Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2020) 44: 50 – 84
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doi:10.3906/mat-1905-91

Nonabelian cocycles and the spectrum of a symmetric monoidal category

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Received: 23.05.2019 • **Accepted/Published Online:** 21.10.2019 • **Final Version:** 20.01.2020

Abstract: We present an Eilenberg–MacLane-type description for the first, second and third spaces of the spectrum defined by a symmetric monoidal category.

Key words: Symmetric monoidal category, nonabelian cocycles, classifying space, loop space, spectrum, higher category

1. Introduction

The homotopy theory of categorical structures is nowadays a relevant part of the machinery in algebraic topology and algebraic K-theory. To briefly refer to the historical background in which this paper fits, let us recall that Thomason [27] proved that the classifying space functor from the small categories to spaces, $\mathscr{A} \mapsto B\mathscr{A}$, establishes an equivalence of model categories and therefore induces an equivalence between the corresponding homotopy categories. When a small category $\mathscr{A}$ is equipped with a symmetric monoidal structure, then the group completion of its classifying space $B\mathscr{A}$ is an infinite loop space. More precisely, thanks to its symmetric monoidal structure, the category $\mathscr{A}$ has associated a connective spectrum

$$\mathrm{Spt}\mathscr{A} = \big(\mathrm{Spt}_n\mathscr{A},\ \jmath : \mathrm{Spt}_n\mathscr{A} \simeq \Omega\,\mathrm{Spt}_{n+1}\mathscr{A}\big)_{n\geq 0},$$

endowed with a group completion map $\jmath : B\mathscr{A} \to \Omega\mathrm{Spt}_0\mathscr{A}$, which is usually called the algebraic K-theory of the symmetric monoidal category $\mathscr{A}$. The following are relevant examples: The spectrum associated to a skeleton of the category of finitely generated projective modules and isomorphisms over a unitary ring, under direct sum, is Quillen's algebraic K-theory of the ring. The spectrum associated to an abelian group regarded as a discrete symmetric monoidal category is its Eilenberg–MacLane spectrum. The spectrum associated to a skeleton of the category of finite sets and isomorphisms, under disjoint union, is isomorphic to the sphere spectrum in the stable homotopy category.

There are several constructions of an algebraic K-theory functor $\mathrm{Spt} : \mathscr{A} \mapsto \mathrm{Spt}\mathscr{A}$, from the category of symmetric monoidal small categories to the category of connective spectra, satisfying that the 0th space of $\mathrm{Spt}\mathscr{A}$ is the group completion of the classifying space $B\mathscr{A}$, for any symmetric monoidal category $\mathscr{A}$. These constructions are known as infinite loop space machines. For instance, we have those of Segal [22], May [17, 18], or Thomason [26], but in fact all of them are shown to be naturally equivalent when they are considered in the stable homotopy category of connective spectra [18, 19, 28]. Nevertheless, there is a problem with the spaces $\mathrm{Spt}_n\mathscr{A}$ since their construction, by means of any known infinite loop space machine, produces huge

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2010 *AMS Mathematics Subject Classification:* 18D05,18D10,19D23,55P47

CW-complexes having many cells with little apparent intuitive connection with the data of the symmetric monoidal category. This leads one to search for any smaller simplicial sets that geometrically realize the spaces $\mathrm{Spt}_n\mathscr{A}$ and whose cells give a logical meaning to the data of the symmetric monoidal category. In this paper we provide an Eilenberg–MacLane-style response for $n = 0, 1, 2$, which, summarizing, is as follows.

With the "guiding delooping hypotheses of higher category theory" in mind, for each positive integer $n \leq 3$, we associate to a symmetric monoidal category $\mathscr{A}$ a weak $(n+1)$-category with only one i-cell for $i < n$ and whose hom-category of endomorphisms of its $(n-1)$-cell is $\mathscr{A}$. This is denoted by $\Sigma^n\mathscr{A}$ and called the nth suspension of the symmetric monoidal category $\mathscr{A}$. Then, following Street [24], for each positive integer $n \leq 4$, we introduce (nonabelian, normalized) n*-cocycles* of a small category C with coefficients in a symmetric monoidal category $\mathscr{A}$ to be normalized lax functors $C \to \Sigma^{n-1}\mathscr{A}$, from the category C to the $(n-1)$-suspension n-category $\Sigma^{n-1}\mathscr{A}$.

Thereafter, for $1 \leq n \leq 4$, we define pointed simplicial sets

$$K(\mathscr{A}, n) : \Delta^{op} \to \mathbf{Set}_*, \qquad [p] \mapsto Z^n([p], \mathscr{A}),$$

whose p-simplices are the n-cocycles, with coefficients in $\mathscr{A}$, of the category defined by the ordered set $[p] = \{0 < \cdots < p\}$.

The more striking instance is for $\mathscr{A}$ the one-object symmetric monoidal category whose morphisms are the elements of an abelian group A since, in this case, $K(\mathscr{A}, n) = K(A, n)$ is just the nth Eilenberg–MacLane minimal complex defined by the abelian group. For arbitrary symmetric monoidal categories $\mathscr{A}$, these simplicial sets $K(\mathscr{A}, n)$ have the desired size, properties and geometrical realizations, as we elucidate in the paper. Thus, for example, $K(\mathscr{A}, n)$ is a $(n+1)$-coskeletal simplicial set. It has only one m-simplex for $m < n-1$. Its $(n-1)$-simplices are the objects a of $\mathscr{A}$ and its n-simplices are morphisms of $\mathscr{A}$ of the form

$$\begin{array}{rl} a_1 \to a_0, & \text{if} \quad n = 1, \\ a_1 \to a_0 + a_2, & \text{if} \quad n = 2, \\ a_3 + a_1 \to a_0 + a_2, & \text{if} \quad n = 3, \\ a_3 + a_1 \to a_0 + a_2 + a_4, & \text{if} \quad n = 4, \end{array}$$

with the object a_i as corresponding i-face, for $0 \leq i \leq n$.

For $n = 1$, we have that $K(\mathscr{A}, 1)$ is $= \mathrm{Ner}\mathscr{A}$, the ordinary nerve of the underlying category, and its geometric realization $|K(\mathscr{A}, 1)|$ is $B\mathscr{A}$, the classifying space of the category. For $n = 2, 3, 4$, there are natural homotopy equivalences

$$|K(\mathscr{A}, n)| \simeq \mathrm{Spt}_{n-2}(\mathscr{A}).$$

It is worth pointing out that, for now, we are not able to establish a similar description of spaces $\mathrm{Spt}_n(\mathscr{A})$ for $n \geq 3$. This is because the constructions and arguments we use in this paper for the cocycle description of $\mathrm{Spt}_n(\mathscr{A})$ need of combinatorial notions and facts concerning weak $(n+2)$-categories and, to go further with n, we are faced with the obstacle that finding an explicit combinatorial definition of weak m-categories for $m \geq 5$, as far as we know, is an ongoing research topic.

The plan of the paper, briefly, is as follows. After this introductory first section, the rest is organized in four sections. We dedicate the second section to the notion of n-cocycle of a small category with coefficients in a symmetric monoidal category, for $1 \leq n \leq 4$. In the third section we describe the simplicial sets $K(\mathscr{A}, n)$ associated to a symmetric monoidal category $\mathscr{A}$, define the suspension maps $S : \Sigma K(\mathscr{A}, n) \to K(\mathscr{A}, n+1)$, and state the main result of the paper, namely, Theorem 3.2, whose proof we give in the fifth section, after a preparatory fourth section, where we show the weak $(n-1)$-categories that the n-cocycles form.

For simplicity, we have written the paper in terms of permutative categories, that is, symmetric strict monoidal categories. For our purposes, there is no real loss of generality in dealing with permutative categories, as every symmetric monoidal category is monoidally equivalent to a permutative one by Isbell's theorem [14].

2. Cocycles

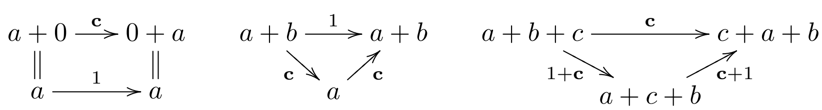
Throughout this paper, $\mathscr{A} = (\mathscr{A}, +, 0, \mathbf{C})$ denotes a (small) permutative category, that is, a small category $\mathscr{A}$ with a functor $+ : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ and an object $0 \in \mathrm{Ob}\mathscr{A}$ such that

$$a + (b + c) = (a + b) + c \quad \text{and} \quad 0 + a = a = a + 0$$

naturally for objects $a, b, c \in \mathrm{Ob}\mathscr{A}$, together with a natural family of isomorphisms

$$\mathbf{c} = \mathbf{c}_{a,b} : a + b \xrightarrow{\cong} b + a$$

such that the diagrams below commute.



Symmetric monoidal categories are closely related to higher categories, see the studies by Cheng and Gurski [6–8] on the hypothesized Periodic Table of n-categories by Baez and Dolan [1]. In particular, a permutative category $\mathscr{A}$ gives rise to a one-object (strict) 2-category, denoted by $\Sigma\mathscr{A}$, whose hom-category is the underlying category to the permutative category. This is usually called the "suspension" or "delooping" 2-category of the underlaying category $\mathscr{A}$ defined by its monoidal structure, see Kapranov and Voevodsky [16] or Street [25] for the terminology. Going higher, thanks to its symmetric monoidal structure, $\mathscr{A}$ also produces an one-object one-1-cell (semistrict, aka Gray-category) 3-category [11, 12], denoted by $\Sigma^2\mathscr{A}$ and called its "double suspension", whose hom 2-category is the suspension 2-category $\Sigma\mathscr{A}$, as well as an one-object one-1-cell one-2-cell (semistrict) 4-category [13]*, denoted by $\Sigma^3\mathscr{A}$ and called its "triple suspension", whose hom 3-category is the double suspension 3-category $\Sigma^2\mathscr{A}$.

Next, following Street [24], we introduce (nonabelian, normalized) n-cocycles of a small category C with coefficients in a permutative category $\mathscr{A}$, for $n \leq 4$.

Definition 2.1 *For $1 \leq n \leq 4$, an n-cocycle of a small category C with coefficients in a permutative $\mathscr{A}$ is a unitary lax functor from C to $\Sigma^{n-1}\mathscr{A}$. Let $Z^n(C, \mathscr{A})$ denote the set of such n-cocycles.*

We unpack below the definition of n-cocycle.

⧫ A 1-*cocycle* $F = (F, f) \in Z^1(C, \mathscr{A})$ is a functor from C to the underlaying category $\mathscr{A}$, so it consists of

- objects $f(c)$ of $\mathscr{A}$, one for each object c of C,

*See also the unpublished paper by T. Trimble: Notes on tetracategories 2006. Available at: math. ucr. edu/home/baez/trimble/tetracategories. html.

- morphisms $F(\sigma): f(c_0) \to f(c_1)$, one for each morphism $\sigma: c_0 \to c_1$ of C,

such that

- for any object c of C, $\quad F(1_c) = 1_{f(c)}$,

- for any two composable morphisms $c_0 \overset{\sigma_1}{\to} c_1 \overset{\sigma_2}{\to} c_2$ in C, $\quad F(\sigma_2\sigma_1) = F(\sigma_2)F(\sigma_1)$.

The set $Z^1(C, \mathscr{A})$ is pointed by the *zero* 1*-cocycle* $0 = (1_0, 0)$, which is defined by $0(c) = 0$, for any object c of C, and $1_0(\sigma) = 1_0$, for any morphism σ of C.

♦ A 2*-cocycle* $G = (G, g) \in Z^2(C, \mathscr{A})$ consists of

- objects $g(\sigma)$ of $\mathscr{A}$, one for each morphism $c_0 \overset{\sigma}{\to} c_1$ of C,

- morphisms $G(\sigma_1, \sigma_2): g(\sigma_2\sigma_1) \to g(\sigma_2) + g(\sigma_1)$, one for each pair of composable morphisms $c_0 \overset{\sigma_1}{\to} c_1 \overset{\sigma_2}{\to} c_2$,

such that

- for any object c of C, $\quad g(1_c) = 0$,

- for any morphism $c_0 \overset{\sigma}{\to} c_1$ of C, $\quad G(1, \sigma) = 1_{g(\sigma)} = G(\sigma, 1)$,

- for any three composable morphisms $c_0 \overset{\sigma_1}{\to} c_1 \overset{\sigma_2}{\to} c_2 \overset{\sigma_3}{\to} c_3$ of C, the diagram below commutes.

$$\begin{array}{ccc} g(\sigma_3\sigma_2\sigma_1) & \xrightarrow{G(\sigma_2\sigma_1, \sigma_3)} & g(\sigma_3) + g(\sigma_2\sigma_1) \\ {\scriptstyle G(\sigma_1, \sigma_3\sigma_2)} \downarrow & & \downarrow {\scriptstyle 1 + G(\sigma_1, \sigma_2)} \\ g(\sigma_3\sigma_2) + g(\sigma_1) & \xrightarrow{G(\sigma_2, \sigma_3) + 1} & g(\sigma_3) + g(\sigma_2) + g(\sigma_1) \end{array}$$

The set $Z^2(C, \mathscr{A})$ is pointed by the *zero* 2*-cocycle* $0 = (1_0, 0)$, which is defined by $0(\sigma) = 0$, for any morphism σ of C, and $1_0(\sigma_1, \sigma_2) = 1_0$, for any pair of composable morphisms of C.

♦ A 3*-cocycle* $H = (H, h) \in Z^3(C, \mathscr{A})$ consists of

- objects $h(\sigma_1, \sigma_2)$ of $\mathscr{A}$, one for each two composable morphisms $c_0 \overset{\sigma_1}{\to} c_1 \overset{\sigma_2}{\to} c_2$ of C,

- morphisms $H(\sigma_1, \sigma_2, \sigma_3): h(\sigma_1, \sigma_2) + h(\sigma_2\sigma_1, \sigma_3) \to h(\sigma_2, \sigma_3) + h(\sigma_1, \sigma_3\sigma_2)$, one for each three composable morphisms $c_0 \overset{\sigma_1}{\to} c_1 \overset{\sigma_2}{\to} c_2 \overset{\sigma_3}{\to} c_3$ of C,

satisfying

- for any morphism $c_0 \overset{\sigma}{\to} c_1$ of C, $\quad h(1, \sigma) = 0 = h(\sigma, 1)$,

- for any pair of composable morphisms $c_0 \overset{\sigma_1}{\to} c_1 \overset{\sigma_2}{\to} c_2$ of C,

$$H(\sigma_1, \sigma_2, 1) = H(1, \sigma_1, \sigma_2) = H(\sigma_1, 1, \sigma_2) = 1_{h(\sigma_1, \sigma_2)},$$

- for any four composable morphisms $c_0 \stackrel{\sigma_1}{\to} c_1 \stackrel{\sigma_2}{\to} c_2 \stackrel{\sigma_3}{\to} c_3 \stackrel{\sigma_4}{\to} c_4$ of C, the following diagram commutes.

$$\begin{array}{ccc}
h(\sigma_1,\sigma_2)+h(\sigma_2\sigma_1,\sigma_3)+h(\sigma_3\sigma_2\sigma_1,\sigma_4) & \xrightarrow{1+H(\sigma_2\sigma_1,\sigma_3,\sigma_4)} & h(\sigma_1,\sigma_2)+h(\sigma_3,\sigma_4)+h(\sigma_2\sigma_1,\sigma_4\sigma_3) \\
\Big\downarrow{\scriptstyle H(\sigma_1,\sigma_2,\sigma_3)+1} & & \Big\downarrow{\scriptstyle \mathbf{C}+1} \\
h(\sigma_2,\sigma_3)+h(\sigma_1,\sigma_3\sigma_2)+h(\sigma_3\sigma_2\sigma_1,\sigma_4) & & h(\sigma_3,\sigma_4)+h(\sigma_1,\sigma_2)+h(\sigma_2\sigma_1,\sigma_4\sigma_3) \\
\Big\downarrow{\scriptstyle 1+H(\sigma_1,\sigma_3\sigma_2,\sigma_4)} & & \Big\downarrow{\scriptstyle 1+H(\sigma_1,\sigma_2,\sigma_4\sigma_3)} \\
h(\sigma_2,\sigma_3)+h(\sigma_3\sigma_2,\sigma_4)+h(\sigma_1,\sigma_4\sigma_3\sigma_2) & \xrightarrow{H(\sigma_2,\sigma_3,\sigma_4)+1} & h(\sigma_3,\sigma_4)+h(\sigma_2,\sigma_4\sigma_3)+h(\sigma_1,\sigma_4\sigma_3\sigma_2)
\end{array}$$

The set $Z^3(C,\mathscr{A})$ is pointed by the *zero* 3-*cocycle* $0=(1_0,0)$ which is defined by $0(\sigma_1,\sigma_2)=0$, for any pair of composable morphisms in C, and $1_0(\sigma_1,\sigma_2,\sigma_3)=1_0$, for any triplet of composable arrows.

♦ A 4-*cocycle* $T=(T,t)\in Z^4(C,\mathscr{A})$ consists of

- objects $t(\sigma_1,\sigma_2,\sigma_3)$ of $\mathscr{A}$, one for each triplet of morphisms $c_0 \stackrel{\sigma_1}{\to} c_1 \stackrel{\sigma_2}{\to} c_2 \stackrel{\sigma_3}{\to} c_3$ in C,
- morphisms of $\mathscr{A}$

$$t(\sigma_1,\sigma_2,\sigma_4\sigma_3)+t(\sigma_2\sigma_1,\sigma_3,\sigma_4)\xrightarrow{T(\sigma_1,\sigma_2,\sigma_3,\sigma_4)} t(\sigma_2,\sigma_3,\sigma_4)+t(\sigma_1,\sigma_3\sigma_2,\sigma_4)+t(\sigma_1,\sigma_2,\sigma_3)\ ,$$

  one for each four composable morphisms $c_0 \stackrel{\sigma_1}{\to} c_1 \stackrel{\sigma_2}{\to} c_2 \stackrel{\sigma_3}{\to} c_3 \stackrel{\sigma_4}{\to} c_4$ of C,

such that

- for any two composable morphisms $c_0 \stackrel{\sigma_1}{\to} c_1 \stackrel{\sigma_2}{\to} c_2$ in C,

$$t(1,\sigma_1,\sigma_2)=0=t(\sigma_1,1,\sigma_2)=t(\sigma_1,\sigma_2,1),$$

- for any three composable morphisms $c_0 \stackrel{\sigma_1}{\to} c_1 \stackrel{\sigma_2}{\to} c_2 \stackrel{\sigma_3}{\to} c_3$ in C,

$$T(1,\sigma_1,\sigma_2,\sigma_3)=1_{t(\sigma_1,\sigma_2,\sigma_3)}=T(\sigma_1,1,\sigma_2,\sigma_3)=T(\sigma_1,\sigma_2,1,\sigma_3)=T(\sigma_1,\sigma_2,\sigma_3,1),$$

- for any five morphisms $c_0 \stackrel{\sigma_1}{\to} c_1 \stackrel{\sigma_2}{\to} c_2 \stackrel{\sigma_3}{\to} c_3 \stackrel{\sigma_4}{\to} c_4 \stackrel{\sigma_5}{\to} c_5$ in C, the diagram below commutes.

$$\begin{array}{ccccccc}
A_1 & \xrightarrow{1+T(\sigma_2\sigma_1,\sigma_3,\sigma_4,\sigma_5)} & A_2 & \xrightarrow{\mathbf{C}+1} & A_3 & \xrightarrow{1+T(\sigma_1,\sigma_2,\sigma_4\sigma_3,\sigma_5)+1} & A_4 \\
\Big\downarrow{\scriptstyle T(\sigma_1,\sigma_2,\sigma_3,\sigma_5\sigma_4)+1} & & & & & & \Big\downarrow{\scriptstyle 1+T(\sigma_1,\sigma_2,\sigma_3,\sigma_4)} \\
A'_1 & & & & & & \\
\Big\downarrow{\scriptstyle 1+\mathbf{C}} & & & & & & \\
A'_2 & \xrightarrow{1+T(\sigma_1,\sigma_3\sigma_2,\sigma_4,\sigma_5)+1} & A'_3 & \xrightarrow{T(\sigma_2,\sigma_3,\sigma_4,\sigma_5)+1} & A'_4 & \xrightarrow{1+\mathbf{C}+1} & A_5
\end{array}$$

$A_1=t(\sigma_1,\sigma_2,\sigma_5\sigma_4\sigma_3)+t(\sigma_2\sigma_1,\sigma_3,\sigma_5\sigma_4)+t(\sigma_3\sigma_2\sigma_1,\sigma_4,\sigma_5),$

$A_2=t(\sigma_1,\sigma_2,\sigma_5\sigma_4\sigma_3)+t(\sigma_3,\sigma_4,\sigma_5)+t(\sigma_2\sigma_1,\sigma_4\sigma_3,\sigma_5)+t(\sigma_2\sigma_1,\sigma_3,\sigma_4),$

$A_3=t(\sigma_3,\sigma_4,\sigma_5)+t(\sigma_1,\sigma_2,\sigma_4\sigma_3,\sigma_5)+t(\sigma_2\sigma_1,\sigma_4\sigma_3,\sigma_5)+t(\sigma_2\sigma_1,\sigma_3,\sigma_4),$

$A_4=t(\sigma_3,\sigma_4,\sigma_5)+t(\sigma_2,\sigma_4\sigma_3,\sigma_5)+t(\sigma_1,\sigma_4\sigma_3\sigma_2,\sigma_5)+t(\sigma_1,\sigma_2,\sigma_4\sigma_3)+t(\sigma_2\sigma_1,\sigma_3,\sigma_4),$

$A_5=t(\sigma_3,\sigma_4,\sigma_5)+t(\sigma_2,\sigma_4\sigma_3,\sigma_5)+t(\sigma_1,\sigma_4\sigma_3\sigma_2,\sigma_5)+t(\sigma_2,\sigma_3,\sigma_4)+t(\sigma_1,\sigma_3\sigma_2,\sigma_4)+t(\sigma_1,\sigma_2,\sigma_3),$

$$A'_1 = t(\sigma_2,\sigma_3,\sigma_5\sigma_4) + t(\sigma_1,\sigma_3\sigma_2,\sigma_5\sigma_4) + t(\sigma_1,\sigma_2,\sigma_3) + t(\sigma_3\sigma_2\sigma_1,\sigma_4,\sigma_5),$$
$$A'_2 = t(\sigma_2,\sigma_3,\sigma_5\sigma_4) + t(\sigma_1,\sigma_3\sigma_2,\sigma_5\sigma_4) + t(\sigma_3\sigma_2\sigma_1,\sigma_4,\sigma_5) + t(\sigma_1,\sigma_2,\sigma_3),$$
$$A'_3 = t(\sigma_2,\sigma_3,\sigma_5\sigma_4) + t(\sigma_3\sigma_2,\sigma_4,\sigma_5) + t(\sigma_1,\sigma_4\sigma_3\sigma_2,\sigma_5) + t(\sigma_1,\sigma_3\sigma_2,\sigma_4) + t(\sigma_1,\sigma_2,\sigma_3),$$
$$A'_4 = t(\sigma_3,\sigma_4,\sigma_5) + t(\sigma_2,\sigma_4\sigma_3,\sigma_5) + t(\sigma_2,\sigma_3,\sigma_4) + t(\sigma_1,\sigma_4\sigma_3\sigma_2,\sigma_5) + t(\sigma_1,\sigma_3\sigma_2,\sigma_4) + t(\sigma_1,\sigma_2,\sigma_3).$$

The set $Z^4(C,\mathscr{A})$ is pointed by the *zero 4-cocycle* $0=(1_0,0)$ defined by $0(\sigma_1,\sigma_2,\sigma_3)=0$, for any three composable morphisms in C, and $1_0(\sigma_1,\sigma_2,\sigma_3,\sigma_4)=1_0$, for any four composable arrows in C.

