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Non-Abelian anomalous (super)fluids in thermal equilibrium from differential geometry

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Abstract: We apply differential geometry methods to the computation of the anomaly-induced hydrodynamic equilibrium partition function. Implementing the imaginary-time prescription on the Chern-Simons effective action on a stationary background, we obtain general closed expressions for both the invariant and anomalous part of the partition function. This is applied to the Wess-Zumino-Witten action for Goldstone modes, giving the equilibrium partition function of superfluids. In all cases, we also study the anomaly-induced gauge currents and energy-momentum tensor, providing explicit expressions for them.

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JHEP11(2018)076

Contents

1 Introduction

Quantum anomalies play a central role in high energy physics (see [1–5] for reviews), being the cause behind important physical phenomena. They also impose severe constraints on physically viable theories, and their dual infrared/ultraviolet character can be exploited to extract nonperturbative information in a variety of physically interesting situations. In recent years it has been realized that anomalies are also relevant for hydrodynamics [6, 7]. In the presence of anomalous currents coupling to external, nondynamical gauge fields, parity is broken and additional tensor structures in the constitutive relations are allowed, associated with new transport coefficients. Preservation of the second law of thermodynamics leads to a number of identities to be satisfied by these additional coefficients, ensuring the nondissipative character of these anomaly-induced effects. This is of much physical interest, since anomalous hydrodynamics is related to transport phenomena associated with chiral imbalance, such as the chiral magnetic and vortical effects [8, 9] (more details can be found in the reviews [10–12]). There are also important connections to a variety of other condensed matter systems [13], giving rise to experimental signatures [14].

Alternatively, equilibrium hydrodynamics can be studied without resorting to the entropy current [15–18]. The underlying idea is the construction of an effective action for the hydrodynamic sources on a generic stationary spacetime, with the appropriate number of derivatives and consistent with all relevant symmetries. In the case of anomalous hydrodynamics, this effective action functional include additional terms which, upon gauge transformations of the external sources, render the right value of the anomaly. The equilibrium thermal partition function is obtained from this effective action, with the inverse temperature identified with the length of the compactified Euclidean time. Correlation functions of currents and other quantities are then computed by functional differentiation with respect to external sources.

Since the 1980s, differential geometry has revealed itself as a very powerful tool in the analysis of anomalies in quantum field theory [1]. The reason for its success lies in the fact that anomalies are originated in the topological structure of the gauge bundle, so they are determined, up to a global normalization, from topological invariant quatities. This also extends to the case of systems with spontaneously symmetry breaking, where the anomaly very much constraints the dynamics of Goldstone bosons [19–21]. One of the big advantages of using differential geometry methods in the analysis of anomalies is the possibility of constructing effective actions rather straightforwardly in terms of the Chern-Simons forms derived from the appropriate anomaly polynomial. Using homotopy methods [1], it is possible in many cases to find closed expressions for the Chern-Simons form and its descent quantities. Apart from this important fact, a further benefit of the differential geometry approach is that it exhausts all perturbative contributions to the anomaly, which in the non-Abelian case include not just the triangle, but also the square and pentagon one-loop diagrams. These techniques were employed in refs. [17, 18, 22] to study various aspects of the physics of anomalous fluids in thermal equilibrium.

In this article we carry out a systematic construction of the equilibrium partition functions for fluids with non-Abelian chiral anomalies in arbitrary even dimension, $D = 2n - 2$, using differential geometry methods along the lines of [18]. Our aim is to provide explicit expressions that could be easily applied to the study of nondissipative anomalous fluids. To this end, we give a general prescription to obtain the anomaly-induced part of the partition function by performing a dimensional reduction on the time circle of the Chern-Simons effective action defined on a $(2n-1)$-dimensional stationary spacetime. With this we immediately show that the partition function splits into a *local* anomalous piece and a *nonlocal* invariant part, as pointed out in [18]. Moreover, we provide operative expressions for both contributions to the partition function that can be used in the analysis of generic theories. Their use is illustrated with various examples in four dimensions ($n = 3$).

One of the main targets of our analysis is the study of hydrodynamics in the presence of spontaneous symmetry breaking [23–25], extending the analysis of [18] to this setup. The equilibrium partition function for these systems is built from the dimensional reduction of the Wess-Zumino action describing the dynamics of the Goldstone modes on a generic stationary background. Unlike the unbroken case, here we see that the effective action only depends on the local anomalous part of the action. The general expressions obtained this way might be of relevance for non-Abelian anomalous superfluids [26–30].

In all cases, we provide explicit expression for both the gauge currents and the energy-momentum tensor. When the symmetry remains unbroken, the covariant current is computed from the variation of the nonlocal invariant part of the action, whereas the consistent current is obtained from the anomalous local piece [18]. In the presence of spontaneous symmetry breaking, on the other hand, we find that both the covariant and the consistent currents can be written in terms of the Bardeen-Zumino current interpolating between the two. We thus compute the currents on a stationary background from the dimensional reduction of the Bardeen-Zumino current. Moreover, an evaluation of the leading anomaly-induced energy-momentum tensor gives a vanishing result, due to the cancellation between the contributions of gauge fields and Goldstone modes.

The power of the techniques presented here lie on their systematic and wide applicability, providing a fast and efficient way of computing the equilibrium partition function and the transport coefficients in generic models. We will illustrate this point with some sample applications. A systematic analysis of particular models, including full details, will be presented in a future work [31].

The remainder of the article is organized as follows. In section 2 we offer a quick review of the basic aspects of the differential geometry approach to the construction of anomalous effective actions. We pay special attention to the case of two fermions with different chirality coupled to independent external gauge fields and, using the generalized transgression formula [32], we compute the effective action giving the anomaly in the Bardeen form. The transgression formula is further used in section 3 to define the currents associated with the Chern-Simons effective action. In section 4, we construct the fluid equilibrium partition function on a generic stationary background by dimensional reduction of the Chern-Simons effective action on the time cycle. In this way, we show that the anomalous part of the dimensionally-reduced partition function becomes local. Our results are then applied to the four-dimensional Bardeen anomaly and, as a particular case, we obtain the partition function of a two-flavor hadronic fluid.

In section 5 we study the gauge currents and the energy-momentum tensor derived from the anomaly-induced partition function on a general stationary background. We obtain the Wess-Zumino term relating the consistent and the covariant currents. The expressions found in this section are illustrated with the results for two-flavor QCD coupled to an external electromagnetic field on a nontrivial stationary background. Section 6 is devoted to the study of anomalous fluids in theories with spontaneous symmetry breaking. We construct the partition function at leading order in derivatives, starting from the Wess-Zumino-Witten effective action of Goldstone bosons on a stationary metric and implementing the imaginary time prescription. After discussing the issue of the currents in

JHEP11(2018)076

this setup, we compute the partition function in four-dimensions for a two-flavor hadronic superfluid where the global flavor group is broken down to its vector subgroup. We also compute the associated currents and anomaly-induced energy-momentum tensor. Finally, in section 7 we summarize our results and discuss future lines of work.

To make the article self-contained, a short review of the generalized transgression formula is presented in appendix A, whereas some explicit expressions are deferred to appendix B. Finally, in appendix C we have summarized some relevant trace identities for the group U(2).

2 Anomalies and differential geometry

Differential geometry is a very powerful tool in the study of quantum field theory anomalies. One of its advantages is its power in providing very general prescriptions to construct quantum effective actions. In this section we are going to review basic aspects of the differential geometry approach to quantum anomalies to be applied in the rest of the paper. More details can be found in the reviews [1–5].

2.1 Chiral anomalies and effective actions

We begin by studying the theory of a chiral fermion coupled to an external gauge field $\mathcal{A}_\mu \equiv \mathcal{A}^a_\mu t_a$ described by the Lagrangian

$$\mathscr{L}_{\text{YM}} = i\overline{\psi}\gamma^\mu\left(\partial_\mu - it_a\mathcal{A}^a_\mu\right)\psi, \tag{2.1}$$

where $t_a = t_a^\dagger$ are the Hermitian generators of the Lie algebra $\mathfrak{g} = \text{Lie}(\mathcal{G})$ satisfying the commutation relations $[t_a, t_b] = if_{abc}t_c$. Here and in the following, the gauge coupling constant g is absorbed into the gauge field $\mathcal{A}^a_\mu$. It is convenient to introduce the anti-Hermitian, Lie algebra valued one-form[1]

$$\mathcal{A} = -i\mathcal{A}^a_\mu t_a dx^\mu \equiv -i\mathcal{A}_\mu dx^\mu, \tag{2.2}$$

while the associated field strength two-form is defined by

$$\mathcal{F} = d\mathcal{A} + \mathcal{A}^2 \equiv -\frac{i}{2}\mathcal{F}_{\mu\nu}dx^\mu dx^\nu, \tag{2.3}$$

with components

$$\mathcal{F}_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu - i[\mathcal{A}_\mu, \mathcal{A}_\nu]. \tag{2.4}$$

The field strength two-form satisfies the Bianchi identity

$$d\mathcal{F} = [\mathcal{F}, \mathcal{A}]. \tag{2.5}$$

Finite gauge transformations are implemented by the Lie group elements

$$g = e^{-it_a u^a} \equiv e^u, \tag{2.6}$$

[1]In this paper we systematically omit the wedge symbol $\wedge$ in the exterior product of differential forms. Otherwise, we follow the conventions of ref. [33].

JHEP11(2018)076

where $u(x)$ is a Lie algebra valued function depending on the spacetime point. The gauge field one-form $\mathcal{A}$ transforms as a connection

$$\mathcal{A}_g = g^{-1}\mathcal{A}g + g^{-1}dg, \tag{2.7}$$

whereas the field strength transforms covariantly as an adjoint field

$$\mathcal{F}_g = g^{-1}\mathcal{F}g. \tag{2.8}$$

The corresponding infinitesimal transformations are obtained by expanding the previous expressions to leading order in the gauge function $u(x)$, to give

$$\begin{aligned} \delta_u\mathcal{A} &= du + [\mathcal{A}, u] \equiv Du, \\ \delta_u\mathcal{F} &= [\mathcal{F}, u], \end{aligned} \tag{2.9}$$

where we have introduced the adjoint covariant derivative Du.

To study gauge anomalies it is convenient to work with the fermion effective action functional obtained by integrating out the fermion field

$$e^{i\Gamma[\mathcal{A}]} \equiv \int \mathscr{D}\overline{\psi}\mathscr{D}\psi\, e^{iS_{\mathrm{YM}}[\mathcal{A},\psi,\overline{\psi}]}. \tag{2.10}$$

The gauge anomaly is then signalled by the transformation of this effective action under gauge transformation. Indeed, under a general shift $\mathcal{A}^a_\mu \to \mathcal{A}^a_\mu + \delta\mathcal{A}^a_\mu$, the consistent current is defined from the first order variation of $\Gamma[\mathcal{A}]$ by

$$\delta\Gamma[\mathcal{A}] = \int d^Dx\, \delta\mathcal{A}^a_\mu(x) J^\mu_a(x), \tag{2.11}$$

whereas the anomaly is given by the failure of the effective functional to be invariant under gauge transformations

$$\delta_u\Gamma[\mathcal{A}] = -\int d^Dx\, u^a(x) G_a[\mathcal{A}(x)]_{\mathrm{cons}}, \tag{2.12}$$

where $G_a[\mathcal{A}(x)]_{\mathrm{cons}}$ is the consistent anomaly. Particularizing now eq. (2.11) to gauge transformations, $\delta\mathcal{A}^a_\mu = (D_\mu u)^a$, we arrive at the anomalous (non)conservation law for the consistent gauge current

$$D_\mu J^\mu_a(x)_{\mathrm{cons}} = G_a[\mathcal{A}(x)]_{\mathrm{cons}}. \tag{2.13}$$

This form of the anomaly as well as the current defined in eq. (2.11) are called *consistent* because the anomaly satisfies the Wess-Zumino consistency condition implied by the closure of the commutator of two infinitesimal gauge transformations

$$\delta_u\delta_v - \delta_v\delta_u = \delta_{[u,v]}. \tag{2.14}$$

Indeed, from the definition of the consistent anomaly in terms of the gauge variation of the effective functional given in (2.12), we automatically arrive at the Wess-Zumino consistency conditions [19] by applying eq. (2.14) to the effective action

$$\int d^Dx\, v^a\delta_u G_a[\mathcal{A}] - \int d^Dx\, u^a\delta_v G_a[\mathcal{A}] = \int d^Dx\, [u,v]^a G_a[\mathcal{A}]. \tag{2.15}$$

JHEP11(2018)076

In fact, the gauge variation of any functional of the gauge field will automatically satisfy the Wess-Zumino consistency conditions. This applies to the consistent anomaly, which being obtained as the gauge variation of $\Gamma[\mathcal{A}]$ fulfills them naturally. However, not every solution to the consistency condition (2.15) gives an anomaly. For a theory to be anomalous, the anomaly should be given as the gauge variation of a *nonlocal* functional of the gauge field. Otherwise the violation in the conservation of the gauge current can be cancelled by adding a local counterterm to the action (i.e., by imposing an appropriate renormalization condition).

