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Renormalization of vector fields with mass-like coupling in curved spacetime

C. Garcia-Recio[a], **L. L. Salcedo**[b]

Departamento de Física Atómica, Molecular y Nuclear and Instituto Carlos I de Física Teórica y Computacional, Universidad de Granada, 18071 Granada, Spain

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Abstract Using the method of covariant symbols we compute the divergent part of the effective action of the Proca field with non-minimal mass term. Specifically a quantum abelian vector field with a non-derivative coupling to an external tensor field in curved spacetime in four dimensions is considered. Relatively explicit expressions are obtained which are manifestly local but non polynomial in the external fields. Our result is shown to reproduce existing ones in all particular cases considered. Internal consistency with Weyl invariance is also verified.

Contents

1 Introduction

Although models involving scalar fields are the most commonly considered in applications of relativistic gravity and cosmology, e.g. for inflation or $f(R)$ gravity, vector fields also attract considerable interest [1–3]. As regards to introducing a persistent anisotropy after inflation, it has been pointed out that minimally coupled vector fields would not suffice so non-minimally coupled models have been considered [4]. In such non minimal models the vector field can be coupled to a mass-like term $M^{\mu\nu}(x)$ with a possibly local dependence and a possibly non trivial tensor structure (see Eq. (2.1)). Most of these studies are at the classical level and it is only natural to investigate the effect of quantum fluctuations. As it turns out, the evaluation of the quantum fluctuations of vector fields with a non minimal coupling is not entirely straightforward. For scalar or minimal vector theories, the ultraviolet (UV) divergent part of the effective action, $\Gamma^{\rm div}$, is local (hence a polynomial with respect to the covariant derivatives) and also a polynomial in the external fields. A notorious exception to this rule is the metric, due to its coupling to the kinetic energy term in the action. Nevertheless, locality still requires that terms involving derivatives of the metric are a polynomial in the curvature and derivatives of it. At variance with this, for a generalized Proca field locality is preserved but $\Gamma^{\rm div}$ is no longer a polynomial in $M^{\mu\nu}(x)$.[1]

The peculiar behavior is due to the kinetic energy term. As is well-known, in a direct Lorentz covariant formulation, the

[a] e-mail: g_recio@ugr.es
[b] e-mail: salcedo@ugr.es

[1] Unless, $M^{\mu\nu}(x) = m^2 g^{\mu\nu}$ with constant m. In this case $\Gamma^{\rm div}$ is indeed a polynomial in m^2.

kinetic-energy term of an abelian vector field displays U(1) gauge invariance. This implies that the quantum fluctuations are not efficiently quenched for all polarizations, resulting in a propagator with a singular kernel. A mass term breaks such gauge invariance and changes the number of propagating degrees of freedom, but the leading (i.e., most UV divergent) term of the action is still singular. The mass term introduces a penalty to large amplitude fluctuations of the vector field, but large wavenumbers are not suppressed for the longitudinal polarization. When this issue is resolved, removing spurious degrees of freedom, one finds that $M^{\mu\nu}(x)$ behaves as an additional metric field.

Early studies of non-minimally coupled vector fields were undertaken in [5] at the classical level and in [6] at the quantum level. The first explicit attempt to a calculation of Γ^{div} for the action in Eq. (2.1) have been addressed in [7] using the local momentum approach [8]. To cope with the above mentioned singular-kernel problem, the canonical quantization scheme of Faddeev and Jackiw [9] was used. The results in [7] are partial because only the ultrastatic case is considered in detail. It is correctly concluded that UV divergences can not be removed by a local and polynomial (in $M^{\mu\nu}$) counterterm Lagrangian. A new attempt was taken in [10]. Unlike the canonical quantization approach, manifest relativistic covariance was preserved by using the Stückelberg's method [11,12] to transform the action into one with exact gauge symmetry. The gauge is then fixed, including the usual compensating Faddeev–Popov term. The new action contains now a vector field and a scalar field (plus a ghost field that is completely decoupled from the other fields). The approach of [10] was to diagonalize the vector–scalar action using a non-local kernel. Unfortunately, as noted in [13], the detailed implementation of this step is questionable, and the resulting divergent part of the effective action turned out to be non local. A complete and impeccable calculation of Γ^{div} has been carried out in [13], where also the various types of generalized Proca fields are classified. The calculation there, besides using Stückelberg's method, exploits the Weyl invariance of the action (see Sect. 2). In this way the problem is transformed into one where the external fields are two metric fields and a heat kernel approach is then applied. The final result is expressed in terms of the two metric fields (and their corresponding connection and curvature structures). It is fully local, although not polynomial in $M^{\mu\nu}(x)$, and several cross-checks are satisfied.

In this work, we also carry out a calculation of the functional $\Gamma^{\text{div}}[g_{\mu\nu}, M^{\mu\nu}]$ for the action in Eq. (2.1), starting from the Stückelberg formulation introduced in [10]. The difference between our calculation and that in [13] is that we use throughout the original metric $g_{\mu\nu}$, with the exception of a term for which a different metric is clearly superior, and in any case just one metric is present in each single term of the final result. Another difference is that, instead of the heat kernel, we use the method of covariant symbols, which seems quite appropriate for this kind of problems. The method was introduced in [14] for flat spacetime and extended to curved spacetime in [15], and also to finite temperature in [16]. It has been applied to fermions [17–20] and to obtain a strict derivative expansion of the heat kernel in curved spacetime [21]. The method of covariant symbols is related to the method of symbols (of pseudodifferential operators) as described in [22,23], where the shift $\nabla_\mu \to \nabla_\mu + p_\mu$ is applied and p_μ represents the momentum of the particle running in the quantum loop. However, in the method of covariant symbols results are manifestly covariant (i.e., the covariant derivative appears only in the form $[\nabla_\mu,\ ]$) and in this sense it is closer to the momentum space approach of [8]. Indeed, p_μ is introduced in such a way that any pseudodifferential operator constructed out of ∇_μ and other multiplicative operators is mapped to a covariant operator which is multiplicative with respect to ∇_μ (although it may contain derivatives with respect to p_μ). This guarantees that all the expressions are local throughout the calculation and the UV divergence is controlled by the integration over the loop momentum p_μ. Moreover, the map is an algebra homomorphism, hence the covariant symbol of any operator is immediately obtained from the covariant symbols of its building blocks (e.g., ∇_μ and $M^{\mu\nu}$). Our final result avoids the use of a bimetric formulation yet it agrees with previous results in the literature and in particular it correctly reproduces those in [13].

The paper is organized as follows. In Sect. 2 we discuss the formulation of the problem to make the kernel a regular one at the price of introducing the Stückelberg scalar field. The expansion organizing the calculation is spelled out, and the Weyl symmetry of the problem is noted. In Sect. 3 we present the calculation of the terms which are elementary. Also there we summarize the method of covariant symbols, which is already applied in that section for some of the terms. In Sect. 4 the remaining terms are computed through a systematic use of the method of covariant symbols. Rather explicit expressions are obtained involving only the original metric. The number of terms has been minimized using integration by parts. Some elliptic integrals are left implicit, as more detailed expressions would not be helpful. Several checks of the result are done in Sect. 5, considering particular cases or expansions and the validity of Weyl invariance of the final expression. Our conclusions are presented in Sect. 6. Further details regarding conventions, proving auxiliary results, or summarizing covariant symbols properties are presented in the appendices.

2 Formulation of the problem

The goal is to obtain the divergent part of the effective action, Γ^{div}, of an abelian vector field $A_\mu(x)$ in curved spacetime coupled to an external tensor field. The divergent part of the effective action will be extracted using dimensional regularization and Euclidean signature is used throughout.

The action is given by

$$S = \int d^4x\sqrt{g}\left(\frac{1}{4}\mathscr{F}^{\mu\nu}\mathscr{F}_{\mu\nu} + \frac{1}{2}M^{\mu\nu}A_\mu A_\nu\right) \tag{2.1}$$

where

$$\mathscr{F}_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu. \tag{2.2}$$

The connection is the Levi-Civita connection for the Riemannian metric $g_{\mu\nu}$, hence $\mathscr{F}_{\mu\nu}$ coincides with $\partial_\mu A_\nu - \partial_\nu A_\mu$. Unless otherwise stated $g_{\mu\nu}$ is used to raise, lower and contract world indices. $M^{\mu\nu}(x)$ is an abelian symmetric tensor field which is assumed to be positive definite, so that the Gaussian functional integration over $A_\mu(x)$ converges for large amplitude fluctuations.

The kinetic term is gauge invariant, implying that fluctuations with large wavenumbers are not suppressed for the longitudinal polarization. To cope with this problem we follow [10] and apply Stückelberg's method. A new scalar field φ is introduced and the field B_μ is defined through the change of variables

$$A_\mu = B_\mu + \frac{1}{m}\nabla_\mu\varphi. \tag{2.3}$$

The mass m is arbitrary and is introduced so that φ has the standard dimensions. Since it can be reabsorbed in the field and its value has no effect on the final result (as is readily verified) we set $m = 1$ from now on. In the new variables the action takes the form

$$S = \int d^4x\sqrt{g}\left(\frac{1}{4}\mathscr{F}^{\mu\nu}\mathscr{F}_{\mu\nu} + \frac{1}{2}M^{\mu\nu}B_\mu B_\nu + M^{\mu\nu}B_\mu\nabla_\nu\varphi + M^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi\right) \tag{2.4}$$

and $\mathscr{F}_{\mu\nu} = \nabla_\mu B_\nu - \nabla_\nu B_\mu$. The whole action is now gauge invariant (namely, under $B_\mu \to B_\mu + \partial_\mu\Lambda$, $\varphi \to \varphi - \Lambda$) since A_μ is. The next step is to fix the gauge. A convenient choice is obtained by adding the term

$$S_{\rm gf} = \int d^4x\sqrt{g}\,\frac{1}{2}(\nabla^\mu B_\mu)^2 \tag{2.5}$$

as well as the compensating Faddeev–Popov ghost term

$$S_{\rm gh} = \int d^4x\sqrt{g}\,\nabla^\mu\omega^*\nabla_\mu\omega \tag{2.6}$$

where $\omega(x)$ is a scalar complex fermionic field.

The total action $S_{\rm tot} = S + S_{\rm gf} + S_{\rm gh}$ can be expressed as

$$S_{\rm tot} = S_{\rm gh} + \int d^4x\sqrt{g}\,\frac{1}{2}\phi^\dagger\hat{K}\phi \tag{2.7}$$

with

$$\phi = \begin{pmatrix} B_\mu \\ \varphi \end{pmatrix}, \quad \hat{K} = \begin{pmatrix} \hat{F}^{\mu\nu} & \hat{H}^\mu \\ \hat{H}^{\dagger\mu} & \hat{G} \end{pmatrix}. \tag{2.8}$$

The differential operators $\hat{F}$, $\hat{H}$ and $\hat{G}$ are given by

$$\hat{F}^{\mu\nu} = -g^{\mu\nu}\Box + \mathscr{R}^{\mu\nu} + M^{\mu\nu}, \quad \hat{G} = -\nabla_\mu M^{\mu\nu}\nabla_\nu,$$
$$\hat{H}^\mu = M^{\mu\nu}\nabla_\nu, \quad \hat{H}^{\dagger\mu} = -\nabla_\nu M^{\mu\nu}. \tag{2.9}$$

where $\Box \equiv \nabla^\mu\nabla_\mu$ and $\mathscr{R}_{\mu\nu}$ is Ricci's tensor. This tensor is generated from $\frac{1}{4}\mathscr{F}^2_{\mu\nu} + \frac{1}{2}(\nabla^\mu B_\mu)^2$, using integration by parts to give $-\frac{1}{2}B^\mu\Box B_\mu - \frac{1}{2}B_\mu[\nabla^\mu, \nabla^\nu]B_\nu$ up to boundary terms. The operator $\hat{F}$ acts on the space of vectors, while $\hat{G}$ acts on the space of scalars.

