Research Article

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The Gelfand problem for the 1-homogeneous p-Laplacian

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Abstract: In this paper, we study the existence of viscosity solutions to the Gelfand problem for the 1-homogeneous p-Laplacian in a bounded domain $\Omega \subset \mathbb{R}^N$, that is, we deal with

$$-\frac{1}{p-1}|\nabla u|^{2-p}\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda e^u$$

in Ω with $u = 0$ on $\partial\Omega$. For this problem we show that, for $p \in [2, \infty]$, there exists a positive critical value $\lambda^* = \lambda^*(\Omega, N, p)$ such that the following holds:

- If $\lambda < \lambda^*$, the problem admits a minimal positive solution w_λ.
- If $\lambda > \lambda^*$, the problem admits no solution.

Moreover, the branch of minimal solutions $\{w_\lambda\}$ is increasing with λ. In addition, using degree theory, for fixed p we show that there exists an unbounded continuum of solutions that emanates from the trivial solution $u = 0$ with $\lambda = 0$, and for a small fixed λ we also obtain a continuum of solutions with $p \in [2, \infty]$.

Keywords: Gelfand problem, elliptic equations, viscosity solutions

MSC 2010: 35J15, 35J60, 35J70

1 Introduction

This paper deals with the Gelfand problem corresponding to the 1-homogeneous p-Laplacian,

$$\begin{cases} -\Delta_p^N u = \lambda e^u & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{$P_{\lambda,p}$}$$

where $\Omega \subset \mathbb{R}^N$ is a regular bounded domain, $p \in [2, \infty]$ and the operator Δ_p^N is the 1-homogeneous p-Laplacian (it is also called the normalized p-Laplacian) defined, for $p < \infty$, by

$$\Delta_p^N u := \frac{1}{p-1}|\nabla u|^{2-p}\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \alpha\Delta u + \beta\Delta_\infty u,$$

with $\alpha = 1/(p-1)$ and $\beta = (p-2)/(p-1)$, and for $p = \infty$,

$$\Delta_\infty u \equiv \Delta_\infty^N u = \frac{\nabla u}{|\nabla u|} \cdot \Big(D^2 u \frac{\nabla u}{|\nabla u|}\Big)$$

is the 1-homogeneous infinity Laplacian. These kinds of elliptic operators for $2 \le p < \infty$ have 1 and $1/(p-1)$ as ellipticity constants, hence there is a lack of uniform ellipticity when we let $p \to \infty$. Therefore, the theory

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of uniformly elliptic operators can not be applied. Moreover, we remark the lack of variational structure and differentiability of this operator, in contrast to what happens with the classical p-Laplacian. This fact implies that the theory concerning "stable solutions" can not be applied to our problem.

Note that the 1-homogeneous p-Laplacian is a convex combination of Laplacian and infinity Laplacian operators since $\alpha + \beta = 1$. Moreover, $\alpha = 1, \beta = 0$ if $p = 2$, and $\alpha \to 0, \beta \to 1$ as $p \to \infty$. This operator appears when one considers Tug-of-War games with noise; see [18, 23, 24], where the Poisson problem is studied. Moreover, the sublinear problem and the eigenvalue problem associated to the 1-homogeneous p-Laplacian, namely, the problem with right-hand side λu^q for $0 < q \leq 1$, has been studied in [20] and [19]. In view of these two references it seems natural to deal with the superlinear case (which for this operator is challenging due to the fact that there is no variational structure and no Sobolev spaces framework).

Concerning the Gelfand problem, since it is a classical problem, there is a large number of references. We refer to [2, 4, 5, 10, 21] and the references therein for the Laplacian, and to [25] for the fractional Laplacian.

Our first result for this problem reads as follows.

Theorem 1.1. *For every fixed $p \in [2, +\infty]$, there exists a positive extremal parameter $\lambda^* = \lambda^*(\Omega, N, p)$ such that*
- *if $\lambda < \lambda^*$, problem $(P_{\lambda,p})$ admits a minimal positive solution w_λ;*
- *if $\lambda > \lambda^*$, problem $(P_{\lambda,p})$ has no positive solution.*

Moreover, the branch of minimal solutions $\{w_\lambda\}$ is increasing with λ. Even more, in the case of a ball, $\Omega = B_r$, the minimal solution is radial.

One of our main tools for the proof of this result is a comparison principle (that we prove here) adapted to the particular structure of the 1-homogeneous p-Laplacian (see Theorem 3.3). This result generalizes previous ones in [3, 20]. We believe that this comparison principle is of independent interest.

Using arguments from degree theory, we can obtain the following result concerning solutions that are not necessarily the minimal one. Remark that we even obtain a continuum of solutions for a fixed p using λ as parameter, or for fixed λ small taking p as parameter. More precisely, for fixed p we denote by $\mathscr{S}_p$ the solution set

$$\mathscr{S}_p = \{(\lambda, u) \in [0, \lambda^*(\Omega, N, p)] \times \mathcal{C}(\overline{\Omega}) : u \text{ solves } (P_{\lambda,p})\}.$$

Analogously, for fixed λ we denote by $\mathcal{S}_\lambda$ the solution set

$$\mathcal{S}_\lambda = \{(p, u) \in [2, \infty] \times \mathcal{C}(\overline{\Omega}) : u \text{ solves } (P_{\lambda,p})\}.$$

Theorem 1.2. *For every fixed $p \in [2, \infty]$, there exists an unbounded continuum of solutions $\mathcal{C} \subset \mathscr{S}_p$ that emanates from $\lambda = 0$, $u = 0$, i.e., $(0, 0) \in \mathcal{C}$. Moreover, for every fixed $\lambda < \lambda_0 = \min\{\lambda^*(\Omega, N, 2), (2d^2 e)^{-1}\}$, where d is the diameter of Ω, there exists a continuum of solutions $\mathcal{D} \subset \mathcal{S}_\lambda$ with $\mathrm{Proj}_{[2,+\infty]}\mathcal{D} = [2, +\infty]$ and $\|u\|_\infty \leq 1$ for all $(p, u) \in \mathcal{D}$.*

We remark that, as a consequence of the previous theorem, there is a lower bound for the extremal parameter found in Theorem 1.1: $0 < \lambda_0 \leq \lambda^*(\Omega, N, p)$ for every $p \in [2, +\infty]$.

The use of degree theory is new for this kind of operators. Here we perform homotopies both in the parameters λ and p. The deformation in p is needed in order to start the argument with the trivial solution $u = 0$ for the problem with $p = 2$ and $\lambda = 0$, $\Delta u = 0$, which is known to have degree 1. Note that, due to the nonsmoothness of the operator, there is a nontrivial difficulty in the computation of the degree of the trivial solution to $\Delta_p^N u = 0$. Also note that the necessary compactness is nontrivial; we rely here on results from [7].

Remark 1.3. Our results can be generalized to handle the equation

$$-\Delta_p^N u = \lambda f(u),$$

with a general continuous nonlinearity f that verifies

$$f(0) > 0, \quad f(s) \text{ is increasing}, \quad \frac{f(s)}{s} \geq k > 0.$$

To simplify the exposition we just write the details for $f(s) = e^s$ and we make a comment at the end of the paper on how to deal with this general case.

The rest of this paper is organized as follows: In Section 2, we collect some preliminaries and state the definition of a viscosity solution to our equation. In Section 3, we prove our comparison result. Finally, in Sections 4 and 5 we prove our main results concerning the Gelfand problem.