Example 2.2 Let ΣA denote the one-object permutative category whose morphisms are the elements of an abelian group A and where both composition and addition are given by the addition in A. Then, for any small category C and $1\leq n\leq 4$, $Z^n(C,\Sigma A)=Z^n(C,A)$ is the usual set of normalized n-cocycles of C with coefficients in the abelian group A.

3. The simplicial sets $K(\mathscr{A},n)$

Hereafter, we regard the ordered sets $[p]=\{0,1,\dots,p\}$ of the simplicial category Δ as categories with only one morphism $i\to j$ whenever $i\leq j$, so that a weakly order-preserving map $\alpha:[p]\to[q]$ in Δ is the same as functor, which we usually identify with the list $(\alpha(0),\dots,\alpha(p))$. Thus, a p-simplex of the simplicial standard q-simplex $\Delta[q]=\Delta(-,[q])$ is also the same as a list of integers $(i_0,i_1,\cdots,i_p)$ with $0\leq i_0\leq\cdots\leq i_p\leq q$. The geometric realization $|\Delta[q]|=\Delta^q$ is the oriented topological standard q-simplex, whose vertices we denote simply by $0,1,\dots,q$, and whose oriented p-face with vertices $i_0,\dots,i_p$, where $0\leq i_0<\cdots<i_p\leq q$, we denote by $(i_0,i_1,\cdots,i_p)$. The generating codegeneracy and coface maps in the simplicial category Δ are denoted, for $0\leq m\leq p$, by

$$s^m:[p+1]\to[p],\quad i\mapsto\begin{cases} i & \text{if } i\leq m\\ i-1 & \text{if } i>m\end{cases},\qquad d^m:[p-1]\to[p],\ i\mapsto\begin{cases} i & \text{if } i<m\\ i+1 & \text{if } i\geq m\end{cases}.$$

If $X:\Delta^{op}\to\mathbf{Sets}$ is a simplicial set, its degeneracy and face maps $X(s^m)$ and $X(d^m)$ are denoted as usually by s_m and d_m, respectively.

For any given permutative category $\mathscr{A}$ and each integer n, with $1\leq n\leq 4$, the assignment $C\mapsto Z^n(C,\mathscr{A})$ that carries each small category C to the pointed set of n-cocycles of C in $\mathscr{A}$, is clearly functorial on the category C, so we can define the pointed simplicial sets

$$K(\mathscr{A},n):\Delta^{op}\to\mathbf{Set}_*,\qquad [q]\mapsto Z^n([q],\mathscr{A}).$$

Example 3.1 Let ΣA be the permutative category defined by an abelian group A, as in Example 2.2. Then, for $1\leq n\leq 4$, $K(\Sigma A,n)=K(A,n)$ is the nth Eilenberg–MacLane minimal complex.

These simplicial sets $K(\mathscr{A},n)$ have the following pleasing geometric descriptions and properties.

♦ $K(\mathscr{A},1)$ is just $\mathrm{Ner}\mathscr{A}$, the ordinary nerve of the underlying category, here pointed by 0, the zero object of $\mathscr{A}$. Hence, its geometric realization

$$|K(\mathscr{A},1)|=B\mathscr{A}$$

is the usual classifying space of the underlying category [21, 22]. This is a 2-coskeletal simplicial set, whose q-simplices $F=(F,f)$ are geometrically represented as the 1-skeleton of an oriented standard q-simplex Δ^q, with an object $f(i_0)$ of $\mathscr{A}$ placed on each i_0-vertex, $0 \le i_0 \le q$, and a morphism of $\mathscr{A}$

$$f(i_0) \xrightarrow{F(i_0,i_1)} f(i_1) \ ,$$

placed on each 1-face (i_0, i_1), $0 \le i_0 < i_1 \le q$, with the 1-cocycle requirement that every 2-face triangle commutes, that is, $F(i_1, i_2)\, F(i_0, i_1) = F(i_0, i_2)$, for every $0 \le i_0 < i_1 < i_2 \le q$.

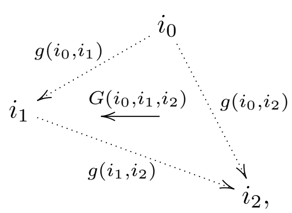
♦ $K(\mathscr{A}, 2)$ is the geometric nerve of the suspension 2-category $\Sigma\mathscr{A}$, see [2, 9, 25]. It is a 3-coskeletal reduced (one vertex) simplicial set, whose q-simplices $G = (G, g)$ are geometrically represented as the 2-skeleton of the oriented standard q-simplex with an object $g(i_0, i_1)$ of $\mathscr{A}$ placed on each 1-face (i_0, i_1), $0 \le i_0 < i_1 \le q$,

$$i_0 \overset{g(i_0,i_1)}{\dashrightarrow} i_1 \ ,$$

and a morphism of $\mathscr{A}$

$$g(i_0, i_2) \xrightarrow{G(i_0,i_1,i_2)} g(i_1, i_2) + g(i_0, i_1)$$

placed on the inside of each 2-face (i_0, i_1, i_2), $0 \le i_0 < i_1 < i_2 \le q$,

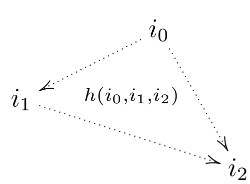


(see Street's 2nd oriental [24])

with the 2-cocycle requirement that every 3-face tetrahedron is commutative, that is, for any $0 \le i_0 < i_1 < i_2 < i_3 \le q$, the following diagram in $\mathscr{A}$ commutes

$$\begin{array}{ccc} g(i_0,i_3) & \xrightarrow{G(i_0,i_2,i_3)} & g(i_2,i_3)+g(i_0,i_2) \\ {\scriptstyle G(i_0,i_1,i_3)}\downarrow & & \downarrow{\scriptstyle 1+G(i_0,i_1,i_2)} \\ g(i_1,i_3)+g(i_0,i_1) & \xrightarrow{G(i_1,i_2,i_3)+1} & g(i_2,i_3)+g(i_1,i_2)+g(i_0,i_1). \end{array}$$

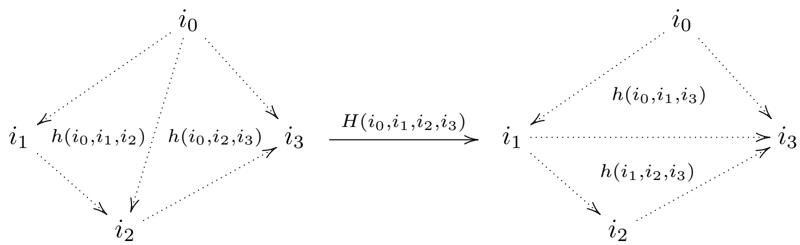
♦ $K(\mathscr{A}, 3)$ is the geometric nerve of the double suspension 3-category $\Sigma^2\mathscr{A}$, see [4, 5]. This is a 4-coskeletal 1-reduced (only one 1-simplex) simplicial set, whose q-simplices $H = (H, h)$ are geometrically represented as the 3-skeleton of the oriented standard q-simplex Δ^q, with an object $h(i_0, i_1, i_2)$ of $\mathscr{A}$ placed on each 2-face (i_0, i_1, i_2), $0 \le i_0 < i_1 < i_2 \le q$,



and a morphism of $\mathscr{A}$

$$h(i_0, i_1, i_2) + h(i_0, i_2, i_3) \xrightarrow{H(i_0,i_1,i_2,i_3)} h(i_1, i_2, i_3) + h(i_0, i_1, i_3)$$

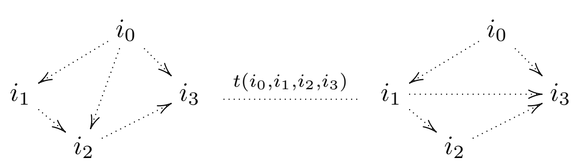
placed inside of each 3-face (i_0, i_1, i_2, i_3), $0 \leq i_0 < i_1 < i_2 < i_3 \leq q$, (see Street's third oriental [24]),



with the 3-cocycle requirement that every 4-face is commutative. That is, the following diagram in $\mathscr{A}$ commutes, for any $0 \leq i_0 < i_1 < i_2 < i_3 < i_4 \leq q$.

$$\begin{array}{ccc}
h(i_0,i_1,i_2)+h(i_0,i_2,i_3)+h(i_0,i_3,i_4) & \xrightarrow{1+H(i_0,i_2,i_3,i_4)} & h(i_0,i_1,i_2)+h(i_1,i_2,i_3,i_4)+h(i_0,i_2,i_4) \\
\downarrow{\scriptstyle H(i_0,i_1,i_2,i_3)+1} & & \downarrow{\scriptstyle \mathbf{C}+1} \\
h(i_1,i_2,i_3)+h(i_0,i_1,i_3)+h(i_0,i_3,i_4) & & h(i_2,i_3,i_4)+h(i_0,i_1,i_2)+h(i_0,i_2,i_4) \\
\downarrow{\scriptstyle 1+H(i_0,i_1,i_3,i_4)} & & \downarrow{\scriptstyle 1+H(i_0,i_1,i_2,i_4)} \\
h(i_1,i_2,i_3)+h(i_1,i_3,i_4)+h(i_0,i_1,i_4) & \xrightarrow{H(i_1,i_2,i_3,i_4)+1} & h(i_2,i_3,i_4)+h(i_1,i_2,i_4)+h(i_0,i_1,i_4)
\end{array}$$

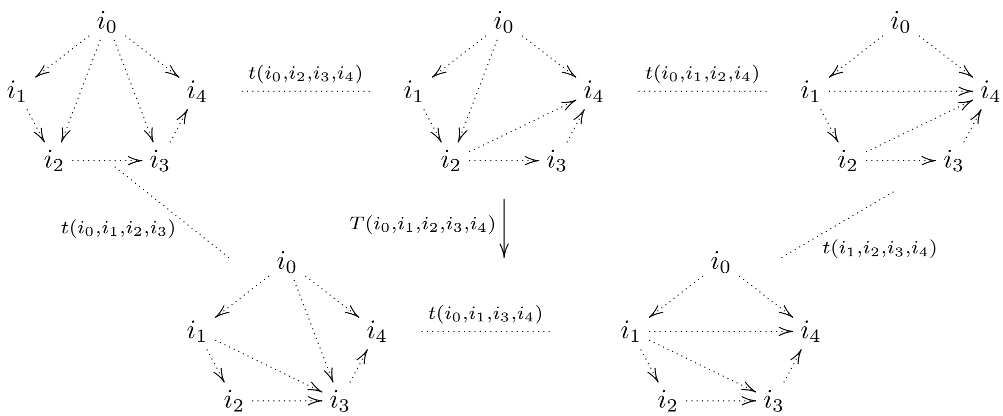
♦ $K(\mathscr{A}, 4)$ is a 5-coskeletal 2-reduced (i.e, with only one 2-simplex) simplicial set, whose q-simplices $T = (T, t)$ are geometrically represented as the 4-skeleton of the oriented standard q-simplex, with an object $t(i_0, i_1, i_2, i_3)$ of $\mathscr{A}$ placed inside each 3-face (i_0, i_1, i_2, i_3) of Δ^q



and a morphism of $\mathscr{A}$

$$t(i_0, i_1, i_2, i_4) + t(i_0, i_2, i_3, i_4) \xrightarrow{T(i_0,i_1,i_2,i_3,i_4)} t(i_1, i_2, i_3, i_4) + t(i_0, i_1, i_3, i_4) + t(i_0, i_1, i_2, i_3)$$

placed on the inside of each oriented 4-face $(i_0, i_1, i_2, i_3, i_4)$, $0 \le i_0 < i_1 < i_2 < i_3 < i_4 \le q$,



(see Street's fourth oriental [24])

with the 4-cocycle requirement that every 5-face is commutative, that is, the diagram in $\mathscr{A}$ below commutes for any $0 \le i_0 < i_1 < i_2 < i_3 < i_4 < i_5 \le q$.

$$\begin{array}{ccccccc}
A_1 & \xrightarrow{1+T(i_0,i_2,i_3,i_4,i_5)} & A_2 & \xrightarrow{\mathbf{C}+1} & A_3 & \xrightarrow{1+T(i_0,i_1,i_2,i_4,i_5)+1} & A_4 \\
{\scriptstyle T(i_0,i_1,i_2,i_3,i_5)+1}\downarrow & & & & & & \\
A'_1 & & & & & & \downarrow{\scriptstyle 1+T(i_0,i_1,i_2,i_3,i_4)} \\
{\scriptstyle 1+\mathbf{C}}\downarrow & & & & & & \\
A'_2 & \xrightarrow{1+T(i_0,i_1,i_3,i_4,i_5)+1} & A'_3 & \xrightarrow{T(i_1,i_2,i_3,i_4,i_5)+1} & A'_4 & \xrightarrow{1+\mathbf{C}+1} & A_5
\end{array}$$

$$\begin{aligned}
A_1 &= t(i_0,i_1,i_2,i_5) + t(i_0,i_2,i_3,i_5) + t(i_0,i_3,i_4,i_5),\\
A_2 &= t(i_0,i_1,i_2,i_5) + t(i_2,i_3,i_4,i_5) + t(i_0,i_2,i_4,i_5) + t(i_0,i_2,i_3,i_4),\\
A_3 &= t(i_2,i_3,i_4,i_5) + t(i_0,i_1,i_2,i_5) + t(i_0,i_2,i_4,i_5) + t(i_0,i_2,i_3,i_4),\\
A_4 &= t(i_2,i_3,i_4,i_5) + t(i_1,i_2,i_4,i_5) + t(i_0,i_1,i_4,i_5) + t(i_0,i_1,i_2,i_4) + t(i_0,i_2,i_3,i_4),\\
A_5 &= t(i_2,i_3,i_4,i_5) + t(i_1,i_2,i_4,i_5) + t(i_0,i_1,i_4,i_5) + t(i_1,i_2,i_3,i_4) + t(i_0,i_1,i_3,i_4) + t(i_0,i_1,i_2,i_3),\\
A'_1 &= t(i_1,i_2,i_3,i_5) + t(i_0,i_1,i_3,i_5) + t(i_0,i_1,i_2,i_3) + t(i_0,i_3,i_4,i_5),\\
A'_2 &= t(i_1,i_2,i_3,i_5) + t(i_0,i_1,i_3,i_5) + t(i_0,i_3,i_4,i_5) + t(i_0,i_1,i_2,i_3),\\
A'_3 &= t(i_1,i_2,i_3,i_5) + t(i_1,i_3,i_4,i_5) + t(i_0,i_1,i_4,i_5) + t(i_0,i_1,i_3,i_4) + t(i_0,i_1,i_2,i_3),\\
A'_4 &= t(i_2,i_3,i_4,i_5) + t(i_1,i_2,i_4,i_5) + t(i_1,i_2,i_3,i_4) + t(i_0,i_1,i_4,i_5) + t(i_0,i_1,i_3,i_4) + t(i_0,i_1,i_2,i_3).
\end{aligned}$$

3.1. Suspensions

For each $n = 1, 2, 3$, there are suspension pointed maps

$$S : Z^n([q], \mathscr{A}) \to Z^{n+1}([q+1], \mathscr{A}) \qquad (q \ge 0), \tag{3.1}$$

carrying q-simplices of $K(\mathscr{A}, n)$ to $(q+1)$-simplices de $K(\mathscr{A}, n+1)$, which are defined as follows:

- ⧫ If $F = (F, f) \in Z^1([q], \mathscr{A})$, then its suspension $SF = (SF, Sf) \in Z^2([q+1], \mathscr{A})$ is given by

$$(Sf)(i_0, i_1) = \begin{cases} f(i_0) & \text{if } i_1 = q+1, \\ 0 & \text{if } i_1 < q+1, \end{cases} \qquad (SF)(i_0, i_1, i_2) = \begin{cases} F(i_0, i_1) & \text{if } i_2 = q+1, \\ 1_0 & \text{if } i_2 < q+1. \end{cases}$$

♦ If $G=(G,g)\in Z^2([q],\mathscr{A})$, then $SG=(SG,Sg)\in Z^3([q+1],\mathscr{A})$ is defined by

$$(Sg)(i_0,i_1,i_2)=\begin{cases} g(i_0,i_1) & \text{if } i_2=q+1,\\ 0 & \text{if } i_2<q+1,\end{cases}\quad (SG)(i_0,i_1,i_2,i_3)=\begin{cases} G(i_0,i_1,i_2) & \text{if } i_3=q+1,\\ 1_0 & \text{if } i_3<q+1.\end{cases}$$

♦ If $H=(H,h)\in Z^3([q],\mathscr{A})$, then $SH=(SH,Sh)\in Z^4([q+1],\mathscr{A})$ is given by

$$(Sh)(i_0,i_1,i_2,i_3)=\begin{cases} h(i_0,i_1,i_2) & \text{if } i_3=q+1,\\ 0 & \text{if } i_3<q+1,\end{cases}$$

$$(SH)(i_0,i_1,i_2,i_3,i_4)=\begin{cases} H(i_0,i_1,i_2,i_3) & \text{if } i_4=q+1,\\ 1_0 & \text{if } i_4<q+1.\end{cases}$$

For any $q\geq 0$, the suspension maps (3.1) satisfy the following simplicial equalities

$$\begin{cases} d_{q+1}S &=& 0\\ d_jS &=& Sd_j, & \text{for } 0\leq j\leq q,\\ s_jS &=& Ss_j, & \text{for } 0\leq j\leq q.\end{cases}$$

Hence, they define natural simplicial maps

$$S:\Sigma K(\mathscr{A},n)\to K(\mathscr{A},n+1)$$

one for each $n=1,2,3$, from the Kan-suspension [15] of $K(\mathscr{A},n)$ to $K(\mathscr{A},n+1)$. By composing the induced map on geometric realizations $S:|\Sigma K(\mathscr{A},n)|\to|K(\mathscr{A},n+1)|$ with the natural homeomorphism $\Sigma|K(\mathscr{A},n)|\cong|\Sigma K(\mathscr{A},n)|$, between the ordinary (reduced) suspension of the geometric realization of $K(\mathscr{A},n)$ and the geometric realization of its Kan-suspension [15, Proposition 2.3], we get an induced natural map $S:\Sigma|K(\mathscr{A},n)|\to|K(\mathscr{A},n+1)|$. Let

$$\jmath:|K(\mathscr{A},n)|\to\Omega|K(\mathscr{A},n+1)| \tag{3.2}$$

denote its adjoint map. The following is the main result in this paper, whose proof is given in the subsequent Section 5.

