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Hesitant Fuzzy Linguistic Analytic Hierarchical Process With Prioritization, Consistency Checking, and Inconsistency Repairing

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ABSTRACT Analytic hierarchy process (AHP), as one of the most important methods to tackle multiple criteria decision-making problems, has achieved much success over the past several decades. Given that linguistic expressions are much closer than numerical values or single linguistic terms to a human way of thinking and cognition, this paper investigates the AHP with comparative linguistic expressions. After providing the snapshot of classical AHP and its fuzzy extensions, we propose the framework of hesitant fuzzy linguistic AHP, which shows how to yield a decision for qualitative decision-making problems with complex linguistic expressions. First, the comparative linguistic expressions over criteria or alternatives are transformed into hesitant fuzzy linguistic elements and then the hesitant fuzzy linguistic preference relations (HFLPRs) are constructed. Considering that HFLPRs may be inconsistent, we conduct consistency checking and improving processes after obtaining priorities from the HFLPRs based on a linear programming method. Regarding the consistency-improving process, we develop a new way to establish a perfectly consistent HFLPR. The procedure of the hesitant fuzzy linguistic AHP is given in stepwise. Finally, a numerical example concerning the used-car management in a lemon market is given to illustrate the efficiency of the proposed hesitant fuzzy linguistic AHP method.

INDEX TERMS Analytic hierarchical process, hesitant fuzzy linguistic term set, hesitant fuzzy linguistic preference relation, multiple criteria decision making, used-car management.

I. INTRODUCTION

As an important research branch of decision-making theory, Multiple Criteria Decision Making (MCDM) has gained great success in management science and operations research [1]. The goal of a MCDM problem is to select the best alternative from an infinite alternative set based on some determined decision matrices. In this sense, how to establish such decision matrices is very important and this determines the reasonability degree of final results. Due to the complexity of the problem and the limitation of Decision-Makers' (DMs') knowledge and cognition, DMs may prefer to use linguistic terms or expressions to directly express their subjective feelings and perceptions, especially over the qualitative criteria. To overcome the inability of traditional fuzzy linguistic approach [2], Rodríguez *et al.* [3] introduced the Hesitant Fuzzy Linguistic Term Set (HFLTS) together with context-free grammars to represent cognitive complex linguistic information of DMs. With the HFLTS, DMs can provide their uncertain assessments by means of several linguistic terms or comparative linguistic expressions [4], [5]. The HFLTS opens a door for the possibility of developing comprehensive and powerful decision theories and methodologies based on complex linguistic knowledge [6]. In this

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paper, we focus on investigating the qualitative MCDM problems in which the assessments of DMs are given as several linguistic terms or comparative linguistic expressions represented by Hesitant Fuzzy Linguistic Elements (HFLEs) [4].

Analytical Hierarchy Process (AHP) [7] is one of the most useful methods to solve MCDM problems. As can be seen from Section II, many achievements have been obtained over the past several decades. However, it is worth noting that the classical AHP is not designed for capturing uncertain preferences of human perceptions [8]. Krejčí [9] illustrated by an example that, ignoring the uncertainty of preference intensities leads to significant information loss in MCDM problems. In other words, neglecting imprecision in preference relations may cause irrational results. To avoid this drawback, scholars have extended the classical AHP into different contexts [10]–[18]. The significant difference between these extensional AHP models comes from different representation forms of preference intensities of DMs.

When uncertainty is allowed in evaluations, the matrices given by DMs are often inconsistent. The consistency properties are important intrinsic characteristics of preference relations, which is indispensable to guarantee that the final ranking results are reasonable. That is to say, the consistency of the preference information provided by DMs is necessary in decision-making process. The additive consistency or multiplicative consistency plays a key role in deriving priorities from preference relations [19]. If the given information were not consistent, the result deduced from such inconsistent information would be incorrect. To avoid producing wrong results, the consistency improving procedure for inconsistent preference information is needed. We note that there are some achievements about the consistency of hesitant fuzzy linguistic preference relations (HFLPRs) [20]–[22]. However, all the above achievements regarding the consistency of HFLPRs were based on normalizing the HFLPRs by artificially adding elements for the shorter HFLEs. Such a normalization process may lead to original information loss and biased. To overcome this disadvantage in HFLPRs, in Section III-B, we introduce a new definition of normalized HFLPRs based on continuous intervals.

Prioritization is the most important step in AHP method. There are several widely used methods to derive priorities from preference relations in AHP method, such as the eigenvector method [7], the least squares method [23], the logarithmic least square method [24] and the linear programming method [25]. Mikhailov [25] proved that in uncertain situations, the linear programming method performs better than other three methods. Thus, the linear programming method is investigated to derive priorities from HFLPRs in Section IV-B.

Given that the HFLTS is a recently developed and useful tool in representing cognitive complex linguistic information, scholars [26], [27] have started to investigate the hierarchical MCDM problems with hesitant linguistic information. However, there are potential improvements by analyzing these models:

(1) The models in Refs [26], [27] did not consider the consistency checking and repairing processes for the HFLPRs.

(2) These two models had an assumption that the midpoint of each HFLE in an HFLPR is the most possible value to denote the DM's preference.

(3) The models in Refs [26], [27] obtained priorities by simple normalizing of the summations of each row of the HFLPRs. As we know, this priority-determining method is only valid for multiplicative preference relations on condition that the multiplicative preference relation has totally multiplicative consistency. Since the HFLPR is not multiplicative preference relation and the consistency property of the HFLPR has not been checked in the models proposed in Ref. [26], [27], such a priority-deriving method is questionable in some situations.

To fill the above-mentioned gaps and challenges, this paper aims to propose an integrated framework of hesitant fuzzy linguistic analytic hierarchical process, named as HFL-AHP for short, to broaden the application scope of traditional AHP method. Specifically, we try to achieve following goals:

(1) We introduce a new definition of HFLPR based on intervals, which retains the continuous semantics of cognitive linguistic expressions given by DMs, rather than artificially adds meaningless terms in the intervals to normalize the HFLEs with different lengths.

(2) We introduce an interesting way to build the multiplicative consistent HFLPR during the consistency repairing process, which considers the confidence level of evaluations given by DMs and maintains the originally-confident evaluations as much as possible.

(3) We use a linear programming method to derive the priorities from HFLPRs in the HFL-AHP method, which prevents the priorities with low consistency and ensures the high reliability of final results.

The rapid development of the used-car market happens in China. How to identify a high quality used-car is an interesting and practical decision-making problem. In this paper, we apply the proposed HFL-AHP method to choose used-cars to show the validity and efficiency of the proposed method.

The reminder of this paper is organized as follows: Section II shows a snapshot of AHP method and its extensions. Section III reviews some concepts related to HFLPRs and proposes a new definition of HFLPR based on intervals. Section IV gives the framework of the HFL-AHP and its detailed procedure. A case study is given in Section V and some comparative analyses with the previous hesitant fuzzy linguistic hierarchical methods are provided to illustrate the reasonability of the proposed method. The paper ends in Section VI with some concluding remarks.

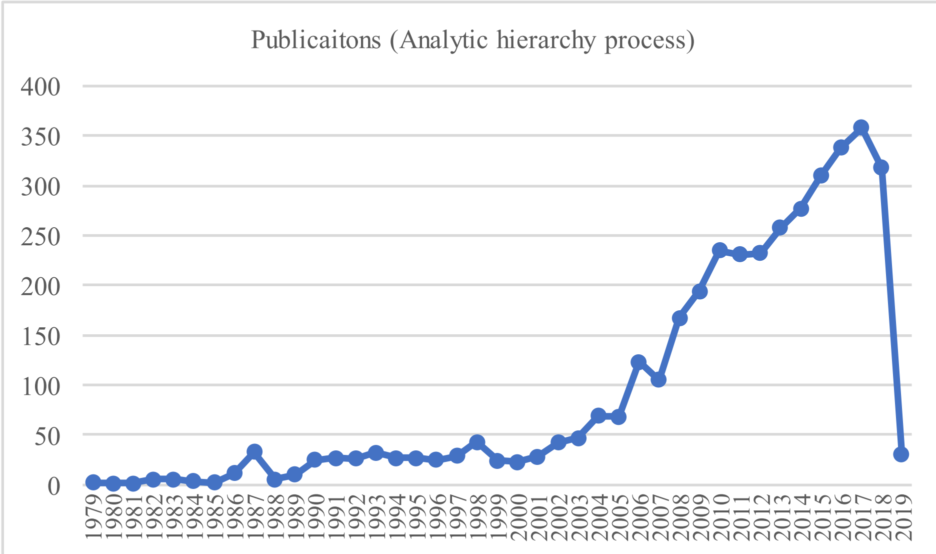

FIGURE 1. The number of publications by year related to AHP from 1979 to 2019 (February 16).

