s_{0<o_1>}\} & \{s_{0<o_0>}\} & \{s_{0<o_{-3}>}\} \\ \{s_{-1<o_1>}, s_{-2<o_3>}\} & \{s_{2<o_1>}, s_{1<o_{-2}>}\} & \{s_{0<o_3>}\} & \{s_{0<o_0>}\} \end{pmatrix}$$

$$\tilde{H}^{7}_{S_O} = \begin{pmatrix} \{s_{0<o_0>}\} & \{s_{-1<o_1>}, s_{0<o_2>}\} & \{s_{0<o_2>}, s_{1<o_3>}\} & \{s_{-1<o_{-2}>}\} \\ \{s_{1<o_{-1}>}, s_{0<o_{-2}>}\} & \{s_{0<o_0>}\} & \{s_{0<o_{-1}>}, s_{1<o_1>}\} & \{s_{-2<o_1>}, s_{-1<o_2>}\} \\ \{s_{0<o_{-2}>}, s_{-1<o_{-3}>}\} & \{s_{0<o_1>}, s_{-1<o_{-1}>}\} & \{s_{0<o_0>}\} & \{s_{1<o_1>}, s_{2<o_3>}\} \\ \{s_{1<o_2>}\} & \{s_{2<o_{-1}>}, s_{1<o_{-2}>}\} & \{s_{-1<o_{-1}>}, s_{-2<o_{-3}>}\} & \{s_{0<o_0>}\} \end{pmatrix} \quad \tilde{H}^{8}_{S_O} = \begin{pmatrix} \{s_{0<o_0>}\} & \{s_{1<o_3>}\} & \{s_{-1<o_2>}, s_{-1<o_3>}\} & \{s_{-1<o_{-1}>}, s_{-1<o_{-1}>}\} \\ \{s_{1<o_3>}\} & \{s_{0<o_0>}\} & \{s_{-1<o_1>}\} & \{s_{-1<o_1>}\} \\ \{s_{1<o_{-2}>}, s_{1<o_{-3}>}\} & \{s_{-1<o_1>}\} & \{s_{0<o_0>}\} & \{s_{1<o_{-1}>}\} \\ \{s_{1<o_1>}, s_{1<o_1>}\} & \{s_{-1<o_1>}\} & \{s_{1<o_{-1}>}\} & \{s_{0<o_0>}\} \end{pmatrix}$$

$$\tilde{H}^{9}_{S_O} = \begin{pmatrix} \{s_{0<o_0>}\} & \{s_{1<o_{-1}>}\} & \{s_{2<o_2>}, s_{3<o_1>}\} & \{s_{0<o_3>}\} \\ \{s_{-1<o_1>}\} & \{s_{0<o_0>}\} & \{s_{1<o_{-1}>}, s_{2<o_1>}\} & \{s_{-1<o_3>}, s_{0<o_2>}\} \\ \{s_{-2<o_{-2}>}, s_{-3<o_{=1}>}\} & \{s_{-1<o_1>}, s_{-2<o_{-1}>}\} & \{s_{0<o_0>}\} & \{s_{-2<o_1>}, s_{-1<o_2>}\} \\ \{s_{0<o_{-3}>}\} & \{s_{1<o_{-3}>}, s_{0<o_{-2}>}\} & \{s_{2<o_{-1}>}, s_{1<o_{-2}>}\} & \{s_{0<o_0>}\} \end{pmatrix} \quad \tilde{H}^{10}_{S_O} = \begin{pmatrix} \{s_{0<o_0>}\} & \{s_{1<o_{-2}>}, s_{2<o_{-1}>}\} & \{s_{1<o_{-2}>}\} & \{s_{0<o_2>}, s_{1<o_1>}\} \\ \{s_{-1<o_2>}, s_{-2<o_1>}\} & \{s_{0<o_0>}\} & \{s_{-1<o_1>}, s_{0<o_{-1}>}\} & \{s_{-2<o_{-2}>}, s_{-1<o_2>}\} \\ \{s_{-1<o_2>}\} & \{s_{1<o_{-1}>}, s_{0<o_1>}\} & \{s_{0<o_0>}\} & \{s_{0<o_{-2}>}\} \\ \{s_{0<o_{-2}>}, s_{-1<o_{-1}>}\} & \{s_{2<o_2>}, s_{1<o_{-2}>}\} & \{s_{0<o_2>}\} & \{s_{0<o_0>}\} \end{pmatrix}$$

$$\tilde{H}^{11}_{S_O} = \begin{pmatrix} \{s_{0<o_0>}\} & \{s_{2<o_1>}\} & \{s_{-1<o_2>}, s_{-1<o_2>}\} & \{s_{-2<o_{-1}>}, s_{-1<o_{-2}>}\} \\ \{s_{-2<o_{-1}>}\} & \{s_{0<o_0>}\} & \{s_{0<o_{-1}>}, s_{0<o_1>}\} & \{s_{-1<o_0>}, s_{-1<o_1>}\} \\ \{s_{1<o_{-2}>}, s_{1<o_{-2}>}\} & \{s_{0<o_1>}, s_{0<o_{-1}>}\} & \{s_{0<o_0>}\} & \{s_{1<o_{-2}>}, s_{2<o_3>}\} \\ \{s_{2<o_1>}, s_{1<o_2>}\} & \{s_{1<o_0>}, s_{1<o_{-1}>}\} & \{s_{-1<o_2>}, s_{-2<o_{-3}>}\} & \{s_{0<o_0>}\} \end{pmatrix} \quad \tilde{H}^{12}_{S_O} = \begin{pmatrix} \{s_{0<o_0>}\} & \{s_{2<o_{-2}>}\} & \{s_{2<o_2>}, s_{3<o_{-1}>}\} & \{s_{0<o_2>}\} \\ \{s_{-2<o_2>}\} & \{s_{0<o_0>}\} & \{s_{1<o_{-1}>}, s_{2<o_{-1}>}\} & \{s_{-1<o_3>}, s_{0<o_2>}\} \\ \{s_{-2<o_{-2}>}, s_{-3<o_1>}\} & \{s_{-1<o_1>}, s_{-2<o_1>}\} & \{s_{0<o_0>}\} & \{s_{-2<o_1>}, s_{-1<o_3>}\} \\ \{s_{0<o_{-2}>}\} & \{s_{1<o_{-3}>}, s_{0<o_{-2}>}\} & \{s_{2<o_{-1}>}, s_{1<o_{-3}>}\} & \{s_{0<o_0>}\} \end{pmatrix}$$

$$\tilde{H}^{13}_{S_O} = \begin{pmatrix} \{s_{0<o_0>}\} & \{s_{1<o_{-2}>}, s_{2<o_{-2}>}\} & \{s_{1<o_{-2}>}\} & \{s_{0<o_2>}, s_{1<o_{-1}>}\} \\ \{s_{-1<o_2>}, s_{-2<o_2>}\} & \{s_{0<o_0>}\} & \{s_{-1<o_{-1}>}, s_{0<o_1>}\} & \{s_{-2<o_1>}, s_{-1<o_2>}\} \\ \{s_{-1<o_2>}\} & \{s_{1<o_1>}, s_{0<o_{-1}>}\} & \{s_{0<o_0>}\} & \{s_{0<o_{-3}>}\} \\ \{s_{0<o_{-2}>}, s_{-1<o_1>}\} & \{s_{2<o_{-1}>}, s_{1<o_{-2}>}\} & \{s_{0<o_3>}\} & \{s_{0<o_0>}\} \end{pmatrix} \quad \tilde{H}^{14}_{S_O} = \begin{pmatrix} \{s_{0<o_0>}\} & \{s_{1<o_3>}\} & \{s_{-1<o_2>}, s_{-1<o_3>}\} & \{s_{-1<o_{-1}>}\} \\ \{s_{-1<o_{-3}>}\} & \{s_{0<o_0>}\} & \{s_{-1<o_1>}\} & \{s_{-1<o_{-1}>}, s_{-1<o_3>}\} \\ \{s_{1<o_{-2}>}, s_{1<o_{-3}>}\} & \{s_{1<o_{-1}>}\} & \{s_{0<o_0>}\} & \{s_{1<o_{-3}>}, s_{1<o_3>}\} \\ \{s_{1<o_1>}\} & \{s_{1<o_1>}, s_{1<o_{-3}>}\} & \{s_{-1<o_3>}, s_{-1<o_{-3}>}\} & \{s_{0<o_0>}\} \end{pmatrix}$$

$$\tilde{H}^{15}_{S_O} = \begin{pmatrix} \{s_{0<o_0>}\} & \{s_{-1<o_{-1}>}\} & \{s_{1<o_{-2}>}, s_{2<o_1>}\} & \{s_{2<o_{-3}>}\} \\ \{s_{1<o_1>}\} & \{s_{0<o_0>}\} & \{s_{2<o_1>}, s_{3<o_{-1}>}\} & \{s_{3<o_3>}\} \\ \{s_{-1<o_2>}, s_{-2<o_{-1}>}\} & \{s_{-2<o_{-1}>}, s_{-3<o_1>}\} & \{s_{0<o_0>}\} & \{s_{-1<o_3>}\} \\ \{s_{-2<o_3>}\} & \{s_{-3<o_{-3}>}\} & \{s_{1<o_{-3}>}\} & \{s_{0<o_0>}\} \end{pmatrix} \quad \tilde{H}^{16}_{S_O} = \begin{pmatrix} \{s_{0<o_0>}\} & \{s_{1<o_1>}\} & \{s_{3<o_{-2}>}, s_{4<o_{-3}>}\} & \{s_{0<o_2>}\} \\ \{s_{-1<o_{-1}>}\} & \{s_{0<o_0>}\} & \{s_{1<o_0>}, s_{2<o_2>}\} & \{s_{-1<o_3>}, s_{0<o_2>}\} \\ \{s_{-3<o_2>}, s_{4<o_3>}\} & \{s_{-1<o_0>}, s_{-2<o_{-2}>}\} & \{s_{0<o_0>}\} & \{s_{-2<o_2>}, s_{-1<o_3>}\} \\ \{s_{0<o_{-2}>}\} & \{s_{1<o_{-3}>}, s_{0<o_{-2}>}\} & \{s_{2<o_{-2}>}, s_{1<o_{-3}>}\} & \{s_{0<o_0>}\} \end{pmatrix}$$

$$\tilde{H}_{S_O}^{17}=\begin{pmatrix} \{s_{0<o_0>}\} & \{s_{-1<o_2>},s_{0<o_1>}\} & \{s_{1<o_{-3}>},s_{2<o_3>}\} & \{s_{0<o_2>}\} \\ \{s_{1<o_{-2}>},s_{0<o_{-1}>}\} & \{s_{0<o_0>}\} & \{s_{0<o_{-1}>},s_{1<o_1>}\} & \{s_{-2<o_1>},s_{-1<o_2>}\} \\ \{s_{-1<o_3>},s_{-2<o_{-3}>}\} & \{s_{0<o_1>},s_{-1<o_{-1}>}\} & \{s_{0<o_0>}\} & \{s_{2<o_{-2}>},s_{3<o_2>}\} \\ \{s_{0<o_{-2}>}\} & \{s_{2<o_{-1}>},s_{1<o_{-2}>}\} & \{s_{-2<o_2>},s_{-3<o_{-2}>}\} & \{s_{0<o_0>}\} \end{pmatrix} \quad \tilde{H}_{S_O}^{18}=\begin{pmatrix} \{s_{0<o_0>}\} & \{s_{1<o_1>}\} & \{s_{2<o_2>},s_{3<o_1>}\} & \{s_{0<o_3>}\} \\ \{s_{-1<o_{-1}>}\} & \{s_{0<o_0>}\} & \{s_{1<o_{-1}>},s_{2<o_{-1}>}\} & \{s_{-1<o_3>},s_{0<o_1>}\} \\ \{s_{-2<o_{-2}>},s_{-3<o_{-1}>}\} & \{s_{-1<o_1>},s_{-2<o_1>}\} & \{s_{0<o_0>}\} & \{s_{-2<o_1>},s_{-1<o_3>}\} \\ \{s_{0<o_{-3}>}\} & \{s_{1<o_{-3}>},s_{0<o_{-1}>}\} & \{s_{2<o_{-1}>},s_{1<o_{-3}>}\} & \{s_{0<o_0>}\} \end{pmatrix}$$

$$\tilde{H}_{S_O}^{19}=\begin{pmatrix} \{s_{0<o_0>}\} & \{s_{-1<o_0>}\} & \{s_{1<o_{-2}>},s_{2<o_{-1}>}\} & \{s_{2<o_{-3}>}\} \\ \{s_{1<o_0>}\} & \{s_{0<o_0>}\} & \{s_{2<o_1>},s_{3<o_1>}\} & \{s_{3<o_3>}\} \\ \{s_{-1<o_2>},s_{-2<o_1>}\} & \{s_{-2<o_{-1}>},s_{-3<o_{-1}>}\} & \{s_{0<o_0>}\} & \{s_{0<o_{-2}>}\} \\ \{s_{-2<o_3>}\} & \{s_{-3<o_{-3}>}\} & \{s_{0<o_2>}\} & \{s_{0<o_0>}\} \end{pmatrix} \quad \tilde{H}_{S_O}^{20}=\begin{pmatrix} \{s_{0<o_0>}\} & \{s_{2<o_2>}\} & \{s_{-1<o_2>},s_{0<o_{-1}>}\} & \{s_{-2<o_{-1}>},s_{-1<o_{-3}>}\} \\ \{s_{-2<o_{-2}>}\} & \{s_{0<o_0>}\} & \{s_{0<o_{-1}>},s_{0<o_{-1}>}\} & \{s_{-1<o_1>},s_{-1<o_2>}\} \\ \{s_{1<o_{-2}>},s_{0<o_1>}\} & \{s_{0<o_1>},s_{0<o_1>}\} & \{s_{0<o_0>}\} & \{s_{1<o_{-3}>},s_{2<o_3>}\} \\ \{s_{2<o_1>},s_{1<o_3>}\} & \{s_{1<o_{-1}>},s_{1<o_{-2}>}\} & \{s_{-1<o_3>},s_{-2<o_{-3}>}\} & \{s_{0<o_0>}\} \end{pmatrix}$$

4.2. Solving the LSGDM problem

Utilizing the model discussed in Subsection 3.4 to deal with this LSGDM problem:

Step 1. Based on Eqs. (21-23), check whether all decision makers reach the given consensus threshold. The consensus degrees of the pair of alternatives $cpda^{(0)}$, the alternatives $cda^{(0)}$ and the overall consensus degree of preference relations $ocd^{(0)}$ can be obtained:

$$cpda^{(0)}=\begin{pmatrix} 1 & 0.8535 & 0.8356 & 0.8497 \\ 0.8535 & 1 & 0.8413 & 0.8275 \\ 0.8356 & 0.8413 & 1 & 0.8160 \\ 0.8497 & 0.8275 & 0.8160 & 1 \end{pmatrix}, \quad cda^{(0)}=\{0.8463, 0.8408, 0.8310, 0.8311\} \text{ and}$$

$$ocd^{(0)}=0.8310.$$

In this LSGDM problem, the given consensus threshold is $\xi=0.85$ and $ocd^{(0)}<\xi$. So all decision makers do not reach group consensus and go to Step 2.

Step 2. Based on Subsection 3.1, we cluster all decision makers into several small groups. The clustering process can be shown as follows:

Firstly, based on Eq. (12), the overall similarity matrix $OSM=\left(osm^{ab}\right)_{20\times 20}$ is established:

$$\left(\begin{array}{cccccccccccccccccccc}
1.0000 & 0.8698 & 0.9219 & 0.8177 & 0.8646 & 0.8620 & 0.9505 & 0.9141 & 0.8099 & 0.8646 & 0.9036 & 0.8438 & 0.8750 & 0.9193 & 0.7318 & 0.8047 & 0.9010 & 0.8099 & 0.7318 & 0.8958 \\
 & 1.0000 & 0.8646 & 0.8802 & 0.9661 & 0.9922 & 0.8359 & 0.8776 & 0.8620 & 0.9844 & 0.8307 & 0.8177 & 0.9948 & 0.8724 & 0.7422 & 0.8620 & 0.8594 & 0.8724 & 0.7526 & 0.8281 \\
 & & 1.0000 & 0.8021 & 0.8411 & 0.8568 & 0.9557 & 0.8464 & 0.7943 & 0.8594 & 0.8568 & 0.8594 & 0.8646 & 0.8516 & 0.7370 & 0.7891 & 0.9583 & 0.7943 & 0.7370 & 0.8385 \\
 & & & 1.0000 & 0.8724 & 0.8724 & 0.7943 & 0.7943 & 0.9714 & 0.8750 & 0.7630 & 0.9115 & 0.8854 & 0.7891 & 0.7891 & 0.9714 & 0.8333 & 0.9818 & 0.7891 & 0.7604 \\
 & & & & 1.0000 & 0.9635 & 0.8125 & 0.8854 & 0.8438 & 0.8609 & 0.8385 & 0.8151 & 0.9609 & 0.8802 & 0.7240 & 0.8490 & 0.8359 & 0.8646 & 0.7344 & 0.8359 \\
 & & & & & 1.0000 & 0.8281 & 0.8698 & 0.8594 & 0.9870 & 0.8229 & 0.8099 & 0.9870 & 0.8646 & 0.7500 & 0.8542 & 0.8516 & 0.8698 & 0.7604 & 0.8203 \\
 & & & & & & 1.0000 & 0.8646 & 0.7865 & 0.8307 & 0.8854 & 0.8516 & 0.8411 & 0.8698 & 0.7240 & 0.7813 & 0.9401 & 0.7865 & 0.7240 & 0.8672 \\
 & & & & & & & 1.0000 & 0.7656 & 0.8776 & 0.9375 & 0.8099 & 0.8828 & 0.9948 & 0.6667 & 0.7813 & 0.8151 & 0.7865 & 0.6667 & 0.9401 \\
 & & & & & & & & 1.0000 & 0.8516 & 0.7344 & 0.9036 & 0.8672 & 0.7604 & 0.8021 & 0.9740 & 0.8255 & 0.9792 & 0.8021 & 0.7318 \\
 & & & & & & & & & 1.0000 & 0.8307 & 0.8177 & 0.9782 & 0.8724 & 0.7474 & 0.8516 & 0.8542 & 0.8672 & 0.7578 & 0.8281 \\
 & & & & & & & & & & 1.0000 & 0.8151 & 0.8359 & 0.9427 & 0.6354 & 0.7500 & 0.8255 & 0.7552 & 0.6354 & 0.9766 \\
 & & & & & & & & & & & 1.0000 & 0.8229 & 0.8151 & 0.7526 & 0.9193 & 0.8906 & 0.9193 & 0.7422 & 0.8073 \\
 & & & & & & & & & & & & 1.0000 & 0.8776 & 0.7474 & 0.8672 & 0.8646 & 0.8776 & 0.7578 & 0.8333 \\
 & & & & & & & & & & & & & 1.0000 & 0.6615 & 0.7760 & 0.8203 & 0.7813 & 0.6615 & 0.9453 \\
 & & & & & & & & & & & & & & 1.0000 & 0.7969 & 0.7578 & 0.7865 & 0.9792 & 0.6328 \\
 & & & & & & & & & & & & & & & 1.0000 & 0.8203 & 0.9792 & 0.7969 & 0.7474 \\
 & & & & & & & & & & & & & & & & 1.0000 & 0.8255 & 0.7474 & 0.8073 \\
 & & & & & & & & & & & & & & & & & 1.0000 & 0.7865 & 0.7526 \\
 & & & & & & & & & & & & & & & & & & 1.0000 & 0.6328 \\
 & & & & & & & & & & & & & & & & & & & 1.0000
\end{array}\right)$$

Additionally, we rank all different elements of the upper triangular matrix of OSM, and then calculate the rate of threshold shown in Table 4.1.

