COMPLEX ARGUMENTATION IN ELEMENTARY SCHOOL

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This paper describes a study of mathematical argumentation in primary school. The principal aim is to explore the nature of complex argumentation at a structural level. The context of the study was a teaching experiment involving nine tasks that promoted argumentation among fifth graders. We use the framework and method of reconstructing complex argumentation in the classroom proposed by Knipping (2008). The findings show that complex argumentation at a structural level in the context of refuting conclusions is characterized by a source-like structure with the addition of a new refutation argument element.

Keywords: Argumentation structures; Complex argumentation; Mathematics; Primary school; Refutation; Teaching experiment.

Argumentación compleja en Educación Primaria

Este artículo describe un estudio de argumentación matemática en educación primaria. El objetivo principal es explorar la naturaleza de la argumentación compleja en un nivel estructural. El contexto del estudio fue un experimento de enseñanza con nueve tareas que promovieron la argumentación entre estudiantes de quinto grado. Usamos el marco teórico y metodológico para reconstruir la argumentación compleja en el salón de clase propuesto por Knipping (2003). Los resultados muestran que la argumentación compleja a nivel estructural en el contexto de refutar conclusiones se caracteriza por ser una estructura de fuente con el agregado de un argumento de refutación.

Términos clave: Argumentación compleja; Educación primaria; Estructuras argumentativas; Experimento de enseñanza; Matemáticas; Refutación

Knipping (2008) argues that reconstructing classroom argumentation structures provides a window into the underlying rationale of the proving process. In other words, such reconstructions give us insight into a teacher's conscious and

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unconscious goals in guiding the argumentation in a particular way, as well as other contextual constraints that might also affect the proving process. In this article we describe an argumentation structure that emerged in a grade five classroom, in a lesson focused on refuting a conjecture. This is a novel application of Knipping's approach, both because of the focus on refutations and because Knipping's approach has not previously been applied to proving processes of younger students. Most studies have focused on middle and high school classes.

Several researchers in the field of Mathematics Education have studied mathematical argumentation in the classroom (e.g., Cervantes-Barraza & Cabañas-Sanchez, 2018; Knipping & Reid, 2015; Krummheuer, 1995, 2000, 2015; Reid, Knipping, & Crosby, 2011; Whitenack & Yackel, 2002). Mathematical argument is recognized as a social, cognitive activity and an outcome of a proving process that encourages students to critique or refute others' arguments, and to convince them of an argument's validity (Knipping & Reid, 2015; Rumsey & Langrall, 2016). In the mathematics classroom, argumentation emerges as a collective process of conversation through which teacher and students work together in order to reach a conclusion, which is a new learning for the students (Krummheuer, 1995, 2015).

Mathematical argumentation can be reconstructed and analysed at the level of content or at a structural level. We focus on structures emerging in complex argumentation that reveal the function and meaning of mathematical statements in mathematics classroom talk. Krummheuer (1995) was one of the first researchers to study argumentation in primary school. He adapted Toulmin's (2003) scheme to reconstruct and analyse elementary school students' arguments and described a basic argumentative structure, which he called the core, including data leading to conclusions, supported by warrants. Krummheuer (2015) and Knipping and Reid (2015) describe chains of such core arguments, in which conclusions can be used as data in subsequent arguments. These chains are basically linear, however, argumentation in elementary school is often not a linear process (Knipping & Reid, 2015; Rumsey & Langrall, 2016). In fact, argumentation in an elementary mathematics classroom can be quite complex, as the teacher makes argumentative contributions, and supplies data or warrants (Conner, Singletary, Smith, Wagner, & Francisco, 2014), refutes student's conclusions and guides them towards a mathematical consensus (Potari, Zachariades, & Zaslavsky, 2010). These complex, non-linear, argumentation structures are our focus here.

Complex argumentation has been studied from two perspectives, one centred on the students and the second on the teacher. Knipping (2008) studied complex argumentation at secondary school in a proving context in which students and the teacher contribute, paying attention to emerging structures. She identified two complex argumentation structures. In other studies, Knipping and Reid (2015) used the same methods to reconstruct complex argumentation in other classroom proving processes and identified two other structures. In a teaching context, Erkek and Bostan (2018) indicated that future mathematics teachers "resort frequently to simple global argumentation structures since their mathematical reasoning was insufficient" (p. 1).

According to Knipping (2008), analysing complex argumentation permits one to understand the rationale and the contextual constraints that shape these argumentations and can help teachers to improve their efforts in teaching (p. 429). Following the same idea, Erkek and Bostan (2018) inferred that prospective middle school mathematics teachers need to be challenged about argumentation and about how to facilitate argumentation effectively in their future classroom experiences. Furthermore, the National Council of Teachers of Mathematics points out that argumentation should be promoted from primary school and not ignored until upper school (NCTM, 2000). However, based on our review of the literature, we find no studies centred on exploring the nature of complex argumentation in primary school. This study aims to partly fill this gap, by examining the structure of complex argumentations in fifth grader's arguments. More specifically, here we are interested in this question: What are the characteristics of the complex argumentation structures emerging in a fifth grade mathematics classroom, while students are refuting conclusions?

FRAMEWORK

This research is based theoretically on a framework for studying complex argumentation in classrooms. This allows us to analyse and understand what we mean by complex argumentation, and the key features of argumentation structures reported in the literature.

Argumentation

The concept of argumentation from the position of Toulmin (2003), as embodied in his work, *The uses of argument*, refers to the central activity of presenting conclusions, reasons that support them, receiving criticisms or refutations that question the validity of the conclusions, and providing further reasons based on criticism. Toulmin proposed a skeleton of argumentation composed of six elements. The data (D) are the set of information or evidence on which the conclusion is based. The conclusion (C) is the thesis established by the arguer, the relationship between the data. The conclusion is justified by the warrant (W) and presents rules, particular cases, invariant characteristics of mathematical objects, and mathematical properties. The warrant has a support called backing (B), an element whose function is to present support for the warrant through formulas, mathematical theorems, and/or axioms. The modal qualifier (Q) is the element that indicates the security of the argument, by means of phrases like always, never, and probably, for all x, etc. The rebuttal (R) has the function of indicating exceptions to the claim (see Figure 1).

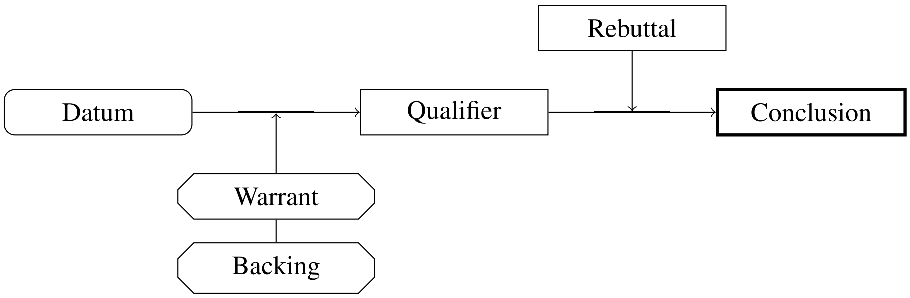

Figure 1. Argumentation model of Toulmin (2003)

This argumentation scheme has been used to analyse mathematical argumentation in several ways: to study collective mathematical argumentation in primary classrooms (Krummheuer, 1995, 2015), to study the socio-mathematical norms in classroom (Yackel, 2002), to reconstruct mathematicians' arguments (Inglis & Mejía-Ramos, 2005), to analyse proof processes at secondary school (Pedemonte, 2007; Reid, Knipping, & Crosby, 2011) and to study mathematical teachers' argumentation (Conner, 2008, 2017; Solar & Deulofeu, 2016).