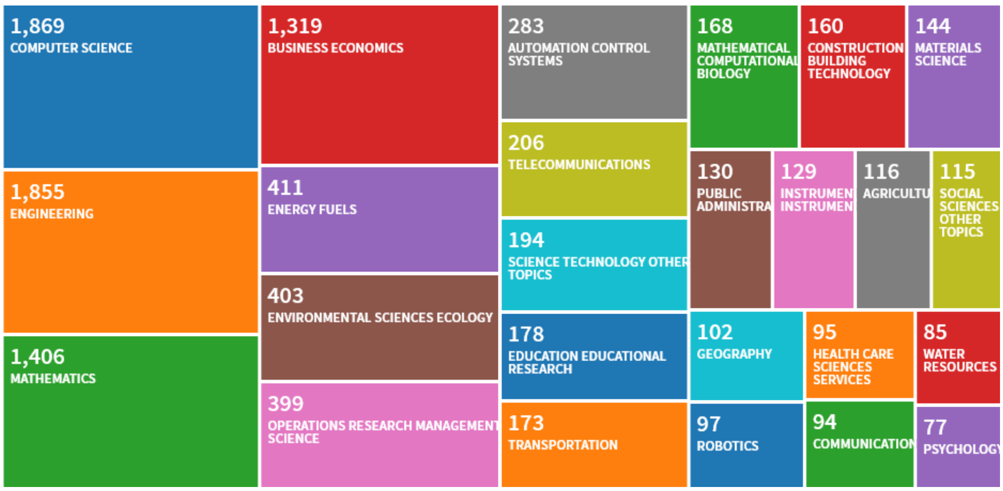

FIGURE 2. Top 25 different application areas of AHP.

II. A SNAPSHOT OF ANALYTIC HIERARCHY PROCESS AND ITS EXTENSIONS

In this section, we introduce a snapshot of AHP in terms of the classical AHP and its fuzzy extensions.

A. THE CLASSICAL AHP

AHP was proposed by Saaty [7] to tackle the problems with either qualitative or quantitative information. Basically, five main steps are designed in the AHP method: (1) problem decomposition, (2) pairwise comparison, (3) priorities deduction, (4) priorities synthesis and (5) alternatives ranking.

Firstly, the problems should be decomposed into the goal level, the criterion level, the sub-criterion level if necessary, and the alternatives level. Secondly, according to the criterion level and the sub-criterion level, pairwise comparisons can be made between criteria and alternatives. Thirdly, priorities of criteria and alternatives are then derived by some priorities-deriving methods. Consistency checking is a key process before deducing priorities. Then, the final scores of alternatives can be obtained by aggregating the priorities of criteria and the ratings of alternatives over all criteria. Finally, the ranking of alternatives is deduced according to the final scores of alternatives.

As AHP method is flexible in handling complex and comprehensive decision-making problems, it has attracted many scholars' attention and achieved great success over the past several decades [8]. More importantly, the number of publications by year related to AHP is almost constantly increasing year by year. Figure 1 shows the number of publications by year related to AHP. From Fig. 1, we can find that AHP is still a hot research topic at present.

Figure 2 illustrates the top 25 application areas of AHP. In Fig. 2, the areas of colored rectangles show the numbers of applications with the AHP method. As can be seen, the most

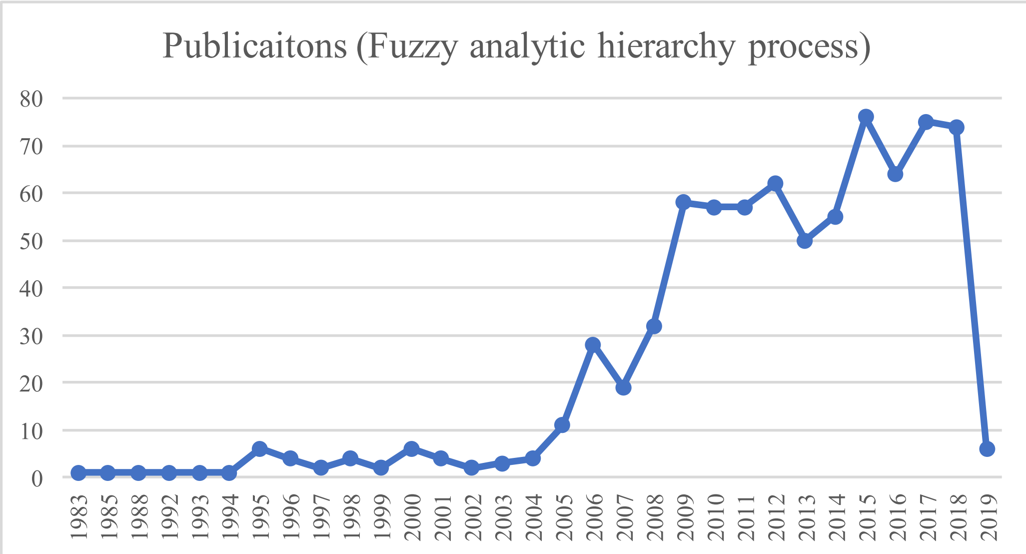

FIGURE 3. The number of publications by year related to fuzzy AHP from 1979 to 2019 (February 16).

popular application domains are computer science, engineering, mathematics, and business economics.

B. FUZZY EXTENSIONS OF AHP

Nonetheless, in many situations, DMs might be reluctant or unable to determine crisp evaluation values to the pairwise comparison judgments due to their limited knowledge, or the subjectivity of qualitative evaluation criteria, or variations of individual judgments in group decision making. The classical AHP method has been extended to fuzzy environment, which has become the most popular research direction regarding the AHP method [8]. The annual trend of fuzzy AHP-related publications is presented in Fig. 3 based on the retrieval results from Web of Science. Figure 3 reveals the gradually increasing popularity of fuzzy extensions on AHP.

Fuzzy extension of AHP was firstly proposed by Laarhoven and Pedrycz [10] using triangular fuzzy numbers to depict the uncertain information. Buckley [11] extended the AHP with trapezoidal fuzzy numbers and obtained the fuzzy weights by geometric mean method. After that, Chang [12] used a row mean method to derive priorities for comparison ratios within the context of triangular fuzzy numbers. Since fuzzy set contains only the membership information, it cannot fully characterize people's degree of uncertainty about things. Atanassov [28] held the view that uncertainty can be composed of membership, non-membership and hesitancy, and then proposed the intuitionistic fuzzy set to align humans' thoughts. Given the efficiency of intuitionistic fuzzy set, Xu and Liao [13] presented intuitionistic fuzzy extension of AHP method to precisely depict uncertainty from three aspects. Afterwards, Liao *et al.* [14] further proposed the intuitionistic fuzzy analytic network process. Using a single value range from 0 to 1 to describe the ambiguity or uncertainty may not be applicable in some situations. Therefore, Torra [29] proposed the hesitant fuzzy set. With the hesitant fuzzy set, a set of multiple possible values can be used to represent the degree of membership. Based on more precise description of preference degree described by hesitant fuzzy sets, Zhu and Xu proposed hesitant AHP [15] and hesitant group AHP methods [16].

Motivated by Refs. [15], [26], [27] and the efficiency of HFLTS in representing cognitive complex linguistic information, this paper concentrates on improving the AHP with the qualitative linguistic expressions expressed as HFLTSs. The first and last steps of the HFL-AHP method do decomposition and synthesis, respectively. These steps are the same as the previous HFL-AHP methods. The most difference appears in remaining steps: pairwise comparison, consistency checking and improving processes, and priorities deducing. Hence, in this paper, we pay more attention to pairwise comparison process in Section III. Furthermore, consistency properties and prioritization of HFLPRs are investigated in Section IV.

III. HESITANT FUZZY LINGUISTIC PREFERENCE RELATION AND ITS CONSISTENCY

In this section, we introduce some concepts about HFLPRs and a new definition of HFLPRs based on intervals.

A. HFLPR

Generally speaking, there are two forms of information regarding the evaluation values in decision matrices: quantitative and qualitative information. For quantitative information, numerical values are used to represent the judgments of Decision-Makers (DMs) over the alternatives with respect to different criteria. Nevertheless, due to the complexity of the problem and the limitation of DMs' knowledge and cognition, it is sometimes difficult for DMs to provide crisp values as their judgments, especially over the qualitative criteria.

DMs may prefer to use linguistic terms or words to directly express their subjective feelings and perceptions, for example, "*low*" cost, "*high*" quality, "*good*" performance. Therefore, the decision theories and methods based on linguistic terms [2] have wide application potentials.

Traditional fuzzy linguistic approach uses only singleton linguistic term to represent the value of a linguistic variable, but cannot represent inaccurate and ambiguous linguistic expressions which often exist in DMs' assessments. For example, when evaluating a research proposal, a DM may say "*it is between good and excellent*" as he/she is uncertain or hesitant about his/her opinion. To denote the hesitancy over several linguistic terms, Rodríguez *et al.* [3] firstly defined the HFLTS as an ordered finite subset of the consecutive linguistic terms of a linguistic term set. For better understanding and broader use of HFLTS, Liao *et al.* [4] redefined the concept of HFLTS in mathematical form to enhance the computing with linguistic expressions and gave the score function of the HFLE.