It is well known that the non-Abelian gauge anomaly $G_a[\mathcal{A}]$ for a chiral fermion can be obtained from the corresponding anomaly polynomial by using the Stora-Zumino descent equations [1]. For a right-handed fermion in $D = 2n - 2$ spacetime dimensions, the corresponding anomaly polynomial is given by

$$\mathcal{P}_n(\mathcal{F}) = c_n \mathrm{Tr}\, \mathcal{F}^n, \tag{2.16}$$

where the correct normalization c_n is found by applying the Atiyah-Singer index theorem to certain Dirac operator in $2n = D + 2$ dimensions [34]

$$c_n = \frac{1}{n!} \frac{i^n}{(2\pi)^{n-1}}. \tag{2.17}$$

Using the Bianchi identity (2.5) it is easy to show that the anomaly polynomial is a closed form

$$d\, \mathrm{Tr}\, \mathcal{F}^n = 0, \tag{2.18}$$

whereas the gauge transformation (2.9) implies that it is gauge invariant

$$\delta_u \mathrm{Tr}\, \mathcal{F}^n = 0. \tag{2.19}$$

Invoking the Poincaré lemma, we conclude from the identity (2.18) that $\mathrm{Tr}\, \mathcal{F}^n$ is locally exact. Thus, we write

$$\mathrm{Tr}\, \mathcal{F}^n = d\omega^0_{2n-1}(\mathcal{A}), \tag{2.20}$$

where $\omega^0_{2n-1}(\mathcal{A})$ is the Chern-Simons form. Equation (2.19), on the other hand, implies also that the gauge variation of ω^0_{2n-1} is, locally, a total differential

$$\delta_u \omega^0_{2n-1}(\mathcal{A}) = d\omega^1_{2n-2}(u, \mathcal{A}). \tag{2.21}$$

This identity is the key ingredient to identify the consistent anomaly. The fermion effective action is constructed as the integral of the Chern-Simons form

$$\Gamma[\mathcal{A}]_{\mathrm{CS}} = c_n \int_{\mathcal{M}_{2n-1}} \omega^0_{2n-1}(\mathcal{A}), \tag{2.22}$$

where $\mathcal{M}_{2n-1}$ is a $(2n-1)$-dimensional manifold whose boundary $\partial\mathcal{M}_{2n-1}$ corresponds to the $(2n-2)$-dimensional spacetime manifold. Its gauge variation is computed using (2.21)

JHEP11(2018)076

and the Stokes theorem to give

$$\delta_u\Gamma[\mathcal{A}]_{\rm CS} = c_n \int\limits_{\mathcal{M}_{2n-1}} \delta_u\omega^0_{2n-1}(\mathcal{A}) = c_n \int\limits_{\mathcal{M}_{2n-1}} d\omega^1_{2n-2}(u,\mathcal{A})$$
$$= c_n \int\limits_{\partial\mathcal{M}_{2n-1}} \omega^1_{2n-2}(u,\mathcal{A}). \tag{2.23}$$

Comparing with eq. (2.12), we find that $\omega^1_{2n-2}(u,\mathcal{A})$ yields the properly normalized anomaly for a right-handed fermion as

$$-\,c_n\,\omega^1_{2n-2}(u,\mathcal{A}) = u^a G_a[\mathcal{A}] \equiv \mathrm{Tr}\,\Big(uG[\mathcal{A}(x)]_{\rm cons}\Big). \tag{2.24}$$

Moreover, since it results from a gauge variation of a functional, it automatically satisfies the Wess-Zumino consistency condition, hence the subscript on the right-hand side of this equation. The crucial element in this analysis is that, being an integral over a higher-dimensional manifold, the effective action (2.22) is *nonlocal* in dimension $D = 2n-2$ and provides a nontrivial solution to the consistency condition. Incidentally, eq. (2.24) is also valid for a left-handed fermion with the replacement $c_n \to -c_n$.

From the previous discussion, we see that the computation of the gauge anomaly boils down to the evaluation of the Chern-Simons form. This object can be expressed in a closed form using the homotopy formula

$$\omega^0_{2n-1}(\mathcal{A}) = n\int_0^1 dt\,\mathrm{Tr}\,\Big(\mathcal{A}\mathcal{F}_t^{n-1}\Big), \tag{2.25}$$

where $\mathcal{F}_t = d\mathcal{A}_t + \mathcal{A}_t^2$ and $\mathcal{A}_t = t\mathcal{A}$ is a family of gauge connections continuously interpolating between $\mathcal{A}_0 = 0$ and $\mathcal{A}_1 = \mathcal{A}$. The same interpolation can be used to obtain the expression [1]

$$\omega^1_{2n-2}(u,\mathcal{A}) = n\int_0^1 dt(1-t)\,\mathrm{Tr}\,\Big[ud\Big(\mathcal{A}\mathcal{F}_t^{n-2} + \mathcal{F}_t\mathcal{A}\mathcal{F}_t^{n-3} + \ldots + \mathcal{F}_t^{n-2}\mathcal{A}\Big)\Big]. \tag{2.26}$$

In $D = 4$ spacetime dimensions ($n = 3$), the above formulae yield the five-dimensional Chern-Simons form

$$\omega^0_5(\mathcal{A}) = \mathrm{Tr}\,\left(\mathcal{A}\mathcal{F}^2 - \frac{1}{2}\mathcal{A}^3\mathcal{F} + \frac{1}{10}\mathcal{A}^5\right), \tag{2.27}$$

and the consistent anomaly

$$\mathrm{Tr}\,\Big(uG[\mathcal{A}(x)]_{\rm cons}\Big) = \frac{i}{24\pi^2}\omega^1_4(u,\mathcal{A}) = \frac{i}{24\pi^2}\mathrm{Tr}\,\left[ud\left(\mathcal{A}\mathcal{F} - \frac{1}{2}\mathcal{A}^3\right)\right]. \tag{2.28}$$

2.2 The Bardeen form of the anomaly

In this paper we are particularly interested in a more general theory with chiral fermions ψ_L, ψ_R respectively coupled to two external gauge fields that we denote by $\mathcal{A}_L$ and $\mathcal{A}_R$

$$\mathscr{L}_{\rm YM} = i\overline{\psi}_L\gamma^\mu\Big(\partial_\mu - it_a\mathcal{A}^a_{L\mu}\Big)\psi_L + i\overline{\psi}_R\gamma^\mu\Big(\partial_\mu - it_a\mathcal{A}^a_{R\mu}\Big)\psi_R. \tag{2.29}$$

Taking into account that left-handed fermions contribute with a relative minus sign, the anomaly polynomial is given by

$$\mathcal{P}_n(\mathcal{F}_R, \mathcal{F}_L) = c_n\Big(\operatorname{Tr}\mathcal{F}_R^n - \operatorname{Tr}\mathcal{F}_L^n\Big), \tag{2.30}$$

where the normalization constant c_n is given in eq. (2.17) and $\mathcal{F}_{R,L}$ are defined in terms of $\mathcal{A}_{R,L}$ by the usual relation (2.3).

Since the field strengths for the left and right gauge fields satisfy the Bianchi identity (2.5), the anomaly polynomial is closed. Thus, the Chern-Simons form must be locally exact

$$\operatorname{Tr}\mathcal{F}_R^n - \operatorname{Tr}\mathcal{F}_L^n = d\omega_{2n-1}^0(\mathcal{A}_R, \mathcal{A}_L). \tag{2.31}$$

Using the generalized transgression formula derived in [32] (see appendix A for a quick review of the technique), the Chern-Simons form in the right-hand side of this equation can be written as [cf. eq. (2.25)]

$$\omega_{2n-1}^0(\mathcal{A}_{R,L}, \mathcal{F}_{R,L}) = n\int_0^1 dt\,\operatorname{Tr}\Big(\dot{\mathcal{A}}_t\mathcal{F}_t^{n-1}\Big), \tag{2.32}$$

where $\mathcal{F}_t$ is the field strength associated with a one-parameter family of connections $\mathcal{A}_t$ interpolating between $\mathcal{A}_0 = \mathcal{A}_L$ and $\mathcal{A}_1 = \mathcal{A}_R$. The dot represents the derivative with respect to t.

At this point, we have different choices for the family of connections. One possibility is to take

$$\mathcal{A}_t = (1-2t)\vartheta\left(\frac{1}{2}-t\right)\mathcal{A}_L + (2t-1)\vartheta\left(t-\frac{1}{2}\right)\mathcal{A}_R, \tag{2.33}$$

with $\vartheta(x)$ the Heaviside step function. Computing the integral in (2.32), we arrive at the result

$$\omega_{2n-1}^0(\mathcal{A}_R, \mathcal{A}_L) = \omega_{2n-1}^0(\mathcal{A}_R) - \omega_{2n-1}^0(\mathcal{A}_L), \tag{2.34}$$

from which we construct the left-right symmetric form of the anomaly. Our original Lagrangian (2.29) is invariant under $\mathcal{G}_R\times\mathcal{G}_L$ gauge transformations generated by (u_R, u_L) and acting independently on $(\mathcal{A}_R, \mathcal{A}_L)$. The solution (2.34), however, leads to an effective action $\Gamma[\mathcal{A}_R, \mathcal{A}_L]_{\rm CS}$ which does not remain invariant under vector gauge transformations, defined as those for which $u_R = u_L \equiv u_V$. As it is well known from the general theory of anomalies, we can impose the conservation of the vector current as a renomalization condition. In the language of differential geometry, this ambiguity amounts to adding a exact differential on the right-hand side of (2.34)

$$\widetilde{\omega}_{2n-1}^0(\mathcal{A}_R, \mathcal{A}_L) = \omega_{2n-1}^0(\mathcal{A}_R) - \omega_{2n-1}^0(\mathcal{A}_L) + dS_{2n-2}(\mathcal{A}_R, \mathcal{A}_L), \tag{2.35}$$

which does not modify the identity (2.31). The explicit expression of the Bardeen counterterm S_{2n-2} leading to the preservation of vector gauge transformations can be found in appendix B.