Table 4.1. The rate of threshold.

C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}
0.0013	0.0026	0.0052	0.0026	0.0007	0.0013	0.0104	0.0039	0.0026	0.0052	0.0312

Fig. 5 can be drawn to describe the cluster process:

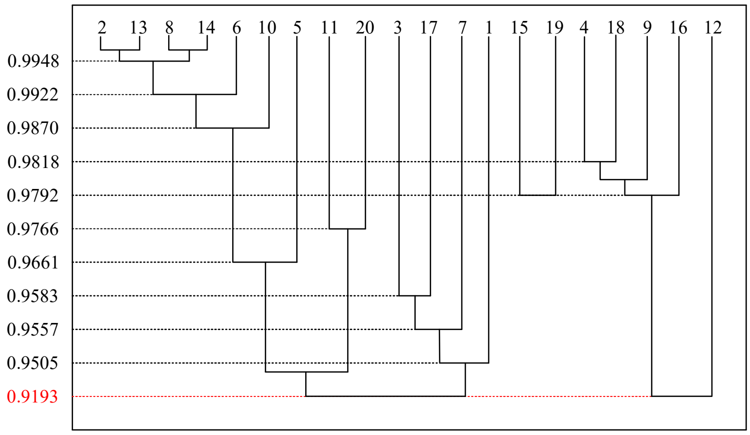

Fig. 5. The clustering process.

Therefore, all decision makers can be divided into three groups:

$$\{\{1,2,3,5,6,7,8,10,11,13,14,17,20\},\{15,19\},\{4,9,12,16,18\}\}$$

Then go to Step 3 and start the first round of consensus reaching process:

Round 1.

Step 3¹. Based on $cpda^{(0)}$, $cda^{(0)}$, $ocd^{(0)}$, and Eqs. (24-29), all alternatives need to be improved. In this round, we only discuss A_3 firstly. The decision makers and the parts of alternatives that need to repair their consensus degrees can be shown as follows:

(1) The decision makers e^9, e^{16} and e^{18} need to improve their preference information in pair of alternative $(1,3)$;

(2) The decision maker e^{19} needs to improve their preference information in pair of alternatives $(2,3)$;

(3) The decision maker e^{17} needs to improve their preference information in pair of alternatives $(3,4)$.

And then we calculate all decision makers' weights based on Eqs. (15-19):

$$w^{(1)} = (0.0654, 0.0657, 0.0654, 0.0253, 0.0661, 0.0657, 0.0654, 0.0657, 0.0253, 0.0658, 0.0657, 0.0251, 0.0657, 0.0657, 0.0101, 0.0253, 0.0655, 0.0253, 0.0101, 0.0658)^T.$$

Furthermore, based on Eq. (30), we can obtain the group DHHFLPR $\left(\tilde{H}_{S_O}^{c}\right)^{(1)}=\left(\left(h_{S_{O_{ij}}}^{c}\right)_{m\times m}\right)^{(1)}$:

$$\left(\tilde{H}_{S_O}^{c}\right)^{(1)}=\begin{pmatrix} \left\{s_{0<o_0>}\right\} & \left\{s_{0.95<o_{0.42}>}\right\} & \left\{s_{0.67<o_{0.28}>}\right\} & \left\{s_{-0.25<o_{0.14}>}\right\} \\ \left\{s_{-0.95<o_{-0.42}>}\right\} & \left\{s_{0<o_0>}\right\} & \left\{s_{0.01<o_{0.12}>}\right\} & \left\{s_{-1.13<o_{1.15}>}\right\} \\ \left\{s_{-0.67<o_{-0.28}>}\right\} & \left\{s_{-0.01<o_{-0.12}>}\right\} & \left\{s_{0<o_0>}\right\} & \left\{s_{0.76<o_{-0.82}>}\right\} \\ \left\{s_{0.25<o_{-0.14}>}\right\} & \left\{s_{1.13<o_{-1.15}>}\right\} & \left\{s_{-0.76<o_{0.82}>}\right\} & \left\{s_{0<o_0>}\right\} \end{pmatrix}$$

Then, the moderator feeds all information obtained in this round back to the three groups as a reference. Each group discusses whether adjusts the corresponding decision maker's evaluation information and how to adjust them. Finally, all corresponding experts agree to change and the changed information is listed below:

(1) For the pair of alternatives $(1,3)$, the decision maker e^9 decreases $\left\{s_{2<o_2>},s_{3<o_1>}\right\}$ into $\left\{s_{0<o_2>}\right\}$, e^{16} decreases $\left\{s_{3<o_{-2}>},s_{4<o_{-3}>}\right\}$ into $\left\{s_{1<o_{-1}>}\right\}$, and e^{18} decreases $\left\{s_{2<o_2>},s_{3<o_1>}\right\}$ into $\left\{s_{2<o_{-2}>}\right\}$;

(2) For the pair of alternatives $(2,3)$, the decision maker e^{19} decreases $\left\{s_{2<o_1>},s_{3<o_1>}\right\}$ into $\left\{s_{1<o_{-1}>}\right\}$;

(3) For the pair of alternatives $(3,4)$, the decision maker e^{17} decreases $\left\{s_{2<o_{-2}>},s_{3<o_2>}\right\}$ into $\left\{s_{0<o_{-1}>}\right\}$.

Step 4^1. Collect all modified evaluation information, and go back to Step 1. Check whether all decision makers' reach the given consensus threshold again. The consensus degrees for the pair of alternatives $cpda^{(1)}$, the alternatives $cda^{(1)}$ and $cdpr^{(1)}$ are obtained:

$$cpda^{(1)}=\begin{pmatrix} 1 & 0.8535 & 0.8780 & 0.8497 \\ 0.8535 & 1 & 0.8594 & 0.8275 \\ 0.8780 & 0.8594 & 1 & 0.8315 \\ 0.8497 & 0.8275 & 0.8315 & 1 \end{pmatrix},\quad cda^{(1)}=\left\{0.8604,0.8468,0.8563,0.8362\right\}\ \text{and}$$

$$ocd^{(1)}=0.8362.$$

Obviously, all decision makers still do not reach the given consensus threshold. Then we need to go to Step 4 again and start the second round of consensus reaching process:

Round 2:

Step 3^2. Based on $cpda^{(1)}$, $cda^{(1)}$ and $ocd^{(1)}$, and Eqs. (24-29), we need to adjust the alternatives A_2 and A_4. In this round, we only discuss A_4. The decision makers and the parts of alternatives need to improve their consensus degrees, which can be shown as follows:

(1) The decision maker e^{20} needs to improve his/her preference information in pair of alternatives $(1,4)$;

(2) The decision makers e^{15} and e^{19} need to improve their preference information in pair of alternatives $(2,4)$;

(3) The decision maker e^{3} needs to improve his/her preference information in pair of alternatives $(3,4)$,

And then, we calculate all decision makers' weights again:

$$w^{(2)} = (0.0654, 0.0657, 0.0654, 0.0254, 0.0661, 0.0657, 0.0654, 0.0657, 0.0251, 0.0658, 0.0657, 0.0253, 0.0657, 0.0657, 0.0102, 0.0252, 0.0656, 0.0252, 0.0100, 0.0658)^T.$$

Then we can obtain the group DHHFLPR $\left(\tilde{H}^c_{S_O}\right)^{(2)} = \left(\left(h^c_{S_{O_{ij}}}\right)_{m\times m}\right)^{(2)}$:

$$\left(\tilde{H}^c_{S_O}\right)^{(2)} = \begin{pmatrix} \{s_{0<o_0>}\} & \{s_{0.95<o_{0.42}>}\} & \{s_{0.53<o_{0.24}>}\} & \{s_{-0.25<o_{0.14}>}\} \\ \{s_{-0.95<o_{-0.42}>}\} & \{s_{0<o_0>}\} & \{s_{-0.01<o_{0.09}>}\} & \{s_{-1.13<o_{1.15}>}\} \\ \{s_{-0.53<o_{-0.24}>}\} & \{s_{0.01<o_{-0.09}>}\} & \{s_{0<o_0>}\} & \{s_{0.6<o_{-0.89}>}\} \\ \{s_{0.25<o_{-0.14}>}\} & \{s_{1.13<o_{-1.15}>}\} & \{s_{-0.6<o_{0.89}>}\} & \{s_{0<o_0>}\} \end{pmatrix}$$

In this round, the decision maker e^{20} disagree to change, therefore, this group discuss and decide to improve his preference relation based on Eq. (32) randomly. The rest corresponding experts agree to change and the changed information is listed below:

(1) For the pair of alternatives $(1,4)$, the decision maker e^{20} increases $\{s_{-2<o_{-1}>}, s_{-1<o_{-3}>}\}$ into $\{s_{-1<o_3>}\}$;

(2) For the pair of alternatives $(2,4)$, the decision maker e^{15} decreases $\{s_{3<o_3>}\}$ into $\{s_{-1<o_3>}\}$, and e^{19} decreases $\{s_{3<o_3>}\}$ into $\{s_{2<o_1>}\}$;

(3) For the pair of alternatives $(3,4)$, the decision maker e^{3} decreases $\{s_{2<o_{-1}>}\}$ into $\{s_{1<o_2>}\}$.

Step 4[2]. Collect all evaluation information, and go back to Step 1. Then the consensus degrees for the pair of alternatives $cpda^{(2)}$, the alternatives $cda^{(2)}$ and $ocd^{(2)}$ are obtained:

$$cpda^{(2)}=\begin{pmatrix} 1 & 0.8535 & 0.8780 & 0.8638 \\ 0.8535 & 1 & 0.8594 & 0.8505 \\ 0.8780 & 0.8594 & 1 & 0.8394 \\ 0.8638 & 0.8505 & 0.8394 & 1 \end{pmatrix}, \quad cda^{(2)}=\{0.8651,0.8545,0.8589,0.8512\} \text{ and}$$

$$ocd^{(2)}=0.8512.$$

Obviously, we obtain $ocd^{(2)}=0.8512>0.85$. Therefore, all decision makers reach the given consensus threshold. Then go to Step 5.

Step 5. Calculate all decision makers' weights

$$w^{*}=(0.0654,0.0657,0.0653,0.0254,0.0661,0.0657,0.0654,0.0657,0.0251,0.0658,0.0657,0.0253,0.0657, 0.0657,0.0102,0.0252,0.0656,0.0252,0.01,0.0658)^{T}$$

and obtain the final group DHHFLPR:

$$*\tilde{H}_{S_O}^{c}=\begin{pmatrix} \{s_{0<o_0>}\} & \{s_{0.95<o_{0.42}>}\} & \{s_{0.53<o_{0.24}>}\} & \{s_{-0.22<o_{0.46}>}\} \\ \{s_{-0.95<o_{-0.42}>}\} & \{s_{0<o_0>}\} & \{s_{0.01<o_{0.1}>}\} & \{s_{-1.14<o_{1.11}>}\} \\ \{s_{-0.53<o_{-0.24}>}\} & \{s_{-0.01<o_{-0.1}>}\} & \{s_{0<o_0>}\} & \{s_{0.5<o_{-0.69}>}\} \\ \{s_{0.22<o_{-0.46}>}\} & \{s_{1.14<o_{-1.11}>}\} & \{s_{-0.5<o_{0.69}>}\} & \{s_{0<o_0>}\} \end{pmatrix}$$

Then the synthetical value of each alternative is $SV(A)=\{2.1926,1.7622,1.9649,2.0803\}$. Therefore, the ranking order is $A_1 \succ A_4 \succ A_3 \succ A_2$. We can get the result that Chengdu is the optimal city in the process of the implementation status evaluations of the water resources management policies.

Step 6. End.

4.3. Comparison analyses

We can transform all DHHFLPRs into HFLPRs by deleting the second hierarchy linguistic information of all DHHFLEs. Then we deal with this LSGDM problem based on the model discussed in this paper.

Step 1. Calculate the consensus degrees of the pair of alternatives $cpda'^{(0)}$, the alternatives $cda'^{(0)}$ and the overall consensus degree of preference relations $ocd'^{(0)}$:

$$cpda'^{(0)} = \begin{pmatrix} 1 & 0.8635 & 0.8171 & 0.8536 \\ 0.8635 & 1 & 0.8421 & 0.8579 \\ 0.8171 & 0.8421 & 1 & 0.8059 \\ 0.8536 & 0.8579 & 0.8059 & 1 \end{pmatrix}, \quad cda'^{(0)} = \{0.8447, 0.8545, 0.8217, 0.8391\} \text{ and}$$

$$ocd'^{(0)} = 0.8217 < 0.85.$$

So the decision makers do not reach group consensus and go to Step 2.

Step 2. Cluster all decision makers into several small groups:

$$\{\{1,3,7,17\},\{2,5,6,10,13\},\{4,9,12,16,18\},\{8,11,14,20\},\{15,19\}\}$$

Then go to Step 3 and start the first round of consensus reaching process:

Round 1.

Step 3[1]. Based on $cpda'^{(0)}$, $cda'^{(0)}$, $ocd'^{(0)}$, the alternative A_3 needs to be improved in this round. The decision makers and the parts of alternatives that need to repair their consensus degrees can be obtained and the modified results can be shown as follows:

(1) The decision maker e^{16} needs to improve their preference information in pair of alternatives $(1,3)$ by decreasing $\{s_3, s_4\}$ into $\{s_2\}$;

(2) The decision maker e^{15} needs to improve their preference information in pair of alternatives $(2,3)$ by decreasing $\{s_2, s_3\}$ into $\{s_1\}$;

(3) The decision maker e^{17} needs to improve their preference information in pair of alternatives $(3,4)$ by increasing $\{s_{-2}, s_{-3}\}$ into $\{s_{-1}\}$.

Step 4[1]. Collect all modified evaluation information, and go back to Step 1. The consensus degrees for the pair of alternatives $cpda'^{(1)}$, the alternatives $cda'^{(1)}$ and $cdpr'^{(1)}$ are obtained:

$$cpda'^{(1)} = \begin{pmatrix} 1 & 0.8635 & 0.8332 & 0.8536 \\ 0.8635 & 1 & 0.8556 & 0.8579 \\ 0.8332 & 0.8556 & 1 & 0.8188 \\ 0.8536 & 0.8579 & 0.8188 & 1 \end{pmatrix}, \quad cda'^{(1)} = \{0.8501, 0.8590, 0.8359, 0.8434\} \text{ and}$$

$$ocd'^{(1)} = 0.8359.$$

Obviously, all decision makers still do not reach the given consensus threshold. Then we need to go to Step 3 again and start the second round of consensus reaching process:

Round 2:

Step 3[2]. In this round, we also discuss A_3. Then we obtain

(1) The decision makers e^{8}, e^{11} and e^{14} need to improve their preference information in pair

of alternatives $(1,3)$. e^8 increases $\{s_{-1}\}$ into $\{s_0\}$, e^{11} increases $\{s_{-1}\}$ into $\{s_1\}$, and e^{14} increases $\{s_{-1}\}$ into $\{s_1\}$,

(2) The decision maker e^3 needs to improve their preference information in pair of alternatives $(3,4)$ by decreasing $\{s_2, s_3\}$ into $\{s_1\}$.

Step 4². Collect all modified evaluation information, and go back to Step 1. The consensus degrees for the pair of alternatives $cpda'^{(2)}$, the alternatives $cda'^{(2)}$ and $cdpr'^{(2)}$ are obtained:

$$cpda'^{(1)} = \begin{pmatrix} 1 & 0.8635 & 0.8793 & 0.8536 \\ 0.8635 & 1 & 0.8556 & 0.8579 \\ 0.8793 & 0.8556 & 1 & 0.8336 \\ 0.8536 & 0.8579 & 0.8336 & 1 \end{pmatrix}, \quad cda'^{(1)} = \{0.8655, 0.8590, 0.8561, 0.8484\} \text{ and}$$

$$ocd'^{(2)} = 0.8484.$$

All decision makers also do not reach the given consensus threshold. Then we need to go to Step 3 again and start the third round of consensus reaching process:

Round 3:

Step 3³. In this round, we need to discuss A_4. Then we obtain that the decision makers e^1, e^4, e^9, e^{16} and e^{18} need to improve their preference information in pair of alternatives $(3,4)$. e^1 decreases $\{s_2\}$ into $\{s_1\}$, e^4 increases $\{s_{-2}, s_{-1}\}$ into $\{s_0\}$, e^9 increases $\{s_{-2}, s_{-1}\}$ into $\{s_{-1}\}$, e^{16} increases $\{s_{-1}, s_0\}$ into $\{s_0\}$, and $\{e^{18}\}$ increases $\{s_{-2}, s_{-1}\}$ into $\{s_0\}$.

