Salvador López-Martínez*

A singularity as a break point for the multiplicity of solutions to quasilinear elliptic problems

https://doi.org/10.1515/anona-2020-0056
Received March 18, 2019; accepted July 31, 2019.

Abstract: In this paper we deal with the elliptic problem

$$\begin{cases} -\Delta u = \lambda u + \mu(x)\dfrac{|\nabla u|^q}{u^\alpha} + f(x) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $0 \lneqq \mu \in L^\infty(\Omega)$, $0 \lneqq f \in L^{p_0}(\Omega)$ for some $p_0 > \frac{N}{2}$, $1 < q < 2$, $\alpha \in [0, 1]$ and $\lambda \in \mathbb{R}$. We establish existence and multiplicity results for $\lambda > 0$ and $\alpha < q-1$, including the non-singular case $\alpha = 0$. In contrast, we also derive existence and uniqueness results for $\lambda > 0$ and $q - 1 < \alpha \leq 1$. We thus complement the results in [1, 2], which are concerned with $\alpha = q-1$, and show that the value $\alpha = q-1$ plays the role of a break point for the multiplicity/uniqueness of solution.

Keywords: Nonlinear elliptic equations, Singular gradient terms, Multiplicity of solutions, Uniqueness of solution

MSC: 35A01, 35A02, 35J25, 35J62, 35J75

1 Introduction

In this paper we deal with the following boundary value problem:

$$\begin{cases} -\Delta u = \lambda u + \mu(x)\dfrac{|\nabla u|^q}{u^\alpha} + f(x) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{P_λ}$$

Here, Ω is a bounded domain of $\mathbb{R}^N$ ($N \geq 3$) with boundary $\partial\Omega$ smooth enough, $0 \lneqq \mu \in L^\infty(\Omega)$, $0 \lneqq f \in L^{p_0}(\Omega)$ for some $p_0 > \frac{N}{2}$, $1 < q < 2$, $0 \leq \alpha \leq 1$ and $\lambda \in \mathbb{R}$. A solution to (P_λ) is a function $0 < u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ which satisfies the equation in (P_λ) in the usual weak sense (we will be more precise about the concept of solution in Definition 3.1 below). Observe that, if $\alpha > 0$, then the lower order term presents a singularity as u approaches zero, i.e., as x approaches $\partial\Omega$. Our goal is to study the existence, nonexistence, uniqueness and multiplicity of solutions to (P_λ), specially for $\lambda > 0$.

***Corresponding Author: Salvador López-Martínez,** Departamento de Análisis Matemático, Universidad de Granada, Facultad de Ciencias, Avenida Fuentenueva s/n, 18071, Granada, Spain, E-mail: salvadorlopez@ugr.es

The first motivation for dealing with this problem comes from the non-singular case $\alpha = 0$, i.e.,

$$\begin{cases} -\Delta u = \lambda u + \mu(x)|\nabla u|^q + f(x) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{R_λ}$$

Non-singular problems with lower order terms having natural growth in the gradient have been extensively studied since the pioneering works by Boccardo, Murat and Puel in the '80s and '90s (see [3–5] and references therein) and, in particular, problem (R_λ) is very well understood for $1 < q \le 2$ and $\lambda \le 0$. Indeed, it is well-known from classical results (see [3, 5]) that problem (R_λ) admits at least one solution for all $\lambda < 0$. Concerning the uniqueness of solution, it was first dealt with in [6], and their results have been improved in several directions since then (see [7] and references therein). In particular, it has been recently proved in [7] that uniqueness holds for all $\lambda \le 0$. However, the existence of solution for $\lambda = 0$ is not always guaranteed. Roughly speaking, if $\|f\|_{L^{p_0}(\Omega)}$ or $\|\mu\|_{L^\infty(\Omega)}$ are small enough (if $1 < q < 2$, then one needs to ask both to be small, see Proposition 3.4 below), then there exists a unique solution to (R_0), as it is shown for instance in [8] (see also [9] and references therein). Conversely, it is proved in [10] (see also [11]) that, if f or μ are large in some sense, there exists no solution to (R_0); in consequence, $\lambda = 0$ is a bifurcation point from infinity. Concerning this last case, a very precise description of the blow-up of the solutions at $\lambda = 0$, and also a necessary and sufficient condition for the existence of solution to (R_0) in terms of the corresponding ergodic problem, are given in [12] under slightly stronger hypotheses on f and μ.

The scenario in which (R_0) has a solution is not so well understood and has risen interest in the recent years. In this case one expects to find solutions to (R_λ) for small $\lambda > 0$ by a continuation argument. However, the uniqueness and multiplicity problems are harder to deal with for $\lambda > 0$, and very few results are known in this direction. In fact, up to our knowledge, the literature contains results concerning only the quadratic case $q = 2$. In this regard, the first advances can be found in [13] for $\mu > 0$ constant. Shortly after that, some improvements appeared in [14], where $\lambda = \lambda(x)$ is allowed to change sign but μ is still constant. These two works employ variational techniques. Going further, topological degree and bifurcation are used in [15] to handle problem (R_λ) with $\lambda > 0$ and $\mu \in L^\infty(\Omega)$ such that $\mu_1 \le \mu \le \mu_2$ for some constants $\mu_2 > \mu_1 > 0$. We also quote [16], where functions $0 \lneqq \mu \in L^\infty(\Omega)$ vanishing on $\partial\Omega$, and even with compact support, are permitted at the expense of imposing $N \le 3$ (the cases $N = 4, 5$ are also handled provided $\lambda = \lambda(x)$ satisfies extra hypotheses). Very recently, a similar problem to (R_λ) with the p -Laplacian as principal operator has been considered in [17], while sign-changing coefficients (including μ) are allowed in [18].

In all these works, the authors prove that, if there is a solution to (R_0), then problem (R_λ) admits at least two different solutions for all $\lambda > 0$ small enough, and it was first shown in [15] that the branch of positive solutions bifurcates from infinity to the right of the axis $\lambda = 0$ (see [19] for a more complete picture when different sign conditions on f are imposed). We stress again that all the mentioned papers have in common the assumption $q = 2$. Indeed, the techniques employed for $q = 2$ usually involve exponential test functions which somehow remove the dependence on the gradient in the equation. For instance, this idea allows the authors of [13] to study the problem variationally, while in [15] it is essential in order to find a priori estimates for $\lambda > 0$. However, this idea fails for $1 < q < 2$ as the gradient term can not be removed when one looks for a priori estimates satisfied by supersolutions to (R_λ). Up to our knowledge, the multiplicity or uniqueness of solutions for $\lambda > 0$ is an open problem if $1 < q < 2$.

Turning back to (P_λ), another motivation for studying this problem comes from the very recent paper [1]. In such a work, problem (P_λ) is studied in the singular case $\alpha > 0$, mostly in the special case $\alpha = q - 1$. Elliptic problems with singularities at $u = 0$ have become of remarkable interest since the seminal papers [20–22]. Without the aim of being exhaustive, some related references dealing with this kind of singularities (with or without lower order terms with natural growth in the gradient) are [23–35]. The interested reader is referred to [35] and references therein, where a rather complete background on singular problems can be found.

Focusing specifically on problem (P_λ), in Remark 6.1 of [1] the authors observe that, if $q = 2$ and $0 < \alpha < q - 1 = 1$, the techniques in [15] can be adapted to derive again a multiplicity result for $\lambda > 0$.

Hence, roughly speaking, mild singularities at zero do not alter the behavior of the solutions, as far as the multiplicity for $\lambda > 0$ is concerned. Nonetheless, the main result in that paper shows that multiplicity fails for $1 < q \leq 2$ and $\alpha = q - 1$ (see [33] for $q = 2$ and μ constant). To be precise, the authors prove under natural hypotheses on μ and f that, if $\alpha = q - 1$, there exists $\lambda^\star \in (0, \lambda_1]$ (where $\lambda_1 = \inf_{v \in H_0^1(\Omega)\setminus\{0\}} \int_\Omega |\nabla v|^2 / \int_\Omega v^2$) such that problem (P_λ) has a solution if and only if $\lambda < \lambda^\star$, and in this case, the solution is unique (see also [2] for a similar existence result when f and u may change sign). In particular, one has existence and uniqueness for $\lambda > 0$ small. Since this result is true for $1 < q \leq 2$, it is natural to wonder whether $\alpha = q - 1$ is a break point for the multiplicity of solutions not only in the case $q = 2$, but also for $1 < q < 2$.

In the present work we contribute to these topics by proving that, if there is a solution to (P_0), then there are at least two different solutions to (P_λ) for all $\lambda > 0$ small enough provided q and α satisfy certain relations involving also the dimension N. We prove also that the branch of positive solutions bifurcates from infinity to the right of the axis $\lambda = 0$.

To be more precise, we consider the following set of hypotheses:

$$\begin{cases} \Omega \subset \mathbb{R}^N \text{ is a bounded domain of class } \mathcal{C}^2, \\ \mu \in L^\infty(\Omega) \text{ satisfies that } \mu \geq \mu_0 \text{ in } \Omega \text{ for some constant } \mu_0 > 0, \\ 0 \lneqq f \in L^{p_0}(\Omega) \text{ for some } p_0 > \frac{N}{2}, \\ q \in (1, 2), \\ \alpha \in [0, q-1). \end{cases} \tag{H1}$$

Observe that μ is bounded away from zero but not necessarily constant. We introduce here the main result of this paper:

Theorem 1.1. *Assume that* (H1) *holds and that* (P_0) *admits a solution* u_0. *If* $q > \frac{N}{N-1}$, *suppose also that*

$$\frac{q - 1 - \alpha}{q - 2\alpha} \leq \frac{q - \alpha}{N - q + 1}. \tag{1.1}$$

Then, there exists $\bar{\lambda} \in (0, \lambda_1)$ *such that problem* (P_λ) *admits at least two different solutions for all* $\lambda \in (0, \bar{\lambda}]$. *Moreover, zero is the unique bifurcation point from infinity to problem* (P_λ).

Even though this result deals only with the range $\lambda > 0$, in order to make a more complete picture we will gather and prove in Section 3 some existence, nonexistence and uniqueness results about problem (P_λ) for $\lambda \leq 0$. We stress that the uniqueness result for $\lambda \leq 0$, apart from being new in the literature, shows that $\lambda = 0$ is a critical point beyond which the nature of the problem changes drastically, as in the well-known case $q = 2$ and $\alpha = 0$.

Concerning the proof of Theorem 1.1, the idea is to derive a priori estimates of the solutions to (P_λ) for all $\lambda > \lambda_0$ which are independent of $\lambda > 0$. This idea first appeared in [15] for $q = 2$ and $\alpha = 0$, but the approach for deriving the estimates does not work in our framework. For our purposes, it is more convenient to use the arguments developed in [16], which allow us to find L^p estimates of supersolutions. After that, we establish a bootstrap argument, which works thanks to some results in [9], that yields an L^∞ estimate. Actually, these results are valid only in the nonsingular case $\alpha = 0$, so we will extend some parts of them to our singular framework. After writing the present work, it came to the author's knowledge that similar results extending [9] to a more general setting have been recently obtained in [36].

Hypothesis (1.1) in Theorem 1.1 deserves some comments. It appears in the proof as a result of the combination of the mentioned techniques from [16] and the bootstrap from [9]. However, we presume that this is a technical assumption forced by the tools we employed, so the theorem might admit some improvements. In order to clarify the meaning of this condition, we derive two corollaries below in which simpler conditions assuring (1.1) are imposed. For instance, if we consider the sequence

$$Q_n = \begin{cases} 2 & \forall n \leq 4, \\ \dfrac{n + 2 - \sqrt{n^2 - 4n - 4}}{4} & \forall n \geq 5, \end{cases} \tag{1.2}$$

then $q \in (1, Q_N] \setminus \{2\}$ implies (1.1), with no extra hypotheses on α apart from $0 \leq \alpha < q - 1$ (see Corollary 3.17). Observe that $Q_n > 1$ but $\lim_{n\to\infty} Q_n = 1$. This means that, if N is large, then q has to be chosen close to 1. However, one would expect a multiplicity result for any $q \in (1, 2)$ and any N. This still remains as an open problem. In any case, Corollary 3.17 represents a remarkable advance, in particular, about the nonsingular problem (R_λ). Changing the point of view, we give in Corollary 3.18 below a condition on α that is sufficient for applying Theorem 1.1 even for q close to 2 and for N large.

With the aim of having a deeper insight into problem (P_λ), we also consider in this work the case $q-1 < \alpha \leq 1$. In contrast to the previous situation $(0 \leq \alpha < q-1)$, we will prove that existence and *uniqueness* hold for $\lambda > 0$ small enough. For this purpose, we will need the following assumption on Ω:

$$\begin{cases} \text{There exist } r_0, \theta_0 > 0 \text{ such that, if } x \in \partial\Omega \text{ and } 0 < r < r_0, \text{ then} \\ |\Omega_r| \leq (1-\theta_0)|B_r(x)| \text{ for every connected component } \Omega_r \text{ of } \Omega \cap B_r(x). \end{cases} \tag{A}$$

Note that, if $\partial\Omega$ is Lipschitz, then Ω satisfies (A) (see [7]), so this represents only a mild restriction. The precise hypotheses that we need are gathered here:

$$\begin{cases} \Omega \subset \mathbb{R}^N \text{ is a bounded domain satisfying condition (A)}, \\ 0 \lneqq \mu \in L^\infty(\Omega), \\ 0 \lneqq f \in L^{p_0}(\Omega) \text{ for } p_0 > \frac{N}{2}, \\ q \in (1,2), \\ q - 1 < \alpha \leq 1. \end{cases} \tag{H2}$$

We emphasize that μ is allowed to vanish in subsets of Ω with nonzero measure.

The statement of the main result in the $q - 1 < \alpha \leq 1$ case is the following:

Theorem 1.2. *Assume that* (H2) *holds. Then there exists a solution to* (P_λ) *for all* $\lambda < \lambda_1$*, and there exists no solution to* (P_λ) *for all* $\lambda \geq \lambda_1$*. Moreover, the solution is unique for all* $\lambda \leq 0$ *and, if f satisfies that*

$$\forall \omega \subset\subset \Omega \quad \exists c_\omega > 0 : \quad f \geq c_\omega \quad \text{in } \omega,$$

then the solution is unique for all $\lambda < \lambda_1$*. Finally,* λ_1 *is the unique bifurcation point from infinity to problem* (P_λ)*.*

Even though we are specially interested in the uniqueness part, the existence statement in Theorem 1.2 deserves also attention. Observe that one has existence of solution if and only if $\lambda < \lambda_1$. This suggests that the nonlinear term does not play an essential role in this case, since the situation is analogous to the linear problem $(\mu \equiv 0)$. Recall that this is not the case when $\alpha = q - 1$, for which one has existence if and only if $\lambda < \lambda^*$, where $\lambda^* < \lambda_1$ provided $\mu > 0$ (see [1, Remark 6.3]).

The proof of the existence of solution in Theorem 1.2 is performed by passing to the limit in certain family of approximate nonsingular problems. We will derive Hölder continuous a priori estimates on the solutions to such a family, which will allow us to pass to the limit. For proving such estimates, the assumption $\alpha \leq 1$ is essential (see Remark 3.3 below). Moreover, the continuity of the solutions is also essential to prove their uniqueness. Indeed, we state and prove in Section 2 two comparison principles valid for continuous lower and upper solutions to singular equations. As far as we know, these two results are new, and they are interesting by themselves as only few uniqueness results for singular equations are known (see [1, 33, 37–39]). We follow in their proofs the arguments in [7] and [1].

As a summary, our results contribute to the theory of equations with subquadatic growth in the gradient, extending what it is known about the multiplicity of solutions in the quadratic case. On the other hand, they can be seen as a link between the singular and nonsingular theory, in the sense that they show that the presence or not of a singularity is determining only if it is strong enough. Finally, new existence and uniqueness results are given for strong singularities, where the uniqueness part is specially remarkable.

We organize the paper as follows: in Section 2 we deal with the mentioned comparison principles; we devote Section 3 to prove Theorem 1.1 as well as some auxiliary results and some consequences of the mentioned theorem; Section 4 contains the proof of Theorem 1.2, and Section 5 is an appendix where we prove a continuation result needed in the proof of Theorem 1.1.

Acknowledgments

The author wants to thank warmly T. Leonori and J. Carmona for their helpful contributions to this work. He wants to thank also A.J. Fernández and M. Magliocca for useful conversations and suggestions.