Definition 1 [4]: Let $S = \{s_\alpha | \alpha \in \{-\tau, \cdots, 0, \cdots, \tau\}\}$ be a linguistic term set. An HFLTS on X, H_S, is in mathematical form of $H_S = \{\langle x, h_S(x)\rangle | x \in X\}$ where $h_S(x) = \{s_{\phi_l}(x) | s_{\phi_l}(x) \in S,\ \phi_l \in \{-\tau, \cdots, 0, \cdots, \tau\} l = 1, 2, \cdots, L(x)\}$ with $L(x)$ being the number of linguistic terms in $h_S(x)$ and $s_{\phi_l}(x)$ $(l = 1, 2, \cdots L(x))$ being the continuous terms in S. $h_S(x)$ represents the set of possible degrees of the linguistic variable x to S and is named as the HFLE.

The score function of the HFLE h_S is [30]

$$M(h_S) = s_{\bar{t}},\ \text{where } \bar{t} = \frac{1}{L}\sum\nolimits_{l=1}^{L} \phi_l \tag{1}$$

with L being the number of linguistic terms in h_S, and ϕ_l being the subscript of the linguistic term s_{ϕ_l} in h_S. Other score functions about the HFLE can be found in Ref. [31] and a state-of-the art survey on decision making with HFLTSs can be seen in Refs. [5], [6]. Other extensions of the HFLTS have been proposed recently [32]–[34].

Transformation rules [3] were given to convert linguistic expressions into HFLEs. After transformations, Rodríguez *et al.* [3] proposed the envelope form of HFLE as an interval of a linguistic variable by considering continuous semantics in human perceptions. The envelope form is mathematically given as

$$env\,(h_S) = \left[h_S^-,\ h_S^+\right] \tag{2}$$

where h_S^- and h_S^+ are the lower and upper bounds of h_S, respectively. The transformation rules and the corresponding envelopes of HFLEs are listed in Table 1 for clear understanding.

Preference relation is a useful tool to depict the preference intensities of pairwise comparisons over a set of objects. Considering the cognitive complex linguistic information for pairwise comparisons as HFLTS, Rodríguez *et al.* [3] firstly introduced the concept HFLPR. Later, Zhu and Xu [20] defined the HFLPR in mathematical form.

TABLE 1. The transformation rules from linguistic expressions to HFLEs and their envelopes.

Transformation rules	Envelopes of HFLEs
$TR_1(s_q) = \{s_q \mid s_q \in S\}$	$env(h_S) = [s_q, s_q]$
$TR_2(\textit{less than } s_j) = \{s_q \mid s_q \in S \textit{ and } s_q \le s_j\}$	$env(h_S) = [s_{-\tau}, s_j]$
$TR_3(\textit{greater than } s_i) = \{s_q \mid s_q \in S \textit{ and } s_q \ge s_i\}$	$env(h_S) = [s_i, s_\tau]$
$TR_4(\textit{between } s_i \textit{ and } s_j) = \{s_q \mid s_q \in S \textit{ and } s_i \le s_q \le s_j\}$	$env(h_S) = [s_i, s_j]$

Definition 2 [20]: An HFLPR is presented by a matrix $H = (h_S^{kj})_{n\times n} \subset X \times X$, where $h_S^{kj} = \{h_S^{kj\rho(l)} \big| l = 1, 2, \cdots, L_{h_S^{kj}}\}$ ($L_{h_S^{kj}}$ is the number of linguistic terms in h_S^{kj}) is an HFLE, representing the hesitant linguistic degrees to which x_k is preferred to x_j. For all $k, j = 1, 2, \cdots, n$, $h_S^{kj} (k < j)$ should meet the conditions as follows:

$$\begin{aligned} h_S^{kj\rho(l)} + h_S^{jk\rho(l)} &= s_0, h_{kk} = s_0, L_{h_S^{kj}} = L_{h_S^{jk}}, \\ h_S^{kj\rho(l)} &< h_S^{kj\rho(l+1)}, h_S^{jk\rho(l+1)} < h_S^{jk\rho(l)} \end{aligned} \tag{3}$$

where $\rho(l)$ is the permutation of $1, 2, \cdots, L_{h_S^{kj}}$ to guarantee that $h_S^{kj\rho(l)}$ and $h_S^{jk\rho(l)}$ are the lth linguistic terms in h_S^{kj} and h_S^{jk}, respectively.

The prerequisite of Eq. (3) is that the length of h_S^{kj} and h_S^{jk} should be equal. To achieve this goal, a normalization scheme is introduced to add linguistic terms to make the lengths be equal, shown as follows:

$$\hat{h}_S^{kj} = \theta \cdot h_S^{kj+} + (1-\theta) \cdot h_S^{kj-} \tag{4}$$

where h_S^{kj+} and h_S^{kj-} are the maximal and minimal elements in h_S^{kj}, respectively. $\theta \in [0, 1]$ is a control parameter to denote the attitude of DMs [20].

Consistency is an important property for preference relations. Additive consistency and multiplicative consistency are two common properties of preference relations based on transitivity. For example, Zhu and Xu [20] investigated the additive consistency of HFLPRs based on normalizations. Since the additive consistency is sometimes unreasonable [19], Zhang and Wu [21] gave a definition of multiplicative consistent HFLPR based on the normalized HFLPR $\bar{H} = (\bar{h}_S^{kj})_{n\times n}$ on the linguistic term set $S = \{s_\alpha | \alpha \in \{0, 1, 2, \cdots, g, \cdots, 2g\}\}$.

Definition 3 [21]: Let $H = (h_S^{kj})_{n\times n}$ be an HFLPR and $\bar{H} = (\bar{h}_S^{kj})_{n\times n}$ be a normalized HFLPR with θ. $I\left(\bar{h}_S^{kj\rho(l)}\right)$ means the subscript of the lth linguistic term in $\bar{h}_S^{kj}$. If

$$\begin{aligned} &I\left(\bar{h}_S^{kt\rho(l)}\right) I\left(\bar{h}_S^{tj\rho(l)}\right) I\left(\bar{h}_S^{jk\rho(l)}\right) \\ &\quad = I\left(\bar{h}_S^{tk\rho(l)}\right) I\left(\bar{h}_S^{kj\rho(l)}\right) I\left(\bar{h}_S^{jt\rho(l)}\right), \quad k, t, j = 1, 2, \ldots, n \end{aligned} \tag{5}$$

then $H = (h_S^{kj})_{n\times n}$ is a multiplicative consistent HFLPR with θ.

Furthermore, Tang *et al.* [22] defined the incomplete HFLPR and developed a method to estimate unknown elements of the incomplete HFLPR based on the additive consistency properties.

Note: The consistency properties of HFLPR in Definition 3 [21] and in Refs. [20], [22] are based on the normalized HFLPR. The HFLPR is normalized by adding elements artificially until the lengths of all elements in the HFLPR are equal. Such a normalization method is rude and may change the original information given by the DM. To overcome this deficiency, the HFLPR based on intervals without adding additional terms is proposed in next subsection.

B. NEW DEFINITION OF HFLPR BASED ON ENVELOPES

To avoid numerous meaningless elements in HFLPRs and retain the uncertainty of cognitive information provided by DMs, we can convert the cognitive linguistic expressions into intervals, which also represent the continuous semantics of the cognitive linguistic expressions. The semantic intervals can be obtained by the envelops of HFLEs. In this way, a new form of HFLPR based on intervals is defined as follows:

$$\begin{pmatrix} env\left(h_S^{11}\right) & env\left(h_S^{12}\right) & \cdots & env\left(h_S^{1n}\right) \\ env\left(h_S^{21}\right) & env\left(h_S^{22}\right) & \cdots & env\left(h_S^{2n}\right) \\ \vdots & \vdots & \cdots & \vdots \\ env\left(h_S^{n1}\right) & env\left(h_S^{n2}\right) & \cdots & env\left(h_S^{nn}\right) \end{pmatrix} = \begin{pmatrix} \left[h_S^{11-}, h_S^{11+}\right] & \left[h_S^{12-}, h_S^{12+}\right] & \cdots & \left[h_S^{1n-}, h_S^{1n+}\right] \\ \left[h_S^{21-}, h_S^{21+}\right] & \left[h_S^{22-}, h_S^{22+}\right] & \cdots & \left[h_S^{2n-}, h_S^{2n+}\right] \\ \vdots & \vdots & \cdots & \vdots \\ \left[h_S^{n1-}, h_S^{n1+}\right] & \left[h_S^{n2-}, h_S^{n2+}\right] & \cdots & \left[h_S^{nn-}, h_S^{nn+}\right] \end{pmatrix} \quad (6)$$

where the elements in the HFLPR are intervals and the normalization of the HFLPR is no necessary.