2 Preliminaries

In this section, we introduce the notion of a viscosity solution for problem $(P_{\lambda,p})$. Actually, we give the definition for a more general family of nonlinearities and we consider the following boundary value problem:

$$\begin{cases} -\Delta_p^N u = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Since the normalized infinity Laplacian

$$\Delta_\infty u = \frac{\nabla u}{|\nabla u|} \cdot \Big(D^2 u \frac{\nabla u}{|\nabla u|} \Big)$$

is not well defined at the points where $|\nabla u(x)| = 0$, we have to use the semicontinuous envelopes of the operator

$$(\xi, X) \mapsto \frac{\xi}{|\xi|} \cdot \Big(X \frac{\xi}{|\xi|} \Big), \quad \xi \in \mathbb{R}^N,\ X \in \mathbb{S}_N,$$

in order to define viscosity solutions for problem (2.1) (see [8, 9]). To this end, we denote the largest and the smallest eigenvalue for $A \in \mathbb{S}_N$ by $M(A)$ and $m(A)$, respectively. That is,

$$M(A) = \max_{|\eta|=1} \eta \cdot (A\eta), \quad m(A) = \min_{|\eta|=1} \eta \cdot (A\eta).$$

Let us denote by $\mathrm{USC}(\omega)$ the set of upper semicontinuous functions $u : \omega \subset \mathbb{R}^N \to \mathbb{R}$, and we denote by $\mathrm{LSC}(\omega)$ the set of lower semicontinuous functions.

Definition 2.1. (i) $\underline{u} \in \mathrm{USC}(\Omega)$ is a viscosity subsolution of the equation $-\Delta_p^N u = \lambda f(x,u)$ if whenever $x_0 \in \Omega$ and $\varphi \in \mathcal{C}^2(\Omega)$ such that $\varphi(x_0) = \underline{u}(x_0)$ and $\varphi - \underline{u} > 0$ in $\Omega \setminus \{x_0\}$, then

$$\begin{cases} -\Delta_p^N \varphi(x_0) \le \lambda f(x_0, \varphi(x_0)) & \text{if } \nabla\varphi(x_0) \ne 0, \\ -\alpha\Delta\varphi(x_0) - \beta M(D^2\varphi(x_0)) \le \lambda f(x_0, \varphi(x_0)) & \text{if } \nabla\varphi(x_0) = 0. \end{cases}$$

If, in addition, $\underline{u} \in \mathrm{USC}(\overline{\Omega})$ and $\underline{u} \le 0$ on $\partial\Omega$, we say that $\underline{u}$ is a subsolution of (2.1).

(ii) $\overline{u} \in \mathrm{LSC}(\Omega)$ is a viscosity supersolution of the equation $-\Delta_p^N u = \lambda f(x,u)$ if whenever $x_0 \in \Omega$ and $\psi \in \mathcal{C}^2(\Omega)$ such that $\psi(x_0) = \overline{u}(x_0)$ and $\overline{u} - \psi > 0$ in $\Omega \setminus \{x_0\}$, then

$$\begin{cases} -\Delta_p^N \psi(x_0) \ge \lambda f(x_0, \psi(x_0)) & \text{if } \nabla\psi(x_0) \ne 0, \\ -\alpha\Delta\psi(x_0) - \beta m(D^2\psi(x_0)) \ge \lambda f(x_0, \psi(x_0)) & \text{if } \nabla\psi(x_0) = 0. \end{cases}$$

If, in addition, $\overline{u} \in \mathrm{LSC}(\overline{\Omega})$ and $\overline{u} \ge 0$ on $\partial\Omega$, we say that $\overline{u}$ is a supersolution of (2.1).

(iii) A continuous function $u : \overline{\Omega} \to \mathbb{R}$ is a viscosity solution of (2.1) if it is both a viscosity supersolution and a viscosity subsolution.

In what follows, φ stands for test functions whose graph touches the graph of u from above, and ψ denotes test functions whose graph touches the graph of u from below. Notice that the inequalities $\varphi - \underline{u} > 0$ and $\overline{u} - \psi > 0$ have to be satisfied in a neighborhood of $\{x_0\}$ and not necessarily in the whole $\Omega \setminus \{x_0\}$.

Remark 2.2. Let u be a classical subsolution of (2.1), that is, $u \in \mathcal{C}^2(\overline{\Omega})$, $u \le 0$ on $\partial\Omega$ and for every $x \in \Omega$ it satisfies

$$\begin{cases} -\Delta_p^N u(x) \le \lambda f(x, u(x)) & \text{if } \nabla u(x) \ne 0, \\ -\alpha\Delta u(x) - \beta M(D^2 u(x)) \le \lambda f(x, u(x)) & \text{if } \nabla u(x) = 0. \end{cases}$$

Then u is a viscosity subsolution. Indeed, let $\varphi \in \mathcal{C}^2(\Omega)$ be such that $\varphi(x_0) = u(x_0)$ and $\varphi - u > 0$ in $\Omega \setminus \{x_0\}$; then $\nabla(\varphi - u)(x_0) = 0$ and $D^2(\varphi - u)(x_0)$ is a positive definite $N \times N$ matrix. Therefore,

$$\eta \cdot (D^2\varphi(x_0)\eta) \geq \eta \cdot (D^2 u(x_0))\eta, \quad \eta \in \mathbb{R}^N,$$

and $\operatorname{tr}(D^2\varphi(x_0)) \geq \operatorname{tr}(D^2 u(x_0))$ (i.e., $\Delta\varphi(x_0) \geq \Delta u(x_0)$). Hence, if $\nabla u(x_0) \neq 0$, we obtain

$$-\alpha\Delta\varphi(x_0) - \beta\Delta_\infty^N\varphi(x_0) \leq -\alpha\Delta u(x_0) - \beta\Delta_\infty^N u(x_0) \leq \lambda f(x_0, \varphi(x_0)).$$

Finally, using that $M(D^2\varphi(x_0)) \geq M(D^2 u(x_0))$ for $\nabla u(x_0) = 0$, it follows that u is a viscosity subsolution. We can proceed analogously with the supersolution case. Thus, classical solutions of (2.1) are solutions in the viscosity sense.

Let us observe that $\underline{u} \in \operatorname{USC}(\Omega)$ is a viscosity subsolution of $-\Delta_p^N u = \lambda f(x, u)$ if

$$\begin{cases} -\alpha \operatorname{tr}(X) - \beta\dfrac{\eta}{|\eta|} \cdot \Big(X\dfrac{\eta}{|\eta|}\Big) \leq \lambda f(x_0, \varphi(x_0)) & \text{if } \eta \neq 0, \\ -\alpha \operatorname{tr}(X) - \beta M(X) \leq \lambda f(x_0, \varphi(x_0)) & \text{if } \eta = 0, \end{cases} \tag{2.2}$$

whenever $x_0 \in \Omega$ and $(\eta, X) = (\nabla\varphi(x_0), D^2\varphi(x_0)) \in \mathbb{R}^N \times \mathbb{S}_N$ for some $\varphi \in \mathcal{C}^2(\Omega)$ such that $\varphi(x_0) = \underline{u}(x_0)$ and $\varphi - \underline{u} > 0$ in $\Omega \setminus \{x_0\}$. Thus, as in [9], we can characterize viscosity sub- and supersolutions by using the concept of upper and lower semijets in the sense of the following definition.