Theorem 3.2 *For $n=2,3$, the map $\jmath$ in (3.2) is a homotopy equivalence; thus,*

$$|K(\mathscr{A},2)|\simeq\Omega|K(\mathscr{A},3)|$$
$$|K(\mathscr{A},3)|\simeq\Omega|K(\mathscr{A},4)|,$$

while $\jmath:|K(\mathscr{A},1)|\to\Omega|K(\mathscr{A},2)|$ is a group completion map, that is, it induces isomorphism on homology after inverting the action of the monoid π_0 of components of $|K(\mathscr{A},1)|=B\mathscr{A}$,

$$(\pi_0)^{-1}H_*(|K(\mathscr{A},1)|)\cong H_*(\Omega|K(\mathscr{A},2)|).$$

Therefore, there are natural homotopy equivalences

$$|K(\mathscr{A},2)|\simeq\mathrm{Spt}_0(\mathscr{A}),\quad |K(\mathscr{A},3)|\simeq\mathrm{Spt}_1(\mathscr{A}),\quad |K(\mathscr{A},4)|\simeq\mathrm{Spt}_2(\mathscr{A}).$$

4. The $(n-1)$-categories of n-cocycles

This auxiliary section prepares for the proof of Theorem 3.2 we give in the next section. Here we mainly show that, for any permutative category $\mathscr{A}$, any small category C, and each integer $n = 2, 3, 4$, the n-cocycles of C with coefficients in $\mathscr{A}$ are the objects of a $(n-1)$-category.

★ $Z^1(C, \mathscr{A})$ *is a monoid*, where the addition $F + F' = (F + F', f + f')$ of two 1-cocycles F and F' is the 1-cocycle given by $(f + f')(c) = f(c) + f'(c)$, for any object c of C, and $(F + F')(\sigma) = F(\sigma) + F'(\sigma)$, for any morphism σ of C.

★ $Z^2(C, \mathscr{A})$ *is the set of objects of a category*, denoted by $\mathcal{Z}^2(C, \mathscr{A})$, where

⧫ a *morphism* $F : G \to G'$ is a relative lax transformation, which consists of

- morphisms $F(\sigma) : g(\sigma) \to g'(\sigma)$ in $\mathscr{A}$, one for each morphism $c_0 \xrightarrow{\sigma} c_1$ of C,

such that

- for any object c of C, $\quad F(1_c) = 1_0$,
- for any two composable morphisms $c_0 \xrightarrow{\sigma_1} c_1 \xrightarrow{\sigma_2} c_2$ in C, the diagram below commutes.

$$\begin{array}{ccc} g(\sigma_2\sigma_1) & \xrightarrow{G(\sigma_1,\sigma_2)} & g(\sigma_2) + g(\sigma_1) \\ {\scriptstyle F(\sigma_2\sigma_1)}\downarrow & & \downarrow{\scriptstyle F(\sigma_2)+F(\sigma_1)} \\ g'(\sigma_2\sigma_1) & \xrightarrow{G'(\sigma_1,\sigma_2)} & g'(\sigma_2) + g'(\sigma_1) \end{array}$$

the composition of two morphisms $F : G \to G'$ and $F' : G' \to G''$ is the morphism $F'F : G \to G''$ given by $(F'F)(\sigma) = F'(\sigma)\,F(\sigma)$, for any morphism σ of C. The identity of a 2-cocycle G is the morphism denoted by $1_g : G \to G$ which is given by $1_g(\sigma) = 1_{g(\sigma)}$, for each morphism $\sigma : c_0 \to c_1$ of C.

Let us stress the identification of monoids

$$\mathrm{End}_{\mathcal{Z}^2(C,\mathscr{A})}(0) = Z^1\big(C, \Sigma\,\mathrm{End}_{\mathscr{A}}(0)\big), \qquad 0 \xrightarrow{F} 0 \equiv (F, 0),$$

where $\Sigma\,\mathrm{End}_{\mathscr{A}}(0)$ is the one-object permutative category defined by the commutative monoid of endomorphisms of the zero object in $\mathscr{A}$.

★ $Z^3(C, \mathscr{A})$ *is the set of objects of a (strict)* 2*-category*, denoted by $\mathcal{Z}^3(C, \mathscr{A})$, where

⧫ a 1*-cell* $G = (G, g) : H \to H'$ is a relative lax transformation, which consists of

- objects $g(\sigma)$ of $\mathscr{A}$, one for each morphism $c_0 \xrightarrow{\sigma} c_1$ of C,
- morphisms $G(\sigma_1, \sigma_2) : h(\sigma_1, \sigma_2) + g(\sigma_2\sigma_1) \to g(\sigma_2) + g(\sigma_1) + h'(\sigma_1, \sigma_2)$, one for each two composable morphisms $c_0 \xrightarrow{\sigma_1} c_1 \xrightarrow{\sigma_2} c_2$ of C,

such that

- for any object c of C, $\quad g(1_c) = 0$,

- for any morphism $c_0 \xrightarrow{\sigma} c_1$ of C, $\quad G(1,\sigma) = 1_{g(\sigma)} = G(\sigma,1)$,

- for any three composable morphisms $c_0 \xrightarrow{\sigma_1} c_1 \xrightarrow{\sigma_2} c_2 \xrightarrow{\sigma_3} c_3$ of C, the diagram below commutes.

$$\begin{array}{ccc}
h(\sigma_1,\sigma_2)+h(\sigma_2\sigma_1,\sigma_3)+g(\sigma_3\sigma_2\sigma_1) & \xrightarrow{1+G(\sigma_2\sigma_1,\sigma_3)} & h(\sigma_1,\sigma_2)+g(\sigma_3)+g(\sigma_2\sigma_1)+h'(\sigma_2\sigma_1,\sigma_3) \\
\downarrow{\scriptstyle H(\sigma_1,\sigma_2,\sigma_3)+1} & & \downarrow{\scriptstyle \mathbf{C}+1} \\
h(\sigma_2,\sigma_3)+h(\sigma_1,\sigma_3\sigma_2)+g(\sigma_3\sigma_2,\sigma_1) & & g(\sigma_3)+h(\sigma_1,\sigma_2)+g(\sigma_2\sigma_1)+h'(\sigma_2\sigma_1,\sigma_3) \\
\downarrow{\scriptstyle 1+G(\sigma_1,\sigma_3\sigma_2)} & & \downarrow{\scriptstyle 1+G(\sigma_1,\sigma_2)+1} \\
h(\sigma_2,\sigma_3)+g(\sigma_3\sigma_2)+g(\sigma_1)+h'(\sigma_1,\sigma_3\sigma_2) & & g(\sigma_3)+g(\sigma_2)+g(\sigma_1)+h'(\sigma_1,\sigma_2)+h'(\sigma_2\sigma_1,\sigma_3) \\
\downarrow{\scriptstyle G(\sigma_2,\sigma_3)+1} & & \downarrow{\scriptstyle 1+H'(\sigma_1,\sigma_2,\sigma_3)} \\
g(\sigma_3)+g(\sigma_2)+h'(\sigma_2,\sigma_3)+g(\sigma_1)+h'(\sigma_1,\sigma_3\sigma_2) & \xrightarrow{1+\mathbf{C}+1} & g(\sigma_3)+g(\sigma_2)+g(\sigma_1)+h'(\sigma_2,\sigma_3)+h'(\sigma_1,\sigma_3\sigma_2)
\end{array}$$

♦ a 2-*cell* $H \underset{G'}{\overset{G}{\Downarrow F}} H'$ is a relative modification, which consists of

- morphisms $F(\sigma): g(\sigma) \to g'(\sigma)$ of $\mathscr{A}$, one for each morphism $c_0 \xrightarrow{\sigma} c_1$ of C,

such that

- for any object c of C, $\quad F(1_c) = 1_0$,

- for any two composable morphisms $c_0 \xrightarrow{\sigma_1} c_1 \xrightarrow{\sigma_2} c_2$ in C, the diagram below commutes.

$$\begin{array}{ccc}
h(\sigma_1,\sigma_2)+g(\sigma_2\sigma_1) & \xrightarrow{G(\sigma_1,\sigma_2)} & g(\sigma_2)+g(\sigma_1)+h'(\sigma_1,\sigma_2) \\
\downarrow{\scriptstyle 1+F(\sigma_2\sigma_1)} & & \downarrow{\scriptstyle F(\sigma_2)+F(\sigma_1)+1} \\
h(\sigma_1,\sigma_2)+g'(\sigma_2\sigma_1) & \xrightarrow{G'(\sigma_1,\sigma_2)} & g'(\sigma_2)+g'(\sigma_1)+h'(\sigma_1,\sigma_2)
\end{array}$$

♦ the vertical composition of 2-cells $H \overset{G}{\underset{G''}{\Downarrow F \; G' \; \Downarrow F'}} H' \mapsto H \underset{G''}{\overset{G}{\Downarrow F'F}} H'$, is given by pointwise composition of arrows in $\mathscr{A}$, that is, for any $\sigma: c_0 \to c_1$ in C,

$$(F'F)(\sigma) = F'(\sigma)\, F(\sigma) : g(\sigma) \to g''(\sigma),$$

and the identity of a 1-cell $G = (G, g) : H \to H''$ is the 2-cell denoted by $1_g : G \Rightarrow G$ which is given by $1_g(\sigma) = 1_{g(\sigma)}$, the identity morphism of $g(\sigma)$ in $\mathscr{A}$, for any $\sigma: c_0 \to c_1$ of C.

♦ the horizontal composition of two 1-cells $G: H \to H'$ and $G': H' \to H''$ is the 1-cell

$$G' \dotplus G = (G' \dotplus G, g' + g) : H \to H'',$$

where, for each $c_0 \overset{\sigma_1}{\to} c_1$ of C, $(g'+g)(\sigma) = g'(\sigma) + g(\sigma)$ and, for any $c_0 \overset{\sigma_1}{\to} c_1 \overset{\sigma_2}{\to} c_2$ in C, the morphism $(G' \dotplus G)(\sigma_1,\sigma_2)$ is the dotted one in the commutative diagram

$$\begin{array}{ccc}
h(\sigma_1,\sigma_2)+g'(\sigma_2\sigma_1)+g(\sigma_2\sigma_1) & \overset{(G'\dotplus G)(\sigma_1,\sigma_2)}{\cdots\cdots\cdots\cdots\rightarrow} & g'(\sigma_2)+g(\sigma_2)+g'(\sigma_1)+g(\sigma_1)+h''(\sigma_1,\sigma_2) \\
{\scriptstyle 1+\mathbf{C}}\big\downarrow & & \big\uparrow{\scriptstyle 1+\mathbf{C}+1} \\
h(\sigma_1,\sigma_2)+g(\sigma_2\sigma_1)+g'(\sigma_2\sigma_1) & & g'(\sigma_2)+g(\sigma_2)+g(\sigma_1)+g'(\sigma_1)+h''(\sigma_1,\sigma_2) \\
{\scriptstyle G(\sigma_1,\sigma_2)+1}\big\downarrow & & \big\uparrow{\scriptstyle \mathbf{C}+1} \\
g(\sigma_2)+g(\sigma_1)+h'(\sigma_1,\sigma_2)+g'(\sigma_2\sigma_1) & \xrightarrow{1+G'(\sigma_1,\sigma_2)} & g(\sigma_2)+g(\sigma_1)+g'(\sigma_2)+g'(\sigma_1)+h''(\sigma_1,\sigma_2),
\end{array}$$

and the horizontal composition of 2-cells $H \underset{G_1}{\overset{G}{\Downarrow F}} H' \underset{G'_1}{\overset{G'}{\Downarrow F'}} H'' \;\mapsto\; H \underset{G'_1 \dotplus G_1}{\overset{G'\dotplus G}{\Downarrow F'+F}} H''$ is given by pointwise addition of objects in $\mathscr{A}$, that is, for any $\sigma : c_0 \to c_1$ in $\mathscr{A}$,

$$(F'+F)(\sigma) = F'(\sigma) + F(\sigma) : g'(\sigma) + g(\sigma) \longrightarrow g'_1(\sigma) + g_1(\sigma).$$

The identity of a 3-cocycle $H \in Z^3(C,\mathscr{A})$ is the 1-cell $(1_h, 0) : H \to H$,where $0(\sigma) = 0$, for any $\sigma : c_0 \to c_1$ of C and $1_h(\sigma_1,\sigma_2) = 1_{h(\sigma_1,\sigma_2)}$, the identity of $h(\sigma_1,\sigma_2)$ in $\mathscr{A}$, for any $c_0 \overset{\sigma_1}{\to} c_1 \overset{\sigma_2}{\to} c_2$ in C.

A quite straightforward verification proves that $\mathcal{Z}^3(C,\mathscr{A})$ is actually a strict 2-category.

Let us point out the identification of categories (actually, of monoidal categories)

$$\mathcal{Z}^2(C,\mathscr{A}) = \mathbf{End}_{\mathcal{Z}^3(C,\mathscr{A})}(0), \qquad G \overset{F}{\to} G' \;\equiv\; 0 \underset{G'}{\overset{G}{\Downarrow F}} 0\,. \tag{4.1}$$

★ *$Z^4(C,\mathscr{A})$ is the set of objects of a (semistrict) 3-category* (i.e. a strict, cubical tricategory [11], also known as a Gray category), denoted by $\mathcal{Z}^4(C,\mathscr{A})$, which is as follows.

♦ a *1-cell* $H = (H,h) : T \to T'$ is a relative lax transformation; it consists of

- objects $h(\sigma_1,\sigma_2)$ of $\mathscr{A}$, one for each two composable morphisms $c_0 \overset{\sigma_1}{\to} c_1 \overset{\sigma_2}{\to} c_2$ of C,
- morphisms

$$t(\sigma_1,\sigma_2,\sigma_3) + h(\sigma_1,\sigma_2) + h(\sigma_2\sigma_1,\sigma_3) \xrightarrow{H(\sigma_1,\sigma_2,\sigma_3)} h(\sigma_2,\sigma_3) + h(\sigma_1,\sigma_3\sigma_2) + t'(\sigma_1,\sigma_2,\sigma_3)\ ,$$

  one for each three composable morphisms $c_0 \overset{\sigma_1}{\to} c_1 \overset{\sigma_2}{\to} c_2 \overset{\sigma_3}{\to} c_3$ of C,

such that

- for any morphism $c_0 \overset{\sigma}{\to} c_1$ of C, $\quad h(1,\sigma) = 0 = h(\sigma,1)$,
- for any pair of composable morphisms $c_0 \overset{\sigma_1}{\to} c_1 \overset{\sigma_2}{\to} c_2$ of C,

$$H(\sigma_1,\sigma_2,1) = 1_{h(\sigma_1,\sigma_2)} = H(1,\sigma_1,\sigma_2) = H(\sigma_1,1,\sigma_2),$$

- for any morphisms $c_0 \overset{\sigma_1}{\to} c_1 \overset{\sigma_2}{\to} c_2 \overset{\sigma_3}{\to} c_3 \overset{\sigma_4}{\to} c_4$ of C, the following diagram commutes.

$$\begin{array}{ccccccccccc}
A_1 & \xrightarrow{1+\mathbf{C}+1} & A_2 & \xrightarrow{1+H(\sigma_2\sigma_1,\sigma_3,\sigma_4)} & A_3 & \xrightarrow{1+\mathbf{C}+1} & A_4 & \xrightarrow{H(\sigma_1,\sigma_2,\sigma_4\sigma_3)+1} & A_5 & \xrightarrow{1+\mathbf{C}+1} & A_6 \\
{\scriptstyle T(\sigma_1,\sigma_2,\sigma_3,\sigma_4)+1}\downarrow & & & & & & & & & & \downarrow{\scriptstyle 1+T'(\sigma_1,\sigma_2,\sigma_3,\sigma_4)} \\
A'_1 & & & & & & & & & & \\
{\scriptstyle 1+H(\sigma_1,\sigma_2,\sigma_3)+1}\downarrow & & & & & & & & & & \\
A'_2 & \xrightarrow{1+\mathbf{C}+1+\mathbf{C}} & A'_3 & \xrightarrow{1+H(\sigma_1,\sigma_3\sigma_2,\sigma_4)+1} & A'_4 & \xrightarrow{H(\sigma_2,\sigma_3,\sigma_4)+1} & A'_5 & \xrightarrow{1+\mathbf{C}+1} & A'_6 & \xrightarrow{\mathbf{C}+1} & A_7
\end{array}$$

$A_1 = t(\sigma_1,\sigma_2,\sigma_4\sigma_3)+t(\sigma_2\sigma_1,\sigma_3,\sigma_4)+h(\sigma_1,\sigma_2)+h(\sigma_2\sigma_1,\sigma_3)+h(\sigma_3\sigma_2\sigma_1,\sigma_4)$, $A_2 = t(\sigma_1,\sigma_2,\sigma_4\sigma_3)+h(\sigma_1,\sigma_2)+t(\sigma_2\sigma_1,\sigma_3,\sigma_4)+h(\sigma_2\sigma_1,\sigma_3)+h(\sigma_3\sigma_2\sigma_1,\sigma_4)$, $A_3 = t(\sigma_1,\sigma_2,\sigma_4\sigma_3)+h(\sigma_1,\sigma_2)+h(\sigma_3,\sigma_4)+h(\sigma_2\sigma_1,\sigma_4\sigma_3)+t'(\sigma_2\sigma_1,\sigma_3,\sigma_4)$, $A_4 = t(\sigma_1,\sigma_2,\sigma_4\sigma_3)+h(\sigma_1,\sigma_2)+h(\sigma_2\sigma_1,\sigma_4\sigma_3)+h(\sigma_3,\sigma_4)+t'(\sigma_2\sigma_1,\sigma_3,\sigma_4)$, $A_5 = h(\sigma_2,\sigma_4\sigma_3)+h(\sigma_1,\sigma_4\sigma_3\sigma_2)+t'(\sigma_1,\sigma_2,\sigma_4\sigma_3)+h(\sigma_3,\sigma_4)+t'(\sigma_2\sigma_1,\sigma_3,\sigma_4)$, $A_6 = h(\sigma_2,\sigma_4\sigma_3)+h(\sigma_1,\sigma_4\sigma_3\sigma_2)+h(\sigma_3,\sigma_4)+t'(\sigma_1,\sigma_2,\sigma_4\sigma_3)+t'(\sigma_2\sigma_1,\sigma_3,\sigma_4)$, $A_7 = h(\sigma_2,\sigma_4\sigma_3)+h(\sigma_1,\sigma_4\sigma_3\sigma_2)+h(\sigma_3,\sigma_4)+t'(\sigma_2,\sigma_3,\sigma_4)+t'(\sigma_1,\sigma_3\sigma_2,\sigma_4)+t'(\sigma_1,\sigma_2,\sigma_3)$, $A'_1 = t(\sigma_2,\sigma_3,\sigma_4)+t(\sigma_1,\sigma_3\sigma_2,\sigma_4)+t(\sigma_1,\sigma_2,\sigma_3)+h(\sigma_1,\sigma_2)+h(\sigma_2\sigma_1,\sigma_3)+h(\sigma_3\sigma_2\sigma_1,\sigma_4)$, $A'_2 = t(\sigma_2,\sigma_3,\sigma_4)+t(\sigma_1,\sigma_3\sigma_2,\sigma_4)+h(\sigma_1,\sigma_2)+h(\sigma_1,\sigma_3\sigma_2)+t'(\sigma_1,\sigma_2,\sigma_3)+h(\sigma_3\sigma_2\sigma_1,\sigma_4)$, $A'_3 = t(\sigma_2,\sigma_3,\sigma_4)+h(\sigma_2,\sigma_3)+t(\sigma_1,\sigma_3\sigma_2,\sigma_4)+h(\sigma_1,\sigma_3\sigma_2)+h(\sigma_3\sigma_2\sigma_1,\sigma_4)+t'(\sigma_1,\sigma_2,\sigma_3)$, $A'_4 = t(\sigma_2,\sigma_3,\sigma_4)+h(\sigma_2,\sigma_3)+h(\sigma_3\sigma_2,\sigma_4)+h(\sigma_1,\sigma_4\sigma_3\sigma_2)+t'(\sigma_1,\sigma_3\sigma_2,\sigma_4)+t'(\sigma_1,\sigma_2,\sigma_3)$, $A'_5 = h(\sigma_3,\sigma_4)+h(\sigma_2,\sigma_4\sigma_3)+t'(\sigma_2,\sigma_3,\sigma_4)+h(\sigma_1,\sigma_4\sigma_3\sigma_2)+t'(\sigma_1,\sigma_3\sigma_2,\sigma_4)+t'(\sigma_1,\sigma_2,\sigma_3)$, $A'_6 = h(\sigma_3,\sigma_4)+h(\sigma_2,\sigma_4\sigma_3)+h(\sigma_1,\sigma_4\sigma_3\sigma_2)+t'(\sigma_2,\sigma_3,\sigma_4)+t'(\sigma_1,\sigma_3\sigma_2,\sigma_4)+t'(\sigma_1,\sigma_2,\sigma_3)$.