Refutations

Knipping and Reid (2015) added an element to the Toulmin scheme, which they call *refutation*. They describe it as follows:

> *A refutation differs from a rebuttal in that a rebuttal is local to a step in an argument and specifies exceptions to the conclusion. A refutation completely negates some part of the argument. In a finished argumentation refuted conclusions would have no place, but as we are concerned with representing the entire argumentation that occurred, it is important for us to include refutations and the arguments they refute, as part of the context of the remainder of the argumentation, even if there is no direct link to be made between the refuted argument and other parts of the argumentation* (p. 82).

As Reid, Knipping and Crosby (2011) discuss, refutations can refute a datum, a conclusion, a warrant, or the logic underlying the argument itself. One example they give of a refutation of the logic underlying an argument is an exchange between a teacher and student, in which the student proposes that they divide by three "because there's three numbers". It is true that there are three numbers, and that they should divide by three, but the teacher rejects the link between the two as it is not based on a deduction but on an analogy. He says, "It's not a great reason" and calls on another student to answer. Reid, Knipping and Crosby (2011)'s diagram for this exchange is shown in Figure 2.

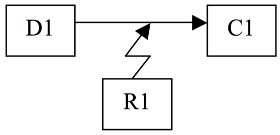

Figure 2. Refutation of an argument (Reid, Knipping, & Crosby, 2011, p. 7)

The refutations described by Reid, Knipping and Crosby (2011) consist of a single statement. Reid, Knipping and Crosby (2011) describe refutations with backings, and refutations that involve an argument from a datum to a conclusion, but in general the refutations represented using Toulmin's scheme in the mathematics education literature consist of only one or two statements.

Complex Argumentation

In the classroom context, argumentation is characterized by being a social activity whose purpose is to convince an audience with reasons presented in a logical manner (Goizueta & Planas, 2013; Solar & Deulofeu, 2016). Argumentation in this context does not always occur in a linear form or chain of reasoning. Reid and Knipping (2010) indicate that argumentations in classrooms are interconnected in complex ways at the structure level. Toulmin's model is not enough to reconstruct such complex argumentations, so Knipping (2003) developed an extended argumentation model based on Toulmin's work to describe complex argumentation in classrooms.

Argumentation in classrooms occurs at two levels, local and global. The first refers to the steps that make up an argument (i.e., argumentation stream AS), while the global argumentation refers to the structure of the argumentation made up by interconnected local arguments (Knipping & Reid, 2015). According to Knipping and Reid (2015), complex argumentation leads to structures that reflect the proof processes in the classroom, in this case mathematical argumentation. The analysis of complex argumentation structures allows the reconstruction of the meaning of mathematical statements in terms of data, warrants, refutations, or conclusions and provides a global picture of the argumentation.

Knipping and Reid (2015) have described four types of argumentation structures in the literature: source, spiral, reservoir, and gathering structures. Here we focus only on the source and spiral structures. Knipping and Reid (2015) describe the source structure as having the following characteristics:

- ♦ Parallel arguments. These are independent arguments leading to support a single conclusion (Knipping, 2003). For example, in Figure 3, AS-1 and AS-2 both lead to the same conclusion.
- ♦ Conclusions based on more than one piece of information (datum). For example, in Figure 3 the first conclusion in AS-8 depends on data from AS-4, AS-5 and AS-7.
- ♦ Refutations. In Figure 3 there are refutations of data in AS-3 and AS-6.

On the other hand, spiral structures (see Figure 4) are argumentation structures that have the following characteristics (Knipping & Reid, 2015):

- Parallel arguments used to prove a final (or nearly final) conclusion in several different ways. In Figure 4 AS-A, AS-D and AS-E all share the same conclusion.
- Refutations. In Figure 4 a refutation of the logic of an argument occurs in AS-D.

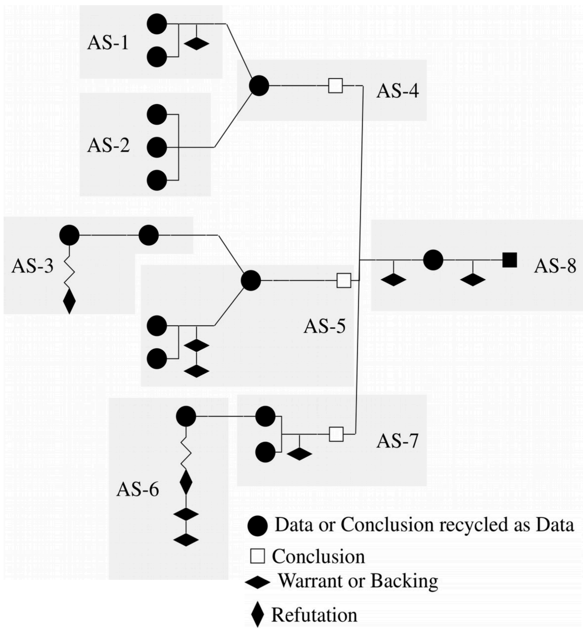

Figure 3. Source structure (Reid, Knipping, & Crosby, 2011, p. 185)

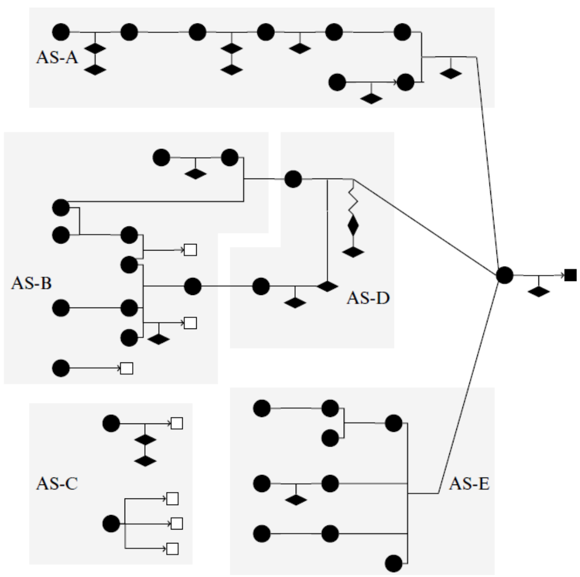

Figure 4. Spiral structure (Knipping & Reid, 2015, p. 94)

METHOD

Context and Participants

The research context was a teaching experiment (Steffe & Thompson, 2000) in a fifth-grade mathematics class. Participants were 22 students around 9-11 years old and the two first authors took on the role of teacher. The experiment made use of a set of tasks designed by the research group to foster mathematical argumentation and took place over 4 lessons with a duration of 60 minutes each. Each lesson involved twenty minutes in which the students explored a task individually, followed by forty minutes in which they presented their conclusions orally to the class.