Notation

- For every $x \in \mathbb{R}^N$, the distance from x to $\partial\Omega$ will be denoted as $\delta(x)$. Furthermore, for $p \geq 1$ we will denote as $L^p(\Omega, \delta)$ the space of measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\|u\|_{L^p(\Omega,\delta)} := \left(\int_\Omega |u(x)|^p \delta(x) dx \right)^{\frac{1}{p}} < +\infty,$$

  identifying functions equal up to a set of zero measure.
- For $p \geq 1$, we will denote the usual Marcinkiewicz space as $\mathcal{M}^p(\Omega)$, i.e., the space of measurable functions $u : \Omega \to \mathbb{R}$ for which there exists $c > 0$ such that $|\{|u| > k\}|k^p \leq c$ for all $k > 0$. In this case, we denote

$$\|u\|_{\mathcal{M}^p(\Omega)} := \left(\inf\{c > 0 : |\{|u| > k\}|k^p \leq c \text{ for all } k > 0\right)^{\frac{1}{p}}.$$

- For $k \geq 0$, the usual truncation functions will be written as $T_k(s) = \max\{-k, \min\{s, k\}\}$ and $G_k(s) = s - T_k(s)$ for all $s \in \mathbb{R}$.
- The principal eigenvalue of the $-\Delta$ operator in Ω under zero Dirichlet boundary conditions will be denoted as λ_1. In other words, λ_1 is the unique real number satisfying that the equation $-\Delta\varphi = \lambda_1\varphi$ has a solution $0 < \varphi \in H_0^1(\Omega)$. We will write φ_1 for the positive eigenfunction associated with λ_1 such that $\|\varphi_1\|_{L^\infty(\Omega)} = 1$.

2 Comparison principles

We start with a comparison principle valid for singular equations. The proof basically follows the steps of a similar result in [7]. However, up to our knowledge this is the first time that a comparison result has been proved including a general positive singular lower order term on the right hand side of the equation (see the comparison results in [1], where a specific 1-homogeneous singular term is considered).

Theorem 2.1. *Let $1 < q \leq 2$, $\lambda \leq 0$, $h \in L^1_{loc}(\Omega)$ and $g : \Omega \times (0, +\infty) \to [0, +\infty)$ satisfying*

$$\begin{aligned} s \mapsto g(x, s) &\quad \textit{is nonincreasing for a.e. } x \in \Omega, \\ x \mapsto g(x, s) &\quad \textit{is locally essentially bounded for all } s > 0. \end{aligned}$$

Let $u, v \in C(\Omega) \cap W^{1,N}_{loc}(\Omega)$, with $u, v > 0$ in Ω, be such that

$$\int_\Omega \nabla u \nabla\phi \leq \lambda \int_\Omega u\phi + \int_\Omega g(x, u)|\nabla u|^q\phi + \int_\Omega h(x)\phi \quad \textit{and} \tag{2.1}$$

$$\int_\Omega \nabla v \nabla\phi \geq \lambda \int_\Omega v\phi + \int_\Omega g(x, v)|\nabla v|^q\phi + \int_\Omega h(x)\phi \tag{2.2}$$

for every $0 \leq \phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ with compact support. Suppose also that the following boundary condition holds:

$$\limsup_{x \to x_0}(u(x) - v(x)) \leq 0 \quad \forall x_0 \in \partial\Omega. \tag{2.3}$$

Then, $u \leq v$ in Ω.

Remark 2.2. Theorem 2.1 is valid for a wide class of lower order terms. For instance, the model example is

$$g(x,s)=\frac{\mu(x)}{s^\alpha}\quad \text{a.e. } x\in\Omega,\ \forall s>0,$$

for any $\alpha > 0$ and $0 \le \mu \in L^\infty_{\mathrm{loc}}(\Omega)$. In particular, the growth of the singularity is irrelevant in the proof. Nonetheless, the comparison principle does not work for $\lambda > 0$. Indeed, as we pointed out in the Introduction, if the singularity is mild enough in some sense, then a multiplicity phenomenon appears for $\lambda > 0$. Thus, for the model case, the comparison result is sharp in terms of the sign of λ.

Remark 2.3. In Theorem 2.1, $u, v \in C(\Omega)$ are not assumed to be continuous up to $\partial\Omega$, so a suitable ordering condition on the boundary is given by (2.3). However, if $u, v \in C(\overline{\Omega})$, then (2.3) is equivalent to the usual and more natural condition $u(x_0) \le v(x_0)$ for all $x_0 \in \partial\Omega$.

Proof of Theorem 2.1. Let us denote $w = u - v$. For $k > 0$, we consider the function $\phi = (w - k)^+$, and we also denote

$$A_k=\{x\in\Omega : w(x)\ge k\}.$$

Notice that $\operatorname{supp}(\phi) \subset A_k$. Moreover, condition (2.3) implies that $A_k \subset\subset \Omega$, so ϕ has compact support. In particular, $\phi \in H^1_0(\Omega)\cap L^\infty(\Omega)$, so it can be taken as test function in (2.1) and (2.2), obtaining that

$$\int_\Omega \nabla u\nabla(w-k)^+ \le \lambda\int_\Omega u(w-k)^+ + \int_\Omega g(x,u)|\nabla u|^q(w-k)^+ + \int_\Omega h(x)(w-k)^+ \tag{2.4}$$

and

$$\int_\Omega \nabla v\nabla(w-k)^+ \ge \lambda\int_\Omega v(w-k)^+ + \int_\Omega g(x,v)|\nabla v|^q(w-k)^+ + \int_\Omega h(x)(w-k)^+. \tag{2.5}$$

Subtracting (2.5) from (2.4) we get

$$\int_\Omega |\nabla(w-k)^+|^2 \le \lambda\int_\Omega ((w-k)^+)^2 + \lambda k\int_\Omega (w-k)^+ + \int_\Omega (g(x,u)|\nabla u|^q - g(x,v)|\nabla v|^q)(w-k)^+.$$

Since $\lambda \le 0$, we deduce that

$$\int_\Omega |\nabla(w-k)^+|^2 \le \int_\Omega (g(x,u)|\nabla u|^q - g(x,v)|\nabla v|^q)(w-k)^+. \tag{2.6}$$

Assume in order to achieve a contradiction that $w^+ \not\equiv 0$, and let $k_0 \in (0, \|w^+\|_{L^\infty(\Omega)})$. Let also $\omega \subset\subset \Omega$ be an open set such that $A_{k_0} \subset \omega$. Observe that $A_k \subset A_{k_0}$ for all $k \ge k_0$. Then, using the properties of g, it is clear that

$$g(x,u)\le g(x,v)\le g(x,\inf_\omega(v))\le \Big\|g(\cdot,\inf_\omega(v))\Big\|_{L^\infty(\omega)}$$

in A_k for every $k \in [k_0, \|w^+\|_{L^\infty(\Omega)}]$. Therefore, from (2.6) we deduce that

$$\begin{aligned}\int_\Omega |\nabla(w-k)^+|^2 &\le \int_\Omega g(x,v)\big||\nabla u|^q - |\nabla v|^q\big|(w-k)^+ \\ &\le \Big\|g(\cdot,\inf_\omega(v))\Big\|_{L^\infty(\omega)}\int_{A_k}\big||\nabla u|^q - |\nabla v|^q\big|(w-k)^+\end{aligned} \tag{2.7}$$

for every $k \in [k_0, \|w^+\|_{L^\infty(\Omega)}]$.

For every $j \in \mathbb{R}$, let us denote $\Omega_j = \{x \in \Omega : |w(x)| = j\}$, and consider also the set $J = \{j \in \mathbb{R} : |\Omega_j| \ne 0\}$. Since $|\Omega| < \infty$, then J is at most countable, which implies that the set $\bigcup_{j\in J}\Omega_j$ is measurable, and we also have that

$$\nabla w = 0 \quad \text{in } \bigcup_{j\in J}\Omega_j \implies |\nabla u_1| = |\nabla v_1| \quad \text{in } \bigcup_{j\in J}\Omega_j.$$

Hence, if we define the set $Z = \Omega \setminus \bigcup_{j\in J} \Omega_j$, we deduce from (2.7) that

$$\int_\Omega |\nabla (w-k)^+|^2 \le \left\| g(\cdot, \inf_\omega (v)) \right\|_{L^\infty(\omega)} \int_{A_k\cap Z} \left(\int_0^1 \frac{d}{dt}(|t\nabla u + (1-t)\nabla v|^q) dt \right) (w-k)^+. \tag{2.8}$$

Taking into account that $u, v \in W^{1,N}_{\mathrm{loc}}(\Omega)$ and $A_k \subset\subset \Omega$, we have that

$$|t\nabla u + (1-t)\nabla v| \le |\nabla u| + |\nabla v| + 1 \equiv \eta \in L^N(A_k \cap Z).$$

Hence, from (2.8) we derive that

$$\begin{aligned} \|(w-k)^+\|^2_{H^1_0(\Omega)} &\le C \int_{A_k\cap Z} \left(\int_0^1 |t\nabla u + (1-t)\nabla v|^{q-2}(t\nabla u + (1-t)\nabla v)\nabla w dt \right)(w-k)^+ \\ &\le C \int_{A_k\cap Z} \eta^{q-1} |\nabla w| (w-k)^+ \le C \int_{A_k\cap Z} \eta |\nabla (w-k)^+| (w-k)^+ \\ &\le C \|\eta\|_{L^N(A_k\cap Z)} \|(w-k)^+\|_{H^1_0(\Omega)} \|(w-k)^+\|_{L^{2^\star}(\Omega)} \\ &\le C \|\eta\|_{L^N(A_k\cap Z)} \|(w-k)^+\|^2_{H^1_0(\Omega)}. \end{aligned} \tag{2.9}$$

Let us now define the function $F : [k_0, \|w^+\|_{L^\infty(\Omega)}] \to \mathbb{R}$ by

$$F(k) = \|\eta\|_{L^N(A_k\cap Z)} = \||\nabla u| + |\nabla v| + 1\|_{L^N(A_k\cap Z)} \quad \forall k \in [k_0, \|w^+\|_{L^\infty(\Omega)}),$$

and $F(\|w^+\|_{L^\infty(\Omega)}) = 0$. It is clear that F is nonincreasing and continuous. Thus, choosing k close enough to $\|w^+\|_{L^\infty(\Omega)}$, we deduce from (2.9) that $(w-k)^+ \equiv 0$. That is to say, $w \le k$ in Ω. But this is not possible since $k < \|w^+\|_{L^\infty(\Omega)} = \sup_\Omega(w)$.

In conclusion, we have proved that $w^+ \equiv 0$, i.e., $w \le 0$ in Ω. □

Next theorem is another comparison principle which works for $\lambda > 0$. In turn, one has to impose stronger hypotheses on g and h. The proof is similar to the one above combined with some ideas in [1].

Theorem 2.4. *Let* $1 < q \le 2$, $\lambda \in \mathbb{R}$, $0 \le h \in L^1_{loc}(\Omega)$ *and* $g : \Omega \times (0, +\infty) \to [0, +\infty)$ *satisfying*

$$\begin{aligned} &s \mapsto s^{q-1} g(x,s) \quad \text{is nonincreasing for a.e. } x \in \Omega, \\ &x \mapsto g(x,s) \qquad \text{is locally essentially bounded for all } s > 0. \end{aligned}$$

If $\lambda > 0$, *assume also that*

$$\forall \omega \subset\subset \Omega \quad \exists c_\omega > 0 : \quad h \ge c_\omega \text{ in } \omega. \tag{2.10}$$

Let $u, v \in C(\Omega) \cap W^{1,N}_{loc}(\Omega)$, *with* $u, v > 0$ *in* Ω, *satisfying respectively* (2.1) *and* (2.2) *for every* $0 \le \phi \in H^1_0(\Omega) \cap L^\infty(\Omega)$ *with compact support. Suppose also that, for every* $\varepsilon > 0$, *the following boundary condition holds:*

$$\limsup_{x\to x_0} \left(\frac{u(x)}{v(x)+\varepsilon} \right) \le 1 \quad \forall x_0 \in \partial\Omega. \tag{2.11}$$

Then, $u \le v$ *in* Ω.

Remark 2.5. The observation made in Remark 2.3 is valid also for Theorem 2.4 substituting condition (2.3) with (2.11).

Proof of Theorem 2.4. For every $\varepsilon > 0$, let us consider the function

$$w_\varepsilon = \log\left(\frac{u}{v+\varepsilon}\right).$$

We claim that $w_\varepsilon^+ \equiv 0$ for any $\varepsilon > 0$. Suppose by contradiction that there exists $\varepsilon_0 > 0$ such that $w_{\varepsilon_0}^+ \not\equiv 0$. Let us fix $k_0 \in \big(0, \|w_{\varepsilon_0}^+\|_{L^\infty(\Omega)}\big)$ and $\varepsilon \in (0, \varepsilon_0)$, the latter to be chosen small enough later. It is clear that $w_{\varepsilon_0} \le w_\varepsilon$ in Ω, so $w_\varepsilon^+ \not\equiv 0$.

For $k \in [k_0, \|w_\varepsilon^+\|_{L^\infty(\Omega)}]$, let us denote

$$A_k = \{x \in \Omega : w_\varepsilon(x) \ge k\} = \{x \in \Omega : u(x) \ge e^k(v(x) + \varepsilon)\}.$$

Notice that $\operatorname{supp}(w_\varepsilon - k)^+ \subset A_k$. By (2.11), we also have that $\limsup_{x\to x_0} w_\varepsilon(x) \le 0$ for all $x_0 \in \partial\Omega$, which implies that $A_k \subset\subset \Omega$. Then, the function $(w_\varepsilon - k)^+$ has compact support, and in particular, $(w_\varepsilon - k)^+ \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Therefore, we may take $\frac{(w_\varepsilon-k)^+}{u}$ as test function in (2.1), and $\frac{(w_\varepsilon-k)^+}{v+\varepsilon}$ in (2.2), obtaining

$$\begin{aligned}\int_\Omega \frac{\nabla u}{u}\nabla(w_\varepsilon - k)^+ \le{}& \int_\Omega \frac{|\nabla u|^2}{u^2}(w_\varepsilon - k)^+ + \lambda\int_\Omega (w_\varepsilon - k)^+\\ &+ \int_\Omega u^{q-1}g(x,u)\frac{|\nabla u|^q}{u^q}(w_\varepsilon - k)^+ + \int_\Omega \frac{h(x)}{u}(w_\varepsilon - k)^+\end{aligned} \tag{2.12}$$

and, using that $g \ge 0$,

$$\begin{aligned}\int_\Omega \frac{\nabla v}{v+\varepsilon}\nabla(w_\varepsilon - k)^+ \ge{}& \int_\Omega \frac{|\nabla v|^2}{(v+\varepsilon)^2}(w_\varepsilon - k)^+ + \lambda\int_\Omega \frac{v}{v+\varepsilon}(w_\varepsilon - k)^+\\ &+ \int_\Omega v^{q-1}g(x,v)\frac{|\nabla v|^q}{v^{q-1}(v+\varepsilon)}(w_\varepsilon - k)^+ + \int_\Omega \frac{h(x)}{v+\varepsilon}(w_\varepsilon - k)^+\\ \ge{}& \int_\Omega \frac{|\nabla v|^2}{(v+\varepsilon)^2}(w_\varepsilon - k)^+ + \lambda\int_\Omega (w_\varepsilon - k)^+ - \int_\Omega \frac{\lambda\varepsilon}{v+\varepsilon}(w_\varepsilon - k)^+\\ &+ \int_\Omega v^{q-1}g(x,v)\frac{|\nabla v|^q}{(v+\varepsilon)^q}(w_\varepsilon - k)^+ + \int_\Omega \frac{h(x)}{v+\varepsilon}(w_\varepsilon - k)^+.\end{aligned} \tag{2.13}$$

Let $\omega \subset\subset \Omega$ be an open set such that $A_{k_0} \subset \omega$. Observe that $A_k \subset A_{k_0}$ for all $k \ge k_0$. Then, it is clear that

$$u^{q-1}g(x,u) \le v^{q-1}g(x,v) \le \sup_\omega(v)^{q-1}g(x,\inf_\omega(v)) \le \sup_\omega(v)^{q-1}\Big\|g(\cdot,\inf_\omega(v))\Big\|_{L^\infty(\omega)}$$

in A_k for every $k \in [k_0, \|w_\varepsilon^+\|_{L^\infty(\Omega)}]$. Therefore,

$$\begin{aligned}&\int_\Omega \left(u^{q-1}g(x,u)\frac{|\nabla u|^q}{u^q} - v^{q-1}g(x,v)\frac{|\nabla v|^q}{(v+\varepsilon)^q}\right)(w_\varepsilon - k)^+\\ &\le \sup_\omega(v)^{q-1}\Big\|g(\cdot,\inf_\omega(v))\Big\|_{L^\infty(\omega)}\int_\Omega \left|\frac{|\nabla u|^q}{u^q} - \frac{|\nabla v|^q}{(v+\varepsilon)^q}\right|(w_\varepsilon - k)^+.\end{aligned}$$

Moreover, we have that

$$h\left(\frac{1}{u} - \frac{1}{v+\varepsilon}\right) + \frac{\lambda\varepsilon}{v+\varepsilon} \le 0 \quad \text{in } A_k \text{ for every } k \in [k_0, \|w_\varepsilon^+\|_{L^\infty(\Omega)}] \tag{2.14}$$

whenever $\lambda \le 0$. On the other hand, if $\lambda > 0$, let us take

$$\varepsilon < \min\left\{\varepsilon_0, \frac{1-e^{-k_0}}{\lambda}c_\omega\right\},$$

where c_ω is the constant given by (2.10). With this choice, it is straightforward to deduce that (2.14) holds again.