In the present formalism, this choice of S_{2n-2} amounts to selecting a particular interpolating curve in space of gauge connections, which turns out to be

$$\mathcal{A}_t = (1-t)\mathcal{A}_L + t\mathcal{A}_R. \tag{2.36}$$

JHEP11(2018)076

Thus, the corresponding Chern-Simons form is

$$\widetilde{\omega}^0_{2n-1}(\mathcal{A}_R,\mathcal{A}_L) = n\int_0^1 dt\,\mathrm{Tr}\left[(\mathcal{A}_R-\mathcal{A}_L)\mathcal{F}_t^{n-1}\right], \tag{2.37}$$

where in this case $\mathcal{F}_t$ is explicitly given by

$$\mathcal{F}_t = (1-t)\mathcal{F}_L + t\mathcal{F}_R + t(t-1)(\mathcal{A}_R-\mathcal{A}_L)^2. \tag{2.38}$$

Since both $\mathcal{A}_R-\mathcal{A}_L$ and $\mathcal{F}_t$ transform covariantly as adjoint fields under vector gauge transformations, the effective action constructed from (2.37) is automatically invariant under this subgroup. The anomaly can be obtained now by applying a generic gauge transformation to $\widetilde{\omega}^0_{2n-1}(\mathcal{A}_R,\mathcal{A}_L)$

$$\delta_{u_R,u_L}\widetilde{\omega}^0_{2n-1}(\mathcal{A}_R,\mathcal{A}_L) = d\widetilde{\omega}^1_{2n-2}(u_R,u_L,\mathcal{A}_R,\mathcal{A}_L). \tag{2.39}$$

Using again the generalized transgression formula, it is possible to write $\widetilde{\omega}^1_{2n-2}$ in a compact expression

$$\begin{aligned}\widetilde{\omega}^1_{2n-2}(u_{L,R},\mathcal{A}_{L,R},\mathcal{F}_{L,R}) = n\int_0^1 dt\,\mathrm{Tr}\Bigg\{(u_R-u_L)\Bigg[&\mathcal{F}_t^{n-1}\\ &+t(t-1)\sum_{k=0}^{n-2}\Big\{\mathcal{A}_R-\mathcal{A}_L,\mathcal{F}_t^{n-k-2}(\mathcal{A}_R-\mathcal{A}_L)\mathcal{F}_t^k\Big\}\Bigg]\Bigg\},\end{aligned} \tag{2.40}$$

which gives the anomaly in the Bardeen or conserved vector form [35]. The dependence on the gauge functions makes it manifest that the anomaly vanishes for vector gauge transformations $u_R = u_L$.

Later in the paper we will consider the Bardeen form of the anomaly written in terms of vector and axial-vector gauge fields $(\mathcal{V},\mathcal{A})$, defined in terms of $(\mathcal{A}_R,\mathcal{A}_L)$ by

$$\begin{aligned}\mathcal{A}_R &\equiv \mathcal{V}+\mathcal{A},\\ \mathcal{A}_L &\equiv \mathcal{V}-\mathcal{A},\end{aligned} \tag{2.41}$$

while the corresponding field strengths are related by

$$\begin{aligned}\mathcal{F}_R &\equiv d\mathcal{A}_R+\mathcal{A}_R^2 = \mathcal{F}_V+\mathcal{F}_A,\\ \mathcal{F}_L &\equiv d\mathcal{A}_L+\mathcal{A}_L^2 = \mathcal{F}_V-\mathcal{F}_A.\end{aligned} \tag{2.42}$$

Here $\mathcal{F}_V$ and $\mathcal{F}_A$ are respectively given by

$$\begin{aligned}\mathcal{F}_V &= -\frac{i}{2}\mathcal{V}_{\mu\nu}dx^\mu dx^\nu = d\mathcal{V}+\mathcal{V}^2+\mathcal{A}^2,\\ \mathcal{F}_A &= -\frac{i}{2}\mathcal{A}_{\mu\nu}dx^\mu dx^\nu = d\mathcal{A}+\mathcal{A}\mathcal{V}+\mathcal{V}\mathcal{A},\end{aligned} \tag{2.43}$$

where the components of the corresponding field strengths take the form

$$\begin{aligned}\mathcal{V}_{\mu\nu} &= \partial_\mu\mathcal{V}_\nu-\partial_\nu\mathcal{V}_\mu-i[\mathcal{V}_\mu,\mathcal{V}_\nu]-i[\mathcal{A}_\mu,\mathcal{A}_\nu],\\ \mathcal{A}_{\mu\nu} &= \partial_\mu\mathcal{A}_\nu-\partial_\nu\mathcal{A}_\mu-i[\mathcal{V}_\mu,\mathcal{A}_\nu]-i[\mathcal{A}_\mu,\mathcal{V}_\nu].\end{aligned} \tag{2.44}$$

JHEP11(2018)076

In order to write the original Lagrangian (2.29) in terms of the vector and axial-vector gauge fields, we combine left- and right-handed Weyl spinors ψ_L, ψ_R into a single Dirac field ψ as

$$\begin{aligned}\psi_R &= \frac{1}{2}(1+\gamma_5)\psi, \\ \psi_L &= \frac{1}{2}(1-\gamma_5)\psi,\end{aligned} \tag{2.45}$$

so the Lagrangian becomes

$$\mathscr{L}_{\mathrm{YM}} = i\overline{\psi}\gamma^\mu\Big(\partial_\mu - it_a\mathcal{V}_\mu^a - i\gamma_5 t_a\mathcal{A}_\mu^a\Big)\psi. \tag{2.46}$$

In terms of these fields, the Chern-Simons form in (2.37) is recast as

$$\widetilde{\omega}_{2n-1}^0(\mathcal{V},\mathcal{A}) = 2n\int_0^1 dt\,\mathrm{Tr}\left(\mathcal{A}\mathcal{F}_t^{n-1}\right), \tag{2.47}$$

with

$$\mathcal{F}_t = \mathcal{F}_V + (2t-1)\mathcal{F}_A + 4t(t-1)\mathcal{A}^2. \tag{2.48}$$

Similarly, eq. (2.40) giving the Bardeen anomaly can be rewritten in terms of axial and vector gauge fields in the following closed form

$$\widetilde{\omega}_{2n-2}^1(u_A,\mathcal{V},\mathcal{A}) = 2n\int_0^1 dt\,\mathrm{Tr}\left\{u_A\left[\mathcal{F}_t^{n-1} + 4t(t-1)\sum_{k=0}^{n-2}\left\{\mathcal{A},\mathcal{F}_t^{n-k-2}\mathcal{A}\mathcal{F}_t^k\right\}\right]\right\}, \tag{2.49}$$

where we have defined

$$\begin{aligned}u_R &= u_V + u_A, \\ u_L &= u_V - u_A.\end{aligned} \tag{2.50}$$

These functions generate infinitesimal vector and axial-vector gauge transformations given by

$$\begin{aligned}\delta_{V,A}\mathcal{V} &= du_V + [\mathcal{V},u_V] + [\mathcal{A},u_A], \\ \delta_{V,A}\mathcal{A} &= du_A + [\mathcal{A},u_V] + [\mathcal{V},u_A].\end{aligned} \tag{2.51}$$

In particular, $\mathcal{A}$, $\mathcal{F}_V$, and $\mathcal{F}_A$ transform as adjoint fields under vector gauge transformations ($u_A = 0$). Thus, the Chern-Simons form (2.47) remains invariant under them, while the anomaly (2.49) only depends on u_A.

For later applications, we particularize eqs. (2.47) and (2.49) to the four-dimensional case ($n = 3$). In terms of the fields $(\mathcal{V},\mathcal{A})$, the relevant anomaly polynomial is given by

$$\mathcal{P}_3(\mathcal{F}_V,\mathcal{F}_A) = -\frac{i}{12\pi^2}\mathrm{Tr}\left(\mathcal{F}_A^3 + 3\mathcal{F}_A\mathcal{F}_V^2\right). \tag{2.52}$$

After a few manipulations, the Chern-Simons form preserving the vector Ward identity is obtained as

$$\widetilde{\omega}_5^0(\mathcal{A},\mathcal{F}_V,\mathcal{F}_A) = 6\mathrm{Tr}\left(\mathcal{A}\mathcal{F}_V^2 + \frac{1}{3}\mathcal{A}\mathcal{F}_A^2 - \frac{4}{3}\mathcal{A}^3\mathcal{F}_V + \frac{8}{15}\mathcal{A}^5\right), \tag{2.53}$$

JHEP11(2018)076

while the celebrated Bardeen anomaly [35] is obtained from

$$\widetilde{\omega}_4^1(u_A,\mathcal{A},\mathcal{F}_V,\mathcal{F}_A)=6\mathrm{Tr}\left\{u_A\left[\mathcal{F}_V^2+\frac{1}{3}\mathcal{F}_A^2 -\frac{4}{3}\Big(\mathcal{A}^2\mathcal{F}_V+\mathcal{A}\mathcal{F}_V\mathcal{A}+\mathcal{F}_V\mathcal{A}^2\Big)+\frac{8}{3}\mathcal{A}^4\right]\right\}. \tag{2.54}$$

As already pointed out, the absence of u_V in this expression makes explicit the conservation of the consistent vector current $J_V^\mu(x)_{\mathrm{cons}}$.

In the case of a two flavor hadronic fluid, the chiral group is $\mathrm{U}(2)_L\times\mathrm{U}(2)_R$ and traces can be computed using the identities given in appendix C. For the Chern-Simons form (2.53), we have

$$\begin{aligned}\widetilde{\omega}_5^0(\mathcal{A},\mathcal{F}_V,\mathcal{F}_A)=&\frac{3}{2}(\mathrm{Tr}\,\mathcal{F}_V)^2(\mathrm{Tr}\,\mathcal{A})+3(\mathrm{Tr}\,\widehat{\mathcal{F}}_V^2)(\mathrm{Tr}\,\mathcal{A})+6\mathrm{Tr}\,(\widehat{\mathcal{A}}\widehat{\mathcal{F}}_V)(\mathrm{Tr}\,\mathcal{F}_V)\\&+\frac{1}{2}(\mathrm{Tr}\,\mathcal{F}_A)^2(\mathrm{Tr}\,\mathcal{A})+(\mathrm{Tr}\,\widehat{\mathcal{F}}_A^2)(\mathrm{Tr}\,\mathcal{A})+2\mathrm{Tr}\,(\widehat{\mathcal{A}}\widehat{\mathcal{F}}_A)(\mathrm{Tr}\,\mathcal{F}_A)\\&-4\mathrm{Tr}\,(\widehat{\mathcal{A}}^2\widehat{\mathcal{F}}_V)(\mathrm{Tr}\,\mathcal{A})-4(\mathrm{Tr}\,\widehat{\mathcal{A}}^3)(\mathrm{Tr}\,\mathcal{F}_V),\end{aligned} \tag{2.55}$$

where the hat indicates the projection onto the SU(2) factors [see eq. (C.3)]. We notice here the absence of terms containing one single trace over SU(2) indices. The reason is that SU(2) is a safe group, so anomalies can only appear from one-loop diagrams containing at least one vertex coupling to a U(1) factor. We can assume that the axial-vector external field $\mathcal{A}$, as well as the associated field strength $\mathcal{F}_A$, lies on the $\mathrm{SU}(2)_A$ factor. This means that $\mathrm{Tr}\,\mathcal{A}=\mathrm{Tr}\,\mathcal{F}_A=0$ and the Chern-Simons form simplifies to

$$\widetilde{\omega}_5^0(\mathcal{A},\mathcal{F}_V,\mathcal{F}_A)=2\mathrm{Tr}\left(3\widehat{\mathcal{A}}\widehat{\mathcal{F}}_V-2\widehat{\mathcal{A}}^3\right)(\mathrm{Tr}\,\mathcal{F}_V). \tag{2.56}$$

This expression makes it explicit that two terms come respectively from the $\mathrm{SU}(2)^2\mathrm{U}(1)$ triangle and the $\mathrm{SU}(2)^3\mathrm{U}(1)$ square diagrams. Playing the same game with the traces in eq. (2.54), we retrieve the expression leading to the anomaly found in ref. [36] for two flavor QCD

$$\widetilde{\omega}_4^1(u_A,\mathcal{A},\mathcal{F}_V)=6\mathrm{Tr}\left[\widehat{u}_A\Big(\widehat{\mathcal{F}}_V-2\widehat{\mathcal{A}}^2\Big)\right](\mathrm{Tr}\,\mathcal{F}_V), \tag{2.57}$$

where, by consistency, the axial-vector gauge function u_A is taken not to have components on $\mathrm{U}(1)_A$.