Teaching Experiment

The aim of the teaching experiment was to promote argumentation in the mathematics classroom based on refuting conclusions. We selected the topic of triangle classification according to angles, due to research indicating that students have difficulties classifying triangles based on the kind of angles (Gal & Linchesky, 2010). The tasks included questions about the existence of equilateral, isosceles, and scalene triangles, including an angle of ninety-degree, or one greater or less than ninety-degree (Table 1).

The design of the tasks in the preparation of a teaching experiment is significant and considered a starting point in the achievement of learning objectives (Kieran, Doorman, & Ohtani, 2015; Steffe & Thompson, 2000). Well-designed tasks provide opportunities for students to develop deep levels of understanding. To design the tasks, we considered several principles to promote collective argumentation through refutation. Attention is given to the manner of proposing and conducting the argumentation in the classroom, and the participation of students in the construction of valid arguments was encouraged. All tasks were designed according to five design principles, thought to promote refutations in collective argumentation:

(P1) High cognitive-demand level: requires complex and non-algorithmic thinking; a predictable, well-known approach is not explicitly suggested by the task, instructions or an example (Smith & Stein, 1998);

(P2) Open tasks: contains a significant degree of indeterminacy in the initial information (data); that is, there is no emphasis on indicating the information on which the student has to base their answers/conclusion (Ponte, 2005);

(P3) Introduce false conclusions: creates an opportunity for students to develop their own ideas and to have the confidence to validate their own conclusions and those of other students (Rumsey & Langrall, 2016);

(P4) Generate cognitive conflict: confronts the students with contradictory information (Limón, 2001);

(P5) Management of the confrontation of positions: includes questions oriented to manage the conflict, that is to say there is more than one position students can take and opportunities to provide refutations of other positions (Solar & Deulofeu, 2016).

Table 1
Teaching Experiment Tasks

Equilateral triangle (block 1)	Isosceles triangle (block 2)	Scalene triangle (block 3)
T1: Are there equilateral triangles with a 90° angle? Justify your response.	T4: Are there isosceles triangles with a 90° angle? Justify your response.	T7: Are there scalene triangles with a 90° angle? Justify your response.
T2: Are there equilateral triangles with one angle less than 90°? Justify your response.	T5: Are there isosceles triangles with one angle less than 90°? Justify your response.	T8: Are there scalene triangle with one angle less than 90°? Justify your response.
T3: Are there equilateral triangles with one angle greater than 90°? Justify your response.	T6: Are there isosceles triangles with one angle greater than 90°? Justify your response.	T9: Are there scalene triangle with one angle greater than 90°? Justify your response.

All the tasks were designed according to principles P1 and P2: the students must solve the task without algorithmic procedures and the given information does not guide the solution. Task 1 is the focus of the analysis in this article. It states a false conclusion (P3), the existence of an equilateral triangle with an angle of ninety degrees, so this question promotes cognitive conflict (P4) and allows teacher management of the argumentation (P5).

Reconstruction of Complex Argumentation

Toulmin's model allows the reconstruction of argumentation at the local level, that is to say, the reconstruction of basic argumentation steps involving data, conclusions, and warrants. For the reconstruction of complex argumentation, this model is not adequate as it cannot capture the global structure that develops in classroom conversations (Knipping, 2008). Toulmin's model does not consider several elements present in a mathematics class such as teacher interventions, implicit statements, and refutations, and it does not represent the interconnection of several argumentation steps. To address this need, Knipping (2008) and Knipping and Reid (2015) adapted Toulmin's model to reconstruct complex argumentation in what they call global argumentation structures. Knipping (2003) proposed a three-stage process: Reconstruct the argumentation sequence along with the meaning of the conversation in the classroom, analyse argumentation structures, and compare these argumentation structures and reveal their rationale.

The first stage for the reconstruction of the complex argumentation is to divide the activity that took place in the class into episodes. This has the purpose of identifying arguments for analysis. Then, for each episode, the function of the mathematical statements in the argumentation in terms of data, conclusions, warrants, backing, and refutations is reconstructed (Knipping, 2008). Statements that function as data (D) in the argumentation could be, for example, the initial information given by the teacher, which might be conveyed through drawings, explicit statements, equations, or questions. A warrant (W) can be properties or regularities that the students use to support the conclusion (C). The conclusion (C) is the final answer of the student in a mathematical task. Refutation (R) occurs in cases where students deny one part of the argumentation. They can refute the data, the warrant or the claim, or the argument as a whole. We recognize that in the context of complex argumentation, students do not make everything explicit. They use informal language and terminology to describe content related to mathematical properties, invariant characteristics, mathematics relationships or relevant information. Some argumentation elements are left implicit. This implies that the analysis of students' phrases in terms of data, warrants, or conclusions may not capture everything that they intended to communicate.

Table 2

Sample Functional Reconstruction of an Argumentation

#		Transcription	Argumentative Function (A. F.)
1	*Teacher:*	Are there equilateral triangles with a ninety-degree angle?	Data/Conclusion
2	*Andrea:*	Yes, there are.	Conclusion
3	*Teacher:*	What can the class say about Andrea's response?	Teacher Intervention
4	*Andrea:*	Because its angles are equal [She drew a triangle with three ninety-degree angles on the board]	Implicit Warrant/ [drawing as a data]
5	*Kimberly and José:*	It is wrong!	Refutation of data
6	*Kimbery:*	Because the sum of its angles goes over one hundred eighty degree	Warrant

In Table 2 we provide an example of how we reconstructed the mathematical meaning of classroom talk. These data come from a previous intervention with a group of fifth graders who are not considered as part of our analysis in Section 4. The teacher poses Task 1: the existence of an equilateral triangle with ninety-degree angles. Transcriptions of episodes were translated from Spanish into English and the original transcription is in Annex 1.

Note that the time sequence is not the same as the logical sequence of the argument from data to conclusion. Andrea first states her conclusion (Line 2). As she does not express it in a complete sentence, we must look back to the teacher's question, which supplies most of the words of the conclusion: "Yes, there are equilateral triangles with a ninety-degree angle." Andrea supports her conclusion by drawing a triangle and marking its angles "ninety-degree" (similar to Figure 7, below). This is data in her argument, and her statement "because its angles are equal" (Line 4) refers to an implicit warrant, the fact that in general a triangle with three equal angles is equilateral. Kimberly and José refute Andrea's data, using the expression "it is wrong" (Line 5), Kimberly supports her refutation with a warrant, the mathematical property that the sum of internal angles in a triangle is one hundred and eighty degree, identified implicitly in line 6.

Local analysis of argumentation refers to the steps that make up an argument (Figure 5). This allows one to analyse the warrant's content, the relationship between data and conclusion, and refutation that denies some part of the argument. Lines 1, 2, and 4 in Table 2 form a local argument (see Knipping & Reid, 2015) consisting of data, warrant, and conclusion (see Figure 5). In terms of Toulmin's model, Figure 5 represents an argumentation step, Andrea's drawing is the data, from which she arrives at a conclusion stated in terms of the information given by the teacher. Her warrant is the equality of the angles in an equilateral triangle.