Therefore, subtracting (2.12) and (2.13), and taking into account that $u, v \in W_{\mathrm{loc}}^{1,N}(\Omega)$ and also (2.14), we may argue as in the proof of [1, Theorem 3.2] and achieve a contradiction taking k close enough to $\|w_\varepsilon^+\|_{L^\infty(\Omega)}$.

In conclusion, necessarily $w_\varepsilon^+ \equiv 0$ for any $\varepsilon > 0$, i.e., $u \le v + \varepsilon$ in Ω for any $\varepsilon > 0$. Letting $\varepsilon \to 0$ it follows that $u \le v$ in Ω. □

3 Multiplicity for $0 \leq \alpha < q - 1$

In this section we will study problem (P_λ) under condition (H1). In this case observe that, if $0 < u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ and $t > 0$, then

$$\frac{|\nabla tu|^q}{(tu)^\alpha} = t^{q-\alpha}\frac{|\nabla u|^q}{u^\alpha}.$$

Since $\alpha < q - 1$, then $q - \alpha > 1$. That is to say, the lower order term has *superlinear homogeneity*.

The concept of solution we will adopt is gathered in the following definition.

Definition 3.1. Given $\lambda \in \mathbb{R}$, a subsolution to (P_λ) is a function $u \in H^1_0(\Omega) \cap L^\infty(\Omega)$ such that $u > 0$ a.e. in Ω, $\mu\frac{|\nabla u|^q}{u^\alpha} \in L^1_{\mathrm{loc}}(\Omega)$ and

$$\int_\Omega \nabla u \nabla \phi \leq \lambda \int_\Omega u\phi + \int_\Omega \mu(x)\frac{|\nabla u|^q}{u^\alpha}\phi + \int_\Omega f(x)\phi \quad \forall 0 \leq \phi \in C^1_c(\Omega).$$

Reciprocally, a supersolution to (P_λ) is a function $u \in H^1(\Omega) \cap L^\infty(\Omega)$ such that $u > 0$ a.e. in Ω, $\mu\frac{|\nabla u|^q}{u^\alpha} \in L^1_{\mathrm{loc}}(\Omega)$ and satisfies the reverse inequality. Finally, a solution to (P_λ) is a function $u \in H^1_0(\Omega) \cap L^\infty(\Omega)$ which is both a subsolution and a supersolution to (P_λ).

Remark 3.2. Arguing as in [1, Appendix], it can be proved that, if $u \in H^1_0(\Omega) \cap L^\infty(\Omega)$ is a solution to (P_λ), then $\mu\frac{|\nabla u|^q}{u^\alpha}\phi \in L^1(\Omega)$ for all $\phi \in H^1_0(\Omega) \cap L^\infty(\Omega)$ and

$$\int_\Omega \nabla u \nabla \phi = \lambda \int_\Omega u\phi + \int_\Omega \mu(x)\frac{|\nabla u|^q}{u^\alpha}\phi + \int_\Omega f(x)\phi \quad \forall \phi \in H^1_0(\Omega) \cap L^\infty(\Omega).$$

This fact allows us, in particular, to take u itself as test function.

Remark 3.3. Assume that (H1) holds. By taking φ_1 as test function in the weak formulation of (P_λ) one easily deduces that, if u is a solution to (P_λ), then $\lambda < \lambda_1$. Furthermore, since $\alpha \in [0, 1]$, it can be proved as in [1, Appendix], which follows the ideas in [40], that every solution u to (P_λ), for any $\lambda < \lambda_1$, satisfies that $u \in C^{0,\eta}(\overline{\Omega})$ for some $\eta \in (0, 1)$. Finally, since the solutions to (P_λ) are positive in compact subsets of Ω, then it can be seen again as in the mentioned appendix that $u \in W^{1,N}_{\mathrm{loc}}(\Omega)$ for every solution to (P_λ) for any $\lambda < \lambda_1$.

Our first result is concerned with the existence and uniqueness of solution to (P_λ) for $\lambda \leq 0$. The existence is well-known from the works that are quoted in the proof below. However, a precise statement for unbounded datum f is required for our purposes. In any case, the uniqueness is new up to our knowledge.

Proposition 3.4. *Assume that* (H1) *holds. Then, problem* (P_λ) *has a unique solution for all* $\lambda < 0$. *Moreover, assume additionally that either* $\alpha > 0$ *or the following smallness condition holds:*

$$a\left(b + \|f\|_{L^{p_0}(\Omega)}\right) < \left(\frac{2}{N} - \frac{1}{p_0}\right)\frac{N^2|B_1(0)|^{\frac{2}{N}}}{|\Omega|^{\frac{2}{N}-\frac{1}{p_0}}},$$

where $B_1(0)$ *denotes the unit ball in* $\mathbb{R}^N$, *and* $a, b > 0$ *are such that*

$$\|\mu\|_{L^\infty(\Omega)}|s|^q \leq a|s|^2 + b \quad \forall s \in \mathbb{R}.$$

Then (P_0) *has a unique solution.*

Proof. The result for $\alpha = 0$ and $\lambda \leq 0$ is well-known. Indeed, the existence of solution for $\alpha = 0$ and $\lambda < 0$ is proved in [3, 5], the existence for $\alpha = \lambda = 0$ under the smallness condition is proved in [8], and the uniqueness for $\alpha = 0$ and $\lambda \leq 0$, in [7]. Thus, we assume that $\alpha \in (0, q - 1)$.

Observe now that, by Young's inequality, there exist $C_1, C_2 > 0$ such that

$$0 \le \mu(x)\frac{|\xi|^q}{|s|^\alpha} \le C_1\frac{|\xi|^2}{|s|^{\frac{2\alpha}{q}}} + C_2 \tag{3.1}$$

for all $\xi \in \mathbb{R}^N$, for all $s \in \mathbb{R}\setminus\{0\}$ and for a.e. $x \in \Omega$, where

$$\frac{2\alpha}{q} < \frac{2(q-1)}{q} = 2 - \frac{2}{q} < 1. \tag{3.2}$$

Then, the hypotheses of [31, Proposition 4.1] are fulfilled, so there exists a solution $u_0 \in H_0^1(\Omega)\cap L^\infty(\Omega)$ to (P_0) in some weaker sense than Definition 3.1. Nonetheless, since $f \gneqq 0$ in Ω, then the strong maximum principle implies that $u_0 > 0$ in Ω, so u_0 is in fact a solution to (P_λ) in the sense of Definition 3.1.

Concerning the existence for $\lambda < 0$, we argue by approximation as follows. For all $n \in \mathbb{N}$, let us consider the problem

$$\begin{cases} -\Delta u_n = \lambda u_n + \mu(x)\dfrac{|\nabla u_n|^q}{u_n^\alpha} + T_n(f(x)) & \text{in } \Omega, \\ u_n > 0 & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

Since (3.1) and (3.2) hold, we know from [29] that there exists a solution $u_n \in H_0^1(\Omega)\cap L^\infty(\Omega)$ to (3.3) for all n. Notice now that

$$-\Delta u_n \le \mu(x)\frac{|\nabla u_n|^q}{u_n^\alpha} + f(x) \text{ in } \Omega.$$

Hence, Theorem 2.1 applies (see Remark 3.3) and yields

$$u_n \le u_0 \le \|u_0\|_{L^\infty(\Omega)} \text{ in } \Omega.$$

In other words, $\{u_n\}$ is bounded in $L^\infty(\Omega)$. By taking u_n as test function in the weak formulation of (3.3), we immediately deduce that $\{u_n\}$ is also bounded in $H_0^1(\Omega)$. Hence, there exists $u \in H_0^1(\Omega)\cap L^\infty(\Omega)$ such that, passing to a subseqence, $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$ and $u_n \to u$ strongly in $L^p(\Omega)$ for any $p \in [1,\infty)$.

Observe also that, again by comparison, $u_n \ge z$ for all n, where $z \in H_0^1(\Omega)\cap L^\infty(\Omega)$ is the unique solution to

$$\begin{cases} -\Delta z = \lambda z + T_1(f(x)) & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega. \end{cases}$$

Now, the strong maximum principle applied on z implies that

$$\forall \omega \subset\subset \Omega \quad \exists c_\omega > 0: \quad u_n \ge c_\omega \quad \text{a.e. in } \omega, \quad \forall n.$$

Therefore, $\{-\Delta u_n\}$ is bounded in $L^1_{\mathrm{loc}}(\Omega)$. Thus, by virtue of [44, Theorem 2.1], $\nabla u_n \to \nabla u$ strongly in $L^q(\Omega)^N$, up to a subsequence. The convergences we have proved about $\{u_n\}$ and $\{\nabla u_n\}$ are enough to pass to the limit in (3.3). The proof is standard, we refer to the proof of [1, Proposition 5.2] for further details. In sum, u is a solution to (P_λ).

The uniqueness of u is a direct consequence of Theorem 2.1 and Remark 3.3. □

Remark 3.5. For the sake of simplicity, we have assumed (H1) in Proposition 3.4. Nevertheless, as it has been shown in the proof, the condition $\mu \ge \mu_0$ is not needed, only $\mu \ge 0$ is sufficient.

Next result shows that, if $\alpha = 0$, then the existence of solution to (P_0) may fail if f or μ are too large in some sense, in contrast to the case $\alpha > 0$. Thus, the smallness assumption in Proposition 3.4 is justified. This result is basically contained in [10, Theorem 2.1]. We include the statement and proof in our context for completeness.

Proposition 3.6. *Assume that* (H1) *holds with $\alpha = 0$, and suppose that (P_λ) admits a solution for some $\lambda \ge 0$. Then,*

$$\int_\Omega f(x)\phi^{q'} \le \int_\Omega \frac{|\nabla\phi|^{q'}}{((q-1)\mu(x))^{\frac{1}{q-1}}} \quad \forall 0 \le \phi \in W_0^{1,q'}(\Omega)\cap L^\infty(\Omega).$$

Proof. Let u be a solution to (P_λ), and let $0 \le \phi \in W_0^{1,q'}(\Omega) \cap L^\infty(\Omega)$. Since $q' > 2$, then $\phi^{q'} \in H_0^1(\Omega) \cap L^\infty(\Omega)$, so it can be taken as test function in the weak formulation of (P_λ) to obtain, after using Young's inequality, that

$$\begin{aligned}\int_\Omega \left(\lambda u + \mu(x)|\nabla u|^q + f(x)\right)\phi^{q'} &= \int_\Omega \nabla u \nabla(\phi^{q'}) = q' \int_\Omega \phi^{q'-1}\nabla u \nabla \phi \\ &\le \int_\Omega \mu(x)|\nabla u|^q \phi^{q'} + \int_\Omega \frac{|\nabla \phi|^{q'}}{((q-1)\mu(x))^{\frac{1}{q-1}}}.\end{aligned}$$

Hence, it is now clear that the result follows. □

Our aim in the next two subsections is to prove, for a fixed $\lambda_0 > 0$, an L^∞ estimate for the solutions to (P_λ) for all $\lambda > \lambda_0$. Such an estimate implies that zero is the only possible bifurcation point from infinity to problem (P_λ). This fact will be the key to prove multiplicity of solutions to (P_λ) for $\lambda > 0$ small enough.

3.1 A priori L^p estimates

This subsection is devoted to proving an L^p estimate on the supersolutions to (P_λ) for $\lambda > 0$. The techniques employed here have been taken from [16].

The first result of the subsection provides an apparently weak local estimate on the solutions to (P_λ). Notwithstanding, this is the starting point for proving the L^∞ estimate we are aiming at. Concerning the proof, we will argue similarly as in Proposition 3.6.

Lemma 3.7. *Assume that* (H1) *holds. Then, for every* $\lambda_0 > 0$ *and* $\omega \subset\subset \Omega$ *there exists* $C > 0$ *such that*

$$\int_\omega u \le C. \tag{3.4}$$

for every supersolution u to (P_λ) with $\lambda > \lambda_0$.

Proof. Let $\phi \in C_c^1(\Omega)$ be such that $\omega \subset\subset \operatorname{supp}(\phi)$, $0 \le \phi \le 1$ in Ω and $\phi = 1$ in ω. Taking $\phi^\beta \in C_c^1(\Omega)$ for some $\beta > 1$ as test function in (P_λ) and using Young's inequality twice we obtain that

$$\begin{aligned}\int_\Omega \left(\lambda u + \mu(x)\frac{|\nabla u|^q}{u^\alpha} + f(x)\right)\phi^\beta &\le \int_\Omega \nabla u \nabla(\phi^\beta) = \beta \int_\Omega \phi^{\beta-1}\nabla u \nabla \phi \\ &\le \frac{\mu_0}{2}\int_\Omega \frac{|\nabla u|^q}{u^\alpha}\phi^\beta + C\int_\Omega \frac{|\nabla(\phi^\beta)|^{q'}}{\phi^{\beta(q'-1)}} u^{\frac{\alpha}{q-1}} \\ &\le \frac{\mu_0}{2}\int_\Omega \frac{|\nabla u|^q}{u^\alpha}\phi^\beta + \frac{\lambda_0}{2}\int_\Omega u\phi^\beta + C\int_\Omega \left(\frac{|\nabla \phi|}{\phi}\right)^{\frac{q}{q-1-\alpha}}\phi^\beta.\end{aligned}$$

Taking $\beta = \frac{q}{q-1-\alpha}$, the last term in the previous inequality is bounded. Therefore,

$$\int_\Omega \left(\lambda_0 u + \mu_0 \frac{|\nabla u|^q}{u^\alpha} + f(x)\right)\phi^\beta \le \frac{\mu_0}{2}\int_\Omega \frac{|\nabla u|^q}{u^\alpha}\phi^\beta + \frac{\lambda_0}{2}\int_\Omega u\phi^\beta + C,$$

so (3.4) follows by taking into account that $\phi = 1$ in ω. □

The following is a slightly more general version of [41, Lemma 3.2].

Lemma 3.8. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with boundary of class $\mathcal{C}^2$, and let $0 \le h \in L^1(\Omega)$ and $v \in H^1(\Omega)$ be such that $v^- \in H_0^1(\Omega)$ and $-\Delta v \ge h$ in Ω. Then, there exists a constant $C > 0$ depending only on Ω such that*

$$\frac{v}{\delta} \ge C \int_\Omega \delta h \quad \text{a.e. in } \Omega.$$

Proof. Let us consider the following problem for all $n \in \mathbb{N}$:

$$\begin{cases} -\Delta v_n = T_n(h(x)), & x \in \Omega, \\ v_n = 0, & x \in \partial\Omega, \end{cases}$$

It is well-known that it has a unique solution $v_n \in C_0^{1,\nu}(\overline{\Omega})$ for all $\nu \in (0,1)$. Moreover, [41, Lemma 3.2] implies that

$$v_n(x) \ge C\delta(x) \int_\Omega \delta T_n(h) \quad \forall x \in \Omega,$$

for some $C > 0$ depending only on Ω. In particular, it does not depend on n.

On the other hand, by comparison, it is clear that $v_n \le v$ a.e. in Ω, so

$$v \ge C\delta \int_\Omega \delta T_n(h) \quad \text{a.e. in } \Omega.$$

We conclude the proof by letting n tend to infinity. □

Next lemma is an immediate consequence of Lemma 3.8.

Lemma 3.9. *Assume that* (H1) *holds. Then, there exists $C > 0$ such that*

$$u(x) \ge C\delta(x) \int_\Omega \left(\lambda u + \mu(y)\frac{|\nabla u|^q}{u^\alpha} + f(y)\right)\delta(y)dy \quad \text{a.e. } x \in \Omega, \tag{3.5}$$

for every supersolution u to (P_λ) with $\lambda > 0$.

Combining Lemmas 3.7 and 3.9 we obtain in the following result some estimates in weighted Lebesgue spaces.

Lemma 3.10. *Assume that* (H1) *holds. Then, for every $\lambda_0 > 0$ there exists $C > 0$ such that*
1. $\|u\|_{L^p(\Omega,\delta)} \le C \quad \forall p \in \left[1, \frac{N+1}{N-1}\right)$,
2. $\left\|\frac{|\nabla u|^q}{u^\alpha}\right\|_{L^1(\Omega,\delta)} = C\||\nabla u^{1-\frac{\alpha}{q}}|\|_{L^q(\Omega,\delta)} \le C,$

for every supersolution u to (P_λ) with $\lambda > \lambda_0$.