Definition 2.3. For $u \in \operatorname{USC}(\mathcal{O})$ and $x_0 \in \mathcal{O}$, we define the *upper semijet*

$$J_{\mathcal{O}}^{2,+}u(x_0) = \{(\nabla\varphi(x_0), D^2\varphi(x_0)) : \varphi \in \mathcal{C}^2(\mathcal{O}),\ \varphi(x_0) = u(x_0) \text{ and } \varphi - u \text{ has a local minimum at } x_0\}.$$

Analogously, for $u \in \operatorname{LSC}(\mathcal{O})$ and $x_0 \in \mathcal{O}$ we define the *lower semijet*

$$J_{\mathcal{O}}^{2,-}u(x_0) = \{(\nabla\psi(x_0), D^2\psi(x_0)) : \psi \in \mathcal{C}^2(\mathcal{O}),\ \psi(x_0) = u(x_0) \text{ and } \psi - u \text{ has a local maximum at } x_0\}.$$

Finally, we introduce the sets $\bar{J}_{\mathcal{O}}^{2,+}u(x_0)$, $\bar{J}_{\mathcal{O}}^{2,-}u(x_0)$ as follows: $(p, X) \in \bar{J}_{\mathcal{O}}^{2,+}u(x_0)$ if there exist $x_n \in B_r(x_0)$ and $(p_n, X_n) \in J_{\mathcal{O}}^{2,+}u(x_n)$, such that $u(x_n) \to u(x_0)$ and $(x_n, p_n, X_n) \to (x_0, p, X)$ as $n \to \infty$. An analogous statement holds for $\bar{J}_{\mathcal{O}}^{2,-}u(x_0)$.

Remark 2.4. It is clear that $\underline{u} \in \operatorname{USC}(\Omega)$ is a viscosity subsolution of $-\Delta_p^N u = \lambda f(x, u)$ if (2.2) is verified for every $(\eta, X) \in J_{\Omega}^{2,+}\underline{u}(x_0)$. Moreover, if $\underline{u}$ is a subsolution, then (2.2) is verified for every $(\eta, X) \in \bar{J}_{\Omega}^{2,+}\underline{u}(x_0)$. The analogous statement holds for supersolutions.

Remark 2.5. In [12], a parabolic equation of the form

$$u_t = |\nabla u|^\gamma(\Delta u + (p-2)\Delta_\infty^N u)$$

was studied using viscosity solutions. The definition of viscosity solutions given there (inspired by [22]) differs from ours. In fact, in [12] Imbert, Lin and Silvestre restrict the class of test functions in order to give sense to the equation when the gradient vanishes (note that this parabolic problem can be singular or degenerate according to the value of γ). In our definition we do not restrict the test functions but we give a meaning to $\Delta_\infty^N u$ in terms of the largest and the smallest eigenvalue of $D^2 u$ at points where the gradient vanishes. With our definition we can prove a comparison principle in the following section.

3 Comparison principle and uniqueness

In this section, we start giving sufficient conditions on f to prove a comparison principle and hence obtain uniqueness for (2.1).

Definition 3.1. Given a positive function $h \in C^1(0, +\infty)$ such that $h \in L^1(0, 1)$ and $h'(s)/h^2(s)$ is nondecreasing, we say that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the *h-decreasing* condition if for every $x \in \Omega$,

$$h(s)f(x, s) \text{ is decreasing with respect to } s. \tag{3.1}$$

Remark 3.2. Observe that if $f(x,s) = f_0(x) > 0$, that is, f does not depend on s, then f satisfies the h-decreasing condition for $h(s) = 1/s^q$ for any $0 < q < 1$. In addition, when $f(x,s) = f_0(x)s^q > 0$ for some $0 \le q < 1$, then f satisfies the h-decreasing condition for $h(s) = 1/s^{q+\varepsilon}$ for any $0 < \varepsilon < 1 - q$. Moreover, taking a decreasing function h, we obtain that any function $0 < f \in \mathcal{C}^1(\Omega \times \mathbb{R})$ nonincreasing with respect to s also satisfies the *h-decreasing* condition (since $h'(s)f(x,s) + h(s)f'_s(x,s) < 0$ in this case).

Theorem 3.3. *Assume that* $0 < f \in \mathcal{C}(\Omega \times \mathbb{R})$ *satisfies the* h-decreasing *condition. Let* $\underline{u}, \overline{u} \in C(\overline{\Omega})$ *be a sub- and a supersolution, respectively, of* $-\Delta_p^N u = f(x,u)$ *such that* $\overline{u} > 0$ *in* Ω *and* $\underline{u} \le \overline{u}$ *on* $\partial\Omega$. *Then* $\underline{u} \le \overline{u}$ *in* $\overline{\Omega}$.

Proof. We argue by contradiction following closely the ideas in [9]. Suppose that $\Omega^+ = \{x \in \overline{\Omega} : \underline{u}(x) > \overline{u}(x)\}$ is nonempty. Let

$$H(s) = \int_0^s h(t)\,dt$$

for $s \ge 0$. By hypothesis, $\underline{u} \le \overline{u}$ on $\partial\Omega$. Using that $\underline{u}, \overline{u} \in C(\overline{\Omega})$, we have that there exists $\hat{x} \in \Omega^+$ with

$$H(\underline{u}(\hat{x})) - H(\overline{u}(\hat{x})) = \sup_{x\in\Omega^+} H(\underline{u}(x)) - H(\overline{u}(x)) > 0.$$

Since Ω^+ is an open set, we can take $\hat{\Omega}$, an open neighborhood of $\hat{x}$, such that $\overline{\hat{\Omega}} \subset \Omega^+$. Now, let $\underline{w}$ and $\overline{w}$ be the positive functions defined for $x \in \hat{\Omega}$ by

$$\underline{w}(x) = H(\underline{u}(x)) \quad \text{and} \quad \overline{w}(x) = H(\overline{u}(x)).$$

Clearly $\underline{w}, \overline{w} \in C(\overline{\hat{\Omega}})$ and

$$\underline{w}(x) > \overline{w}(x) > 0, \qquad x \in \hat{\Omega}. \tag{3.2}$$

Now, we claim that $\underline{w}$ and $\overline{w}$ are a sub- and a supersolution (in the viscosity sense) of the equation

$$-\Delta_p^N w + \frac{h'(H^{-1}(w))}{h^2(H^{-1}(w))}|\nabla w|^2 = h(H^{-1}(w))f(x, H^{-1}(w)) \quad \text{in } \hat{\Omega}. \tag{Q}$$

Indeed, we proceed to show that $\underline{w}$ is a subsolution (the fact that $\overline{w}$ is a supersolution can be proved in the same way). For every $x_0 \in \hat{\Omega}$, we take $\varphi \in \mathcal{C}^2(\hat{\Omega})$ with $\varphi(x_0) = \underline{w}(x_0)$ and $\varphi(x) > \underline{w}(x)$ for every $x \in \hat{\Omega} \setminus \{x_0\}$. If $\nabla\varphi(x_0) \ne 0$ and we take $\tilde{\varphi} = H^{-1}(\varphi)$, then it is easy to check that

$$\begin{aligned}
-\Delta_p^N \varphi(x_0) + \frac{h'(H^{-1}(\varphi(x_0)))}{h^2(H^{-1}(\varphi(x_0)))}|\nabla\varphi(x_0)|^2 &= -\alpha\Delta\varphi(x_0) - \beta\Delta_\infty\varphi(x_0) + h'(\tilde{\varphi}(x_0))|\nabla\tilde{\varphi}(x_0)|^2 \\
&= -\alpha\Delta\tilde{\varphi}(x_0)h(\tilde{\varphi}(x_0)) - \alpha h'(\tilde{\varphi}(x_0))|\nabla\tilde{\varphi}(x_0)|^2 - \beta\Delta_\infty\tilde{\varphi}(x_0)h(\tilde{\varphi}(x_0)) \\
&\qquad - \beta h'(\tilde{\varphi}(x_0))|\nabla\tilde{\varphi}(x_0)|^2 + h'(\tilde{\varphi}(x_0))|\nabla\tilde{\varphi}(x_0)|^2 \\
&= -\Delta_p^N\tilde{\varphi}(x_0)h(\tilde{\varphi}(x_0)).
\end{aligned}$$