♦ a 2-*cell* $T \underset{H'}{\overset{H}{\Downarrow G}} T'$ is a relative lax modification; it consists of

- objects $g(\sigma)$ of $\mathscr{A}$, one for each morphism $c_0 \overset{\sigma}{\to} c_1$ of C,
- morphisms $G(\sigma_1,\sigma_2) : h(\sigma_1,\sigma_2)+g(\sigma_2\sigma_1) \to g(\sigma_2)+g(\sigma_1)+h'(\sigma_1,\sigma_2)$, one for each two composable morphisms $c_0 \overset{\sigma_1}{\to} c_1 \overset{\sigma_2}{\to} c_2$ of C,

such that

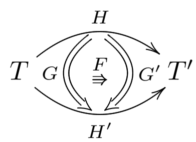
- for any object c of C, $\quad g(1_c) = 0$,
- for any morphism $c_0 \overset{\sigma}{\to} c_1$ of C, $\quad G(1,\sigma) = 1_{g(\sigma)} = G(\sigma,1)$,
- for any three composable morphisms $c_0 \overset{\sigma_1}{\to} c_1 \overset{\sigma_2}{\to} c_2 \overset{\sigma_3}{\to} c_3$ of C, the diagram below commutes.

$$\begin{array}{ccccccccc}
A_1 & \xrightarrow{1+G(\sigma_2\sigma_1,\sigma_3)} & A_2 & \xrightarrow{\mathbf{C}+1} & A_3 & \xrightarrow{1+G(\sigma_1,\sigma_2)+1} & A_4 & \xrightarrow{1+\mathbf{C}+1} & A_5 \\
{\scriptstyle H(\sigma_1,\sigma_2,\sigma_3)+1}\downarrow & & & & & & & & \downarrow{\scriptstyle 1+H'(\sigma_1,\sigma_2,\sigma_3)} \\
A'_2 & \xrightarrow{1+\mathbf{C}} & A'_3 & \xrightarrow{1+G(\sigma_1,\sigma_3\sigma_2)+1} & A'_4 & \xrightarrow{G(\sigma_2,\sigma_3)+1} & A'_5 & \xrightarrow{1+\mathbf{C}+1} & A_6
\end{array}$$

$A_1 = t(\sigma_1,\sigma_2,\sigma_3)+h(\sigma_1,\sigma_2)+h(\sigma_2\sigma_1,\sigma_3)+g(\sigma_3\sigma_2\sigma_1)$, $A_2 = t(\sigma_1,\sigma_2,\sigma_3)+h(\sigma_1,\sigma_2)+g(\sigma_3)+g(\sigma_2\sigma_1)+h'(\sigma_2\sigma_1,\sigma_3)$, $A_3 = g(\sigma_3)+t(\sigma_1,\sigma_2,\sigma_3)+h(\sigma_1,\sigma_2)+g(\sigma_2\sigma_1)+h'(\sigma_2\sigma_1,\sigma_3)$, $A_4 = g(\sigma_3)+t(\sigma_1,\sigma_2,\sigma_3)+g(\sigma_2)+$

$g(\sigma_1)+h'(\sigma_1,\sigma_2)+h'(\sigma_2\sigma_1,\sigma_3)$, $A_5 = g(\sigma_3)+g(\sigma_2)+g(\sigma_1)+t(\sigma_1,\sigma_2,\sigma_3)+h'(\sigma_1,\sigma_2)+h'(\sigma_2\sigma_1,\sigma_3)$, $A_6 = g(\sigma_3)+g(\sigma_2)+g(\sigma_1)+h'(\sigma_2,\sigma_3)+h'(\sigma_1,\sigma_3\sigma_2)+t'(\sigma_1,\sigma_2,\sigma_3)$, $A'_2 = h(\sigma_2,\sigma_3)+h(\sigma_1,\sigma_3\sigma_2)+t'(\sigma_1,\sigma_2,\sigma_3)+g(\sigma_3\sigma_2\sigma_1)$, $A'_3 = h(\sigma_2,\sigma_3)+h(\sigma_1,\sigma_3\sigma_2)+g(\sigma_3\sigma_2\sigma_1)+t'(\sigma_1,\sigma_3\sigma_2)$, $A'_4 = h(\sigma_2,\sigma_3)+g(\sigma_3\sigma_2)+g(\sigma_1)+h'(\sigma_1,\sigma_3\sigma_2)+t'(\sigma_1,\sigma_2,\sigma_3)$, $A'_5 = g(\sigma_3)+g(\sigma_2)+h'(\sigma_2,\sigma_3)+g(\sigma_1)+h'(\sigma_1,\sigma_3\sigma_2)+t'(\sigma_1,\sigma_2,\sigma_3)$.

⧫ a 3-*cell*



is a relative perturbation, which consists of

- morphisms $F(\sigma): g(\sigma) \to g'(\sigma)$ of $\mathscr{A}$, one for each morphism $c_0 \xrightarrow{\sigma} c_1$ of C,

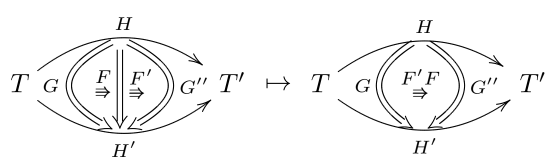
such that

- for any object c of C, $\quad F(1_c) = 1_0$,
- for any two composable morphisms $c_0 \xrightarrow{\sigma_1} c_1 \xrightarrow{\sigma_2} c_2$ in C, the diagram below commutes.

$$\begin{array}{ccc} h(\sigma_1,\sigma_2)+g(\sigma_2\sigma_1) & \xrightarrow{G(\sigma_1,\sigma_2)} & g(\sigma_2)+g(\sigma_1)+H'(\sigma_1,\sigma_2) \\ {\scriptstyle 1+F(\sigma_2\sigma_1)}\downarrow & & \downarrow{\scriptstyle F(\sigma_2)+F(\sigma_1)+1} \\ h(\sigma_1,\sigma_2)+g'(\sigma_2\sigma_1) & \xrightarrow{G'(\sigma_1,\sigma_2)} & g'(\sigma_2)+g'(\sigma_1)+h'(\sigma_1,\sigma_2) \end{array}$$

⧫ compositions and identities in the hom 2-category $\mathbf{Hom}_{\mathcal{Z}^4(C,\mathscr{A})}(T,T')$, are as follows.

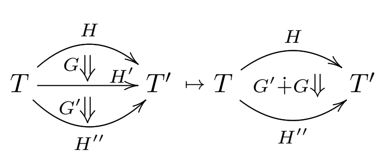
- the horizontal composition of 3-cells



is given by pointwise composition of arrows in $\mathscr{A}$; that is, for any $\sigma: c_0 \to c_1$ in C,

$$(F'F)(\sigma) = F'(\sigma)F(\sigma): g(\sigma) \to g''(\sigma).$$

- the identity 3-cell of a 2-cell $G = (G,g): H \Rightarrow H'$ is the 3-cell denoted by $1_g : G \Rrightarrow G$ which is given by $1_g(\sigma) = 1_{g(\sigma)} : g(\sigma) \to g(\sigma)$, the identity morphism of $g(\sigma)$ in $\mathscr{A}$, for any $\sigma : c_0 \to c_1$ of C.
- the vertical composition of 2-cells

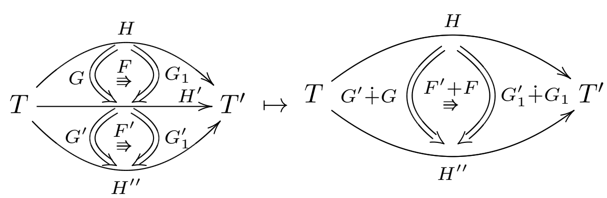


is the 2-cell $G' \dotplus G = (G' \dotplus G, g' + g) : H \Rightarrow H''$ where, for each $c_0 \overset{\sigma_1}{\to} c_1$ of C, $(g'+g)(\sigma) = g'(\sigma) + g(\sigma)$ and, for any $c_0 \overset{\sigma_1}{\to} c_1 \overset{\sigma_2}{\to} c_2$ in C, the morphism $(G' \dotplus G)(\sigma_1, \sigma_2)$ is the dotted one in the commutative diagram

$$\begin{array}{ccc}
h(\sigma_1,\sigma_2) + g'(\sigma_2\sigma_1) + g(\sigma_2\sigma_1) & \overset{(G'\dotplus G)(\sigma_1,\sigma_2)}{\dashrightarrow} & g'(\sigma_2) + g(\sigma_2) + g'(\sigma_1) + g(\sigma_1) + h''(\sigma_1,\sigma_2) \\
{\scriptstyle 1+\mathbf{C}}\downarrow & & \uparrow{\scriptstyle 1+\mathbf{C}+1} \\
h(\sigma_1,\sigma_2) + g(\sigma_2\sigma_1) + g'(\sigma_2\sigma_1) & & g'(\sigma_2) + g(\sigma_2) + g(\sigma_1) + g'(\sigma_1) + h''(\sigma_1,\sigma_2) \\
{\scriptstyle G(\sigma_1,\sigma_2)+1}\downarrow & & \uparrow{\scriptstyle \mathbf{C}+1} \\
g(\sigma_2) + g(\sigma_1) + h'(\sigma_1,\sigma_2) + g'(\sigma_2\sigma_1) & \xrightarrow{1+G'(\sigma_1,\sigma_2)} & g(\sigma_2) + g(\sigma_1) + g'(\sigma_2) + g'(\sigma_1) + h''(\sigma_1,\sigma_2).
\end{array}$$

- the identity 2-cell of a 1-cell $H = (H, h) : T \to T'$ is the 2-cell $(1_h, 0) : H \Rightarrow H$ where $0(\sigma) = 0$, for any $\sigma : c_0 \to c_1$ of C and $1_h(\sigma_1, \sigma_2) = 1_{h(\sigma_1,\sigma_2)}$, for any $c_0 \overset{\sigma_1}{\to} c_1 \overset{\sigma_2}{\to} c_2$ in C.

- the vertical composition of 3-cells



is given by pointwise addition of morphisms in $\mathscr{A}$, that is, for any $\sigma : c_0 \to c_1$ in $\mathscr{A}$,

$$(F' + F)(\sigma) = F'(\sigma) + F(\sigma) : g'(\sigma) + g(\sigma) \longrightarrow g'_1(\sigma) + g_1(\sigma).$$

♦ the composition and the identities in the Gray-category $\mathcal{Z}^4(C, \mathscr{A})$ are as follows.

- the composite of 1-cells $T \overset{H}{\to} T' \overset{H'}{\to} T''$ is the 1-cell $H' \oplus H = (H' \oplus H, h' + h) : T \to T''$, where $(h' + h)(\sigma_1, \sigma_2) = h'(\sigma_1, \sigma_2) + h(\sigma_1, \sigma_2)$, for any $c_0 \overset{\sigma_1}{\to} c_1 \overset{\sigma_2}{\to} c_2$ in C, and, for any $c_0 \overset{\sigma_1}{\to} c_1 \overset{\sigma_2}{\to} c_2 \overset{\sigma_3}{\to} c_3$, $(H' \oplus H)(\sigma_1, \sigma_2, \sigma_3)$ is the dotted morphism in the commutative diagram

$$\begin{array}{ccccc}
A_1 & \xrightarrow{1+\mathbf{C}+1} & A_2 & \xrightarrow{1+\mathbf{C}} & A_3 \\
 & & & & \downarrow{\scriptstyle H(\sigma_1,\sigma_2,\sigma_3)+1} \\
{\scriptstyle (H'\oplus H)(\sigma_1,\sigma_2,\sigma_3)}\vdots & & & & A_4 \\
 & & & & \downarrow{\scriptstyle 1+H'(\sigma_1,\sigma_2,\sigma_3)} \\
A_7 & \xleftarrow{1+\mathbf{C}+1} & A_6 & \xleftarrow{\mathbf{C}+1} & A_5
\end{array}$$

$A_1 = t(\sigma_1,\sigma_2,\sigma_3) + h'(\sigma_1,\sigma_2) + h(\sigma_1,\sigma_2) + h'(\sigma_2\sigma_1,\sigma_3) + h(\sigma_2\sigma_1,\sigma_3)$, $A_2 = t(\sigma_1,\sigma_2,\sigma_3) + h(\sigma_1,\sigma_2) + h'(\sigma_1,\sigma_2) + h'(\sigma_2\sigma_1,\sigma_3) + h(\sigma_2\sigma_1,\sigma_3)$, $A_3 = t(\sigma_1,\sigma_2,\sigma_3) + h(\sigma_1,\sigma_2) + h(\sigma_2\sigma_1,\sigma_3) + h'(\sigma_1,\sigma_2) + h'(\sigma_2\sigma_1,\sigma_3)$, $A_4 = h(\sigma_2,\sigma_3) + h(\sigma_1,\sigma_3\sigma_2) + t'(\sigma_1,\sigma_2,\sigma_3) + h'(\sigma_1,\sigma_2) + h'(\sigma_2\sigma_1,\sigma_3)$, $A_5 = h(\sigma_2,\sigma_3) + h(\sigma_1,\sigma_3\sigma_2) + h'(\sigma_2,\sigma_3) + h'(\sigma_1,\sigma_3\sigma_2) + t''(\sigma_1,\sigma_2,\sigma_3)$, $A_6 = h'(\sigma_2,\sigma_3) + h(\sigma_2,\sigma_3) + h(\sigma_1,\sigma_3\sigma_2) + h'(\sigma_1,\sigma_3\sigma_2) + t''(\sigma_1,\sigma_2,\sigma_3)$, $A_7 = h'(\sigma_2,\sigma_3) + h(\sigma_2,\sigma_3) + h(\sigma_1,\sigma_3\sigma_2) + h'(\sigma_1,\sigma_3\sigma_2) + t''(\sigma_1,\sigma_2,\sigma_3)$.

- the (strict) identity 1-cell of any 4-cocycle $T = (T, t) \in Z^4(C, \mathscr{A})$ is the 1-cell $(1_t, 0) : T \to T$, where $0(\sigma_1, \sigma_2) = 0$, for any $c_0 \overset{\sigma_1}{\to} c_1 \overset{\sigma_2}{\to} c_2$ in C, and $1_t(\sigma_1, \sigma_2, \sigma_3) = 1_{t(\sigma_1,\sigma_2,\sigma_3)}$, for any three composable morphisms $c_0 \overset{\sigma_1}{\to} c_1 \overset{\sigma_2}{\to} c_2 \overset{\sigma_3}{\to} c_3$ of C.

- the composition of 2-cells $T \underset{H_1}{\overset{H}{\Downarrow G}} T' \underset{H'_1}{\overset{H'}{\Downarrow G'}} T'' \;\mapsto\; T \underset{H'_1 \oplus H_1}{\overset{H' \oplus H}{\Downarrow G' \oplus G}} T''$ is the 2-cell

$$G' \oplus G = (G' \oplus G, g' + g) : H' \oplus H \Rightarrow H'_1 \oplus H_1$$

where, for any $c_0 \overset{\sigma}{\to} c_1$ of C, $(g' \oplus g)(\sigma) = g'(\sigma) + g(\sigma)$ and, for any $c_0 \overset{\sigma_1}{\to} c_1 \overset{\sigma_2}{\to} c_2$, the morphism $(G' \oplus G)(\sigma_1, \sigma_2)$ is the dotted one in the commutative diagram

$$\begin{array}{ccccc}
A_1 & \xrightarrow{1+\mathbf{C}+1} & A_2 & \xrightarrow{G'(\sigma_1,\sigma_2)+G(\sigma_1,\sigma_2)} & A_3 \\
{\scriptstyle (G'\oplus G)(\sigma_1,\sigma_2)}\vdots & & & & \downarrow{\scriptstyle 1+\mathbf{C}+\mathbf{C}} \\
A_5 & & \xleftarrow{\quad 1+\mathbf{C}+1 \quad} & & A_4
\end{array}$$

$A_1 = h'(\sigma_1, \sigma_2) + h(\sigma_1, \sigma_2) + g'(\sigma_2\sigma_1) + g(\sigma_2\sigma_1)$,
$A_2 = h'(\sigma_1, \sigma_2) + g'(\sigma_2\sigma_1) + h(\sigma_1, \sigma_2) + g(\sigma_2\sigma_1)$,
$A_3 = g'(\sigma_2) + g'(\sigma_1) + h'_1(\sigma_1, \sigma_2) + g(\sigma_2) + g(\sigma_1) + h_1(\sigma_1, \sigma_2)$,
$A_4 = g'(\sigma_2) + g(\sigma_2) + g(\sigma_1) + g'(\sigma_1) + h'_1(\sigma_1, \sigma_2) + h_1(\sigma_1, \sigma_2)$,
$A_5 = g'(\sigma_2) + g(\sigma_2) + g'(\sigma_1) + g(\sigma_1) + h'_1(\sigma_1, \sigma_2) + h_1(\sigma_1, \sigma_2)$.

- the composition of 3-cells

$$T \underset{H_1}{\overset{H}{G \overset{F}{\Rrightarrow} G_1}} T' \underset{H'_1}{\overset{H'}{G' \overset{F'}{\Rrightarrow} G'_1}} T'' \;\mapsto\; T \underset{H'_1\oplus H_1}{\overset{H'\oplus H}{G'\oplus G \overset{F'+F}{\Rrightarrow} G'_1 \oplus G_1}} T''$$

is the 3-cell $F' + F : G' \oplus G \Rrightarrow G'_1 \oplus G_1$ defined, at any $c_0 \overset{\sigma}{\to} c_1$, by

$$(F' + F)(\sigma) = F'(\sigma) + F(\sigma) : g'(\sigma) + g(\sigma) \longrightarrow g'_1(\sigma) + g_1(\sigma).$$

♦ For $T, T', T'' \in Z^4(C, \mathscr{A})$, the structure constraint of the composition cubical functor

$$\oplus : \mathbf{Hom}_{\mathcal{Z}^4(C,\mathscr{A})}(T, T') \times \mathbf{Hom}_{\mathcal{Z}^4(C,\mathscr{A})}(T', T'') \longrightarrow \mathbf{Hom}_{\mathcal{Z}^4(C,\mathscr{A})}(T, T'')$$

at the cells $T \underset{H_3}{\overset{H_1}{\underset{G_2\Downarrow}{\overset{G_1\Downarrow}{\xrightarrow{H_2}}}}} T' \underset{H'_3}{\overset{H'_1}{\underset{G'_2\Downarrow}{\overset{G'_1\Downarrow}{\xrightarrow{H'_2}}}}} T''$ is the invertible 3-cell

$$(G'_2 \dotplus G'_1) \oplus (G_2 \dotplus G_1) \Rrightarrow (G'_2 \oplus G_2) \dotplus (G'_1 \oplus G_1)$$

defined by the isomorphisms

$$g_2'(\sigma)+g_1'(\sigma)+g_2(\sigma)+g_1(\sigma)\xrightarrow{1+\mathbf{C}+1} g_2'(\sigma)+g_2(\sigma)+g_1'(\sigma)+g_1(\sigma)\qquad (c_0\stackrel{\sigma}{\to}c_1\ \in C).$$

To prove in full details that $\mathscr{Z}^4(C,\mathscr{A})$ is a Gray-category several verifications are necessary. Most of them, however, are immediately apparent from the definitions, whereas all the others are quite straightforward consequences of MacLane's Coherence Theorem on symmetric monoidal categories. This, recall, states that *every diagram in $\mathscr{A}$ whose vertices are permutated instances of $+$ and whose edges are expansions instances of $\mathbf{C}$ commutes.*

Let us stress the identification of 2-categories (actually, of monoidal 2-categories)

$$\mathscr{Z}^3(C,\mathscr{A})=\mathbf{End}_{\mathscr{Z}^4(C,\mathscr{A})}(0),\qquad H\underset{G'}{\overset{G}{\Downarrow_F}}H' \ \equiv\ 0\ \underset{H'}{\overset{H}{G\overset{F}{\Rrightarrow}G'}}\ 0. \tag{4.2}$$

4.1. Unitary lax functors $D\to\mathscr{Z}^n(C,\mathscr{A})$

For later use in the following section, we fix below what we mean by a unitary lax functor from a small category D to the $(n-1)$-category of n-cocycles of a small category C with coefficients in the permutative category $\mathscr{A}$, for $n=2,3,4$.