Step 4². Collect all modified evaluation information, and go back to Step 1. The consensus degrees for the pair of alternatives $cpda'^{(2)}$, the alternatives $cda'^{(2)}$ and $cdpr'^{(2)}$ are obtained:

$$cpda'^{(3)} = \begin{pmatrix} 1 & 0.8635 & 0.8793 & 0.8536 \\ 0.8635 & 1 & 0.8556 & 0.8579 \\ 0.8793 & 0.8556 & 1 & 0.8852 \\ 0.8536 & 0.8579 & 0.8852 & 1 \end{pmatrix}, \quad cda'^{(3)} = \{0.8655, 0.8590, 0.8734, 0.8656\} \text{ and}$$

$$ocd'^{(3)} = 0.8590.$$

Step 5. Calculate all decision makers' weights and obtain the final group DHHFLPR. Then the synthetical value of each alternative is $SV(A) = \{2.2886, 1.7522, 1.8648, 2.0944\}$. Therefore, the ranking order also is $A_1 \succ A_4 \succ A_3 \succ A_2$. We can also get the result that Chengdu is the optimal city in the process of the implementation status evaluations of the water resources management policies.

Step 6. End.

4.4. Discussion

Some comparative analyses can be shown as follows:

Firstly, based on the consensus reaching processes and the decision making results discussed in Subsection 4.2 and Subsection 4.3, some analyses are summarized as follows:

a) It is obvious that some preference information will be lost if we transform DHHFLPRs into HFLPRs by deleting the second hierarchy linguistic information of all DHHFLEs. Therefore, even though both methods obtain the same decision making result $A_1 \succ A_4 \succ A_3 \succ A_2$, the DHHFLPRs can describe linguistic information more correctly than HFLPRs in this LSGDM problem.

b) Because the linguistic information is changed by this transformation, we get different clustering results in these two subsections. The clustering result in Subsection 4.3 is more decentralized than that in Subsection 4.2. The main reason is that the diversification among all HFLPRs is low and some preferences are very similar but have great differences with other's category.

c) Clearly, in these two methods, the decision makers who need to repair preference information are different, and the numbers of iterations are also different, which is related to the linguistic information transformation.

Secondly, for the clustering method, we utilize the information entropy to cluster the decision makers. The main advantages are listed as follows: (1) By utilizing the rate of threshold change to determine the optimal classification threshold, our method can give a reasonable clustering for some decision makers with the high similarity degrees. (2) Our method can make the clustering process clearer by the dynamic description with a clustering figure.

Of course, there are some other clustering methods, such as the k-means clustering method [30], the fuzzy c-mean clustering method [22], and the interval type-2 fuzzy equivalence clustering analysis [29], etc. The main shortcoming of k-means clustering method and fuzzy c-mean algorithm is the selections of the cluster centers K and N respectively considering that there is no any theoretic guidelines for setting K and N. Therefore, these two methods will waste lots of time on this point.

Thirdly, compared with other weights-determining methods [9,11,13,17,26,33,38,39], the developed double hierarchy information entropy-based weights-determining method can be used to obtain three kinds of weight information: the weight of each group, the weights of decision makers included in each group, and the weights of all decision makers. Therefore, we have great flexibility to choose different weights when dealing with some particular problems. Additionally, this method is very simple and reasonable, so we can save lots of time in this stage.

Finally, in the consensus reaching process, we choose to do only one clustering process at the beginning of improving consensus degree. However, it is clear that the clustering may be changed when we finish every round of consensus degree improving. But our choices have two advantages: 1) If we do not change the cluster result, the decision makers included in each group can know each other better and then they can finish the consensus reaching process more efficiently; 2) On the contrary, if we cluster the decision makers at each round, then the decision makers included in every group need to know each other again and again. This process will waste lots of time.

5. Conclusions and future research directions

In this paper, we have discussed the consensus reaching processes for LSGDM with DHHFLPRs. The main contributions and innovations can be summarized: 1) We have developed a consensus reaching process for dealing with the LSGDM problems with DHHFLPRs. 2) We have proposed some novel methods including the similarity degree-based clustering method, the double hierarchy information entropy-based weights-determining method, the consensus measures, and the LSGDM model for dealing with the LSGDM problems with DHHFLPRs information. 3) We have applied our method to a practical LSGDM problem that is to evaluate Sichuan water resource management, and we have made comparative analyses with some existing methods.

However, there still exist some shortcomings in this paper:

(1) When we need to collect all preference information together, it is very common that there will exist some same linguistic terms in a set. Therefore, maybe we will lose these important information. Next, one kind of DHLTS with probability needs to be studied.

(2) In the consensus reaching process, it is necessary to study some methods for managing the minority opinions and noncooperative behaviors.

In the future, we need to deeply discuss the consensus reaching process with DHHFLPRs in some practical LSGDM problems. For example, it is necessary to discuss the situation about the uncooperative decision makers, establish some novel clustering methods, and develop some methods to deal with incomplete DHHFLPRs, etc.

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References

[1] N. Capuano, F. Chiclana, H. Fujita, E. Herrera-Viedma, V. Loia, Fuzzy group decision making with incomplete information guided by social influence, IEEE Transaction on Fuzzy Systems 26(3) (2018) 1704-1718.

[2] Y.C. Dong, C.C. Li, F. Herrera, Connecting the linguistic hierarchy and the numerical scale for the 2-tuple linguistic model and its use to deal with hesitant unbalanced linguistic information, Information Sciences 367-368 (2016) 259-278.

[3] D. Dubois, The role of fuzzy sets indecision sciences: Old techniques and new directions, Fuzzy Sets and Systems 184 (2011)3-28.

[4] X.J. Gou, H.C. Liao, Z.S. Xu, R. Min, F. Herrera, Group decision making with double hierarchy hesitant fuzzy linguistic preference relations: consistency based measures, index and repairing algorithms and decision model, Technical report (2018).

[5] X.J. Gou, Z.S. Xu, H.C. Liao, F. Herrera, Double hierarchy hesitant fuzzy linguistic MULTIMOORA method for evaluating the implementation status of haze controlling measures, Information Fusion 38 (2017) 22-34.

[6] Z.N. Hao, Z.S. Xu, H. Zhao, H. Fujita, A dynamic weight determination approach based on the intuitionistic fuzzy Bayesian network and its application to emergency decision making, IEEE Transaction on Fuzzy Systems 26(4) (2018) 1893-1907.

[7] F. Herrera, L. Martínez, A 2-tuple fuzzy linguistic representation model for computing with words, IEEE Transactions on Fuzzy Systems 8 (2000) 746-752.

[8] E. Herrera-Viedma, F.J. Cabrerizo, J. Kacprzy, W. Pedrycz, A review of soft consensus models in a fuzzy environment, Information Fusion 17 (2014) 4-13.

[9] H.M. Hsu, C.T. Chen, Aggregation of fuzzy opinions under group decision making, Fuzzy Sets and Systems 79(3) (1996) 279-285.

[10] Á. Labella, Y. Liu, R.M. Rodríguez, L. Martínez. Analyzing the performance of classical consensus models in largescale group decision making: A comparative study, Applied Soft Computing 67 (2018) 677-690.

[11] H.S. Lee, Optimal consensus of fuzzy opinions under group decision making environment, Fuzzy Sets and Systems 132(3) (2000) 303-315.

[12] C.C. Li, Y.C. Dong, F. Herrera, E. Herrera-Viedma, L. Martínez, Personalized individual semantics in computing with words for supporting linguistic group decision making. An application on consensus reaching, Information Fusion 33 (2017) 29-40.

[13] H. Linstone, M. Turoff, The Delphi Method: Techniques and Applications, Addison-Wesley

Publishing Company, London, 1975.

[14] B.S. Liu, T.F. Huo, P.C. Liao, J. Gong, B. Xue, A group decision-making aggregation model for contractor selection in large scale construction projects based on two-stage partial least squares (PLS) path modeling, Group Decision and Negotiation 24 (2015) 855-883.

[15] B.S. Liu, Y.H. Shen, X.H. Chen, Y. Chen, X.Q. Wang, A partial binary tree DEA-DA cyclic classification model for decision makers in complex multi-attribute large-group interval-valued intuitionistic fuzzy decision making problems, Information Fusion 18(1) (2014) 119-130.

[16] B.S. Liu, Y.H. Shen, X.H. Chen, H. Sun, Y. Chen, A complex multi-attribute large-group PLS decision-making method in the interval-valued intuitionistic fuzzy environment, Applied Mathematical Modelling 38 (2014) 4512-4527.

[17] B.S. Liu, Y.H. Shen, Y. Chen, X.H. Chen, Y.M. Wang, A two-layer weight determination method for complex multi-attribute large-group decision making experts in a linguistic environment, Information Fusion 23 (2015) 156-165.

[18] B.S. Liu, Y.H. Shen, W. Zhang, X.H. Chen, X.Q. Wang, An interval-valued intuitionistic fuzzy principal component analysis model-based method for complex multi-attribute large-group decision-making, European Journal of Operational Research 245(1) (2015) 209-225.

[19] R. Liu, X.H. Chen, Improved clustering algorithm and its application in complex huge group decision-making, Systems Engineering & Electronics 28 (2006) 1695-1699.

[20] Y. Liu, Z.P. Fan, X. Zhang, A method for large group decision making based on evaluation information provided by participators from multiple groups, Information Fusion 29(C) (2016) 132-141.

[21] I. Palomares, F.J. Estrella, L. Martínez. F. Herrera, Consensus under a fuzzy context: Taxonomy, analysis framework AFRYCA and experimental case of study, Information Fusion 20 (2014) 252-271.

[22] I. Palomares, L. Martínez, F. Herrera, A consensus model to detect and manage noncooperative behaviors in large-scale group decision making, IEEE Transactions on Fuzzy Systems 22(3) (2014) 516-530.

[23] I. Palomares, L. Martínez, F. Herrera, MENTOR: A graphical monitoring tool of preferences evolution in large-scale group decision making, Knowledge-Based Systems 58 (2014) 66-74.

[24] Q. Pang, H. Wang, Z.S. Xu, Probabilistic linguistic term sets in multi-attribute group decision making, Information Sciences 369 (2016) 128-143.

[25] F.J. Quesada, I. Palomares, L. Martínez, Managing experts behavior in large-scale consensus reaching processes with uninorm aggregation operators, Applied Soft Computing 35 (2015) 873-887.

[26] R. Ramanathan, L.S. Ganesh, Group preference aggregation methods employed in AHP: an evaluation and an intrinsic process for deriving members' weightages, European Journal of Operational Research 79(2) (1994) 249-265.
[27] S. Saint, J.R. Lawson, Rules for reaching consensus: A modern approach to decision making, Pfeiffer & Company, San Diego (Jossey-Bass) (1994).
[28] J. Wu, L.F. Dai, F. Chiclana, H. Fujita, E. Herrera-Viedma, A minimum adjustment cost feedback mechanism based consensus model for group decision making under social network with distributed linguistic trust, Information Fusion 41 (2018) 232-242.
[29] T. Wu, X.W. Liu, An interval type-2 fuzzy clustering solution for large-scale multiple-criteria group decision-making problems, Knowledge-Based Systems 114 (2016) 118-127.
[30] Z.B. Wu, J.P. Xu, A consensus model for large-scale group decision making with hesitant fuzzy information and changeable clusters, Information Fusion 41 (2018) 217-231.
[31] M.M. Xia, Z.S. Xu, Hesitant fuzzy information aggregation in decision making, International Journal of Approximate Reasoning 52 (2011) 395-407.
[32] X.H. Xu, Z.J. Du, X.H. Chen, Consensus model for multi-criteria large-group emergency decision making considering non-cooperative behaviors and minority opinions, Decision Support Systems 79 (2015) 150-160.
[33] X.H. Xu, L.Y. Zhang, Q.F. Wan, A variation coefficient similarity measure and its application in emergency group decision-making, Systems Engineering Procedia 5 (2012) 119-124.
[34] Z.S. Xu, Deviation measures of linguistic preference relations in group decision making, Omega 33(3) (2005) 249-254.
[35] Z.S. Xu, A method based on linguistic aggregation operators for group decision making with linguistic preference relations, Information Sciences 166(1-4) (2004) 19-30.
[36] Z.S. Xu, J. Chen, Ordered weighted distance measure, Journal of Systems Science and Systems Engineering 16 (2008) 529-555.
[37] Z.S. Xu, H. Wang, On the syntax and semantics of virtual linguistic terms for information fusion in decision making, Information Fusion 34 (2017) 43–48.
[38] Z.L. Yue, An extended TOPSIS for determining weights of decision experts with interval numbers, Knowledge-Based Systems 24(1) (2011) 146-153.
[39] Z.L. Yue, Approach to group decision making based on determining the weights of experts by using projection method, Applied Mathematical Modelling 36(7) (2012) 2900-2910.
[40] H.J. Zhang, Y.C. Dong, E. Herrera-Viedma, Consensus building for the heterogeneous LSGDM with the individual concerns and satisfactions, IEEE Transactions on Fuzzy Systems 26(2) (2018) 884-898.

[41] X.L. Zhang, A novel probabilistic linguistic approach for large-scale group decision making with incomplete weight information, International Journal of Fuzzy Systems 20(7) (2018) 2245-2256.

[42] Z. Zhang, C.H. Guo, L. Martínez, Managing multigranular linguistic distribution assessments in large-scale multiattribute group decision making, IEEE Transactions on Systems, Man, and Cybernetics: Systems 47(11) (2017) 3063-3076.

[43] J. Zhu, S.T. Zhang, Y. Chen, L.L. Zhang, A hierarchical clustering approach based on three-dimensional gray relational analysis for clustering a large group of decision makers with double information, Group Decision & Negotiation 25(2) (2016) 325-354.

5 Managing minority opinions and non-cooperative behaviors in large-scale group decision making under DHLPRs: A consensus model

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Managing Minority Opinions and Non-Cooperative Behaviors in Large-Scale Group Decision Making under DHLPRs: A Consensus Model

Xunjie Gou, *student member*, *IEEE*, Huchang Liao, *Senior Member*, *IEEE*, Zeshui Xu, *Fellow*, *IEEE*, Francisco Herrera, *Senior Member*, *IEEE*.

***Abstract*—With the rapid development of society and continual progress of science and technology, large-scale group decision-making (LSGDM) problems are very commonly encountered in actual life. Considering that people's cognition process and decision making information are more and more complex, double hierarchy linguistic term set (DHLTS) can be used to express complex linguistic information reasonably and intuitively. In LSGDM, sometimes some experts do not modify their preferences or even do it on the contrary way to the remaining experts, and some different opinions or minority preferences are often cited as obstacles to decision making. Therefore, this paper gives a concept of double hierarchy linguistic preference relation (DHLPR) and develops a consensus model to manage minority opinions and non-cooperative behaviors in LSGDM with DHLPRs. Additionally, to establish the consensus model, some basic tools such as distance-based cluster method, weight-determining method, and comprehensive adjustment coefficient-determining method are developed. Finally, a practical LSGDM problem is set up to prove that the proposed consensus model is feasible and effective, and some comparative analyses are made to highlight the advantages of these methods and models and analyze current deficiencies.**

***Index Terms*—Double hierarchy linguistic preference relation; Large-scale group decision making; Consensus model; Minority opinions; Non-cooperative behaviors**

I. INTRODUCTION

With the rapid development of society and continual progress of science and technology, more and more decision making problems need a lot more people to participate in. Therefore, large-scale group decision making (LSGDM) was proposed with a condition when the number of the DMs reaches or exceeds 20 [1]. Now, the LSGDM problems have attracted comprehensive studies over the last decade [2-18], including consensus reaching process (CRP) [2-7, 11, 13-16], cluster algorithms [8, 9, 15, 16, 18], graphical monitoring tool [3], and the managements of minority opinions and non-cooperative behaviors [2,5], etc.

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X. J. Gou and H. C. Liao are with the Business School, Sichuan University, Chengdu 610064, China and are with the Andalusian Research Institute in Data Science and Computational Intelligence (DaSCI), University of Granada, Granada 18071, Spain (e-mail: gouxunjie@qq.com; liaohuchang@163.com).

Z. S. Xu is with the Business School, Sichuan University, Chengdu 610064, China (e-mail: xuzeshui@263.net).

F. Herrera is with the Andalusian Research Institute in Data Science and Computational Intelligence (DaSCI), University of Granada, Granada 18071, Spain and is with the Faculty of Computing and Information Technology, King Abdulaziz University, Jeddah, Saudi Arabia (e-mail: herrera@decsai.ugr.es)

Additionally, there exists a key point when dealing with the LSGDM problems, namely, what kind of evaluation information form can be much better and more accurately used to represent the real thoughts of DMs? Considering that natural languages are more in line with the real thoughts of people because they usually utilize natural languages to talk with others, express emotions or comment on something, etc. Then, the research of qualitative information is becoming more and more popular in recent years, especially the study of linguistic information. Therefore, the use of linguistic labels in decision making is a useful tool and a natural way to represent preferences [19]. In recent years, lots of linguistic models have been developed such as type-2 fuzzy sets [20], hesitant fuzzy linguistic term set (HFLTS) [21], 2-tuple linguistic model [22], virtual linguistic model [23] and linguistic terms with weakened hedges [24], etc.