There are two kinds of scales associated to numerical scores [35] with respect to different types of consistency. One is $[-g, g]$ scale or $[0, 2g]$ scale based on additive consistency of preference relations. These two scales are interchangeable. The $[0, 2g]$ scale can be transformed into the $[-g, g]$ scale. For simplicity of notations, we use the $[-g, g]$ scale for illustration. In the $[-g, g]$ scale, if a DM holds the view that object A_i is better than A_j, then the preference degree should be greater than zero. The other scale is the $\left[a^{-g}, a^{g}\right]$ exponential scale based on multiplicative consistency, where a indicates the difference between two adjacent evaluation levels and a is usually greater than 1. Saaty [35] illustrated that the $\left[a^{-g}, a^{g}\right]$ exponential scale aligns human perceptions much closer than the $[-g, g]$ scale by an example about "Judge chairs' lightness in visual". Besides, the geometrical scale $\left[a^{-g}, a^{g}\right]$ is a transitive scale by exponential operations, which is a good alternative to solve the criticism of the original 1-9 scale's intransitivity in AHP [36].

TABLE 2. The basic linguistic terms ($g = 7$) and their corresponding utility values.

Sensations/Perceptions	Linguistic terms	$[-g, g]$ scale	$\left[a^{-g}, a^{g}\right]$ exponential scale
Extremely bad	s_{-7}	-7	a^{-7}
Very bad	s_{-5}	-5	a^{-5}
Bad	s_{-3}	-3	a^{-3}
Slightly bad	s_{-1}	-1	a^{-1}
Equal	s_0	0	a^{0}
Slightly good	s_1	1	a^{1}
Good	s_3	3	a^{3}
Very good	s_5	5	a^{5}
Extremely good	s_7	7	a^{7}

Table 2 tabulates the basic linguistic terms associated to different scales. The relation between perceptions and the $[-g, g]$ scale is linear mapping, and the mapped values of the $[-g, g]$ scale are utility values in humans' thoughts based on additive consistency. While the relation between sensations and the $\left[a^{-g}, a^{g}\right]$ scale is exponential mapping, and the mapped values of the $\left[a^{-g}, a^{g}\right]$ scale are utility values in humans' thoughts in physics based on multiplicative consistency.

Based on continuous semantic intervals of cognitive linguistic expressions, the cognitions given by DMs are understandable in calculation process since we maintain the continuous intervals rather than multiple discrete and meaningless terms in these intervals. This is the advantage of the new definition of HFLPR based on intervals comparing with Definition 2.

Then, motivated by the multiplicative consistency of intervals [37], a new definition of multiplicative consistent HFLPR is introduced as follows:

Definition 4: Let $H_S = (h_S^{kj})_{n\times n}$ be an HFLPR and $env\,(H_S)_{n\times n} = env\left(h_S^{kj}\right)_{n\times n}$ be the envelope form of the HFLPR. $U\left(h_S^{kj}\right) = \left[U\left(h_S^{kj-}\right), U\left(h_S^{kj+}\right)\right]$ means the interval utility value of the envelope $env\left(h_S^{kj}\right)$ corresponding to the HFLE h_S^{kj}. If

$$U\left(h_S^{kj-}\right)U\left(h_S^{kj+}\right) = U\left(h_S^{kt-}\right)U\left(h_S^{kt+}\right)U\left(h_S^{tj-}\right)U\left(h_S^{tj+}\right), \quad k, t, j = 1, 2, \ldots, n \quad (7)$$

then $H = (h_S^{kj})_{n\times n}$ is a multiplicative consistent HFLPR.

For the continuous interval definition of HFLPR based on transitivity, if no uncertain information is provided or only one linguistic term is given in each evaluation, the interval definition would reduce to the crisp situation, i.e., $U\left(h_S^{kj}\right) = U\left(h_S^{kt}\right) \times U\left(h_S^{tj}\right)$. For the evaluations of two objects, $U\left(h_S^{kj}\right) \times U\left(h_S^{jk}\right) = a^{kj} \times a^{jk} = a^{kj+jk} = a^{kj-kj} = a^0 = 1$ can be obtained.

Different types of consistency properties of HFLPR were investigated based on different scales. Considering that the exponential scale aligns much closer to human thoughts and the additive consistency is unreasonable at times [19], in this paper, after obtaining the interval utility values of HFLEs, the multiplicative consistency of HFLPR with intervals is investigated and employed to derive the priorities from the HFLPR.

Motivated by the relationship between multiplicative consistent intuitionistic fuzzy preference relation and its corresponding priorities [38], the relation between multiplicative consistent HFLPR and its corresponding weights based on Definition 4 can be presented as follows:

Definition 5: Let $H_S = (h_S^{kj})_{n\times n}$ be an HFLPR and $env\,(H_S)_{n\times n} = env\left(h_S^{kj}\right)_{n\times n}$ be its envelope form. $U\left(h_S^{kj}\right) = \left[U\left(h_S^{kj-}\right), U\left(h_S^{kj+}\right)\right]$ denotes the interval utility value of $env\left(h_S^{kj}\right)$. If there exists a vector $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$, such that

$$U\left(h_S^{kj-}\right) \le \omega_k/\omega_j \le U\left(h_S^{kj+}\right), \quad k, j = 1, 2, \cdots, n \tag{8}$$

where $\omega_j \ge 0$, $\sum_{j=1}^{n} \omega_j = 1$, then H_S is a multiplicative consistent HFLPR.

IV. THE FRAMEWORK OF THE HFL-AHP METHOD

This section develops the complete framework of the HFL-AHP method. We first make a conceptualization about the MCDM problems with cognitive linguistic information in Section IV-A. Next, we introduce an interval linear programming method in Section IV-B to derive priorities from the HFLPR whose corresponding utility values are denoted by the $\left[a^{-g}, a^{g}\right]$ exponential scale. Meanwhile, the consistency index could be deduced within the interval linear programming method. An inconsistency repairing process is introduced in Section IV-C for implementing HFL-AHP method. The procedure of the HFL-AHP method are summarized in Section IV-D. We should note that the existing literature such as Refs. [26], [27] did not take the consistency checking and repairing process into consideration and their priorities deducing methods are questionable in some situations.

A. DESCRIPTION OF THE HESITANT FUZZY LINGUISTIC MCDM PROBLEMS

The steps for disaggregation of AHP contain constructing the decision-making model and determining the pairwise comparison matrices under the hesitant fuzzy linguistic environment. The first step of AHP is to decompose a problem into three levels: the goal level, the criteria level (sometimes may include the sub-criteria level) and the alternative level.

Suppose that a MCDM problem can be denoted as a goal G. Let $A = \{A_1, A_2, \cdots, A_i, \cdots, A_m\}$ be a set of alternatives and $C = \left\{C_1, C_2, \cdots, C_j, \cdots, C_n\right\}$ be a set of criteria. DMs are asked to evaluate alternatives over each criterion, and the pairwise preference intensities of alternatives on each criterion form the individual preference relation. As we mentioned above, the preference intensities can be represented by linguistic expressions. Regarding the criteria, their relative importance can also be denoted by linguistic expressions. Based on the transformation function [3], these linguistic expressions can be transformed to HFLEs.

To simplify the presentation, below we take the HFLPR about the criteria as an example. The analyses regarding the HFLPRs on alternatives over the criteria are similar. We suppose that the linguistic expressions on criteria are denoted as ll_{kj} $(k, j = 1, 2, \cdots, n)$. Then, by the transformation function [3], an HFLPR on criteria is obtained as follows:

$$LL_S = \begin{pmatrix} ll_S^{11} & ll_S^{12} & \cdots & ll_S^{1n} \\ ll_S^{21} & ll_S^{22} & \cdots & ll_S^{2n} \\ \vdots & \vdots & \cdots & \vdots \\ ll_S^{n1} & ll_S^{n2} & \cdots & ll_S^{nn} \end{pmatrix} \overset{\text{transformation function}}{\longrightarrow} H_S$$
$$= \begin{pmatrix} h_S^{11} & h_S^{12} & \cdots & h_S^{1n} \\ h_S^{21} & h_S^{22} & \cdots & h_S^{2n} \\ \vdots & \vdots & \cdots & \vdots \\ h_S^{n1} & h_S^{n2} & \cdots & h_S^{nn} \end{pmatrix} \overset{\text{bound measure}}{\longrightarrow} env\,(H_S) \tag{9}$$

B. LINEAR PROGRAMMING MEHTOD FOR PRIORITIZATION OF HFLPR

The priorities of a preference relation depict the intrinsic importance in human's thoughts. In this regard, Mikhailov [25] developed a fuzzy programming method to derive crisp weights from a fuzzy preference relation. Motivated by this idea, a linear programming method for prioritization of HFLPR with intervals is developed in this section.

According to Definition 5, when the ratio of two criteria's weights, ω_i, ω_j, belongs to $\left[U\left(h_S^{ij-}\right), U\left(h_S^{ij+}\right)\right]$, the corresponding HFLPR is multiplicative consistent. As we can see, Eq. (8) is equivalent to Eq. (10) by multiply ω_j:

$$\begin{cases} \omega_j \times U\left(h_S^{ij-}\right) - \omega_i \le 0 \\ \omega_i - \omega_j \times U\left(h_S^{ij+}\right) \le 0, \end{cases}$$
$$i = 1, 2, \cdots, n-1; \quad j = 2, \cdots, n \tag{10}$$

where the number of inequations in Eq. (10) is $n\,(n-1)$ because n inequations of the evaluations on n objects and themselves are not necessary by knowing constant values of their weights.