♦ A unitary lax functor $\mathcal{G}=(\mathcal{G},\mathcal{F}):D\to\mathscr{Z}^2(C,\mathscr{A})$ is simply a functor, so it provides

- a 2-cocycle $\mathcal{G}_d=(\mathcal{G}_d,g_d)$, for each object d of D,
- a morphism $\mathcal{F}_\delta:\mathcal{G}_{d_0}\to\mathcal{G}_{d_1}$, for each morphism $d_0\stackrel{\delta}{\to}d_1$ of D,

satisfying

- for any object d of D, $\quad\mathcal{F}_{1_d}=1_{\mathcal{G}_d}$,
- for any morphisms $d_0\stackrel{\delta_1}{\to}d_1\stackrel{\delta_2}{\to}d_2$ in D, $\quad\mathcal{F}_{\delta_2\delta_1}=\mathcal{F}_{\delta_2}\,\mathcal{F}_{\delta_1}$.

♦ A unitary lax functor $\mathcal{H}=(\mathcal{H},\mathcal{G},\mathcal{F}):D\to\mathscr{Z}^3(C,\mathscr{A})$ consists of

- a 3-cocycle $\mathcal{H}_d=(\mathcal{H}_d,h_d)$, for each object d of D,
- a 1-cell $\mathcal{G}_\delta=(\mathcal{G}_\delta,g_\delta):\mathcal{H}_{d_0}\to\mathcal{H}_{d_1}$, for each morphism $d_0\stackrel{\delta}{\to}d_1$ of D,
- a 2-cell

$$\begin{array}{ccc}\mathcal{H}_{d_0} & \xrightarrow{\ \mathcal{G}_{\delta_2\delta_1}\ } & \mathcal{H}_{d_2}\\ & \Downarrow\mathcal{F}_{\delta_1,\delta_2} & \\ {\scriptstyle\mathcal{G}_{\delta_1}}\searrow & & \nearrow{\scriptstyle\mathcal{G}_{\delta_2}}\\ & \mathcal{H}_{d_1}, & \end{array}$$

for each $d_0\stackrel{\delta_1}{\to}d_1\stackrel{\delta_2}{\to}d_2$ in D, all subject to

- for any object d of D, $\quad \mathcal{G}_{1_d} = 1_{\mathcal{H}_d}$,
- for any morphism $d_0 \stackrel{\delta}{\to} d_1$ of D, $\quad \mathcal{F}_{1,\delta} = 1_{\mathcal{G}_\delta} = \mathcal{F}_{\delta,1}$,
- for any morphisms $d_0 \stackrel{\delta_1}{\to} d_1 \stackrel{\delta_2}{\to} d_2 \stackrel{\delta_3}{\to} d_3$ in D, the following diagram commutes.

$$\begin{array}{ccc} \mathcal{G}_{\delta_3\delta_2\delta_1} & \xRightarrow{\mathcal{F}_{\delta_1,\delta_3\delta_2}} & \mathcal{G}_{\delta_3\delta_2} \dotplus \mathcal{G}_{\delta_1} \\ {\scriptstyle \mathcal{F}_{\delta_2\delta_1,\delta_3}}\Downarrow & & \Downarrow{\scriptstyle \mathcal{F}_{\delta_2,\delta_3}\dotplus 1} \\ \mathcal{G}_{\delta_3} \dotplus \mathcal{G}_{\delta_2\delta_1} & \xRightarrow[1\dotplus\mathcal{F}_{\delta_1,\delta_2}]{} & \mathcal{G}_{\delta_3} \dotplus \mathcal{G}_{\delta_2} \dotplus \mathcal{G}_{\delta_1} \end{array}$$

♦ A unitary lax functor $\mathcal{T} = (\mathcal{T}, \mathcal{H}, \mathcal{G}, \mathcal{F}) : D \to \mathcal{Z}^4(C, \mathscr{A})$ consists of

- a 4-cocycle $\mathcal{T}_d = (\mathcal{T}_d, t_d)$, for each object d of D,
- a 1-cell $\mathcal{H}_\delta = (\mathcal{H}_\delta, h_\delta) : \mathcal{T}_{d_0} \to \mathcal{T}_{d_1}$, for each morphism $d_0 \stackrel{\delta}{\to} d_1$ of D,
- a 2-cell

$$\begin{array}{ccccc} & & \mathcal{T}_{d_1} & & \\ & {\scriptstyle \mathcal{H}_{\delta_1}}\nearrow & \Downarrow{\scriptstyle \mathcal{G}_{\delta_1,\delta_2}} & \searrow{\scriptstyle \mathcal{H}_{\delta_2}} & \\ \mathcal{T}_{d_0} & & \xrightarrow[\mathcal{H}_{\delta_2\delta_1}]{} & & \mathcal{T}_{d_2}, \end{array}$$

  for each $d_0 \stackrel{\delta_1}{\to} d_1 \stackrel{\delta_2}{\to} d_2$ in D,
- a 3-cell

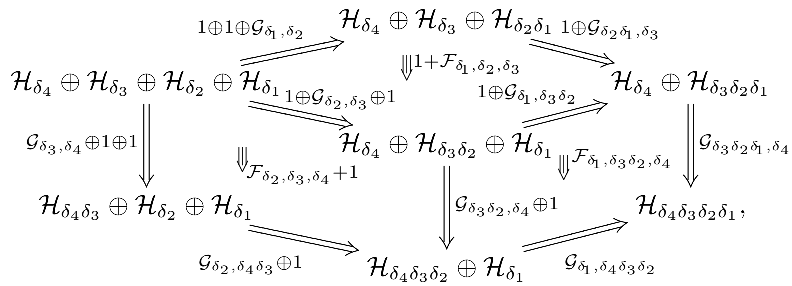
$$\begin{array}{ccc} \mathcal{H}_{\delta_3} \oplus \mathcal{H}_{\delta_2} \oplus \mathcal{H}_{\delta_1} & \xRightarrow{\mathcal{G}_{\delta_2,\delta_3}\oplus 1} & \mathcal{G}_{\delta_3\delta_2} \oplus \mathcal{G}_{\delta_1} \\ {\scriptstyle 1\oplus\mathcal{G}_{\delta_1,\delta_2}}\Downarrow & \stackrel{\mathcal{F}_{\delta_1,\delta_2,\delta_3}}{\Rrightarrow} & \Downarrow{\scriptstyle \mathcal{G}_{\delta_1,\delta_3\delta_2}} \\ \mathcal{H}_{\delta_3} \oplus \mathcal{H}_{\delta_2\delta_1} & \xRightarrow[\mathcal{G}_{\delta_2\delta_1,\delta_3}]{} & \mathcal{H}_{\delta_3\delta_2\delta_1}, \end{array}$$

  for each $d_0 \stackrel{\delta_1}{\to} d_1 \stackrel{\delta_2}{\to} d_2 \stackrel{\delta_3}{\to} d_3$ in D,

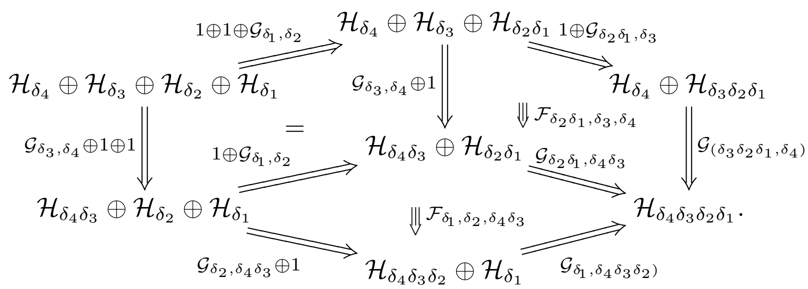
all subject to

- for any object d of D, $\quad \mathcal{H}_{1_d} = 1_{\mathcal{T}_d}$,
- for any morphism $d_0 \stackrel{\delta}{\to} d_1$ of D, $\quad \mathcal{G}_{1,\delta} = 1_{\mathcal{H}_\delta} = \mathcal{G}_{\delta,1}$,
- for any morphisms $d_0 \stackrel{\delta_1}{\to} d_1 \stackrel{\delta_2}{\to} d_2$ in D, $\quad \mathcal{F}_{(1,\delta_1,\delta_2)} = 1_{\mathcal{G}_{\delta_1,\delta_2}} = \mathcal{F}_{\delta_1,1,\delta_2} = \mathcal{F}_{\delta_1,\delta_2,1}$,
- for any morphisms $d_0 \stackrel{\delta_1}{\to} d_1 \stackrel{\delta_2}{\to} d_2 \stackrel{\delta_3}{\to} d_3 \stackrel{\delta_4}{\to} d_4$ in D, the equation $A = B$ on pasted 3-cells in $\mathcal{Z}^4(C, \mathscr{A})$ holds, where

$A =$



$B =$



5. Proof of Theorem 3.2

The strategy of the proof is as follows. For each $n = 2, 3, 4$, we first construct a simplicial $(n-1)$-category $\mathcal{K}(\mathscr{A}, n)$, whose simplicial set of objects is $K(\mathscr{A}, n)$. These $\mathcal{K}(\mathscr{A}, n)$ determine reduced simplicial spaces $B\mathcal{K}(\mathscr{A}, n)$. In Proposition 5.2 we show a group completion map $|K(\mathscr{A}, 1)| \to \Omega|B\mathcal{K}(\mathscr{A}, 2)|$, and homotopy equivalences $|K(\mathscr{A}, 2)| \simeq \Omega|B\mathcal{K}(\mathscr{A}, 3)|$ and $|K(\mathscr{A}, 3)| \simeq \Omega|B\mathcal{K}(\mathscr{A}, 4)|$. Next, in Proposition 5.3 we prove that the inclusion maps $K(\mathscr{A}, n) \hookrightarrow \mathcal{K}(\mathscr{A}, n)$ induce homotopy equivalences $|K(\mathscr{A}, n)| \simeq |B\mathcal{K}(\mathscr{A}, n)|$. Hence, the claimed group completion map and homotopy equivalences in Theorem 3.2 follow, since they are, respectively, the compositions

$$|K(\mathscr{A}, 1)| \to \Omega|B\mathcal{K}(\mathscr{A}, 2)| \simeq \Omega|K(\mathscr{A}, 2)|,$$

$$|K(\mathscr{A}, 2)| \simeq \Omega|B\mathcal{K}(\mathscr{A}, 3)| \simeq \Omega|K(\mathscr{A}, 3)|,$$

$$|K(\mathscr{A}, 3)| \simeq \Omega|B\mathcal{K}(\mathscr{A}, 4)| \simeq \Omega|K(\mathscr{A}, 4)|.$$

In the proof, we use that n-categories, for $n \leq 3$, are closely related to spaces by means of a well-understood classifying space construction on them. By [3, Theorem 6.1] and [5, Theorem 5.4], the classifying space $B\mathbf{X}$ of an n-category $\mathbf{X}$ can be realized as $B\mathbf{X} = |\Delta\mathbf{X}|$, the geometric realization of its geometric nerve $\Delta\mathbf{X}$, that is, the simplicial set

$$\Delta\mathbf{X} : \Delta^{op} \to \mathbf{Set}, \qquad [p] \mapsto {}_{\mathrm{ul}}\mathrm{Func}([p], \mathbf{X}),$$

whose p-simplices are the unitary lax functors from $[p]$ to $\mathbf{X}$. For instance, $\Delta\Sigma^n\mathscr{A} = K(\mathscr{A}, n+1)$ and

$$B\Sigma^n\mathscr{A} = |K(\mathscr{A}, n+1)|. \tag{5.1}$$

The assignments $C \mapsto \mathcal{Z}^n(C, \mathscr{A})$, for $1 \leq n \leq 4$, are strictly functorial on the small category C (in the sense of Gurski, see [12, Remark 4.15], when $n = 4$). Then, we have a simplicial $(n-1)$-categories

$$\mathcal{K}(\mathscr{A}, n) : \Delta^{op} \to (n-1)\text{-}\mathbf{Cat}, \qquad [q] \mapsto \mathcal{Z}^n([q], \mathscr{A}),$$

and replacing each $\mathcal{Z}^n([q], \mathscr{A})$ by its classifying space, we obtain simplicial spaces

$$B\mathcal{K}(\mathscr{A}, n) : \Delta^{op} \to \mathbf{Top}, \qquad [q] \mapsto B\mathcal{Z}^n([q], \mathscr{A}) = \big|[p] \mapsto {}_{\mathrm{ul}}\mathrm{Func}\big([p], \mathcal{Z}^n([q], \mathscr{A})\big)\big|, \tag{5.2}$$

one for each $n = 1, 2, 3, 4$, whose geometric realization [22] is denoted by $|B\mathcal{K}(\mathscr{A}, n)|$; that is,

$$|B\mathcal{K}(\mathscr{A}, n)| = \big|[q] \mapsto \big|[p] \mapsto {}_{\mathrm{ul}}\mathrm{Func}\big([p], \mathcal{Z}^n([q], \mathscr{A})\big|\big|.$$

For $n = 1$, we have $\mathcal{K}(\mathscr{A}, 1) = K(\mathscr{A}, 1)$. Hence, $B\mathcal{K}(\mathscr{A}, 1) = K(\mathscr{A}, 1)$ and

$$|B\mathcal{K}(\mathscr{A}, 1)| = |K(\mathscr{A}, 1)| = B\mathscr{A}.$$

Next, we analyze these simplicial spaces $B\mathcal{K}(\mathscr{A}, n)$ for $n = 2, 3, 4$.

♦ For $n = 4$ and at degree 0, we have the classifying space of $\mathcal{Z}^4([0], \mathscr{A})$, which is the final 3-category; that is, it has only one object, the zero 4-cocycle 0, only one 1-cell, the identity $1_0 : 0 \to 0$, only one 2-cell, the identity $1_{1_0} : 1_0 \Rightarrow 1_0$, and only one 3-cell, the identity $1_{1_{1_0}} : 1_{1_0} \Rrightarrow 1_{1_0}$. Hence, its geometric nerve $\Delta\mathcal{Z}^4([0], \mathscr{A}) \cong \Delta[0]$ is the simplicial set with only one simplex in every dimension; therefore, its geometric realization

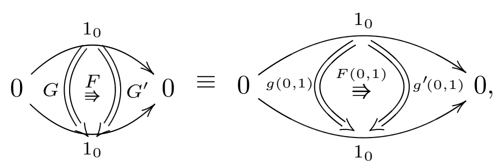
$$B\mathcal{Z}^4([0], \mathscr{A}) = \mathrm{pt} \tag{5.3}$$

is the one point space. Since $\mathcal{Z}^3([0], \mathscr{A}) = \mathbf{End}_{\mathcal{Z}^4([0],\mathscr{A})}(0)$ and, $\mathcal{Z}^2([0], \mathscr{A}) = \mathbf{End}_{\mathcal{Z}^3([0],\mathscr{A})}(0)$, by (4.2) and (4.1), we see that $\mathcal{Z}^3([0], \mathscr{A})$ is the final 2-category and $\mathcal{Z}^2([0], \mathscr{A})$ is the final category. Hence, both $\Delta\mathcal{Z}^3([0], \mathscr{A})$ and $\Delta\mathcal{Z}^2([0], \mathscr{A})$ are the final simplicial set and we also have

$$B\mathcal{Z}^3([0], \mathscr{A}) = \mathrm{pt}, \qquad B\mathcal{Z}^2([0], \mathscr{A}) = \mathrm{pt}. \tag{5.4}$$

♦ For $n = 4$ and at degree 1, we have the classifying space of the 3-category $\mathcal{Z}^4([1], \mathscr{A})$. This has only one object, the zero 4-cocycle 0, and only one 1-cell, the identity $1_0 : 0 \to 0$. Furthermore, a 2-cell $G = (G, g) : 1_0 \Rightarrow 1_0$ in $\mathcal{Z}^4([1], \mathscr{A})$ is entirely equivalent to giving the object $g(0,1)$ of $\mathscr{A}$, and giving a 3-cell $F : G \Rrightarrow G'$ is equivalent to giving the morphism $F(0,1) : g(0,1) \to g'(0,1)$ of $\mathscr{A}$. This way, we have the identification of 3-categories

$$\mathcal{Z}^4([1], \mathscr{A}) = \Sigma^2\mathscr{A},$$



whence, by (5.1),

$$B\mathcal{Z}^4([1], \mathscr{A}) = B\Sigma^2\mathscr{A} = |K(\mathscr{A}, 3)|. \tag{5.5}$$

Furthermore, since

$$\mathcal{Z}^3([1],\mathscr{A}) = \mathbf{End}_{\mathcal{Z}^4([1],\mathscr{A})}(0) = \mathbf{End}_{\Sigma^2\mathscr{A}}(0) = \Sigma\mathscr{A},$$

$$\mathcal{Z}^2([1],\mathscr{A}) = \mathbf{End}_{\mathcal{Z}^3([1],\mathscr{A})}(0) = \mathbf{End}_{\Sigma\mathscr{A}}(0) = \mathscr{A},$$

we also have

$$B\mathcal{Z}^3([1],\mathscr{A}) = B\Sigma\mathscr{A} = |K(\mathscr{A},2)|, \tag{5.6}$$

$$B\mathcal{Z}^2([1],\mathscr{A}) = B\mathscr{A} = |K(\mathscr{A},1)|. \tag{5.7}$$

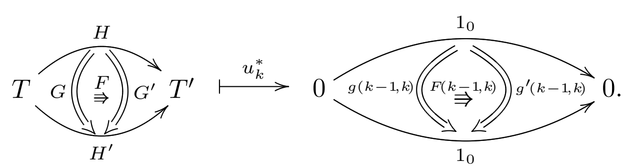
♦ For $n=4$ and at any degree $q\geq 2$, we have the strict functor

$$\mathcal{U} = (u_1^*,\dots,u_q^*) : \mathcal{Z}^4([q],\mathscr{A}) \longrightarrow \mathcal{Z}^4([1],\mathscr{A})^q = \Sigma^2\mathscr{A}^q \tag{5.8}$$

where, for each $1\leq k\leq q$,

$$u_k^* : \mathcal{Z}^4([q],\mathscr{A}) \to \mathcal{Z}^4([1],\mathscr{A}) = \Sigma^2\mathscr{A}$$

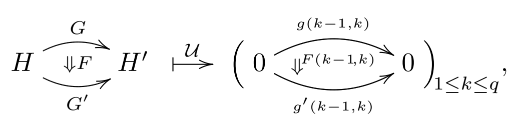
is the the strict functor induced by the map $u_k:[1]\to[q]$ in Δ given by $u_k(0)=k-1$ and $u_k(1)=k$; that is, each u_k^* acts on cells of $\mathcal{Z}^4([q],\mathscr{A})$ by



The functor $\mathcal{U}$ restricts to the corresponding 2-categories of endomorphisms of the zero 4-cocycles, $\mathbf{End}_{\mathcal{Z}^4([q],\mathscr{A})}(0)$ and $\mathbf{End}_{\mathcal{Z}^4([1],\mathscr{A})}(0)^q$, to the strict functor

$$\mathcal{U} = (u_1^*,\dots,u_q^*) : \mathcal{Z}^3([q],\mathscr{A}) \longrightarrow \mathcal{Z}^3([1],\mathscr{A})^q = \Sigma\mathscr{A}^q \tag{5.9}$$

which acts on cells of $\mathcal{Z}^3([q],\mathscr{A})$ by



and likewise this $\mathcal{U}$ restricts to the corresponding categories of endomorphisms of the zero 3-cocycles, $\mathbf{End}_{\mathcal{Z}^3([q],\mathscr{A})}(0)$ and $\mathbf{End}_{\mathcal{Z}^3([1],\mathscr{A})}(0)^q$, to the functor

$$\mathcal{U} = (u_1^*,\dots,u_q^*) : \mathcal{Z}^2([q],\mathscr{A}) \longrightarrow \mathcal{Z}^2([1],\mathscr{A})^q = \mathscr{A}^q, \tag{5.10}$$

which acts by $G \xrightarrow{F} G' \ \overset{\mathcal{U}}{\longmapsto}\ \Big(\, g(k-1,k) \xrightarrow{F(k-1,k)} g'(k-1,k)\,\Big)_{1\leq k\leq q}$.

Lemma 5.1 *For every $q \geq 2$, the functors $\mathcal{U}$ in (5.8), (5.9), and (5.10) induce respective homotopy equivalences on classifying spaces*

$$B\mathcal{Z}^n([q],\mathscr{A}) \simeq |K(\mathscr{A},n-1)|^q \qquad (n=2,3,4).$$

Proof We first prove the case when $n=4$. To do that, we show below a normal lax functor (actually, a pseudo-functor)

$$\mathcal{V}:\Sigma^2\mathscr{A}^q \to \mathcal{Z}^4([q],\mathscr{A}), \tag{5.11}$$

such that $\mathcal{U}\mathcal{V} = id_{\Sigma^2\mathscr{A}^q}$, together with a lax transformation

$$\Theta : id_{\mathcal{Z}^4([q],\mathscr{A})} \Rightarrow \mathcal{V}\mathcal{U}. \tag{5.12}$$

Then, since $B\mathcal{U}\,B\mathcal{V} = id_{|K(\mathscr{A},3)|^q}$ and Θ induces a homotopy equivalence $id_{B\mathcal{Z}^4([q],\mathscr{A})} \simeq B\mathcal{V}\,B\mathcal{U}$, as in [5, Proposition 5.6], it follows that the induced map $B\mathcal{U}: B\mathcal{Z}^4([q],\mathscr{A}) \to |K(\mathscr{A},3)|^q$ is actually a homotopy equivalence.