Now, taking into account that $\tilde{\varphi}(x_0) = \underline{u}(x_0)$ and $(\tilde{\varphi} - \underline{u})(x) > 0$ in $\hat{\Omega} \setminus \{x_0\}$, it follows that $\tilde{\varphi}$ is a test function touching from above u at x_0. Thus, since $\underline{u}$ is a subsolution of $-\Delta_p^N u = f(x,u)$, we get

$$-\Delta_p^N \tilde{\varphi}(x_0) \le f(x_0, H^{-1}(\tilde{\varphi}(x_0))).$$

Consequently,

$$-\Delta_p^N \varphi(x_0) + \frac{h'(H^{-1}(\varphi(x_0)))}{h^2(H^{-1}(\varphi(x_0)))}|\nabla\varphi(x_0)|^2 \le h(H^{-1}(\varphi(x_0)))f(x_0, H^{-1}(\varphi(x_0))).$$

In the case $\nabla\varphi(x_0) = 0$, since $\nabla\tilde{\varphi}(x_0) = 0$ and $D^2\varphi(x_0) = h(\tilde{\varphi}(x_0))D^2\tilde{\varphi}(x_0)$, we have

$$\begin{aligned}
-\alpha\Delta\varphi(x_0) - \beta M(D^2\varphi(x_0)) &= -\alpha\Delta\tilde{\varphi}(x_0)h(\tilde{\varphi}(x_0)) - \beta M(D^2\tilde{\varphi}(x_0))h(\tilde{\varphi}(x_0)) \\
&\le h(H^{-1}(\varphi(x_0)))f(x_0, H^{-1}(\varphi(x_0))).
\end{aligned}$$

Therefore, we conclude that $\underline{w}$ is a subsolution of problem (Q), which was our claim.

Now, consider the sequence of functions

$$\Psi_n(x,y) = \underline{w}(x) - \overline{w}(y) - \frac{n}{4}|x-y|^4, \quad (x,y) \in \overline{\hat{\Omega}} \times \overline{\hat{\Omega}},\ n \in \mathbb{N}.$$

For every $n \in \mathbb{N}$, let $(x_n, y_n) \in \overline{\hat{\Omega}} \times \overline{\hat{\Omega}}$ be such that

$$\Psi_n(x_n, y_n) = \sup_{\overline{\hat{\Omega}} \times \overline{\hat{\Omega}}} \Psi_n(x,y).$$

We note that $\Psi_n(x_n, y_n)$ is finite since $\underline{w} - \overline{w}$ is continuous and $\overline{\hat{\Omega}}$ is compact. Moreover,

$$\Psi_n(x_n, y_n) \geq \Psi(x,x) = \underline{w}(x) - \overline{w}(x) > 0$$

Furthermore, we can assume that $x_n, y_n \to \hat{x}, \hat{y}$, $\underline{w}(x_n) \to \underline{w}(\hat{x})$ and $\overline{w}(y_n) \to \overline{w}(\hat{y})$ as $n \to \infty$, and that $\hat{x} = \hat{y}$ (see [9, Lemma 3.1 and Proposition 3.7]). Next, by [9, Theorem 3.2], there exist $X_n, Y_n \in \mathbb{S}_N$ satisfying
(i) $X_n \leq Y_n$,
(ii) $(\eta_n, X_n) \in \overline{J}^{2,+}_{\hat{\Omega}}(\underline{w}(x_n))$, $(\eta_n, Y_n) \in \overline{J}^{2,-}_{\hat{\Omega}}(\overline{w}(y_n))$,
(iii) $X_n \leq 0 \leq Y_n$ for $x_n = y_n$,
where $\eta_n = n|x_n - y_n|^2(x_n - y_n)$.

Hence, if $x_n \neq y_n$, having in mind that $\underline{w}$ and $\overline{w}$ are sub- and supersolution of (Q) and using Remark 2.4, we obtain that

$$\begin{aligned}
&h(H^{-1}(\overline{w}(y_n)))f(y_n, H^{-1}(\overline{w}(y_n))) \\
&\quad \leq -\alpha \operatorname{tr}(Y_n) - \beta \frac{\eta_n}{|\eta_n|} \cdot \Big(Y_n \frac{\eta_n}{|\eta_n|}\Big) + \frac{h'(H^{-1}(\overline{w}(y_n)))}{h^2(H^{-1}(\overline{w}(y_n)))}|\eta_n|^2 \\
&\quad \leq -\alpha \operatorname{tr}(X_n) - \beta \frac{\eta_n}{|\eta_n|} \cdot \Big(X_n \frac{\eta_n}{|\eta_n|}\Big) + \frac{h'(H^{-1}(\underline{w}(x_n)))}{h^2(H^{-1}(\underline{w}(x_n)))}|\eta_n|^2 + \Big(\frac{h'(H^{-1}(\overline{w}(y_n)))}{h^2(H^{-1}(\overline{w}(y_n)))} - \frac{h'(H^{-1}(\underline{w}(x_n)))}{h^2(H^{-1}(\underline{w}(x_n)))}\Big)|\eta_n|^2 \\
&\quad \leq h(H^{-1}(\underline{w}(x_n)))f(x_n, H^{-1}(\underline{w}(x_n))) + \Big(\frac{h'(H^{-1}(\overline{w}(y_n)))}{h^2(H^{-1}(\overline{w}(y_n)))} - \frac{h'(H^{-1}(\underline{w}(x_n)))}{h^2(H^{-1}(\underline{w}(x_n)))}\Big)|\eta_n|^2.
\end{aligned}$$

Letting $n \to \infty$, by the continuity of $\overline{w}$, $\underline{w}$, f, h, h', and using that h'/h^2 is nondecreasing, we get

$$h(H^{-1}(\overline{w}(\hat{x})))f(\hat{x}, H^{-1}(\overline{w}(\hat{x}))) \leq h(H^{-1}(\underline{w}(\hat{x})))f(\hat{x}, H^{-1}(\underline{w}(\hat{x}))).$$

This is a contradiction to (3.2) since it implies, by using (3.1), that

$$h(H^{-1}(\overline{w}(\hat{x})))f(\hat{x}, H^{-1}(\overline{w}(\hat{x}))) > h(H^{-1}(\underline{w}(\hat{x})))f(\hat{x}, H^{-1}(\underline{w}(\hat{x}))).$$

If $x_n = y_n$ for $n \geq n_0$, then $\eta_n = 0$ and by (iii) we have

$$\begin{aligned}
h(H^{-1}(\overline{w}(y_n)))f(y_n, H^{-1}(\overline{w}(y_n))) &\leq -\alpha \operatorname{tr}(Y_n) - \beta m(Y_n) \\
&\leq -\alpha \operatorname{tr}(X_n) - \beta M(X_n) \\
&\leq h(H^{-1}(\underline{w}(x_n)))f(x_n, H^{-1}(\underline{w}(x_n)));
\end{aligned}$$

arguing as above, this leads to a contradiction. □

Let us extract easy consequences of this comparison principle.