However, because of people's cognition process and the decision making information are more and more complex, sometimes the linguistic models mentioned above cannot describe some complex linguistic terms or linguistic term sets (LTSs) comprehensively and accurately. For example, some DMs may tend to use some complex and detailed uncertain linguistic information to represent their comprehensive opinions such that "*entirely low*", "*just right medium*", "*a little high*", etc. So one question emerged: Can we extend the linguistic computational models to new model using elaborated /enriched linguistic representations? Then Gou et al. [25] defined a double hierarchy linguistic term set (DHLTS), which can be used to handle complex linguistic terms well by dividing them into two simple linguistic hierarchies where the first hierarchy LTS is the main linguistic hierarchy and the second hierarchy LTS is the linguistic feature or detailed supplementary of each linguistic term in the first hierarchy LTS. More explanations are given in Section II.

Considering that more and more DMs prefer to give their preferences by making pairwise comparisons between any two alternatives, meanwhile this kind of preference reflects the relationships between different alternatives intuitively. Therefore, preference relation becomes one of the popular and effective tools. Based on the DHLTS and preference form, this paper gives a double hierarchy linguistic preference relation (DHLPR), and utilizes it to express the evaluation information of all DMs more reasonably in LSGDM.

CRP is the key and focus work when dealing with LSGDM problems, which unifies all DMs' opinions and ensures that the LSGDM problems can be solved smoothly. At present, amounts of studies have been done about the CRP of LSGDM [2-7, 11, 13-16]. For example, some of the consensus models are based on self-organizing maps [2], graphical monitoring tool (MENTOR) [3], expert weighting methodology [4], individual concerns and satisfactions [6], and the feedback mechanism [7], etc. Besides, two typical items are very common and have important reflections in CRP of LSGDM. Firstly, some individuals or subgroups do not want to modify their preferences because they tend to stick to the ideas of themselves and do not want to lose their own interests, which can be denoted as non-cooperative behaviors [2, 5, 26]. The secondly, named as minority opinions [27], contain some cases as a leader, a very experienced expert, a young and aggressive DM, and a noteworthy and independent DM, etc. Although they are only the small fractions in LSGDM, it is likely to determine the direction of the decision making problem. Therefore, we focus on dealing with these preferences provided by DMs or groups reasonably and accurately.

How to identify and manage these two kinds of DMs? To handle this problem, this paper is dedicated to proposing a consensus model to identify and manage minority opinions and non-cooperative behaviors in LSGDM with DHLPRs.

The main contributions are listed as follows:

(1) In LSGDM, clustering all DMs into several groups will be convenient to manage them by using groups as units. This paper develops a novel cluster method. Meanwhile, a flow chart is drawn to show the process of this method more intuitively. Additionally, a weight-determining method and a consensus model are established, respectively.

(2) Develop two methods for CRP in LSGDM to deal with minority opinions and non-cooperative behaviors respectively including identifying, measuring and modifying them.

(3) An algorithm of LSGDM with minority opinions and non-cooperative behaviors is established to promote the CRP.

(4) To adjust the preferences of DMs more reasonably, this paper designs methods to determine the comprehensive adjustment coefficient, which consists of subjective adjustment coefficient and objective adjustment coefficient.

With the rapid development of economy, haze has become a major factor affecting People's Daily life in China. Although the haze treatment of China has achieved initial results and the overall picture has improved, the pollution has not been effectively curbed. Therefore, it is very necessary to determine the most main reason of haze formation and handle it. The proposed methods above are used to solve this case effectively.

The rest sections of this paper are organized as follows: Section II analyzes DHLTS and the minority opinions and non-cooperative behaviors. Section III develops a consensus model. Section IV establishes some methods to manage minority opinions and non-cooperative behaviors. Section V applies the proposed consensus model to a practical case study and makes some detailed comparative analyses. Some concluding remarks are pointed out in Section VI.

A flow chart is drawn to show the framework of this paper:

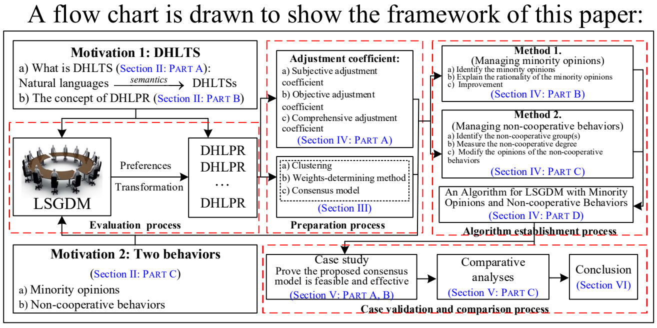

Fig. 1. The framework of this paper.

II. PRELIMINARIES

This section mainly introduces the concepts of DHLTS and DHLPR and analyzes the minority opinions and non-cooperative behaviors.

A. What is DHLTS?

In recent years, Artificial Intelligence (AI) is more and more popular and important in the real life of human being, and consists of many research fields such as language recognition, image recognition, natural language processing and expert system. From the view of AI, how to collect and represent natural languages exactly is one of the most important parts. For dealing with natural languages, Zadeh [28] provided the concept of Computing with Words (CW), and explained it by "*Computing with words is a system of computation in which the objects of computation are words, phrases and propositions drawn from a natural language. The carriers of information are propositions. It is important to note that Computing with words is the only system of computation which offers a capability to compute with information described in a natural language [28].*" Based on CW and at first, some linguistic models, such as LTSs, are proposed to represent simple natural languages. However, some more complex linguistic information is more and more common as sentences, a set of linguistic terms, etc. Then, lots of complex linguistic models mentioned in Section I have been developed by corresponding syntax and semantic rules [20-24].

However, as we discussed in Section I, the existing linguistic models have some gaps. For example, we cannot use them to express some words as "*only a little good*" or linguistic sets as "{*only a little high, just right high*}", etc. Therefore, it is necessary to consider an important issue: Does it make sense if we split each complex linguistic term into two parts with the form of "adverb+adjective" and express them by different kinds of linguistic terms? In fact, Zadeh has explained this idea when he dealing with a CW problem [28]: "*In effect, this is the solution to the problem which I posed to you. As you can see, reduction of the original problem to the solution of a variational problem is not so simple. However, solution of the variational problem to which the original problem is reduced, is well within the capabilities of desktop computers.*"

According to this idea, Wang et al. [24] proposed a concept of linguistic terms with weakened hedges, which regards the "adverbs" as a few weakened hedges expressed by other linguistic labels. However, two gaps are obvious: 1) All weakened hedges are included in a set, which will be inconvenient if different linguistic terms need different sets of weakened hedges. 2) One weakened hedge may have different meanings when embellishing different linguistic terms.

Therefore, to distinguish the sets of different modifiers and give corresponding semantics to modifier exactly, Gou et al. [25] proposed the concept of DHLTS by adding a second hierarchy LTS to each first hierarchy LTS and gave its mathematical form:

Definition 1 [25, 29]. Let $S=\{s_t \,|\, t=-\tau,\ldots,-1,0,1,\ldots,\tau\}$ be the first LTSs, $O^t=\{o_k^t \,|\, k=-\varsigma,\ldots,-1,0,1,\ldots,\varsigma\}$ be the second hierarchy of s_t. Then a DHLTS is denoted by

$$S_O = \{s_{t<o_k^t>} \mid t = -\tau, \ldots, -1, 0, 1, \ldots, \tau;\ k = -\varsigma, \ldots, -1, 0, 1, \ldots, \varsigma\} \quad (1)$$

where $s_{t<o_k^t>}$ is called double hierarchy linguistic term (DHLT). For convenient, the DHLT can be simplified by $s_{t<o_k>}$.

Remark 1. For understanding the Definition 1 better, the syntax rule of DHLT can be given. Let S and O be the first and second LTSs, respectively defined as before. A DHLT, denoted by $s_{t<o_k>}$, is generated by the following rule:

$< Auxiliary\ term > := o_k,\ o_k \in O$;

$< \Pr imary\ term > := s_t,\ s_t \in S$;

$< DHLT > := < Auxiliary\ term >< \Pr imary\ term >$.

In addition, the semantic of DHLT $s_{t<o_k>}$ is based on the linguistic terms s_t and o_k, which can be seen in Fig. 1:

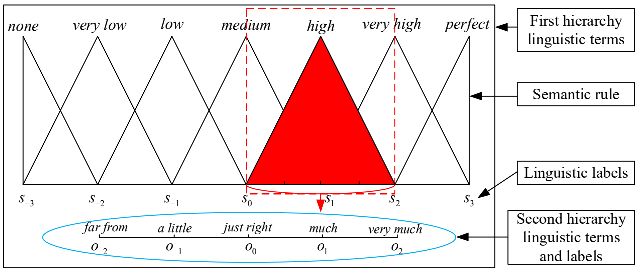

Fig. 2. The second hierarchy LTS of a linguistic term in the first hierarchy LTS.

In Fig. 2, we give a second hierarchy LTS of the first hierarchy linguistic term s_1. In other words, an adjective can be embellished by more than one adverb. Then, four important points are obtained: 1) All elements in DHLTS are expressed by linguistic labels without any numerical scales, which reflect the semantics of original natural languages to a greater extent; 2) The second hierarchy LTS is necessary when the set of adverbs of a first hierarchy linguistic term is large. 3) Each second hierarchy LTS can be regarded as a set of adverbs and extends the linguistic representations (richer vocabularies). 4) Each linguistic terms in the first hierarchy LTS has its own second hierarchy LTS, and usually they are different [29].

Next, some examples are given to understand DHLTS.

1) When a doctor is telling the patient about his illness, he may describe that the patient's blood pressure has *slightly high*.

2) When evaluating a car's performance, people may say that the acceleration of one hundred kilometers is *incredibly fast*.

3) When evaluating a student's grades, a teacher will say that most of students exhibit *significant obvious* improvement.

As these examples, the form of "adverb + adjective" is very common in daily life and the vocabulary of adverbs is also huge. Therefore, it may be a good choice to form a set of adverbs related to a certain adjective. Next, some comparisons between DHLTS and several typical linguistic models are analyzed as:

1) The linguistic model based on type-2 fuzzy set represents the semantics of the linguistic terms by type-2 membership functions which is formed by fuzzy set. Therefore, it is difficult to make an accurate cognition of the meaning of language through numbers. The DHLTS only use linguistic labels to express linguistic information, so the original meaning of natural language is represented clearly.

2) A 2-tuple linguistic term takes use of a linguistic term and a real number to represent linguistic information. Even though this linguistic model also divides linguistic information into two parts, the real number still do not convey the linguistic meaning of the original linguistic information.

3) A HFLTS can be used to express complex linguistic information by taking more than one linguistic terms. However, it can contain only some simple or vague linguistic terms, and cannot represent the form of "adverb + adjective" clearly.

Some other work about DHLTS has also been developed including its extension in hesitant fuzzy environment named as double hierarchy hesitant fuzzy linguistic term set (DHHFLTS) [25], the managing of consensus reaching process for LSGDM with double hierarchy hesitant fuzzy linguistic preference relations [14], the distance and similarity measures of DHHFLTSs [31], and the new concept of free DHHFLTS [29].

B. Minority Opinions and Non-cooperative Behaviors

In the CRP of LSGDM, a noticeable drawback usually found in such large groups is the presence of experts and subgroups of experts who present a behavior that does not contribute to achieve consensus [30], because they are not going to adjust their preferences to reach the consensus. In large groups, it is common that there exist several subgroups or coalitions of experts with similar interests. Some of these subgroups are prone to modify their preferences to achieve an agreement, while some others do not modify their preferences or even do it on the contrary way to the remaining experts [2]. These non-cooperating individuals and subgroups are called non-cooperative behaviors [2,5,26].

Additionally, in spite of different opinions or minority preferences are often cited as obstacles to decision making, appropriate processing for them can make the decision result more reasonable and accurate [27]. DMs who hold the minority opinions in a large-scale group mainly consist of four types [27]: (1) A leader, who is always able to give some unique points of views, and has enough rights to determine the final decision result. (2) A experienced expert, who often has a deep insight about decision making problem, and can propose constructive suggestions. (3) A young and aggressive DM, whose opinion is relatively extreme, and who is rarely influenced by other DMs' opinions. (4) A noteworthy and independent DM, whose view is usually out of the ordinary ones. As we know, the leader and the experienced expert, such as the CEO of a company and the experienced professor, are very powerful and experienced, so the preferences provided by them are also positive in general. On the contrary, the other two kinds of DMs are usually inexperienced or extreme such as the new employee. Therefore, the preference provided by the first two should be given high attention, while the latter two should be considered prudently.

On the studying of minority opinions and non-cooperative behaviors, the existing research have some gaps. Firstly, some research only studied one part of them. Some only dealt with the non-cooperative behaviors [2, 26], and other only discussed the minority views [27]. Therefore, it will result in incomplete information processing. Secondly, Xu et al. [5] developed a consensus model for multi-criteria large-group emergency decision making by dealing with non-cooperative behaviors and minority opinions. However, the cluster method contains too many factors from human and the normalization of individual decision matrices will lose lots of original information. Therefore, It would be a meaningful work to deal with non-cooperative behaviors and minority opinions simultaneously in the CRP of LSGDM with DHLPRs by proposing novel cluster method and consensus model.

As we know, the linguistic information is expressed more precise under double hierarchy linguistic environment. Therefore, the gap between any two DMs becomes more uncertain and will narrow or widen. For instance, the distance between "*only a little low*" and "*a little high*" is closer than that of "*low*" and "*high*", but distance between "*very much low*" and "*extremely high*" is farther than that of "*low*" and "*high*". Considering that the distance between DMs is the basic tool in cluster and CRP. Therefore, how to identify and deal with minority opinions and non-cooperative behaviors with DHLPRs is an important and urgent work. Based on the analyses above, it will be a pressing task to develop a consensus reaching model to deal with them and promote CRP in the LSGDM problems with DHLPRs.

C. The Concept of DHLPR

Before giving the definition of DHLPR, it is necessary to develop the additive and multiplicative operational laws for DHLTs under some conditions:

Definition 2. Let $S_O=\{s_{t<o_k>}\,|\,t=-\tau,\ldots,-1,0,1,\ldots,\tau;\,k=-\varsigma,\ldots,-1,0,1,\ldots,\varsigma\}$ be a DHLTS, $s_{t^1<o_{k^1}>}$, $s_{t^2<o_{k^2}>}$ and $s_{t<o_k>}$ be three DHLTs, $\lambda\,(0<\lambda<1)$ be a real number. Then

(1) $s_{t^1<o_{k^1}>}\oplus s_{t^2<o_{k^2}>}=s_{t^1+t^2<o_{k^1+k^2}>}$, *if* $t^1+t^2\le\tau$ *and* $k^1+k^2\le\varsigma$;

(2) $\lambda s_{t<o_k>}=s_{\lambda t<o_{\lambda k}>}$, $0<\lambda<1$.

In decision making process, let $A=\{A_1,A_2,\ldots,A_m\}$ be a fixed set of alternatives, then an additive DHLPR can be developed:

Definition 3. Let $S_O=\{s_{t<o_k>}\,|\,t=-\tau,\ldots,-1,0,1,\ldots,\tau;\,k=-\varsigma,\ldots,-1,0,1,\ldots,\varsigma\}$ be a DHLTS. An additive DHLPR $\mathbb{R}$ is presented by a matrix $\mathbb{R}=(r_{ij})_{m\times m}\subset A\times A$, where $r_{ij}\in S_O$ $(i,j=1,2,\ldots,m)$ is a DHLT, indicating the degree of A_i is preferred to A_j. For all $i,j=1,2,\ldots,m$, $r_{ij}\,(i<j)$ satisfies the conditions $r_{ij}+r_{ji}=s_{0<o_0>}$ and $r_{ii}=s_{0<O_0>}$.

It is common that some calculations may obtain some results but not included in S_O. a virtual DHLTS (VDHLTS) $\bar{S}_O$ was defined [25] by

$$\bar{S}_O=\{s_{t<o_k>}\,|\,t\in[-\tau,\tau];\ k\in[-\varsigma,\varsigma]\}\tag{2}$$

Based on $\bar{S}_O$ and the discussion of monotonic function [4] and virtual linguistic model [23], a monotonic function was defined [25] for making the equivalent transformation from DHLT to numerical scale. It also provides convenience for using the mathematical expressions to make the operations among DHLTs, and reduces the difficulty of computations.

Definition 4 [25]**.** Let $\bar{S}_O$ be a VDHLTS. Then the subscript (ϕ,φ) of the DHLT $s_{\phi<o_\varphi>}$ that expresses the equivalent information to a numerical scale γ can be transformed to γ by a monotonic function f:

$$f:[-\tau,\tau]\times[-\varsigma,\varsigma]\to[0,1],f(\phi,\varphi)=(\varphi+(\tau+\phi)\varsigma)/2\varsigma\tau=\gamma\tag{3}$$

III. Cluster, Weights-Determining Method and Consensus Measures

This section develops a consensus model to manage minority opinions and non-cooperative behaviors in CRP of LSGDM with DHLPRs, some basic contents are discussed firstly:

a) Cluster. By clustering, all DMs can be classified into several small groups, which makes the CRP much simpler because the communication among small group is smoother. Additionally, the minority opinions can be identified quickly, which is considered as the group with the least number of DMs.

b) Weights-determining method. Weights of DMs and group are very important for aggregating preferences. Meanwhile, the identifying and measuring minority opinions also depend on the weight of each group obtained by the cluster.

c) Consensus measures. By establishing some consensus measures, it is convenient to identify whether all DMs reach the given consensus threshold result or not.

A. The main elements of LSGDM with DHLPRs

An LSGDM problem can be defined as a situation where a large number of DMs provide their preferences by making pairwise comparisons among a set of alternatives. Then the main elements of a typical LSGDM problem with DHLPRs are described as follows:

(1) Let $A=\{A_1,A_2,\ldots,A_m\}(m\ge 3)$ be a discrete finite set of alternatives, it expresses all possible solutions of an LSGDM.