⧫ The lax functor $\mathcal{V}$ in (5.11) is as follows. On the unique 0- and 1-cells of $\Sigma^2\mathscr{A}^q$, $\mathcal{V}$ acts by $\mathcal{V}(0,\dots,0)=0$ and $\mathcal{V}(1_0,\dots,1_0)=1_0$. If $\mathbf{a}=(a_1,\dots,a_q):(1_0,\dots,1_0)\Rightarrow(1_0,\dots,1_0)$ is a 2-cell in $\Sigma^2\mathscr{A}^q$, then $\mathcal{V}(\mathbf{a}) = (\mathcal{V}_{\mathbf{a}}, g_{\mathcal{V}\mathbf{a}}):1_0\Rightarrow 1_0$ is the 2-cell of $\mathcal{Z}^4(C,\mathscr{A})$ defined by

$$g_{\mathcal{V}\mathbf{a}}(i_0,i_1) = \begin{cases} 0 & if \quad i_0 = i_1, \\ a_{i_1}+\cdots+a_{i_0+1} & if \quad i_0 < i_1, \end{cases}$$

for each $0\leq i_0\leq i_1\leq q$, and, for any $0\leq i_0\leq i_1\leq i_2\leq q$,

$$\mathcal{V}_{\mathbf{a}}(i_0,i_1,i_2) = 1_{g_{\mathcal{V}\mathbf{a}}(i_0,i_2)} : g_{\mathcal{V}\mathbf{a}}(i_0,i_2) \longrightarrow g_{\mathcal{V}\mathbf{a}}(i_1,i_2)+g_{\mathcal{V}\mathbf{a}}(i_0,i_1)$$

is the identity morphism. If $\mathbf{u}=(u_1,\dots,u_q):\mathbf{a}\Rrightarrow\mathbf{a}'$ is a 3-cell in $\Sigma^2\mathscr{A}^q$, then $\mathcal{V}(\mathbf{u}):\mathcal{V}(\mathbf{a})\Rrightarrow\mathcal{V}(\mathbf{a}')$ is defined by

$$\mathcal{V}(\mathbf{u})(i_0,i_1) = \begin{cases} 1_0 & \text{if} \quad i_0 = i_1, \\ u_{i_1}+\cdots+u_{i_0+1} & \text{if} \quad i_0 < i_1. \end{cases}$$

For any other 3-cell $\mathbf{u}':\mathbf{a}'\Rrightarrow\mathbf{a}''$ in $\Sigma^2\mathscr{A}^q$, the equality $\mathcal{V}(\mathbf{u}')\,\mathcal{V}(\mathbf{u}) = \mathcal{V}(\mathbf{u}'\mathbf{u})$ is clear. Furthermore, for any two 2-cells $\mathbf{a}$ and $\mathbf{b}$ of $\Sigma^2\mathscr{A}^q$, the structure constraint of $\mathcal{V}$, for both compositions $\dotplus$ and $\oplus$ on 2-cells in $\mathcal{Z}^4(C,\mathscr{A})$,

$$\chi : \mathcal{V}\mathbf{a}\oplus\mathcal{V}\mathbf{b} = \mathcal{V}\mathbf{a}\dotplus\mathcal{V}\mathbf{b} \xrightarrow{\ \cong\ } \mathcal{V}(\mathbf{a}+\mathbf{b})\ ,$$

is given by the isomorphisms $\chi(i_0,i_1):(g_{\mathcal{V}\mathbf{a}}+g_{\mathcal{V}\mathbf{b}})(i_0,i_1)\to g_{\mathcal{V}(\mathbf{a}+\mathbf{b})}(i_0,i_1)$, recursively defined, for any integers $0\leq i_0\leq i_1\leq q$, by $\chi(i_0,i_1)=1_0$ if $i_1=i_0$, and, for $i_0<i_1$, by the commutative diagrams

$$\begin{array}{ccc} a_{i_1}+g_{\mathcal{V}\mathbf{a}}(i_0,i_1-1)+b_{i_1}+g_{\mathcal{V}\mathbf{b}}(i_0,i_1-1) & \xrightarrow{\ \chi(i_0,i_1)\ } & a_{i_1}+b_{i_1}+g_{\mathcal{V}(\mathbf{a}+\mathbf{b})}(i_0,i_1-1) \\ {\scriptstyle 1+\mathbf{C}+1}\searrow & & \nearrow{\scriptstyle 1+\chi(i_0,i_1-1)} \\ & a_{i_1}+b_{i_1}+g_{\mathcal{V}\mathbf{a}}(i_0,i_1-1)+g_{\mathcal{V}\mathbf{b}}(i_0,i_1-1) & \end{array}$$

The structure constraints Ω, δ, and $\mathcal{G}amma$ as in [11, Definition 3.2] for $\mathcal{V}$ are identities.

♦ The equality $\mathcal{UV} = id_{\Sigma^2\mathscr{A}^q}$ is clear.

♦ The lax transformation $\Theta : id_{\mathcal{Z}^4([q],\mathscr{A})} \Rightarrow \mathcal{VU}$ in (5.12) is as follows.

• Its component at a $T = (T,t) \in Z^4([q],\mathscr{A})$ is $\Theta_T = (\Theta_T, \theta_T) : T \to \mathcal{VU}(T) = 0$, where the objects $\theta_T(i_0,i_1,i_2)$ of $\mathscr{A}$, for $0 \leq i_0 \leq i_1 \leq i_2 \leq q$, are recursively defined by

$$\theta_T(i_0,i_1,i_2) = \begin{cases} 0 & if \quad i_1 = i_2, \\ t(i_0,i_1,i_2-1,i_2) + \theta_T(i_0,i_1,i_2-1) & if \quad i_1 < i_2, \end{cases}$$

and, for $0 \leq i_0 \leq i_1 \leq i_2 \leq q$, the morphisms

$$t(i_0,i_1,i_2,i_3) + \theta_T(i_0,i_1,i_2) + \theta_T(i_0,i_2,i_3) \xrightarrow{\Theta_T(i_0,i_1,i_2,i_3)} \theta_T(i_1,i_2,i_3) + \theta_T(i_0,i_1,i_3) \ ,$$

are also defined recursively by the equalities $\Theta_T(i_0,i_1,i_2,i_3) = 1_{\theta_T(i_0,i_1,i_2)}$ if $i_3 = i_2$, and, for $i_2 < i_3$, by the commutative diagrams below.

$$\begin{array}{ccccc} A_1 & & \xrightarrow{\Theta_T(i_0,i_1,i_2,i_3)} & & A_5 \\ {\scriptstyle 1+\mathbf{C}+1}\downarrow & & & & \uparrow{\scriptstyle 1+\mathbf{C}+1} \\ A_2 & \xrightarrow{T(i_0,i_1,i_2,i_3-1,i_3)+1} & A_3 & \xrightarrow{1+\Theta_T(i_0,i_1,i_2,i_3-1)} & A_4 \end{array}$$

$A_1 = t(i_0,i_1,i_2,i_3) + \theta_T(i_0,i_1,i_2) + t(i_0,i_2,i_3-1,i_3) + \theta_T(i_0,i_2,i_3-1)$, $A_2 = t(i_0,i_1,i_2,i_3) + t(i_0,i_2,i_3-1,i_3) + \theta_T(i_1,i_2,i_3-1) + \theta_T(i_0,i_2,i_3-1)$, $A_3 = t(i_1,i_2,i_3-1,i_3) + t(i_0,i_1,i_3-1,i_3) + t(i_0,i_1,i_2,i_3-1) + \theta_T(i_0,i_1,i_2) + \theta_T(i_0,i_2,i_3-1)$, $A_4 = t(i_1,i_2,i_3-1,i_3) + t(i_0,i_1,i_3-1,i_3) + \theta_T(i_1,i_2,i_3-1) + \theta_T(i_0,i_1,i_3-1)$, $A_5 = t(i_1,i_2,i_3-1,i_3) + \theta_T(i_1,i_2,i_3-1) + t(i_0,i_1,i_3-1,i_3) + \theta_T(i_0,i_1,i_3-1)$.

• The naturality component of Θ at a 1-cell $H = (H,h) : T \to T'$ of $\mathcal{Z}^4([q],\mathscr{A})$

$$\begin{array}{ccc} T & \xrightarrow{H} & T' \\ {\scriptstyle \Theta_T}\downarrow & \overset{\Theta_H}{\Rightarrow} & \downarrow{\scriptstyle \Theta_{T'}} \\ 0 & \xrightarrow[\mathcal{VU}(H)=1_0]{} & 0, \end{array}$$

is the 2-cell given by the objects $\theta_H(i_0,i_1)$ of $\mathscr{A}$, for $0 \leq i_0 \leq i_1 \leq q$, recursively defined by

$$\theta_H(i_0,i_1) = \begin{cases} 0 & if \quad i_0 = i_1, \\ h(i_0,i_1-1,i_1) + \theta_H(i_0,i_1-1) & if \quad i_0 < i_1, \end{cases}$$

and the morphisms

$$\theta_T(i_0,i_1,i_2) + \theta_H(i_0,i_2) \xrightarrow{\Theta_H(i_0,i_1,i_2)} \theta_H(i_1,i_2) + \theta_H(i_0,i_1) + \theta_{T'}(i_0,i_1,i_2) + h(i_0,i_1,i_2) \ ,$$

for $0 \leq i_0 \leq i_1 \leq i_2 \leq q$, are also defined recursively by $\Theta_H(i_0,i_1,i_2) = 1_{\theta_H(i_0,i_1)}$ if $i_2 = i_1$, and, for $i_2 > i_1$ by the commutative diagrams below.

$$\begin{array}{ccccccc} A_1 & \xrightarrow{1+\mathbf{C}+1} & A_2 & \xrightarrow{1+\Theta_H(i_0,i_1,i_2-1)} & A_3 & \xrightarrow{1+\mathbf{C}} & A_4 \\ {\scriptstyle \Theta_H(i_0,i_1,i_2)}\downarrow & & & & & & \downarrow{\scriptstyle H(i_0,i_1,i_2-1,i_2)+1} \\ A_7 & & \xleftarrow{1+\mathbf{C}} & A_6 & & \xleftarrow{1+\mathbf{C}+1} & A_5 \end{array}$$

$A_1 = t(i_0,i_1,i_2-1,i_2)+\theta_T(i_0,i_1,i_2-1)+h(i_0,i_2-1,i_2)+\theta_H(i_0,i_2-1)$, $A_2 = t(i_0,i_1,i_2-1,i_2)+h(i_0,i_2-1,i_2)+\theta_T(i_0,i_1,i_2-1)+\theta_H(i_0,i_2-1)$, $A_3 = t(i_0,i_1,i_2-1,i_2)+h(i_0,i_2-1,i_2)+\theta_H(i_1,i_2-1)+\theta_H(i_0,i_1)+\theta_{T'}(i_0,i_1,i_2-1)+h(i_0,i_1,i_2-1)$, $A_4 = t(i_0,i_1,i_2-1,i_2)+h(i_0,i_1,i_2-1)+h(i_0,i_2-1,i_2)+\theta_H(i_1,i_2-1)+\theta_H(i_0,i_1)+\theta_{T'}(i_0,i_1,i_2-1)$, $A_5 = h(i_1,i_2-1,i_2)+h(i_0,i_1,i_2)+t'(i_0,i_1,i_2-1,i_2)+\theta_H(i_1,i_2-1)+\theta_H(i_0,i_1)+\theta_{T'}(i_0,i_1,i_2-1)$, $A_6 = h(i_1,i_2-1,i_2)+h(i_0,i_1,i_2)+\theta_H(i_1,i_2-1)+\theta_H(i_0,i_1)+t'(i_0,i_1,i_2-1,i_2)+\theta_{T'}(i_0,i_1,i_2-1)$, $A_7 = h(i_1,i_2-1,i_2)+\theta_H(i_1,i_2-1)+\theta_H(i_0,i_1)+t'(i_0,i_1,i_2-1,i_2)+\theta_{T'}(i_0,i_1,i_2-1)+h(i_0,i_1,i_2)$.

- The structure invertible 3-cell of Θ at a 2-cell $T \overset{H}{\underset{H'}{\Downarrow G}} T'$

$$\begin{array}{ccc}
\Theta_T = 1_0 \oplus \Theta_T & \xRightarrow{\mathcal{VU}(G)\oplus 1} & 1_0 \oplus \Theta_T = \Theta_T \\
{\scriptstyle \Theta_H}\Big\Downarrow & \overset{\Theta_G}{\Rrightarrow} & \Big\Downarrow{\scriptstyle \Theta_{H'}} \\
\Theta_{T'} \oplus H & \xRightarrow[1\oplus G]{} & \Theta_{T'} \oplus H',
\end{array}$$

is given by the isomorphisms

$$\Theta_G(i_0,i_1) : g(i_0,i_1)+\theta_H(i_0,i_1) \to \theta_{H'}(i_0,i_1)+g_{\mathcal{VU}(G)}(i_0,i_1),$$

recursively defined, for $0 \le i_0 \le i_1 \le q$, by $\Theta_G(i_0,i_1) = 1_0$ if $i_0 = i_1$, and, for $i_0 < i_1$, by the commutative diagrams below, where we use that $\mathcal{U}(G) = (g(0,1),\dots,g(q-1,q))$, so that $g_{\mathcal{VU}(G)}(i_0,i_1) = g(i_1-1,i_1)+g_{\mathcal{VU}(G)}(i_0,i_1-1)$.

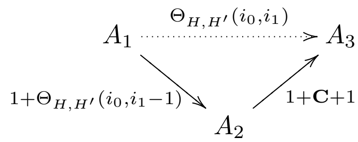
$$\begin{array}{ccccc}
A_1 & \xrightarrow{\mathbf{C}+1} & A_2 & \xrightarrow{G(i_0,i_1-1,i_1)+1} & A_3 \\
{\scriptstyle \Theta_G(i_0,i_1)}\vdots & & & & \big\downarrow{\scriptstyle \mathbf{C}+1} \\
A_6 & \xleftarrow{1+\mathbf{C}+1} & A_5 & \xleftarrow{1+\Theta_G(i_0.i_1-1)} & A_4
\end{array}$$

$A_1 = g(i_0,i_1)+h(i_0,i_1-1,i_1)+\theta_H(i_0,i_1-1)$,
$A_2 = h(i_0,i_1-1,i_1)+g(i_0,i_1)+\theta_H(i_0,i_1-1)$,
$A_3 = g(i_1-1,i_1)+g(i_0,i_1-1)+h'(i_0,i_1-1,i_1)+\theta_H(i_0,i_1-1)$,
$A_4 = h'(i_0,i_1-1,i_1)+g(i_1-1,i_1)+g(i_0,i_1-1)+\theta_H(i_0,i_1-1)$,
$A_5 = h'(i_0,i_1-1,i_1)+g(i_1-1,i_1)+\theta_{H'}(i_0,i_1-1)+g_{\mathcal{VU}(G)}(i_0,i_1-1)$,
$A_6 = h'(i_0,i_1-1,i_1)+\theta_{H'}(i_0,i_1-1)+g(i_1-1,i_1)+g_{\mathcal{VU}(G)}(i_0,i_1-1)$,

- At any two composable 1-cells $T \xrightarrow{H} T' \xrightarrow{H'} T''$ in $\mathcal{Z}^4([q],\mathscr{A})$, the structure invertible 3-cell

$$\begin{array}{ccc}
\Theta_T & \xRightarrow{\Theta_H} & \Theta_{T'} \oplus H \\
 & {\scriptstyle \Theta_{H'\oplus H}} \searrow\!\!\searrow \quad \overset{\Theta_{H,H'}}{\Rrightarrow} & \Big\Downarrow{\scriptstyle \Theta_{H'}\oplus 1_H} \\
 & & \Theta_{T''} \oplus H' \oplus H
\end{array}$$

is given is given by the isomorphisms $\Theta_{H,H'}(i_0,i_1) : \theta_{H'\oplus H}(i_0,i_1) \to \theta_{H'}(i_0,i_1)+\theta_H(i_0,i_1)$, for $0 \le i_0 \le i_1 \le q$, recursively defined by $\Theta_{H,H'}(i_0,i_1)=1_0$ if $i_0=i_1$ and, for $i_0<i_1$, by the commutative diagrams below.



$$A_1 = h'(i_0,i_1-1,i_1)+h(i_0,i_1-1,i_1)+\theta_{H'\oplus H}(i_0,i_1-1)\,,$$
$$A_2 = h'(i_0,i_1-1,i_1)+h(i_0,i_1-1,i_1)+\theta_{H'}(i_0,i_1-1)+\theta_H(i_0,i_1-1)\,,$$
$$A_3 = h'(i_0,i_1-1,i_1)+\theta_{H'}(i_0,i_1-1)+h(i_0,i_1-1,i_1)+\theta_H(i_0,i_1-1)\,.$$

We now prove the homotopy equivalence when $n=3$, that is, the strict functor $\mathcal{U}:\mathcal{Z}^3([q],\mathscr{A})\to\Sigma\mathscr{A}^q$ in (5.9) induces a homotopy equivalence on classifying spaces: The lax functor $\mathcal{V}:\Sigma^2\mathscr{A}^q\to\mathcal{Z}^4([q],\mathscr{A})$ in (5.11), restricts to the corresponding 2-categories of endomorphisms of the zero 4-cocycles, $\mathbf{End}_{\Sigma^2\mathscr{A}}(0)^q$ and $\mathbf{End}_{\mathcal{Z}^4([q],\mathscr{A})}(0)$. Hence, it defines a lax functor

$$\mathcal{V}:\Sigma\mathscr{A}^q\to\mathcal{Z}^3([q],\mathscr{A}). \tag{5.13}$$

Since $\mathcal{U}\mathcal{V}=id_{\Sigma\mathscr{A}^q}$, we have $B\mathcal{U}\,B\mathcal{V}=id_{|K(\mathscr{A},2)|^q}$. On the other hand, the lax transformation Θ in (5.12) verifies that $\Theta_0=1_0$, so that its restriction to the 2-categories of endomorphisms of the zero 4-cocycles gives a lax transformation

$$\Theta:\mathcal{V}\mathcal{U}\Rightarrow id_{\mathcal{Z}^3([q],\mathscr{A})}. \tag{5.14}$$

This, by [3, Proposition 7.1], induces a homotopy equivalence $B\mathcal{V}\,B\mathcal{U}\simeq id_{B\mathcal{Z}^3([q],\mathscr{A})}$, and the result follows.

The proof that the restricted functor $\mathcal{U}:\mathcal{Z}^2([q],\mathscr{A})\to\mathscr{A}^q$ induces a homotopy equivalence on classifying spaces is entirely similar: The homomorphism $\mathcal{V}$ in (5.13) restricts to the corresponding 2-categories of endomorphisms of the zero 3-cocycles, $\mathbf{End}_{\Sigma\mathscr{A}}(0)^q$ and $\mathbf{End}_{\mathcal{Z}^3([q],\mathscr{A})}(0)$, by giving a functor $\mathcal{V}:\mathscr{A}^q\to\mathcal{Z}^2([q],\mathscr{A})$. From the equality $\mathcal{U}\mathcal{V}=id_{\mathscr{A}^q}$, it follows that $B\mathcal{U}\,B\mathcal{V}=id_{|K(\mathscr{A},1)|^q}$. Furthermore, the lax transformation Θ in (5.14) restricts to the categories of endomorphisms of the zero 3-cocycles and gives a transformation $\Theta:\mathcal{V}\mathcal{U}\Rightarrow id_{\mathcal{Z}^2([q],\mathscr{A})}$ which, by [22, Proposition 2.1], induces a homotopy equivalence $B\mathcal{V}\,B\mathcal{U}\simeq id_{B\mathcal{Z}^2([q],\mathscr{A})}$. □

Proposition 5.2 *There is a natural group completion map*

$$|K(\mathscr{A},1)|\to\Omega|B\mathcal{K}(\mathscr{A},2)|$$

and natural homotopy equivalences

$$|K(\mathscr{A},2)|\simeq\Omega|B\mathcal{K}(\mathscr{A},3)|,\quad |K(\mathscr{A},3)|\simeq\Omega|B\mathcal{K}(\mathscr{A},4)|.$$

Proof The simplicial spaces $B\mathcal{K}(\mathscr{A},n):[q]\mapsto B\mathcal{Z}^n([q],\mathscr{A})$, satisfy

- $B\mathcal{Z}^n([0],\mathscr{A})$ is the one-point space, see (5.3), (5.4),
- $B\mathcal{Z}^n([1],\mathscr{A})=|K(\mathscr{A},n-1)|$, see (5.5), (5.6), (5.7),

- for any $q \geq 2$, the Segal projection maps $B\mathcal{U} : B\mathcal{Z}^n([q], \mathscr{A}) \to |K(\mathscr{A}, n-1)|^q$ are homotopy equivalences, by Lemma 5.1,

- $\pi_0|K(\mathscr{A}, 3)| = 0 = \pi_0|K(\mathscr{A}, 2)|$, since $K(\mathscr{A}, 3)$ and $K(\mathscr{A}, 2)$ have only one vertex.