3 Chern-Simons effective actions, currents, and anomaly inflow

In section 2.1 we have seen how the anomalous effective action $\Gamma[\mathcal{A}]_{\mathrm{CS}}$ can be constructed from the Chern-Simons form ω_{2n-1}^0 [see eq. (2.22)]. In the next section, we show how this effective action can be used to generate an anomalous partition function capturing the physical consequences of the anomaly under stationary conditions. But before that, in order to understand how this is possible, we need to study the currents induced by $\Gamma[\mathcal{A}]_{\mathrm{CS}}$. These are obtained by applying a general variation of the gauge field that will be denoted $\delta\mathcal{A}=B$, with B an infinitesimal Lie-algebra valued one-form.

JHEP11(2018)076

The result of this variation on the Chern-Simons form, $\delta_B\omega^0_{2n-1}(\mathcal{A})$, can be efficiently computed with the help of the generalized transgression formula (see [32] and appendix A). More specifically

$$\omega^0_{2n-1}(\mathcal{A}+B)-\omega^0_{2n-1}(\mathcal{A})=\int_0^1 \ell_t\, d\omega^0_{2n-1}(\mathcal{A}_t)+d\int_0^1 \ell_t\,\omega^0_{2n-1}(\mathcal{A}_t) \tag{3.1}$$

where $\mathcal{A}_t=\mathcal{A}+tB$ is a family of connections interpolating between $\mathcal{A}$ and $\mathcal{A}+B$, while the action of the operator ℓ_t is given by

$$\begin{aligned}&\ell_t\mathcal{A}_t=0,\\&\ell_t\mathcal{F}_t=d_t\mathcal{A}_t=dtB.\end{aligned} \tag{3.2}$$

Notice that, in order to compute the currents we only need to evaluate (3.1) to linear order in B, using that for any function of $\mathcal{A}$ and $\mathcal{F}$

$$\int_0^1 \ell_t f(\mathcal{A}_t,\mathcal{F}_t)=\ell f(\mathcal{A},\mathcal{F})+\mathcal{O}(B^2), \tag{3.3}$$

where the operator ℓ, introduced in [37], is defined by $\ell\mathcal{F}=B$, $\ell\mathcal{A}=0$.

Thus, to linear order in B, the first term in the right-hand side of (3.1) evaluates to

$$\begin{aligned}\int_0^1 \ell_t d\omega^0_{2n-1}(\mathcal{A}_t)&=\int_0^1 \ell_t \mathrm{Tr}\,\mathcal{F}_t^n=\ell\,\mathrm{Tr}\,\mathcal{F}^n+\mathcal{O}(B^2)\\&=n\mathrm{Tr}\,(B\mathcal{F}^{n-1})+\mathcal{O}(B^2),\end{aligned} \tag{3.4}$$

while the second one gives

$$\int_0^1 \ell_t\,\omega^0_{2n-1}(\mathcal{A}_t)=\ell\,\omega^0_{2n-1}(\mathcal{A})+\mathcal{O}(B^2). \tag{3.5}$$

Integrating eq. (3.1) over $\mathcal{M}_{2n-1}$ and using the Stokes theorem finally yields the following expression for the variation of the effective action functional

$$\begin{aligned}\delta_B\Gamma[\mathcal{A}]_{\mathrm{CS}}&=\int_{\mathcal{M}_{2n-1}}\ell\,\mathcal{P}_n(\mathcal{F})+c_n\int_{\mathcal{M}_{2n-2}}\ell\omega^0_{2n-1}(\mathcal{A})\\&\equiv\int_{\mathcal{M}_{2n-1}}\mathrm{Tr}\,(B\mathcal{J}_{\mathrm{bulk}})-\int_{\mathcal{M}_{2n-2}}\mathrm{Tr}\,(B\mathcal{J}_{\mathrm{BZ}}),\end{aligned} \tag{3.6}$$

The bulk current term on the second line can be read from (3.4)

$$\mathrm{Tr}\,(B\mathcal{J}_{\mathrm{bulk}})=\ell\,\mathcal{P}_n(\mathcal{F})\quad\Longrightarrow\quad\mathcal{J}_{\mathrm{bulk}}=nc_n\mathcal{F}^{n-1}. \tag{3.7}$$

On the other hand, the action of the operator ℓ on the Chern-Simons form gives the covariant (Bardeen-Zumino) current [37]

$$c_n\ell\,\omega^0_{2n-1}(\mathcal{A})=-\mathrm{Tr}\,(B\mathcal{J}_{\mathrm{BZ}}). \tag{3.8}$$

In fact, $\mathcal{J}_{\rm BZ}$ is a $(2n-3)$-form taking values on the Lie algebra $\mathfrak{g}$, dual to the Bardeen-Zumino one-form current which added to the consistent current gives the covariant one [37]. Henceforth, all currents will be represented by the corresponding dual forms (see the appendix B for details).

The restriction of $\mathcal{J}_{\rm bulk}$ to the boundary $\mathcal{M}_{2n-1}$ turns out to coincide with (minus) the covariant anomaly, in agreement with the anomaly inflow mechanism [18, 22, 38]. To see this, we go back to eq. (2.11) and evaluate (3.6) for a gauge variation of the gauge field, i.e., we set

$$B \equiv \delta_u \mathcal{A} = du + [\mathcal{A}, u] \equiv Du\,. \tag{3.9}$$

Integrating then by parts and using the cyclic property of the trace, together with the Stokes theorem applied on the first term in the right-hand side of (3.6), we have

$$\int\limits_{\mathcal{M}_{2n-1}} \mathrm{Tr}\left[(Du)\mathcal{J}_{\rm bulk}\right] = -\int\limits_{\mathcal{M}_{2n-1}} \mathrm{Tr}\left[u(D\mathcal{J}_{\rm bulk})\right] + \int\limits_{\mathcal{M}_{2n-2}} \mathrm{Tr}\left(u\mathcal{J}_{\rm bulk}\right), \tag{3.10}$$

while the second term can be written as

$$-\int\limits_{\mathcal{M}_{2n-2}} \mathrm{Tr}\left[(Du)\mathcal{J}_{\rm BZ}\right] = \int\limits_{\mathcal{M}_{2n-2}} \mathrm{Tr}\left[u(D\mathcal{J}_{\rm BZ})\right]. \tag{3.11}$$

Finally, noting that the Bianchi identity $D\mathcal{F} = 0$ implies $D\mathcal{J}_{\rm bulk} = 0$, we find that the gauge variation of the Chern-Simons action can be written

$$\begin{aligned} \delta_u \Gamma[\mathcal{A}]_{\rm CS} &= \int\limits_{\mathcal{M}_{2n-2}} \mathrm{Tr}\left[u(\mathcal{J}_{\rm bulk} + D\mathcal{J}_{\rm BZ})\right] \\ &= -\int\limits_{\mathcal{M}_{2n-2}} \mathrm{Tr}\left(u\, G[\mathcal{A}]_{\rm cons}\right), \end{aligned} \tag{3.12}$$

where we have used (2.12) to get the last equality. This implies

$$\mathcal{J}_{\rm bulk}\Big|_{\mathcal{M}_{2n-2}} = -\Big(G[\mathcal{A}]_{\rm cons} + D\mathcal{J}_{\rm BZ}\Big) = -G[\mathcal{A}]_{\rm cov}, \tag{3.13}$$

where $G[\mathcal{A}]_{\rm cov}$ is the covariant anomaly. This completes our identification of the currents induced by the Chern-Simons effective action.

Using eq. (3.8) for $n = 3$, we retrieve the Bardeen-Zumino current for a right-handed fermion in four dimensions

$$\mathcal{J}^R_{\rm BZ} = \frac{i}{24\pi^2}\left(\mathcal{F}_R\mathcal{A}_R + \mathcal{A}_R\mathcal{F}_R - \frac{1}{2}\mathcal{A}_R^3\right), \tag{3.14}$$

while the left-handed current differs just by a relative minus sign. It will be useful to write these expressions also in terms of vector and axial gauge fields. Thus, instead of (3.8) we use

$$c_n \ell\, \widetilde{\omega}^0_{2n-1}(\mathcal{V}, \mathcal{A}) = -\mathrm{Tr}\left(B_V \mathcal{J}^V_{\rm BZ} + B_A \mathcal{J}^A_{\rm BZ}\right), \tag{3.15}$$

where B_V and B_A are the shifts of the vector and axial-vector gauge fields respectively. The operator ℓ annihilate the gauge fields, $\ell\mathcal{V} = \ell\mathcal{A} = 0$, whereas its action on the field strengths is defined by $\ell\mathcal{F}_V = B_V$ and $\ell\mathcal{F}_A = B_A$. Thus, the Bardeen-Zumino currents in four dimensions are given by

$$\mathcal{J}^V_{\rm BZ} = \frac{i}{12\pi^2}\Big[3(\mathcal{F}_V\mathcal{A} + \mathcal{A}\mathcal{F}_V) - 4\mathcal{A}^3\Big], \tag{3.16}$$

$$\mathcal{J}^A_{\rm BZ} = \frac{i}{12\pi^2}\Big(\mathcal{F}_A\mathcal{A} + \mathcal{A}\mathcal{F}_A\Big). \tag{3.17}$$

For the bulk currents, we generalize (3.7) to

$$\ell\,\mathcal{P}_n(\mathcal{V},\mathcal{A}) = \mathrm{Tr}\,\Big(B_V\mathcal{J}^V_{\rm bulk} + B_A\mathcal{J}^A_{\rm bulk}\Big), \tag{3.18}$$

which for $n = 3$ gives the Bardeen form of the bulk currents

$$\begin{aligned}\mathcal{J}^V_{\rm bulk} &= -\frac{i}{24\pi^2}\Big(\mathcal{F}_V^2 + \mathcal{F}_A^2\Big),\\ \mathcal{J}^A_{\rm bulk} &= -\frac{i}{24\pi^2}\Big(\mathcal{F}_V\mathcal{F}_A + \mathcal{F}_A\mathcal{F}_V\Big).\end{aligned} \tag{3.19}$$

At this point it is important to stress that the difference between the left-right symmetric and Bardeen forms of the consistent anomaly stems from the use of different Chern-Simons forms, i.e., (2.34) and (2.37) respectively. On the other hand, the invariant polynomial is unique and so are the bulk currents that, restricted to physical spacetime, give the covariant anomalies. In other words, the covariant anomalies in both the left-right symmetric and Bardeen forms are exactly the same. The consequence is that there is no way of constructing a conserved covariant vector current.

4 The anomalous partition function from dimensional reduction

We turn now to the computation of the partition function of anomalous hydrodynamics using the differential geometry formalism for quantum anomalies reviewed in section 2. Given the anomalous effective functional, the partition function $\log Z$ can be obtained by using the imaginary time prescription: we take all fields to be time-independent and compactify the Euclidean time direction to a circle of length β. This amounts to replacing the integration over time in $\log Z \equiv W[\mathcal{A}] = i\Gamma[\mathcal{A}]$ according to

$$i\int dx^0 \longrightarrow \beta_0 \equiv \frac{1}{T_0}, \tag{4.1}$$

with T_0 the equilibrium temperature.

On the other hand, we know that the gauge variation of the Chern-Simons action $\Gamma[\mathcal{A}]_{\rm CS}$ defined in eq. (2.22) reproduces the consistent anomaly. Thus we may expect the Chern-Simons action evaluated on a time-independent background to be closely related to the anomalous partition function for that background. One potential difficulty is that $\Gamma[\mathcal{A}]_{\rm CS}$ lives in a five-dimensional manifold $\mathcal{M}_5$ and we saw in the previous section that

JHEP11(2018)076

the boundary current induced on four-dimensional spacetime $\mathcal{M}_4 = \partial\mathcal{M}_5$ is the Bardeen-Zumino current, *not* the consistent current one would expect from an effective functional [cf. eq. (2.11)]. Nevertheless, we will see that the dimensional reduction of the Chern-Simons action on a time-independent background yields a satisfactory anomalous partition function.