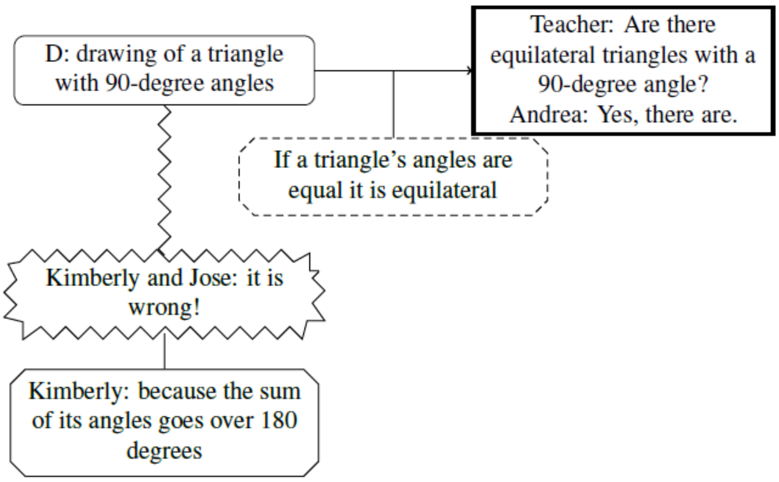

Figure 5. Reconstruction of local argumentation

In representing local argumentations, we use a rectangle with rounded corners to show a datum, one with diagonally cut corners to show a warrant or backing, one with a jagged outline to show a refutation and one with a dark outline to show a conclusion. Implicit statements are shown with dashed outlines.

Analysis at the global level represents the complex structure of the argumentation constituted by interconnected local arguments (Knipping & Reid, 2015). In this research, the representations of the complex argumentation structures have the following conventions (Figure 6): the conclusions are represented by squares, warrants and backings with diamonds, the data with circles, refutations with black rhombuses, and refutations that become conclusions with a rhombus inscribed in a square. Implicit statements are marked in grey.

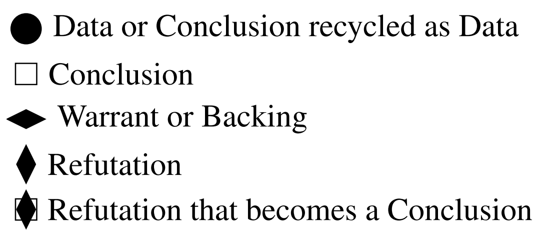

Figure 6. Argumentation structure conventions

The last two conventions are proposed by the authors as an emerging result of previous research related to mathematical argumentation. In this process, we also compare emerging argumentation structures with those reported in the literature to establish them as empirical contributions to research in mathematical argumentation in the classroom.

FINDINGS

In this section we describe an example of the complex argumentation structure in an elementary school mathematics class. Data presented here come from the teaching experiment conducted with fifth-graders. We focus on Task 1 concerning the existence of an equilateral triangle with an angle of ninety degrees, with the aim of exemplifying what happened throughout the teaching experiment.

Complex Argumentation in Task 1

The design of the tasks provides the students two possible conclusions, “yes” or “no”. In Task 1 the teacher started asking (¿?) about existence of equilateral triangles with one ninety-degree angle (see Table 3). This initial question provides two statements that became part of data and conclusions of the students’ argumentation. The first statement is that the triangle must be equilateral (D0). This is initial information that the students must include. The second statement raises the possibility of having an angle of ninety degrees as a part of their conclusion, without stating its truth. Based on these statements, Rene claimed that such equilateral triangles exist (C2), supporting his claim with a drawing (Figure 7) of an equilateral triangle with three ninety degree angles on the board (D1).

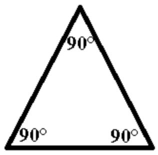

Figure 7. Reconstruction of Rene's drawing of an equilateral triangle with ninety-degree angles

Table 3
Transcription of Argumentation in Episode 1

Line	A.F.	Transcription	
1	¿? D0	*Teacher:*	Are there equilateral triangles with an angle of ninety degree?
2	C2	*Rene:*	Yes!
3	C1/D2 D1	*Rene:*	Because its three angles measure ninety. [He then drew an equilateral triangle with ninety degree angles]
4	C3/D4 C4	*Ezequiel:*	But if we have an angle of ninety would be greater than one hundred and eighty degrees

Note. A.F.= Argumentative Function

Based on Rene's drawing (D1) Ezequiel tells the class what will happen if the equilateral triangle with ninety-degree angles exists. This statement is a conclusion (C4) based on Rene's claim about the existence of an equilateral triangle with ninety-degree angles. We could speculate about Ezequiel's intent in drawing this conclusion, but he does not explicitly refute either Rene's data or his conclusion. This local argumentation <1> can be reconstructed as two chains of arguments (see Figure 8).

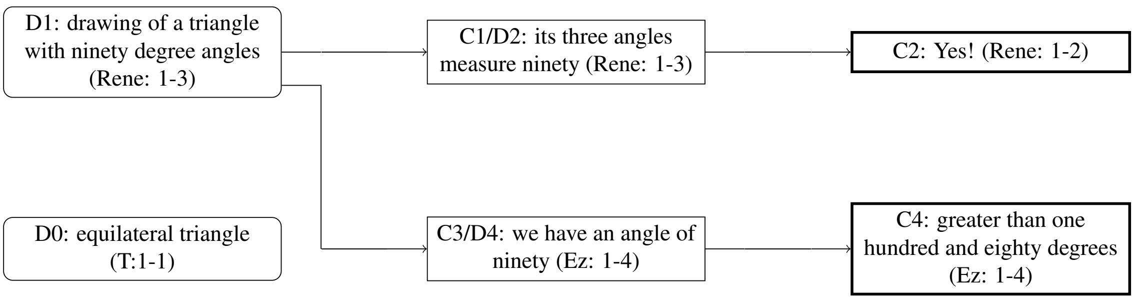

Figure 8. Chains of arguments in episode <1>

In this episode, the arguments leave out both warrants and statements that could connect the data with the conclusions. As Knipping and Reid (2015) note, in many arguments warrants are left implicit. Well known general principles do not have to be stated every time they are used in an argument. Obvious connecting statements, or statements considered to be obvious by the speaker, also are often left implicit. For example, Rene could have also said that the triangle he drew is equilateral, which follows from his statement about its angles (C1/D2).

In episode <2> Juliet presents a conclusion (C6) that is at the same time a refutation of Rene's conclusion (C2) (see Table 4). Her argument uses as a warrant a general property of equilateral triangles that she has learned: the angles always measure sixty degree (W6). She concludes that triangles like the one that Rene has drawn cannot exist.