Functional integration over B_μ, φ and ω provides the effective action

$$\Gamma = \Gamma_K + \Gamma_{\rm gh}, \quad \Gamma_K = \frac{1}{2}{\rm Tr}\,\log\hat{K},$$
$$\Gamma_{\rm gh} = -{\rm Tr}_0\log(-\Box). \tag{2.10}$$

The subindex zero in $\Gamma_{\rm gh}$ indicates to take the functional trace in the space of scalars.

In order to compute Γ_K we split $\hat{K}$ as

$$\hat{K} = \hat{K}_D + \hat{K}_A, \quad \hat{K}_D = \begin{pmatrix} \hat{F} & 0 \\ 0 & \hat{G} \end{pmatrix}, \quad \hat{K}_A = \begin{pmatrix} 0 & \hat{H} \\ \hat{H}^\dagger & 0 \end{pmatrix}. \tag{2.11}$$

This allows to make the expansion

$$\Gamma_K = \sum_{n=0}^{\infty}\Gamma_{K,n}, \quad \Gamma_{K,0} = \frac{1}{2}{\rm Tr}\log\hat{K}_D,$$
$$\Gamma_{K,n} = \frac{(-1)^{n+1}}{2n}{\rm Tr}((\hat{K}_D^{-1}\hat{K}_A)^n) \quad (n > 0). \tag{2.12}$$

In this expansion all terms with odd n vanish since $\hat{K}_A$ has to appear an even number of times to have a non null contribution to the trace. In addition, terms with $n > 4$ are UV convergent, so only $\Gamma_{K,n}$ for $n = 0, 2, 4$ have a contribution to $\Gamma^{\rm div}$:

$$\Gamma^{\rm div}_K = \Gamma^{\rm div}_{K,0} + \Gamma^{\rm div}_{K,2} + \Gamma^{\rm div}_{K,4}. \tag{2.13}$$

The zeroth term can be further expanded as

$$\Gamma_{K,0} = \Gamma_F + \Gamma_G, \quad \Gamma_F = \frac{1}{2}{\rm Tr}_1\log\hat{F}, \quad \Gamma_G = \frac{1}{2}{\rm Tr}_0\log\hat{G}. \tag{2.14}$$

Before finishing this section, let us note the *Weyl symmetry* present in the action, namely S is invariant under the local rescaling

$$g_{\mu\nu}(x) \to g^\Omega_{\mu\nu}(x) = \Omega^2(x)\,g_{\mu\nu}(x),$$
$$M^{\mu\nu}(x) \to (M^\Omega)^{\mu\nu}(x) = \Omega^{-4}(x)\,M^{\mu\nu}(x). \tag{2.15}$$

This symmetry can be secured in the final result by using Weyl-invariant combinations, for instance

$$(g^{\Omega}_{\mu\nu}, (M^{\Omega})^{\mu\nu}) = (\hat{g}_{\mu\nu}, \tilde{g}^{\mu\nu}) \quad \text{with } \Omega = (\det(M^{\alpha}{}_{\beta}))^{1/8}. \tag{2.16}$$

This choice corresponds to the prescription $\det \hat{g}_{\mu\nu} = \det \tilde{g}_{\mu\nu}$, where $\tilde{g}_{\mu\nu}$ stands for the inverse matrix of $\tilde{g}^{\mu\nu}$. This is the approach adopted in [13]. Here we take the alternative route of using directly the original pair of external fields $(g_{\mu\nu}, M^{\mu\nu})$ and Weyl invariance will provide a check of the calculation. An exception is taken in the case of Γ_G since there the advantages of using $\tilde{g}_{\mu\nu}$ are overwhelming.

3 Elementary contributions to Γ^{div}

3.1 Γ_{gh}

The value of $\text{Tr}_0 \log(-\Box)$ is a standard result [24] that can be obtained in many ways. In terms of heat kernel coefficients a well-known relation is

$$\text{Tr} \log(-\Box)\big|_{\text{div}} = \frac{1}{(4\pi)^2}\frac{1}{\epsilon}\int d^4x\sqrt{g}\,\text{tr}(b_2(x)) \tag{3.1}$$

where

$$d = 4 + 2\epsilon \tag{3.2}$$

is the dimension parameter in dimensional regularization.[2] The explicit form of the second Schwinger–DeWitt coefficient is (see e.g. [21])

$$b_2 = \frac{1}{12}Z^2_{\mu\nu} + \frac{1}{72}\mathbf{R}^2 - \frac{1}{180}\mathscr{R}^2_{\mu\nu} + \frac{1}{180}R^2_{\mu\nu\alpha\beta}. \tag{3.3}$$

(See Appendix A for definitions of the symbols and conventions used in this work.) This expression of b_2 holds for any tensor space. In the particular case of the scalar space $Z_{\mu\nu} := [\nabla_\mu, \nabla_\nu]$ vanishes, and $\text{tr}_0(1) = 1$, hence

$$\text{tr}_0(b_2) = \frac{1}{180}\mathscr{G} + \frac{1}{60}\mathscr{R}^2_{\mu\nu} + \frac{1}{120}\mathbf{R}^2, \tag{3.4}$$

where, following [13], we have expressed the result using the topological Gauss–Bonnet term

$$\mathscr{G} = \mathbf{R}^2 - 4\mathscr{R}^2_{\mu\nu} + R^2_{\mu\nu\alpha\beta}. \tag{3.5}$$

The final result for the ghost contribution is therefore

$$\Gamma^{\text{div}}_{\text{gh}} = \frac{1}{32\pi^2\epsilon}\int d^4x\sqrt{g}\left(-\frac{1}{90}\mathscr{G} - \frac{1}{30}\mathscr{R}^2_{\mu\nu} - \frac{1}{60}\mathbf{R}^2\right). \tag{3.6}$$

[2] All our calculations are consistent with the results in [13], up to an overall minus sign. This should indicate that $d = 4 - 2\epsilon$ is being used in that reference.

3.2 Γ_G

The term Γ_G can be identified with the effective action corresponding to the action

$$S_G = \int d^4x\sqrt{g}\,\frac{1}{2}M^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi. \tag{3.7}$$

To deal with this term one approach is that of [10] where $M^{\mu\nu}$ is directly used as an alternative (contravariant) metric. However, simpler expressions are obtained by using as new metric $\tilde{g}_{\mu\nu}$ defined through the condition [13,25]

$$\sqrt{g}\,M^{\mu\nu} = \sqrt{\tilde{g}}\,\tilde{g}^{\mu\nu}, \tag{3.8}$$

hence $\tilde{g}_{\mu\nu}$ is the inverse of $\tilde{g}^{\mu\nu} = M^{\mu\nu}/\sqrt{\det(M^{\lambda}{}_{\sigma})}$. In this way S_G takes the standard form

$$S_G = \int d^4x\sqrt{\tilde{g}}\,\frac{1}{2}\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\varphi\tilde{\nabla}_\nu\varphi. \tag{3.9}$$

This immediately implies that

$$\Gamma_G = \frac{1}{2}\tilde{\text{Tr}}_0 \log(-\tilde{\Box}) \tag{3.10}$$

and in turn

$$\Gamma^{\text{div}}_G = \frac{1}{32\pi^2\epsilon}\int d^4x\sqrt{\tilde{g}}\left(\frac{1}{180}\tilde{\mathscr{G}} + \frac{1}{60}\tilde{\mathscr{R}}^2_{\mu\nu} + \frac{1}{120}\tilde{\mathbf{R}}^2\right), \tag{3.11}$$

and of course, $\tilde{g}_{\mu\nu}$ is used everywhere in this expression instead of $g_{\mu\nu}$.

As noted in [13], the combination $\sqrt{\tilde{g}}\,\tilde{\mathscr{G}}$ can be replaced with $\sqrt{g}\,\mathscr{G}$, since its integral is a topological invariant. Another observation is that this expression is invariant under a global rescaling of the metric. Since $\tilde{g}_{\mu\nu}$ picks up a factor $1/m^2$ when the mass parameter m is not set to unity, the invariance property checks that the value of m is not relevant here.

We emphasize that a single metric (and its derived structures) will be used in any single contribution to the effective action. Only $\tilde{g}_{\mu\nu}$ appears in Γ^{div}_G while only $g_{\mu\nu}$ appears in our formulas for all the remaining terms of Γ^{div}.

3.3 Γ_F (part 1)

The expression for Γ_F in (2.14) also follows from the second heat-kernel coefficient for the operator $\hat{F}^{\mu\nu}$ and could be borrowed directly from the results in the literature, however we will evaluate it here in order to introduce the technique of covariant symbols to be exploited in the computation of $\Gamma_{K,2}$ and $\Gamma_{K,4}$.

Let us split $\hat{F}^{\mu\nu}$ into two terms as

$$\hat{F}^{\mu\nu} = -g^{\mu\nu}\Box + Y^{\mu\nu}, \quad Y^{\mu\nu} = \mathscr{R}^{\mu\nu} + M^{\mu\nu} \tag{3.12}$$

and apply an expansion in powers of $Y^{\mu\nu}$

$$\Gamma_F = \frac{1}{2}\mathrm{Tr}_1 \log(-g^{\mu\nu}\Box + Y^{\mu\nu})$$
$$= \frac{1}{2}\mathrm{Tr}_1 \log(-\Box) - \sum_{n=1}^{\infty} \frac{1}{2n}\mathrm{Tr}_1((\Box^{-1}Y^{\mu}{}_{\nu})^n) \tag{3.13}$$
$$:= \sum_{n=0}^{\infty} \Gamma_{F,n}.$$

The UV divergent part of this series finishes at $n = 2$ since terms $n \geq 3$ are already UV convergent:

$$\Gamma_F^{\mathrm{div}} = \Gamma_{F,0}^{\mathrm{div}} + \Gamma_{F,1}^{\mathrm{div}} + \Gamma_{F,2}^{\mathrm{div}}. \tag{3.14}$$

For $\Gamma_{F,0}^{\mathrm{div}}$ (3.1) applies (with Tr_1 and tr_1) as well as (3.3). There the term with $Z_{\mu\nu}$ no longer vanishes, instead (using (A1))

$$\mathrm{tr}_1(Z_{\mu\nu}^2) = -R_{\mu\nu\alpha\beta}^2. \tag{3.15}$$

Together with $\mathrm{tr}_1(1) = 4$, this yields the result

$$\Gamma_{F,0}^{\mathrm{div}} = \frac{1}{32\pi^2\epsilon}\int d^4x\sqrt{g}\left(-\frac{11}{180}\mathscr{G} - \frac{4}{15}\mathscr{R}_{\mu\nu}^2 + \frac{7}{60}\mathbf{R}^2\right). \tag{3.16}$$

In order to compute the remaining terms $\Gamma_{F,1}^{\mathrm{div}}$ and $\Gamma_{F,2}^{\mathrm{div}}$ we will apply the method of covariant symbols.