Hence, by Segal's criterium [23, Proposition 1,5], the induced maps $|K(\mathscr{A}, 2)| \to \Omega|B\mathcal{K}(\mathscr{A}, 3)|$ and $|K(\mathscr{A}, 3)| \to \Omega|B\mathcal{K}(\mathscr{A}, 4)|$ are both homotopy equivalences, while the induced map $|K(\mathscr{A}, 1)| \to \Omega|B\mathcal{K}(\mathscr{A}, 2)|$ is a group completion map by MacDuff-Segal [20, Proposition 1]. □

Since $K(\mathscr{A}, n)$ is the simplicial set of objects of $\mathcal{K}(\mathscr{A}, n)$, we have an evident simplicial functor of inclusion

$$\iota : K(\mathscr{A}, n) \hookrightarrow \mathcal{K}(\mathscr{A}, n),$$

where $K(\mathscr{A}, n)$ is regarded as a simplicial discrete (i.e. with only identities) $(n-1)$-category.

Proposition 5.3 *The induced maps*

$$\iota_* : |K(\mathscr{A}, n)| \simeq |B\mathcal{K}(\mathscr{A}, n)|$$

are homotopy equivalences.

Proof This is clear for $n = 1$. For $n = 2, 3, 4$, by [21, Lemma in page 86], there is a natural homeomorphism $|B\mathcal{K}(\mathscr{A}, n)| \cong |\mathrm{diag}\overline{\Delta\mathcal{K}}(\mathscr{A}, n)|$ between the geometric realization of the simplicial space $B\mathcal{K}(\mathscr{A}, n)$, see (5.2), and the geometric realization of the simplicial set diagonal of the bisimplicial set

$$\overline{\Delta\mathcal{K}}(\mathscr{A}, n) : \Delta^{op} \times \Delta^{op} \to \mathbf{Set}, \quad ([p], [q]) \mapsto {}_{\mathrm{ul}}\mathrm{Func}\big([p], \mathcal{Z}^n([q], \mathscr{A})\big).$$

In this bisimplicial set we shall interpret the p- (resp. q-) direction as the horizontal (resp. vertical) one, so that the horizontal face and degeneracy operators in $\overline{\Delta\mathcal{K}}(\mathscr{A}, n)$

$${}_{\mathrm{ul}}\mathrm{Func}\big([p+1], \mathcal{Z}^n([q], \mathscr{A})\big) \xleftarrow{s^h_m} {}_{\mathrm{ul}}\mathrm{Func}\big([p], \mathcal{Z}^n([q], \mathscr{A})\big) \xrightarrow{d^h_m} {}_{\mathrm{ul}}\mathrm{Func}\big([p-1], \mathcal{Z}^n([q], \mathscr{A})\big)$$

are the induced by the codegeneracy and coface maps $s^m : [p+1] \to [p]$ and $d^m : [p-1] \to [p]$ of the simplicial category, while the vertical face and degeneracy operators in $\overline{\Delta\mathcal{K}}(\mathscr{A}, n)$

$${}_{\mathrm{ul}}\mathrm{Func}\big([p], \mathcal{Z}^n([q+1], \mathscr{A})\big) \xleftarrow{s^v_\ell} {}_{\mathrm{ul}}\mathrm{Func}\big([p], \mathcal{Z}^n([q], \mathscr{A})\big) \xrightarrow{d^v_\ell} {}_{\mathrm{ul}}\mathrm{Func}\big([p], \mathcal{Z}^n([q-1], \mathscr{A})\big)$$

are induced by codegeneracy and coface maps $s^\ell : [q+1] \to [q]$ and $d^\ell : [q-1] \to [q]$, respectively. For each $p \geq 0$, let

$${}_{\mathrm{ul}}\mathrm{Func}\big([p], \mathcal{K}(\mathscr{A}, n)\big) : \Delta^{op} \to \mathbf{Set}, \quad [q] \mapsto {}_{\mathrm{ul}}\mathrm{Func}\big([p], \mathcal{Z}^n([q], \mathscr{A})\big)$$

denote the vertical simplicial set of $\overline{\Delta\mathcal{K}}(\mathscr{A}, n)$ at degree p.

Note that when one restricts the above constructions to $K(\mathscr{A}, n) \subseteq \mathcal{K}(\mathscr{A}, n)$, the resulting bisimplicial subset $\overline{\Delta K}(\mathscr{A}, n) \subseteq \overline{\Delta\mathcal{K}}(\mathscr{A}, n)$ is just the bisimplicial set, constant in the horizonal direction, defined by the simplicial set $K(\mathscr{A},n)=\mathrm{ulFunc}\big([0],\mathcal{K}(\mathscr{A},n)\big)$. Thus, the induced bisimplicial inclusion $\overline{\Delta K}(\mathscr{A},n)\hookrightarrow\overline{\Delta\mathcal{K}}(\mathscr{A},n)$ is, at each horizontal degree $p\geq 0$, the composite

$$(s_0^h)^p : K(\mathscr{A},n)=\mathrm{ulFunc}\big([0],\mathcal{K}(\mathscr{A},n)\big)\longrightarrow \mathrm{ulFunc}\big([p],\mathcal{K}(\mathscr{A},n)\big) \tag{5.15}$$

of the degeneracy simplicial maps $s_0^h:\mathrm{ulFunc}\big([r-1],\mathcal{K}(\mathscr{A},n)\big)\to\mathrm{ulFunc}\big([r],\mathcal{K}(\mathscr{A},n)\big)$. Since every pointwise weak homotopy equivalence between bisimplicial sets is a diagonal weak homotopy equivalence [10, IV, Proposition 1.7], to prove that the induced map on diagonals

$$K(\mathscr{A},n)=\mathrm{diag}\overline{\Delta K}(\mathscr{A},n)\to\mathrm{diag}\overline{\Delta\mathcal{K}}(\mathscr{A},4)$$

is a weak homotopy equivalence, it suffices to prove that every one of these simplicial maps (5.15) is a weak homotopy equivalence. We now distinguish the cases where n is 2, 3, or 4.

The case $n=2$. Here, we have established the theorem if we prove that every horizontal degeneracy map

$$s_0^h:\mathrm{ulFunc}\big([p-1],\mathcal{K}(\mathscr{A},2)\big)\hookrightarrow\mathrm{ulFunc}\big([p],\mathcal{K}(\mathscr{A},2)\big)\qquad (p\geq 1)$$

is a simplicial homotopy equivalence, with d_0^h as a simplicial homotopy inverse. Since $d_0^hs_0^h=id$, it is enough to exhibit a simplicial homotopy $\Phi: id\Rightarrow s_0^hd_0^h$. Such a homotopy is given by the maps

$$\mathrm{ulFunc}\big([p],\mathcal{Z}^2([q],\mathscr{A})\big)\ \underset{\Phi_q}{\overset{\Phi_0}{\underset{\cdots}{\rightrightarrows}}}\ \mathrm{ulFunc}\big([p],\mathcal{Z}^2([q+1],\mathscr{A})\big)\qquad (q\geq 0)$$

which carry a unitary lax functor $\mathcal{G}=(\mathcal{G},\mathcal{F}):[p]\to\mathcal{Z}^2([q],\mathscr{A})$ (see Subsection 4.1 for the notation) to the unitary lax functors

$$\Phi_m(\mathcal{G})=(\mathcal{G}^m,\mathcal{F}^m):[p]\longrightarrow\mathcal{Z}^2([q+1],\mathscr{A})\qquad (0\leq m\leq q)$$

defined as follows:

⧫ For each integer $0\leq j\leq p$, the 2-cocycle $\mathcal{G}_j^m=(\mathcal{G}_j^m,g_j^m)\in Z^2([q+1],\mathscr{A})$ consists of the objects $g_j^m(i_0,i_1)$ of $\mathscr{A}$, for $0\leq i_0\leq i_1\leq q{+}1$, defined to be

- $g_js^m(i_0,i_1)$, if $j\geq 1$,
- $g_1s^m(i_0,i_1)$, if $j=0$, $i_1\leq m$,,
- $g_0s^m(i_0,i_1)$, if $j=0$, $m<i_1$,

where $s^m:[q{+}1]\to[q]$ is the m-th codegeneracy map, and, for $0\leq i_0\leq i_1\leq i_2\leq q{+}1$, of the morphisms $\mathcal{G}_j^m(i_0,i_1,i_2):g_j^m(i_0,i_1)\to g_j^m(i_1,i_2)+g_j^m(i_0,i_1)$ defined as

- $\mathcal{G}_js^m(i_0,i_1,i_2)$, if $j\geq 1$,
- $\mathcal{G}_1s^m(i_0,i_1,i_2)$, if $j=0$, $i_2\leq m$,
- the dotted arrow in the commutative diagram below, if $j=0$ and $i_1\leq m<i_2$,

$$g_0s^m(i_0,i_2)\xrightarrow{\ \mathcal{G}_0^m(i_0,i_1,i_2)\ } g_0s^m(i_1,i_2)+g_1s^m(i_0,i_1)$$

$$g_0s^m(i_0,i_2)\xrightarrow{\ \mathcal{G}_0s^m(i_0,i_1,i_2)\ } g_0s^m(i_1,i_2)+g_0s^m(i_0,i_1)\xrightarrow{\ 1+\mathcal{F}_{0,1}s^m(i_0,i_1)\ } g_0s^m(i_1,i_2)+g_1s^m(i_0,i_1)$$

- $\mathcal{G}_0 s^m(i_0, i_1, i_2)$, if $j = 0$, $m < i_1$.

♦ For each $0 \leq j_0 \leq j_1 \leq p$, the morphism $\mathcal{F}^m_{j_0,j_1} : \mathcal{G}^m_{j_0} \to \mathcal{G}^m_{j_1}$ in $\mathcal{Z}^2([q{+}1], \mathscr{A})$ consists of the morphisms $\mathcal{F}^m_{j_0,j_1}(i_0, i_1) : g^m_{j_0}(i_0, i_1) \to g^m_{j_1}(i_0, i_1)$, for $0 \leq i_0 \leq i_1 \leq q + 1$, which are defined to be

- $\mathcal{F}_{j_0,j_1} s^m(i_0, i_1)$, if $j_0 \geq 1$,
- $\mathcal{F}_{1,j_1} s^m(i_0, i_1)$, if $j_0 = 0$, $i_1 \leq m$,
- $\mathcal{F}_{0,j_1} s^m(i_0, i_1)$, if $j_0 = 0$, $m < i_1$.

<u>The case $n = 3$</u>. The simplicial map $(s^h_0)^p$ in (5.15) is the same as the composite

$$s^h_0 s^h_1 \cdots s^h_{p-1} : K(\mathscr{A}, 3) = {}_{\mathrm{ul}}\mathrm{Func}\big([0], \mathcal{K}(\mathscr{A}, 3)\big) \longrightarrow {}_{\mathrm{ul}}\mathrm{Func}\big([p], \mathcal{K}(\mathscr{A}, 3)\big).$$

of the horizontal degeneracy simplicial maps $s^h_{r-1} : {}_{\mathrm{ul}}\mathrm{Func}\big([r-1], \mathcal{K}(\mathscr{A}, n)\big) \to {}_{\mathrm{ul}}\mathrm{Func}\big([r], \mathcal{K}(\mathscr{A}, n)\big)$. Then, in this case, we prove that every simplicial map

$$s^h_{p-1} : {}_{\mathrm{ul}}\mathrm{Func}\big([p-1], \mathcal{K}(\mathscr{A}, 3)\big) \hookrightarrow {}_{\mathrm{ul}}\mathrm{Func}\big([p], \mathcal{K}(\mathscr{A}, 3)\big) \qquad (p \geq 1)$$

is a simplicial homotopy equivalence, with d^h_p as a simplicial homotopy inverse. It suffices to show a simplicial homotopy $\Phi : id \Rightarrow s^h_{p-1} d^h_p$, which is given by the maps

$$ {}_{\mathrm{ul}}\mathrm{Func}\big([p], \mathcal{Z}^3([q], \mathscr{A})\big) \overset{\Phi_0}{\underset{\Phi_q}{\overset{\cdots}{\rightrightarrows}}} {}_{\mathrm{ul}}\mathrm{Func}\big([p], \mathcal{Z}^3([q+1], \mathscr{A})\big) \qquad (q \geq 0)$$

which carry a unitary lax functor $\mathcal{H} = (\mathcal{H}, \mathcal{G}, \mathcal{F}) : [p] \to \mathcal{Z}^3([q], \mathscr{A})$ to the unitary lax functors

$$\Phi_m(\mathcal{H}) = (\mathcal{H}^m, \mathcal{G}^m, \mathcal{F}^m) : [p] \longrightarrow \mathcal{Z}^3([q+1], \mathscr{A}) \qquad (0 \leq m \leq q)$$

defined as follows:

♦ For each $0 \leq j \leq p$, $\mathcal{H}^m_j = (\mathcal{H}^m_j, h^m_j) \in \mathcal{Z}^3([q+1], \mathscr{A})$ consists of the objects $h^m_j(i_0, i_1, i_2)$ of $\mathscr{A}$, for $0 \leq i_0 \leq i_1 \leq i_2 \leq q{+}1$, defined to be

- $h_j s^m(i_0, i_1, i_2)$, if $j < p$,
- $h_{p-1} s^m(i_0, i_1, i_2)$, if $j = p$, $i_2 \leq m$,,
- $g_{p-1,p} s^m(i_0, i_1) + h_p s^m(i_0, i_1, i_2)$, if $j = p$, $i_1 \leq m < i_2$,
- $h_p s^m(i_0, i_1, i_2)$, if $j = p$, $m < i_1$,

and, for $0 \leq i_0 \leq i_1 \leq i_2 \leq i_3 \leq q{+}1$, of the morphisms

$$\mathcal{H}^m_j(i_0, i_1, i_2, i_3) : h^m_j(i_0, i_1, i_2) + h^m_j(i_0, i_2, i_3) \to h^m_j(i_1, i_2, i_3) + h^m_j(i_0, i_1, i_3)$$

which are defined to be

- $\mathcal{H}_j s^m(i_0, i_1, i_2, i_3)$, if $j < p$,
- $\mathcal{H}_{p-1} s^m(i_0, i_1, i_2, i_3)$, if $j = p$, $i_3 \leq m$,
- the dotted arrow in the commutative diagram below, if $j = p$, $i_2 \leq m < i_3$,

$$\begin{array}{ccc} A_1 & \xrightarrow{\mathcal{H}^m_p(i_0,i_1,i_2,i_3)} & A_4 \\ {\scriptstyle \mathcal{G}_{p-1,p} s^m(i_0,i_1,i_2)+1}\downarrow & & \uparrow{\scriptstyle 1+\mathbf{C}+1} \\ A_2 & \xrightarrow[1+\mathcal{H}_p(i_0,i_1,i_2,i_3)]{} & A_3 \end{array}$$

$A_1 = h_{p-1}s^m(i_0,i_1,i_2) + g_{p-1,p}s^m(i_0,i_2) + h_p s^m(i_0,i_2,i_3)\,,$
$A_2 = g_{p-1,p}s^m(i_1,i_2) + g_{p-1,p}s^m(i_0,i_1) + h_p s^m(i_0,i_1,i_2) + h_p s^m(i_0,i_2,i_3)\,,$
$A_3 = g_{p-1,p}s^m(i_1,i_2) + g_{p-1,p}s^m(i_0,i_1) + h_p s^m(i_1,i_2,i_3) + h_p s^m(i_0,i_1,i_3)\,,$
$A_4 = g_{p-1,p}s^m(i_1,i_2) + h_p s^m(i_1,i_2,i_3) + g_{p-1,p}s^m(i_0,i_1) + h_p s^m(i_0,i_1,i_3)\,.$

- the dotted arrow in the commutative diagram below, if $j=p$, $i_1 \leq m < i_2$,

$$\begin{array}{ccc} g_{p-1,p}s^m(i_0,i_1)+h_p s^m(i_0,i_1,i_2) & \xrightarrow{\mathcal{H}^m_p(i_0,i_1,i_2,i_3)} & h_p s^m(i_1,i_2,i_3)+g_{p-1,p}s^m(i_0,i_1)+h_p s^m(i_0,i_1,i_3) \\ {\scriptstyle 1+\mathcal{H}_p s^m(i_0,i_1,i_2,i_3)} \searrow & & \nearrow {\scriptstyle \mathbf{C}+1} \\ & g_{p-1,p}s^m(i_0,i_1) + h_p s^m(i_1,i_2,i_3) + h_p s^m(i_0,i_1,i_3) & \end{array}$$

- $\mathcal{H}_p s^m(i_0,i_1,i_2,i_3)$, if $j=p$, $m<i_1$.

♦ For each integers $0 \leq j_0 \leq j_1 \leq p$, the 1-cell $\mathcal{G}^m_{j_0,j_1} = (\mathcal{G}^m_{j_0,j_1}, g^m_{j_0,j_1}) : \mathcal{H}^m_{j_0} \to \mathcal{H}^m_{j_1}$ consists of the objects $g^m_{j_0,j_1}(i_0,i_1)$ of $\mathscr{A}$, for $0 \leq i_0 \leq i_1 \leq q+1$, defined to be

- $g_{j_0,j_1}s^m(i_0,i_1)$, if $j_1 < p$,
- $g_{j_0,p-1}s^m(i_0,i_1)$, if $j_1 = p$, $i_1 \leq m$,
- $g_{j_0,p}s^m(i_0,i_1)$, if $j_1 = p$, $m < i_1$,

and the morphisms, for $0 \leq i_0 \leq i_1 \leq i_2 \leq q+1$,

$$h^m_{j_0}(i_0,i_1,i_2) + g^m_{j_0,j_1}(i_0,i_2) \xrightarrow{\mathcal{G}^m_{j_0,j_1}(i_0,i_1,i_2)} g^m_{j_0,j_1}(i_1,i_2) + g^m_{j_0,j_1}(i_0,i_1) + h^m_{j_1}(i_0,i_1,i_2),$$

defined to be

- $\mathcal{G}_{j_0,j_1}s^m(i_0,i_1,i_2)$, if $j_1 < p$,
- $\mathcal{G}_{j_0,p-1}s^m(i_0,i_1,i_2)$, if $j_1 = p$, $i_2 \leq m$,
- the dotted arrow in the commutative diagram below, if $j_p = 0$, $i_1 \leq m < i_2$,

$$\begin{array}{ccc} A_1 & \xrightarrow{\mathcal{G}^m_{j_0,j_1}(i_0,i_1,i_2)} & A_4 \\ {\scriptstyle \mathcal{G}_{j_0,p}s^m(i_0,i_1,i_2)} \downarrow & & \uparrow {\scriptstyle 1+\mathbf{C}+1} \\ A_2 & \xrightarrow{1+\mathcal{F}_{j_0,p-1,p}(i_0,i_1,i_2)+1} & A_3 \end{array}$$

$A_1 = h_{j_0}s^m(i_0,i_1,i_2) + g_{j_0,p}s^m(i_0,i_2)$, $A_2 = g_{j_0,p}s^m(i_1,i_2) + g_{j_0,p}s^m(i_0,i_1) + h_p s^m(i_0,i_1,i_2)$, $A_3 = g_{j_0,p}s^m(i_1,i_2) + g_{p-1,p}s^m(i_0,i_1) + g_{j_0,p-1}s^m(i_0,i_1) + h_p s^m(i_0,i_1,i_2)$, $A_4 = g_{j_0,p}s^m(i_1,i_2) + g_{j_0,p-1}s^m(i_0,i_1) + g_{p-1,p}s^m(i_0,i_1) + h_p s^m(i_0,i_1,i_2)$.

- $\mathcal{G}_{j_0,p}s^m(i_0,i_1,i_2)$, if $j_1 = p$, $m < i_1$.

♦ For each integers $0 \leq j_0 \leq j_1 \leq j_2 \leq p$, the 2-cell

$$\begin{array}{ccc} \mathcal{H}^m_{j_0} & \xrightarrow{\mathcal{G}^m_{j_0,j_2}} & \mathcal{H}^m_{j_2} \\ & \Downarrow \mathcal{F}^m_{j_0,j_1,j_2} & \\ {\scriptstyle \mathcal{G}^m_{j_0,j_1}} \searrow & & \nearrow {\scriptstyle \mathcal{G}^m_{j_1,j_2}} \\ & \mathcal{H}^m_{j_1} & \end{array}$$

consists of the morphisms $\mathcal{F}^m_{j_0,j_1,j_2}(i_0,i_1) : g^m_{j_0,j_2}(i_0,i_1) \to g^m_{j_1,j_2}(i_0,i_1) + g^m_{j_0,j_1}(i_0,i_1)$, $0 \leq i_0 \leq i_1 \leq q+1$, defined to be

- $\mathcal{F}_{j_0,j_1,j_2} s^m(i_0,i_1)$, if $j_2 < p$,
- $\mathcal{F}_{j_0,j_1,p-1} s^m(i_0,i_1)$, if $j_2 = p$, $i_1 \leq m$,
- $\mathcal{F}_{j_0,j_1,p} s^m(i_0,i_1)$, if $j_2 = p$, $m < i_1$.