The above inequations can be rewritten in the form of a matrix for simplicity:

$$R\omega \le 0 \tag{11}$$

where the matrix R has $n\,(n-1)$ rows and n columns.

Mikhailov [25] proposed a membership function $M\,(\omega)$ for the relations between the evaluations and their corresponding weights' ratios in consistent or inconsistent situations. Fig. 4 shows the relations between the ranges of ω_i/ω_j and its weight's membership degrees. When ω_i/ω_j ranges from $U\left(h_S^{ij-}\right)$ to $U\left(h_S^{ij+}\right)$, there exists consistent

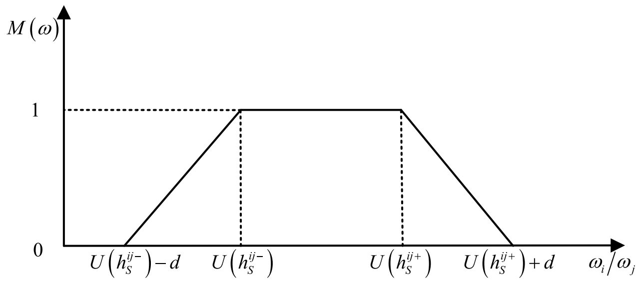

FIGURE 4. The membership function of the ratio of weights for a multiplicative consistent HFLPR.

weights in the given evaluation and the membership degree equals to one. If the value of ω_i/ω_j does not belong to the interval $\left[U\left(h_S^{ij-}\right), U\left(h_S^{ij+}\right)\right]$, the membership degree of weights would decrease. To depict the inconsistent situations, a tolerance parameter d is introduced. When ω_i/ω_j ranges in $\left[U\left(h_S^{ij-}\right)-d, U\left(h_S^{ij-}\right)\right]$ or $\left[U\left(h_S^{ij+}\right), U\left(h_S^{ij+}\right)+d\right]$, the membership degree decreases and varies between 0 and 1. When ω_i/ω_j is ranges in $\left[0, U\left(h_S^{ij-}\right)\right]$ or $\left[U\left(h_S^{ij+}\right)+d, +\infty\right]$, the membership degree decreases to zero.

The membership function of the kth row of Eq. (11) can be written as

$$M\left(R_k\omega\right) = \begin{cases} 1-\dfrac{R_k\omega}{d_k}, & R_k\omega \leq d_k \\ 0, & R_k\omega \geq d_k \end{cases}, k=1,2,\cdots,n(n-1) \tag{12}$$

where d_k is the tolerance degree of the kth row in Eq. (11).

The $n(n-1)$ rows of the matrix R consist of all judgements in the HFLPR except the constant diagonal elements. The membership degree of the matrix R should not be greater than membership degree of each row's constraint. That is

$$M_P(\omega) = \min_{\omega\in S^{n-1}} \left\{M\left(R_1\omega\right), \cdots, M\left(R_{n(n-1)}\omega\right)\right\} \tag{13}$$

where the feasible area P is the intersections of all membership constraints and $S^{n-1} = \left\{(\omega_1,\omega_2,\cdots,\omega_n) | \omega_i > 0, \omega_1+\omega_2+\cdots+\omega_n = 1\right\}$. If there exists one membership degree $M\left(R_k\omega\right) = 0$, $k = 1,2,\cdots,n(n-1)$, then the membership value of the feasible area P should be zero. This implies that the derived priorities do not have consistency at all.

Let $\eta = \max M_P(\omega) = \max \min_{\omega\in S^{n-1}} \{M(R_1\omega), \cdots, M\left(R_{n(n-1)}\omega\right)\}$. Then, the max-min optimization problem can be transformed into a linear programming as follows:

Model 1

$$\begin{aligned} &\max\ \eta \\ &s.t.\ \eta \leq M\left(R_k\omega\right) \end{aligned}$$

Taking the weights' solution space S^{n-1} and Eq. (12) into consideration, Model 1 can be equivalently converted into:

Model 2

$$\begin{aligned} &\max\ \eta \\ &s.t.\ \eta\times d_k + R_k\omega \leq d_k \\ &\quad \omega_1+\omega_2+\cdots+\omega_n = 1 \\ &\quad \omega_i > 0, i = 1,2,\cdots,n \\ &\quad k = 1,2,\cdots,n(n-1) \end{aligned}$$

where the tolerance parameter d_k could be given by DM. In this paper, we set d_k as 1.

The feasible area P is the intersections of linear and convex inequations. Thus, Model 2 has an optimal solution. Solving Model 2, the weights of criteria could be deduced. The optimal value of η denotes the maximum satisfactory degree and thus can be regarded as the consistency index. When η is close to zero, the HFLPR is inconsistent; when η is close to one, the HFLPR shows high consistency.

C. THE IMPROVING PROCESS FOR INCONSISTENT HFLPR

Given that the HFLPRs determined by DMs are always not perfectly consistent, it is necessary to define the acceptable consistency. If an HFLPR is not acceptably consistent, then the inconsistency-improving process is indispensable.

Generally, there are two kinds of inconsistency-repairing methods, which are the automatic improving method and the feedback-based improving method. The former is effective without DMs' involution while the latter needs DMs to revise preference information in the whole process. To obtain an acceptably multiplicative consistent HFLPR efficiently, this paper employs the automatic optimization method to improve the inconsistent HFLPRs.

The main idea of the automatically improving method is to update the HFLPR by combining the original HFLPR given by the DM with its corresponding perfectly multiplicative consistent HFLPR. In this regard, how to establish a perfectly multiplicative consistent HFLPR reasonably based on the HFLPR given by the DM is a critical point.

There are several papers discussing the approaches to build a perfectly consistent HFLPR. Zhu and Xu [20] gave a way to establish a perfectly additive consistent HFLPR based on a normalized HFLPR $H_S^N = \left(h_S^{kjN}\right)_{n\times n}$. Let $\bar{h}_S^{kjN} = \left(\oplus_{t=1}^{n}\left(h_S^{ktN}\right)\oplus\left(h_S^{tjN}\right)\right)\Big/n$ for $k,t,j = 1,2,\cdots,n$; $k \neq t \neq j$. Then, $\bar{H}_S = \left(\bar{h}_S^{kjN}\right)_{n\times n}$ is a perfectly additive consistent HFLPR. Then, using the arithmetic mean operator, the elements in the HFLPR $H_S = \left(h_S^{kj}\right)_{n\times n}$ can be revised. Zhang and Wu [21] provided an approach to build a perfectly multiplicative consistent HFLPR. Let $H_S = \left(h_S^{kj}\right)_{n\times n}$ be an HFLPR on $S = \{s_\alpha | \alpha \in \{0,1,2,\cdots,g,\cdots,2g\}\}$ and

$\bar{H}_S^N = \left(\bar{h}_S^{kjN}\right)_{n\times n}$ be its normalized HFLPR. If

$$I\left(\tilde{h}_S^{kj\rho(l)}\right) = \frac{2g\sqrt[n]{\prod_{t=1}^{n} I_{kt}I_{tj}}}{\sqrt[n]{\prod_{t=1}^{n} I_{kt}I_{tj}} + \sqrt[n]{\prod_{t=1}^{n} \left(2g - I_{kt}\right)\left(2g - I_{tj}\right)}} \tag{14}$$

where $I(\cdot)$ means the subscript of the linguistic term, $I_{kt}I_{tj}$ denotes the items $I\left(\bar{h}_S^{kt\rho(l)}\right)$ and $I\left(\bar{h}_S^{tj\rho(l)}\right)$, respectively, then $\tilde{H}_S = \left(\tilde{h}_S^{kj}\right)_{n\times n}$ is a perfectly multiplicative consistent HFLPR. Similarly, using the geometric average operator, the elements in the HFLPR $H_S = \left(h_S^{kj}\right)_{n\times n}$ are revised.

The original information given by DMs are changed completely by the methods in Refs. [20], [21]. In these cases, the DMs may be reluctant to accept the absolutely different HFLPRs. Liao *et al.* [39] fixed the sub-diagonal elements of a normalized Hesitant Fuzzy Preference Relation (HFPR) and the other elements of the perfectly multiplicative consistent HFPR are deduced based on the sub-diagonal elements as follows:

$$\tilde{h}^{kj\rho(l)} = \begin{cases} \dfrac{\sum_{t=k+1}^{j-1} \frac{\bar{h}^{kt\rho(l)} \times \bar{h}^{tj\rho(l)}}{\bar{h}^{kt\rho(l)} \times \bar{h}^{tj\rho(l)} + \left(1-\bar{h}^{kt\rho(l)}\right) \times \left(1-\bar{h}^{tj\rho(l)}\right)},}{j-k-1} & k+1<j \\ \bar{h}^{kj\rho(l)}, & k+1=j \\ \{0.5\}, & k=j \\ 1-\tilde{h}^{kj\rho(l)}, & k>j \end{cases} \tag{15}$$

where $\bar{h}^{kt\rho(l)}$, $\bar{h}^{tj\rho(l)}$ and $\bar{h}^{kj\rho(l)}$ are the elements of the normalized HFPR $\bar{H} = \left(\bar{h}^{kj}\right)_{n\times n}$. In this way, the secondary diagonal elements are maintained in the perfectly multiplicative consistent HFPR.