As explained in the Introduction, we want to consider a gauge theory on a stationary background. Choosing an appropriate gauge, its line element can be written as [15]

$$ds^2 = -e^{2\sigma(\mathbf{x})}\Big[dt + a_i(\mathbf{x})dx^i\Big]^2 + g_{ij}(\mathbf{x})dx^i dx^j, \tag{4.2}$$

where all ten metric functions $\{\sigma(\mathbf{x}), a_i(\mathbf{x}), g_{ij}(\mathbf{x})\}$ are independent of the time coordinate t. This metric remains invariant under time-independent shifts of the time coordinate, combined with the appropriate transformation of the metric functions $a_i(\mathbf{x})$. Using the notation of refs. [15, 25]

$$\begin{aligned} t &\longrightarrow t' = t + \phi(\mathbf{x}), \\ \mathbf{x} &\longrightarrow \mathbf{x}' = \mathbf{x}, \\ a_i(\mathbf{x}) &\longrightarrow a_i'(\mathbf{x}) = a_i(\mathbf{x}) - \partial_i\phi(\mathbf{x}). \end{aligned} \tag{4.3}$$

For obvious reasons, this isometry is referred to as Kaluza-Klein (KK) gauge transformations. In a theory with static gauge fields, their components

$$\mathcal{A}_\mu = \big(\mathcal{A}_0(\mathbf{x}), \mathcal{A}_i(\mathbf{x})\big), \tag{4.4}$$

transform under (4.3) according to

$$\begin{aligned} \mathcal{A}_0(\mathbf{x}) &\longrightarrow \mathcal{A}_0(\mathbf{x}), \\ \mathcal{A}_i(\mathbf{x}) &\longrightarrow \mathcal{A}_i'(\mathbf{x}) = \mathcal{A}_i(\mathbf{x}) - \mathcal{A}_0(\mathbf{x})\partial_i\phi(\mathbf{x}). \end{aligned} \tag{4.5}$$

This implies the invariance of the following combinations

$$A_i(\mathbf{x}) \equiv \mathcal{A}_i(\mathbf{x}) - \mathcal{A}_0(\mathbf{x})a_i(\mathbf{x}), \tag{4.6}$$

so $A_\mu = (\mathcal{A}_0, A_i)$ denote the KK-invariant gauge fields, which on the other hand behave as standard gauge fields under time-independent gauge transformations. The use of the gauge (4.2), together with the systematic implementation of KK invariance, leads to explicit results that can be easily applied to hydrodynamics, as shown in ref. [15].

In the language of differential forms, the original gauge field can be decomposed into KK-invariant quantities by writing

$$\begin{aligned} \mathcal{A}(\mathbf{x}) &= \mathcal{A}_0(\mathbf{x})\Big[dx^0 + a_i(\mathbf{x})dx^i\Big] + \Big[\mathcal{A}_i(\mathbf{x}) - \mathcal{A}_0(\mathbf{x})a_i(\mathbf{x})\Big]dx^i \\ &= \mathcal{A}_0(\mathbf{x})\theta(\mathbf{x}) + A_i(\mathbf{x})dx^i \equiv \mathcal{A}_0(\mathbf{x})\theta(\mathbf{x}) + \boldsymbol{A}(\mathbf{x}), \end{aligned} \tag{4.7}$$

where we have defined the one-forms

$$\begin{aligned} a(\mathbf{x}) &= a_i(\mathbf{x})dx^i, \\ \theta(\mathbf{x}) &= dx^0 + a(\mathbf{x}). \end{aligned} \tag{4.8}$$

JHEP11(2018)076

Note that the one-forms $\mathcal{A}_0$, θ, and $\boldsymbol{A}$ all remain invariant under KK transformations. The corresponding decomposition of the field strength in terms of these quantities reads

$$\mathcal{F} = \boldsymbol{D}(\mathcal{A}_0\theta) + \boldsymbol{F}, \tag{4.9}$$

where $\boldsymbol{D}$ is the covariant derivative associated with $\boldsymbol{A}$, acting on p-forms according to

$$\boldsymbol{D}\omega_p \equiv d\omega_p + \boldsymbol{A}\omega_p - (-1)^p\omega_p\boldsymbol{A}, \tag{4.10}$$

and we have introduced the field strength associated to $\boldsymbol{A}$

$$\boldsymbol{F} = d\boldsymbol{A} + \boldsymbol{A}^2. \tag{4.11}$$

Incidentally, these metric and field theoretical functions are related to hydrodynamic quantities such as the local temperature $T(\mathbf{x})$, the chemical potential $\mu(\mathbf{x})$, and the fluid velocity $u^\mu(\mathbf{x})$ by [15, 18]

$$\begin{aligned} T(\mathbf{x}) &= T_0\, e^{-\sigma(\mathbf{x})}, \\ \mu(\mathbf{x}) &= e^{-\sigma(\mathbf{x})}\mathcal{A}_0(\mathbf{x}), \\ u(\mathbf{x}) &\equiv u_\mu(\mathbf{x})dx^\mu = -e^{\sigma(\mathbf{x})}\theta(\mathbf{x}). \end{aligned} \tag{4.12}$$

In terms of them, the decomposition (4.7) reads

$$\mathcal{A} = \boldsymbol{A} - \mu u. \tag{4.13}$$

Thus, our field $\boldsymbol{A}$ corresponds to the hatted connection of ref. [18]. Moreover, from the identity $du(\mathbf{x}) = -u(\mathbf{x})\mathfrak{a}(\mathbf{x}) + 2\omega(\mathbf{x})$, we identify the acceleration $\mathfrak{a}(\mathbf{x})$ and vorticity $\omega(\mathbf{x})$ to be

$$\begin{aligned} \mathfrak{a}(\mathbf{x}) &= d\sigma(\mathbf{x}), \\ \omega(\mathbf{x}) &= -\frac{1}{2}e^{\sigma(\mathbf{x})}da(\mathbf{x}). \end{aligned} \tag{4.14}$$

After all these prolegomena, we are in position to compute the anomaly-induced equilibrium partition function implementing dimensional reduction. Taking into account the decomposition (4.7), we can write the identity

$$\int_{\mathcal{M}_{2n-1}} \omega^0_{2n-1}(\mathcal{A}) = \int_{\mathcal{M}_{2n-1}} \omega^0_{2n-1}(\mathcal{A}_0\theta + \boldsymbol{A}) - \int_{\mathcal{M}_{2n-1}} \omega^0_{2n-1}(\boldsymbol{A}), \tag{4.15}$$

which trivially holds since $\boldsymbol{A}$ is independent of dx^0 and the last integral vanishes upon dimensional reduction. The advantage of writing the effective action in this fashion is that the combination of terms on the right-hand side of this equation can be evaluated with the help of the same generalized homotopy formula already used in (3.1), namely

$$\omega^0_{2n-1}(\mathcal{A}_0\theta + \boldsymbol{A}) - \omega^0_{2n-1}(\boldsymbol{A}) = \int_0^1 \ell_t\, d\omega^0_{2n-1}(\mathcal{A}_t) + d\int_0^1 \ell_t\, \omega^0_{2n-1}(\mathcal{A}_t)\,, \tag{4.16}$$

where now we consider the family of connections $\mathcal{A}_t = \boldsymbol{A} + t\mathcal{A}_0\theta$. The details of the computation are given in appendix A, where it is shown that the first term evaluates to

$$\int_0^1 \ell_t\, d\omega^0_{2n-1}(\mathcal{A}_t) = n\int_0^1 dt\, dx^0\, \mathrm{Tr}\left[\mathcal{A}_0(\boldsymbol{F}+t\mathcal{A}_0 da)^{n-1}\right], \tag{4.17}$$

while the second one yields

$$\int_0^1 \ell_t\, \omega^0_{2n-1}(\mathcal{A}_t) = \int_0^1 dt\, dx^0\, \mathcal{A}_0 \frac{\delta}{\delta\mathcal{F}}\omega^0_{2n-1}(\mathcal{A},\mathcal{F})\bigg|_{\substack{\mathcal{A}\to\boldsymbol{A}\\ \mathcal{F}\to\boldsymbol{F}+t\mathcal{A}_0 da}}. \tag{4.18}$$

Substituting these results into eq. (2.22) we arrive at a very interesting decomposition of the Chern-Simons effective action $\Gamma[\mathcal{A}_0,\boldsymbol{A}]$. Indeed, after implementing the prescription (4.1), we find that the partition function naturally splits into two terms [18]

$$i\Gamma[\mathcal{A}_0,\boldsymbol{A}]_{\mathrm{CS}} = W[\mathcal{A}_0,\boldsymbol{A}]_{\mathrm{inv}} + W[\mathcal{A}_0,\boldsymbol{A}]_{\mathrm{anom}}. \tag{4.19}$$

We have found that the two terms are explicitly given by

$$W[\mathcal{A}_0,\boldsymbol{A}]_{\mathrm{inv}} = \frac{nc_n}{T_0}\int\limits_{D_{2n-2}}\int_0^1 dt\, \mathrm{Tr}\left[\mathcal{A}_0(\boldsymbol{F}+t\mathcal{A}_0 da)^{n-1}\right] \tag{4.20}$$

and

$$W[\mathcal{A}_0,\boldsymbol{A}]_{\mathrm{anom}} = \frac{c_n}{T_0}\int\limits_{S^{2n-3}}\int_0^1 dt\mathcal{A}_0 \frac{\delta}{\delta\mathcal{F}}\omega^0_{2n-1}(\mathcal{A},\mathcal{F})\bigg|_{\substack{\mathcal{A}\to\boldsymbol{A}\\ \mathcal{F}\to\boldsymbol{F}+t\mathcal{A}_0 da}}. \tag{4.21}$$

Note that the imaginary time formalism requires to assume that $\mathcal{M}_{2n-1} = D_{2n-2}\times S^1$ and $\mathcal{M}_{2n-2} = S^{2n-3}\times S^1$, where $\mathcal{M}_{2n-2} = \partial D_{2n-1}$ and S^1 denotes the thermal cycle of length $\beta = T_0^{-1}$. Inspection of eqs. (4.20) and (4.21) shows that whereas W_{inv} is of order $n-1$ in the derivative expansion, the order of the local anomalous piece W_{anom} is $n-2$.

A first important thing to be noticed is that $W[\mathcal{A}_0,\boldsymbol{A}]_{\mathrm{inv}}$ is given as an integral over a $(2n-1)$-dimensional Euclidean manifold, and is therefore nonlocal from the viewpoint of $(2n-2)$-dimensional spacetime. However, $W[\mathcal{A}_0,\boldsymbol{A}]_{\mathrm{inv}}$ is manifestly invariant under time-independent gauge transformations and therefore does not contribute to the gauge anomaly. As a consequence, it is the second piece $W[\mathcal{A}_0,\boldsymbol{A}]_{\mathrm{anom}}$ which completely accounts for the gauge noninvariance of the partition function $i\Gamma[\mathcal{A}_0,\boldsymbol{A}]_{\mathrm{CS}}$ and its gauge variation reproduces the consistent anomaly. As $W[\mathcal{A}_0,\boldsymbol{A}]_{\mathrm{anom}}$ is given as a local integral over the spatial manifold S^{2n-3}, the rationale behind eqs. (2.11)–(2.13) shows that it must induce the consistent current for the time-independent background. This makes $W[\mathcal{A}_0,\boldsymbol{A}]_{\mathrm{anom}}$ wholly satisfactory as an anomalous partition function. Notice as well that, although the integral of the Chern-Simons form ω^0_{2n-1} is a topological invariant and therefore does not depend on the background metric, after dimensional reduction both the invariant and anomalous part of the effective action pick up a dependence on the metric function a_i. From eqs. (4.20) and (4.21), we see that this dependence always comes through the KK-invariant combination da.