Proof. Integrating both sides of inequality (3.5) over any open set $\omega \subset\subset \Omega$ and using the estimate (3.4) we deduce that

$$\int_\Omega (-\Delta u)\delta = \int_\Omega \left(\lambda u + \mu(x)\frac{|\nabla u|^q}{u^\alpha} + f(x)\right)\delta \le C\left(\int_\omega u\right)\left(\int_\omega \delta\right)^{-1} \le C.$$

In particular,

$$\int_\Omega \frac{|\nabla u|^q}{u^\alpha}\delta \le C,$$

and this is equivalent to item (2). Regarding item (1), observe that

$$\|\Delta u\|_{L^1(\Omega,\delta)} \le C.$$

Hence, by [42, Proposition 2.2] we obtain directly item (1). □

We finish the subsection with the best L^p estimate for supersolutions that we obtain with these techniques.

Lemma 3.11. *Assume that* (H1) *holds. Then, for every $\lambda_0 > 0$ there exists $C > 0$ such that*

$$\|u\|_{L^m(\Omega)} \le C \tag{3.6}$$

for every supersolution u to (P_λ) with $\lambda > \lambda_0$, where $m = \frac{(q-\alpha)N}{N-q+1} \in (q-\alpha, (q-\alpha)^\star)$.

Proof. Let us denote $v = u^{1-\frac{\alpha}{q}}$. Since $1 - \frac{\alpha}{q} > \frac{1}{2}$, we can argue as in [1, Lemma 2.6] to prove that $v \in H_0^1(\Omega)$. Then, [16, Proposition 2] implies that

$$\int_\Omega v^q \delta^{-(q-1)} \le C \left(\int_\Omega v\delta \right)^q + C \left(\int_\Omega |\nabla v|^q \delta \right),$$

and

$$\left(\int_\Omega v^{q^\star} \delta^{\frac{N}{N-q}} \right)^{q/q^\star} \le C \left(\int_\Omega v\delta \right)^q + C \left(\int_\Omega |\nabla v|^q \delta \right).$$

Hence, by Lemma 3.10 we derive that

$$\int_\Omega v^q \delta^{-(q-1)} \le C \quad \text{and} \quad \int_\Omega v^{q^\star} \delta^{\frac{N}{N-q}} \le C. \tag{3.7}$$

Now, [16, Lemma 3] implies that

$$\int_\Omega v^b \delta^y \le C \left(\int_\Omega v^q \delta^{-(q-1)} \right)^\theta \left(\int_\Omega v^{q^\star} \delta^{\frac{N}{N-q}} \right)^{1-\theta}, \tag{3.8}$$

where

$$b = \frac{qN}{N-q+1}, \quad \theta = \frac{q^\star - b}{q^\star - q} \in (0,1) \quad \text{and} \quad y = \frac{N}{N-q} - \frac{(q^\star - b)(q - 1 + \frac{N}{N-q})}{q^\star - q}.$$

It is easy to check that, in fact, $y = 0$. Therefore, recalling that $m = b\left(1 - \frac{\alpha}{q}\right)$, by (3.8) and (3.7) we conclude that

$$\int_\Omega v^b = \int_\Omega u^m \le C,$$

and the result holds true. □

3.2 A priori L^∞ estimates

In this subsection we will show how to obtain L^∞ estimates on the solutions to (P_λ) for $\lambda > 0$ by combining the L^p estimate given by Lemma 3.11 and a bootstrapp argument. We will make use of several results in [9]. In fact, the ideas in such a paper will be used also to derive some new results which provide analogous estimates in our singular framework.

We start the subsection with the easier case $\alpha = 0$, which is interesting itself; we will deal with the singular case $\alpha \in (0, q-1)$ later. Thus we state and prove the following

Proposition 3.12. *Assume that* (H1) *holds with $\alpha = 0$, and consider the sequence $\{Q_n\}$ defined by* (1.2), *i.e.,*

$$Q_n = \begin{cases} 2 & \forall n \le 4, \\ \dfrac{n + 2 - \sqrt{n^2 - 4n - 4}}{4} & \forall n \ge 5. \end{cases}$$

Then, for every $q \in (1, Q_N] \setminus \{2\}$ and every $\lambda_0 > 0$, there exists $C > 0$ such that

$$\|u\|_{L^\infty(\Omega)} \le C \tag{3.9}$$

for every solution u to (P_λ) with $\lambda > \lambda_0$.

Proof. In this proof, C denotes a positive constant independent of u and λ whose value may vary from line to line.

We start by assuming that $1 < q < \frac{N}{N-1}$. Observe that $\frac{N}{N-1} < Q_N$, so $q \le Q_N$ is not a restriction in this case. Let us denote $h(x) = (\lambda + 1)u + f(x)$. Then, u satisfies

$$\begin{cases} u - \Delta u = \mu(x)|\nabla u|^q + h(x), & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases}$$

We know from Lemma 3.11 that $\|u\|_{L^m(\Omega)} \le C$, where $m = \frac{(q-\alpha)N}{N-q+1}$, so $\|h\|_{L^p(\Omega)} \le C$, where $p = \min\{m, p_0\}$. If $m > \frac{N}{2}$, and taking into account that $p_0 > \frac{N}{2}$, then [9, Theorem 5.8, item (i)] implies that $\|u\|_{L^\infty(\Omega)} \le C$.

Let us assume now that $m = \frac{N}{2}$. Then, [9, Theorem 5.8, item (ii)] implies that $\|u\|_{L^p(\Omega)} \le C$ for all $p < \infty$. In particular, $\|h\|_{L^{p_0}(\Omega)} \le C$. Since $p_0 > \frac{N}{2}$, then again item (i) of the same mentioned theorem yields the L^∞ estimate.

Suppose now that $(2^\star)' < m < \frac{N}{2}$. Let us define the sequence $\{m_n\}$ inductively as

$$m_n = m_{n-1}^{\star\star} = \frac{Nm_{n-1}}{N - 2m_{n-1}} \quad \forall n \in \mathbb{N},$$

where $m_0 = m$. This is clearly an increasing sequence. Moreover, using one more time [9, Theorem 5.8, item (iii)], it is easy to see that $\|u\|_{L^{m_n}(\Omega)} \le C$ for $n \in \mathbb{N}$ as long as $m_n < \frac{N}{2}$. In particular, the same holds for h.

Assume by contradiction that $m_n < \frac{N}{2}$ for all $n \in \mathbb{N}$. Since $\{m_n\}$ is increasing and bounded from above, there exists $l \le \frac{N}{2}$ such that, passing to a not relabeled subsequence, $m_n \to l$. Consequently,

$$l = \frac{Nl}{N - 2l}.$$

From this equality we deduce that $l = 0$. But this is a contradiction because $m_0 > 0$ and the sequence is increasing. Therefore, $m_n \ge \frac{N}{2}$ for some $n \in \mathbb{N}$, so the previous cases imply that $\|u\|_{L^\infty(\Omega)} \le C$.

It only remains to consider the case $1 < m \le (2^\star)'$. Now, item (iv) of the same theorem implies that

$$\|(1+u)^{\tau-1}u\|_{L^{2^\star}(\Omega)} \le C, \quad \text{where} \quad \tau = \frac{m(N-2)}{2(N-2m)} = \frac{m^{\star\star}}{2^\star} \le 1.$$

On the other hand, it is straightforward to prove that, for any $a \in (0, 1)$, there exists a constant $b > 0$ such that

$$as^\tau \le \frac{s}{(1+s)^{1-\tau}} + b \quad \forall s \ge 0.$$

Then, with $m_n = m_{n-1}^{\star\star}$ and $m_0 = m$, as before,

$$\|u\|_{L^{m_1}(\Omega)} = \|u\|_{L^{2^\star\tau}(\Omega)} \le C(\|(1+u)^{\tau-1}u\|_{L^{2^\star}(\Omega)} + 1) \le C.$$

In particular, $\|h\|_{L^{m_1}(\Omega)} \le C$. It can be proved inductively that $\|u\|_{L^{m_n}(\Omega)} \le C$ as long as $m_n \le (2^\star)'$. Arguing as above, we deduce that $\{m_n\}$ is increasing and divergent. Hence, $m_n > (2^\star)'$ for some $n \in \mathbb{N}$, and the proof concludes using the previous cases.

We now turn to the range $\frac{N}{N-1} < q < 2$. The procedure is the same as above, but in this case, instead of Theorem 5.8, one has to apply (a finite number of times) either [9, Theorem 4.9] or [9, Theorem 3.8], depending on the value of q. In both cases, one has to verify in the first step of the bootstrap that $h \in L^{\frac{(q-1)N}{q}}(\Omega)$ so that the hypotheses of both theorems are satisfied. We know by virtue of Lemma 3.11 that $h \in L^m(\Omega)$, so we have to impose that

$$\frac{N(q-1)}{q} \leq \frac{qN}{N-q+1}.$$

One can easily check that the previous inequality is satisfied if and only if $q \leq Q_N$.

It is left to consider the case $q = \frac{N}{N-1}$. Since $\frac{N}{N-1} < Q_N$, we can take $\varepsilon > 0$ small enough so that

$$\frac{N}{N-1} < q + \varepsilon < Q_N.$$

Moreover, we have by Young's inequality that

$$\mu(x)|\xi|^q + h(x) \leq \mu(x)|\xi|^{q+\varepsilon} + h_\varepsilon(x) \quad \forall \xi \in \mathbb{R}^N, \text{ a.e. } x \in \Omega,$$

where $h(x) = (\lambda+1)u + f(x)$ and $h_\varepsilon(x) = h(x) + C_\varepsilon$ for some $C_\varepsilon > 0$. Therefore, the previous case can be applied and the proof concludes. □

We deal now with the singular case. For this purpose, it is necessary to derive results similar to the ones from [9] mentioned in the previous proof, but valid for singular equations. Even though our results are not proper extensions in the whole generality (as in [9] the solutions are weaker than ours and the terms in their equation are not explicit and only satisfy growth restrictions), they are new in considering singular terms.

The mentioned results will be concerned with the following auxiliary problem:

$$\begin{cases} \beta u - \Delta u = \mu(x)\dfrac{|\nabla u|^q}{u^\alpha} + h(x) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.10}$$

where the parameters satisfy

$$1 < q < 2, \quad \alpha \in [0, q-1), \quad \beta > 0, \quad 0 \lneqq \mu \in L^\infty(\Omega). \tag{3.11}$$

For any $p \in \left(1, \frac{N}{2}\right)$, let us denote

$$\sigma = \frac{(N-2)p}{2(N-2p)} \in \left(\frac{1}{2}, +\infty\right). \tag{3.12}$$

The following result provides estimates on solutions to (3.10) when q is large and h has enough summability.

Proposition 3.13. *Assume that q, α, β, μ satisfy* (3.11)*, and assume in addition that*

$$q > \frac{N}{N-1}.$$

Then, for all $M > 0$ and $p \geq \frac{N(q-1-\alpha)}{q-2\alpha}$, $p > 1$, there exists $C > 0$ such that, for any $h \in L^p(\Omega)$ with $\|h\|_{L^p(\Omega)} \leq M$ and for any solution u to problem (3.10)*, the following holds:*

1. *If $p < \frac{2N}{N+2}$, then $\sigma \in \left(\frac{1}{2}, 1\right)$ and $\|u(u+1)^{\sigma-1}\|_{H_0^1(\Omega)} \leq C$, where σ is defined by* (3.12)*;*
2. *if $\frac{2N}{N+2} \leq p < \frac{N}{2}$, then $\sigma \geq 1$ and $\|u\|_{H_0^1(\Omega)} + \|u^\sigma\|_{H_0^1(\Omega)} \leq C$, where σ is defined by* (3.12)*;*
3. *if $p = \frac{N}{2}$, then $\|u^\tau\|_{H_0^1(\Omega)} \leq C$ for all $\tau < \infty$, and*
4. *if $p > \frac{N}{2}$, then $\|u\|_{L^\infty(\Omega)} \leq C$.*

Proof. **Proof of** (1).

First of all, note that $\sigma \in \left(\frac{1}{2}, 1\right)$ if and only if $p \in \left(1, \frac{2N}{N+2}\right)$.

Observe also that, if $1+\frac{2}{N}+\frac{N-2}{N}\alpha \le q < 2$, then $\frac{N(q-1-\alpha)}{q-2\alpha} \ge \frac{2N}{N+2}$, so the condition in item (1) may be fulfilled only if

$$\frac{N}{N-1} < q < 1+\frac{2}{N}+\frac{N-2}{N}\alpha.$$

We will assume consequently that q belongs to such an interval. In fact, we will divide the proof of this item into several steps, considering different ranges for p and q. It can be easily checked that each of these ranges is nonempty.

Case 1: $\frac{N}{N-1} < q < 1+\frac{2}{N}$ and $\frac{N(q-1-\alpha)}{q-2\alpha} \le p \le \frac{N(q-1)}{q}$, $p > 1$.

In this case, there exists $\theta \in \left[0, \frac{N-1}{N-2}\left(q-\frac{N}{N-1}\right)\right) \cap [0,\alpha]$ such that $p = \frac{N(q-1-\theta)}{q-2\theta}$. Then, it is clear that the following relation is satisfied:

$$\frac{2}{2-q}(2\sigma-1-\theta-q(\sigma-1)) = 2^\star\sigma. \tag{3.13}$$

Let us now consider the following functions defined for every $t \ge 0$:

$$\phi(t) = \frac{1}{(\zeta+t)^{1-\sigma}}\left(\frac{t}{\zeta+t}\right)^{\frac{1}{2}},$$

$$\Phi_1(t) = \int_0^t \phi(s)ds,$$

$$\Phi_2(t) = \int_0^t \phi(s)^2 ds,$$

where $\zeta > 0$ will be fixed later. First of all observe that

$$\nabla v \nabla \Phi_2(v) = |\nabla \Phi_1(v)|^2$$

for any $v \in H_0^1(\Omega)$. Moreover, using (3.13) and also that $2\sigma-1 = \frac{2^\star\sigma}{p'}$, it can be proved respectively that

$$\left(t^{-\theta}\phi(t)^{-q}\Phi_2(t)\right)^{\frac{2}{2-q}} \le C\left(\Phi_1(t)^{2^\star} + \zeta^{2^\star\sigma}\right) \quad \forall t \ge 0. \tag{3.14}$$

and

$$\Phi_2(t) \le C\Phi_1(t)^{\frac{2^\star}{p'}} \quad \forall t \ge 0. \tag{3.15}$$

For $k > 0$, let us take $\Phi_2(G_k(u)) \in H_0^1(\Omega) \cap L^\infty(\Omega)$ as test function in the weak formulation of (3.10), so that we obtain

$$\beta\int_\Omega u\Phi_2(G_k(u)) + \int_\Omega |\nabla\Phi_1(G_k(u))|^2 = \int_\Omega \left(\mu(x)\frac{|\nabla G_k(u)|^q}{u^\alpha} + h(x)\right)\Phi_2(G_k(u)). \tag{3.16}$$

Let us now estimate the nonlinear term. Thanks to (3.14) we derive that

$$\begin{aligned}
\int_\Omega \mu(x)\frac{|\nabla G_k(u)|^q}{u^\alpha}\Phi_2(G_k(u)) &\le \frac{\|\mu\|_{L^\infty(\Omega)}}{k^{\alpha-\theta}}\int_{\{u\ge k\}} |\nabla\Phi_1(G_k(u))|^q \frac{\Phi_2(G_k(u))}{G_k(u)^\theta\phi(G_k(u))^q} \\
&\le C\left(\int_\Omega |\nabla\Phi_1(G_k(u))|^2\right)^{\frac{q}{2}}\left(\int_{\{u\ge k\}}\left(\frac{\Phi_2(G_k(u))}{G_k(u)^\theta\phi(G_k(u))^q}\right)^{\frac{2}{2-q}}\right)^{1-\frac{q}{2}} \\
&\le C\left(\int_\Omega |\nabla\Phi_1(G_k(u))|^2\right)^{\frac{q}{2}}\left(\int_\Omega\left(\Phi_1(G_k(u))^{2^\star} + \zeta^{2^\star\sigma}\right)\right)^{1-\frac{q}{2}} \\
&\le C\left(\int_\Omega |\nabla\Phi_1(G_k(u))|^2\right)^{\frac{q}{2}}\left(\left(\int_\Omega |\nabla\Phi_1(G_k(u))|^2\right)^{\frac{2^\star}{2}(1-\frac{q}{2})} + \zeta^{2^\star\sigma(1-\frac{q}{2})}\right).
\end{aligned}$$

We now focus on the last term in (3.16). Using (3.15) we deduce that

$$\begin{aligned}\int_\Omega |h(x)|\Phi_2(G_k(u)) &= \int_{\{|h(x)|\le\beta u\}} |h(x)|\Phi_2(G_k(u)) + \int_{\{|h(x)|>\beta u\}} |h(x)|\Phi_2(G_k(u))\\ &\le \beta\int_\Omega u\Phi_2(G_k(u)) + C\int_{\{|h(x)|>\beta k\}} |h(x)|\Phi_1(G_k(u))^{\frac{2^\star}{p'}}\\ &\le \beta\int_\Omega u\Phi_2(G_k(u)) + C\left(\int_{\{|h(x)|\ge\beta k\}} |h(x)|^p\right)^{\frac{1}{p}}\left(\int_\Omega \Phi_1(G_k(u))^{2^\star}\right)^{\frac{1}{p'}}\\ &\le \beta\int_\Omega u\Phi_2(G_k(u)) + C\left(\int_{\{|h(x)|\ge\beta k\}} |h(x)|^p\right)^{\frac{1}{p}}\left(\int_\Omega |\nabla\Phi_1(G_k(u)|^2\right)^{\frac{2^\star}{2p'}}.\end{aligned}$$

If we denote $Y_k = \|\Phi_1(G_k(u))\|_{H_0^1(\Omega)}$, we have proved so far that

$$Y_k^2 \le CY_k^q\left(Y_k^{2^\star(1-\frac{q}{2})} + \zeta^{2^\star\sigma(1-\frac{q}{2})}\right) + C\|h\chi_{\{|h(x)|\ge\beta k\}}\|_{L^p(\Omega)}Y_k^{\frac{2^\star}{p'}}.$$

Hence, using Young's inequality we obtain that

$$\frac{1}{2}Y_k^2 \le CY_k^{q+2^\star(1-\frac{q}{2})} + C\zeta^{2^\star\sigma} + C\|h\chi_{\{|h(x)|\ge\beta k\}}\|_{L^p(\Omega)}^{2\sigma},$$

or equivalently,

$$C_1Y_k^2 - C_2Y_k^{q+2^\star(1-\frac{q}{2})} \le \zeta^{2^\star\sigma} + \|h\chi_{\{|h(x)|\ge\beta k\}}\|_{L^p(\Omega)}^{2\sigma}, \tag{3.17}$$

for some $C_1, C_2 > 0$ independent of k and ζ.