3.4 Aside: covariant symbols

For an operator $\hat{\mathscr{O}} = \mathscr{O}(Y, \nabla_\mu)$ constructed with one or more multiplicative operators $Y(x)$ and the covariant derivative ∇_μ (which may include all kind of connections, gauge or other) its covariant symbol is defined as

$$\overline{\mathscr{O}} := e^{-\frac{1}{2}\{\nabla_\mu,\partial^\mu\}} e^{-\xi^\alpha p_\alpha} \hat{\mathscr{O}} e^{\xi^\beta p_\beta} e^{\frac{1}{2}\{\nabla_\nu,\partial^\nu\}}\Big|_{\xi^\mu=0}. \tag{3.17}$$

Here $\{,\}$ denotes the anticommutator, ξ^μ are the Riemann coordinates with origin at the point x and corresponding to the connection in ∇_μ, although we will only consider the Levi-Civita connection here. In addition

$$\partial^\mu := \frac{\partial}{\partial p_\mu} \tag{3.18}$$

and p_μ is a momentum variable to be used as integration variable. For convenience, in order to avoid a proliferation of factors i, we use a purely imaginary p_μ, hence $p_\mu = ik_\mu$ and k_μ (real) is the actual integration variable (still we use d^dp as notation). Of course, the operator $\hat{\mathscr{O}}$ itself is assumed not to depend on p_μ or ∂^μ.

The covariant symbols were introduced in [14] for flat spacetime and extended to curved spacetime in [15]. The relevant properties of the covariant symbols are (see [15] for details):

1. $\overline{\mathscr{O}}$ is a covariant multiplicative operator with respect to x, although contains derivatives with respect to p_μ. In addition $\overline{\mathscr{O}^\dagger} = (\overline{\mathscr{O}})^\dagger$, hence when $\mathscr{O}$ is hermitian its covariant symbol is also hermitian.
2. The map $\hat{\mathscr{O}} \to \overline{\mathscr{O}}$ is an algebra homomorphism, since it is defined from a similarity transformation. This implies that
$$\overline{f(\mathscr{O}_1, \ldots, \mathscr{O}_n)} = f(\overline{\mathscr{O}_1}, \ldots, \overline{\mathscr{O}_n}), \tag{3.19}$$
and in particular $\overline{\mathscr{O}} = \mathscr{O}(\overline{Y}, \overline{\nabla}_\mu)$. Note also that $\overline{g}_{\mu\nu} = g_{\mu\nu}$ for the Levi-Civita connection, hence $\overline{(g^{\mu\nu}A_\nu)} = g^{\mu\nu}\overline{A}_\nu$, etc.
3. The diagonal matrix elements can be rewritten as
$$\langle x|\hat{\mathscr{O}}|x\rangle = \frac{1}{\sqrt{g(x)}}\int \frac{d^dp}{(2\pi)^d}\overline{\mathscr{O}}(x, p), \tag{3.20}$$
where it is understood that all ∂^μ at the rightmost position vanish (and also at the leftmost position, from integration by parts). Therefore, the relation
$$\mathrm{Tr}(\hat{\mathscr{O}}) = \int d^4x\sqrt{g(x)}\,\mathrm{tr}(\langle x|\hat{\mathscr{O}}|x\rangle) \tag{3.21}$$
implies
$$\mathrm{Tr}(\hat{\mathscr{O}}) = \int \frac{d^4x\, d^dp}{(2\pi)^d}\mathrm{tr}(\overline{\mathscr{O}}(x, p)). \tag{3.22}$$
Equations (3.20) and (3.22) are key relations in the covariant symbol technique, allowing to compute diagonal matrix elements of operators or functional traces as those appearing in the effective action.

For short we will introduce the notations

$$\langle f\rangle_x \equiv \int d^4x\sqrt{g}\, f, \quad \langle f\rangle_p \equiv \frac{1}{\sqrt{g}}\int \frac{d^dp}{(2\pi)^d} f \tag{3.23}$$

as well as $\langle f\rangle_{x,p} \equiv \langle\langle f\rangle_p\rangle_x$, so that

$$\langle x|\hat{\mathscr{O}}|x\rangle = \left\langle\overline{\mathscr{O}}\right\rangle_p, \quad \mathrm{Tr}(\hat{\mathscr{O}}) = \left\langle\mathrm{tr}(\overline{\mathscr{O}})\right\rangle_{x,p}. \tag{3.24}$$

The explicit form of the covariant symbols for basic operators has been obtained in [15] in a covariant derivative expansion up to two derivatives for a general connection and to four derivatives when the Levi-Civita connection in the world sector is selected (but still arbitrary with respect to gauge or internal indices). The following results are useful

$$\overline{Y} = Y - Y_\alpha\partial^\alpha + \frac{1}{2!}Y_{\alpha\beta}\partial^\alpha\partial^\beta - \frac{1}{3!}Y_{\alpha\beta\gamma}\partial^\alpha\partial^\beta\partial^\gamma + \cdots \tag{3.25}$$

Here Y is any operator that is multiplicative with respect to x, i.e., not containing "free" ∇_μ (all derivatives appear in the form $[\nabla_\mu,\ ]$) and not containing "free" $Z_{\mu_1\ldots\mu_n}$ (see Appendix A for notational conventions). Y may have world indices and we have used the convention of adding new indices to the left to indicate covariant derivatives. So if e.g. $Y = A_\alpha$, $Y_{\mu\nu}$ would be $A_{\mu\nu\alpha} \equiv [\nabla_\mu, [\nabla_\nu, A_\alpha]]$. Furthermore

$$\overline{\nabla}_\mu = p_\mu + \frac{1}{2} Z_{\mu\alpha}\partial^\alpha + \frac{1}{6}\mathscr{R}_{\mu\alpha}\partial^\alpha + \frac{1}{6}R_{\mu\alpha\lambda\beta}p^\lambda\partial^\alpha\partial^\beta + \cdots \tag{3.26}$$

$$\overline{\Box} = p^\mu p_\mu + \frac{1}{6}\mathbf{R} + Z_{\lambda\alpha}p^\lambda\partial^\alpha - \frac{1}{3}\mathscr{R}_{\lambda\alpha}p^\lambda\partial^\alpha + \frac{1}{3}R_{\lambda\alpha\sigma\beta}p^\lambda p^\sigma\partial^\alpha\partial^\beta + \cdots \tag{3.27}$$

Fuller expressions can be found in Appendix B and in [15].

The expansions just presented can be organized by the number of covariant derivatives so that, for instance $Z_{\mu\nu}$, $R_{\mu\nu\alpha\beta}$, $\mathscr{R}_{\mu\nu}$ and $\mathbf{R}$ count as second order, $g_{\mu\nu}$ as zeroth order, etc. Alternatively one can grade a term counting the number N_p of p_μ minus the number of ∂^μ in that term. Hence the expansions for $\overline{Y}$, $\overline{\nabla}_\mu$ and $\overline{\Box}$ start at orders $N_p = 0, 1, 2$ and have been made explicit through orders $N_p = -3, -1$, and 0, respectively. With this convention one can write, for instance,

$$\overline{Y} = (Y)_0 + (Y)_{-1} + (Y)_{-2} + (Y)_{-3} + O(p^{-4}) \tag{3.28}$$

with

$$(Y)_{-n} = \frac{(-1)^n}{n!} Y_{\alpha_1\ldots\alpha_n}\partial^{\alpha_1}\cdots\partial^{\alpha_n}. \tag{3.29}$$

N_p is an additive index related to the degree of UV divergence of a term.

Within the covariant symbols technique there are no free ∇_μ as the covariant symbols are multiplicative, however, there are $Z_{\mu\nu}$ or more generally $Z_{\mu_1\ldots\mu_n}$. These quantities are multiplicative with respect to x but act on world indices, hence they do not commute with p_μ and ∂^μ, instead

$$[Z_{\mu_1\ldots\mu_n}, p_\alpha] = R_{\mu_1\ldots\mu_n\alpha\lambda}p^\lambda, \quad [Z_{\mu_1\ldots\mu_n}, \partial_\alpha] = R_{\mu_1\ldots\mu_n\alpha\lambda}\partial^\lambda. \tag{3.30}$$

An often convenient tool to deal with the momentum integral in $\langle f\rangle_p$ is to introduce a tetrad field $e_a^\mu(x)$ to make a change of variables from p_μ to k_a:

$$g^{\mu\nu} = \delta_{ab}e_a^\mu e_b^\nu, \quad \delta_{ab} = e_\mu^a e_b^\mu, \quad \det(e_\mu^a) = \sqrt{g}, \quad p_\mu = ik_a e_\mu^a. \tag{3.31}$$

In this way, if $f(p, X)$ is an expression tensorially constructed out of p_μ and tensors $X(x)$ (the operators ∂^μ are assumed to be no longer present),

$$\langle f(p,X)\rangle_p = \frac{1}{\sqrt{g}}\int \frac{d^dp}{(2\pi)^d} f(p,X) = \int \frac{d^dk}{(2\pi)^d} f(k,X,e), \tag{3.32}$$

where $f(k, X, e)$ is tensorially constructed out of X and e_μ^a, and the scalars k_a, checking that $\langle f\rangle_p$ is indeed a tensor. Upon integration over k_a the result does not depend on the concrete choice of vierbein field.

Another related observation (not discussed in [15]) refers to derivatives of p_μ. In an expression of the type $\langle f(p,X)\rangle_p$ where f no longer contains ∂_μ and is constructed entirely with p_μ and other world tensors X, the derivative

$$\nabla_\mu\langle f(p,X)\rangle_p \tag{3.33}$$

is obtained by applying ∇_μ only to X (the other tensors in f) and not to p_μ. So, for instance

$$\nabla_\mu\left\langle p_\nu F(p_\alpha p_\beta M^{\alpha\beta})\right\rangle_p = \left\langle p_\nu p_\alpha p_\beta M_\mu{}^{\alpha\beta}F'\right\rangle_p. \tag{3.34}$$

This observation is useful if one wants to apply integration by parts (with respect to ∇_μ) in an expression of the type $\langle f\rangle_{x,p}$.

The statement would seem rather trivial as p_μ is an integration variable. On the other hand, since p_μ does not commute with $Z_{\mu\nu} = [\nabla_\mu, \nabla_\nu]$, it follows that p_μ does not commute with ∇_μ either. Nevertheless the statement holds. This follows from (3.32): the covariant derivative acts on X and e_μ^a, but for any given point one can choose the tetrad field so that $\nabla_\mu e_\nu^a$ vanishes at that point, so the correct result is obtained by applying the derivative only to the tensor fields X, and this holds at all points. A more elaborated proof is presented in Appendix C.

3.5 Γ_F (part 2)

We are now in a position to compute $\Gamma_{F,1}^{\text{div}}$ and $\Gamma_{F,2}^{\text{div}}$ using the method of covariant symbols.