Proposition 3.4 (Uniqueness). *Assume that* $0 < f \in C(\Omega \times \mathbb{R})$ *satisfies the* h-decreasing *condition. Then there exists at most one positive viscosity solution of*

$$\begin{cases} -\Delta_p^N u(x) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{P}$$

Proof. Suppose that there exist two solutions $u_1, u_2 \geq 0$ of (P). Using Theorem 3.3 twice, we obtain that $u_1 \leq u_2$ and $u_2 \leq u_1$, and we conclude that $u_1 = u_2$. □

The next result improves [20], where a starshaped condition on the domain Ω was required.

Corollary 3.5. *As a particular case, we can assert that there exists a unique positive solution of*

$$\begin{cases} -\Delta_p^N u(x) = \lambda u^q & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

for every $\lambda > 0$ and $0 < q < 1$. Moreover, for $\lambda = 0$, the problem admits as unique solution $u = 0$.

Proof. For $\lambda > 0$, the uniqueness is due to Proposition 3.4 and the existence due to [20, Theorem 3.1] (which can be extended to the case $p = \infty$ by using the same iterative procedure as in [20, Theorem 3.1]). For $\lambda = 0$, we observe that u is a solution of $-\Delta_p^N u = 0$ if and only if $-\Delta_p u = 0$ in the viscosity sense, (this holds since it is enough to test the equation $-\Delta_p u = 0$ with test functions with $\nabla\varphi \neq 0$; see [15]). Thus, the trivial solution $u = 0$ is the unique solution when $\lambda = 0$. □

4 Existence of minimal solutions for the Gelfand problem

The first result of this section shows how one can pass to the limit in a sequence of viscosity solutions of a sequence of problems to obtain a viscosity solution of the limit problem.

Lemma 4.1. *Let $u_n, f_n \in \mathcal{C}(\Omega)$ and $p_n \in [2, \infty]$ be three sequences satisfying*

$$-\Delta_{p_n}^N u_n = f_n, \tag{4.1}$$

in the viscosity sense, such that $f_n \to f$, $u_n \to u$ uniformly for every $\omega \Subset \Omega$ and $p_n \to p \in [2, \infty]$. Then u is a viscosity solution to the problem

$$-\Delta_p^N u = f.$$

Proof. First, we prove that u is a subsolution. For every $x_0 \in \Omega$, we take $\varphi \in \mathcal{C}^2(\Omega)$ such that $\varphi(x_0) = u(x_0)$ and $\varphi - u > 0$ in $\Omega \setminus \{x_0\}$. Fix $\delta > 0$ such that $\overline{B_\delta(x_0)} \subset \Omega$, and for every $n \in \mathbb{N}$ we consider x_n as the strict minimum point (not necessarily unique) of $\varphi - u_n$ in $\overline{B_\delta(x_0)}$, i.e.,

$$(\varphi - u_n)(x_n) \leq (\varphi - u_n)(x) \quad \text{for all } x \in \overline{B_\delta(x_0)}.$$

Up to a subsequence, we can assume that $x_n \to x^* \in \overline{B_\delta(x_0)}$. Using that u_n is continuous and that the sequence u_n uniformly converges to u, we deduce that $u_n(x_n) \to u(x^*)$. We obtain, taking limits in the above inequality, that

$$(\varphi - u)(x^*) \leq (\varphi - u)(x) \quad \text{for all } x \in \overline{B_\delta(x_0)},$$

and we can assert that $x^* = x_0$. We set

$$\varphi_n(x) = \varphi(x) + u_n(x_n) - \varphi(x_n) + \|x - x_n\|^4, \quad x \in \overline{B_\delta(x_0)}.$$

It is easy to check that φ_n satisfies

$$\varphi_n(x_n) = u_n(x_n), \quad \nabla\varphi_n(x_n) = \nabla\varphi(x_n), \quad D^2\varphi_n(x_n) = D^2\varphi(x_n), \quad (\varphi_n - u_n)(x) > 0$$

in a neighborhood of x_n. Thus, using that u_n is a subsolution of (4.1) and taking φ_n as test function, we obtain the following:

(i) If $\nabla\varphi_n(x_n) \neq 0$, then $-\alpha_n\Delta\varphi_n(x_n) - \beta_n\Delta_\infty\varphi_n(x_n) \leq f_n(x_n)$, and thus

$$-\alpha_n\Delta\varphi(x_n) - \beta_n\Delta_\infty\varphi(x_n) \leq f_n(x_n). \tag{4.2}$$

(ii) If $\nabla\varphi_n(x_n) = 0$, then $-\alpha_n\Delta\varphi_n(x_n) - \beta_n M(D^2\varphi_n(x_n)) \leq f_n(x_n)$, and thus

$$-\alpha_n\Delta\varphi(x_n) - \beta_n M(D^2\varphi(x_n)) \leq f_n(x_n), \tag{4.3}$$

where $\alpha_n = 1/(p_n - 1)$, $\beta_n = (p_n - 2)/(p_n - 1)$ if $p_n < +\infty$, and $\alpha_n = 0$, $\beta_n = 1$ if $p_n = \infty$.

Now, denoting $\alpha = 1/(p - 1)$, $\beta = (p - 2)/(p - 1)$ if $p < +\infty$, and $\alpha = 0$, $\beta = 1$ in the other case, we distinguish three different cases.

Case 1: $\nabla\varphi(x_0) \neq 0$. In this case, we can suppose that, up to a subsequence, $\nabla\varphi_n(x_n) \neq 0$ for $n \geq n_0$ and, taking into account that $\varphi \in \mathcal{C}^2$ and the continuity and uniform convergence of f_n, we can pass to the limit in (4.2) as $n \to \infty$ to obtain

$$-\alpha\Delta\varphi(x_0) - \beta\Delta_\infty\varphi(x_0) \leq f(x_0).$$

Case 2: $\nabla\varphi(x_0) = 0$ and, up to a subsequence, $\nabla\varphi_n(x_n) \neq 0$ for $n \geq n_0$. In this case, since

$$\Delta_\infty\varphi(x_n) \leq M(D^2\varphi(x_n)),$$

replacing in (4.2), we get (4.3). Taking limits, we obtain the desired inequality

$$-\alpha\Delta\varphi(x_0) - \beta M(D^2\varphi(x_0)) \leq f(x_0). \tag{4.4}$$

Case 3: $\nabla\varphi(x_0) = \nabla\varphi_n(x_n) = 0$ for $n \geq n_0$. We obtain (4.4) directly from (4.3).

On the other hand, to prove that u is a supersolution, we argue in a similar way. To be more specific, for every $x_0 \in \Omega$ we take the test function $\psi \in \mathcal{C}^2(\Omega)$ satisfying that $u - \psi$ has a strict minimum at x_0 with $\psi(x_0) = u(x_0)$. Now, taking x_n, the strict minimum of $u_n - \psi$ in $\overline{B_\delta(x_0)} \subset \Omega$, we set

$$\psi_n(x) = \psi(x) + u_n(x_n) - \psi(x_n) - \|x - x_n\|^4$$

as the test function in (4.1) touching the graph of u_n from below in x_n. The rest of the proof runs as before. □

Now we can prove the existence of minimal solutions of $(P_{\lambda,p})$ for λ small and the nonexistence of solutions for λ large, that is, we prove Theorem 1.1.