Below we introduce an approach to choose the fixed elements among the upper triangular elements of a given HFLPR. The uncertainty index $\varphi\left(h_S^{kj}\right)$ of a linguistic evaluation h_S^{kj} is defined as follows based on the interval utility value of the linguistic evaluation:

$$\varphi\left(h_S^{kj}\right) = U\left(h_S^{kj+}\right) \Big/ U\left(h_S^{kj-}\right) - 1 \tag{16}$$

$\varphi\left(h_S^{kj}\right)$ equals to zero when h_S^{kj} has only one specific linguistic term, which implies that the DM is very certain about this evaluation. The uncertainty index $\varphi(H_S)$ of an HFLPR can be calculated as follows:

$$\varphi(H_S) = 2\sum_{k=1}^{n}\sum_{j>k}^{n} \varphi\left(h_S^{kj}\right) \tag{17}$$

It is observed that only the upper triangular elements are considered due to the reciprocity of pairwise comparisons and the lower triangular elements can be deduced by the upper triangular elements.

The main idea to establish a perfectly consistent HFLPR is to build an acceptably consistent incomplete HFLPR first. Then, by Eq. (15), we can get the perfectly consistent complete HFLPR. As we know, to get the complete HFLPR, the incomplete HFLPR should have at least $n-1$ known judgments with at least one known element in each row or each column.

The algorithm to find the fixed elements and build a perfectly consistent HFLPR is given as follows:

Algorithm 1 Establish the Perfectly Multiplicative Consistent HFLPR)

Input: An HFLPR given by DM.

Output: A perfectly multiplicative consistent HFLPR.

Step 1. Calculate the uncertainty indices of the upper triangular elements of the given HFLPR by Eq. (16), and then rank these uncertainty degrees in ascending order.

Step 2. Find the element with the smallest uncertainty degree and record its position in the HFLPR. If there are multiple minimal uncertainty degrees, arbitrarily choose one of them.

Step 3. Delete the row or column of the element found in Step 2. Then, go to Step 2.

Step 4. Establish an incomplete HFLPR with $n-1$ elements in each row and an incomplete HFLPR with $n-1$ elements in each column.

Step 5. Calculate the uncertainty index of the two incomplete HFLPRs established in Step 4 by Eq. (17) and choose the incomplete HFLPR with the smaller uncertainty degree. If the uncertainty degrees are equal, choose any one of them.

Step 6. Establish the perfectly multiplicative consistent HFLPR by Eq. (15) for the incomplete HFLPR found in Step 5. Output the established perfectly multiplicative consistent HFLPR and end the algorithm.

It is noted that Step 6 can be replaced by the additive consistency alternatively. Then, the output of Algorithm 1 is a perfectly additive consistent HFLPR.

The perfectly multiplicative consistent HFLPR established by Algorithm 1 maintains the certain evaluations given by DMs rather than just fixing the secondary diagonal elements as obtained in Ref. [39].

D. PROCEDURE OF THE HFL-AHP METHOD

In summary, we describe the procedure of the HFL-AHP in detail for solving the MCDM problems with cognitive linguistic information. Figure 5 illustrates the procedure intuitively.

Step 1, Step 5 and Step 6 are the same as those in traditional AHP method. Step 3 deduces crisp priorities in uncertain conditions. It shows better performance than other prioritization methods. Step 4 includes the consistency checking and inconsistency-repairing processes, which are distinct from other hierarchical methods with hesitant fuzzy linguistic information in Refs. [26], [27]. The parameter λ denotes the proportion of information retained by the DM. The smaller

Algorithm 2 (HFL-AHP)

Step 1. Decompose the problem into three levels: goal, criteria and alternatives.

Step 2. Establish the HFLPRs based on the hesitant fuzzy linguistic judgements over alternatives and criteria.

Step 3. Use the linear programming to derive the priority vectors. The consistency index of each HFLPR can be calculated at the same time.

Step 4. Check the consistency of each HFLPR. If the HFLPRs are acceptably consistent, go to Step 5; otherwise, combine them with their corresponding perfectly multiplicative consistent HFLPR $\tilde{H}_S = (\tilde{h}_S^{ij})_{n\times n}$ established by Algorithm 1 to improve their consistency by

$$U\left(\hat{h}_S^{ij}\right) = \lambda \times U\left(h_S^{ij}\right) + (1-\lambda) \times U\left(\tilde{h}_S^{ij}\right) \tag{18}$$

Then go to Step 4.

Step 5. Aggregate the priorities of alternatives on each criterion and the priorities of the criteria to get the final scores of alternatives.

Step 6. Rank the final scores of alternatives and end the procedure.

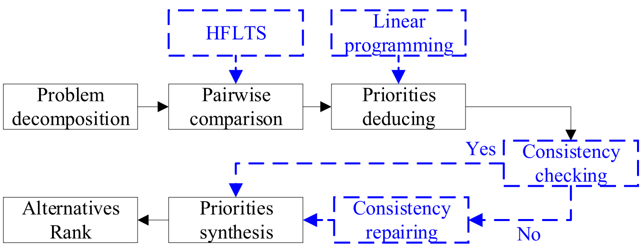

FIGURE 5. The procedure of the HFL-AHP method.

the value of λ is, the faster the HFLPR will reach an acceptable consistency. In Step 6, the ranking result of alternatives can be obtained by the synthetic scores. The top-ranked alternative has the highest synthetic priorities.

V. CASE STUDY: HOW TO CHOOSE A GOOD PEACH IN THE "LEMON" MARKET?

In this section, we use a case study concerning the selection of used-car to illustrate the validity and efficiency of the proposed HFL-AHP method. To verify the advantages of the proposed HFL-AHP method, we compare it with two existing HFL-AHP models in Refs. [26], [27], respectively.

A. CASE DESCRIPTION

Akerlof [40] first proposed the concept of "lemon" market and took the used-car market for example to illustrate the model of "lemon" market. There is no doubt that the used-car market is a "lemon" market because of asymmetric information between the seller and the buyer. If somebody buys a used-car, he/she may not have the ability to distinguish the quality of a used-car. But after a period of time, he/she could judge the quality of the used-car based on the frequency of breakdown time and maintenance of the used-car.

The number of households that have the ability to purchase cars is increasing with the development of China's economy. The rapid development of the used-car market happens in China, too. There are several reasons for people to buy used-cars. People with limited income can own a car earlier because of low prices of the used-cars, and people who just got a driver's license wants to buy a used-car to practice driving skills. Furthermore, people want to drive more luxury cars or their favorite cars with limited money. In addition, the space for depreciation of second-hand cars is much smaller than that of new cars. Used-car means legal identity and remaining useful life. However, since each vehicle has different manufacturing quality, intensity of use and maintenance, it would inevitably cause differences in quality. Identifying a high quality used-car is not an easy task. Hence, we try to use the HFL-AHP method to tackle this issue.

The set of criteria contains four indicators for choosing a used-car: mileage, closeness of the used-car's price to corresponding new cars' prices, serviceability, and reputation of brand.

(1) **Mileage.** Mileage shows precise miles that the used-car drive. The mileage represents the degree of use of the previous owner, and the greater the value of the mileage is, the more severe the car wear out. According to Motor Vehicle Compulsory Scrap Standards in China, the largest mileage of automobiles is 600,000 kilometers. When the value of mileage of a car arrives 600,000 kilometers, the car must be scrapped in force. Hence, the smaller the value of mileage is, the stronger the willing to buy is. In this sense, mileage is a cost criterion. Usually in a MCDM problem, we need to normalize the benefit criterion and cost criterion. However, no normalization is necessary when using the cognitive linguistic information. For instance, the linguistic term "*very good*" denotes the good performance of an alternative no matter what the criterion type is.

(2) **Closeness of the used-car's price to corresponding new cars' prices.** Different people have distinct views about the closeness of the used-car's price to corresponding new cars' prices. Someone may think that the higher the closeness is, the better the used car's quality should be. Someone wants to buy a used-car with a big difference from the new car's price, in case the purchased used-car depreciates sharply.

(3) **Serviceability.** People who do not have knowledge about maintenance are more willing to go to the car repair center to maintain and check the car in periodicity. Serviceability means the distribution density and professionalism of the car repair center.

(4) **Reputation of brand.** Steve Jobs, the chairman, chief executive officer, and a co-founder of Apple Inc., once said: "*The brand lies in trust.*" The reputation of the used-car brand is an important factor for buyers to consider. Most consumers will consider the recognition

of the brand and its influence in the automotive industry and the automotive market when they buy used-cars. If the brand has a good reputation, then, accordingly, its brand value will be improved. Unless it is a major event, the brand value is generally not easy to devalue.

Based on the above description, we decompose this problem into three levels:

- ✓ A goal = {buy a used-car};
- ✓ Criteria = {mileage, closeness of the used-car's price to corresponding new cars' prices, serviceability, reputation of brand};
- ✓ Alternatives = {Audi, Volvo, HONDA, Land Rover}.