In four-dimensional spacetime ($n = 3$), the anomalous, local piece of the partition function $W[\mathcal{A}_0, \boldsymbol{A}]_{\text{anom}}$ is explicitly given by

$$W[\mathcal{A}_0, \boldsymbol{A}]_{\text{anom}} = -\frac{i}{24\pi^2 T_0} \int\limits_{S^3} \text{Tr}\left[\mathcal{A}_0 \left(\boldsymbol{F}\boldsymbol{A} + \boldsymbol{A}\boldsymbol{F} - \frac{1}{2}\boldsymbol{A}^3\right) + \mathcal{A}_0^2\, \boldsymbol{A} da\right]. \tag{4.22}$$

It is very important to keep in mind that the fact that $W[\mathcal{A}_0, \boldsymbol{A}]_{\text{inv}}$ is not contributing to the anomaly does not mean that it can be discarded as a useless byproduct of dimensional reduction. As we will see in the next section, the most efficient way to compute the covariant currents and the energy-momentum tensor involves $W[\mathcal{A}_0, \boldsymbol{A}]_{\text{inv}}$. In a four-dimensional spacetime this term reads

$$W[\mathcal{A}_0, \boldsymbol{A}]_{\text{inv}} = -\frac{i}{8\pi^2 T_0} \int\limits_{D_4} \text{Tr}\left[\mathcal{A}_0 \boldsymbol{F}^2 + da\, \mathcal{A}_0^2\, \boldsymbol{F} + \frac{1}{3}(da)^2 \mathcal{A}_0^3\right]. \tag{4.23}$$

It should be stressed again that D_4 is a four-dimensional Euclidean hypersurface containing an extra spatial coordinate, so it is nonlocal from the viewpoint of four-dimensional spacetime.

With applications to hadronic fluids in mind, it is convenient to write the Bardeen form of the anomalous part of the partition function in terms of vector and axial gauge fields $\mathcal{V}_\mu$ and $\mathcal{A}_\mu$, whose components are denoted respectively by

$$\begin{aligned} \mathcal{V}_\mu &= \big(\mathcal{V}_0(\mathbf{x}), \mathcal{V}_i(\mathbf{x})\big), \\ \mathcal{A}_\mu &= \big(\mathcal{A}_0(\mathbf{x}), \mathcal{A}_i(\mathbf{x})\big). \end{aligned} \tag{4.24}$$

Again, we can define the new gauge fields

$$\begin{aligned} V_\mu(\mathbf{x}) &\equiv \big(V_0(\mathbf{x}), V_i(\mathbf{x})\big) = \big(\mathcal{V}_0(\mathbf{x}), \mathcal{V}_i(\mathbf{x}) - V_0(\mathbf{x}) a_i(\mathbf{x})\big), \\ A_\mu(\mathbf{x}) &\equiv \big(A_0(\mathbf{x}), A_i(\mathbf{x})\big) = \big(\mathcal{A}_0(\mathbf{x}), \mathcal{A}_i(\mathbf{x}) - A_0(\mathbf{x}) a_i(\mathbf{x})\big). \end{aligned} \tag{4.25}$$

which remain invariant under KK transformations (4.3). Repeating the analysis leading to eq. (4.21), we find in this case

$$\begin{aligned} W[\mathcal{V}_0, \mathcal{A}_0, \boldsymbol{V}, \boldsymbol{A}]_{\text{anom}} = \frac{c_n}{T_0} \int_{S^{2n-3}} \int_0^1 dt \Bigg\{ & \left[\left(\mathcal{V}_0 \frac{\delta}{\delta \mathcal{F}_V}\right) \widetilde{\omega}^0_{2n-1}(\mathcal{V}, \mathcal{A})\right]\Bigg|_{\blacksquare \to \blacksquare_t} \\ & + \left[\left(\mathcal{A}_0 \frac{\delta}{\delta \mathcal{F}_A}\right) \widetilde{\omega}^0_{2n-1}(\mathcal{V}, \mathcal{A})\right]\Bigg|_{\blacksquare \to \blacksquare_t} \Bigg\}, \end{aligned} \tag{4.26}$$

where we have used the compact notation

$$\blacksquare \to \blacksquare_t : \begin{cases} \mathcal{V} \longrightarrow \boldsymbol{V}, \\ \mathcal{A} \longrightarrow \boldsymbol{A} \\ \mathcal{F}_V \longrightarrow \boldsymbol{F}_V + t V_0 da, \\ \mathcal{F}_A \longrightarrow \boldsymbol{F}_A + t A_0 da \end{cases}. \tag{4.27}$$

Evaluating this expression for $n = 3$, we find the anomalous partition function in four dimensions to be

$$W[\mathcal{V}_0, \mathcal{A}_0, \boldsymbol{V}, \boldsymbol{A}]_{\rm anom} = -\frac{i}{4\pi^2 T_0}\int_{S^3} \mathrm{Tr}\left[\mathcal{V}_0\left(\boldsymbol{F}_V\boldsymbol{A} + \boldsymbol{A}\boldsymbol{F}_V - \frac{4}{3}\boldsymbol{A}^3\right)\right.$$
$$\left. + \frac{1}{3}\mathcal{A}_0\Big(\boldsymbol{F}_A\boldsymbol{A} + \boldsymbol{A}\boldsymbol{F}_A\Big) + \frac{1}{3}da\Big(3\mathcal{V}_0^2 + \mathcal{A}_0^2\Big)\boldsymbol{A}\right]. \quad (4.28)$$

On the other hand, $W[\mathcal{V}_0, \mathcal{A}_0, \boldsymbol{V}, \boldsymbol{A}]_{\rm inv}$ is directly constructed from the corresponding invariant polynomial and is therefore the same for both the left-right symmetric and Bardeen form of the anomaly. This means that $W[\mathcal{V}_0, \mathcal{A}_0, \boldsymbol{V}, \boldsymbol{A}]_{\rm inv}$ can be simply obtained by performing a mere change of variables on $W[\mathcal{A}_{R0}, \boldsymbol{A}_R]_{\rm inv} - W[\mathcal{A}_{L0}, \boldsymbol{A}_L]_{\rm inv}$. The resulting partition function in four dimensions ($n = 3$) is

$$W[\mathcal{V}_0, \mathcal{A}_0, \boldsymbol{V}, \boldsymbol{A}]_{\rm inv} = -\frac{i}{4\pi^2 T_0}\int_{D_4} \mathrm{Tr}\left\{\mathcal{A}_0\Big(\boldsymbol{F}_V^2 + \boldsymbol{F}_A^2\Big) + \mathcal{V}_0\Big(\boldsymbol{F}_V\boldsymbol{F}_A + \boldsymbol{F}_A\boldsymbol{F}_V\Big)\right.$$
$$+ da\Big[\Big(\mathcal{V}_0^2 + \mathcal{A}_0^2\Big)\boldsymbol{F}_A + \Big(\mathcal{V}_0\mathcal{A}_0 + \mathcal{A}_0\mathcal{V}_0\Big)\boldsymbol{F}_V\Big]$$
$$\left. + \frac{1}{3}(da)^2\Big(\mathcal{A}_0^3 + 3\mathcal{A}_0\mathcal{V}_0^2\Big)\right\}. \quad (4.29)$$

Equations (4.28) and (4.29) are the main results of this section.

We can particularize the anomalous partition function (4.28) for the gauge group $\mathrm{U}(2)_L \times \mathrm{U}(2)_R$, relevant for a two-flavor hadronic fluid. In this case, the four generators t_a (with $a = 0, 1, 2, 3$) are given by

$$t_0 = \frac{1}{2}\mathbb{1}, \qquad t_i = \frac{1}{2}\sigma_i, \quad (4.30)$$

with σ_i the Pauli matrices ($i = 1, 2, 3$). Due to the properties of the U(2) generators, most traces factorize (see appendix C) and the anomalous part of the effective action takes the form

$$W[\mathcal{V}_0, \mathcal{A}_0, \boldsymbol{V}, \boldsymbol{A}]_{\rm anom}^{\mathrm{U}(2)\times\mathrm{U}(2)} = -\frac{i}{4\pi^2 T_0}\int_{S^3}\left\{(\mathrm{Tr}\,\mathcal{V}_0)\mathrm{Tr}\left(\boldsymbol{F}_V\boldsymbol{A} - \frac{2}{3}\boldsymbol{A}^3\right) + (\mathrm{Tr}\,\boldsymbol{F}_V)\mathrm{Tr}\,(\mathcal{V}_0\boldsymbol{A})\right.$$
$$+ \Big[\mathrm{Tr}\,(\mathcal{V}_0\boldsymbol{F}_V) - (\mathrm{Tr}\,\mathcal{V}_0)(\mathrm{Tr}\,\boldsymbol{F}_V)\Big](\mathrm{Tr}\,\boldsymbol{A}) + \frac{1}{3}(\mathrm{Tr}\,\boldsymbol{F}_A)\mathrm{Tr}\,(\mathcal{A}_0\boldsymbol{A})$$
$$+ \frac{1}{3}\Big[\mathrm{Tr}\,(\mathcal{A}_0\boldsymbol{F}_A) - (\mathrm{Tr}\,\mathcal{A}_0)(\mathrm{Tr}\,\boldsymbol{F}_V)\Big](\mathrm{Tr}\,\boldsymbol{A}) + \frac{1}{3}(\mathrm{Tr}\,\mathcal{A}_0)\mathrm{Tr}\,(\boldsymbol{F}_A\boldsymbol{A})$$
$$- \frac{2}{3}(\mathrm{Tr}\,\boldsymbol{A})\mathrm{Tr}\,(\mathcal{V}_0\boldsymbol{A}^2) - \frac{1}{2}da\Big[(\mathrm{Tr}\,\mathcal{V}_0)^2 - \mathrm{Tr}\,\mathcal{V}_0^2\Big](\mathrm{Tr}\,\boldsymbol{A}) \quad (4.31)$$
$$- \frac{1}{6}da\Big[(\mathrm{Tr}\,\mathcal{A}_0)^2 - \mathrm{Tr}\,\mathcal{A}_0^2\Big](\mathrm{Tr}\,\boldsymbol{A}) + da(\mathrm{Tr}\,\mathcal{V}_0)\mathrm{Tr}\,(\mathcal{V}_0\boldsymbol{A})$$
$$\left. + \frac{1}{3}da(\mathrm{Tr}\,\mathcal{A}_0)\mathrm{Tr}\,(\mathcal{A}_0\boldsymbol{A})\right\}.$$

JHEP11(2018)076

The expression gets much simpler when axial-vectors fields do not have components on the $\mathrm{U}(1)_A$ factor, so their traces vanish. In this case, we have

$$W[\mathcal{V}_0, \mathcal{A}_0, \boldsymbol{V}, \boldsymbol{A}]^{\mathrm{U}(2)\times\mathrm{U}(2)}_{\mathrm{anom}} = -\frac{i}{4\pi^2 T_0}\int_{S^3}\left\{(\mathrm{Tr}\,\mathcal{V}_0)\mathrm{Tr}\left(\boldsymbol{F}_V\boldsymbol{A} - \frac{2}{3}\boldsymbol{A}^3\right) + (\mathrm{Tr}\,\boldsymbol{F}_V)\mathrm{Tr}\left(\mathcal{V}_0\boldsymbol{A}\right)\right.$$
$$\left. + da\big(\mathrm{Tr}\,\mathcal{V}_0\big)\mathrm{Tr}\left(\mathcal{V}_0\boldsymbol{A}\right)\right\}. \tag{4.32}$$

This form of the partition function accounts for the two-flavor anomalies of QCD when the global chiral symmetry $\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R\times\mathrm{U}(1)_V$ is not broken, but as we will show, it may also be used to obtain the anomalous partition function when a SU(2) multiplet of pions is included. An inspection of this expression reveals its diagrammatic origin: the terms emerge from the $\mathrm{U}(1)\mathrm{SU}(2)^2$ triangle and the $\mathrm{U}(1)\mathrm{SU}(2)^3$ square diagrams. The latter are required to guarantee the Wess-Zumino consistency equations, although previous studies [7] only considered the effects of anomalous triangle diagrams through $\mathrm{Tr}\big(t_a\{t_b, t_c\}\big)$.