Proof of Theorem 1.1. Let $z \in \mathcal{C}^2([0,1])$ be a classical solution to the problem

$$\begin{cases} -z''(r) - \alpha(N-1)\dfrac{z'(r)}{r} = \lambda e^{z(r)}, & r \text{ in } (0,1), \\ z(1) = 0, & z'(0) = 0, \end{cases} \tag{4.5}$$

with

$$\alpha = \frac{1}{p-1} \quad \text{if } p < +\infty \text{ and } \quad \alpha = 0 \text{ in the other case.}$$

Then $u(x) := z(|x|)$ is a solution to the problem

$$\begin{cases} -\Delta_p^N u = \lambda e^u & \text{in } B_1, \\ u > 0 & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1, \end{cases} \tag{4.6}$$

in the sense of Definition 2.1 (iii) (see also Remark 2.2). Due to [14], it is well known that there exists a positive number $\tilde{\lambda}(B_1)$, depending only on p, N, such that problem (4.5) has no solution for $\lambda > \tilde{\lambda}(B_1)$. Moreover, for every $0 \leq \lambda < \tilde{\lambda}(B_1)$ there exists a classical solution $z \in \mathcal{C}^2([0,1])$ (see also [13] for a complete description of the multiplicity of solutions). Observe that for any classical solution $z \in \mathcal{C}^2([0,1])$, $\lambda \geq 0$, of (4.5) it holds that $\lambda \leq \tilde{\lambda}(B_1)$ (we refer again to [13] for a complete description of the multiplicity of solutions).

Note also that the relationship between classical solutions of (4.5) and viscosity radial solutions of (4.6) is bidirectional. Given a solution $u \in \mathcal{C}(\bar{B_1})$ of (4.6) radially symmetric and decreasing, then $z(r) = u(|x|)$ for some $x \in \Omega$ with $|x| = r$ satisfies (4.5) in the weak sense (which is equivalent to be a classical solution in this case).

By taking into account Remark 2.2, u is also a solution to our problem in the viscosity sense.

Now, for any fixed $R > 0$, we can rescale the problem and consider

$$v(r) := z\Big(\frac{r}{R}\Big).$$

It is easy to check that we arrive at the ODE

$$\begin{cases} -v''(r) - \alpha(N-1)\dfrac{v'(r)}{r} = \dfrac{\lambda}{R^2}e^{v(r)} & \text{in } (0,R), \\ v(R) = 0, & v'(0) = 0. \end{cases} \tag{4.7}$$

Summarizing, we have that there exists a positive value

$$\tilde{\lambda}(B_R) = \frac{\tilde{\lambda}(B_1)}{R^2} > 0,$$

which is decreasing with respect to R, such that problem $(P_{\lambda,p})$ admits at least a solution for every $\lambda < \tilde{\lambda}(B_R)$ in the ball of radius R, $\Omega = B_R$.

Let now Ω be a bounded domain and $R_1 > 0$ given by

$$R_1 = \min\{R > 0 : \Omega \subset B_R\}.$$

Notice that if u_{R_1} is a solution in B_{R_1} for some $\Lambda < \tilde{\lambda}(B_{R_1})$, then it is a supersolution in Ω for $\lambda \leq \Lambda < \tilde{\lambda}(B_{R_1})$. We claim that there exists a solution of problem $(P_{\lambda,p})$ with $\lambda = \Lambda$. Indeed, to prove this fact we use a standard monotone iteration argument: let $w_0 = 0$, and for every $n \geq 1$ we define the recurrent sequence $\{w_n\}$ by

$$\begin{cases} -\Delta_p^N w_n = \lambda e^{w_{n-1}} & \text{in } \Omega, \\ w_n > 0 & \text{in } \Omega, \\ w_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{Q_n}$$

The sequence $\{w_n\} \in \mathcal{C}(\bar{\Omega})$ is well defined by [18, 24]; see also [17]. Note that we are solving a problem of the form $-\Delta_p^N w_n = f$ in Ω, with $f > 0$ and $w_n = 0$ on $\partial\Omega$ as boundary condition. Then the existence is a consequence of a limit procedure involving game theory (in this problem the right-hand side, f, enters into the problem as a running payoff and the boundary condition $w_n = 0$ as the final payoff). The existence of such a solution can also be proved directly by using Perron's method thanks to our general comparison principle.

Moreover, the sequence $\{w_n\}$ is increasing with n. Indeed, taking into account that $0 < w_1$, we obtain $\lambda e^{w_0} \leq \lambda e^{w_1}$, and by using the comparison principle in Theorem 3.3, it follows that $w_1 \leq w_2$. By an inductive argument, we get $0 < w_1 \leq w_2 \leq \cdots \leq w_n$ for all $n \geq 1$. From the fact that u_{R_1} is a supersolution of problem $(P_{\lambda,p})$, with a similar inductive argument, we prove that $w_n \leq u_{R_1}$ for every $n \in \mathbb{N}$.

Since $u_{R_1} \in L^\infty(\Omega)$, the sequence $\{w_n(x)\}$ is increasing and bounded by $u_{R_1}(x)$; therefore, there exists

$$w_\lambda(x) := \lim_{n\to\infty} w_n(x).$$

In addition, thanks to the subtle Krylov–Safonov $\mathcal{C}^{0,\alpha}$-estimates of w_n for every $p \in [2, \infty]$ (here we refer to [6, 7]), we obtain that $w_n \to w_\lambda$ uniformly. Taking $f_n = \lambda e^{w_{n-1}}$ and $p_n = p$ in Lemma 4.1, we get that w_λ is a solution of problem $(P_{\lambda,p})$.

To prove that the obtained solution w_λ is minimal let v_λ be a solution of problem $(P_{\lambda,p})$. By a similar argument, using the comparison principle and induction in n, we have $w_n \leq v_\lambda$ for all $n \in \mathbb{N}$. As $w_\lambda(x) = \lim_{n\to\infty} w_n(x)$ (we use again comparison here), we obtain $w_\lambda \leq v_\lambda$.

We have thus proved that for every $\lambda < \tilde{\lambda}(B_{R_1})$ there exists a minimal solution w_λ of problem $(P_{\lambda,p})$. In particular,

$$0 < \tilde{\lambda}(B_{R_1}) \leq \lambda^*(\Omega, N, p) = \sup\{\lambda > 0 : \exists \text{ a minimal solution of } (P_{\lambda,p})\} \leq \infty.$$

Now to ensure that $\lambda^*(\Omega, N, p) < \infty$ let

$$R_2 = \max\{R > 0 : B_R \subset \Omega\};$$

we remark that without loss of generality we can assume that $0 \in \Omega$. In that way, taking w_λ, the minimal solution in Ω, as a supersolution in B_{R_2} and applying the above argument again, with Ω replaced by B_{R_2}, we obtain that $\lambda^*(\Omega, N, p) \leq \lambda^*(B_{R_2}, N, p)$.

Note that in the case $\Omega = B_r$ we can perform the previous argument starting with $w_0 = 0$ and obtain that the minimal solution is radial. In fact, by uniqueness, in this case w_n is radial for every n. Remark that in this case the unique minimal solution leads to a solution of the ODE (4.7), and thus $\lambda^*(B_{R_2}, N, p) \leq \tilde{\lambda}(B_{R_2})$. □

Remark 4.2. The arguments used in the previous proof show that the extremal parameter verifies

$$\begin{aligned} \lambda^*(\Omega, N, p) &= \sup\{\lambda > 0 : \text{there exists a minimal solution of } (P_{\lambda,p})\} \\ &= \sup\{\lambda > 0 : \text{there exists a solution of } (P_{\lambda,p})\} \\ &= \sup\{\lambda > 0 : \text{there exists a nonnegative supersolution of } (P_{\lambda,p})\}. \end{aligned}$$

Also note that

$$\lambda^*(\Omega_1, N, p) \le \lambda^*(\Omega_2, N, p) \quad \text{when } \Omega_2 \subset \Omega_1,$$

and that the extremal value for a ball, $\Omega = B_R$, is the one that corresponds to the existence of a radial solution; we refer to [13, 14] for the analysis of the resulting ODE.