Let us define the function $F : [0, +\infty) \to \mathbb{R}$ by

$$F(Y) = C_1Y^2 - C_2Y^{q+2^\star(1-\frac{q}{2})} \quad \forall Y \ge 0.$$

Since $q < 2$, it easy to see that

$$2 < q + 2^\star\left(1 - \frac{q}{2}\right).$$

This means that F is positive near zero, negative far from zero, and has a unique maximum $F^\star > 0$ with a corresponding unique maximizer $Z^\star > 0$.

We now choose $\zeta = \min\left\{1, \left(\frac{F^\star}{2}\right)^{\frac{1}{2^\star\sigma}}\right\}$. Thus,

$$\max_{Y\ge0}(F(Y) - \zeta^{2^\star\sigma}) = F^\star - \zeta^{2^\star\sigma} \ge \frac{F^\star}{2} > 0.$$

Let us now consider

$$k^\star = \inf\left\{k > 0 : \|h\chi_{\{|h(x)|\ge\beta k\}}\|_{L^p(\Omega)}^{2\sigma} < F^\star - \zeta^{2^\star\sigma}\right\}.$$

Hence, for any $\rho > 0$, the equation $F(Y) = \zeta^{2^\star\sigma} + \|h\chi_{\{h(x)\ge\beta(k^\star+\rho)\}}\|_{L^p(\Omega)}^{2\sigma}$ has two roots Z_1 and Z_2 such that $Z_1 < Z^\star < Z_2$. By virtue of inequality (3.17), it holds that for every $k \ge k^\star + \rho$, either $Y_k \le Z_1$ or $Y_k \ge Z_2$. But the function $k \mapsto Y_k$ is continuous and tends to zero as k tends to infinity. Therefore,

$$Y_{k^\star+\rho} \le Z_1 < Z^\star.$$

If we let now ρ tend to zero, we obtain that

$$Y_{k^\star} = \|\Phi_1(G_{k^\star}(u))\|_{H_0^1(\Omega)} \le Z^\star.$$

Notice that

$$\begin{aligned}\|\Phi_1(G_k(u))\|^2_{H_0^1(\Omega)} &= \int_\Omega \frac{|\nabla G_k(u)|^2 G_k(u)}{(\zeta + G_k(u))^{2(1-\sigma)+1}} \geq \int_\Omega \frac{|\nabla G_k(u)|^2 G_k(u)}{(1 + G_k(u))^{2(1-\sigma)+1}}\\ &\geq \int_\Omega \frac{|\nabla u|^2 (u-k)}{(1+u-k)^{2(1-\sigma)+1}} \chi_{\{u\geq k+1\}}\\ &\geq \frac{1}{2}\int_\Omega \frac{|\nabla u|^2}{(1+u-k)^{2(1-\sigma)}} \chi_{\{u\geq k+1\}}\\ &\geq \frac{1}{2^{2(1-\sigma)+1}}\int_\Omega \frac{|\nabla G_{k+1}(u)|^2}{(G_{k+1}(u)+1)^{2(1-\sigma)}}.\end{aligned}$$

Hence, we have that

$$\int_\Omega \frac{|\nabla G_k(u)|^2}{(G_k(u)+1)^{2(1-\sigma)}} \leq C \quad \forall k \geq k^* + 1. \tag{3.18}$$

For $k \geq k^* + 1$, estimate (3.18) implies that

$$\begin{aligned}\left\|\frac{u}{(1+u)^{1-\sigma}}\right\|^2_{H_0^1(\Omega)} &= \int_\Omega \frac{|\nabla u|^2}{(1+u)^{2(1-\sigma)}}\left(\frac{1+\sigma u}{1+u}\right)^2\\ &= \int_\Omega \frac{|\nabla G_k(u)|^2}{(1+u)^{2(1-\sigma)}}\left(\frac{1+\sigma u}{1+u}\right)^2 + \int_\Omega \frac{|\nabla T_k(u)|^2}{(1+u)^{2(1-\sigma)}}\left(\frac{1+\sigma u}{1+u}\right)^2\\ &\leq C + \int_\Omega \frac{|\nabla T_k(u)|^2}{(1+u)^{2(1-\sigma)}}\left(\frac{1+\sigma u}{1+u}\right)^2.\end{aligned}$$

We claim now that

$$\int_\Omega \frac{|\nabla T_k(u)|^2}{(1+u)^{2(1-\sigma)}}\left(\frac{1+\sigma u}{1+u}\right)^2 \leq C. \tag{3.19}$$

Indeed, let us define the real functions for all $t \geq 0$:

$$\begin{aligned} z(t) &= \frac{1}{(1+t)^{2(1-\sigma)}}\left(\frac{1+\sigma t}{1+t}\right)^2,\\ y(t) &= \frac{1}{t}\int_0^t z(s)ds.\end{aligned}$$

It is easy to see that

$$ty'(t) + y(t) = z(t) \quad \forall t \geq 0,$$

and also that

$$y(t) \leq Cz(t) \quad \forall t \geq 0, \text{ for some } C > 0.$$

Now we take $T_k(u)y(u)$ as test function in the weak formulation of (3.10) and get

$$\int_\Omega y(u)|\nabla T_k(u)|^2 + \int_\Omega T_k(u)y'(u)|\nabla u|^2 = \int_\Omega \left(\mu(x)\frac{|\nabla u|^q}{u^\alpha} + h(x) - \beta u\right) T_k(u)y(u). \tag{3.20}$$

Concerning the left hand side of (3.20), observe that

$$\int_\Omega y(u)|\nabla T_k(u)|^2 + \int_\Omega T_k(u)y'(u)|\nabla u|^2 = \int_\Omega z(u)|\nabla T_k(u)|^2 + k\int_\Omega y'(u)|\nabla G_k(u)|^2, \tag{3.21}$$

where, by virtue of (3.18),

$$-k\int_\Omega y'(u)|\nabla G_k(u)|^2 \le \int_\Omega \frac{ky(u)}{u}|\nabla G_k(u)|^2 \le C\int_\Omega z(u)|\nabla G_k(u)|^2 \le C. \tag{3.22}$$

Gathering (3.20), (3.21) and (3.22) together we deduce that

$$\int_\Omega z(u)|\nabla T_k(u)|^2 \le C(k^\star+1)\left(\int_\Omega y(u)|\nabla u|^q + 1\right) \tag{3.23}$$

$$\le C(k^\star+1)\left(\int_\Omega z(u)|\nabla T_k(u)|^q + \int_\Omega z(u)|\nabla G_k(u)|^q + 1\right). \tag{3.24}$$

We will show now that there exists $k_0 > 0$ independent of $\|h\|_{L^p(\Omega)}$ such that $k^\star \le k_0$. Indeed, the absolute continuity of the integral implies that there exists $\rho > 0$ such that, if $|\{|h(x)| \ge \beta k_0\}| < \rho$ for some $k_0 > 0$, then$\|h\chi_{\{|h(x)|\ge\beta k_0\}}\|^{2\sigma}_{L^p(\Omega)} < F^\star - \zeta^{2^\star\sigma}$, i.e., $k^\star \le k_0$. Observe that, if $k_0 > \frac{M|\Omega|^{\frac{1}{p'}}}{\beta\rho}$, where $\|h\|_{L^p(\Omega)} \le M$, then $|\{|h(x)| \ge \beta k_0\}| < \rho$ and k_0 does not depend on $\|h\|_{L^p(\Omega)}$, as we wanted to show.

Therefore, we can estimate $k^\star$ in (3.24) and, by virtue of (3.18), we obtain that

$$\int_\Omega z(u)|\nabla T_k(u)|^2 \le C\left(\int_\Omega z(u)|\nabla T_k(u)|^q + 1\right).$$

We finally arrive at (3.19) by using Young's inequality and by the fact that z is a bounded function. This concludes the proof of Case 1.

Case 2: $\frac{N}{N-1} < q < 1+\frac{2}{N}$ and $\frac{N(q-1)}{q} < p < \frac{2N}{N+2}$.

Observe that, in this range, one has in particular that $\|h\|_{L^r(\Omega)} \le |\Omega|^{\frac{p-r}{p}}M$, where $r = \frac{N(q-1)}{q}$. Then, Case 1 can be applied for $\theta = 0$. We will use this fact later. Let us also denote $\sigma_r = \frac{(N-2)r}{2(N-2r)} = \frac{(N-2)(q-1)}{2(2-q)} \in \left(\frac{1}{2}, 1\right)$.

Recalling the definitions of ϕ, Φ_1 and Φ_2 in the previous case, for some $k > 0$ we take $\Phi_2(G_k(u))$ as test function in the weak formulation of (3.10), so that we obtain

$$\int_\Omega |\nabla\Phi_1(G_k(u))|^2 \le C\int_\Omega \left(\frac{|\nabla G_k(u)|^q}{u^\alpha} + |h(x)|\right)\Phi_2(G_k(u)). \tag{3.25}$$

It can be easily proved that

$$\Phi_2(t) \le C\phi(t)\Phi_1(t) \quad \forall t \ge 0,$$

for some $C > 0$. Thus, using this inequality in the singular term of (3.25), we deduce that

$$\begin{aligned}\int_\Omega \frac{|\nabla G_k(u)|^q}{u^\alpha}\Phi_2(G_k(u)) &\le C\int_\Omega |\nabla G_k(u)|^{q-1}|\nabla G_k(u)|\phi(G_k(u))\Phi_1(G_k(u))\\ &\le C\left(\int_\Omega |\nabla G_k(u)|^{N(q-1)}\right)^{\frac{1}{N}}\left(\int_\Omega |\nabla G_k(u)|^2\phi(G_k(u))^2\right)^{\frac{1}{2}}\left(\int_\Omega \Phi_1(G_k(u))^{2^*}\right)^{\frac{1}{2^*}} \qquad (3.26)\\ &\le C\left(\int_\Omega |\nabla G_k(u)|^{N(q-1)}\right)^{\frac{1}{N}}\int_\Omega |\nabla\Phi_1(G_k(u))|^2.\end{aligned}$$

Now we claim that

$$\int_\Omega |\nabla G_k(u)|^{N(q-1)} \le C$$

for some $k > 0$ large enough. Indeed, since $q < 1 + \frac{2}{N}$, we can apply Hölder's inequality with exponent $\frac{2}{N(q-1)} > 1$ and obtain that, for any $k > 0$,

$$\int_\Omega |\nabla G_k(u)|^{N(q-1)} \le C \int_\Omega \left|\nabla \frac{G_k(u)}{(1+G_k(u))^{1-\sigma_r}}\right|^{N(q-1)} (1+G_k(u))^{(1-\sigma_r)N(q-1)}$$

$$\le C \left(\int_\Omega \left|\nabla \frac{G_k(u)}{(1+G_k(u))^{1-\sigma_r}}\right|^2\right)^{\frac{N(q-1)}{2}} \left(\int_\Omega (1+G_k(u))^{\left(\frac{2}{N(q-1)}\right)'(1-\sigma_r)N(q-1)}\right)^{1-\frac{N(q-1)}{2}}.$$

Therefore, by Case 1 and Sobolev's inequality,

$$\int_\Omega |\nabla G_k(u)|^{N(q-1)} \le C + C\left(\int_\Omega |\nabla G_k(u)|^{N(q-1)}\right)^{1-\sigma_r}.$$

Hence, the fact that $\sigma_r < 1$ implies that

$$\int_\Omega |\nabla G_k(u)|^{N(q-1)} \le C \quad \forall k \ge k^\star + 1,$$

and the proof of the claim is done. As a consequence, it can be shown, again by virtue of the absolute continuity of the integral, that the limit

$$\lim_{k\to\infty}\left(\int_\Omega |\nabla G_k(u)|^{N(q-1)}\right)^{\frac{1}{N}} = 0,$$

is uniform in u. Hence, from (3.26) we deduce that there exists $k_0 > 0$ independent of u such that

$$\int_\Omega \frac{|\nabla G_k(u)|^q}{u^\alpha}\Phi_2(G_k(u)) \le \frac{1}{2}\int_\Omega |\nabla \Phi_1(G_k(u))|^2 \quad \forall k \ge k_0.$$

Then, we derive from (3.25) that

$$\int_\Omega |\nabla \Phi_1(G_k(u))|^2 \le C\int_\Omega |h(x)|\Phi_2(G_k(u)) \quad \forall k \ge k_0.$$

By virtue of (3.15) we immediately obtain the estimate

$$\int_\Omega |\nabla \Phi_1(G_{k_0}(u))|^2 \le C.$$

We conclude this case similarly as Case 1.

Case 3: $1 + \frac{2}{N} \le q < 1 + \frac{2}{N} + \frac{N-2}{N}\alpha$ and $\frac{N(q-1-\alpha)}{q-2\alpha} \le p \le \frac{2N}{N-2}$, $p > 1$.

In this case, it is clear that $\frac{2N}{N+2} \le \frac{N(q-1)}{q}$. Thus, the proof of Case 1 can be reproduced here.

We conclude this way the proof of item (1).

Proof of (2)**.**

Case 1: $1 + \frac{2}{N} \le q < 2$ and $\frac{2N}{N+2} \le p \le \frac{N(q-1)}{q}$.

In this case, there exists $\theta \in \left[0, \left(q-1-\frac{2}{N}\right)\frac{N}{N-2}\right] \cap [0,\alpha]$ such that $p = \frac{N(q-1-\theta)}{q-2\theta}$. Then, (3.13) holds.

Now, for $k > 0$, let us take $G_k(u)^{2\sigma-1}$ as test function in the weak formulation of (3.10). Notice that this choice is valid since $\sigma > 1$. Then, following the arguments of the proof of Case 1 of item (1) we obtain that

$$\|G_{k^\star}(u)^\sigma\|_{H_0^1(\Omega)} \le C,$$

where $k^\star = \inf\left\{k > 0 : \|h\chi_{\{|h(x)|\geq\beta k\}}\|_{L^p(\Omega)} < F^\star\right\}$ and $F^\star > 0$ is the unique maximum of the function $F(Y) = CY^{2-\frac{2^\star}{p'}} - Y^{q+2^\star(1+\frac{q}{2})-\frac{2^\star}{p'}}$, $Y \geq 0$, for some $C > 0$.