Table 4
Transcription of Juliet's Argument in Episode 2

Line	A.F.		Transcription
1	¿?	*Teacher:*	What can you tell [Juliet]?
2	C6/R1	*Juliet:*	No, there are not![such triangles]
3	W6	*Juliet:*	Because its angles are sixty
4	C5/D6	*Juliet:*	And there [in Rene's drawing] its angles are ninety and do not exist…

Note. A.F.= Argumentative Function

Elsewhere it has been reported that a conclusion can act as a future datum (Krummheuer, 1995, 2015; Knipping & Reid, 2015). Here a conclusion acts as a refutation (C3/R1). Juliet's refutation (R1) is explicit, she refutes Rene's conclusion (C1) pointing out equilateral triangles with ninety-degree angles do not exist and putting in doubt Rene's drawing (D1) (see Figure 9).

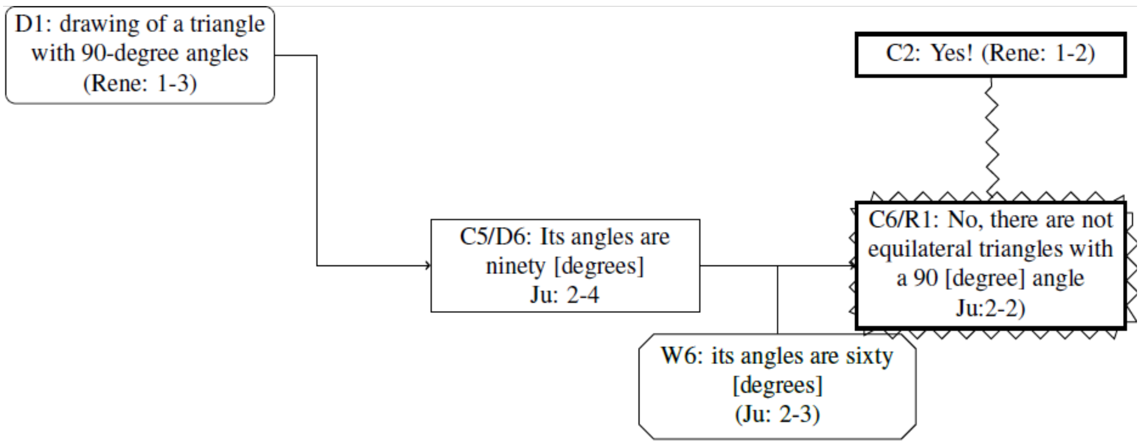

Figure 9. Refutation of Rene's conclusion

In episode <3> students also refute Rene's conclusion directly. In Mia's argument, she concludes that an equilateral triangle with ninety-degree angles would have an angle sum of two hundred and seventy degree (C7/D8). She does not state this conclusion explicitly, but it follows from her warrant (W7). Kenya and other students then make explicit that the angle sum is too big, and the teacher's question prompts them to be more precise, that the sum is bigger than one hundred and eighty degree. This echoes Ezequiel's earlier conclusion (C4) that the angle sum is more than one hundred and eighty degree. Mia's refutation (C9/R2) of (C1) follows from their conclusion, implicitly making use of the general rule that the angle sum is one hundred and eighty degrees as a warrant (see Figure 10). As they all know the general rule, we feel confident including it as an implicit warrant here.

Table 5
Mia's Refutation Argument in Episode 3

Line	A.F.		Transcription
1	C9/R2	*Mia:*	Not!
2	C8/D9	*Mia:*	Because it does not give one hundred and eighty
3	W7	*Mia:*	And three times ninety is two hundred and seventy
4	C8/D9	*Kenya:*	Teacher it is too big
5	¿?	*Teacher:*	Bigger than what?
6	C8/D9	*Students:*	One hundred and eighty

Note. A.F.= Argumentative Function

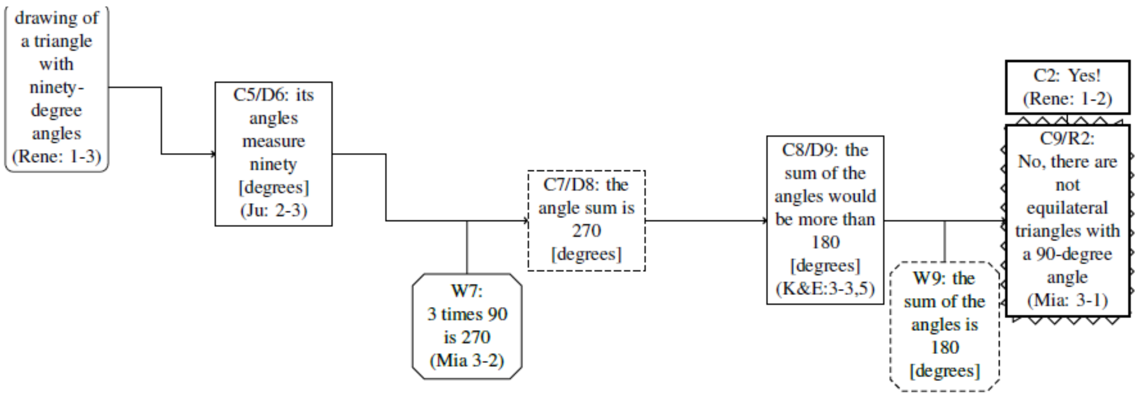

Figure 10. *Mia's refutation of the conclusion*

Connecting argumentations episodes <1>, <2> and <3> provide us with a part of the whole argumentation structure of Task 1 (see Figure 11). One thing visible in this argumentation structure is that the argumentation in episode <3> and Ezequiel's argumentation in episode <1> are similar. Only one statement is the same in both (marked "a" in Figure 11), "greater than one hundred and eighty degrees" (Ezequiel) "it is bigger than one hundred and eighty" (Kenya and other students), but other statements made explicitly in episode <3>, like "three times ninety is two hundred and seventy" (W7) and the refutation "Not!" (C9/R2), correspond to implicit statements that make Ezequiel's argumentation more complete (marked "b"). One of Ezequiel's statements "we have an angle of ninety" (D4) is not made explicitly in episode <3> but as it had just been repeated by Juliet in episode <2> "there [in Rene's drawing] its angles are ninety" (D6) we include it in episode <3>.

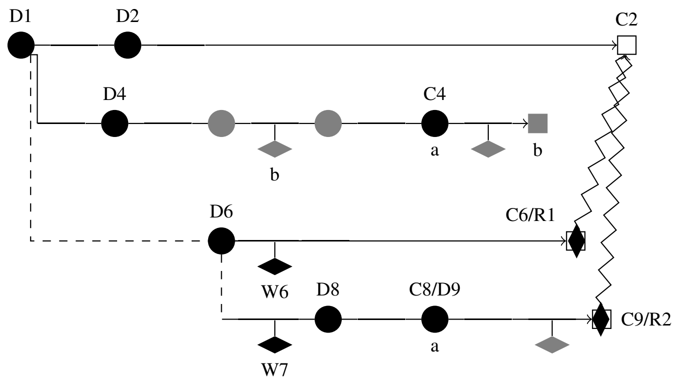

Figure 11. The structure of the argumentation in episodes <1>, <2>, and <3>

After this round of refutations, the students still discussed the existence of equilateral triangles with ninety-degree angles (see Table 6). The teacher started by asking the class about the initial information (data), types of angles, and characteristics of equilateral triangles. In episodes <4> and <5> the student's arguments are more sophisticated than their first arguments. They arrive at the final conclusion and make explicit properties they used in an implicit way earlier.