To compute the UV divergent part of

$$\Gamma_{F,1} = -\frac{1}{2}\text{Tr}_1(\Box^{-1}Y^\mu{}_\nu), \tag{3.35}$$

we use (3.24) to transform it into

$$\Gamma_{F,1} = -\frac{1}{2}\left\langle \text{tr}_1\left(\overline{\Box}^{-1}\overline{Y}^\mu{}_\nu\right)\right\rangle_{x,p}. \tag{3.36}$$

The expansions in (3.25) for $Y^\mu{}_\nu$ and in (3.27) for $\Box$ apply:

$$\overline{Y}_{\mu\nu} = (\overline{Y}_{\mu\nu})_0 + (\overline{Y}_{\mu\nu})_{-1} + (\overline{Y}_{\mu\nu})_{-2} + O(p^{-3}), \quad \overline{\Box} = (\overline{\Box})_2 + (\overline{\Box})_0 + (\overline{\Box})_{-1} + O(p^{-2}), \tag{3.37}$$

and hence

$$\overline{\Box}^{-1} = (\overline{\Box})_2^{-1} - (\overline{\Box})_2^{-1}(\overline{\Box})_0(\overline{\Box})_2^{-1} + O(p^{-5}). \tag{3.38}$$

One can then expand the product $\overline{\Box}^{-1}\overline{Y}^\mu{}_\nu$. Clearly all terms with odd degree N_p vanish within $\langle\,\rangle_p$ due to parity. Also terms with $N_p < -4$ are UV convergent. In fact within dimensional regularization only the terms with $N_p = -4$ can have a non null contribution. That is[3]

$$\Gamma_{F,1}^{\text{div}} = -\frac{1}{2}\left\langle \text{tr}_1\left((\overline{\Box}^{-1}\overline{Y}^\mu{}_\nu)_{-4}\right)\right\rangle_{x,p} \tag{3.39}$$

[3] Strictly speaking the right-hand side of (3.39) produces $\Gamma(-\epsilon) = -1/\epsilon + O(1)$. The UV part is defined as the pole term.

with

$$(\overline{\Box}^{-1}\overline{Y}_{\mu\nu})_{-4} = (\overline{\Box})_2^{-1}(\overline{Y}_{\mu\nu})_{-2} - (\overline{\Box})_2^{-1}(\overline{\Box})_0(\overline{\Box})_2^{-1}(\overline{Y}_{\mu\nu})_0 = -N_g\frac{1}{2!}Y_{\alpha\beta\mu\nu}\partial^\alpha\partial^\beta - N_g(\overline{\Box})_0\, N_g Y_{\mu\nu}. \tag{3.40}$$

Here we have introduced the quantity

$$N_g := (-g^{\mu\nu}p_\mu p_\nu)^{-1} \tag{3.41}$$

which is positive definite. A further simplification occurs because terms of the type $\langle X\partial^\mu\rangle_p$, as well as $\langle \partial^\mu X\rangle_p$ vanish identically, hence the first term in $(\overline{\Box}^{-1}\overline{Y}_{\mu\nu})_{-4}$ drops off:

$$\Gamma^{\rm div}_{F,1} = \frac{1}{2}\left\langle \mathrm{tr}_1\left(N_g(\overline{\Box})_0\, N_g Y^\mu{}_\nu\right)\right\rangle_{x,p}. \tag{3.42}$$

The form of $(\overline{\Box})_0$ can be read off from (3.27), namely,

$$(\overline{\Box})_0 = \frac{1}{6}\mathbf{R} + Z_{\lambda\alpha}p^\lambda\partial^\alpha - \frac{1}{3}\mathscr{R}_{\lambda\alpha}p^\lambda\partial^\alpha + \frac{1}{3}R_{\lambda\alpha\sigma\beta}p^\lambda p^\sigma\partial^\alpha\partial^\beta. \tag{3.43}$$

The method here is to move the ∂^μ to the right or to the left, to exploit the properties $0 = \langle X\partial^\mu\rangle_p = \langle\partial^\mu X\rangle_p$. This can be conveniently done using relations of the type

$$[\partial^\alpha, N_g] = 2p^\alpha N_g^2,\quad [\partial^\alpha,[\partial^\beta,N_g]] = 2g^{\alpha\beta}N_g^2 + 8p^\alpha p^\beta N_g^3, \tag{3.44}$$

as well as standard angular averages of the type

$$\left\langle p_\mu p_\nu N_g^3\right\rangle_p = -\frac{1}{4}g_{\mu\nu}\left\langle N_g^2\right\rangle_p. \tag{3.45}$$

These manipulations produce

$$\Gamma^{\rm div}_{F,1} = \frac{1}{12}\left\langle \mathbf{R}Y^\mu{}_\mu N_g^2\right\rangle_{x,p}. \tag{3.46}$$

The quantity $\langle N_g^2\rangle_p$ is UV divergent and can be reduced to a standard flat-space form using a vierbein field, as previously discussed around (3.31). Hence

$$\langle N_g^2\rangle_p = \frac{1}{\sqrt{g}}\int\frac{d^dp}{(2\pi)^d}\frac{1}{(-p_\mu^2)^2} = \int\frac{d^dk}{(2\pi)^d}\frac{1}{(k^2)^2} = \frac{\Gamma(-\epsilon)}{(4\pi)^2} = -\frac{1}{(4\pi)^2\epsilon} + O(1). \tag{3.47}$$

Here we have simplified the calculation anticipating the result for the UV divergent part. A more rigorous treatment would use a denominator $(-p_\mu^2+m^2)$ with $m^2>0$, to avoid infrared divergences. The contribution from m^2 goes into the $O(1)$ terms, as it should since the effect of m^2 is subleading in the UV region. This justifies the prescription of taking directly the terms of order p^{-4} to isolate the UV divergent contributions. It is worth noticing that $\langle N_g^2\rangle_p$ correctly goes to $+\infty$ as $d\to 4$ from $d<4$, or equivalently $\epsilon\to 0^-$. This checks that the sign of our calculations is the correct one.

Hence

$$\Gamma^{\rm div}_{F,1} = \frac{1}{32\pi^2\epsilon}\left\langle -\frac{1}{6}\mathbf{R}Y^\mu{}_\mu\right\rangle_x. \tag{3.48}$$

The remaining term is also readily computed:

$$\Gamma_{F,2} = -\frac{1}{4}\left\langle \mathrm{tr}_1\left((\overline{\Box}^{-1}\overline{Y}^\mu{}_\nu)^2\right)\right\rangle_{x,p}, \tag{3.49}$$

and selecting the terms of order p^{-4}

$$\Gamma^{\rm div}_{F,2} = -\frac{1}{4}\langle N_g^2 Y_{\mu\nu}^2\rangle_{x,p} = \frac{1}{32\pi^2\epsilon}\left\langle\frac{1}{2}Y_{\mu\nu}^2\right\rangle_x. \tag{3.50}$$

In summary, for Γ_F one obtains

$$\Gamma^{\rm div}_F = \frac{1}{32\pi^2\epsilon}\left\langle -\frac{11}{180}\mathscr{G} - \frac{4}{15}\mathscr{R}_{\mu\nu}^2 + \frac{7}{60}\mathbf{R}^2 - \frac{1}{6}\mathbf{R}Y^\mu{}_\mu + \frac{1}{2}Y_{\mu\nu}^2\right\rangle_x, \tag{3.51}$$

and in terms of $M_{\mu\nu}$, using $Y_{\mu\nu} = \mathscr{R}_{\mu\nu} + M_{\mu\nu}$,

$$\Gamma^{\rm div}_F = \frac{1}{32\pi^2\epsilon}\int d^4x\sqrt{g}\left(-\frac{11}{180}\mathscr{G} + \frac{7}{30}\mathscr{R}_{\mu\nu}^2 - \frac{1}{20}\mathbf{R}^2 - \frac{1}{6}\mathbf{R}M^\mu{}_\mu + \mathscr{R}_{\mu\nu}M^{\mu\nu} + \frac{1}{2}M_{\mu\nu}^2\right). \tag{3.52}$$

This result reproduces that in Eq. (26) of [13] and agrees with previous literature [26,27]. This completes the calculation of $\Gamma^{\rm div}_{K,0}$.

For convenience we also make explicit the combined result of $\Gamma^{\rm div}_{\rm gh}$ and $\Gamma^{\rm div}_F$:

$$\Gamma^{\rm div}_{{\rm gh}+F} = \frac{1}{32\pi^2\epsilon}\int d^4x\sqrt{g}\left(-\frac{13}{180}\mathscr{G} + \frac{1}{5}\mathscr{R}_{\mu\nu}^2 - \frac{1}{15}\mathbf{R}^2 - \frac{1}{6}\mathbf{R}M^\mu{}_\mu + \mathscr{R}_{\mu\nu}M^{\mu\nu} + \frac{1}{2}M_{\mu\nu}^2\right). \tag{3.53}$$

4 Remaining contributions to $\Gamma^{\rm div}$

As shown in Sect. 2 the divergent part of the effective action can be split as

$$\Gamma^{\rm div} = \Gamma^{\rm div}_{\rm gh} + \Gamma^{\rm div}_F + \Gamma^{\rm div}_G + \Gamma^{\rm div}_{K,2} + \Gamma^{\rm div}_{K,4}. \tag{4.1}$$

The computation of the last two terms will be undertaken here.

The term $\Gamma^{\rm div}_{K,2}$ is defined in Eq. (2.12) as

$$\Gamma_{K,2} = -\frac{1}{4}\mathrm{Tr}((\hat{K}_D^{-1}\hat{K}_A)^2). \tag{4.2}$$

Expanding in terms of the matrices $\hat{K}_D^{-1}$ and $\hat{K}_A$, and exploiting the cyclic property of the trace, which is justified for the

UV divergent component, this expression can be brought to the form

$$\Gamma_{K,2} = -\frac{1}{2}\mathrm{Tr}(\hat{G}^{-1}\hat{H}^{\dagger}\hat{F}^{-1}\hat{H}), \tag{4.3}$$

where we have chosen to use an operator acting on the space of scalars. The method of covariant symbols then yields

$$\Gamma_{K,2} = -\frac{1}{2}\left\langle \mathrm{tr}_0\left(\overline{G^{-1}}\;\overline{H^{\dagger\mu}}\,\overline{F^{-1}_{\mu\nu}}\;\overline{H}^{\nu}\right)\right\rangle_{x,p}, \tag{4.4}$$

and for the UV divergent part

$$\Gamma^{\mathrm{div}}_{K,2} = -\frac{1}{2}\left\langle \mathrm{tr}_0\left(\overline{G^{-1}}\;\overline{H^{\dagger\mu}}\,\overline{F^{-1}_{\mu\nu}}\;\overline{H}^{\nu}\right)_{-4}\right\rangle_{x,p}, \tag{4.5}$$

where the subindex -4 indicates to retain only the terms of order p^{-4}. The method to select those terms is to use the known expansions of the covariant symbols for the basic blocks ∇_μ, $\Box$, $M^{\mu\nu}$, etc, to obtain

$$\begin{aligned}\overline{F}^{\mu\nu} &= (\overline{F}^{\mu\nu})_2 + (\overline{F}^{\mu\nu})_0 + O(p^{-1}),\\ \overline{G} &= (\overline{G})_2 + (\overline{G})_1 + (\overline{G})_0 + O(p^{-1}),\\ \overline{H}^{\mu} &= (\overline{H}^{\mu})_1 + (\overline{H}^{\mu})_0 + (\overline{H}^{\mu})_{-1} + O(p^{-2}).\end{aligned} \tag{4.6}$$

Furthermore

$$\begin{aligned}(\overline{F}^{\mu\nu})_2 &= -p_\alpha p^\alpha g^{\mu\nu} = g^{\mu\nu}N_g^{-1},\\ (\overline{G})_2 &= -p_\mu p_\nu M^{\mu\nu} \equiv N_M^{-1}.\end{aligned} \tag{4.7}$$

The quantity N_g was defined in (3.41) while N_M has been newly defined here and is also positive. This gives

$$\begin{aligned}\overline{F^{-1}_{\mu\nu}} &= (\overline{F^{-1}_{\mu\nu}})_{-2} + (\overline{F^{-1}_{\mu\nu}})_{-4} + O(p^{-5}),\\ \overline{G^{-1}} &= (\overline{G^{-1}})_{-2} + (\overline{G^{-1}})_{-3} + (\overline{G^{-1}})_{-4} + O(p^{-5}),\end{aligned} \tag{4.8}$$

with

$$\begin{aligned}(\overline{F^{-1}_{\mu\nu}})_{-2} &= g_{\mu\nu}N_g\\ (\overline{F^{-1}_{\mu\nu}})_{-4} &= -N_g(\overline{F}_{\mu\nu})_0 N_g,\\ (\overline{G^{-1}})_{-2} &= N_M\\ (\overline{G^{-1}})_{-3} &= -N_M(\overline{G})_1 N_M\\ (\overline{G^{-1}})_{-4} &= -N_M(\overline{G})_0 N_M + N_M(\overline{G})_1 N_M(\overline{G})_1 N_M.\end{aligned} \tag{4.9}$$

Substitution of these expressions in Eq. (4.5) selects seven terms of the type $(\overline{G^{-1}})_l(\overline{H^{\dagger}})_m(\overline{F^{-1}})_n(\overline{H})_p$ with (l, m, n, p) taking values $(-4, 1, -2, 1)$, $(-3, 1, -2, 0)$, $(-3, 0, -2, 1)$, $(-2, -1, -2, 1)$, $(-2, 0, -2, 0)$, $(-2, 1, -4, 1)$, and $(-2, 1, -2, -1)$. The last term is actually vanishing since $(\overline{H}^{\mu})_{-1}$ is of the form $X\partial^\nu$.