B. SOLVE THE CASE BY THE PROPOSED HFL-AHP METHOD

Below we use the HFL-AHP method to solve this problem. Since Step 1 is given above, we start the calculation process from Step 2.

Step 2: Establish HFLPRs $H_S = (h_S^{ij})_{n\times n}$ based on the hesitant fuzzy linguistic judgments over alternatives and criteria.

After communicating with a DM, the pairwise judgments over four criteria are converted into following HFLPR by transformation rules. The HFLPR with respect to the importance of four criteria is denoted as H_S^C, associated with its corresponding envelop form $env\left(H_S^C\right)$ and utility matrix $U\left(H_S^C\right)$. The HFLPRs of the alternatives over each criterion is given as $H_S^{C_j}$ $(j = 1, 2, 3, 4)$.

$$H_S^C = \begin{pmatrix} \{s_0\} & \{s_{-7}, s_{-6}, s_{-5}\} & \{s_2, s_3\} & \{s_2\} \\ - & \{s_0\} & \{s_1, s_2\} & \{s_0\} \\ - & - & \{s_0\} & \{s_{-5}, s_{-4}, s_{-3}, s_{-2}\} \\ - & - & - & \{s_0\} \end{pmatrix}$$

$$env\left(H_S^C\right) = \begin{pmatrix} [s_0, s_0] & [s_{-7}, s_{-5}] & [s_2, s_3] & [s_2, s_2] \\ - & [s_0, s_0] & [s_1, s_2] & [s_0, s_0] \\ - & - & [s_0, s_0] & [s_{-5}, s_{-2}] \\ - & - & - & [s_0, s_0] \end{pmatrix}$$

$$U\left(H_S^C\right) = \begin{pmatrix} [1, 1] & [0.0884, 0.1768] & [2, 2.8284] & [2, 2] \\ - & [1, 1] & [1.4142, 2] & [1, 1] \\ - & - & [1, 1] & [0.1768, 0.5] \\ - & - & - & [1, 1] \end{pmatrix}$$

$$H_S^{C_1} = \begin{pmatrix} \{s_0\} & \{s_1\} & \{s_1, s_2\} & \{s_1, s_2\} \\ - & \{s_0\} & \{s_2\} & \{s_1\} \\ - & - & \{s_0\} & \{s_1, s_2\} \\ - & - & - & \{s_0\} \end{pmatrix}$$

$$H_S^{C_2} = \begin{pmatrix} \{s_0\} & \{s_0\} & \{s_1, s_2\} & \{s_2, s_3\} \\ - & \{s_0\} & \{s_0, s_1\} & \{s_2, s_3\} \\ - & - & \{s_0\} & \{s_4\} \\ - & - & - & \{s_0\} \end{pmatrix}$$

$$H_S^{C_3} = \begin{pmatrix} \{s_0\} & \{s_1\} & \{s_3\} & \{s_1, s_2\} \\ - & \{s_0\} & \{s_2\} & \{s_3, s_4\} \\ - & - & \{s_0\} & \{s_1, s_2\} \\ - & - & - & \{s_0\} \end{pmatrix}$$

$$H_S^{C_4} = \begin{pmatrix} \{s_0\} & \{s_0\} & \{s_1\} & \{s_0, s_1\} \\ - & \{s_0\} & \{s_1, s_2\} & \{s_4, s_5\} \\ - & - & \{s_0\} & \{s_1\} \\ - & - & - & \{s_0\} \end{pmatrix}$$

where the value of a is set as $\sqrt{2}$ which is the same as in Ref. [30]. Since the lower triangular elements in these matrices can be obtained by reciprocity property of comparisons, we only list the upper triangular elements in the HFLPRs.

Step 3: Use the linear programming to derive the priority vector ω. The consistency index η of these HFLPRs can be calculated at the same time.

As for $U\left(H_S^C\right)$, based on Model 2, we can obtain Model 2-1. In this paper, without loss of generality, the value of the tolerance parameter d_k is set to one.

Model 2-1

$$\begin{aligned} &\max\ \eta \\ &s.t.\ \eta + \omega_2 \times 0.0884 - \omega_1 \le 1;\ \eta + \omega_1 - \omega_2 \times 0.1768 \le 1 \\ &\quad \eta + \omega_3 \times 2 - \omega_1 \le 1;\ \eta + \omega_1 - \omega_3 \times 2.8284 \le 1 \\ &\quad \eta + \omega_4 \times 2 - \omega_1 \le 1;\ \eta + \omega_1 - \omega_4 \times 2 \le 1 \\ &\quad \eta + \omega_3 \times 1.4142 - \omega_2 \le 1;\ \eta + \omega_2 - \omega_3 \times 2 \le 1 \\ &\quad \eta + \omega_4 \times 1 - \omega_2 \le 1;\ \eta + \omega_2 - \omega_4 \times 1 \le 1 \\ &\quad \eta + \omega_4 \times 0.1768 - \omega_3 \le 1;\ \eta + \omega_3 - \omega_4 \times 0.5 \le 1 \\ &\quad \omega_1 + \omega_2 + \omega_3 + \omega_4 = 1;\ \omega_i > 0, i = 1, 2, 3 \end{aligned}$$

Solving Model 2-1, we can get that the weights of four criteria are 0.2325, 0.3675, 0.2000, 0.2000 and the consistency index η is 0.8325.

Step 4: Let the consistency threshold be 0.9. Since $\eta = 0.832$ is lower than the consistency threshold, it is necessary to improve the consistency degree. We can build a perfectly multiplicative consistent HFLPR by Algorithm 1. The input of Algorithm 1 is H_S^C. We calculate the uncertainty values of triangular elements in H_S^C by Eq. (16) and obtain:

$$\begin{aligned} \varphi\left(h_S^{12}\right) &= U\left(h_S^{12+}\right)/U\left(h_S^{12-}\right) - 1 = 1; \\ \varphi\left(h_S^{13}\right) &= U\left(h_S^{13+}\right)/U\left(h_S^{13-}\right) - 1 = 0.4142; \\ \varphi\left(h_S^{14}\right) &= U\left(h_S^{14+}\right)/U\left(h_S^{14-}\right) - 1 = 0; \\ \varphi\left(h_S^{23}\right) &= U\left(h_S^{23+}\right)/U\left(h_S^{23-}\right) - 1 = 0.4142; \\ \varphi\left(h_S^{24}\right) &= U\left(h_S^{24+}\right)/U\left(h_S^{24-}\right) - 1 = 0; \\ \varphi\left(h_S^{34}\right) &= U\left(h_S^{34+}\right)/U\left(h_S^{34-}\right) - 1 = 1.8284. \end{aligned}$$

Ranking these uncertainty values in an ascending order, we obtain $\varphi(h_S^{14}) = \varphi\left(h_S^{24}\right) < \varphi\left(h_S^{13}\right) = \varphi\left(h_S^{23}\right) < \varphi\left(h_S^{12}\right) < \varphi\left(h_S^{34}\right)$. The elements with smallest uncertainty value are h_S^{14} and h_S^{24}, and we choose h_S^{14} to revise. There are two ways to find the other elements with the smallest uncertainty degree: delete row by row or delete column by column. For row way, the first row of h_S^{14} should be deleted. Then, the rest ascending ranking is $\varphi\left(h_S^{24}\right) < \varphi\left(h_S^{23}\right) < \varphi\left(h_S^{34}\right)$ and go to Step 3. Repeat Steps 3-4, the elements h_S^{24} and h_S^{34} are found in the row way. For column way, the first column of h_S^{14} should be deleted. Then, the rest ascending ranking is $\varphi\left(h_S^{13}\right) = \varphi\left(h_S^{23}\right) < \varphi\left(h_S^{12}\right)$ and go to Step 3. Repeat Steps 3-4, the elements h_S^{13} and h_S^{12} are found in the column way.