Table 6
Transcription of Episode 4

Line	F. A.		Transcription
1	¿?	*Teacher:*	Well, we already know that their sides are...
2	C10	*Students:*	Equal
3	¿?	*Teacher:*	Now we know, what it the measure of their angles is?
4	C11	*Students:*	Sixty…
5	¿?	*Teacher:*	It can measure more than sixty?
6	C13	*Students:*	No!
7	C12/D13 W13	*Augustine:*	Because the sum of interior angles is going to be greater than one hundred eighty! [Implicit warrant]

Note. A.F.= Argumentative Function

The teacher's first two questions in this episode (in Lines 1 and 3) implicitly refer back to the starting datum (D0), that they are talking about equilateral triangles. She invites the students to tell what they know about equilateral triangles. One of these conclusions has already been used (by Juliet in episode <2>) as a warrant (W6). The teacher's next question, "It can measure more than sixty?", functions as a hypothesis (H12), a datum that is not taken as true, but is instead investigated. Augustine draws a conclusion (C12) from this hypothesis, which contradicts the general rule about the sum of the angles in a triangle that they have already used as an implicit warrant earlier. From this contradiction they conclude that the hypothesis is false (Figure 12).

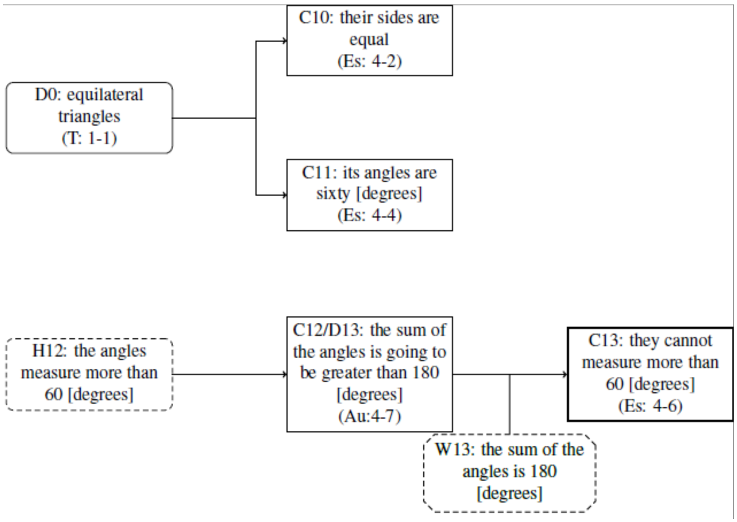

Figure 12. Argumentation stream of episode 4

In the next episode, the students again use a contradiction as a way of refutation (see Table 7) in order to conclude that equilateral triangles do not have ninety-degree angles (C18). To support this conclusion, students conclude from a hypothesis that the angles measure ninety, that the angle sum will be two hundred and seventy degree (C17), which contradicts the triangle internal angle sum property stated explicitly by some students (W18).

Table 7
Transcription of Episode 5

Line	A.F.		Transcription
1	¿?	*Teacher:*	Then, when I asked, are there equilateral triangles with a ninety-degree angle? Is there such a triangle?
2	C18/R3	*Students:*	No!
3	¿?	*Teacher:*	Well, whoever told me no, give me a justification!
4	W17	*Leonel:*	No, because if we make a multiplication of ninety times three it gives us two hundred and seventy
5	¿? C14/D15	*Teacher:*	And why multiply by three?
6	D14	*Leonel:*	Because there are three angles...
7	D16, C16	*Teacher:*	Then if one angle measures ninety the others should measure ninety, and it would then say...how would it be?

Table 7
Transcription of Episode 5

Line	A.F.		Transcription
8	C17/D18	*Students:*	Two hundred and seventy
9	¿?	*Teacher:*	And what is the sum of the angles?
10	W18	*Students:*	One hundred and eighty

Note. A.F.= Argumentative Function

This argumentation is similar to that in episode <3> and Ezequiel's in episode <1>. The datum (D16) in the teacher's argument in Line 7 corresponds to Ezequiel's C3/D4. Her conclusion makes explicit (C3) that Ezequiel left implicit, that all three angles measure ninety (which Ezequiel may not have mentioned as Rene had just said it (C1/D2)). Leonel, in Line 4, and other students, in Line 8, make explicit the warrant (W17) and conclusion (C17) that connect the angle measures to Ezequiel's conclusion (C4). Here, however, rather than first concluding that the angle sum would be more than one hundred and eighty degree, as Ezequiel did in C4 and other students did in episode <3> (C8), the students observe the contradiction to the angle sum property immediately and conclude (C18) that equilateral triangles with ninety-degree angles cannot exist. This conclusion is the third direct refutation of Rene's (C2).

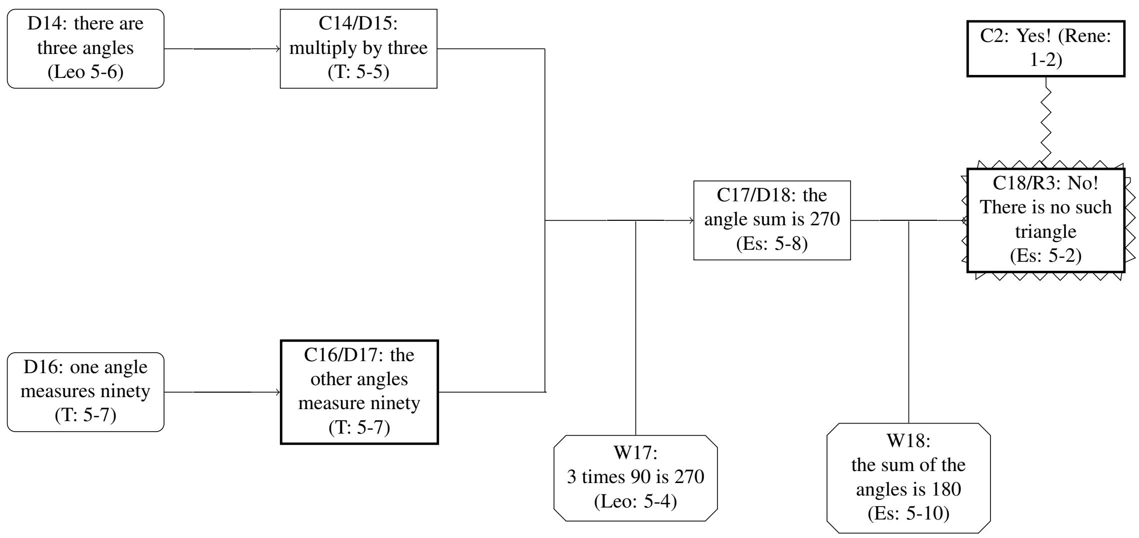

Figure 13. Argumentation stream of episode 5

The teacher followed episode <5> with a last question, which prompted a new argument <6> that leads to a slightly different conclusion, that a triangle with a ninety-degree angle cannot be equilateral (C19).