For $\Gamma_{K,4}$ one has similarly

$$\Gamma_{K,4} = -\frac{1}{4}\mathrm{Tr}((\hat{G}^{-1}\hat{H}^{\dagger}\hat{F}^{-1}\hat{H})^2). \tag{4.10}$$

In this case there is just one term of $O(p^{-4})$, namely,

$$\Gamma^{\mathrm{div}}_{K,4} = -\frac{1}{2}\left\langle \mathrm{tr}_0\left(\left((\overline{G^{-1}})_{-2}(\overline{H^{\dagger}})_1(\overline{F^{-1}})_{-2}(\overline{H})_1\right)^2\right)\right\rangle_{x,p}. \tag{4.11}$$

The calculation proceeds[4] by carrying out the derivatives ∂^μ, either to the right, or to the left when this gives a lower number of terms. Next the operators $Z_{\mu_1\ldots\mu_m}$ are also moved to the right or the left to exploit the properties $\mathrm{tr}_0(XZ_{\mu_1\ldots\mu_m}) = \mathrm{tr}_0(Z_{\mu_1\ldots\mu_m}X) = 0$. This produces an expression involving only p_μ inside momentum integrals with powers of N_g and N_M, and other tensors constructed with $M^{\mu\nu}$ and its derivatives and the Riemann tensor and its derivatives.

Integration by parts can be applied both with respect to x^μ and with respect to p_μ in order to reduce the number of terms in the final expression. We have chosen to remove terms having $M^{\mu\nu}$ with more than one covariant derivative. Likewise the identities

$$\partial^\mu N_g^n = 2nN_g^{n+1}p^\mu, \quad \partial^\mu N_M^n = 2nN_M^{n+1}M^{\mu\nu}p_\nu, \tag{4.12}$$

have been applied in order to bring the expression to one involving only a few independent momentum integrals. Such procedure yields the following result[5]

$$\begin{aligned}\Gamma^{\mathrm{div}}_{K,2+4} = \frac{1}{32\pi^2\epsilon}\langle & I^{1,1}T^{1,1} + I^{1,2}_{\mu\nu}T^{1,2}_{\mu\nu}\\ &+ I^{1,3}_{\mu\nu\alpha\beta}T^{1,3}_{\mu\nu\alpha\beta} + I^{3,2}_{\mu\nu\alpha\beta\rho\sigma}T^{3,2}_{\mu\nu\alpha\beta\rho\sigma}\rangle_x.\end{aligned} \tag{4.13}$$

The tensors $T^{n,m}_{\mu_1\ldots\mu_k}$ take the following form

$$\begin{aligned}T^{1,1} = &-\frac{1}{8}M_{\mu\mu\nu}M_{\alpha\alpha\nu} - \frac{1}{8}M_{\mu\nu\alpha}M_{\nu\mu\alpha} + \frac{1}{8}M_{\mu\nu}M_{\mu\alpha}M_{\nu\alpha}\\ &+ \frac{1}{12}M_{\mu\nu}M_{\alpha\beta}R_{\mu\alpha\nu\beta} - \frac{1}{24}M_{\mu\nu}M_{\mu\nu}\mathbf{R},\end{aligned} \tag{4.14}$$

$$\begin{aligned}T^{1,2}_{\mu\nu} = &-\frac{1}{12}M_{\mu\alpha}M_{\alpha\nu\beta}M_{\beta\rho\rho} - \frac{2}{3}M_{\mu\alpha}M_{\alpha\nu\beta}M_{\rho\rho\beta}\\ &+ \frac{1}{4}M_{\mu\alpha}M_{\alpha\beta\beta}M_{\rho\rho\nu}\\ &- \frac{1}{3}M_{\mu\alpha}M_{\alpha\beta\rho}M_{\beta\nu\rho} - \frac{1}{96}M_{\alpha\alpha}M_{\beta\mu\nu}M_{\beta\rho\rho}\\ &- \frac{1}{24}M_{\alpha\alpha}M_{\beta\mu\nu}M_{\rho\rho\beta}\\ &- \frac{1}{48}M_{\alpha\alpha}M_{\beta\mu\rho}M_{\beta\nu\rho} + \frac{1}{6}M_{\alpha\alpha}M_{\beta\mu\rho}M_{\rho\nu\beta}\end{aligned}$$

[4] The manipulations have been carried out using code in Mathematica written by the authors.

[5] Of course there is some ambiguity in writing the result due to integration by parts. The result presented has 32 terms grouped into four structures as regards to momentum integrals. Shorter expressions, still with four structures, could exist. There are shorter expressions, namely with 26 terms, but involving a larger number of different momentum integrals.

$$+\frac{1}{3}M_{\alpha\beta}M_{\alpha\mu\nu}M_{\rho\rho\beta}$$
$$-\frac{1}{4}M_{\alpha\beta}M_{\alpha\mu\rho}M_{\beta\nu\rho}+\frac{1}{8}M_{\alpha\beta}M_{\alpha\rho\rho}M_{\beta\mu\nu}$$
$$+\frac{1}{8}M_{\mu\alpha}M_{\nu\alpha}M_{\beta\rho}M_{\beta\rho}-\frac{1}{8}M_{\mu\alpha}M_{\alpha\beta}M_{\nu\beta}M_{\rho\rho}$$
$$+\frac{1}{4}M_{\mu\alpha}M_{\alpha\beta}M_{\beta\rho}M_{\nu\rho}$$
$$+\frac{1}{12}M_{\mu\alpha}M_{\nu\alpha}M_{\beta\rho}\mathscr{R}_{\beta\rho}-\frac{1}{12}M_{\mu\alpha}M_{\alpha\beta}M_{\nu\beta}\mathbf{R}, \tag{4.15}$$

$$T^{1,3}_{\mu\nu\alpha\beta}=+\frac{1}{2}M_{\mu\rho}M_{\nu\rho}M_{\sigma\alpha\lambda}M_{\lambda\beta\sigma}-M_{\mu\rho}M_{\nu\sigma}M_{\rho\alpha\lambda}M_{\sigma\beta\lambda}$$
$$-\frac{2}{3}M_{\mu\rho}M_{\nu\sigma}M_{\rho\alpha\lambda}M_{\lambda\beta\sigma}$$
$$+\frac{1}{2}M_{\mu\rho}M_{\nu\sigma}M_{\rho\lambda\lambda}M_{\sigma\alpha\beta}-\frac{1}{24}M_{\mu\rho}M_{\sigma\sigma}M_{\lambda\nu\rho}M_{\lambda\alpha\beta}$$
$$+\frac{2}{3}M_{\mu\rho}M_{\sigma\lambda}M_{\rho\nu\sigma}M_{\lambda\alpha\beta}$$
$$-\frac{1}{6}M_{\mu\rho}M_{\sigma\lambda}M_{\sigma\nu\rho}M_{\lambda\alpha\beta}+\frac{1}{24}M_{\rho\rho}M_{\sigma\lambda}M_{\sigma\mu\nu}M_{\lambda\alpha\beta}$$
$$-\frac{1}{12}M_{\rho\sigma}M_{\rho\lambda}M_{\sigma\mu\nu}M_{\lambda\alpha\beta}, \tag{4.16}$$

$$T^{3,2}_{\mu\nu\alpha\beta\rho\sigma}=-\frac{1}{3}M_{\mu\lambda}M_{\lambda\nu\alpha}M_{\beta\rho\sigma}+\frac{1}{12}M_{\lambda\lambda}M_{\mu\nu\alpha}M_{\beta\rho\sigma}. \tag{4.17}$$

On the other hand the integrals $I^{n,m}_{\mu_1\ldots\mu_k}$ are defined from the relation

$$\left\langle N_g^n N_M^m\, p_{\mu_1}\cdots p_{\mu_k}\right\rangle_p=\frac{1}{32\pi^2\epsilon}\,I^{n,m}_{\mu_1\ldots\mu_k}. \tag{4.18}$$

In our case $k=2(n+m)-4\geq 0$ and the $I^{n,m}_{\mu_1\ldots\mu_k}$ are UV finite.

The integrals $I^{n,m}_{\mu_1\ldots\mu_k}$ can be represented in several ways and are subject to relations among them. However, these elliptic integrals do not admit a simple closed form. A straightforward way to extract the UV finite factor is by using 4-spherical coordinates, with radial coordinate $r=N_g^{-1/2}$. In this case

$$I^{n,m}_{\mu_1\ldots\mu_{2n+2m-4}}=\frac{(-1)^{2n+2m+1}}{\pi^2}\int d^3\Omega_{\hat{k}}\,\frac{\hat{k}_{\mu_1}\cdots\hat{k}_{\mu_{2n+2m-4}}}{(\hat{k}_\mu\hat{k}_\nu M^{\mu\nu})^m}, \tag{4.19}$$

with $\hat{k}_a^2=1$, $\hat{k}_\mu=\hat{k}_a e^a_\mu$ and $g_{\mu\nu}=\delta_{ab}e^a_\mu e^b_\nu$. Some simplification is obtained by going to the local frame in which $M^\mu{}_\nu(x)$ is diagonal. In this case the relevant integrals become

$$\hat{I}^{n,m}_{a_1\ldots a_{2n}}=\int d^3\Omega_{\hat{k}}\,\frac{\hat{k}_{a_1}\cdots\hat{k}_{a_{2n}}}{(\sum_a M_a\hat{k}_a^2)^m}, \tag{4.20}$$

where M_a are the eigenvalues of $M^\mu{}_\nu$. All these integrals follow from applying derivatives with respect to the M_a to the generating integral

$$\hat{I}(z)=\int d^3\Omega_{\hat{k}}\,(z-\sum_a M_a\hat{k}_a^2)^{-1}. \tag{4.21}$$

Another explicit expression, closer to that in [13], is derived in Appendix D, namely,

$$I^{n,m}_{\mu_1\ldots\mu_{2n+2m-4}}=\frac{(-2)^{3-n-m}}{\Gamma(n)\Gamma(m)}\int_0^\infty dt\,\frac{t^{m-1}}{\sqrt{\det((M_t)^\mu{}_\nu)}}\times[(M_t^{-1})^{n+m-2}]_{\mu_1\ldots\mu_{2n+2m-4}}. \tag{4.22}$$

Here we have defined

$$(M_t)^{\mu\nu}=g^{\mu\nu}+tM^{\mu\nu}, \tag{4.23}$$

$(M_t^{-1})_{\mu\nu}$ denotes the inverse matrix of $(M_t)^{\mu\nu}$ and $[(M_t^{-1})^n]_{\mu_1\ldots\mu_{2n}}$ stands for the symmetrized product of n factors $(M_t^{-1})_{\mu\nu}$ (hence a total of $(n-1)!!$ terms). E.g.

$$[M_t^{-1}]_{\mu\nu}=(M_t^{-1})_{\mu\nu},$$
$$[(M_t^{-1})^2]_{\mu\nu\alpha\beta}=(M_t^{-1})_{\mu\nu}(M_t^{-1})_{\alpha\beta}+(M_t^{-1})_{\mu\alpha}(M_t^{-1})_{\nu\beta}+(M_t^{-1})_{\mu\beta}(M_t^{-1})_{\nu\alpha}. \tag{4.24}$$

Equation (4.22) assumes $n,m\geq 1$. The cases $m=0$ and $n=0$ can be worked out separately, or obtained from the same formulas with the replacements, respectively,

$$\frac{1}{\Gamma(m)}\to t\delta(t),\qquad \frac{1}{\Gamma(n)}\to\frac{1}{t}\delta(1/t), \tag{4.25}$$

and in both cases the Dirac deltas have their support at 0^+. This prescription yields, for instance, $I^{2,0}=-2$, in agreement with Eq. (3.47).