Observe that

$$G_{k^\star}(u) = u - k^\star \geq 1 \quad \text{in the set } \{u \geq k^\star + 1\}.$$

Therefore,

$$\int_\Omega |\nabla G_{k^\star+1}(u)|^2 = \int_\Omega \chi_{\{u\geq k^\star+1\}}|\nabla u|^2 \leq \int_\Omega \chi_{\{u\geq k^\star+1\}}|\nabla u|^2 G_{k^\star}(u)^{2(\sigma-1)}$$
$$\leq \int_\Omega \chi_{\{u\geq k^\star\}}|\nabla u|^2 G_{k^\star}(u)^{2(\sigma-1)} = \frac{1}{\sigma^2}\int_\Omega |\nabla G_{k^\star}(u)^\sigma|^2 \leq C.$$

Now we take $T_{k^\star+1}(u)$ as test function in the weak formulation of (3.10) so we get

$$\int_\Omega |\nabla T_{k^\star+1}(u)|^2 = \int_\Omega \left(\mu(x)\frac{|\nabla u|^q}{u^\alpha} + h(x) - \beta u\right) T_{k^\star+1}(u)$$
$$\leq C(k^\star+1)^{1-\alpha}\int_\Omega |\nabla u|^q + (k^\star+1)\int_\Omega |h(x)|$$
$$\leq C(k^\star+1)\left(\int_\Omega |\nabla T_{k^\star+1}(u)|^q + \int_\Omega |\nabla G_{k^\star+1}(u)|^q + 1\right)$$
$$\leq C(k^\star+1)\left(\int_\Omega |\nabla T_{k^\star+1}(u)|^q + 1\right).$$

Again, the absolute continuity of the integral implies that $k^\star \leq k_0$ for some $k_0 > 0$ independent of $\|h\|_{L^p(\Omega)}$. Thus we can estimate $k^\star$ in the last inequality and, using Young's inequality, deduce that

$$\int_\Omega |\nabla T_{k^\star+1}(u)|^2 \leq C.$$

Summarizing, $\int_\Omega |\nabla u|^2 \leq C$, which proves the first part of item (2). Moreover,

$$\int_\Omega |\nabla u^\sigma|^2 = \int_\Omega |\nabla G_{k^\star}(u)^\sigma|^2 + \int_\Omega |\nabla T_{k^\star}(u)^\sigma|^2$$
$$\leq C + \sigma^2\int_\Omega T_{k^\star}(u)^{2(\sigma-1)}|\nabla T_{k^\star}(u)|^2 \leq C + \sigma^2(k^\star)^{2(\sigma-1)}\int_\Omega |\nabla u|^2 \leq C.$$

Thus, the proof of Case 1 is concluded.

Case 2: $1 + \frac{2}{N} \leq q < 2$ and $\frac{N(q-1)}{q} < p < \frac{N}{2}$.

Let us denote, as above, $r = \frac{N(q-1)}{q}$ and $\sigma_r = \frac{(N-2)(q-1)}{2(2-q)} \geq 1$. It is clear that $\|h\|_{L^r(\Omega)} \leq |\Omega|^{\frac{p-r}{p}}M$ and $\sigma_r = \frac{(N-2)r}{2(N-2r)}$, so Case 1 of item (2) can be applied.

For some $k > 0$, we take $G_k(u)^{2\sigma-1}$ as test function in the weak formulation of (3.10), so we obtain

$$\beta\int_\Omega uG_k(u)^{2\sigma-1} + \frac{2\sigma-1}{\sigma^2}\int_\Omega |\nabla G_k(u)^\sigma|^2 = \int_\Omega \left(\mu(x)\frac{|\nabla G_k(u)|^q}{u^\alpha} + h(x)\right) G_k(u)^{2\sigma-1}. \tag{3.27}$$

In order to estimate the nonlinear term, notice that

$$\frac{q}{2} + \frac{2-q}{2^\star} + \frac{2-q}{N} = 1.$$

Hence, we can use Hölder inequality with those three exponents, and we deduce that

$$\begin{aligned}\int_\Omega \frac{|\nabla G_k(u)|^q}{u^\alpha} G_k(u)^{2\sigma-1} &\le C\int_\Omega |\nabla G_k(u)|^q G_k(u)^{(\sigma-1)q} G_k(u)^{(2-q)\sigma} G_k(u)^{q-1}\\ &\le C\left(\int_\Omega |\nabla G_k(u)^\sigma|^2\right)^{\frac{q}{2}} \left(\int_\Omega G_k(u)^{\sigma 2^\star}\right)^{\frac{2-q}{2^\star}} \left(\int_\Omega G_k(u)^{2^\star\sigma_r}\right)^{\frac{2-q}{N}}\\ &\le C\|G_k(u)\|^{q-1}_{L^{2^\star\sigma_r}(\Omega)} \int_\Omega |\nabla G_k(u)^\sigma|^2.\end{aligned}$$

Now, thanks to Case 1 of item (2) and the absolute continuity of the integral, there exists $k_0 > 0$ independent of u such that

$$C\|G_k(u)\|^{q-1}_{L^{2^\star\sigma_r}(\Omega)} < \frac{2\sigma-1}{\sigma^2} \quad \forall k \ge k_0.$$

Then, from (3.27) we derive that

$$C\int_\Omega |\nabla G_k(u)^\sigma|^2 \le \|h\|_{L^p(\Omega)} \left(\int_\Omega G_k(u)^{(2\sigma-1)p'}\right)^{\frac{1}{p'}} \quad \forall k \ge k_0.$$

Since $(2\sigma-1)p' = 2^\star\sigma$, we conclude that

$$C\int_\Omega |\nabla G_k(u)^\sigma|^2 \le \left(\int_\Omega |\nabla G_k(u)^\sigma|^2\right)^{\frac{2^\star}{2p'}} \quad \forall k \ge k_0.$$

Clearly, $\frac{2^\star}{2p'} = \frac{2\sigma-1}{2\sigma} < 1$, so we deduce that

$$\int_\Omega |\nabla G_{k_0}(u)^\sigma|^2 \le C.$$

Finally, using that u is bounded in $H_0^1(\Omega)$ (from Case 1), we obtain that

$$\begin{aligned}\int_\Omega |\nabla u^\sigma|^2 &= \int_\Omega |\nabla G_{k_0}(u)^\sigma|^2 + \int_\Omega |\nabla T_{k_0}(u)^\sigma|^2\\ &\le C + \sigma^2 \int_\Omega T_{k_0}(u)^{2(\sigma-1)} |\nabla T_{k_0}(u)|^2 \le C + \sigma^2 k_0^{2(\sigma-1)} \int_\Omega |\nabla u|^2 \le C.\end{aligned}$$

This proves Case 2.

Case 3: $\frac{N}{N-1} < q < 1 + \frac{2}{N}$ and $\frac{2N}{N+2} \le p < \frac{N}{2}$.

Here one can argue as in Case 2 of the proof of item (1), but considering this time $\phi(s) = s^{\sigma-1}$ for all $s \ge 0$.

Proof of (3)**.**

Since $\sigma = \frac{(N-2)p}{2(N-2p)} \to +\infty$ as $p \to \frac{N}{2}$, item (3) is a clear consequence of item (2).

Proof of (4)**.**

Let us take $G_k(u)$ as test function in the weak formulation of (3.10) for some $k > 0$, so we obtain this time, removing the term with β,

$$\int_\Omega |\nabla G_k(u)|^2 \le C\int_\Omega |\nabla G_k(u)|^q G_k(u) + \int_\Omega |h(x)| G_k(u). \tag{3.28}$$

We consider now two different cases.

Case 1: $1 + \frac{2}{N} \le q < 2$.

In this case, we have that $r = \frac{N(q-1)}{q} \in \left[\frac{2N}{N+2}, \frac{N}{2}\right)$, so $\sigma_r = \frac{(N-2)r}{2(N-2r)} \geq 1$. On the other hand, it can be checked that

$$\left(1 - \frac{2}{N}\right) 2^\star + \frac{2}{N} 2^\star \sigma_r = \frac{2}{2-q}.$$

Then, we can use Hölder's inequality in such a way that

$$\begin{aligned}\int_\Omega |\nabla G_k(u)|^q G_k(u) &\leq \left(\int_\Omega |\nabla G_k(u)|^2\right)^{\frac{q}{2}} \left(\int_\Omega G_k(u)^{\frac{2}{2-q}}\right)^{1-\frac{q}{2}} \\ &\leq \left(\int_\Omega |\nabla G_k(u)|^2\right)^{\frac{q}{2}} \left(\int_\Omega G_k(u)^{2^\star}\right)^{\frac{2-q}{2^\star}} \|G_k(u)\|^{q-1}_{L^{2^\star \sigma_r}(\Omega)} \\ &\leq C \|G_k(u)\|^{q-1}_{L^{2^\star \sigma_r}(\Omega)} \int_\Omega |\nabla G_k(u)|^2.\end{aligned}$$

Next, by item (2) we can take $k \geq k_0$, with k_0 independent of u, so that $\|G_k(u)\|^{q-1}_{L^{2^\star \sigma_r}(\Omega)}$ is small enough. Then, from (3.28) we deduce that

$$C \int_\Omega |\nabla G_k(u)|^2 \leq \int_\Omega |h(x)| G_k(u).$$

We conclude by using the Stampacchia's method in a direct way.

Case 2: $\frac{N}{N-1} < q < 1 + \frac{2}{N}$.

In this case, Hölder's inequality yields

$$\int_\Omega |\nabla G_k(u)|^q G_k(u) \leq \left(\int_\Omega |\nabla G_k(u)|^{N(q-1)}\right)^{\frac{1}{N}} \left(\int_\Omega |\nabla G_k(u)|^2\right)^{\frac{1}{2}} \left(\int_\Omega G_k(u)^{2^\star}\right)^{\frac{1}{2^\star}}.$$

By Case 2 of item (2), we can take $k \geq k_0$, with k_0 independent of u, such that $\int_\Omega |\nabla G_k(u)|^{N(q-1)}$ is small enough. Then, from (3.28) we deduce that

$$C \int_\Omega |\nabla G_k(u)|^2 \leq \int_\Omega |h(x)| G_k(u),$$

and we can apply again Stampacchia's method.

The proof is now concluded. □

We prove now a result analogous to Proposition 3.13 for q small.

Proposition 3.14. *Assume that q, α, β, μ satisfy* (3.11)*, and assume in addition that*

$$q < \frac{N}{N-1}.$$

Then, for all $M > 0$ and $p \geq 1$, there exists $C > 0$ such that, for any $h \in L^p(\Omega)$ with $\|h\|_{L^p(\Omega)} \leq M$ and for any solution u to problem (3.10)*, the following holds:*

1. *If $p = 1$, then $\|u\|_{\mathcal{M}^{\frac{N}{N-2}}(\Omega)} + \||\nabla u|\|_{\mathcal{M}^{\frac{N}{N-1}}(\Omega)} \leq C$;*
2. *if $1 < p < \frac{2N}{N+2}$, then $\|u(1+u)^{\sigma-1}\|_{H_0^1(\Omega)} \leq C$, where σ is defined by* (3.12)*;*
3. *if $\frac{2N}{N+2} \leq p < \frac{N}{2}$, then $\|u^\sigma\|_{H_0^1(\Omega)} \leq C$, where σ is defined by* (3.12)*;*
4. *if $p = \frac{N}{2}$, then $\|u^\tau\|_{H_0^1(\Omega)} \leq C$ for all $\tau < \infty$, and*

5. *if* $p > \frac{N}{2}$, *then* $\|u\|_{L^\infty(\Omega)} \le C$.

Proof. We will prove first item (1). Thus, for $j, k > 0$, let us take $T_j(G_k(u))$ as test function in the weak formulation of (3.10), so we obtain

$$\beta \int_\Omega u T_j(G_k(u)) + \int_\Omega \nabla u \nabla T_j(G_k(u)) = \int_\Omega \left(\mu(x) \frac{|\nabla G_k(u)|^q}{u^\alpha} + |h(x)|\right) T_j(G_k(u)). \tag{3.29}$$

On the one hand, it is clear that

$$\int_\Omega \nabla u \nabla T_j(G_k(u)) = \int_\Omega \left|\nabla T_j(G_k(u))\right|^2.$$

On the other hand, concerning the right hand side of (3.29), we obtain that

$$\int_\Omega \left(\mu(x) \frac{|\nabla G_k(u)|^q}{u^\alpha} + |h(x)|\right) T_j(G_k(u)) \le jC \left(\int_\Omega |\nabla G_k(u)|^q + \int_{\{|h(x)| \ge \beta k\}} |h(x)|\right) + \beta \int_\Omega u T_j(G_k(u)).$$

In sum, we deduce that

$$\int_\Omega |\nabla T_j(G_k(u))|^2 \le jC \left(\int_\Omega |\nabla G_k(u)|^q + \int_{\{|h(x)| \ge \beta k\}} |h(x)|\right).$$

Then, we apply [43, Lemma 4.2], so that we deduce that

$$\|\nabla G_k(u)\|_{\mathcal{M}^{\frac{N}{N-1}}(\Omega)} \le C \left(\int_\Omega |\nabla G_k(u)|^q + \int_{\{|h(x)| \ge \beta k\}} |h(x)|\right).$$

Since $q < \frac{N}{N-1}$, we have the immersions

$$\mathcal{M}^{\frac{N}{N-1}}(\Omega) \subset L^{\frac{N}{N-1}}(\Omega) \subset L^q(\Omega).$$

Therefore,

$$C\|\nabla G_k(u)\|_{L^q(\Omega)} \le \int_\Omega |\nabla G_k(u)|^q + \int_{\{|h(x)| \ge \beta k\}} |h(x)|.$$

We now consider the function $F : [0, \infty) \to \mathbb{R}$ defined as

$$F(Y) = CY - Y^q \quad \forall Y \ge 0,$$

and we denote

$$Y_k = \|\nabla G_k(u)\|_{L^q(\Omega)}.$$

Thus we have proved that

$$F(Y_k) \le \|h\chi_{\{|h(x)| \ge \beta k\}}\|_{L^1(\Omega)}.$$

The proof of this part concludes as in the previous proposition.

The proofs of the rest of the items follow the same arguments of Proposition 3.13. We only stress that the estimate

$$\int_\Omega |\nabla G_k(u)|^{N(q-1)} \le C \quad \forall k \ge k_0$$

is proved in a different way. Indeed, since $q < \frac{N}{N-1}$, then $N(q-1) < \frac{N}{N-1}$, so we deduce that

$$\int_\Omega |\nabla G_k(u)|^{N(q-1)} \leq C \left(\int_\Omega |\nabla G_k(u)|^{\frac{N}{N-1}} \right)^{(N-1)(q-1)}.$$

Therefore, the estimate holds by virtue of item (1). □

The same arguments of the proof of Proposition 3.12 (but using Propositions 3.13 and 3.14 instead of the results in [9]) are valid also for proving the main result of this subsection.

Proposition 3.15. *Assume that* (H1) *holds. If* $q > \frac{N}{N-1}$, *suppose also that* (1.1) *is satisfied. Then, for every* $\lambda_0 > 0$, *there exists* $C > 0$ *such that*

$$\|u\|_{L^\infty(\Omega)} \leq C$$

for every solution u *to* (P_λ) *with* $\lambda > \lambda_0$.

Remark 3.16. Notice that, in principle, one can not apply Propositions 3.13 nor 3.14 to prove Proposition 3.15 in the case $q = \frac{N}{N-1}$. However, for $\varepsilon > 0$ small, we have that $\frac{N}{N-1} + \varepsilon < 1 + \frac{2}{N}$ and

$$\frac{|\nabla u|^{\frac{N}{N-1}}}{u^\alpha} \chi_{\{u \geq k\}} \leq \frac{|\nabla u|^{\frac{N}{N-1}+\varepsilon}}{u^\alpha} \chi_{\{u \geq k\}} + C_\varepsilon$$

for any $k > 0$ and any solution u to (P_λ). Hence, the conclusions of Proposition 3.13 hold for $q = \frac{N}{N-1} + \varepsilon$.

3.3 Proof of the main result and consequences

We prove now the main result of the paper.

Proof of Theorem 1.1. Since there is a solution u_0 to (P_0), then Proposition 5.2 (see also Remark 5.3) implies that there exists an unbounded connected set Σ^+ such that

$$(0, u_0) \in \Sigma^+ \subset ([0, +\infty) \times L^\infty(\Omega)) \cap \Sigma,$$

where

$$\Sigma = \{(\lambda, u) \in \mathbb{R} \times L^\infty(\Omega) : u \text{ is a solution to } (P_\lambda)\}.$$

We claim that Σ^+ bifurcates from infinity to the right of the axis $\lambda = 0$. Indeed, since (P_λ) does not have any solution for $\lambda \geq \lambda_1$ (see Remark 3.3), then $\Sigma^+ \subset ([0, \lambda_1) \times L^\infty(\Omega)) \cap \Sigma$. Therefore, since Σ^+ is unbounded, then its projection onto $L^\infty(\Omega)$ is unbounded. Now, Proposition 3.15 implies that $\Sigma^+ \cap ((\lambda_0, \lambda_1) \times L^\infty(\Omega))$ is bounded for all $\lambda_0 \in (0, \lambda_1)$. That is to say, $\Sigma^+ \cap ((0, \lambda_0) \times L^\infty(\Omega))$ is unbounded for all $\lambda_0 > 0$, and our claim is true.