In addition, we note that, if we have a solution to our problem, then the inequality

$$-\Delta_p^N u = \lambda e^u \ge \lambda u$$

holds. Therefore, we must have $\lambda \le \lambda_{1,p}(\Omega)$, where $\lambda_{1,p}(\Omega)$ is the first eigenvalue of the operator $-\Delta_p^N$ with Dirichlet boundary conditions. We conclude that

$$\lambda^*(\Omega, N, p) \le \lambda_{1,p}(\Omega).$$

5 Unbounded continua of solutions

For the reader's convenience, we recall the following general results from the theory of global continua of solutions using degree theory, which will be essential for our analysis. For the proofs we refer to [1, 16, 26].

Theorem 5.1 (Continuation theorem of Leray–Schauder). *Let X be a real Banach space, $\mathcal{O}$ an open bounded subset of X and assume that $T : \mathbb{R} \times X \to X$ is completely continuous (i.e., relatively compact and continuous). Furthermore, assume that for $\lambda = \lambda_0$ we have that $u \ne T(\lambda_0, u)$ for every $u \in \partial\mathcal{O}$ and $\deg(I - T(\lambda_0, \cdot), \mathcal{O}, 0) \ne 0$. Let*

$$\Sigma = \{(\lambda, u) \in [\lambda_0, \infty) \times X : u = T(\lambda, u)\}.$$

Then there exists a maximal connected and closed $\mathcal{C} \subset \Sigma$. Moreover, the following statements are valid:
(i) $\mathcal{C} \cap \{\lambda_0\} \times \mathcal{O} \ne \emptyset$.
(ii) *Either $\mathcal{C}$ is unbounded or $\mathcal{C} \cap \{\lambda_0\} \times X \setminus \overline{\mathcal{O}} \ne \emptyset$.*

Theorem 5.2 (Homotopy property). *Let X be a real Banach space, let $\mathcal{O}$ be an open subset of X and let $T \in \mathcal{C}([0, 1] \times \overline{\mathcal{O}}, X)$ be completely continuous in $[0, 1] \times \overline{\mathcal{O}}$. If $b : [0, 1] \to X$ is continuous and $b(t) \ne u - T(t, u)$ in $[0, 1] \times \partial\mathcal{O}$, then $\deg(I - T, \mathcal{O}, b(t))$ remains constant for all $t \in [0, 1]$.*

Theorem 5.3 (Classical Leray–Schauder's theorem). *Let X be a real Banach space, let $\mathcal{O} \subset X$ be an open and bounded subset of X and let $\Phi : [a, b] \times \overline{\mathcal{O}} \to X$ be given by $\Phi(t, u) = u - T(t, u)$, with T being completely continuous. We also assume that*

$$\Phi(t, u) \ne u \quad \text{for all } (t, u) \in [a, b] \times \partial\mathcal{O}.$$

Then, if $\deg(\Phi(a, \cdot), \mathcal{O}, 0) \ne 0$, the following assertions hold:
(i) *The equation $\Phi(t, u) = 0$ with $u \in X$ has a solution in $\mathcal{O}$ for every $a \le t \le b$.*
(ii) *There exists a closed and connected set $\Sigma_{a,b} \subset \{(t, u) \in [a, b] \times X : u = T(t, u)\}$ that intersects $t = a$ and $t = b$.*

Let us consider the operator

$$K : [0, 1] \times \mathbb{R} \times \mathcal{C}(\overline{\Omega}) \to \mathcal{C}(\overline{\Omega})$$

by defining $u := K(t, \lambda, w)$, for every $t \in [0, 1]$, $\lambda \in \mathbb{R}$ and $w \in \mathcal{C}(\overline{\Omega})$, as the unique solution in $\mathcal{C}(\overline{\Omega})$ of the problem

$$\begin{cases} -\Delta_{p(t)}^N u = \lambda^+ e^{w^+} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$p(t) = \frac{t-2}{t-1}.$$

That is, $-\Delta_{p(t)}^N u = -(1-t)\Delta u - t\Delta_\infty u$. Notice that every $p(t) \in [2,\infty]$ is labeled by a unique $t \in [0,1]$ (and conversely), thus K is well defined.

Now, we prove that K is completely continuous, which allows us to apply the Leray–Schauder degree techniques (see [16]), in order to study the existence of "continua of solutions" of $(P_{\lambda,p})$, i.e., connected and closed subsets in the solution set

$$\mathscr{S}_p = \Big\{(\lambda, u) \in [0,\infty) \times \mathcal{C}(\overline{\Omega}) : K\Big(\frac{p-2}{p-1}, \lambda, u\Big) = u\Big\}$$

for every fixed $p \in [2, +\infty]$, or, if we fixed λ instead, in

$$\mathcal{S}_\lambda = \Big\{(p, u) \in [2,\infty] \times \mathcal{C}(\overline{\Omega}) : K\Big(\frac{p-2}{p-1}, \lambda, u\Big) = u\Big\}.$$

Lemma 5.4. *Let us assume that $u_n \in \mathcal{C}(\overline{\Omega})$ satisfies*

$$\begin{cases} -\Delta_{p(t_n)}^N u_n = \lambda_n e^{w_n} & \textit{in } \Omega, \\ u_n = 0 & \textit{on } \partial\Omega, \end{cases}$$

with $t_n \in [0,1]$ and $0 \le \lambda_n$, w_n bounded in $\mathbb{R} \times \mathcal{C}(\overline{\Omega})$. Then, up to a subsequence, u_n is strongly convergent to $u \in \mathcal{C}(\overline{\Omega})$. If, in addition, $\lambda_n \to \lambda$, $t_n \to t$ and w_n converges in $\mathcal{C}(\overline{\Omega})$ to w, then u is a solution of the problem

$$\begin{cases} -\Delta_{p(t)}^N u = \lambda e^{w} & \textit{in } \Omega, \\ u = 0 & \textit{on } \partial\Omega. \end{cases}$$

Proof. If $\lambda_n = 0$, then $u_n = 0$ is the unique solution (Corollary 3.5) and the proof is immediate. In the other case, since $0 < \lambda_n e^{w_n} \le C$ for some positive constant, u_n is a subsolution of the problem

$$\begin{cases} -\Delta_{p(t_n)}^N v = C & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

It is well known, by the theory of uniformly elliptic fully nonlinear equations, that, for every fixed $n \in \mathbb{N}$, $u_n \in \mathcal{C}^{0,\nu(n)}(\overline{\Omega})$ whenever $2 \le p(t_n) \le M$ for some M sufficiently large (for instance, greater than the dimension N), with $0 < \nu(n) < 1$ (see [6, 11]). We stress that this Hölder estimates depend on the ratio between the ellipticity constants, which in this case is $p(t_n) - 1$ and, consequently, it blows-up as $p(t_n) \to \infty$. However, for $p(t_n) \in [M, \infty]$ it is shown in [7, Theorem 7] that

$$u_n \in \mathcal{C}^{0,\rho(n)}(\overline{\Omega}) \quad \text{for } \rho(n) = \frac{p(t_n) - N}{p(t_n) - 1}.$$

Thus, we can assert that the sequence $u_n \in \mathcal{C}^{0,\gamma}(\overline{\Omega})$, where $\gamma = \min\{\nu(n), \rho(n) : n \in \mathbb{N}\}$. Hence, the Ascolí–Arzelá theorem gives that u_n possesses a subsequence converging in $\mathcal{C}(\overline{\Omega})$, which concludes the first part of the lemma. Finally, the second part is a direct consequence of the uniqueness of solutions by Proposition 3.4 and Lemma 4.1. □

Proof of Theorem 1.2. For fixed $R > 0$, let $\mathcal{O}_R$ be the open ball of radius R of $\mathcal{C}(\overline{\Omega})$, and we fix some λ_R with

$$0 < \lambda_R < \frac{R}{2d^2 e^R},$$

where d is the diameter of Ω.