We establish an incomplete HFLPR with $n-1$ elements in each row and an incomplete HFLPR with $n-1$ elements in each column. Then, the acceptable incomplete HFLPRs by row H_{S-row}^C and column $H_{S-column}^C$ are shown as follows:

$$H_{S-row}^C = \begin{pmatrix} \{s_0\} & - & - & \{s_2\} \\ - & \{s_0\} & - & \{s_0\} \\ - & - & \{s_0\} & \{s_{-5}, s_{-4}, s_{-3}, s_{-2}\} \\ - & - & - & \{s_0\} \end{pmatrix};$$

$$H_{S-column}^C = \begin{pmatrix} \{s_0\} & \{s_{-7}, s_{-6}, s_{-5}\} & \{s_2, s_3\} & \{s_2\} \\ - & \{s_0\} & - & - \\ - & - & \{s_0\} & - \\ - & - & - & \{s_0\} \end{pmatrix}$$

The uncertainty values of H_{S-row}^C and $H_{S-column}^C$ can be calculated by Eq. (17), shown as:

$$\varphi\left(H_{S-row}^C\right) = 2\left[\varphi\left(h_S^{14}\right) + \varphi\left(h_S^{24}\right) + \varphi\left(h_S^{34}\right)\right] = 3.6568;$$
$$\varphi\left(H_{S-column}^C\right) = 2\left[\varphi\left(h_S^{12}\right) + \varphi\left(h_S^{13}\right) + \varphi\left(h_S^{14}\right)\right] = 2.8284.$$

It is not hard to find that $\varphi\left(H_{S-column}^C\right) < \varphi(H_{S-row}^C)$ and the acceptable incomplete HFLPR $H_{S-column}^C$ should be regarded as the basis of establishing a perfectly multiplicative consistent HFLPR. Using Eq. (7), a perfectly multiplicative consistent HFLPR $\tilde{H}_S^C$ could be established as:

$$\tilde{H}_S^C = \begin{pmatrix} \{s_0\} & \{s_{-7}, s_{-6}, s_{-5}\} & \{s_2, s_3\} & \{s_2\} \\ - & \{s_0\} & \{s_8, s_9\} & \{s_7, s_8, s_9\} \\ - & - & \{s_0\} & \{s_1, s_0\} \\ - & - & - & \{s_0\} \end{pmatrix}$$

Let $\lambda = 0.9$. Using Eq. (18), the interval utility value of the combined HFLPR $\hat{H}_S^C$ can be calculated as $U\left(\hat{H}_S^C\right)$, as shown at the bottom of this page.

Using linear programming method to derive the priorities of $U\left(\hat{H}_S^C\right)$, the value of the consistency index $\eta\left(\hat{H}_S^C\right)$ is 0.9076, which is greater than the consistency threshold. Therefore, the weights of criteria derived from $U\left(\hat{H}_S^C\right)$ are reliable. The weights of four criteria are 0.1868, 0.534, 0.1396, 0.1396.

In analogous, we can derive the priorities of alternatives over each criterion from $H_S^{C_1}$, $H_S^{C_2}$, $H_S^{C_3}$ and $H_S^{C_4}$. The results are listed in Table 3.

TABLE 3. The priorities of criteria and alternatives.

Alternatives	C_1 - 0.1868	C_2 - 0.534	C_3 - 0.1396	C_4 - 0.1396	Final scores
A_1	0.3565	0.3184	0.3721	0.2637	0.3254
A_2	0.2971	0.3184	0.3227	0.3635	0.3213
A_3	0.1783	0.2761	0.1613	0.257	0.2391
A_4	0.1681	0.0871	0.1439	0.1158	0.1142

Step 5: The final scores of alternatives can be aggregated by multiplying weights of criteria and the priorities of alternatives on each criterion. The last column of Table 3 shows the results.

Step 6: The results show that the ranking of these four alternatives is $A_1 > A_2 > A_3 > A_4$. That is Audi>Volvo> HONDA>Land Rover.

C. COMPARATIVE ANALYSES AND DISCUSSIONS

There are several papers concerning the hierarchical hesitant fuzzy linguistic models [26], [27]. The most obvious imperfections of these two methods are without consistency checking, consistency improving and prioritization processes. In these two papers, the HFLEs were converted into intervals. Then, interval vectors were established to get alternatives' overall preferences. Eventually, the midpoints of these interval vectors were used to rank the alternatives.

Below we use the method in Refs. [26], [27] to solve the case.

Step 1: Firstly, we transform the HFLPR H_S^C in $[-g, g]$ scale to the HFLPR $\vec{H}_S^C$ in $[0, 2g]$ scale. Using the method in Refs. [26], [27], the summation of each row can be calculated and regarded as the interval utilities.

The midpoints of the overall interval utilities can be viewed as the weights of criteria. After normalization these weights, the weights of four criteria are 0.2562, 0.2512, 0.2020, 0.2906. As lacking of consistency checking process, the weights of criteria derived by the method in

$$U\left(\hat{H}_S^C\right) = \begin{pmatrix} [1, 1] & [0.0884, 0.1768] & [2, 2.8284] & [2, 2] \\ - & [1, 1] & [2.8728, 4.0627] & [2.0314, 3.1627] \\ - & - & [1, 1] & [0.2298, 0.55] \\ - & - & - & [1, 1] \end{pmatrix}$$

$$\vec{H}_S^C = \begin{pmatrix} \{s_7\} & \{s_0, s_1, s_2\} & \{s_9, s_{10}\} & \{s_9\} \\ \{s_{12}, s_{13}, s_{14}\} & \{s_7\} & \{s_8, s_9\} & \{s_7\} \\ \{s_4, s_5\} & \{s_5, s_6\} & \{s_7\} & \{s_2, s_3, s_4, s_5\} \\ \{s_5\} & \{s_7\} & \{s_9, s_{10}, s_{11}, s_{12}\} & \{s_7\} \end{pmatrix}$$

$$\vec{H}_S^C \xrightarrow{\text{row summation}} \begin{pmatrix} [25], [27] \\ [34], [37] \\ [18], [23] \\ [28], [31] \end{pmatrix}$$

Refs. [26], [27] are different from those deduced by our method.

Step 2: In analogous, we can obtain the interval utilities, midpoints and normalized weights of alternatives on each criterion, which are tabulated in Table 4.

TABLE 4. The normalized priorities of alternatives.

Normalized priorities	A_1	A_2	A_3	A_4
C_1	0.2857	0.2679	0.2321	0.2143
C_2	0.2857	0.2768	0.2679	0.1696
C_3	0.2991	0.2902	0.2188	0.1920
C_4	0.2634	0.3036	0.2366	0.1964

Step 3: Aggregating all the priorities into final scores of alternatives, the final scores are shown in Table 5.

TABLE 5. The final scores of alternatives by two methods.

Normalized priorities	A_1	A_2	A_3	A_4	Final rankings
Our method	0.3254	0.3213	0.2391	0.1142	$A_1 > A_2 > A_3 > A_4$
Method in Refs. [25,26]	0.2819	0.2850	0.2397	0.1934	$A_2 > A_1 > A_3 > A_4$

From Table 5, we can find that the ranking of A_1 and A_2 deduced by the method in Refs. [26], [27] is different from that obtained by our proposed method. The four HFLPRs about the priorities of alternatives on each criterion are acceptably consistent after checked by our method. Hence, the difference among the importance of criteria is the key factor influencing the ranking results between A_1 and A_2. The different rankings appear because of distinct weights of criteria. By our proposed method, the ranking of the weights of criteria is $C_2 > C_1 > C_3 = C_4$ while by the method in Refs. [25], [26], the ranking of the weights of criteria is $C_4 > C_1 > C_2 > C_3$. That is to say, the DM thinks that "Closeness of the used-car's price to corresponding new cars' prices" is the most important factor among these four criteria. Without consistency checking process and inconsistency repairing process, the priorities of criteria would change dramatically.

To show the clear distinction, we use the weights of criteria deduced by our method and the priorities of alternatives calculated by the methods in Refs. [26], [27] to obtain the final scores of alternatives. In this way, the final scores of alternatives are 0.2845, 0.2807, 0.2500 and 0.1848, which shows the ranking result of four alternatives is $A_1 > A_2 > A_3 > A_4$, which is the same as that obtained by our method. This denotes the importance of consistency checking process and inconsistency repairing process. The traditional hesitant linguistic hierarchical methods use the summation of rows in evaluation matrix without consistency checking and improving process may lead to misleading results.

In conclusion, the advantages of the proposed HFL-AHP with respect to the existing methods can be summarized from three aspects: consistency checking and improving processes, continuous semantics of linguistic expression, and adaptable programming method to derive crisp weights.

(1) The proposed approach considers the importance of the consistency checking and improving processes, which were neglected in Refs. [26], [27]. This guarantees the reliability of the final results.

(2) The uncertain information in linguistic expressions are kept in the analyzing procedure of the proposed method instead of only choosing the midpoints of intervals.

(3) The priority-determining model in the proposed approach deduces crisp weights by using all evaluations given by the DM.

VI. CONCLUSIONS

AHP has been applied in many fields due to its efficiency, and fuzzy extensions of AHP has been a diverting research area with challenges because of the unavoidable uncertainty in decision-making process. In this context, this paper investigated a hesitant fuzzy linguistic extension of AHP, which broadened the applications of the AHP method. The advantages of the proposed HFL-AHP method can be summarized as follows:

1) The consistency checking and inconsistency repairing algorithms were proposed in this paper, which filled the gaps of previous hierarchical hesitant fuzzy linguistic methods in Refs. [26], [27].
2) The interval form of HFLPRs were used to avoid discretization of the continuous semantic interval given by the DM, which is beneficial for DM to understand and apply the proposed method.

3) A new approach to establish a perfectly consistent HFLPR was given, which preserves the initial evaluations of DM as much as possible.

There are some future research topics related to HFL-AHP. For example, the HFL-AHP method combined with Choquet integral may be an interesting research issue with challenges. Methods for group decision-making with HFL-AHP by Maclaurin symmetric mean operators could be a good research topic [41], [42].

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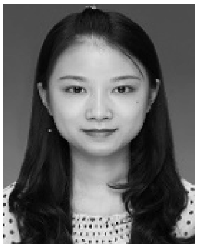

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