5 Gauge currents and energy-momentum tensor in stationary backgrounds

Once we have arrived at a general prescription to obtain the equilibrium partition function, we undertake the construction of the anomaly-induced consistent and covariant gauge currents, as well as the energy-momentum tensor. By expressing them in terms of the appropriate fluid fields listed in eqs. (4.12) and (4.14), the hydrodynamic constitutive relations can be obtained.

5.1 Consistent and covariant currents

From eq. (3.6), we know that the physical spacetime current induced by the Chern-Simons action $\Gamma[\mathcal{A}]_{\mathrm{CS}}$ equals (minus) the Bardeen-Zumino current. On the other hand, we have argued that $W[\mathcal{A}_0, \boldsymbol{A}]_{\mathrm{anom}}$, which is the boundary component of the dimensional reduction of $i\Gamma[\mathcal{A}]_{\mathrm{CS}}$ on a time-independent background, induces the consistent current. These two results can be mutually consistent only if the boundary current $\mathcal{X}$ induced by $W[\mathcal{A}_0, \boldsymbol{A}]_{\mathrm{inv}}$

$$\delta_B W[\mathcal{A}_0, \boldsymbol{A}]_{\mathrm{inv}} = \int_{S^{2n-3}} \mathrm{Tr}\left(B\mathcal{X}\right) + \text{bulk contribution}, \tag{5.1}$$

satisfies

$$\mathcal{J}_{\mathrm{cons}} + \mathcal{X} = -\mathcal{J}_{\mathrm{BZ}} \quad\Longrightarrow\quad \mathcal{X} = -(\mathcal{J}_{\mathrm{cons}} + \mathcal{J}_{\mathrm{BZ}}) = -\mathcal{J}_{\mathrm{cov}}. \tag{5.2}$$

This shows that the physical spacetime current induced by $W[\mathcal{A}_0, \boldsymbol{A}]_{\mathrm{inv}}$ is (minus) the covariant current. Exploiting this fact, we can obtain a very simple expression for $\mathcal{J}_{\mathrm{cov}}$, as shown in the following.

Let us analyze this issue in more detail. From (4.7), a general variation of the gauge field $\mathcal{A}$ admits the following Kaluza-Klein invariant decomposition

$$B \equiv \delta_B\mathcal{A} = \delta_B\mathcal{A}_0\theta + \delta_B\boldsymbol{A} \equiv \mathcal{B}_0\theta + \boldsymbol{B} \tag{5.3}$$

where $\mathcal{B}_0$ and $\boldsymbol{B}$ are respectively conjugate to the KK invariant currents $\mathcal{J}_0$ and $\boldsymbol{J}$ (see [15] and appendix B for details). As shown in [18], the covariant version of these currents, $\mathcal{J}_{0,\text{cov}}$ and $\boldsymbol{J}_{\text{cov}}$, can be obtained from the variation of the nonanomalous part of the effective action. Given that this is a nonlocal functional in $2n-2$ dimensions, it is enough to extract the terms proportional to $d\mathcal{B}_0$ and $d\boldsymbol{B}$ from a general variation (5.3), since these are the only terms that will give boundary contributions upon integration by parts of $\delta_B W[\mathcal{A}_0, \boldsymbol{A}]_{\text{inv}}$. Now, from the general expression for this functional given in eq. (4.20), it is obvious that $\delta_B W[\mathcal{A}_0, \boldsymbol{A}]_{\text{inv}}$ is independent of $d\mathcal{A}_0$, with the consequence that $\mathcal{J}_{0,\text{cov}} = 0$. This fact notwithstanding, we will see in the next section that this is no longer true in the presence of spontaneous symmetry breaking, where we will find a nonvanishing anomalous contribution to $\mathcal{J}_{0,\text{cov}}$.

On the other hand, the only dependence of $\delta_B W[\mathcal{A}_0, \boldsymbol{A}]_{\text{inv}}$ on $d\boldsymbol{B}$ is through

$$\delta_B \boldsymbol{F} = d\boldsymbol{B} + \{\boldsymbol{A}, \boldsymbol{B}\}. \tag{5.4}$$

Thus, integrating by parts $\delta_B W[\mathcal{A}_0, \boldsymbol{A}]_{\text{inv}}$, and taking into account a minus sign from the Stokes theorem, yields the following expression for the anomalous covariant current in a stationary background (cf. [18])

$$\begin{aligned} \mathcal{J}_{0,\text{cov}} &= 0, \\ \boldsymbol{J}_{\text{cov}} &= T_0 \frac{\delta}{\delta \boldsymbol{F}} W[\mathcal{A}_0, \boldsymbol{F}, da]_{\text{inv}}, \end{aligned} \tag{5.5}$$

where W_{inv} is now considered a functional of $\mathcal{A}_0$, $\boldsymbol{F}$, and da.

The anomalous consistent current can be obtained by varying W_{anom}

$$\begin{aligned} \mathcal{J}_{0,\text{cons}} &= T_0 \frac{\delta}{\delta \mathcal{A}_0} W[\mathcal{A}_0, \boldsymbol{A}, da]_{\text{anom}}, \\ \boldsymbol{J}_{\text{cons}} &= T_0 \frac{\delta}{\delta \boldsymbol{A}} W[\mathcal{A}_0, \boldsymbol{A}, da]_{\text{anom}}, \end{aligned} \tag{5.6}$$

where W_{anom} has also to be considered as a functional of $\mathcal{A}_0$, $\boldsymbol{A}$, and da. Although the current $\boldsymbol{J}_{\text{cov}}$ could also be computed by adding $\mathcal{J}_{\text{BZ}}$ to $\mathcal{J}_{\text{cons}}$, it is far simpler to obtain it directly from eq. (5.5). Since only the covariant currents are relevant in hydrodynamics, we will give explicit expressions for them in four dimensions. Using (5.5) with (4.23) yields

$$\begin{aligned} \mathcal{J}_{0,\text{cov}} &= 0, \\ \boldsymbol{J}_{\text{cov}} &= -\frac{i}{8\pi^2}\Big(\mathcal{A}_0 \boldsymbol{F} + \boldsymbol{F}\mathcal{A}_0 + \mathcal{A}_0^2 da\Big). \end{aligned} \tag{5.7}$$

As usual, this formula directly gives $\boldsymbol{J}_R(\mathcal{A}_R)_{\text{cov}}$, as well as $\boldsymbol{J}_L(\mathcal{A}_L)_{\text{cov}}$ with a relative minus sign.

The fact that the covariant currents can be computed directly from W_{inv}, which is obtained from the anomaly polynomial and is completely independent of the particular Chern-Simons form being used, has an important consequence: in a theory with left and right gauge fields, $(\mathcal{A}_R, \mathcal{A}_L)$ or their combinations $(\mathcal{V}, \mathcal{A})$, the covariant currents are the same independently of whether the anomalies are given in the left-right symmetric or the

Bardeen form. Thus, there are two ways to compute the covariant currents in terms of vector and axial gauge fields, either from the expressions for $\boldsymbol{J}_{R,\text{cov}}, \boldsymbol{J}_{L,\text{cov}}$ given by eq. (5.7)

$$
\begin{aligned}
\boldsymbol{J}_V(\mathcal{V},\mathcal{A})_{\text{cov}} &= \boldsymbol{J}_R(\mathcal{V}+\mathcal{A})_{\text{cov}} + \boldsymbol{J}_L(\mathcal{V}-\mathcal{A})_{\text{cov}}, \\
\boldsymbol{J}_A(\mathcal{V},\mathcal{A})_{\text{cov}} &= \boldsymbol{J}_R(\mathcal{V}+\mathcal{A})_{\text{cov}} - \boldsymbol{J}_L(\mathcal{V}-\mathcal{A})_{\text{cov}},
\end{aligned}
\tag{5.8}
$$

or directly from (4.29), using

$$
\begin{aligned}
\boldsymbol{J}_{V,\text{cov}} &= T_0 \frac{\delta}{\delta \boldsymbol{F}_V} W[\mathcal{V}_0,\mathcal{A}_0,\boldsymbol{F}_V,\boldsymbol{F}_A,da]_{\text{inv}}, \\
\boldsymbol{J}_{A,\text{cov}} &= T_0 \frac{\delta}{\delta \boldsymbol{F}_A} W[\mathcal{V}_0,\mathcal{A}_0,\boldsymbol{F}_V,\boldsymbol{F}_A,da]_{\text{inv}}.
\end{aligned}
\tag{5.9}
$$

Either way, the final result in four dimensions is

$$
\begin{aligned}
\boldsymbol{J}_{V,\text{cov}} &= -\frac{i}{4\pi^2}\Big[\mathcal{V}_0\boldsymbol{F}_A + \boldsymbol{F}_A\mathcal{V}_0 + \mathcal{A}_0\boldsymbol{F}_V + \boldsymbol{F}_V\mathcal{A}_0 + da(\mathcal{V}_0\mathcal{A}_0 + \mathcal{A}_0\mathcal{V}_0)\Big], \\
\boldsymbol{J}_{A,\text{cov}} &= -\frac{i}{4\pi^2}\Big[\mathcal{V}_0\boldsymbol{F}_V + \boldsymbol{F}_V\mathcal{V}_0 + \mathcal{A}_0\boldsymbol{F}_A + \boldsymbol{F}_A\mathcal{A}_0 + da(\mathcal{V}_0^2 + \mathcal{A}_0^2)\Big].
\end{aligned}
\tag{5.10}
$$

Note that, as mentioned above, eq. (3.19) implies that neither current is conserved.

5.2 The energy-momentum tensor

The Chern-Simons action (2.22) is topological, i.e independent of the gravitational background, so it cannot induce an anomalous energy-momentum tensor. In particular, there is no Bardeen-Zumino energy-momentum tensor induced by $\Gamma[\mathcal{A}]_{\text{CS}}$ on $\mathcal{M}_{2n-2} = \partial\mathcal{M}_{2n-1}$. On the other hand, we have seen that upon dimensional reduction on a stationary background, both the invariant and anomalous part of the partition function W_{inv} and W_{anom} pick up a dependence on the metric functions a_i, which are conjugate to the Kaluza-Klein invariant components ${T_0}^i$ of the energy-momentum tensor (see [15] and appendix B for details). These are the only nonvanishing components of the energy-momentum tensor, as the partition function is independent of the other metric functions σ and g_{ij}. We denote them by its dual $(2n-4)$-form $\boldsymbol{T}$.