Table 8
Final Argument

Line	A.F.	Transcription	
1	¿?	*Teacher:*	Who would like to give another answer
2	D19	*Juliet:*	If they are ninety
3	C19	*Juliet:*	Would not be an equilateral triangle...
4	W19	*Juliet:*	The interior angles are sixty

Note. A.F.= Argumentative Function

In the beginning of the class Rene supported his conclusion with a drawing of an equilateral triangle with ninety-degree angles (D1). In order to conclude that equilateral triangles with ninety-degree angles do not exist (C19), Juliet takes from Rene's drawing that the angles are ninety-degree angles (D19) and which contradicts the warrant (W19) that the measure of the internal angles of an equilateral triangle must be sixty degrees.

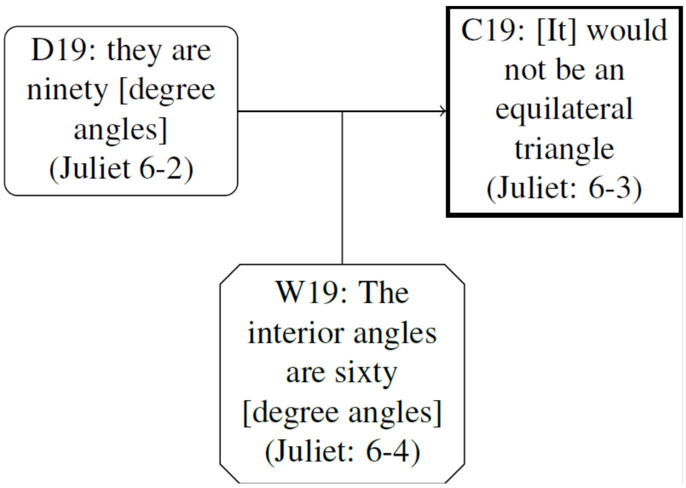

Figure 14. Argumentation stream of Juliet's argument

Analysing the Global Argumentation of the Task

The complex argumentation structure for Task 1 (Figure 15) shows the interconnections between the students' arguments. The global argumentation can be analysed by putting together all the episodes or argumentation streams <1>, <2>, <3>, <4>, <5>, and <6>.

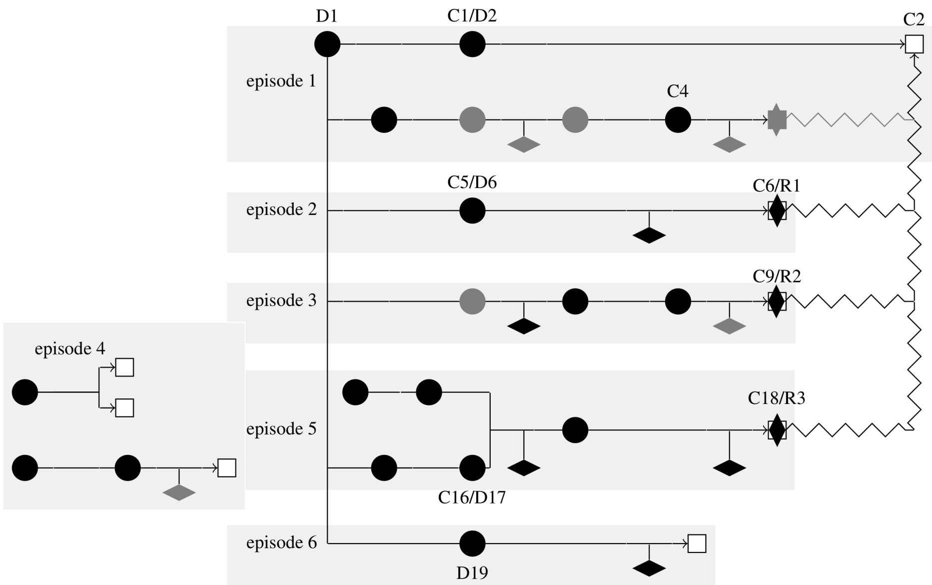

Figure 15. The argumentation structure for Task 1

The global argumentation structure shows the development of the argumentation through six argumentation streams. In the first part of the class (episodes <1>, <2>, and <3>) students critiqued Rene's conclusion using refutations, but often with implicit warrants and data. In the later episodes the argumentation has fewer implicit data, and in some cases warrants are present, although not for all steps.

Most of the streams begin with Rene's picture (D1) as initial data and end in a refutation of his conclusion (C2). Recall that in earlier literature about refutations analised using the Toulmin scheme the refutations were single statements (sometimes with warrants or backing) that refuted data or arguments. Here we see refutations that are conclusions of more extended streams involving multiple steps, and refuting a conclusion. Such streams, in which the refutation is the conclusion of an argument involving its own data and warrants, is a new type of argument that emerged in the analysis of this teaching experiment.

The parallel arguments visible in Figure 15 are similar to those in the source structure and the spiral structure (see Figures 2 & 3). But there are important differences. Unlike in the source structure, the three conclusions (C6, C9), and (C18), are not distinct statements each of which are used as data in a later argument. In fact, the three conclusions (which are also the refutations R1, R2 and R3) take the same form, and could be considered the same statement. Hence the situation is more like that in the spiral structure where several distinct arguments lead to the same conclusion. However, here the difference is that the arguments themselves are not distinct.

We have arranged the diagram to show equivalent statements in vertical rows. For example, Rene's statement that there are three ninety degree angles in his picture (C1/D2) is repeated by Juliet in episode <2> (C5/D6), by the teacher in episode <5> (C16/D17) and by Juliet again in episode <6> (D19). It is implicit in Ezequiel's argument in episode <1> and in episode <3>. This arrangement makes it possible to see that argumentation stream 3 and Ezequiel's argument in episode 1 have the same statements in the same sequence, although some are implicit in both arguments. The argumentation stream from episode <5> includes most of the same statements explicitly, omitting only Ezequiel's conclusion that the sum is greater than one hundred and eighty degrees (C4) which is not necessary. This structure of equivalent arguments repeated with increasing explicitness differs from the source and spiral structures in which different arguments come together.

However, episode 2 includes a different argument for the same conclusion, and so its relationship to episode <5> can be seen as similar to the parallel argument in the spiral structure. Interestingly, the argumentation stream in episode <6> includes the same statements as in the argumentation stream in episode <2>, but the conclusions are different (The conclusion in episode <6> is the converse of the conclusion in episode <2>). But as children (and many adults) tend to treat conditional statements and converses as equivalent (O'Brien, Spapiro, & Reali, 1971), Juliet (who made both statements) may have simply misspoken.

Moreover, the argumentation streams in episodes <2>, <3>, and <5> all end in refutations of Rene's conclusions (C2), so while they are *parallel* to each other, they can be considered *perpendicular* to Rene's argument. As Reid and Knipping (2010) note, the argumentation structure is likely to be closely tied to the teacher's goals and the nature of the task. In the context of this teaching experiment, where creating contexts for refutations was a goal, the teacher's questions orientated the whole collective argumentation, by providing parts of the data, warrants or conclusions, and also by repeatedly asking the class about the conclusions of other students and other possible answers.