We have proved that there exists a sequence $\{(\lambda_n, u_n)\} \subset \Sigma^+$ such that $\lambda_n \to 0$ and $\|u_n\|_{L^\infty(\Omega)} \to +\infty$ as $n \to +\infty$. We will show now that this fact and the connection of Σ^+ are enough to proof multiplicity of solutions for all $\lambda > 0$ small enough. Indeed, assume by contradiction that there exists another sequence $\{(\mu_n, v_n)\} \subset \Sigma^+$ such that $\mu_n \to 0$ as $n \to \infty$ and (P_{μ_n}) admits no other solution but v_n for all n. On the other hand, using that $(0, u_0) \in \Sigma^+$ and Σ^+ is connected, it is clear that $\Sigma^+ \cap B_r((0, u_0)) \setminus \{(0, u_0)\} \neq \emptyset$ for all $r > 0$, where $B_r((0, u_0))$ denotes the open ball in $\mathbb{R} \times L^\infty(\Omega)$ centered at $(0, u_0)$ with radius r. Hence, since v_n is unique and $\mu_n \to 0$, we have that, for all $r > 0$, there exists $n_r \in \mathbb{N}$ such that, if $n \geq n_r$, then $(\mu_n, v_n) \in \Sigma^+ \cap B_r((0, u_0)) \setminus \{(0, u_0)\}$. In other words, $v_n \to u_0$ in $L^\infty(\Omega)$ as $n \to +\infty$. Let us now take a not relabeled subsequence $\{(\mu_n, v_n)\}$ such that $\mu_{n+1} < \lambda_n < \mu_n$ for all n. Let us also fix $\eta > \|u_0\|_{L^\infty(\Omega)}$, and take n large enough so that $\max\{\|v_n\|_{L^\infty(\Omega)}, \|v_{n+1}\|_{L^\infty(\Omega)}\} < \eta < \|u_n\|_{L^\infty(\Omega)}$. We claim that there exists $(\nu_n, w_n) \in \Sigma^+$ such that $\nu_n \in (\mu_{n+1}, \mu_n)$ and $\|w_n\|_{L^\infty(\Omega)} = \eta$.

Indeed, let us consider the set

$$A_{n,\eta} = \{(\lambda, u) \in \Sigma : \lambda \in (\mu_{n+1}, \mu_n), \|u\|_{L^\infty(\Omega)} = \eta\}.$$

Arguing by contradiction, assume that $\Sigma^+ \cap A_{n,\eta} = \emptyset$. Let us define also

$$B_{n,\eta} = \{(\lambda, u) \in \Sigma : \lambda \in \{\mu_{n+1}, \mu_n\}, \|u\|_{L^\infty(\Omega)} > \eta\}.$$

On the one hand, the uniqueness of v_n and the fact that $\max\{\|v_n\|_{L^\infty(\Omega)}, \|v_{n+1}\|_{L^\infty(\Omega)}\} < \eta$ imply that $\Sigma^+ \cap B_{n,\eta} = \emptyset$. On the other hand, if we consider the set

$$U_{n,\eta} = \{(\lambda, u) \in \Sigma^+ : \lambda \in (\mu_{n+1}, \mu_n), \|u\|_{L^\infty(\Omega)} > \eta\},$$

then it is clear that $U_{n,\eta}$ is open in Σ^+, $(\lambda_n, u_n) \in U_{n,\eta}$ and $\partial U_{n,\eta} = A_{n,\eta} \cup B_{n,\eta}$. Hence, denoting $V_{n,\eta} = \Sigma^+ \setminus \overline{U_{n,\eta}}$, we deduce that $V_{n,\eta}$ is also nonempty and open in Σ^+, $U_{n,\eta} \cap V_{n,\eta} = \emptyset$ and $\Sigma^+ = U_{n,\eta} \cup V_{n,\eta}$. This contradicts that Σ^+ is connected.

Therefore, we have found a sequence $\{(v_n, w_n)\} \subset \Sigma^+$ such that $v_n \to 0$ as $n \to +\infty$ and $\|w_n\|_{L^\infty(\Omega)} = \eta$ for all n large enough. In particular, $\{w_n\}$ is bounded in $L^\infty(\Omega)$. Then, we can argue as in the proof of Proposition 3.4 in order to pass to the limit in (P_{v_n}). Thus, there exists $w \in H_0^1(\Omega) \cap L^\infty(\Omega)$ such that $w_n \rightharpoonup w$ weakly in $H_0^1(\Omega)$, $w_n \to w$ strongly in $L^\infty(\Omega)$ and w is a solution to (P_0). But $\|w\|_{L^\infty(\Omega)} = \eta > \|u_0\|_{L^\infty(\Omega)}$. This is a contradiction, as u_0 is unique by virtue of Theorem 2.1 and Remark 3.3. The proof in now concluded. □

We conclude the section by stating and proving two corollaries of Theorem 1.1. The first one provides multiplicity of solutions for q small, but for any $\alpha \in [0, q-1)$.

Corollary 3.17. *Assume that* (H1) *holds with $q \in (1, Q_N] \setminus \{2\}$, where Q_N is defined in* (1.2). *Assume also that there exists a solution to (P_0). Then, the conclusions of Theorem 1.1 hold true.*

Proof. Consider the function $z : [0, q-1) \to \mathbb{R}$ given by

$$z(s) = \frac{q-s}{N-q+1} - \frac{q-1-s}{q-2s} \quad \forall s \in [0, q-1).$$

It can be proved that z is increasing. Indeed,

$$\begin{aligned} Nz'(s) &= -\frac{1}{N-q+1} + \frac{2-q}{(q-2s)^2} \\ &= \frac{4}{(N-q+1)(q-2s)^2}\left(s - \frac{q-\sqrt{(2-q)(N+1-q)}}{2}\right)\left(\frac{q+\sqrt{(2-q)(N+1-q)}}{2} - s\right). \end{aligned}$$

Using that $N \geq 3$ and $q < 2$, it is straightforward to deduce that

$$\frac{q-\sqrt{(2-q)(N+1-q)}}{2} < 0 \quad \text{and} \quad \frac{q+\sqrt{(2-q)(N+1-q)}}{2} > q-1,$$

which means that $z'(s) > 0$ for all $s \in [0, q-1)$. Moreover, since $q \leq Q_N$, then $z(0) \geq 0$ (see Proposition 3.12). Thus, $z(\alpha) \geq 0$, or equivalently, condition (1.1) holds and Theorem 1.1 can be applied. □

The second corollary gives multiplicity of solutions for all $q \in (1,2)$ at the expense of taking α close to $q-1$.

Corollary 3.18. *Assume that* (H1) *holds holds and that there exists a solution to (P_0). If $q > \frac{N}{N-1}$, suppose also that $\alpha \geq \left(q - \frac{N}{N-1}\right)\frac{N-1}{N-2}$. Then, the conclusions of Theorem 1.1 hold true.*

Proof. One only has to notice that, if $\alpha \geq \left(q - \frac{N}{N-1}\right)\frac{N-1}{N-2}$, then $\frac{N(q-1-\alpha)}{q-2\alpha} \leq 1$. But $\frac{(q-\alpha)N}{N-q+1} > 1$, that is to say, (1.1) holds and Theorem 1.1 can be applied. □

4 Uniqueness for $q - 1 < \alpha \le 1$

We will consider in this section problem (P_λ) under condition (H2). Observe that if $0 < u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ and $t > 0$, then

$$\frac{|\nabla tu|^q}{(tu)^\alpha} = t^{q-\alpha}\frac{|\nabla u|^q}{u^\alpha}.$$

In this case, $\alpha > q - 1$, so $q - \alpha < 1$. That is to say, the lower order term has *sublinear homogeneity*.

Remark 4.1. The conclusions of Remark 3.3 are valid also under hypothesis (H2).

We will prove the existence of solution to (P_λ) after deriving certain a priori estimates on an approximate problem and passing eventually to the limit, in a way that such a limit will be the solution we look for. Thus, consider the following approximate problem:

$$\begin{cases} -\Delta u_n = \lambda u_n + \mu(x)\dfrac{T_n(|\nabla u_n|^q)}{\left(|u_n| + \frac{1}{n}\right)^\alpha} + T_n(f(x)) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

In the next lemma we show that problem (4.1) admits a solution.

Lemma 4.2. *Assume that* (H2) *holds. Then there exists a solution $u_n \in H^1_0(\Omega) \cap L^\infty(\Omega)$ to problem* (4.1) *for all $n \in \mathbb{N}$ and for all $\lambda < \lambda_1$.*

Proof. Fix $n \in \mathbb{N}$ and $\lambda < \lambda_1$. Then, the following linear problem has a solution $0 < \overline{\psi} \in H^1_0(\Omega) \cap L^\infty(\Omega)$:

$$\begin{cases} -\Delta u = \lambda u + n^{1+\alpha}\mu(x) + n & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Clearly, $\overline{\psi}$ is a supersolution to (4.1). Moreover, $\underline{\psi} = 0$ is a subsolution to (4.1). Since $\underline{\psi} \le \overline{\psi}$, then there exists a solution $u_n \in H^1_0(\Omega) \cap L^\infty(\Omega)$ to (4.1) (see [4]). ☐

We prove now the key estimates for proving the existence of solution to problem (P_λ).

Proposition 4.3. *Assume that* (H2) *holds, and let $\lambda < \lambda_1$. Then there exist $\eta \in (0, 1)$ and $C > 0$ such that*

$$\|u_n\|_{H^1_0(\Omega)} + \|u_n\|_{C^{0,\eta}(\overline{\Omega})} \le C$$

for every solution u_n to (4.1) *and for every n.*

Proof. **Step 1: H^1_0 estimate.**

Let us take u_n as test function in the weak formulation of (4.1). Then we obtain by using Poincaré's and Hölder's inequalities that

$$\begin{aligned} \int_\Omega |\nabla u_n|^2 &\le \lambda \int_\Omega u_n^2 + \|\mu\|_{L^\infty(\Omega)} \int_\Omega |\nabla u_n|^q u_n^{1-\alpha} + \int_\Omega f(x) u_n \\ &\le \frac{\lambda}{\lambda_1}\int_\Omega |\nabla u_n|^2 + C\left(\int_\Omega |\nabla u_n|^2\right)^{\frac{q}{2}}\left(\int_\Omega u_n^{\frac{2(1-\alpha)}{2-q}}\right)^{1-\frac{q}{2}} + C\left(\int_\Omega u_n^{2^*}\right)^{\frac{1}{2^*}}. \end{aligned}$$

Now, since $\alpha > q - 1$, then $\frac{2(1-\alpha)}{2-q} < 2 < 2^*$. Hence, we can apply Sobolev's inequality to get that

$$\left(1 - \frac{\lambda}{\lambda_1}\right)\int_\Omega |\nabla u_n|^2 \le C\left(\int_\Omega |\nabla u_n|^2\right)^{\frac{q+1-\alpha}{2}} + C\left(\int_\Omega |\nabla u_n|^2\right)^{\frac{1}{2}}.$$

Observe now that $\frac{q+1-\alpha}{2} < 1$. Therefore, we deduce that $\|u_n\|_{H_0^1(\Omega)} \le C$.

Step 2: L^∞ estimate.

Assume now, in order to achieve a contradiction, that $\{\|u_n\|_{L^\infty(\Omega)}\}_{n\in\mathbb{N}}$ is unbounded, and choose a not relabeled divergent subsequence. Then, the function $v_n = \frac{u_n}{\|u_n\|_{L^\infty(\Omega)}}$ satisfies

$$\begin{cases} -\Delta v_n = \lambda v_n + \dfrac{\mu(x)T_n(|\nabla u_n|^q)}{\|u_n\|_{L^\infty(\Omega)}\left(u_n + \frac{1}{n}\right)^{q-1+\alpha}} + \dfrac{f(x)}{\|u_n\|_{L^\infty(\Omega)}} & \text{in } \Omega, \\ v_n > 0 & \text{in } \Omega, \\ v_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

Notice that $\|v_n\|_{L^\infty(\Omega)} = 1$ for all n, and also that

$$0 \le \frac{\mu(x)T_n(|\nabla u_n|^q)}{\|u_n\|_{L^\infty(\Omega)}\left(u_n + \frac{1}{n}\right)^{q-1+\alpha}} \le \frac{\|\mu\|_{L^\infty(\Omega)}|\nabla v_n|^q}{\|u_n\|^{\alpha}_{L^\infty(\Omega)} v_n^{q-1+\alpha}}. \tag{4.3}$$

Then, it is standard to prove that $\|v_n\|_{C^{0,\eta}(\overline{\Omega})} \le C$ for all n and for some $\eta \in (0,1)$ independent of n following the arguments in [40] (see [1, Appendix]). Hence, by Arzelà-Ascoli theorem, there exists $v \in C(\overline{\Omega})$ such that, up to a subsequence, $v_n \to v$ uniformly in $\overline{\Omega}$. Necessarily, $\|v\|_{L^\infty(\Omega)} = 1$, so $v \not\equiv 0$. Moreover, by using the strong maximum principle conveniently, $v > 0$ in Ω. This last fact combined with the uniform convergence implies that,

$$\forall \omega \subset\subset \Omega \ \exists c_\omega > 0 : \ v_n \ge c_\omega \text{ in } \omega.$$

See the proof of [1, Proposition 5.2] for more details.

Let now $\phi \in C_c^1(\Omega)$ be such that $\operatorname{supp}(\phi) \subset \omega$ for some open set $\omega \subset\subset \Omega$. Then, from (4.3) we deduce that

$$\left| \int_\Omega \frac{\mu(x)T_n(|\nabla u_n|^q)\phi}{\|u_n\|_{L^\infty(\Omega)}\left(u_n + \frac{1}{n}\right)^{q-1+\alpha}} \right| \le \frac{\|\mu\phi\|_{L^\infty(\Omega)}}{\|u_n\|^{\alpha}_{L^\infty(\Omega)} c_\omega^{q-1+\alpha}} \int_\omega |\nabla v_n|^q.$$

Using now that $\{v_n\}_{n\in\mathbb{N}}$ is bounded in $H_0^1(\Omega)$, we conclude that

$$\left| \int_\Omega \frac{\mu(x)T_n(|\nabla u_n|^q)\phi}{\|u_n\|_{L^\infty(\Omega)}\left(u_n + \frac{1}{n}\right)^{q-1+\alpha}} \right| \to 0$$

as $n \to \infty$.

Finally, we pass to the limit in (4.2) and obtain that

$$\begin{cases} -\Delta v = \lambda v & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

This contradicts the fact that $\lambda < \lambda_1$.

Step 3: Hölder estimate. Using that $\alpha \le 1$ ant the previous step, one can easily prove following [1, Appendix] that $\|u_n\|_{C^{0,\eta}(\overline{\Omega})} \le C$ for all n and for some $C > 0$, $\eta \in (0,1)$. ☐

We are ready now to prove the main theorem of this section.

Proof of Theorem 1.2. Concerning the existence of solution, one has only to pass the limit in (4.1) using the a priori estimates in Proposition 4.3. The proof is similar to the one of Proposition 3.4. The nonexistence of solution comes from Remark 3.3.

On the other hand, the uniqueness of solution is a direct consequence of Theorem 2.4 and Remark 3.3.

Finally, similar arguments as in the proof of Step 2 in Proposition 4.3 can be used to prove that λ_1 is the only possible bifurcation point from infinity. Actually, reasoning by contradiction and using that there is no solution to (P_{λ_1}), it is also standard to prove that λ_1 is, indeed, a bifurcation point from infinity. ☐

5 Appendix: Existence of an unbounded continuum

For every $w \in L^\infty(\Omega)$ and $\lambda \in \mathbb{R}$, let us consider the following problem:

$$\begin{cases} -\Delta u + u = \mu(x)\dfrac{|\nabla u|^q}{u^\alpha} + f(x) + (\lambda^+ + 1)w^+ & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

If (H1) is satisfied, it is clear from Proposition 3.4 that there exists a unique solution $u_{\lambda,w} \in H_0^1(\Omega) \cap L^\infty(\Omega)$ to (5.1). Hence, we are allowed to define the map

$$K : \mathbb{R} \times L^\infty(\Omega) \to L^\infty(\Omega), \quad (\lambda, w) \mapsto K(\lambda, w) = u_{\lambda,w}.$$

We will prove next that K is a completely continuous operator, i.e., it is continuous and maps bounded sets to relatively compact sets.

Proposition 5.1. *Assume that* (H1) *holds. Then, the operator K is completely continuous.*

Proof. We first prove that K is continuous. Indeed, let $\{(\lambda_n, w_n)\}$ be a sequence in $\mathbb{R} \times L^\infty(\Omega)$ such that $(\lambda_n, w_n) \to (\lambda, w)$ for some $(\lambda, w) \in \mathbb{R} \times L^\infty(\Omega)$. Let us denote $u_n = K(\lambda_n, w_n)$, and let $B > 0$ be such that $(\lambda_n^+ + 1)w_n^+ \le B$. We know from Proposition 3.4 that there exists $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$ such that

$$\begin{cases} -\Delta v + v = \mu(x)\dfrac{|\nabla v|^q}{v^\alpha} + f(x) + B & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Hence, by virtue of Theorem 2.1 (see also Remark 3.3), we deduce that

$$u_n \le v \le \|v\|_{L^\infty(\Omega)}.$$

In particular, $\{u_n\}$ is bounded in $L^\infty(\Omega)$.