(2) Let $E=\{e^1,e^2,\ldots,e^n\}$ be a set of DMs, they express their preferences by making pairwise comparisons among the set of A. In general, a decision making problem can be called an LSGDM problem when the number of DMs meets or exceeds 20. The weight vector of DMs is denoted by $w=(w_1,w_2,\ldots,w_n)^T$, where $0\le w_i\le 1$ and $\sum_{i=1}^{n}w_i=1$ $(i=1,2,\ldots,n)$.

(3) The preferences of all DMs can be collected and contained into DHLPRs $\mathbb{R}^a=(r_{ij}^a)_{m\times m}$ $(a=1,2,\ldots,n)$.

B. Cluster Method

For an LSGDM problem with DHLPRs, considering that so many DMs and it is very difficult to manage them very well in a big group. Therefore, clustering them into some small groups is a very effective method. In recent years, lots of cluster methods have been developed as k-means cluster method [7], fuzzy c-mean cluster method [2], interval type-2 fuzzy equivalence cluster analysis [8], and the partial binary tree DEA-DA cyclic classification model [9], etc. However, considering that reducing the subjective factors is more beneficial to obtain accurate cluster results, and the distance measure can reflect the relation between any two DHLPRs. A double hierarchy linguistic distance-based cluster method can be developed.

Firstly, based on [31], a distance measure between two DHLPRs is given.

Definition 5. Let $\mathbb{R}^a=(r_{ij}^a)_{m\times m}$ and $\mathbb{R}^b=(r_{ij}^b)_{m\times m}$ be two DHLPRs provided by the DMs e^a and e^b, respectively, then

$$d(\mathbb{R}^a,\mathbb{R}^b)=\sqrt{\frac{2}{m(m-1)}\sum_{i=1}^{m}\sum_{i<j}^{m}(f(r_{ij}^a)-f(r_{ij}^b))^2} \quad (4)$$

is called the distance measure between $\mathbb{R}^a$ and $\mathbb{R}^b$.

The smaller distance two DMs have, the greater possibility they are in a same group. The cluster method is developed:

Step 1. Establish the overall distance matrix. Based on Eq. (4), an overall distance matrix $ODM=(odm^{ab})_{n\times n}$ associated with all pairs of DMs is obtained, where

$$odm^{ab}=d(\mathbb{R}^a,\mathbb{R}^b)\ (a,b=1,2,\ldots,n) \quad (5)$$

Step 2. Choose the classification threshold. Ranking all the different elements of the upper triangular matrix of ODM (except the diagonal elements) following the ascending order, denoted by $\Delta_1<\Delta_2<\cdots<\Delta_p<\cdots<\Delta_q$, where Δ_i is the i-th small value and $q\le n(n-1)/2$.

Step 3. Determine the optimal classification threshold Δ^*. Let TC_p be the rate of threshold change, obtained by

$$TC_p=(\Delta_p-\Delta_{p-1})/(n_p-n_{p-1}) \quad (6)$$

where n_p and n_{p-1} are the numbers of the p-th and $(p-1)$-th classifications, respectively. When $n_p=n$, the calculation process is over and all TC_p are collected. If

$$TC_\mu=\max_p\{TC_p\} \quad (7)$$

then the μ-th classification threshold can be called the optimal classification threshold, namely, $\Delta^*=\Delta_\mu$.

Step 4. Determine the cluster result. Firstly, all pairs of DMs (e^a,e^b) are classified into the overall groups as $\Im_1,\Im_2,\ldots,\Im_\zeta$ where $odm^{ab}\le\Delta^*$ following the ascending order of odm^{ab}. If $\Im_{\zeta_i}\cap\Im_{\zeta_j}\ne\varnothing$ $(\zeta_i\ne\zeta_j;\zeta_i,\zeta_j=1,2,\ldots,\zeta)$, then these elements of the overall group can be combined into a group. Finally, the cluster result $G_\phi(\phi=1,2,\ldots,\Phi)$ is obtained when $\Im_{\zeta_i}\cap\Im_{\zeta_j}=\varnothing$.

Example 1. Let $S_O=\{s_{t<o_k>}|t=-4,\ldots,-1,0,1,\ldots,4;k=-4,\ldots,-1,0,1,\ldots,4\}$ be a DHLTS, and fours DMs propose their preferences and the overall distance matrix is calculated as:

$$ODM=(odm^{ab})_{4\times4}=\begin{pmatrix}0&0.3687&0.1060&0.3409\\0.3687&0&0.4050&0.3487\\0.1060&0.4050&0&0.3407\\0.3409&0.3487&0.3407&0\end{pmatrix}$$

and then, all the different elements of the upper triangular matrix of ODM are ranked, and the optimal classification threshold is calculated as $\Delta^*=\Delta_2=0.3407$. Therefore, the over group is classified as $\Im_1=(e^1,e^3)$ and $\Im_2=(e^3,e^4)$. Based on Step 4, the clustering results are obtained as: $G_1=\{e^1,e^3,e^4\}$ and $G_2=\{e^2\}$.

C. Weights-determining Method

This paper investigates the CRP in an LSGDM based on the cluster result, but each group's weight is also the essential element. Suppose that all DMs are classified into $\Phi\ (1\le\Phi\le n)$ groups. Each group's weight at the beginning of decision can be obtained by satisfying two hypotheses: (1) The DMs in same group can be given the same weight because of their preferences are very close and can be considered that there is no difference among them. Specially, the experienced DM, as the leader and the experienced expert, should be given a larger weight. But the young or aggressive DM should be assigned a smaller weight. (2) The group with a larger number of DMs should be given a larger weight based on the majority principle.

Therefore, let η_ϕ be the number of DMs in a group $G_\phi(\phi=1,2,\ldots,\Phi)$. Then the weights of the DMs $e^a(a=1,2,\ldots,\eta_\phi)$ in group G_ϕ $(\phi=1,2,\ldots,\Phi)$ is obtained by

$$\omega_\phi^a=1/\eta_\phi\ (a=1,2,\ldots,\eta_\phi,\ \phi=1,2,\ldots,\Phi) \quad (8)$$

Furthermore, based on the number of DMs in a group, the weight of each group G_ϕ is obtained as:

$$w_\phi=\eta_\phi\Big/\sum_{\phi=1}^{\Phi}\eta_\phi \quad (9)$$

There are $0\le w_\phi\le1$ and $\sum_{\phi=1}^{\Phi}w_\phi=1$. Then, the weight of every DM in overall group can be got by $\omega^a=\omega_\phi^a\bullet w_\phi$.

D. A Consensus Model for LSGDM with DHLPRs

In LSGDM, the ideal result of the CRP is a stable state where each DM completely agrees all others' preferences. However, it is very difficult and unattainable considering the differences among people. Therefore, setting a consensus threshold value is very reasonable and necessary, that is, the CRP can be considered to be over when their overall consensus degree reaches or exceeds the given threshold value. Let ξ be the given consensus threshold value, which can be used to decide whether the CRP can be carried out. The consensus threshold is usually set to be smaller than 0.9 [32, 33]. Besides, the overall consensus degree can be calculated by the similarity measure among the DMs' preferences.

As the basis of CRP, a double hierarchy linguistic weighted average (DHLWA) operator needs to be developed firstly:

Definition 6. Let $\mathbb{R}=(\mathbb{R}^1,\mathbb{R}^2,\ldots,\mathbb{R}^n)$ be a set of DHLPRs provided by the DMs $e^a(a=1,2,\ldots,n)$, then all DHLPRs can be aggregated into a preference relation, denoted as $\mathbb{R}^*=(r_{ij}^*)_{m\times m}$, and its basic element r_{ij}^* can be obtained by

$$r_{ij}^*=DHLWA(r_{ij}^1,r_{ij}^2,\ldots,r_{ij}^n)=\sum_{a=1}^{n}\delta_a r_{ij}^a \quad (10)$$

where $\delta=(\delta_1,\delta_2,\ldots,\delta_n)^T$ is the weight vector of all DMs.

Based on the cluster result and the DHLWA operator, the group preference matrix $\mathbb{R}^{G_\phi}=(r_{ij}^{G_\phi})_{m\times m}$ of group G_ϕ is obtained, where $r_{ij}^{G_\phi}=\sum_{a=1}^{\eta_\phi}\omega_\phi^a\bullet r_{ij}^a$. Similarly, the overall preference matrix $\mathbb{R}^*=(r_{ij}^*)_{m\times m}$ is got, where $r_{ij}^*=\sum_{a=1}^{n}\omega^a\bullet r_{ij}^a$.

Then the consensus degree (CD) between a group preference matrix $\mathbb{R}^{G_\phi}$ and the overall preference matrix $\mathbb{R}^*$ is defined:

$$CD(\mathbb{R}^{G_\phi})=1-d(\mathbb{R}^{G_\phi},\mathbb{R}^*) \quad (11)$$

where $d(\mathbb{R}^{G_\phi},\mathbb{R}^*)$ is the distance between $\mathbb{R}^{G_\phi}$ and $\mathbb{R}^*$.

The overall consensus degree (OCD) can be obtained by

$$OCD=(\sum_{\phi=1}^{\Phi}CD(\mathbb{R}^{G_\phi}))\Big/\Phi \quad (12)$$

Clearly, $0 \le OCD \le 1$, and the bigger the value of OCD is, the higher consensus degree among all DMs will be. If $OCD \ge \xi$, then the consensus degree of all DMs is sufficiently high and the CRP is over. Otherwise, some changes about preferences or weights need to be made to improve the consensus degree and reach the given consensus threshold value. In Section IV, some methods are developed to improve the consensus degree by identifying and managing minority opinions and non-cooperative behaviors.

IV. Managing Minority Opinions and Non-Cooperative Behaviors

As we discussed in Section II, minority opinions and non-cooperative behaviors are very important in CRP and should be taken into consideration in LSGDM. This section develops a method to determine some necessary parameters in the CRP, and incorporates minority opinions and non-cooperative behaviors into the consensus model and develops an algorithm to manage them in LSGDM with DHLPRs.

A. Determination of Comprehensive Adjustment Coefficient

In the CRP of an LSGDM, it is common that DMs may face some internal and external pressures, so there exist uncertainty and subjectivity in the opinion adjustment coefficients provided by the DMs [5]. Therefore, some adjustment coefficients need to be developed to improve decision credibility. Firstly, subjective and objective adjustment coefficients are discussed. Then, the comprehensive adjustment coefficient can be obtained based on two rules.

(I) Subjective adjustment coefficient

Suppose that $\mathbb{R}^{G_\phi(\mathbb{Z})} = (r_{ij}^{G_\phi(\mathbb{Z})})_{m\times m}$ is the group preference matrix of the group G_ϕ and $\mathbb{R}^{*(\mathbb{Z})} = (r_{ij}^{*(\mathbb{Z})})_{m\times m}$ is the overall preference matrix in the $\mathbb{Z}$-th iteration. If $OCD < \xi$, then it means that the consensus is not reached. Let G_{ϕ^*} be the group that has the largest difference from all groups, namely, $CD(\mathbb{R}^{G_{\phi^*}}) = \min\{CD(\mathbb{R}^{G_\phi}) | \phi = 1,2,\ldots,\Phi\}$. Considering group consensus degree and practical situation, G_{ϕ^*} can provide an adjustment coefficient, denoted as $\vartheta_{G_\phi}^{S(\mathbb{Z})} (0 \le \vartheta_{G_\phi}^{S(\mathbb{Z})} \le 1)$, to modify its preference. Because of the adjustment coefficient provided by group G_{ϕ^*} reflects its subjective attitude towards the group consensus degree and the opinions of modifications, it can be called subjective adjustment coefficient.

(II) Objective adjustment coefficient

In general, the larger the difference between a group G_ϕ and overall group, the more this group needs to be improved to reach consensus threshold value ξ. That is, the lower the consensus degree of the group G_ϕ, the more correspondingly objective adjustment coefficient $\vartheta_{G_\phi}^{O(\mathbb{Z})}$. The objective adjustment coefficient can be calculated by

$$\vartheta_{G_\phi}^{O(\mathbb{Z})} = 1 - (1-\xi) / (1 - CD(\mathbb{R}^{G_\phi(\mathbb{Z})})) \tag{13}$$

Clearly, $0 \le \vartheta_{G_\phi}^{O(\mathbb{Z})} \le 1$. From Eq. (13), it is clear and logical that the higher the given consensus threshold value ξ, the greater the effort the DMs need to make.

(III) Comprehensive adjustment coefficient

Combining the subjective adjustment coefficient and the objective adjustment coefficient, the comprehensive adjustment coefficient, denoted as $\vartheta_{G_\phi}^{(\mathbb{Z})}$, can be obtained based on the following rules:

1) If $\vartheta_{G_\phi}^{S(\mathbb{Z})} \ge \vartheta_{G_\phi}^{O(\mathbb{Z})}$, then $\vartheta_{G_\phi}^{(\mathbb{Z})} = \vartheta_{G_\phi}^{S(\mathbb{Z})}$.

2) If $\vartheta_{G_\phi}^{S(\mathbb{Z})} < \vartheta_{G_\phi}^{O(\mathbb{Z})}$, then $\vartheta_{G_\phi}^{(\mathbb{Z})} = \sigma\vartheta_{G_\phi}^{S(\mathbb{Z})} + (1-\sigma)\vartheta_{G_\phi}^{O(\mathbb{Z})}$, where $0 \le \vartheta_{G_\phi}^{(\mathbb{Z})} \le 1$, $\sigma (0 \le \sigma \le 1)$ is a parameter which reflects the importance degree of the subjective adjustment coefficient.

Based on the comprehensive adjustment coefficient, the preferences of group G_{ϕ^*} is improved by

$$\mathbb{R}^{G_{\phi^*}(\mathbb{Z}+1)} = \vartheta_{G_\phi}^{(\mathbb{Z})}\mathbb{R}^{*(\mathbb{Z})} + (1-\vartheta_{G_\phi}^{(\mathbb{Z})})\mathbb{R}^{G_{\phi^*}(\mathbb{Z})} \tag{14}$$

Motivated by Xu [34], it is convenient to improve the group consensus degree and reach the given consensus threshold value by utilizing Eq. (14).

B. Managing Minority Opinions

This subsection develops a method to deal with minority opinions, and it consists of three parts: Identifying the minority opinions, making a discussion among the DMs and adjusting the corresponding weight information.

Method 1. Identify and manage minority opinions

Part 1. Identify the minority opinions. A group can be identified as a minority subgroup if it satisfies two conditions:

a) The consensus degree of the group should be the smallest;

b) The group consists of only one or a few DM(s).

Let $E = \{e^1, e^2, \ldots, e^n\}$ be a set of DMs, and all of them are classified into $\Phi (1 \le \Phi \le n)$ groups. Suppose that a group G_{ϕ^*} (n_{ϕ^*} is the number of DMs in this group) has the biggest difference from all groups (smallest consensus degree), and $\tilde{n} = [n/\Phi]$ ([] is a bracket function) is the threshold which is used to determine which group belongs to the minority opinion group. If $n_{\phi^*} \le \tilde{n}$, then G_{ϕ^*} is called minority opinion group.

Part 2. Explain the rationality of the minority opinion and make a discussion among all groups.

First, the group with minority opinion explains the rationality of its opinion, then a discussion about the group with minority opinion is put into force among the remaining groups. Based on the principle that the minority opinion should be considered fully and treated reasonably, each group should make an wide-ranging discussion and give its attitude and opinion.

Collecting the attitudes and opinions of the remaining groups, if more than half of them think that the opinion of the group G_{ϕ^*} is worth consideration, namely, $\tilde{n} \ge n/2$, then it is necessary to increase the weight of this group for enhancing its importance degree on overall groups. Meanwhile, the adjustment function should be closely with the number of the groups who support the minority opinion group. The more groups support the minority opinion group, the higher weight the group should be given.

Part 3. Improvement.

Based on the analyses above, a weight-improving method can be developed for the minority opinion group. Firstly, ranking the weight vector of all groups in ascending order, denoted as $w'^{(\mathbb{Z})}=(w_1'^{(\mathbb{Z})},w_2'^{(\mathbb{Z})},\ldots,w_\Phi'^{(\mathbb{Z})})^T$, where $w_\phi'^{(\mathbb{Z})}(\phi=1,2,\ldots,\Phi)$ is the ϕ-th smallest weight. Then the difference value, denoted as $dv_{\phi^*}^{M1(\mathbb{Z})}$, between the number of the groups who support the group G_{ϕ^*} (denoted as $n_{\phi^*}^{M1(\mathbb{Z})}$) and the half of the number of the remaining groups can be obtained by

$$dv_{\phi^*}^{M1(\mathbb{Z})}=\begin{cases} round(n_{\phi^*}^{M1(\mathbb{Z})}-(\Phi-1)/2), & if\ \Phi\ is\ a\ even\ number \\ n_{\phi^*}^{M1(\mathbb{Z})}-(\Phi-1)/2, & if\ \Phi\ is\ a\ odd\ number \end{cases} \quad (15)$$

where $round(\cdot)$ is the round operation.

Then a weight improvement function is defined as follows:

Definition 7. Let $w^{(\mathbb{Z})}=(w_1^{(\mathbb{Z})},w_2^{(\mathbb{Z})},\ldots,w_\Phi^{(\mathbb{Z})})^T$ be the weight vector of all groups in the $\mathbb{Z}$-th iteration, and $w'^{(\mathbb{Z})}=(w_1'^{(\mathbb{Z})},w_2'^{(\mathbb{Z})},\ldots,w_\Phi'^{(\mathbb{Z})})^T$ be the weight vector in ascending order. Suppose that G_{ϕ^*} is the group with minority opinion and its weight is the υ-th smallest weight, namely, $w_{\phi^*}^{(\mathbb{Z})}=w_{\upsilon^*}'^{(\mathbb{Z})}$, then the weight improvement function can be developed as:

$$w_{\phi^*}^{M1(\mathbb{Z})}=\min\{\max\{w_\phi^{(\mathbb{Z})}\,|\,\phi=1,2,\ldots,\Phi\},w'^{(\mathbb{Z})}_{\upsilon^*+dv_{\phi^*}^{M1(\mathbb{Z})}}\} \quad (16)$$

where $w_{\phi^*}^{M1(\mathbb{Z})}$ is the adjusted weight and the weight of the group G_{ϕ^*} becomes the $(\upsilon^*+dv_{\phi^*}^{M1(\mathbb{Z})})$-th smallest weight in the new weight vector.