Discussion and Conclusions

In our research, we have identified several features of complex collective argumentation in an elementary school classroom. These include distinct functions that statements can take on the argumentation, an increasing development of the sophistication of the argumentation over time, a novel argumentation structure having some features in common with others described in the literature, as well as some unique features, and the key role of teacher in promoting argumentation.

We identified different functions of statements in the reconstruction of the complex argumentation, including teacher interventions that acted as both data and potential conclusions, implicit warrants, and conclusions that act as refutations. We observed (as did Conner, Singletary, Smith, Wagner, & Francisco, 2014) that the teacher's initial questions can provide, implicitly, part of the data the students use in their arguments. We also observed that students' conclusions could be quickly and succinctly expressed as responses to the teacher's initial questions. The prevalence of implicit warrants in classroom argumentation has been noted before (e.g., Knipping & Reid, 2015), and in our case this seems to occur most often when the needed warrant is a general rule known to all the students, such as the definition of 'equilateral triangle'. In some cases, these implicit warrants are later made explicit. Finally, we observed argumentation streams in which the conclusion acts as a refutation of the conclusion of another stream. Refutations of warrants or data have been reported elsewhere (Reid, Knipping, & Crosby, 2011), but usually having a simple form, consisting of one or two statements. The extended arguments leading to refutations observed here are presumably related to the nature of the task, which was intended to elicit refutations.

The global argumentation structure captures how the students' arguments become more sophisticated from the beginning to the end of the class. In all, four argumentation streams lead to conclusions that refute Rene's initial claim. Three of these involve the same chain of steps (containing, sometimes implicitly, the core elements of data, conclusion and warrant). At first, almost all the elements are implicit, including the conclusion. Then, the conclusion, one warrant and most of the data statements are made explicit. Finally, all the data/conclusion pairs are made explicit, as are the warrants.

The argumentation structure that we have reconstructed from the teaching experiment can be seen as consisting of several parallel refutation arguments perpendicular to the argument for the conclusion they refute. The argumentation structure differs from a spiral structure (Knipping & Reid, 2015), in which different parallel arguments prove the final conclusion, in two ways. First, some of the perpendicular arguments are repeated in more sophisticated ways. Second, the refutations in the spiral structure challenge elements of the parallel arguments and provide support to the whole argumentation, while here the refutations are conclusions of the parallel arguments. The teaching focus is also different, as the nature of the task allows students to present their refutations as new conclusions and provides a way of learning mathematics collectively based on refutations.

In collective argumentation contexts, the teacher plays a key role in the promotion and evolution of students' argumentations. Here we have seen how the teacher guides the argumentation though questions, such as "Who would like to give another answer?" and "What can you tell about this…?" Such questions are used to foster other students' participation. As noted by Rumsey and Langrall (2016), students in the elementary grades are able to look at patterns, make mathematical conjectures, and modify them on the basis of feedback from others. Through questioning, the teacher can create a situation where students can make their conclusions or warrants explicit, supported by others' participation. In the same way, we recognized that the teacher's interventions prompted the evolution of the argumentation's completeness and explicitness. Specific questions like, "Bigger than what?" "Can it measure more than sixty?" prompt students to describe relevant characteristics of triangles. Information that was implicit can become explicit through questions like "Why multiply by three?" Similar to these questions, instructional strategies such as providing language support and providing students with a common background in mathematics (i.e., mathematical terms, invariant characteristics) promote argumentation in elementary school classes (Rumsey & Langrall, 2016).

This research makes important contributions to the learning of mathematics in the context of argumentation in elementary school. Students' arguments can evolve guided by teacher interventions and refutations from other students. Also, refutation arguments promote awareness of the validity of arguments, and permit students to identify mistakes in others' arguments (Solar & Deulofeu, 2016) and also give them the opportunity to improve their understanding of mathematical concepts.

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ANNEX

Table 9
Transcription from the Earlier Intervention in Spanish (Table 2)

#		Transcription
1	*Profesor:*	¿Existen triángulos equiláteros con un ángulo de noventa grados?
2	*Andrea:*	Sí, sí existen.
3	*Profesor:*	¿Qué puede decir la clase acerca de la respuesta de Andrea?
4	*Andrea:*	Porque sus ángulos son iguales [ella dibujó un triángulo

Table 9
Transcription from the Earlier Intervention in Spanish (Table 2)

#		Transcription
		equilátero con tres ángulos de noventa grados]
5	*Kimberly y José:*	¡Está mal!
6	*Kimberly:*	Porque la suma de sus ángulos da más de ciento ochenta grados

Table 10
Transcription of the Class, Task 1 in the Teaching Experiment, in Spanish (Tables 3-8)

<>		Transcription
1	*Profesor:*	¿Existen triángulos equiláteros con un ángulo de noventa grados?
	Rene:	¡Sí!
	Rene:	Porque sus tres ángulos miden noventa grados
	Ezequiel	Pero si tenemos un ángulo de noventa se pasaría de ciento ochenta
2	*Profesor:*	¿Qué puedes decir [Julieta]?
	Julieta:	¡No! , ¡Que no existen!
	Julieta:	Porque sus ángulos son de sesenta y ahí viene siendo que sus ángulos son de noventa y no existen porque sus ángulos son de sesenta
3	*Mía:*	¡No!
	Mía:	Porque no da ciento ochenta y tres por noventa da doscientos setenta.
	Kenia:	¡Se pasa profe!
	Profesor:	¿Se pasa de cuánto?
	Estudiantes:	De ciento ochenta
	Profesor:	Bueno ya sabemos que sus lados son…
	Estudiantes:	Iguales
	Profesor:	Ahora, ¿podemos saber cuánto miden sus ángulos?
4	*Estudiantes:*	Sesenta
	Profesor:	¿Puede medir más de sesenta?
	Estudiantes:	¡No!
	Agustín:	¡Porque se pasa de ciento ochenta! [se refiere a la suma de los

Table 10
Transcription of the Class, Task 1 in the Teaching Experiment, in Spanish (Tables 3-8)

<>		Transcription
		ángulos interiores]
	Profesor:	Entonces cuando les preguntaron; ¿existen triángulos equiláteros con un ángulo de noventa grados?, ¿Existe un triángulo?
	Estudiantes:	¡No!
	Profesor:	Bueno los que me dijeron que no, díganme una justificación. Vamos a escuchar a Leonel.
	Leonel:	¡No!, porque si ponemos una multiplicación de noventa por tres nos da ciento setenta.
5	*Profesor:*	¿Y por qué multiplicar por tres?
	Leonel:	Porque son tres ángulos …
	Profesor:	Entonces si uno midiera noventa el otro me debería medir noventa, sí, y entonces se pasaría dicen ustedes… ¿cuánto sería?
	Estudiantes:	Doscientos setenta
	Profesor:	¿Y cuánto es la suma de los ángulos?
	Estudiantes:	Ciento ochenta
6	*Profesor:*	¿Quién más puede dar otra respuesta?
	Julieta:	Si son de noventa
	Julieta:	Ya no sería un triángulo equilátero…
	Julieta:	Los ángulos interiores son de sesenta

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