Now we can argue as in [1, Appendix] to prove that $\{u_n\}$ is, in fact, bounded in $C^{0,\eta}(\overline{\Omega})$ for some $\eta \in (0, 1)$. Therefore, Arzelà-Ascoli theorem implies that $\{u_n\}$ admits a uniformly convergent subsequence. Say, up to a not relabeled subsequence, $u_n \to u$ uniformly in $\overline{\Omega}$ for some $u \in C(\overline{\Omega})$.

On the other hand, taking u_n as test function in the weak formulation of (5.1) yields

$$\int_\Omega |\nabla u_n|^2 + \int_\Omega u_n^2 = \int_\Omega \mu(x)|\nabla u_n|^q u_n^{1-\alpha} + \int_\Omega (f(x) + (\lambda_n^+ + 1)w_n^+.$$

Using that $\{u_n\}$ and $\{(\lambda_n, w_n)\}$ are bounded in $L^\infty(\Omega)$ and in $\mathbb{R}\times L^\infty(\Omega)$, and also that $\alpha < q-1 < 1$, the previous equality clearly implies that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Then, $u \in H_0^1(\Omega)$ and, up to a new subsequence, $u_n \rightharpoonup u$ in $H_0^1(\Omega)$. Moreover, by [44], $\nabla u_n \to \nabla u$ strongly in $L^q(\Omega)^N$. Furthermore, a lower local estimate on $\{u_n\}$ can be derived by comparison in the usual way. With all these estimates and convergences, the passing to the limit in (5.1) is standard.

Therefore, $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is the unique solution to (5.1). This means that $K(\lambda, w) = u$. Thus, we have proved that, up to a subsequence, $K(\lambda_n, w_n) \to K(\lambda, w)$ strongly in $L^\infty(\Omega)$. Actually, since (λ, w) was fixed from the beginning, the whole sequence, and not just a subseqence, converges to (λ, w). That is to say, K is continuous.

It is left to prove that K maps bounded sets to relatively compact sets. In other words, that for every sequence $\{(\lambda_n, w_n)\}$ bounded in $\mathbb{R} \times L^\infty(\Omega)$, there exists $(\lambda, w) \in \mathbb{R} \times L^\infty(\Omega)$ such that, up to a subsequence, $K(\lambda_n, w_n) \to K(\lambda, w)$ strongly in $L^\infty(\Omega)$. Indeed, it is well-known that, up to a subsequence, $\lambda_n \to \lambda$ in $\mathbb{R}$

and $w_n \to w$ weakly* in $L^\infty(\Omega)$ for some $(\lambda, w) \in \mathbb{R} \times L^\infty(\Omega)$. This convergence is enough to pass to the limit in the term with w_n. In the rest of the terms, we pass to limit arguing as above. Thus, up to a subsequence, $K(\lambda_n, w_n) \to K(\lambda, w)$, and the proof is finished. □

Let us define $\Phi(\lambda, u) = u - K(\lambda, u)$, and

$$\Sigma = \{(\lambda, u) \in \mathbb{R} \times L^\infty(\Omega) : \Phi(\lambda, u) = 0\}.$$

For any $\lambda_0 \in \mathbb{R}$ and any isolated solution $u_0 \in L^\infty(\Omega)$ to the equation $\Phi(\lambda_0, u) = 0$, the Leray-Schauder degree $\deg(\Phi(\lambda_0, \cdot), B_r(u_0), 0)$ is well defined and is constant for $r > 0$ small enough. Thus it is possible to define the so called *index* as

$$i(\Phi(\lambda_0, \cdot), u_0) = \lim_{r\to 0} \deg(\Phi(\lambda_0, \cdot), B_r(u_0), 0).$$

Proposition 5.2. *Assume that* (H1) *holds, and suppose also that* (P_0) *has a solution* u_0*. Then, there exist two unbounded connected sets* $\Sigma^-, \Sigma^+ \subset \Sigma$ *such that* $\Sigma^- \subset (-\infty, 0] \times L^\infty(\Omega)$, $\Sigma^+ \subset [0, \infty) \times L^\infty(\Omega)$ *and* $(0, u_0) \in \Sigma^- \cap \Sigma^+$.

Remark 5.3. Observe that, if $\lambda \geq 0$, solving the equation $\Phi(\lambda, u) = 0$ is equivalent to finding a solution to (P_λ). In particular, the projection of Σ^+ onto $L^\infty(\Omega)$ is actually made of solutions to (P_λ).

Proof of Proposition 5.2. By virtue of Proposition 5.1, K is completely continuous. Moreover, since (P_0) admits at most one solution (by virtue of [7]), then u_0 is the unique solution to $\Phi(0, u) = 0$ (see Remark 5.3). In particular, it is isolated. We will prove now that $i(\Phi(0, \cdot), u_0) \neq 0$ by using the properties of the Leray-Schauder degree.

Indeed, let $T : [0, 1] \times L^\infty(\Omega) \to L^\infty(\Omega)$ be defined as $T(t, w) = u$, where $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is the unique solution to the problem

$$\begin{cases} -\Delta u + u = (1 - t)\mu(x)\dfrac{|\nabla u|^q}{u^\alpha} + f(x) + w^+ & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

It is easy to prove that T is continuous and $T(t, \cdot) : L^\infty(\Omega) \to L^\infty(\Omega)$ is completely continuous arguing as in the proof of Proposition 5.1. Moreover, for any $t \in [0, 1]$, the unique solution $u_t \in H_0^1(\Omega) \cap L^\infty(\Omega)$ to $T(t, u_t) = u_t$ satisfies, thanks to Theorem 2.1 (see also Remark 3.3), that $u_t \leq u_0 \leq \|u_0\|_{L^\infty(\Omega)}$. Hence, if we set $\Psi_t(u) = u - T(t, u)$ and $R = 2\|u_0\|_{L^\infty(\Omega)}$, we have that $\Psi_t(u) \neq 0$ for every $t \in [0, 1]$ and every $u \in \partial B_R(0) = \partial\{v \in L^\infty(\Omega) : \|v\|_{L^\infty(\Omega)} < R\}$. Therefore, the homotopy property of the degree shows that

$$\deg(\Psi_0, B_R(0), 0) = \deg(\Psi_1, B_R(0), 0) \neq 0.$$

On the other hand, let $r > 0$ be small enough so that $B_r(u_0) \subset\subset B_R(0)$. Let us denote the following open, bounded and disjoint subsets of $B_R(0)$ as $A_1 = B_r(u_0)$ and $A_2 = B_R(0) \setminus \overline{B_r(u_0)}$. Since u_0 is unique, then $\Psi_0(u) \neq 0$ for all $u \in \overline{B_R(0)} \setminus (A_1 \cup A_2) = \partial B_R(0) \cup \partial B_r(u_0)$. Then, the additivity property of the degree implies that

$$\deg(\Psi_0, B_R(0), 0) = \deg(\Psi_0, A_1, 0) + \deg(\Psi_0, A_2, 0).$$

Now, again by the uniqueness of u_0, we have that $\Psi_0(u) \neq 0$ for all $u \in A_2$. Thus the solution property of the degree says that $\deg(\Psi_0, A_2, 0) = 0$. That is to say,

$$\deg(\Psi_0, B_R(0), 0) = \deg(\Psi_0, B_r(u_0), 0).$$

Putting all together, we have proved that

$$i(\Phi(0, \cdot), u_0) = \deg(\Phi(0, \cdot), B_r(u_0), 0) = \deg(\Psi_0, B_r(u_0), 0) \neq 0.$$

In conclusion, we can now apply [15, Theorem 2.2], which is essentially [45, Theorem 3.2], and the proof is finished. □

Acknowledgement: Research supported by MINECO-FEDER grant MTM2015-68210-P, Junta de Andalucía FQM-116 and Programa de Contratos Predoctorales del Plan Propio de la Universidad de Granada.

References

[1] J. Carmona, T. Leonori, S. López-Martínez and P.J. Martínez-Aparicio, *Quasilinear elliptic problems with singular and homogeneous lower order terms*. *Nonlinear Anal.* **179** (2019), 105-130.

[2] J. Carmona, S. López-Martínez and P.J. Martínez-Aparicio, *Singular quasilinear elliptic problems with changing sign datum: existence and homogenization. Rev. Mat. Complut.* https://doi.org/10.1007/s13163-019-00313-2.

[3] L. Boccardo, F. Murat and J.-P. Puel, *Existence de solutions faibles pour des équations elliptiques quasi-lináires à croissance quadratique. Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. IV (Paris, 1981/1982)*, 19–73, volume 84 of *Res. Notes in Math.*, Pitman, Boston, Mass.-London, 1983.

[4] L. Boccardo, F. Murat and J.-P. Puel, *Quelques propriétés des opérateurs elliptiques quasi linéaires*. *C. R. Acad. Sci. Paris Sér. I Math.* **307** (1988), 749–752.

[5] L. Boccardo, F. Murat and J.-P. Puel, *L^∞ estimate for some nonlinear elliptic partial differential equations and application to an existence result. SIAM J. Math. Anal.* **23** (1992), no. 2, 326–333.

[6] G. Barles and F. Murat, *Uniqueness and the maximum principle for quasilinear elliptic equations with quadratic growth conditions. Arch. Rational Mech. Anal.* **133** (1995), no. 1, 77–101.

[7] D. Arcoya, C. de Coster, L. Jeanjean and K. Tanaka, *Remarks on the uniqueness for quasilinear elliptic equations with quadratic growth conditions. J. Math. Anal. Appl.* **420** (2014), 772–780.

[8] V. Ferone, M.R. Posteraro and J.M. Rakotoson, *L^∞-estimates for nonlinear elliptic problems with p-growth in the gradient. J. Inequal. Appl.* **3** (1999), no. 2, 109–125.

[9] N. Grenon, F. Murat and A. Porretta, *A priori estimates and existence for elliptic equations with gradient dependent terms. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **13** (2014), no. 1, 137–205.

[10] N.E. Alaa and M. Pierre, *Weak solutions of some quasilinear elliptic equations with data measures. SIAM J. Math. Anal.* **24** (1993), no. 1, 23–35.

[11] K. Hansson, V. Maz'ya and I. Verbitsky, *Criteria of solvability for multidimensional Riccati equations. Ark. Mat.* **37** (1999), no. 1, 87–120.

[12] A. Porretta, *The "ergodic limit" for a viscous Hamilton-Jacobi equation with Dirichlet conditions. Rend. Lincei Mat. Appl.* **21** (2010), 59–78.

[13] L. Jeanjean and B.Sirakov, *Existence and Multiplicity for Elliptic Problems with Quadratic Growth in the Gradient. Comm. Partial Differential Equations* **38** (2013), 244–264.

[14] L. Jeanjean and H. Ramos Quoirin, *Multiple solutions for an indefinite elliptic problem with critical growth in the gradient. Proc. Amer. Math. Soc.* **144** (2016), no. 2, 575–586.

[15] D. Arcoya, C. de Coster, L. Jeanjean and K. Tanaka, *Continuum of solutions for an elliptic problem with critical growth in the gradient. J. Funct. Anal.* **268** (2015), 2298–2335.

[16] Ph. Souplet, *A priori estimates and bifurcation of solutions for an elliptic equation with semidefinite critical growth in the gradient. Nonlinear Anal.* **121** (2015), 412–423.

[17] C. de Coster and A.J. Fernández, *Existence and multiplicity for elliptic p-Laplacian problems with critical growth in the gradient. Calc. Var. Partial Differential Equations* **57** (2018), no. 3, Art. 89, 42 pp.

[18] C. de Coster, A.J. Fernández and L. Jeanjean, *A priori bounds and multiplicity of solutions for an indefinite elliptic problem with critical growth in the gradient. J. Math. Pures Appl.*, to appear.

[19] C. de Coster and L. Jeanjean, *Multiplicity results in the non-coercive case for an elliptic problem with critical growth in the gradient. J. Differential Equations* **262** (2017), no. 10, 5231–5270.

[20] C.A. Stuart, *Existence and approximation of solutions of non-linear elliptic equations. Math. Z.* **147** (1976), no. 1, 53-63.

[21] M.G. Crandall, P.H. Rabinowitz and L. Tartar, *On a Dirichlet problem with a singular nonlinearity. Comm. Partial Differential Equations* **2** (1977), no. 2, 193-222.

[22] A.C. Lazer and P.J. McKenna, *On a singular nonlinear elliptic boundary-value problem. Proc. Amer. Math. Soc.* **111** (1991), no. 3, 721-730.

[23] M. Ghergu and V. Rădulescu, *Multi-parameter bifurcation and asymptotics for the singular Lane-Emden-Fowler equation with a convection term. Proc. Roy. Soc. Edinburgh Sect. A* **135** (2005), no. 1, 61-83.

[24] L. Dupaigne, M. Ghergu and V. Rădulescu, *Lane-Emden-Fowler equations with convection and singular potential. J. Math. Pures Appl. (9)* **87** (2007), no. 6, 563-581.

[25] L. Boccardo, *Dirichlet problems with singular and gradient quadratic lower order terms. ESAIM Control Optim. Calc. Var.* **14** (2008), no. 3, 411–426.

[26] M. Ghergu and V. Rădulescu, *Singular elliptic problems: bifurcation and asymptotic analysis.* Oxford Lecture Series in Mathematics and its Applications, 37. *The Clarendon Press, Oxford University Press, Oxford,* 2008.

[27] D. Arcoya, S. Barile and P.J. Martínez-Aparicio, *Singular quasilinear equations with quadratic growth in the gradient without sign condition. J. Math. Anal. Appl.* **350** (2009), no. 1, 401–408.
[28] D. Arcoya, J. Carmona, T. Leonori, P.J. Martínez-Aparicio, L. Orsina and F. Petitta, *Existence and nonexistence of solutions for singular quadratic quasilinear equations. J. Differential Equations* **246** (2009), no. 10, 4006–4042.
[29] D. Giachetti and F. Murat, *An elliptic problem with a lower order term having singular behaviour. Boll. Unione Mat. Ital. (9)* **2** (2009), 349–370.
[30] D. Arcoya, L. Boccardo, T. Leonori and A. Porretta, *Some elliptic problems with singular natural growth lower order terms. J. Differential Equations* **249** (2010), no. 11, 2771–2795.
[31] D. Giachetti, F. Petitta and S. Segura de León, *Elliptic equations having a singular quadratic gradient term and a changing sign datum. Commun. Pure Appl. Anal.* **11** (2012), 1875–1895.
[32] D. Giachetti, F. Petitta and S. Segura de León, *A priori estimates for elliptic problems with a strongly singular gradient term and a general datum. Differential Integral Equations* **26** (2013), no. 9–10, 913–948.
[33] D. Arcoya and L. Moreno-Mérida, *The effect of a singular term in a quadratic quasi-linear problem. J. Fixed Point Theory Appl.* **19** (2017), 815–831.
[34] M. Chhetri, P. Drábek and R. Shivaji, *Analysis of positive solutions for classes of quasilinear singular problems on exterior domains. Adv. Nonlinear Anal.* **6** (2017), no. 4, 447-459.
[35] R. Durastanti, *Asymptotic behaviour and existence of solutions for singular elliptic equations. Preprint*.
[36] M. Latorre, M. Magliocca and S. Segura de León, *Regularizing effects concerning elliptic equations with a superlinear gradient term. Preprint*.
[37] D. Arcoya and S. Segura de León, *Uniqueness of solutions for some elliptic equations with a quadratic gradient term*. *ESAIM Control Optim. Calc. Var.* **16** (2010), 327–336.
[38] D. Arcoya, J. Carmona and P. J. Martínez-Aparicio, *Comparison principle for elliptic equations in divergence with singular lower order terms having natural growth. Commun. Contemp. Math.* **19** (2017), 1650013, 11 pp.
[39] J. Carmona and T. Leonori, *A uniqueness result for a singular elliptic equation with gradient term. Proc. Roy. Soc. Edinburgh Sect. A* **148** (2018), no. 5, 983–994.
[40] O. Ladyzenskaya and N. Ural'tseva, *Linear and quasilinear elliptic equations.* Translated from the Russian by Scripta Technica, *Academic Press, New York-London* (1968), xviii+495 pp.
[41] H. Brezis and X. Cabré, *Some simple nonlinear PDE's without solutions. Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8)* **1** (1998), no. 2, 223–262.
[42] M. Fila, Ph. Souplet and F. Weissler, *Linear and nonlinear heat equations in L^q_δ spaces and universal bounds for global solutions. Math. Ann.* **320** (2001), 87–113.
[43] P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre and J.L. Vázquez, *An L^1 theory of existence and uniqueness of solutions of nonlinear elliptic equations. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **22** (1995), 241–273.
[44] L. Boccardo and F. Murat, *Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations. Nonlinear Anal.* **19** (1992), 581–597.
[45] P.H. Rabinowitz, *A global theorem for nonlinear eigenvalue problems and applications. Contributions to nonlinear functional analysis (Proc. Sympos., Math. Res. Center, Univ. Wisconsin, Madison, Wis., 1971)*, pp. 11–36. *Academic Press, New York*, 1971.