Based on the method discussed above, the consensus measure will be repeated. However, if there exists no more than half of the groups in favor of the minority opinion group, which means that most DMs hold opposite opinions about the rationality of the opinion given by the minority opinion group, so both the weight improving process and the processing of minority opinions are over.

C. Handling Non-Cooperative Behaviors

As we mentioned above, this subsection is committed to developing a method to handle the non-cooperative behaviors.

Method 2. Identify and manage non-cooperative behaviors

Part 1. Identify the non-cooperative group(s)

According to the opinion of the group G_{ϕ^*}, the remaining groups $G_{\phi'}(\phi'=1,2,\ldots,\Phi;\phi'\neq\phi^*)$ provide their adjustment suggestions, denoted as $\vartheta_{G_{\phi'}G_{\phi^*}}^{(\mathbb{Z})}(0\le\vartheta_{G_{\phi'}G_{\phi^*}}^{(\mathbb{Z})}\le 1)$. Based on Eq. (13), the objective adjustment coefficient is obtained. Then, the expected adjustment suggestion interval is got and denoted as $\bar{\vartheta}_{G_{\phi^*}}^{(\mathbb{Z})}=[\min\{\vartheta_{G_{\phi'}G_{\phi^*}}^{(\mathbb{Z})},\vartheta_{G_{\phi^*}}^{O(\mathbb{Z})}\},\max\{\vartheta_{G_{\phi'}G_{\phi^*}}^{(\mathbb{Z})},\vartheta_{G_{\phi^*}}^{O(\mathbb{Z})}\}]$. If the subjective adjustment coefficient of this group is included in or smaller than the left boundary of interval $\bar{\vartheta}_{G_{\phi^*}}^{(\mathbb{Z})}$, then the group G_{ϕ^*} belongs to a non-cooperative group.

Part 2. Measure the non-cooperative degree

To determine the degree of a group who is unwilling to repair its opinion, the non-cooperative degree should be defined:

Definition 8. Let $\vartheta_{G_{\phi^*}}^{S(\mathbb{Z})}$ be the subjective adjustment coefficient provided by the group G_{ϕ^*}, and can be written by an interval form, i.e., $\bar{\vartheta}_{G_{\phi^*}}^{S(\mathbb{Z})}=[\vartheta_{G_{\phi^*}}^{S(\mathbb{Z})L},\vartheta_{G_{\phi^*}}^{S(\mathbb{Z})U}]$ with $\vartheta_{G_{\phi^*}}^{S(\mathbb{Z})}=\vartheta_{G_{\phi^*}}^{S(\mathbb{Z})L}=\vartheta_{G_{\phi^*}}^{S(\mathbb{Z})U}$. Then based on the possibility degree p proposed in Ref. [35], the non-cooperative degree of the group G_{ϕ^*} is obtained by

$$\Delta^{(\mathbb{Z})}(G_{\phi^*})=1-P(\bar{\vartheta}_{G_{\phi^*}}^{S(\mathbb{Z})}\ge\bar{\vartheta}_{G_{\phi^*}}^{(\mathbb{Z})}) \quad (17)$$

where $0\le\Delta^{(\mathbb{Z})}(G_{\phi^*})\le 1$.

Part 3. Modify the non-cooperative behaviors

a) If $\Delta^{(\mathbb{Z})}(G_{\phi^*})=0$, then G_{ϕ^*} can be regarded as a completely cooperative group. Therefore, it is not necessary to change the weight of G_{ϕ^*}, and the comprehensive adjustment coefficient is only used to repair its preference directly.

b) If $\Delta^{(\mathbb{Z})}(G_{\phi^*})=1$, then G_{ϕ^*} can be regarded as a completely non-cooperative group. It will waste lots of time if improving this group. So the best choice is to remove it.

c) If $0<\Delta^{(\mathbb{Z})}(G_{\phi^*})<1$, then G_{ϕ^*} can be regarded as a partly non-cooperative group. Therefore, firstly it is necessary to adjust its weight for reducing its reflection, and then utilize the comprehensive adjustment coefficient to repair its preference.

Xu et al. [5] developed a non-cooperative degree-based staircase weight adjustment function but it is not very precise. Therefore, a new weight adjustment function is developed:

$$w_{\phi^*}^{M2(\mathbb{Z})}=\begin{cases} w_{\phi^*}^{(\mathbb{Z})}, & \Delta^{(\mathbb{Z})}(G_{\phi^*})\in[0,0.1) \\ w_{\phi^*}^{(\mathbb{Z})}\times 0.9, & \Delta^{(\mathbb{Z})}(G_{\phi^*})\in[0.1,0.2) \\ w_{\phi^*}^{(\mathbb{Z})}\times 0.8, & \Delta^{(\mathbb{Z})}(G_{\phi^*})\in[0.2,0.3) \\ w_{\phi^*}^{(\mathbb{Z})}\times 0.7, & \Delta^{(\mathbb{Z})}(G_{\phi^*})\in[0.3,0.4) \\ w_{\phi^*}^{(\mathbb{Z})}\times 0.6, & \Delta^{(\mathbb{Z})}(G_{\phi^*})\in[0.4,0.5) \\ w_{\phi^*}^{(\mathbb{Z})}\times 0.5, & \Delta^{(\mathbb{Z})}(G_{\phi^*})\in[0.5,0.6) \\ w_{\phi^*}^{(\mathbb{Z})}\times 0.4, & \Delta^{(\mathbb{Z})}(G_{\phi^*})\in[0.6,0.7) \\ w_{\phi^*}^{(\mathbb{Z})}\times 0.3, & \Delta^{(\mathbb{Z})}(G_{\phi^*})\in[0.7,0.8) \\ w_{\phi^*}^{(\mathbb{Z})}\times 0.2, & \Delta^{(\mathbb{Z})}(G_{\phi^*})\in[0.8,0.9) \\ w_{\phi^*}^{(\mathbb{Z})}\times 0.1, & \Delta^{(\mathbb{Z})}(G_{\phi^*})\in[0.9,1) \\ 0, & \Delta^{(\mathbb{Z})}(G_{\phi^*})=1 \end{cases} \quad (18)$$

where $w_{\phi^*}^{(\mathbb{Z})}$ is the weight of the group G_{ϕ^*} and $w_{\phi^*}^{M2(\mathbb{Z})}$ expresses the adjusted weight in the $\mathbb{Z}$-th iteration.

D. An Algorithm for LSGDM with Minority Opinions and Non-Cooperative Behaviors

By the methods proposed above, an algorithm is established to handle LSGDM with DHLPRs, and shown as follows:

Input: Preference matrices $\mathbb{R}^a$ $(a=1,2,\ldots,n)$, iteration number $\mathbb{Z}$, and the given threshold value ξ.

Output: The final overall preference matrix $\mathbb{R}^{*(\mathbb{Z}^*)}$, and the rank of all DMs.

Step 1. Cluster all DMs into Φ groups $G_\phi(\phi = 1,2,\ldots,\Phi)$. and calculate the weight vector of all groups by Eq. (9). Then the group preference matrix $\mathbb{R}^{G_\phi(\mathbb{Z})}$ of each group is calculated based on Eq. (10). Let $\mathbb{Z}=0$ and go to Step 2.

Step 2. Aggregate all group preference matrices into the overall preference matrix $\mathbb{R}^{*(\mathbb{Z})}$ based on Eq. (10).

Step 3. Calculate the consensus degree of each group preference matrix, i.e., $CD(\mathbb{R}^{G_\phi(\mathbb{Z})})$ based on Eq. (11), and obtain the overall consensus degree OCD based on Eq. (12). If $OCD \ge \xi$, then go to Step 5; otherwise, go to Step 4.

Step 4. Consensus improvement process

1) Use Method 1 to identify and manage the group with minority opinion and determine whether the weight of this group needs to be repaired. If so, Eq. (16) is used to modify it, let $\mathbb{Z}=\mathbb{Z}+1$ and go back to Step 2; otherwise, go to Step 4 (II).

2) Use Method 2 to identify whether there exists the group with non-cooperative behavior. If so, firstly it is necessary to decrease its weight based on Eq. (18), and then calculate the comprehensive adjustment coefficient and use it to repair its preference; Otherwise, we can only calculate the comprehensive adjustment coefficient and repair its preference. Let $\mathbb{Z}=\mathbb{Z}+1$ and go back to Step 2.

Step 5. Let $\mathbb{Z}^*=\mathbb{Z}$. Output the group preference matrix $\mathbb{R}^{G_\phi(\mathbb{Z}^*)}$ of each group and the overall preference matrix $\mathbb{R}^{*(\mathbb{Z}^*)}$.

Step 6. Sum all preference results of each row of $\mathbb{R}^{*(\mathbb{Z}^*)}$ based on Eq. (3), and rank alternatives based on the expected values of alternatives: $E(A_i)=\left(\sum_{j=1}^{m} f(r_{ij}^{*(\mathbb{Z}^*)})\right)\Big/m, (i=1,2,\ldots,m)$.

A figure can be drawn to shown this algorithm:

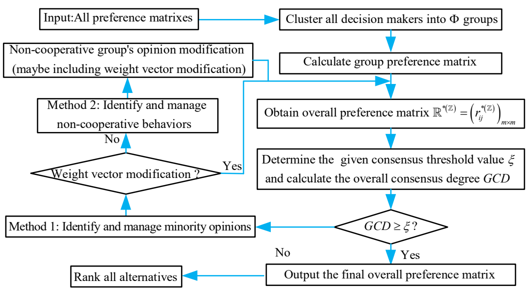

Fig. 3. The CRP in LSGDM.

V. CASE STUDY

This section applies the proposed algorithm to deal with a practical LSGDM problem that is to determine the main reason of haze pollution in a city of China. Firstly, the background about the reasons of haze pollution is described, and then the proposed algorithm is used to deal with this LSGDM problem, finally some comparative analyses are made.

A. Background: The Reasons of Haze Pollution

In recent years, haze remains an important issue in China's development process. The pollution has not been effectively curbed, and local air pollution remains serious such as Henan province, Shandong province, and Shanxi province, etc. Therefore, the situation is not optimistic and China still faces with many problems and challenges. Four main reasons can be summarized: 1) Economic restructuring is lagging behind. 2) Energy consumption structure dominated by fossil energy. 3) Environmental responsibilities in some areas are weakened. 4) Regional coordination and governance mechanism still needs to be further deepened. Suppose that a city of China needs to determine the most main reason of haze formation. Let the above four reasons are the alternatives $A=\{A_1,A_2,A_3,A_4\}$, 20 experts ($E=\{e^1,e^2,\ldots,e^{20}\}$) are invited to provide their preferences, which can be expressed by DHLPRs $\mathbb{R}^a=(r_{ij}^a)_{4\times 4}$ $(a=1,2,\ldots,20)$ with the DHLTS $S_O=\{s_{t<o_k>} \mid t=-4,\ldots,-1,0,1,\ldots,4;\ k=-4,\ldots,-1,0,1,\ldots,4\}$, where

$$S=\{s_{-4}=extremely\ bad, s_{-3}=very\ bad, s_{-2}=bad, s_{-1}=slightly\ bad, s_0=medium,$$
$$s_1=slightly\ good, s_2=good, s_3=very\ good, s_4=extremely\ good\}$$

$$O=\{o_{-4}=far\ from, o_{-3}=scarcely, o_{-2}=only\ a\ little, o_{-1}=a\ little, o_0=just\ right,$$
$$o_1=much, o_2=very\ much, o_3=extremely\ much, o_4=entirely\}$$

$$\mathbb{R}^1=\begin{pmatrix} \{s_{0<o_0>}\} & \{s_{0<o_1>}\} & \{s_{1<o_2>}\} & \{s_{-2<o_{-1}>}\} \\ \{s_{0<o_{-1}>}\} & \{s_{0<o_0>}\} & \{s_{1<o_0>}\} & \{s_{-2<o_2>}\} \\ \{s_{-1<o_{-2}>}\} & \{s_{-1<o_0>}\} & \{s_{0<o_0>}\} & \{s_{2<o_{-3}>}\} \\ \{s_{2<o_1>}\} & \{s_{2<o_{-2}>}\} & \{s_{-2<o_3>}\} & \{s_{0<o_0>}\} \end{pmatrix} \quad \mathbb{R}^2=\begin{pmatrix} \{s_{0<o_0>}\} & \{s_{1<o_{-2}>}\} & \{s_{-1<o_{-2}>}\} & \{s_{1<o_{-1}>}\} \\ \{s_{1<o_2>}\} & \{s_{0<o_0>}\} & \{s_{-3<o_1>}\} & \{s_{-1<o_2>}\} \\ \{s_{-1<o_2>}\} & \{s_{3<o_{-1}>}\} & \{s_{0<o_0>}\} & \{s_{-2<o_{-3}>}\} \\ \{s_{-1<o_1>}\} & \{s_{1<o_{-2}>}\} & \{s_{2<o_3>}\} & \{s_{0<o_0>}\} \end{pmatrix}$$

$$\mathbb{R}^3=\begin{pmatrix} \{s_{0<o_0>}\} & \{s_{0<o_{-1}>}\} & \{s_{1<o_{-1}>}\} & \{s_{-2<o_3>}\} \\ \{s_{0<o_1>}\} & \{s_{0<o_0>}\} & \{s_{2<o_2>}\} & \{s_{-2<o_2>}\} \\ \{s_{-1<o_1>}\} & \{s_{-2<o_{-2}>}\} & \{s_{0<o_0>}\} & \{s_{2<o_{-1}>}\} \\ \{s_{2<o_{-3}>}\} & \{s_{2<o_{-2}>}\} & \{s_{-2<o_1>}\} & \{s_{0<o_0>}\} \end{pmatrix} \quad \mathbb{R}^4=\begin{pmatrix} \{s_{0<o_0>}\} & \{s_{0<o_1>}\} & \{s_{3<o_{-1}>}\} & \{s_{2<o_3>}\} \\ \{s_{0<o_{-1}>}\} & \{s_{0<o_0>}\} & \{s_{1<o_{-1}>}\} & \{s_{1<o_2>}\} \\ \{s_{-3<o_1>}\} & \{s_{-1<o_1>}\} & \{s_{0<o_0>}\} & \{s_{-2<o_1>}\} \\ \{s_{-1<o_{-3}>}\} & \{s_{-1<o_{-2}>}\} & \{s_{2<o_{-1}>}\} & \{s_{0<o_0>}\} \end{pmatrix}$$

$$\mathbb{R}^5=\begin{pmatrix} \{s_{0<o_0>}\} & \{s_{1<o_2>}\} & \{s_{0<o_{-1}>}\} & \{s_{1<o_1>}\} \\ \{s_{-1<o_{-2}>}\} & \{s_{0<o_0>}\} & \{s_{-2<o_{-1}>}\} & \{s_{-1<o_2>}\} \\ \{s_{0<o_1>}\} & \{s_{2<o_1>}\} & \{s_{0<o_0>}\} & \{s_{-3<o_{-3}>}\} \\ \{s_{-1<o_{-1}>}\} & \{s_{1<o_{-2}>}\} & \{s_{3<o_3>}\} & \{s_{0<o_0>}\} \end{pmatrix} \quad \mathbb{R}^6=\begin{pmatrix} \{s_{0<o_0>}\} & \{s_{1<o_{-2}>}\} & \{s_{-1<o_2>}\} & \{s_{1<o_{-1}>}\} \\ \{s_{-1<o_2>}\} & \{s_{0<o_0>}\} & \{s_{-2<o_{-1}>}\} & \{s_{-2<o_{-1}>}\} \\ \{s_{1<o_{-2}>}\} & \{s_{2<o_1>}\} & \{s_{0<o_0>}\} & \{s_{-2<o_{-3}>}\} \\ \{s_{-1<o_1>}\} & \{s_{2<o_1>}\} & \{s_{2<o_3>}\} & \{s_{0<o_0>}\} \end{pmatrix}$$

$$\mathbb{R}^7=\begin{pmatrix} \{s_{0<o_0>}\} & \{s_{-1<o_1>}\} & \{s_{1<o_3>}\} & \{s_{-1<o_{-1}>}\} \\ \{s_{1<o_{-1}>}\} & \{s_{0<o_0>}\} & \{s_{0<o_{-1}>}\} & \{s_{-1<o_2>}\} \\ \{s_{-1<o_{-3}>}\} & \{s_{0<o_1>}\} & \{s_{0<o_0>}\} & \{s_{1<o_1>}\} \\ \{s_{1<o_2>}\} & \{s_{1<o_{-2}>}\} & \{s_{-1<o_{-1}>}\} & \{s_{0<o_0>}\} \end{pmatrix} \quad \mathbb{R}^8=\begin{pmatrix} \{s_{0<o_0>}\} & \{s_{1<o_3>}\} & \{s_{-1<o_2>}\} & \{s_{-1<o_{-1}>}\} \\ \{s_{1<o_3>}\} & \{s_{0<o_0>}\} & \{s_{-1<o_1>}\} & \{s_{-1<o_1>}\} \\ \{s_{1<o_{-2}>}\} & \{s_{1<o_{-1}>}\} & \{s_{0<o_0>}\} & \{s_{1<o_{-1}>}\} \\ \{s_{1<o_1>}\} & \{s_{-1<o_1>}\} & \{s_{0<o_0>}\} & \{s_{0<o_0>}\} \end{pmatrix}$$