The case $n=4$. As in the case $n=2$, we prove here that that every horizontal degeneracy map

$$s^h_0 : \mathrm{ulFunc}\big([p-1],\mathcal{K}(\mathscr{A},4)\big) \hookrightarrow \mathrm{ulFunc}\big([p],\mathcal{K}(\mathscr{A},4)\big) \qquad (p \geq 1)$$

is a simplicial homotopy equivalence, with d^h_0 as a simplicial homotopy inverse, by exhibiting the homotopy $\Phi : id \Rightarrow s^h_0 d^h_0$ given by the maps

$$\mathrm{ulFunc}\big([p],\mathcal{Z}^4([q],\mathscr{A})\big) \underset{\Phi_q}{\overset{\Phi_0}{\underset{\cdots}{\rightrightarrows}}} \mathrm{ulFunc}\big([p],\mathcal{Z}^4([q+1],\mathscr{A})\big) \qquad (q \geq 0)$$

which carry a unitary lax functor $\mathcal{T} = (\mathcal{T},\mathcal{H},\mathcal{G},\mathcal{F}) : [p] \to \mathcal{Z}^4([q],\mathscr{A})$ to the unitary lax functors

$$\Phi_m(\mathcal{T}) = (\mathcal{T}^m,\mathcal{H}^m,\mathcal{G}^m,\mathcal{F}^m) : [p] \longrightarrow \mathcal{Z}^4([q+1],\mathscr{A}) \qquad (0 \leq m \leq q)$$

defined as follows:

⧫ For each integer $0 \leq j \leq p$, the 4-cocycle $\mathcal{T}^m_j = (\mathcal{T}^m_j, t^m_j) \in Z^4([q+1],\mathscr{A})$ consists of the objects $t^m_j(i_0,i_1,i_2,i_3)$ of $\mathscr{A}$, for $0 \leq i_0 \leq i_1 \leq i_2 \leq i_3 \leq q+1$, defined to be

- $t_j s^m(i_0,i_1,i_2,i_3)$, if $j \geq 1$,
- $t_1 s^m(i_0,i_1,i_2,i_3)$, if $j = 0$, $i_3 \leq m$,,
- $t_0 s^m(i_0,i_1,i_2,i_3) + h_{0,1} s^m(i_0,i_1,i_2)$, if $j=0$, $i_2 \leq m < i_3$,
- $t_0 s^m(i_0,i_1,i_2,i_3)$, if $j = 0$, $m < i_2$,

and, for $0 \leq i_0 \leq i_1 \leq i_2 \leq i_3 \leq i_4 \leq q+1$, of the morphisms of $\mathscr{A}$

$$t^m_j(i_0,i_1,i_2,i_4) + t^m_j(i_0,i_2,i_3,i_4) \xrightarrow{\mathcal{T}^m_j(i_0,i_1,i_2,i_3,i_4)} t^m_j(i_1,i_2,i_3,i_4) + t^m_j(i_0,i_1,i_3,i_4) + t^m_j(i_0,i_1,i_2,i_3)$$

defined as

- $\mathcal{T}_j s^m(i_0,i_1,i_2,i_3,i_4)$, if $j \geq 1$,
- $\mathcal{T}_1 s^m(i_0,i_1,i_2,i_3,i_4)$, if $j = 0$, $i_4 \leq m$,
- the dotted arrow in the commutative diagram below, if $j = 0$ and $i_3 \leq m < i_4$,

$$\begin{array}{ccccc}
A_1 & & \xrightarrow{\ \mathcal{T}^m_0(i_0,i_1,i_2,i_3,i_4)\ } & & A_5 \\
{\scriptstyle 1+\mathbf{C}+1}\downarrow & & & & \uparrow{\scriptstyle 1+\mathbf{C}+1} \\
A_2 & \xrightarrow{\mathcal{T}_0 s^m(i_0,i_1,i_2,i_3,i_4)+1} & A_3 & \xrightarrow{1+\mathcal{H}_{0,1}s^m(i_0,i_1,i_2,i_3)} & A_4
\end{array}$$

$A_1 = t_0 s^m(i_0,i_1,i_2,i_4) + h_{0,1} s^m(i_0,i_1,i_2) + t_0 s^m(i_0,i_2,i_3,i_4) + h_{0,1} s^m(i_0,i_2,i_3)$,
$A_2 = t_0 s^m(i_0,i_1,i_2,i_4) + t_0 s^m(i_0,i_2,i_3,i_4) + h_{0,1} s^m(i_0,i_1,i_2) + h_{0,1} s^m(i_0,i_2,i_3)$,

$A_3 = t_0s^m(i_0,i_1,i_2,i_4) + t_0s^m(i_0,i_1,i_3,i_4) + t_0s^m(i_0,i_1,i_2,i_3) + h_{0,1}s^m(i_0,i_1,i_2) + h_{0,1}s^m(i_0,i_2,i_3)$,
$A_4 = t_0s^m(i_0,i_1,i_2,i_4) + t_0s^m(i_0,i_1,i_3,i_4) + h_{0,1}s^m(i_1,i_2,i_3) + h_{0,1}s^m(i_0,i_1,i_3) + t_1s^m(i_0,i_1,i_2,i_3)$,
$A_5 = t_0s^m(i_1,i_2,i_3,i_4) + h_{0,1}s^m(i_1,i_2,i_3) + t_0s^m(i_0,i_1,i_3,i_4) + h_{0,1}s^m(i_0,i_1,i_3) + t_1s^m(i_0,i_1,i_2,i_3)$.

- the dotted arrow in the commutative diagram below, if $j=0$, $i_2 \le m < i_3$,

$$\begin{array}{ccccc} A_1 & & \xrightarrow{\mathcal{T}_0^m(i_0,i_1,i_2,i_3,i_4)} & & A_3 \\ & \searrow^{1+\mathbf{C}} & & \nearrow_{1+\mathcal{T}_0s^m(i_0,i_1,i_2,i_3,i_4)} & \\ & & A_2 & & \end{array}$$

$A_1 = t_0s^m(i_0,i_1,i_2,i_4) + h_{0,1}s^m(i_0,i_1,i_2) + t_0s^m(i_0,i_2,i_3,i_4)$,
$A_2 = t_0s^m(i_0,i_1,i_2,i_4) + t_0s^m(i_0,i_2,i_3,i_4) + h_{0,1}s^m(i_0,i_1,i_2)$,
$A_3 = t_0s^m(i_1,i_2,i_3,i_4) + t_0s^m(i_0,i_1,i_3,i_4) + h_{0,1}s^m(i_0,i_1,i_2)$.

- $\mathcal{T}_0s^m(i_0,i_1,i_2,i_3,i_4)$, if $j=0$, $m<i_2$.

♦ For each $0\le j_0\le j_1\le p$, the 1-cell $\mathcal{H}^m_{j_0,j_1}=(\mathcal{H}^m_{j_0,j_1},h^m_{j_0,j_1}):\mathcal{T}^m_{j_0}\to\mathcal{T}^m_{j_1}$ in $\mathcal{Z}^4([q+1],\mathscr{A})$ consists of the objects $h^m_{j_0,j_1}(i_0,i_1,i_2)$ of $\mathscr{A}$, for $0\le i_0\le i_1\le i_2\le q{+}1$, defined to be

- $h_{j_0,j_1}s^m(i_0,i_1,i_2)$, if $j_0\ge 1$,
- $h_{1,j_1}s^m(i_0,i_1,i_2)$, if $j_0=0$, $i_2\le m$,,
- $h_{0,j_1}s^m(i_0,i_1,i_2)+g_{0,1,j_1}s^m(i_0,i_1)$, if $j_0=0$, $i_1\le m<i_2$,
- $h_{0,j_1}s^m(i_0,i_1,i_2)$, if $j_0=0$, $m<i_1$,

and, for $0\le i_0\le i_1\le i_2\le i_3\le q{+}1$, of the morphisms

$$\begin{array}{c} t^m_{j_0}(i_0,i_1,i_2,i_3)+h^m_{j_0,j_1}(i_0,i_1,i_2) \\ +h^m_{j_0,j_1}(i_0,i_2,i_3) \end{array} \xrightarrow{\mathcal{H}^m_{j_0,j_1}(i_0,i_1,i_2,i_3)} \begin{array}{c} h^m_{j_0,j_1}(i_1,i_2,i_3)+h^m_{j_0,j_1}(i_0,i_1,i_3) \\ +t^m_{j_1}(i_0,i_1,i_2,i_3) \end{array}$$

which are defined to be

- $\mathcal{H}_{j_0,j_1}s^m(i_0,i_1,i_2,i_3)$, if $j_0\ge 1$,
- $\mathcal{H}_{1,j_1}s^m(i_0,i_1,i_2,i_3)$, if $j_0=0$, $i_3\le m$,
- the dotted arrow in the commutative diagram below, if $j_0=0$, $i_2\le m<i_3$,

$$\begin{array}{ccccccc} A_1 & \xrightarrow{1+\mathbf{C}+\mathbf{C}} & A_2 & \xrightarrow{1+\mathcal{G}_{0,1,j_1}s^m(i_0,i_1,i_2)+1} & A_3 & \xrightarrow{\mathbf{C}+1} & A_4 \\ {\scriptstyle \mathcal{H}^m_{0,j_1}(i_0,i_1,i_2,i_3)}\downarrow & & & & & & \downarrow{\scriptstyle 1+\mathcal{H}_{0,j_1}s^m(i_0,i_1,i_2,i_3)} \\ A_7 & & \xleftarrow{1+\mathbf{C}+1} & A_6 & \xleftarrow{\mathbf{C}+1} & & A_5 \end{array}$$

$A_1 = t_0s^m(i_0,i_1,i_2,i_3) + h_{0,1}s^m(i_0,i_1,i_2) + h_{1,j_1}s^m(i_0,i_1,i_2) + h_{0,j_1}s^m(i_0,i_2,i_3) + g_{0,1,j_1}s^m(i_0,i_2)$, $A_2 = t_0s^m(i_0,i_1,i_2,i_3)+h_{1,j_1}s^m(i_0,i_1,i_2)+h_{0,1}s^m(i_0,i_1,i_2)+g_{0,1,j_1}s^m(i_0,i_2)+h_{0,j_1}s^m(i_0,i_2,i_3)$, $A_3 = t_0s^m(i_0,i_1,i_2,i_3)+g_{0,1,j_1}s^m(i_1,i_2)+g_{0,1,j_1}s^m(i_0,i_1)+h_{0,j_1}s^m(i_0,i_1,i_2)+h_{0,j_1}s^m(i_0,i_2,i_3)$, $A_4 = g_{0,1,j_1}s^m(i_1,i_2)+g_{0,1,j_1}s^m(i_0,i_1)+t_0s^m(i_0,i_1,i_2,i_3)+h_{0,j_1}s^m(i_0,i_1,i_2)+h_{0,j_1}s^m(i_0,i_2,i_3)$, $A_5 = g_{0,1,j_1}s^m(i_1,i_2)+g_{0,1,j_1}s^m(i_0,i_1)+h_{0,j_1}s^m(i_1,i_2,i_3)+h_{0,j_1}s^m(i_0,i_1,i_3)+t_{j_1}s^m(i_0,i_1,i_2,i_3)$, $A_6 = h_{0,j_1}s^m(i_1,i_2,i_3)+g_{0,1,j_1}s^m(i_1,i_2)+g_{0,1,j_1}s^m(i_0,i_1)+h_{0,j_1}s^m(i_0,i_1,i_3)+t_{j_1}s^m(i_0,i_1,i_2,i_3)$, $A_7 = h_{0,j_1}s^m(i_1,i_2,i_3)+g_{0,1,j_1}s^m(i_1,i_2)+h_{0,j_1}s^m(i_0,i_1,i_3)+g_{0,1,j_1}s^m(i_0,i_1)+t_{j_1}s^m(i_0,i_1,i_2),i_3)$.

- the dotted arrow in the commutative diagram below, if $j_0 = 0,\ i_1 \leq m < i_2$,

$$\begin{array}{ccc} A_1 & \xrightarrow{\mathcal{H}^m_{0,j_1}(i_0,i_1,i_2,i_3)} & A_4 \\ {\scriptstyle 1+\mathbf{C}}\downarrow & & \uparrow{\scriptstyle \mathbf{C}+1} \\ A_2 & \xrightarrow{\mathcal{H}_{0,j_1}s^m(i_0,i_1,i_2,i_3)+1} & A_3 \end{array}$$

$A_1 = t_0 s^m(i_0,i_1,i_2,i_3) + h_{0,j_1}s^m(i_0,i_1,i_2) + h_{0,j_1}s^m(i_0,i_1) + h_{0,j_1}s^m(i_0,i_2,i_3)$, $A_2 = t_0 s^m(i_0,i_1,i_2,i_4) + h_{0,j_1}s^m(i_0,i_1,i_2) + h_{0,j_1}s^m(i_0,i_2,i_3) + h_{0,j_1}s^m(i_0,i_1)$, $A_3 = h_{0,j_1}s^m(i_1,i_2,i_3) + h_{0,j_1}s^m(i_0,i_1,i_3) + t_{j_1}s^m(i_0,i_1,i_2,i_3) + h_{0,j_1}s^m(i_0,i_1)$, $A_4 = h_{0,j_1}s^m(i_1,i_2,i_3) + h_{0,j_1}s^m(i_0,i_1,i_3) + h_{0,j_1}s^m(i_0,i_1) + t_{j_1}s^m(i_0,i_1,i_2,i_3)$.

- $\mathcal{H}_{0,j_1}s^m(i_0,i_1,i_2,i_3)$, if $j_0 = 0,\ m < i_1$.

♦ For each integers $0 \leq j_0 \leq j_1 \leq j_2 \leq p$, the 2-cell $\mathcal{G}^m_{j_0,j_1,j_2} = (\mathcal{G}^m_{j_0,j_1,j_2}, g^m_{j_0,j_1,j_2})$,

$$\begin{array}{ccc} & t^m_{j_1} & \\ {\scriptstyle \mathcal{H}^m_{j_0,j_1}}\nearrow & \Downarrow \mathcal{G}^m_{j_0,j_1,j_2} & \searrow{\scriptstyle \mathcal{H}^m_{j_1,j_2}} \\ t^m_{j_0} & \xrightarrow[\mathcal{H}^m_{j_0,j_2}]{} & t^m_{j_2}, \end{array}$$

consists of the objects $g^m_{j_0,j_1,j_2}(i_0,i_1)$ of $\mathscr{A}$, for $0 \leq i_0 \leq i_1 \leq q+1$, defined to be

- $g_{j_0,j_1,j_2}s^m(i_0,i_1)$, if $j_0 \geq 1$,
- $g_{1,j_1,j_2}s^m(i_0,i_1)$, if $j_0 = 0,\ i_1 \leq m$,
- $g_{0,j_1,j_2}s^m(i_0,i_1)$, if $j_0 = 0,\ m < i_1$,

and of the morphisms, for $0 \leq i_0 \leq i_1 \leq i_2 \leq q+1$,

$$\begin{array}{c} h^m_{j_1,j_2}(i_0,i_1,i_2) + h^m_{j_0,j_1}(i_0,i_1,i_2) \\ +g^m_{j_0,j_1,j_2}(i_0,i_2) \end{array} \xrightarrow{\mathcal{G}^m_{j_0,j_1,j_2}(i_0,i_1,i_2)} \begin{array}{c} g^m_{j_0,j_1,j_2}(i_1,i_2) + g^m_{j_0,j_1,j_2}(i_0,i_1) \\ +h^m_{j_0,j_2}(i_0,i_1,i_2), \end{array}$$

defined to be

- $\mathcal{G}_{j_0,j_1,j_2}s^m(i_0,i_1,i_2)$, if $j_0 \geq 1$,
- $\mathcal{G}_{1,j_1,j_2}s^m(i_0,i_1,i_2)$, if $j_0 = 0,\ i_2 \leq m$,
- the dotted arrow in the commutative diagram below, if $j_0 = 0,\ i_1 \leq m < i_2$,

$$\begin{array}{ccccc} A_1 & \xrightarrow{1+\mathbf{C}} & A_2 & \xrightarrow{\mathcal{G}_{0,j_1,j_2}s^m(i_0,i_1,i_2)+1} & A_3 \\ {\scriptstyle \mathcal{G}^m_{0,j_1,j_2}(i_0,i_1,i_2)}\downarrow & & & & \downarrow{\scriptstyle 1+\mathbf{C}+1} \\ A_6 & \xleftarrow{1+\mathbf{C}} & A_5 & \xleftarrow{1+\mathcal{F}_{0,1,j_1,j_2}(i_0,i_1)} & A_4 \end{array}$$

$A_1 = h_{j_1,j_2}s^m(i_0,i_1,i_2) + h_{0,j_1}s^m(i_0,i_1,i_2) + g_{0,1,j_1}s^m(i_0,i_1) + g_{0,j_1,j_2}s^m(i_0,i_2)$,

$A_2 = h_{j_1,j_2}s^m(i_0,i_1,i_2) + h_{0,j_1}s^m(i_0,i_1,i_2) + g_{0,j_1,j_2}s^m(i_0,i_2) + g_{0,1,j_1}s^m(i_0,i_1)$,

$A_3 = g_{0,j_1,j_2}s^m(i_1,i_2) + g_{0,j_1,j_2}s^m(i_0,i_1) + h_{0,j_2}s^m(i_0,i_1,i_2) + g_{0,1,j_1}s^m(i_0,i_1)$,

$A_4 = g_{0,j_1,j_2}s^m(i_1,i_2) + h_{0,j_2}s^m(i_0,i_1,i_2) + g_{0,j_1,j_2}s^m(i_0,i_1) + g_{0,1,j_1}s^m(i_0,i_1)$,

$A_5 = g_{0,j_1,j_2} s^m(i_1,i_2) + h_{0,j_2} s^m(i_0,i_1,i_2) + g_{0,1,j_2} s^m(i_0,i_1) + g_{1,j_1,j_2} s^m(i_0,i_1)\,,$

$A_6 = g_{0,j_1,j_2} s^m(i_1,i_2) + g_{1,j_1,j_2} s^m(i_0,i_1) + h_{0,j_2} s^m(i_0,i_1,i_2) + g_{0,1,j_2} s^m(i_0,i_1)\,.$

- $\mathcal{G}_{0,j_1,j_2} s^m(i_0,i_1,i_2)$, if $j_0 = 0,\ m < i_1$.

♦ For each $0 \le j_0 \le j_1 \le j_2 \le j_3 \le p$, the 3-cell

$$\begin{array}{ccc} \mathcal{H}^m_{j_2,j_3} \oplus \mathcal{H}^m_{j_1,j_2} \oplus \mathcal{H}^m_{j_0,j_1} & \overset{\mathcal{G}^m_{j_1,j_2,j_3}\oplus 1}{\Longrightarrow} & \mathcal{H}^m_{j_1,j_3} \oplus \mathcal{H}^m_{j_0,j_1} \\ {\scriptstyle 1\oplus\mathcal{G}^m_{j_0,j_1,j_2}}\Downarrow & \overset{\mathcal{F}^m_{j_0,j_1,j_2,j_3}}{\Rrightarrow} & \Downarrow{\scriptstyle \mathcal{G}^m_{j_0,j_1,j_3}} \\ \mathcal{H}^m_{j_2,j_3} \oplus \mathcal{H}^m_{j_0,j_2} & \underset{\mathcal{G}^m_{j_0,j_2,j_3}}{\Longrightarrow} & \mathcal{H}^m_{j_0,j_3}, \end{array}$$

consists of the morphisms in $\mathscr{A}$

$$g^m_{j_0,j_2,j_3}(i_0,i_1) + g^m_{j_0,j_1,j_2}(i_0,i_1) \xrightarrow{\mathcal{F}^m_{j_0,j_1,j_2,j_3}(i_0,i_1)} g^m_{j_0,j_1,j_3}(i_0,i_1) + g^m_{j_1,j_2,j_3}(i_0,i_1),$$

for $0 \le i_0 \le i_1 \le q+1$, defined to be

- $\mathcal{F}_{j_0,j_1,j_2,j_3} s^m(i_0,i_1)$, if $j_0 \ge 1$,
- $\mathcal{F}_{1,j_1,j_2,j_3} s^m(i_0,i_1)$, if $j_0 = 0,\ i_1 \le m$,
- $\mathcal{F}_{0,j_1,j_2,j_3} s^m(i_0,i_1)$, if $j_0 = 0,\ m < i_1$. □

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