5 Cross-checks of the calculation

5.1 Terms with zero and four derivatives

The effective action can be decomposed as a sum of terms classified by the number of covariant derivatives. In particular $\Gamma^{\rm div}$ can be decomposed as

$$\Gamma^{\rm div}=\Gamma^{{\rm div}(4)}+\Gamma^{{\rm div}(2)}+\Gamma^{{\rm div}(0)}, \tag{5.1}$$

into terms with 4, 2 and 0 covariant derivatives. Such classification is intrinsic since it corresponds to the response under dilatations. Therefore each term $\Gamma^{{\rm div}(n)}$ is well-defined and should coincide among different calculations. In our calculation $\Gamma^{{\rm div}(4)}$ only gets contributions from $\Gamma_{\rm gh}$, Γ_G and Γ_F, while $\Gamma^{{\rm div}(2)}$ and $\Gamma^{{\rm div}(0)}$ only get contributions from Γ_F and $\Gamma_{K,2+4}$.

For the two simplest cases of zero and four covariant derivatives we have checked that our results reproduce those in [13].

Specifically, for four-derivative terms we find

$$\Gamma^{\mathrm{div}(4)} = \frac{1}{32\pi^2\epsilon}\Big\langle -\frac{1}{15}\mathscr{G} + \chi\left(\frac{1}{60}\tilde{\mathscr{R}}^2_{\mu\nu} + \frac{1}{120}\tilde{\mathbf{R}}^2\right) + \frac{1}{5}\mathscr{R}^2_{\mu\nu} - \frac{1}{15}\mathbf{R}^2\Big\rangle_x, \tag{5.2}$$

where

$$\chi = \sqrt{\det(M^{\mu\nu})\det(g_{\mu\nu})}. \tag{5.3}$$

The result in [13] is expressed in terms of

$$\hat{g}_{\mu\nu} = \chi^{1/2} g_{\mu\nu}, \quad \tilde{g}^{\mu\nu} = \chi^{-1} M^{\mu\nu}, \tag{5.4}$$

and can be written as

$$\Gamma^{\mathrm{div}(4)}_{\mathrm{RS}} = \frac{1}{32\pi^2\epsilon}\Big\langle -\frac{1}{15}\mathscr{G} + \chi\left(\frac{1}{60}\tilde{\mathscr{R}}^2_{\mu\nu} + \frac{1}{120}\tilde{\mathbf{R}}^2 + \frac{1}{5}\hat{\mathscr{R}}^2_{\mu\nu} - \frac{1}{15}\hat{\mathbf{R}}^2\right)\Big\rangle_x, \tag{5.5}$$

noting that $\det(\hat{g}) = \det(\tilde{g}) = \chi^2\det(g)$. The two calculations coincide because under a Weyl transformation $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$, the combination $\mathscr{R}^2_{\mu\nu} - \frac{1}{3}\mathbf{R}^2$ transforms into $\Omega^{-4}(\mathscr{R}^2_{\mu\nu} - \frac{1}{3}\mathbf{R}^2)$ up to a total derivative.

Regarding the term involving no covariant derivatives, the result in [13] can be written as

$$\Gamma^{\mathrm{div}(0)}_{\mathrm{RS}} = \frac{1}{32\pi^2\epsilon}\Big\langle \frac{1}{4}\mathrm{tr}(M^2) + \chi^{1/2} I^{\mu\nu}_{(2,1)}\left(-\frac{1}{4}(M^2)_{\mu\nu} + \frac{1}{8}M_{\mu\nu}\mathrm{tr}(M)\right)\Big\rangle_x. \tag{5.6}$$

The identity

$$\hat{g}_{\mu\nu} + u\tilde{g}_{\mu\nu} = u\chi g_{\mu\alpha}(M_t)^{\alpha\beta}(M^{-1})_{\beta\nu}, \quad t = \frac{1}{u\chi^{1/2}}, \tag{5.7}$$

implies the relation

$$I^{\mu\nu}_{(2,1)} = -2\chi^{-1/2} g^{\mu\alpha} I^{2,1}_{\alpha\beta} M^{\beta\nu}, \tag{5.8}$$

and hence

$$\Gamma^{\mathrm{div}(0)}_{\mathrm{RS}} = \frac{1}{32\pi^2\epsilon}\Big\langle -\frac{1}{8}I^{2,0}\mathrm{tr}(M^2) + I^{2,1}_{\mu\nu}\left(-\frac{1}{4}(M^3)^{\mu\nu} + \frac{1}{8}(M^2)^{\mu\nu}\mathrm{tr}(M)\right)\Big\rangle_x. \tag{5.9}$$

This is to be compared with our result, which receives contributions from Γ_F and $\Gamma_{K,2+4}$:

$$\Gamma^{\mathrm{div}(0)} = \frac{1}{32\pi^2\epsilon}\Big\langle -\frac{1}{4}I^{2,0}\mathrm{tr}(M^2) + \frac{1}{8}I^{1,1}\mathrm{tr}(M^3) + I^{1,2}_{\mu\nu}\left(\frac{1}{8}(M^2)^{\mu\nu}\mathrm{tr}(M^2) - \frac{1}{8}(M^3)^{\mu\nu}\mathrm{tr}(M) + \frac{1}{4}(M^4)^{\mu\nu}\right)\Big\rangle_x. \tag{5.10}$$

The two expressions coincide, as follows from the integration-by-parts identity

$$I^{1,2}_{\mu\nu}(M^n)^{\mu\nu} = -I^{2,1}_{\mu\nu}(M^{n-1})^{\mu\nu} - \frac{1}{2}I^{1,1}\mathrm{tr}(M^{n-1}), \tag{5.11}$$

as well as $M^{\mu\nu} I^{n,m}_{\mu\nu\alpha_1\ldots\alpha_k} = -I^{n,m-1}_{\alpha_1,\ldots\alpha_k}$.

5.2 c-Number $M^{\mu\nu}$

A c-number $M^{\mu\nu}$ refers to the case

$$M^{\mu\nu} = X^2 g^{\mu\nu}. \tag{5.12}$$

As noted in [13] the corresponding effective action can be obtained through a Weyl transformation from the case $M^{\mu\nu} = g^{\mu\nu}$. The divergent part of the latter has been computed in [26,28].

As a check of our results we particularize them to the c-number case. The general expressions for Γ^{div}_F and $\Gamma^{\mathrm{div}}_{K,2+4}$ become

$$\Gamma^{\mathrm{div}}_F = \frac{1}{32\pi^2\epsilon}\left\langle -\frac{11}{180}\mathscr{G} + \frac{7}{30}\mathscr{R}^2_{\mu\nu} - \frac{1}{20}\mathbf{R}^2 + \frac{1}{3}\mathbf{R}X^2 + 2X^4\right\rangle_x$$
$$\Gamma^{\mathrm{div}}_{K,2+4} = \frac{1}{32\pi^2\epsilon}\left\langle \frac{1}{6}\mathbf{R}X^2 - \frac{1}{2}X^4 + 3X^2_\mu\right\rangle_x \tag{5.13}$$

The form of $\Gamma^{\mathrm{div}}_{\mathrm{gh}}$ is unchanged. On the other hand, Γ^{div}_G can be worked out using the relation $\tilde{g}_{\mu\nu} = X^2 g_{\mu\nu}$ which is a Weyl transformation. The expansion of the various curvatures (and the determinant) of $\tilde{g}_{\mu\nu}$ in terms of those of $g_{\mu\nu}$ produces the result

$$\Gamma^{\mathrm{div}}_G = \frac{1}{32\pi^2\epsilon}\left\langle \frac{1}{180}\mathscr{G} + \frac{1}{60}\mathscr{R}^2_{\mu\nu} + \frac{1}{120}\mathbf{R}^2\right\rangle_x + \delta\Gamma^{\mathrm{div}}_G, \tag{5.14}$$

with

$$\delta\Gamma^{\mathrm{div}}_G = \frac{1}{32\pi^2\epsilon}\left\langle \frac{1}{15}\left(2\frac{X_\mu X_\nu}{X^2} - \frac{X_{\mu\nu}}{X}\right)\mathscr{R}_{\mu\nu} - \frac{1}{15}\left(2\frac{X_{\mu\mu}}{X} + \frac{1}{2}\frac{X^2_\mu}{X^2}\right)\mathbf{R} + \frac{1}{15}\frac{X^2_{\mu\nu}}{X^2} + \frac{13}{30}\frac{X^2_{\mu\mu}}{X^2} - \frac{4}{15}\frac{X_{\mu\nu}X_\mu X_\nu}{X^3} + \frac{1}{15}\frac{X_{\mu\mu}X^2_\nu}{X^3} + \frac{3}{15}\frac{(X^2_\mu)^2}{X^4}\right\rangle_x. \tag{5.15}$$

As it turns out this expression can be much simplified through integration by parts and Bianchi identities, namely,

$$\delta\Gamma^{\mathrm{div}}_G = \frac{1}{32\pi^2\epsilon}\left\langle -\frac{1}{6}\frac{X_{\mu\mu}}{X}\mathbf{R} + \frac{1}{2}\frac{X^2_{\mu\mu}}{X^2}\right\rangle_x. \tag{5.16}$$

After collecting the various contributions one obtains

$$\Gamma^{\text{div}} = \frac{1}{32\pi^2\epsilon}\left\langle -\frac{1}{15}\mathscr{G} + \frac{13}{60}\mathscr{R}^2_{\mu\nu} - \frac{7}{120}\mathbf{R}^2 + \frac{1}{2}\mathbf{R}X^2 + \frac{3}{2}X^4 + 3X^2_\mu - \frac{1}{6}\frac{X_{\mu\mu}}{X}\mathbf{R} + \frac{1}{2}\frac{X^2_{\mu\mu}}{X^2}\right\rangle_x, \tag{5.17}$$

in agreement with the result quoted in [13]. Actually, it would have been sufficient to verify just the terms with two covariant derivatives, as it has already been shown the coincidence between the two calculations for terms with zero or four derivatives for arbitrary configurations $(g_{\mu\nu}, M^{\mu\nu})$.

5.3 Perturbative expansion

Here we discuss the perturbative expansion of our result for Γ^{div}, using the form

$$M^{\mu\nu} = m^2 g^{\mu\nu} + Y^{\mu\nu}. \tag{5.18}$$

Terms up to second order in powers of $Y^{\mu\nu}$ are displayed. We consider only those terms with at most two covariant derivatives. These are the most interesting ones to check the result as the calculation of $\Gamma_{K,2+4}$ is the most laborious one. Terms with four derivatives have already been shown to coincide with results in the literature.