$$\mathbb{R}^9=\begin{pmatrix} \{s_{0<o_0>}\} & \{s_{-1<o_{-1}>}\} & \{s_{2<o_2>}\} & \{s_{3<o_3>}\} \\ \{s_{1<o_1>}\} & \{s_{0<o_0>}\} & \{s_{1<o_{-1}>}\} & \{s_{1<o_3>}\} \\ \{s_{-2<o_{-2}>}\} & \{s_{-1<o_1>}\} & \{s_{0<o_0>}\} & \{s_{-2<o_1>}\} \\ \{s_{-3<o_{-3}>}\} & \{s_{-1<o_{-3}>}\} & \{s_{2<o_{-1}>}\} & \{s_{0<o_0>}\} \end{pmatrix} \quad \mathbb{R}^{10}=\begin{pmatrix} \{s_{0<o_0>}\} & \{s_{2<o_{-1}>}\} & \{s_{0<o_{-2}>}\} & \{s_{1<o_1>}\} \\ \{s_{-2<o_1>}\} & \{s_{0<o_0>}\} & \{s_{-2<o_{-1}>}\} & \{s_{-1<o_2>}\} \\ \{s_{0<o_2>}\} & \{s_{2<o_1>}\} & \{s_{0<o_0>}\} & \{s_{-3<o_{-2}>}\} \\ \{s_{-1<o_{-1}>}\} & \{s_{1<o_{-2}>}\} & \{s_{3<o_2>}\} & \{s_{0<o_0>}\} \end{pmatrix}$$

$$\mathbb{R}^{11}=\begin{pmatrix} \{s_{0<o_0>}\} & \{s_{2<o_1>}\} & \{s_{-1<o_2>}\} & \{s_{-2<o_{-1}>}\} \\ \{s_{-2<o_{-1}>}\} & \{s_{0<o_0>}\} & \{s_{0<o_{-1}>}\} & \{s_{-1<o_0>}\} \\ \{s_{1<o_{-2}>}\} & \{s_{0<o_1>}\} & \{s_{0<o_0>}\} & \{s_{2<o_3>}\} \\ \{s_{2<o_1>}\} & \{s_{1<o_0>}\} & \{s_{-2<o_{-3}>}\} & \{s_{0<o_0>}\} \end{pmatrix} \quad \mathbb{R}^{12}=\begin{pmatrix} \{s_{0<o_0>}\} & \{s_{0<o_{-2}>}\} & \{s_{3<o_{-1}>}\} & \{s_{2<o_2>}\} \\ \{s_{0<o_2>}\} & \{s_{0<o_0>}\} & \{s_{1<o_{-1}>}\} & \{s_{2<o_2>}\} \\ \{s_{-3<o_1>}\} & \{s_{-1<o_1>}\} & \{s_{0<o_0>}\} & \{s_{-2<o_1>}\} \\ \{s_{-2<o_{-2}>}\} & \{s_{-2<o_{-2}>}\} & \{s_{2<o_{-1}>}\} & \{s_{0<o_0>}\} \end{pmatrix}$$

$$\mathbb{R}^{13}=\begin{pmatrix} \{s_{0<o_0>}\} & \{s_{2<o_{-2}>}\} & \{s_{-1<o_{-2}>}\} & \{s_{0<o_2>}\} \\ \{s_{-2<o_2>}\} & \{s_{0<o_0>}\} & \{s_{-3<o_1>}\} & \{s_{-1<o_2>}\} \\ \{s_{1<o_2>}\} & \{s_{3<o_{-1}>}\} & \{s_{0<o_0>}\} & \{s_{-2<o_{-3}>}\} \\ \{s_{0<o_{-2}>}\} & \{s_{1<o_{-2}>}\} & \{s_{2<o_3>}\} & \{s_{0<o_0>}\} \end{pmatrix} \quad \mathbb{R}^{14}=\begin{pmatrix} \{s_{0<o_0>}\} & \{s_{1<o_3>}\} & \{s_{-1<o_3>}\} & \{s_{-1<o_{-1}>}\} \\ \{s_{-1<o_{-3}>}\} & \{s_{0<o_0>}\} & \{s_{-1<o_1>}\} & \{s_{-1<o_{-1}>}\} \\ \{s_{1<o_{-3}>}\} & \{s_{1<o_{-1}>}\} & \{s_{0<o_0>}\} & \{s_{1<o_3>}\} \\ \{s_{1<o_1>}\} & \{s_{1<o_1>}\} & \{s_{-1<o_{-3}>}\} & \{s_{0<o_0>}\} \end{pmatrix}$$

$$\mathbb{R}^{15}=\begin{pmatrix} \{s_{0<o_0>}\} & \{s_{-1<o_{-1}>}\} & \{s_{1<o_{-2}>}\} & \{s_{2<o_{-3}>}\} \\ \{s_{1<o_1>}\} & \{s_{0<o_0>}\} & \{s_{3<o_{-1}>}\} & \{s_{3<o_3>}\} \\ \{s_{-1<o_2>}\} & \{s_{-3<o_1>}\} & \{s_{0<o_0>}\} & \{s_{-1<o_3>}\} \\ \{s_{-2<o_3>}\} & \{s_{-3<o_{-3}>}\} & \{s_{1<o_{-3}>}\} & \{s_{0<o_0>}\} \end{pmatrix} \quad \mathbb{R}^{16}=\begin{pmatrix} \{s_{0<o_0>}\} & \{s_{-1<o_1>}\} & \{s_{3<o_{-2}>}\} & \{s_{3<o_2>}\} \\ \{s_{1<o_{-1}>}\} & \{s_{0<o_0>}\} & \{s_{2<o_2>}\} & \{s_{1<o_3>}\} \\ \{s_{-3<o_2>}\} & \{s_{-2<o_{-2}>}\} & \{s_{0<o_0>}\} & \{s_{-1<o_3>}\} \\ \{s_{-3<o_{-2}>}\} & \{s_{-1<o_{-3}>}\} & \{s_{1<o_{-3}>}\} & \{s_{0<o_0>}\} \end{pmatrix}$$

$$\mathbb{R}^{17}=\begin{pmatrix} \{s_{0<o_0>}\} & \{s_{0<o_1>}\} & \{s_{1<o_2>}\} & \{s_{-2<o_{-2}>}\} \\ \{s_{0<o_{-1}>}\} & \{s_{0<o_0>}\} & \{s_{2<o_0>}\} & \{s_{-2<o_2>}\} \\ \{s_{-1<o_{-2}>}\} & \{s_{-2<o_0>}\} & \{s_{0<o_0>}\} & \{s_{3<o_{-3}>}\} \\ \{s_{2<o_2>}\} & \{s_{2<o_{-2}>}\} & \{s_{-3<o_3>}\} & \{s_{0<o_0>}\} \end{pmatrix} \quad \mathbb{R}^{18}=\begin{pmatrix} \{s_{0<o_0>}\} & \{s_{-1<o_1>}\} & \{s_{3<o_1>}\} & \{s_{2<o_3>}\} \\ \{s_{1<o_{-1}>}\} & \{s_{0<o_0>}\} & \{s_{1<o_{-1}>}\} & \{s_{2<o_1>}\} \\ \{s_{-3<o_{-1}>}\} & \{s_{-1<o_1>}\} & \{s_{0<o_0>}\} & \{s_{-2<o_1>}\} \\ \{s_{-2<o_{-3}>}\} & \{s_{-2<o_{-1}>}\} & \{s_{2<o_{-1}>}\} & \{s_{0<o_0>}\} \end{pmatrix}$$

$$\mathbb{R}^{19}=\begin{pmatrix} \{s_{0<o_0>}\} & \{s_{-1<o_0>}\} & \{s_{2<o_{-1}>}\} & \{s_{2<o_{-3}>}\} \\ \{s_{1<o_0>}\} & \{s_{0<o_0>}\} & \{s_{2<o_1>}\} & \{s_{3<o_3>}\} \\ \{s_{-2<o_1>}\} & \{s_{-2<o_{-1}>}\} & \{s_{0<o_0>}\} & \{s_{0<o_{-2}>}\} \\ \{s_{-2<o_3>}\} & \{s_{-3<o_{-3}>}\} & \{s_{0<o_2>}\} & \{s_{0<o_0>}\} \end{pmatrix} \quad \mathbb{R}^{20}=\begin{pmatrix} \{s_{0<o_0>}\} & \{s_{2<o_2>}\} & \{s_{-1<o_2>}\} & \{s_{-1<o_{-3}>}\} \\ \{s_{-2<o_{-2}>}\} & \{s_{0<o_0>}\} & \{s_{0<o_{-1}>}\} & \{s_{-1<o_1>}\} \\ \{s_{1<o_{-2}>}\} & \{s_{0<o_1>}\} & \{s_{0<o_0>}\} & \{s_{2<o_3>}\} \\ \{s_{1<o_3>}\} & \{s_{1<o_{-1}>}\} & \{s_{-2<o_{-3}>}\} & \{s_{0<o_0>}\} \end{pmatrix}$$

B. Solving this LSGDM Problem

Step 1. Cluster all DMs into five groups $G_\phi(\phi=1,2,\ldots,5)$ and calculate the weights of all groups, the result are shown in Fig. 4 and TABLE I.

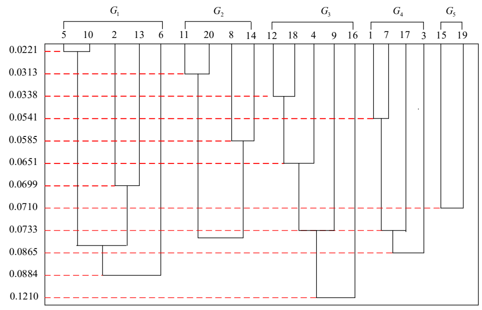

Fig. 4. The cluster result.

TABLE I

THE CLUSTER RESULT AND THE WEIGHT INFORMATION OF EACH GROUP

Group	The DMs in each group	The weight vector of each group	The weight of each group
G_1	$\{e^2,e^5,e^6,e^{10},e^{13}\}$	$(0.2,0.2,0.2,0.2,0.2)^T$	$w_1^{(0)}=0.25$
G_2	$\{e^8,e^{11},e^{14},e^{20}\}$	$(0.25,0.25,0.25,0.25)^T$	$w_2^{(0)}=0.2$
G_3	$\{e^4,e^9,e^{12},e^{16},e^{18}\}$	$(0.2,0.2,0.2,0.2,0.2)^T$	$w_3^{(0)}=0.25$
G_4	$\{e^1,e^3,e^7,e^{17}\}$	$(0.25,0.25,0.25,0.25)^T$	$w_4^{(0)}=0.2$
G_5	$\{e^{15},e^{19}\}$	$(0.5,0.5)^T$	$w_5^{(0)}=0.1$

Step 2. Aggregate all group preference matrices $\mathbb{R}^{G_\phi(0)}$ ($\phi=1,2,\ldots,5$) into the overall preference matrix $\mathbb{R}^{*(0)}$.

Step 3. Suppose that the consensus threshold value is given as $\xi=0.85$. Then the consensus degree of each group $CD(\mathbb{R}^{G_\phi(0)})$ and the overall consensus degree $OCD^{(0)}$ are obtained and shown in TABLE II.

TABLE II

THE CONSENSUS DEGREES OF GROUPS AND OVERALL CONSENSUS DEGREE

	$G_1^{(0)}$	$G_2^{(0)}$	$G_3^{(0)}$	$G_4^{(0)}$	$G_5^{(0)}$
$CD(\mathbb{R}^{G_\phi^{(0)}})$	0.7789	0.797	0.7871	0.7980	0.7598
$OCD^{(0)}$			0.7843		

Clearly, $OCD^{(0)}<\xi$. All groups do not reach the consensus.

Step 4. Consensus reaching.

a) The first consensus iteration process

Firstly, using Method 1 to identify and manage the group with minority opinion, and the group G_5 is called minority opinion group. Next, we have $n_5^{M1(0)}=4$. Based on Eq. (15), the deviation is obtained as $dv_5^{M1(0)}=2$. Based on Eq. (16), the adjusted weight of G_5 is $w_5^{M1(0)}=0.2$. After normalization, the new weight vector is $w^{(1)}=(0.2273,0.1818,0.2273,0.1818,0.1818)^T$.

Additionally, the consensus degree of each group and the overall consensus degree of this round are shown in TABLE III.

TABLE III

THE CONSENSUS DEGREES OF GROUPS AND OVERALL CONSENSUS DEGREE

	$G_1^{(1)}$	$G_2^{(1)}$	$G_3^{(1)}$	$G_4^{(1)}$	$G_5^{(1)}$
$CD(\mathbb{R}^{G_\phi^{(1)}})$	0.7655	0.7856	0.8015	0.7929	0.7816
$OCD^{(1)}$			0.7854		

We have $OCD^{(1)}<\xi$. Therefore, the CRP continues.

b) The second consensus iteration process

Again, identifying the group with minority opinion. And G_1 cannot be regarded as a group with minority opinion. Then, non-cooperative behaviors should be taken into consideration. Suggest that the remaining groups $G_{\phi'}(\phi'=1,2,\ldots,5;\phi'\neq 1)$ provide their adjustment suggestions on the opinions of G_1 as $\vartheta_{G_2G_1}^{(1)}=0.56$, $\vartheta_{G_3G_1}^{(1)}=0.60$, $\vartheta_{G_4G_1}^{(1)}=0.65$, and $\vartheta_{G_5G_1}^{(1)}=0.43$. By Eq. (13), we have $\vartheta_{G_1}^{O(1)}=0.6658$. Then the expected adjustment suggestion interval is $\bar{\vartheta}_{G_1}^{(1)}=[0.43,0.6658]$. Suppose that the DMs in G_1 provide their subjective adjustment coefficient $\vartheta_{G_1}^{S(1)}=0.80$. With Eq. (17), $\Delta^{(1)}(G_1)=0$, which means that G_1 can be regarded as the completely cooperative group. Therefore, it is unnecessary to change the weight of G_1. Considering $\vartheta_{G_1}^{S(1)}>\vartheta_{G_1}^{O(1)}$, the comprehensive adjustment coefficient is $\vartheta_{G_1}^{(1)}=0.80$, and it is utilized to repair the preference of G_1 on the basis of Eq. (14).

Then the consensus degree of each group and the overall consensus degree of this round are shown in TABLE IV.

TABLE IV

THE CONSENSUS DEGREES OF GROUPS AND OVERALL CONSENSUS DEGREE

	$G_1^{(2)}$	$G_2^{(2)}$	$G_3^{(2)}$	$G_4^{(2)}$	$G_5^{(2)}$
$CD(\mathbb{R}^{G_\phi^{(2)}})$	0.9105	0.7801	0.8079	0.8089	0.8062
$OCD^{(2)}$			0.8227		

We also have $OCD^{(2)}<\xi$. Therefore, the CRP continues.

c) The third consensus iteration process

The G_2 can be regarded as a group with minority opinion. However, there is only one group that supports the opinion of the group G_2. Then we need to deal with non-cooperative behaviors. Based on the discussion results and the overall consensus degree, the remaining groups provide their adjustment suggestions on the opinions of G_2 as $\vartheta_{G_1G_2}^{(2)}=0.85$, $\vartheta_{G_3G_2}^{(2)}=0.70$, $\vartheta_{G_4G_2}^{(2)}=0.67$, and $\vartheta_{G_5G_2}^{(2)}=0.90$. By Eq. (13), we have $\vartheta_{G_2}^{O(2)}=0.6682$. Then the expected adjustment suggestion interval is $\bar{\vartheta}_{G_2}^{(2)}=[0.6682,0.90]$. Suppose that the group G_2 provides their subjective adjustment coefficient $\vartheta_{G_2}^{S(2)}=0.85$. Based on Eq. (17), $\Delta^{(2)}(G_2)=0.2157$, which means that G_2 is regarded as a partly non-cooperative group. Therefore, it needs to adjust G_2's weight to reduce its reflection. Based on Eq. (18), we have $w_2^{M2(2)}=0.1454$. After normalization, the new weight vector is $w^{(3)}=(0.2359,0.1509,0.2359,0.1887,0.1887)^T$.

Considering that $\vartheta_{G_2}^{S(2)} > \vartheta_{G_2}^{O(2)}$, the comprehensive adjustment coefficient is $\vartheta_{G_2}^{(2)} = 0.85$, which can be utilized to repair the preference of G_2 on the basis of Eq. (14).

Then the consensus degree of each group and the overall consensus degree of this round are shown in TABLE V.

TABLE V
THE CONSENSUS DEGREES OF GROUPS AND OVERALL CONSENSUS DEGREE

	$G_1^{(3)}$	$G_2^{(3)}$	$G_3^{(3)}$	$G_4^{(3)}$	$G_5^{(3)}$
$CD\,(\mathbb{R}^{G_\phi^{(3)}})$	0.8960	0.9305	0.8399	0.7847	0.8318
$OCD^{(3)}$			0.8566		

We have $OCD^{(3)} > \xi$. Therefore, the CRP is over.

Step 5. Let $\mathbb{Z}^* = 3$. Output the final overall preference matrix $\mathbb{R}^{*(3)} = (r_{ij}^{*(3)})_{m \times m}$.

Step 6. Calculate the expected values of all alternatives and obtain $E(A_1) = 0.5581$, $E(A_2) = 0.5710$, $E(A_3) = 0.4157$ and $E(A_4) = 0.4558$. Then the rank of them is $A_2 \succ A_1 \succ A_4 \succ A_3$, which means that the energy consumption structure dominated by fossil energy is the main reason.