By Lemma 5.4, we obtain that $K \in \mathcal{C}([0,1] \times [0, \lambda_R] \times \overline{\mathcal{O}_R}, \mathcal{C}(\overline{\Omega}))$ and $K(t, \lambda, \cdot)$ is completely continuous for every $(t, \lambda) \in [0,1] \times [0, \lambda_R]$. Now, in order to apply Theorem 5.2 twice for the parameters (t, λ) with $b(t, \lambda) \equiv 0 \in \mathcal{C}(\overline{\Omega})$, we must check an *a priori bound* of the solutions of the equation $u = K(t, \lambda, u)$. That is, $u \ne K(t, \lambda, u)$ in $[0,1] \times [0, \lambda_R] \times \partial\mathcal{O}_R$. In fact, we argue by contradiction: Suppose that $\|u\|_\infty = R$ and there exist $t \in [0,1]$ and $\lambda \in [0, \lambda_R]$ such that u satisfies the equation

$$-\Delta_{p(t)}^N u = \lambda e^u \quad \text{in } \Omega,$$

hence u is a subsolution of problem

$$-\Delta^N_{p(t)} v = \lambda e^R \quad \text{in } \Omega.$$

On the other hand, a simple computation of [7, Theorem 1 and Theorem 3] shows that if $v \in \mathcal{C}(\overline{\Omega})$ is a nonnegative subsolution of the Poisson problem

$$-\Delta^N_p v = f(x) \quad \text{in } \Omega,$$

with $0 \le f \in \mathcal{C}(\overline{\Omega})$ and $p \in [2, \infty]$, then $\|v\|_\infty \le 2d^2\|f\|_\infty$. Applying this last result, we get the following contradiction:

$$R = \|u\|_\infty \le 2d^2\lambda e^R \le 2d^2\lambda_R e^R < R.$$

In this way, due to the homotopy property, we obtain

$$\deg(I - K(t, \lambda, \cdot), \mathcal{O}_R, 0) = \text{const} \quad \text{for all } (t, \lambda) \in [0, 1] \times [0, \lambda_R].$$

Moreover, since

$$K(0, \lambda, w) = (-\Delta)^{-1}(\lambda e^{w^+})$$

is the inverse of the Laplacian operator and it is well known that

$$\deg(I - K(0, 0, \cdot), \mathcal{O}_R, 0) = 1,$$

we get

$$1 = \deg(I - K(0, 0, \cdot), \mathcal{O}_R, 0) = \deg(I - K(t, \lambda, \cdot), \mathcal{O}_R, 0).$$

In order to conclude this proof, we apply the continuation theorem of Leray–Schauder (Theorem 5.1) with $T(\lambda, u) = K(t, \lambda, u)$ for every fixed $t \in [0, 1]$, which is completely continuous (Lemma 5.4). Therefore, using that $\deg(I - T(0, \cdot), \mathcal{O}_R, 0) = 1 \ne 0$, we can assert that there exists a maximal connected subset $\mathcal{C}$ of $\mathscr{S}_p$ that contains $(0, 0)$. Furthermore, $\mathcal{C}$ is not bounded since 0 is the unique solution for $\lambda = 0$. Finally, since for every λ such that there is a solution of $(P_{\lambda,p})$ we can construct a minimal solution, we can state that $\mathcal{C} \subset [0, \lambda^*] \times \mathcal{C}(\overline{\Omega})$.

With the same arguments, using Theorem 5.3 with $T(t, u) = K(t, \lambda, u)$ and $[a, b] = [0, 1]$, for every fixed

$$\lambda \in \Big(0, \lambda_0 = \min\Big\{\lambda^*(\Omega, N, 2), \frac{1}{2d^2 e}\Big\}\Big)$$

we can obtain the existence of a continuum of solutions moving $p \in [2, \infty]$. More precisely, since

$$\deg(I - K(0, \lambda, \cdot), \mathcal{O}_1, 0) = 1,$$

we can apply Theorem 5.3 obtaining the existence of a continuum $\Sigma_{0,1} \subset \{(t, u) \in [0, 1] \times \mathcal{O}_1 : u = T(t, u)\}$ such that $\operatorname{Proj}_{[0,1]}\Sigma_{0,1} = [0, 1]$. Note that the upper bound for λ is used to ensure an a priori bound. Thus, we finish the proof by taking

$$\mathcal{D} = \Big\{\Big(\frac{t-2}{t-1}, u\Big) \in [2, +\infty] \times \mathcal{O}_1 : (t, u) \in \Sigma_{0,1}\Big\}. \qquad \square$$

Remark 5.5. Now we briefly comment on possible extensions for more general nonlinearities. Note that we can also deal with the equation

$$-\Delta^N_p u = \lambda f(u),$$

with a general continuous nonlinearity f that verifies $f(0) > 0$, $f(s)/s \ge k > 0$ and is increasing. In fact, we only need to show the existence and nonexistence of radial solutions (the rest of the arguments can be extended without much difficulties). Hence we arrive at the problem

$$\begin{cases} -z''(r) - \theta\dfrac{z'(r)}{r} = \lambda f(z(r)), & r \in (0, 1), \\ z(r) > 0, & r \in (0, 1), \\ z(1) = z'(0) = 0, \end{cases}$$

where $\theta = (N-1)/(p-1) \in [0, \infty)$ due to the fact that $p \in [2, \infty]$. Multiplying by r^θ and integrating twice, we obtain

$$\begin{aligned} z(r) &= \lambda \int_r^1 \frac{1}{\tau^\theta} \int_0^\tau s^\theta f(z(s))\,ds\,d\tau \\ &\geq \lambda \int_r^1 \frac{1}{\tau^\theta} \int_0^r s^\theta f(z(s))\,ds\,d\tau \\ &\geq \lambda \int_r^1 \frac{1}{\tau^\theta} \int_0^r s^\theta f(z(r))\,ds\,d\tau. \end{aligned}$$

Therefore, for every $r \in (0, 1)$ it must hold that

$$\frac{1}{k} \geq \frac{z(r)}{f(z(r))} \geq \lambda \int_r^1 \int_0^r \Big(\frac{s}{\tau}\Big)^\theta\,ds\,d\tau := \lambda F_\theta(r).$$

As $F_\theta(r)$ is positive in $(0, 1)$ and is bounded above, we conclude that $\lambda \leq 1/(c(\theta)k)$. Hence there is no solution for λ greater than a constant that depends only on p and N.

To look for the existence of solutions for small λ we can use degree theory for the operator

$$T : [0, \infty) \times C([0, 1]) \to C([0, 1])$$

given by

$$T(\lambda, u) = \lambda \int_r^1 \frac{1}{\tau^\theta} \int_0^\tau s^\theta f(u(s))\,ds\,d\tau.$$

Since f is assumed to be continuous, it is easy to check that T is completely continuous. Now, as $T(0, u) = 0$ for every $u \in C([0, 1])$, using Leray–Schauder's theorem, we obtain the existence of a continuum of solutions $\mathcal{C} \subset [0, \infty) \times C([0, 1])$ that is unbounded with $(0, 0) \in \mathcal{C}$. In particular, there exist solutions for values of λ close to 0.

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