Specifically, from Γ_F and $\Gamma^{\text{div}}_{K,2+4}$ we obtain

$$\Gamma^{\text{div}}_F = \frac{1}{32\pi^2\epsilon}\Big\langle 2m^4 + \frac{1}{3}m^2\mathbf{R} + m^2 Y_{\mu\mu} - \frac{1}{6}Y_{\mu\mu}\mathbf{R} + Y_{\mu\nu}\mathscr{R}_{\mu\nu} + \frac{1}{2}Y_{\mu\nu}Y_{\mu\nu} + O(\nabla^4) + O(Y^3)\Big\rangle_x, \tag{5.19}$$

$$\begin{aligned}\Gamma^{\text{div}}_{K,2+4} = \frac{1}{32\pi^2\epsilon}\Big\langle & -\frac{1}{2}m^4 + \frac{1}{6}m^2\mathbf{R} - \frac{1}{4}m^2 Y_{\mu\mu} \\ & + \frac{1}{12}Y_{\mu\mu}\mathbf{R} - \frac{1}{6}Y_{\mu\nu}\mathscr{R}_{\mu\nu} + \frac{1}{16}Y_{\mu\mu}Y_{\nu\nu} - \frac{3}{8}Y_{\mu\nu}Y_{\mu\nu} \\ & + \frac{1}{m^2}\Big(\frac{1}{12}Y_{\mu\mu\nu}Y_{\nu\alpha\alpha} + \frac{1}{48}Y_{\mu\nu\nu}Y_{\mu\alpha\alpha} \\ & - \frac{1}{24}Y_{\mu\nu\alpha}Y_{\mu\nu\alpha} + \frac{1}{4}Y_{\mu\nu\alpha}Y_{\nu\mu\alpha} + \frac{1}{12}Y_{\mu\mu}Y_{\nu\alpha}\mathscr{R}_{\nu\alpha} \\ & - \frac{1}{48}Y_{\mu\mu}Y_{\nu\nu}\mathbf{R} \\ & + \frac{1}{24}Y_{\mu\nu}Y_{\mu\nu}\mathbf{R} - \frac{1}{6}Y_{\mu\nu}Y_{\alpha\beta}R_{\mu\alpha\nu\beta}\Big) \\ & + O(\nabla^4) + O(Y^3)\Big\rangle_x.\end{aligned} \tag{5.20}$$

The total $\Gamma^{\text{div}}_F + \Gamma^{\text{div}}_{K,2+4}$ can be shown to coincide with the result obtained in [13] after using integration by parts there to remove $Y^{\mu\nu}$ with two covariant derivatives.

5.4 Weyl invariance

As noted at the end of Sect. 2 all pairs of external fields in the orbit $(\Omega^2 g_{\mu\nu}, \Omega^{-4}M^{\mu\nu})$ have the same effective action, and such invariance must be present in Γ^{div}. Because $\tilde{g}_{\mu\nu}$ is already Weyl invariant, Γ^{div}_G is also invariant. So we consider the remaining terms.

Since Weyl transformations form a group, it is sufficient to consider the infinitesimal case, namely, $\Omega(x) = 1 + \omega(x)$ and $O(\omega^2)$ is neglected. The infinitesimal variations of the building blocks are readily obtained:

$$\begin{aligned}\delta g_{\mu\nu} &= 2g_{\mu\nu}\omega, \quad \delta M^{\mu\nu} = -4M^{\mu\nu}\omega, \\ \delta R &= -2R\omega - 6\omega_{\mu\mu}, \quad \delta R_{\mu\nu} = -2\omega_{\mu\nu} - g_{\mu\nu}\omega_{\alpha\alpha}, \\ \delta C_{\mu\nu}{}^{\alpha}{}_{\beta} &= 0 \quad \text{(Weyl tensor)}, \\ \delta N_g &= 2N_g\omega, \quad \delta N_M = 4N_M\omega, \\ \delta M_\mu{}^{\alpha\beta} &= -4M_\mu{}^{\alpha\beta}\omega - 2M^{\alpha\beta}\omega_\mu - M_\mu{}^\beta\omega^\alpha - M^\alpha{}_\mu\omega^\beta \\ &\quad + g^\alpha{}_\mu M^{\sigma\beta}\omega_\sigma + g_\mu{}^\beta M^{\alpha\sigma}\omega_\sigma.\end{aligned} \tag{5.21}$$

The terms with ω without derivatives correspond to a global transformation. The invariance of the full expression in the global case is easily checked as it is almost trivial from dimensional counting. Hence the variations contain only ω with derivatives. From integration by parts with respect to x they can be brought to the a form proportional to ω_μ. This procedure gives

$$\begin{aligned}\delta\Gamma^{\text{div}}_{\text{gh}} &= \frac{1}{32\pi^2\epsilon}\left\langle -\frac{1}{3}\mathbf{R}_\mu\omega_\mu\right\rangle_x, \\ \delta\Gamma^{\text{div}}_F &= \frac{1}{32\pi^2\epsilon}\left\langle\left(\frac{1}{3}\mathbf{R}_\mu + 2Y_{\nu\nu\mu}\right)\omega_\mu\right\rangle_x, \\ \delta\Gamma^{\text{div}}_{K,2+4} &= \frac{1}{32\pi^2\epsilon}\left\langle\left(-2Y_{\nu\nu\mu} + \mathscr{O}_\mu\right)\omega_\mu\right\rangle_x.\end{aligned} \tag{5.22}$$

The quantity $\mathscr{O}_\mu$ involves integrals of the type $I^{n,m}_{\mu_1\cdots\mu_{2n+2m-4}}$ and can be shown to vanish identically using integration by parts in momentum space.[6] Hence $\delta\Gamma^{\text{div}} = 0$ is verified.

6 Summary and conclusions

We have carried out a complete calculation of the UV divergent part of the action in Eq. (2.1) within dimensional regularization. The full result is

$$\Gamma^{\text{div}} = \Gamma^{\text{div}}_G + \Gamma^{\text{div}}_{\text{gh}+F} + \Gamma^{\text{div}}_{K,2+4}, \tag{6.1}$$

where Γ^{div}_G is given in Eq. (3.11), $\Gamma^{\text{div}}_{\text{gh}+F}$ in Eq. (3.53), and $\Gamma^{\text{div}}_{K,2+4}$ in Eq. (4.13). We have made use of the method of covariant symbols, instead of the heat-kernel, and avoided

[6] In this case the relations (4.12) were not sufficient and the relation $\partial^\mu \log N_M = 2N_M M^{\mu\nu} p_\nu$ was required.

the use of expressions involving two metric fields in the same term, with the aim of obtaining relatively explicit formulas. Nevertheless the result is involved and this cannot be avoided in any calculation. Our results are fully consistent with those in [13]. Some of our terms are more explicit while those in [13] are more structured (relying on a compact bimetric setting), hence both calculation can be regarded as complementary.

It is noteworthy that the technique used here could have been applied also to the action after making the change (Weyl transformation) from $(g_{\mu\nu}, M^{\mu\nu})$ to $(\hat{g}_{\mu\nu}, \tilde{g}^{\mu\nu})$. The result would have been precisely the same as the one we have already obtained, albeit with $(\hat{g}_{\mu\nu}, \tilde{g}^{\mu\nu})$ playing the role of $(g_{\mu\nu}, M^{\mu\nu})$. However, proceeding in this way we would have missed Weyl invariance as a check of the calculation, since any functional of $(\hat{g}_{\mu\nu}, \tilde{g}^{\mu\nu})$ is Weyl invariant by construction. This also puts the paradox that (our version of the) functional $\Gamma^{\mathrm{div}}[\hat{g}_{\mu\nu}, \tilde{g}^{\mu\nu}]$ is constrained by Weyl invariance, even if the latter is automatically fulfilled. The resolution of the paradox is that a simpler expression can be achieved in terms of the transformed fields using that the two metrics have the same volume element, i.e., $\det(\hat{g}) = \det(\tilde{g})$. Namely, rearranging the derivatives of $\tilde{g}_{\mu\nu}$ to form the corresponding (difference) connection $\delta\Gamma^{\lambda}{}_{\mu\nu}$ and exploiting the property $\delta\Gamma^{\lambda}{}_{\mu\lambda} = 0$ as done in [13]. Instead of doing this we have chosen to use the original fields $(g_{\mu\nu}, M^{\mu\nu})$.

Because the method of covariant symbols works for any gauge or internal index connections, there is no problem of principle to extend this kind of calculations to other cases involving fermions or non abelian vector fields, in the latter case provided the singular kernel problem is suitably dealt with.

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Appendix A: Conventions

1. Riemann tensor

$R^{\mu}{}_{\nu\alpha\beta}$ denotes the Riemann tensor, $\mathscr{R}_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}$ the Ricci tensor and $\mathbf{R} = g^{\mu\nu}\mathscr{R}_{\mu\nu}$ the scalar curvature. Furthermore our convention for the Riemann tensor is such that

$$[\nabla_\mu, \nabla_\nu]A^\alpha = +R_{\mu\nu}{}^{\alpha}{}_{\lambda}A^\lambda. \tag{A1}$$

2. Covariant derivatives

By default indices are raised, lowered and contracted with $g_{\mu\nu}$. (An exception occurs for expressions with tilde in Sect. 3.2 for the computation of Γ_G.) The covariant derivative uses the Levi-Civita connection corresponding to $g_{\mu\nu}$, up to the same exception just noted. Covariant derivatives are indicated *by adding indices to the left*. Hence for instance $A_{\mu\nu\lambda}$ denotes $\nabla_\mu\nabla_\nu A_\lambda$ (meaning $[\nabla_\mu, [\nabla_\nu, A_\lambda]]$), $R_{\lambda\mu\nu\alpha\beta} = \nabla_\lambda R_{\mu\nu\alpha\beta}$, $\mathscr{R}_{\lambda\mu\nu} = \nabla_\lambda\mathscr{R}_{\mu\nu}$, $\mathbf{R}_\lambda = \nabla_\lambda\mathbf{R}$, etc.

3. The operator $Z_{\mu\nu}$

The curvature bundle is defined as

$$Z_{\mu\nu} = [\nabla_\mu, \nabla_\nu]. \tag{A2}$$

It is an antihermitian multiplicative operator with respect to x which acts on world indices. For instance,

$$[Z_{\mu\nu}, A^\alpha{}_\beta] = R_{\mu\nu}{}^{\alpha}{}_{\lambda}A^\lambda{}_\beta - R_{\mu\nu}{}^{\lambda}{}_{\beta}A^\alpha{}_\lambda. \tag{A3}$$

Correspondingly $Z_{\mu\nu}$ commutes with world scalars. Higher rank tensors are defined recursively as

$$Z_{\alpha\mu_1\ldots\mu_n} = [\nabla_\alpha, Z_{\mu_1\ldots\mu_n}] - \frac{1}{2}\{\nabla_\lambda, R_{\mu_1\ldots\mu_n}{}^{\lambda}{}_{\alpha}\}. \tag{A4}$$

The second term in this expression is an exception to our previous to-the-left-indices derivative convention. Such extra term is required to make $Z_{\mu_1\ldots\mu_n}$ a multiplicative operator. These operators fulfill relations analogous to (A3), e.g.

$$[Z_{\mu_1\ldots\mu_n}, A^\alpha] = R_{\mu_1\ldots\mu_n}{}^{\alpha}{}_{\lambda}A^\lambda. \tag{A5}$$

This and similar previous equations assume that A^α has no other indices besides the world index α, otherwise new terms appear at the right-hand side. Eq. (A4) is unchanged.

4. Momentum variables

For convenience we use $p_\mu = ik_\mu$ where k_μ are real but $\int d^d p$ is used to denote $\int d^d k$ as no confusion should arise.