C. Comparative Analyses

(1) Comparison between the proposed method and the existing ones. Gou et al. [14] proposed a CRP for LSGDM with DHHFLPR. Considering that the DHLPR can be regarded as the special situation when elements of DHHFLPR only have one DHLT. Utilizing Gou et al.'s method, the cluster and decision making results are similar as those of the proposed method, but the emphases of them are different. Gou et al.'s method gives a similarity-based cluster technique and a CRP with feedback mechanisms. However, the cluster method discussed in this paper is based on the distance measure directly, and it omits the process of calculating the similarity measure. So it is simpler than Gou et al.'s method. Additionally, Gou et al. only discussed how to find and improve preferences and do not to check whether it belongs to the minority opinions or non-cooperative behaviors, which may cause the consequences of incomplete information analysis. Using the proposed method, the group G_5 belongs to minority opinion and we only need to increase its weight. Therefore, by contrast, the proposed method makes the CRP more sophisticated by dealing with the non-cooperative behaviors and minority opinions.

Moreover, as we mentioned in Section 1, there exist lots of consensus reaching methods for LSGDM and these methods are very useful to handle LSGDM with various types of decision making information. But it is difficult to make comparison between them and the proposed methods with DHLPRs. Simultaneously, some plans will be implemented to utilize these methods to handle LSGDM with DHLPRs.

(2) The existing research have some shortcomings on when studying of minority opinions and non-cooperative behaviors. Firstly, some research only studied one part of them and it will result in incomplete information processing. For instance, Refs. [2, 26] only dealt with the non-cooperative behaviors, and Ref. [27] only discussed the minority views. Secondly, even though Xu et al. [5] developed a consensus model for multi-criteria large-group emergency decision making by dealing with non-cooperative behaviors and minority opinions, the cluster method contains too many factors from human and the normalization of individual decision matrices will lose lots of original information. Therefore, this paper would be better to deal with non-cooperative behaviors and minority opinions simultaneously in the CRP of LSGDM with DHLPRs by proposing novel cluster method and consensus model.

(2) There exist lots of cluster methods as k-means cluster method [7], fuzzy c-mean cluster method [2], interval type-2 fuzzy equivalence cluster analysis [9], the partial binary tree DEA-DA cyclic classification model [10], etc. However, considering that giving subjective factors into the cluster process may change the accuracy of cluster results, also it is better to draw a flow chart to reflect the cluster process. This paper proposes the distance-based cluster methods, which can not only reflect the relation between any two DHLPRs, but also describe the clustering process more detailed and intuitively by a flow chart (As Fig. 4). Additionally, the proposed cluster method is only based on the original preferences and there exist no any subjective factors in the process. Furthermore, many scholars like to utilize similarity measures to develop the cluster methods, however, these similarity measures are usually derived from the distance measures. Therefore, using distance measure to establish the cluster methods can simplify some unnecessary processes.

(3) The weight adjustment method is very important when dealing with the non-cooperative behaviors. This paper improves the method of Xu et al. [5] and develops a novel non-cooperative degree-based staircase weight adjustment function by dividing non-cooperative degrees into some more intervals (As Eq. (18)), which makes the non-cooperative degree more in detail.

(4) The comprehensive adjustment coefficient is vital in the CRP. If only utilizing the subjective adjustment coefficient and supposing that the subjective adjustment coefficient is very small in each round, then the number of iterations will be very big. If only considering the objective adjustment coefficient and neglecting the subjective adjustment coefficient, then the arbitrariness and uncertainty of subjective revision will be reduced, but the DMs' own adjustment coefficients will not be brought to the forefront. Therefore, this situation will violate the original intention of LSGDM. Only with the comprehensive adjustment coefficient, all shortcomings can be overcome and the CRP will be more reasonable.

VI. CONCLUSION

This paper has mainly established a consensus model to manage minority opinions and non-cooperative behaviors in LSGDM under double hierarchy linguistic preference environment. A double hierarchy linguistic distance-based cluster method, a weights-determining method, and a consensus model for LSGDM have been developed. Additionally, this paper has given a CRP in LSGDM which consists of the determination of comprehensive adjustment coefficient, and two methods for managing minority opinions and non-cooperative behaviors. Based on which, an algorithm for LSGDM with minority opinions and non-cooperative behaviors have been established with these proposed methods and models. Furthermore, the algorithm has been applied to a practical case study that is to determine the most main reason of haze formation in a city of China, and some comparative analyses have been made in detail.

Generally, some advantages about the proposed consensus model are summarized: 1) The proposed cluster method is simpler and can be shown in a figure clearly and intuitively. In addition, the cluster process has no any external influence. 2) The proposed consensus model considers the non-cooperative behaviors and minority opinions simultaneously, which is more comprehensive than some existing methods. 3) The proposed weight adjustment method for dealing with the non-cooperative behaviors makes the non-cooperative degree more in detail. 4) The proposed comprehensive adjustment coefficient is vital in the CRP by considering subjective and objective information simultaneously.

As future study, we are dedicated to the study of some more cluster methods and consensus reaching methods under different decision environments. Additionally, it would be also interesting to implement the proposed methods and models to solve some more practical LSGDM problem.

REFERENCES

[1] R. Liu, X. H. Chen, "Improved clustering algorithm and its application in complex huge group decision making," J. Syst. Eng. Electron., vol. 28, no. 11, pp. 1695-1699, 2006.

[2] I. Palomares, L. Martínez, F. Herrera, "A consensus model to detect and manage noncooperative behaviors in large-scale group decision making," IEEE Trans. Fuzzy Syst., vol. 22, no. 3, pp. 516-530, 2014.

[3] I. Palomares, L. Martínez, F. Herrera, "MENTOR: A graphical monitoring tool of preferences evolution in large-scale group decision making," Knowl.-Based Syst., vol. 58, pp. 66-74, 2014.

[4] F. J. Quesada, I. Palomares, L. Martínez, "Managing experts behavior in large-scale consensus reaching processes with uninorm aggregation operators," Appl. Soft Comput., vol. 35, pp. 873-887, 2015.

[5] X. H. Xu, Z. J. Du, X. H. Chen, "Consensus model for multi-criteria large-group emergency decision making considering non-cooperative behaviors and minority opinions," Decis. Support Syst., vol. 79, pp. 150-160, 2015.

[6] H. J. Zhang, Y. C. Dong, E. Herrera-Viedma, "Consensus building for the heterogeneous LSGDM with the individual concerns and satisfactions," IEEE Trans. Fuzzy Syst., vol. 26, no. 2, pp. 884-898, 2018.

[7] Z. B. Wu, J. P. Xu, "A consensus model for large-scale group decision making with hesitant fuzzy information and changeable clusters," Inf. Fusion, vol. 41, pp. 217-231, 2018.

[8] T. Wu, X. W. Liu, "An interval type-2 fuzzy clustering solution for large-scale multiple-criteria group decision making problems," Knowl. Based Syst., vol. 114, pp. 118-127, 2016.

[9] B. S. Liu, Y. H. Shen, X. H. Chen, Y. Chen, X. Q. Wang, "A partial binary tree DEA-DA cyclic classification model for DMs in complex multi-attribute large-group interval-valued intuitionistic fuzzy decision making problems," Inf. Fusion, vol. 18, no. 1, pp. 119-130, 2014.

[10] B. S. Liu, Y. H. Shen, W. Zhang, X. H. Chen, X. Q. Wang, "An interval-valued intuitionistic fuzzy principal component analysis model-based method for complex multi-attribute large-group decision making," Eur. J. Oper. Res., vol. 245, no. 1, pp. 209-225, 2015.

[11] Á. Labella, Y. Liu, R. M. Rodríguez, L. Martínez, "Analyzing the performance of classical consensus models in largescale group decision making: A comparative study," Appl. Soft Comput., vol. 67, pp. 677-690, 2018.

[12] Z. Zhang, C. H. Guo, L. Martínez, "Managing multigranular linguistic distribution assessments in large-scale multiattribute group decision making," IEEE Trans. Syst. Man Cybern.-Syst., vol. 47, no. 11, pp. 3063-3076, 2017.

[13] B. S. Liu, Y. H. Shen, Y. Chen, X. H. Chen, Y. M. Wang, "A two-layer weight determination method for complex multi-attribute large-group decision making experts in a linguistic environment," Inf. Fusion, vol. 23, pp. 156-165, 2015.

[14] X. J. Gou, Z. S. Xu, F. Herrera, "Consensus reaching process for large-scale group decision making with double hierarchy hesitant fuzzy linguistic preference relations," Knowl.-Based Syst., vol. 157, pp. 20-33, 2018.

[15] X. Liu, Y. J. Xu, R. Montes, R. X. Ding, F. Herrera, "Alternative ranking-based clustering and reliability index-based consensus reaching process for hesitant fuzzy large scale group decision making," IEEE Trans. Fuzzy Syst., vol. 27, no. 1, pp. 159-171, 2019.

[16] C. C. Li, Y. C. Dong, F. Herrera, "A consensus model for large-scale linguistic group decision making with a feedback recommendation based on clustered personalized individual semantics and opposing consensus groups," IEEE Trans. Fuzzy Syst., vol. 27, no. 2, pp. 221-233, 2019.

[17] B. S. Liu, Q. Zhou, R. X. Ding, I. Palomares, F. Herrera, "Large-scale group decision making model based on social network analysis: trust relationship-based conflict detection and elimination," Eur. J. Oper. Res., vol. 275, no. 2, pp. 737-754, 2019.

[18] R. C. Ding, X. Wang, K. Shang, B. S. Liu, F. Herrera, "Sparse representation-based intuitionistic fuzzy clustering approach to find the group intra-relations and group leaders for large-scale decision making," IEEE Trans. Fuzzy Syst., vol. 27, no. 3, pp. 559-573, 2019.

[19] L. Martínez, D. Ruan, F. Herrera, "Computing with words in decision support systems: An overview on models and applications," Int. J. Comput. Intell. Syst., vol. 3, no. 4, pp. 382-395, 2010.

[20] C. F. Juang, C. Y. Chen, "Data-driven interval type-2 neural fuzzy system with high learning accuracy and improved model interpretability," IEEE Trans. Cybern., vol. 43, no. 6, pp. 1781-1795, 2013.

[21] R. M. Rodríguez, L. Martínez, F. Herrera, "Hesitant fuzzy linguistic terms sets for decision making," IEEE Trans. Fuzzy Syst., vol. 20, pp. 109-119, 2012.

[22] F. Herrera, L. Martínez, "A 2-tuple fuzzy linguistic representation model for computing with words," IEEE Trans. Fuzzy Syst., vol. 8, pp. 746-752, 2000.

[23] Z. S. Xu, H. Wang, "On the syntax and semantics of virtual linguistic terms for information fusion in decision making," Inf. Fusion, vol. 34, pp. 43-48, 2017.

[24] H. Wang, Z. S. Xu, X. J. Zeng, "Linguistic terms with weakened hedges: A model for qualitative decision making under uncertainty," Inf. Sci., vol. 433, pp. 37-45, 2018.

[25] X. J. Gou, H. C. Liao, Z. S. Xu, F. Herrera, "Double hierarchy hesitant fuzzy linguistic term set and MULTIMOORA method: A case of study to evaluate the implementation status of haze controlling measures," Inf. Fusion, vol. 38, pp. 22-34, 2017.

[26] Y. C. Dong, H. J. Zhang, E. Herrera-Viedma, "Integrating experts' weights generated dynamically into the consensus reaching process and its applications in managing non-cooperative behaviors," Decis. Support Syst., vol. 84, pp. 1-15, 2016.

[27] C. Q. Xiong, D. H. Li, L. H. Jin, "Group consistency analysis for protecting the minority view," Syst. Eng. Theory Pract., vol. 10, pp. 102–107, 2008.

[28] L. A. Zadeh, "Computing with Words: What is computing with words (CWW)?" Springer, Berlin Heidelberg, pp.1-40, 2012.

[29] J. Montserrat-Adell, Z. S. Xu, X. J. Gou, N. Agell, "Free double hierarchy hesitant fuzzy linguistic term sets: An application on raking alternatives in GDM," Inf. Fusion, vol. 47, pp. 45-59, 2019.

[30] R. R. Yager, "Penalizing strategic preference manipulation in multi-agent decision making," IEEE Trans. Fuzzy Syst., vol. 9, no. 3, pp. 393-403, 2001.

[31] X. J. Gou, Z. S. Xu, H. C. Liao, F. Herrera, "Multiple criteria decision making based on distance and similarity measures with double hierarchy hesitant fuzzy linguistic term sets," Comput. Ind. Eng., vol. 126, pp. 516-530, 2018.

[32] E. Herrera-Viedma, L. Martinez, F. Mata, F. Chiclana, "A consensus support system model for group decision making problems with multigranular linguistic preference relations," IEEE Trans. Fuzzy Syst., vol. 13, no. 5, pp. 644-658, 2005.

[33] R. Parreiras, P. Ekel, F. Bernardes, "A dynamic consensus scheme based on a nonreciprocal fuzzy preference relation modeling," Inf. Sci., vol. 211, pp. 1-17, 2012.

[34] Z. S. Xu, "An automatic approach to reaching consensus in multiple attribute group decision making," Comput. Ind. Eng., vol. 56, no. 4, pp. 1369-1374, 2009.

[35] Q. L. Da, X. W. Liu, "Interval number linear programming and the satisfactory solution," Syst. Eng.-Theory Pract., vol. 19, pp. 3-7, 1999.

Bibliography

[AHVFH10] Alonso S., Herrera-Viedma E., F. C., and Herrera F. (2010) A web based consensus support system for group decision making problems and incomplete preferences. *Information Sciences* 180: 4477–4495.

[BMSJ06] Baets B. D., Meyer H. D., Schuymer B. D., and Jenei S. (2006) Cyclic evaluation of transitivity of reciprocal relations. *Social Choice and Welfare* 26(2): 217–238.

[CHVAH09] Chiclana F., Herrera-Viedma E., Alonso S., and Herrera F. (2009) Cardinal consistency of reciprocal preference relations: A characterization of multiplicative transitivity. *IEEE Transactions on Fuzzy Systems* 17(1): 14–23.

[DXL08] Dong Y., Xu Y., and Li H. (2008) On consistency measures of linguistic preference relations. *European Journal of Operational Research* 189(2): 430–444.

[GX16] Gou X. and Xu Z. (2016) Novel basic operational laws for linguistic terms, hesitant fuzzy linguistic term sets and probabilistic linguistic term sets. *Information Sciences* 372: 407–427.

[HM00] Herrera F. and Martínez L. (2000) A 2-tuple fuzzy linguistic representation model for computing with words. *IEEE Transactions on fuzzy systems* 8(6): 746–752.

[LC06] Liu R. and Chen X. (2006) Improved clustering algorithm and its application in complex huge group decision making. *Systems Engineering and Electronics* 28(11): 1695–1699.

[LDC+17] Liu W. Q., Dong Y. C., Chiclana F., Cabrerizo F. J., and Herrera-Viedma E. (2017) Group decision making based on heterogeneous preference relations with self-confidence. *Fuzzy Optimization and Decision Making* 16(4): 429–447.

[LXZ14] Liao H., Xu Z., and Zeng X. (2014) Distance and similarity measures for hesitant fuzzy linguistic term sets and their application in multi-criteria decision making. *Information Sciences* 271: 125–142.

[LXZM15] Liao H., Xu Z., Zeng X., and Merigó (2015) Qualitative decision making with correlation coefficients of hesitant fuzzy linguistic term sets. *Knowledge-Based Systems* 76: 127–138.

[Men02] Mende J. (2002) An architecture for making judgement using computing with words. *International Journal of Applied Mathematics and Computer Science* 12(3): 325–335.

[PEMH14] Palomares I., Estrella F., Martínez L., and Herrera F. (2014) Consensus under a fuzzy context: Taxonomy, analysis framework afryca and experimental case of study. *Information Fusion* 20: 252–271.

[PMH14] Palomares I., Martínez L., and Herrera H. (2014) A consensus model to detect and manage noncooperative behaviors in large-scale group decision making. *IEEE Transactions on Fuzzy Systems* 22(3): 516–530.

[PWX16] Pang Q., Wang H., and Xu Z. (2016) Probabilistic linguistic term sets in multi-attribute group decision making. *Information Sciences* 369: 128–143.

[RMH12] Rodríguez R., Martínez L., and Herrera F. (2012) Hesitant fuzzy linguistic term sets for decision making. *IEEE Transactions on Fuzzy Systems* 20(1): 109–119.

[Tor10] Torra V. (2010) Hesitant fuzzy sets. *International Journal of Intelligent Systems* 25(6): 529–539.

[WXZ18] Wang H., Xu Z., and Zeng X. (2018) Linguistic terms with weakened hedges: A model for qualitative decision making under uncertainty. *Information Sciences* 433-434: 37–45.

[XDC15] Xu X., Du Z., and Chen X. (2015) Consensus model for multi-criteria large-group emergency decision making considering non-cooperative behaviors and minority opinions. *Decision Support Systems* 79: 150–160.

[Xu05] Xu Z. (2005) Deviation measures of linguistic preference relations in group decision making. *Omega* 33(3): 249–254.

[XW17] Xu Z. and Wang H. (2017) On the syntax and semantics of virtual linguistic terms for information fusion in decision making. *Information Fusion* 34: 43–48.

[Yag04] Yager R. (2004) On the retranslation process in zadeh's paradigm of computing with words. *IEEE Transactions on Systems, Man and Cybernetics, Part B: Cybernetics* 34(2): 1184–1195.

[Zad75a] Zadeh L. (1975) The concept of a linguistic variable and its application to approximate reasoning-I. *Information sciences* 8(3): 199–249.

[Zad75b] Zadeh L. (1975) The concept of a linguistic variable and its application to approximate reasoning-II. *Information sciences* 8(4): 301–357.

[Zad75c] Zadeh L. (1975) The concept of a linguistic variable and its application to approximate reasoning-III. *Information sciences* 9(1): 43–80.

[Zad12] Zadeh L. (2012) *Computing with Words: What is computing with words (CWW)?* Springer.

[ZX14] Zhu B. and Xu Z. (2014) Consistency measures for hesitant fuzzy linguistic preference relations. *IEEE Transactions on Fuzzy Systems* 22(1): 35–45.