Appendix B: Some results for covariant symbols

Here we quote expressions for $\overline{\nabla}_\mu$, $\overline{\Box}$ and $\overline{Z}_{\mu\nu}$ up to and including four covariant derivatives. The expression for a multiplicative operator Y not acting on world indices is that in Eq. (3.29). The formulas apply for ∇_μ having an arbitrary connection in gauge or other internal labels, and the Levi-Civita connection for world indices. Of course, $\overline{\Box}$ coincides with $g^{\mu\nu}\overline{\nabla}_\mu\,\overline{\nabla}_\nu$, and $\overline{Z}_{\mu\nu} = [\overline{\nabla}_\mu, \overline{\nabla}_\nu]$. All indices are contracted with the metric $g_{\mu\nu}$ and for clarity we put all world indices as covariant ones, except those in ∂^μ. The covariant symbols have been split as $\overline{\mathscr{O}} = \sum_n (\overline{\mathscr{O}})_n$ where the subindex n in $(\overline{\mathscr{O}})_n$ indicates the value of N_p of the component, i.e., the number of p_μ minus the number of ∂^μ. The components can be equally well be classified by the number of covariant derivatives they contain.

$$\begin{aligned}
(\overline{\nabla}_\mu)_1 &= p_\mu, \\
(\overline{\nabla}_\mu)_0 &= 0 \\
(\overline{\nabla}_\mu)_{-1} &= -\frac{1}{4}\{Z_{\nu\mu}, \partial^\nu\} + \frac{1}{12}\{[Z_{\nu\mu}, p_\alpha], \partial^\nu\partial^\alpha\}, \\
(\overline{\nabla}_\mu)_{-2} &= +\frac{1}{6}\{Z_{\nu\alpha\mu}, \partial^\nu\partial^\alpha\} - \frac{1}{24}\{[Z_{\nu\alpha\mu}, p_\beta], \partial^\nu\partial^\alpha\partial^\beta\}, \\
(\overline{\nabla}_\mu)_{-3} &= -\frac{1}{16}\{Z_{\nu\alpha\beta\mu}, \partial^\nu\partial^\alpha\partial^\beta\} \\
&\quad + \frac{1}{80}\{[Z_{\nu\alpha\beta\mu}, p_\rho], \partial^\nu\partial^\alpha\partial^\beta\partial^\rho\} \\
&\quad + \frac{1}{48}\{Z_{\nu\alpha}, [Z_{\beta\mu}, \partial^\alpha]\partial^\nu\partial^\beta\} \\
&\quad - \frac{7}{720}\{[Z_{\nu\alpha}, p_\beta], [Z_{\rho\mu}, \partial^\alpha]\partial^\nu\partial^\beta\partial^\rho\}.
\end{aligned} \tag{B1}$$

$$\begin{aligned}
(\overline{\Box})_2 &= p_\mu p_\mu, \\
(\overline{\Box})_1 &= 0, \\
(\overline{\Box})_0 &= +\frac{1}{2}\{Z_{\mu\nu}, p_\mu\partial^\nu\} - \frac{1}{3}[[Z_{\mu\nu}, p_\mu], \partial^\nu] \\
&\quad - \frac{1}{6}\{[Z_{\mu\nu}, p_\alpha]p_\mu, \partial^\nu\partial^\alpha\}, \\
(\overline{\Box})_{-1} &= +\frac{1}{6}\{Z_{\mu\nu\alpha}, \{p_\alpha, \partial^\mu\partial^\nu\}\} + \frac{2}{3}[Z_{\mu\mu\nu}, \partial^\nu] \\
&\quad - \frac{1}{12}\{[Z_{\mu\alpha\nu}, p_\beta]p_\nu, \partial^\mu\partial^\alpha\partial^\beta\}, \\
(\overline{\Box})_{-2} &= -\frac{1}{16}\{Z_{\mu\nu\alpha\beta}, \{p_\beta, \partial^\mu\partial^\nu\partial^\alpha\}\} \\
&\quad + \frac{1}{40}\{[Z_{\mu\nu\alpha\beta}, p_\rho]p_\beta, \partial^\mu\partial^\nu\partial^\alpha\partial^\rho\} \\
&\quad - \frac{1}{16}\{Z_{\mu\nu}, \{[Z_{\mu\alpha}, p_\beta], \partial^\nu\partial^\alpha\partial^\beta\}\} \\
&\quad + \frac{1}{8}\{Z_{\mu\nu}Z_{\mu\alpha}, \partial^\nu\partial^\alpha\} \\
&\quad + \frac{1}{30}\{[Z_{\mu\nu}, p_\alpha][Z_{\mu\beta}, p_\rho], \partial^\nu\partial^\alpha\partial^\beta\partial^\rho\} \\
&\quad + \frac{1}{60}[Z_{\mu\nu}, \partial^\nu][Z_{\mu\alpha}, \partial^\alpha] \\
&\quad + \frac{2}{45}[Z_{\mu\nu}, \partial^\alpha][Z_{\mu\alpha}, \partial^\nu] + \frac{2}{45}[Z_{\mu\nu}, \partial^\alpha][Z_{\mu\nu}, \partial^\alpha] \\
&\quad + \frac{1}{3}[Z_{\mu\alpha\nu\alpha}, \partial^\nu]\partial^\mu - \frac{1}{60}[Z_{\mu\nu\mu\alpha}, \partial^\alpha]\partial^\nu \\
&\quad + \frac{1}{40}[Z_{\mu\mu\nu\alpha}, \partial^\alpha]\partial^\nu.
\end{aligned} \tag{B2}$$

$$\begin{aligned}
(\overline{Z}_{\mu\nu})_0 &= Z_{\mu\nu}, \\
(\overline{Z}_{\mu\nu})_{-1} &= -\frac{1}{2}\{Z_{\alpha\mu\nu}, \partial^\alpha\}, \\
(\overline{Z}_{\mu\nu})_{-2} &= +\frac{1}{4}\{Z_{\alpha\beta\mu\nu}, \partial^\alpha\partial^\beta\}.
\end{aligned} \tag{B3}$$

These expressions are written so that hermiticity is manifest (namely, p_μ, ∇_μ, and $Z_{\mu_1\ldots\mu_k}$ are antihermitian while ∂^μ, and $\Box$ are hermitian). Expanded expressions with symbols R, Z, p, ∂ ordered from left to right can be found in [15].

Appendix C: Derivatives of momentum integrated expressions

Here we present an alternative proof of the statement noted at the end of Sect. 3.4, namely, if $f(p, X)$ is tensorially constructed out of p_μ and tensors X (and f no longer contains free ∂_μ), the covariant derivative of $\langle f(p, X)\rangle_p$ follows from applying the derivative only to the tensors X and not to p_μ. The proof relies on the choice of the Levi-Civita connection in the covariant derivative, corresponding to the metric $g_{\mu\nu}$ which also appears in the definition of $\langle f\rangle_p$ through the factor $1/\sqrt{g}$ in (3.23).

Clearly it is sufficient to prove the statement just for the case when the integrand $f(p, X)$ is a scalar. Otherwise, say the integrand is of the form $f_{\mu\nu}(p, X)$ and tensorially constructed out of p_μ and X. Then one can construct a scalar $h = C^{\mu\nu}(x) f_{\mu\nu}$, with a generic tensor $C^{\mu\nu}$, and it is clear that the statement would hold for $\langle f_{\mu\nu}\rangle_p$ if and only if it does for $\langle h\rangle_p$.

For simplicity we consider just the following case

$$I(x) = \frac{1}{\sqrt{g(x)}}\int \frac{d^d p}{(2\pi)^d} f(p_\alpha p_\beta B^{\alpha\beta}(x)) \tag{C1}$$

as it is sufficiently general to illustrate the arguments involved. A (first order) infinitesimal shift $x^\mu \to x^\mu + \epsilon^\mu$, will produce a change $I \to I + \epsilon^\mu \nabla_\mu I$. Similarly in the integrand,

$$\begin{aligned}
B^{\alpha\beta}(x+\epsilon) &= B^{\alpha\beta} + \epsilon^\mu \partial_\mu B^{\alpha\beta} \\
&= B^{\alpha\beta} + \epsilon^\mu(\nabla_\mu B^{\alpha\beta} - \Gamma^\alpha_{\mu\lambda}B^{\lambda\beta} - \Gamma^\beta_{\mu\lambda}B^{\alpha\lambda}).
\end{aligned} \tag{C2}$$

Contraction with $p_\alpha p_\beta$ then gives

$$p_\alpha p_\beta B^{\alpha\beta}(x+\epsilon) = p'_\alpha p'_\beta (B^{\alpha\beta} + \epsilon^\mu \nabla_\mu B^{\alpha\beta}) \tag{C3}$$

with

$$p'_\nu \equiv p_\nu - \epsilon^\mu \Gamma^\lambda_{\mu\nu} p_\lambda \tag{C4}$$

where terms $O(\epsilon^2)$ are neglected everywhere. Changing the integration variables from $p_\mu \to p'_\mu$ gives a Jacobian

$$d^d p = d^d p'(1 + \epsilon^\mu \Gamma^\lambda_{\mu\lambda}) = d^d p'(1 + \epsilon^\mu \partial_\mu \log \sqrt{g}). \tag{C5}$$

This factor exactly cancels with that produced by the shift $x^\mu \to x^\mu + \epsilon^\mu$ in $1/\sqrt{g(x)}$. In summary,

$$\nabla_\mu \langle p_\alpha p_\beta B^{\alpha\beta}\rangle_p = \langle p_\alpha p_\beta \nabla_\mu B^{\alpha\beta}\rangle_p \tag{C6}$$

as advertised.

Appendix D: Momentum integrals

Let us justify Eq. (4.22). The momentum integrals are

$$\tilde{I}^{n,m}_{\mu_1\ldots\mu_k} = \frac{1}{\sqrt{g}} \int \frac{d^d p}{(2\pi)^d} N_g^n N_M^m \, p_{\mu_1} \cdots p_{\mu_k} \tag{D1}$$

with $k = 2n + 2m - 4$. Using a Schwinger representation for the propagators

$$\tilde{I}^{n,m}_{\mu_1\ldots\mu_k} = \frac{1}{\sqrt{g}} \int_0^\infty du \int_0^\infty dv \frac{u^{n-1}}{\Gamma(n)} \frac{v^{m-1}}{\Gamma(m)} \times \int \frac{d^d p}{(2\pi)^d} e^{-u(N_g^{-1}+m_0^2)} e^{-v(N_M^{-1}+m_0^2)} p_{\mu_1} \cdots p_{\mu_k}. \tag{D2}$$

To avoid trivial infrared divergences we have introduced a mass $m_0 > 0$. This does not modify the UV divergence. Next we rescale $p_\mu \to p_\mu/\sqrt{u}$, and make a change of variables from v to $t = v/u$. This gives

$$\tilde{I}^{n,m}_{\mu_1\ldots\mu_k} = \frac{1}{\sqrt{g}} \int_0^\infty dt \frac{t^{m-1}}{\Gamma(n)\Gamma(m)} \int_0^\infty du \frac{e^{-u(1+t)m_0^2}}{u^{1+\epsilon}} \times \int \frac{d^d p}{(2\pi)^d} e^{-N_g^{-1} - t N_M^{-1}} p_{\mu_1} \cdots p_{\mu_k}. \tag{D3}$$

Upon integration over u to yield the UV pole and setting $d \to 4$ in the remaining terms:

$$\tilde{I}^{n,m}_{\mu_1\ldots\mu_k} = \frac{\Gamma(-\epsilon)}{\sqrt{g}} \int_0^\infty dt \frac{t^{m-1}}{\Gamma(n)\Gamma(m)} \times \int \frac{d^4 p}{(2\pi)^4} e^{-N_g^{-1} - t N_M^{-1}} p_{\mu_1} \cdots p_{\mu_k}. \tag{D4}$$

The momentum integral is now standard after Wick's theorem, with exponential factor $\exp(p_\mu p_\nu (g^{\mu\nu} + t M^{\mu\nu}))$. The factor $1/\sqrt{\det(g_{\mu\nu})}$ combines with $1/\sqrt{\det((M_t)^{\mu\nu})}$ to yield the result quoted in Eq. (4.22). Alternatively, one can use a tetrad to integrate over k_a instead of $-ip_\mu$, with the same effect.

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