Thus, in order to compute this anomalous energy-momentum tensor it is enough to take the variation of the anomalous piece of the partition function W_{anom} with respect to a [see eq. (B.10)]. However, as in the case of the covariant gauge currents, it is much simpler to extract it from the boundary contribution of the variation of W_{inv}. The vanishing of the Bardeen-Zumino energy-momentum tensor guarantees that both methods give the same answer. In looking for those terms in W_{inv} depending on da, we should keep in mind that this functional depends explicitly on da, but also implicitly through $\boldsymbol{F} = d\boldsymbol{A} + \boldsymbol{A}^2 = -da\mathcal{A}_0 + \ldots$ Then the same argument leading to (5.5) gives in this case

$$
\boldsymbol{T} = T_0 \left[\frac{\delta}{\delta(da)} - \mathcal{A}_0 \frac{\delta}{\delta \boldsymbol{F}}\right] W[\mathcal{A}_0,\boldsymbol{F},da]_{\text{inv}}. \tag{5.11}
$$

In four dimensions, using eq. (4.23) we find

$$
\boldsymbol{T} = \frac{i}{24\pi^2}\text{Tr}\left(3\mathcal{A}_0^2\boldsymbol{F} + da\mathcal{A}_0^3\right), \tag{5.12}
$$

JHEP11(2018)076

from where we directly read $\boldsymbol{T}(\mathcal{A}_R)$, and $\boldsymbol{T}(\mathcal{A}_L)$ with a relative minus sign. For a theory with vector and axial gauge fields, the anomalous energy-momentum tensor can be computed either from

$$\boldsymbol{T}(\mathcal{V},\mathcal{A}) = \boldsymbol{T}(\mathcal{V}+\mathcal{A}) - \boldsymbol{T}(\mathcal{V}-\mathcal{A})\,, \tag{5.13}$$

or by varying $W_{\rm inv}$

$$\boldsymbol{T}(\mathcal{V},\mathcal{A}) = T_0\left[\frac{\delta}{\delta(da)} - \mathcal{V}_0\frac{\delta}{\delta \boldsymbol{F}_V} - \mathcal{A}_0\frac{\delta}{\delta \boldsymbol{F}_A}\right] W[\mathcal{V}_0,\mathcal{A}_0,\boldsymbol{F}_V,\boldsymbol{F}_A,da]_{\rm inv}. \tag{5.14}$$

Either way, the result in four dimensions is given by

$$\boldsymbol{T}(\mathcal{V},\mathcal{A}) = \frac{i}{4\pi^2}\mathrm{Tr}\left[\boldsymbol{F}_V(\mathcal{V}_0\mathcal{A}_0+\mathcal{A}_0\mathcal{V}_0) + \boldsymbol{F}_A(\mathcal{V}_0^2+\mathcal{A}_0^2) + \frac{1}{3}da(\mathcal{A}_0^3+3\mathcal{A}_0\mathcal{V}_0^2)\right]. \tag{5.15}$$

Before closing this section, we would like to stress that the existence of a nonvanishing anomalous contribution to the energy-momentum tensor is a direct consequence of the requirement of KK invariance of the partition function $W_{\rm anom}$. It is only through the KK invariant decomposition (4.7) that the field $a = a_i\,dx^i$ enters both $W_{\rm anom}$ and $W_{\rm inv}$. Using

$$\mathcal{A} = \mathcal{A}_0 dx^0 + \mathcal{A}_i dx^i \equiv \mathcal{A}_0 dx^0 + \boldsymbol{\mathcal{A}}, \tag{5.16}$$

instead of (4.7) would give $\boldsymbol{T}=0$, but at the price that then neither the partition function nor the gauge currents would be KK invariant. The situation is different in systems with spontaneously broken symmetry where, as we will see in the next section, it is possible to have a non-anomalous energy-momentum tensor while preserving KK invariance.

5.3 Example: vector response in the presence of chiral imbalance

In order to illustrate the techniques presented so far, we construct the currents and energy-momentum tensor for a physically interesting theory: two-flavor QCD coupled to an external electromagnetic field in a nontrivial ($a_j \neq 0$) background. In this example both chiral magnetic and vortical effects can take place. It is known that in the presence of non-Abelian charges, only chemical potentials associated with mutually commuting charges can be considered [39, 40]. Accordingly, we take the external fields taking values on the Cartan subalgebra of U(2) generated by t_0 and t_3 [see eq. (4.30)]. In particular, we consider the field configuration

$$\begin{aligned}\mathcal{V}_0 &= \mathcal{V}_{00}t_0 + \mathcal{V}_{03}t_3,\\ \boldsymbol{V} &= V_0 t_0 + V_3 t_3,\\ \mathcal{A}_0 &= \mathcal{A}_{00}t_0,\\ \boldsymbol{A} &= 0.\end{aligned} \tag{5.17}$$

Since all external fields lie on the Cartan subalgebra, the corresponding field strengths are particularly simple

$$\begin{aligned}\boldsymbol{F}_V &\equiv d\boldsymbol{V} + \boldsymbol{V}^2 + \boldsymbol{A}^2 = t_0 dV_0 + t_3 dV_3,\\ \boldsymbol{F}_A &\equiv d\boldsymbol{A} + \boldsymbol{A}\boldsymbol{V} + \boldsymbol{V}\boldsymbol{A} = 0.\end{aligned} \tag{5.18}$$

The axial chemical potential μ_5, controlling chiral imbalance, is introduced through (4.13), which in components reads

$$
\begin{aligned}
&\mathcal{A}_0 = \mu_5 \mathbb{1} = 2\mu_5 t_0 \qquad \Longrightarrow \qquad \mathcal{A}_{00} = 2\mu_5, \\
&\mathcal{A}_i \equiv A_i + \mathcal{A}_0 a_i = 2 a_i \mu_5 t_0 .
\end{aligned} \tag{5.19}
$$

By assuming μ_5 to be constant, we are just only interested in the one-derivative terms from the vectorial part of the background.

The physical (i.e., KK invariant) electromagnetic gauge field $\mathbb{V}_\mu$ is related to the vector gauge field by the identities

$$
\begin{aligned}
&\mathcal{V}_0 = eQ\mathbb{V}_0, \\
&V_i \equiv \mathcal{V}_i - \mathcal{V}_0 a_i = eQ\mathbb{V}_i .
\end{aligned} \tag{5.20}
$$

Here eQ is the charge matrix for the two light flavors with

$$
Q = \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & -\frac{1}{3} \end{pmatrix} = \frac{1}{3} t_0 + t_3 . \tag{5.21}
$$

Using the expression of the electromagnetic and isospin currents

$$
\begin{aligned}
J^\mu_{\rm em} &= e\overline{\Psi}\gamma^\mu Q \Psi, \\
J^\mu_{\rm iso} &= \overline{\Psi}\gamma^\mu t_3 \Psi,
\end{aligned} \tag{5.22}
$$

we find that the equilibrium currents can be expressed as the following combinations of the covariant currents

$$
\begin{aligned}
\langle J^\mu_{\rm em}\rangle &= \frac{e}{3}\langle J^\mu_{0,{\rm cov}}\rangle + e\langle J^\mu_{3,{\rm cov}}\rangle, \\
\langle J^\mu_{\rm iso}\rangle &= \langle J^\mu_{3,{\rm cov}}\rangle,
\end{aligned} \tag{5.23}
$$

where the current expectation values $\langle J^\mu_{a,{\rm cov}}\rangle$ are given by the first equation in (5.10) by making the replacements

$$
\begin{aligned}
(\mathcal{V}_{00}, V_{0i}) &= \frac{e}{3}(\mathbb{V}_0, \mathbb{V}_i), \\
(\mathcal{V}_{03}, V_{3i}) &= e(\mathbb{V}_0, \mathbb{V}_i).
\end{aligned} \tag{5.24}
$$

Using in addition the expressions for the axial-vector fields in terms of the chemical potential given in eq. (5.19), we find

$$
\begin{aligned}
\langle J^i_{\rm em}\rangle &= -\frac{5e^2 N_c}{18\pi^2}\mu_5 \epsilon^{ijk}\Big(\partial_j \mathbb{V}_k + \mathbb{V}_0 \partial_j a_k\Big), \\
\langle J^i_{\rm iso}\rangle &= -\frac{eN_c}{4\pi^2}\mu_5 \epsilon^{ijk}\Big(\partial_j \mathbb{V}_k + \mathbb{V}_0 \partial_j a_k\Big),
\end{aligned} \tag{5.25}
$$

where N_c is the number of colors. Inspecting these expressions we find contributions from both the chiral magnetic and chiral vortical effects. The first one is associated with the term

JHEP11(2018)076

proportional to the magnetic field $B^i = \epsilon^{ijk}\partial_j \mathbb{V}_k$. Thus, from the first equation in (5.25), we read the chiral magnetic conductivity [12, 41]

$$\sigma_5 = \frac{e^2 N_c \mathrm{Tr}\, Q^2}{2\pi^2}\mu_5 = \frac{5e^2 N_c}{18\pi^2}\mu_5. \tag{5.26}$$

The vortical contribution, on the other hand, is identified as the term proportional to the curl of the KK field, $\epsilon^{ijk}\partial_j a_k$, associated with the vorticity field, as shown in eq. (4.14).

Similarly, we can use eq. (5.15) with the same replacements as above to write the anomaly-induced energy-momentum tensor

$$\begin{aligned}\langle T_0{}^i\rangle &= \frac{(6\mu_5^2 + 5e^2\mathbb{V}_0^2)N_c}{36\pi^2}\mu_5\epsilon^{ijk}\partial_j a_k + \frac{5e^2N_c}{18\pi^2}\mu_5\mathbb{V}_0\epsilon^{ijk}\partial_j\mathbb{V}_k,\\ \langle T_{00}\rangle &= \langle T^{ij}\rangle = 0.\end{aligned} \tag{5.27}$$

Again, the nonvanishing components receive contributions from the chiral magnetic and vortical effects that can be identified as described. To the best of our knowledge, these explicit expressions for the energy-momentum tensor in the non-Abelian theory have not been reported before in the literature. They generalize the result found in [42] for the U(1) case.

6 The Wess-Zumino-Witten partition function

So far we have assumed that the symmetries we have dealt with, albeit maybe anomalous, are preserved by the vacuum. For physical applications, however, it is convenient to consider situations in which these symmetries are spontaneously broken, either total or partially. This is the case, for example, of chiral flavor symmetry in QCD, broken down to its vector subgroup, the electromagnetic gauge symmetry in conventional superconductors or the U(1) global phase in superfluids. Whenever this happens, Goldstone modes appear which couple to the macroscopic external gauge fields and contribute to the anomaly. In this section we study the construction of partition functions for anomalous fluids with spontaneously broken symmetries extending the previous analysis to the Wess-Zumino-Witten (WZW) action.

6.1 Goldstone modes and the WZW action

The WZW action describes the anomaly-induced interactions between the external gauge field $\mathcal{A}$ and the Goldstone bosons ξ^a. It admits a very simple expression [1] in terms of the Chern-Simons action introduced in eq. (2.22)

$$\Gamma[\mathcal{A}, g]_{\mathrm{WZW}} = \Gamma[\mathcal{A}]_{\mathrm{CS}} - \Gamma[\mathcal{A}_g]_{\mathrm{CS}}, \tag{6.1}$$

where $\mathcal{A}_g$ is the gauge transformed of $\mathcal{A}$ by the gauge group element $g \equiv \exp[-i\xi^a t_a]$

$$\mathcal{A}_g = g^{-1}\mathcal{A}g + g^{-1}dg. \tag{6.2}$$

Under a gauge transformation $h = e^u$, the Goldstone fields transform non-linearly according to

$$g_h = h^{-1}g. \tag{6.3}$$

This transformation makes $\mathcal{A}_g$ gauge invariant, namely

$$(\mathcal{A}_h)_{g_h} = \mathcal{A}_g \qquad \Longrightarrow \qquad \delta_u \mathcal{A}_g = 0, \tag{6.4}$$

and, as a consequence, $\delta_u \Gamma[\mathcal{A}_g] = 0$. Thus, the gauge variation of the WZW action immediately gives the consistent anomaly

$$\delta_u \Gamma[\mathcal{A}, g]_{\rm WZW} = \delta_u \Gamma[\mathcal{A}]_{\rm CS} = -\int\limits_{\mathcal{M}_{2n-2}} \mathrm{Tr}\left(uG[\mathcal{A}(x)]_{\rm cons}\right). \tag{6.5}$$

The structure of the WZW action can be better understood by using the transformation property of the Chern-Simons form ω^0_{2n-1} under finite gauge transformations [1]

$$\omega^0_{2n-1}(\mathcal{A}_g, \mathcal{F}_g) = \omega^0_{2n-1}(\mathcal{A}, \mathcal{F}) + \omega^0_{2n-1}(dgg^{-1}, 0) + d\alpha_{2n-2}(\mathcal{A}, \mathcal{F}, g). \tag{6.6}$$

Here, $\omega^0_{2n-1}(dgg^{-1}, 0)$ is a $(2n-1)$-form proportional to $(dgg^{-1})^{2n-1}$ whereas $\alpha_{2n-2}(\mathcal{A}, \mathcal{F}, g)$ is a $(2n-2)$-form whose explicit expressions are given in appendix B. Then, eq. (6.1) can be rewritten as

$$\begin{aligned}\Gamma[\mathcal{A}, g]_{\rm WZW} &= c_n \int\limits_{\mathcal{M}_{2n-1}} \left[\omega^0_{2n-1}(\mathcal{A}) - \omega^0_{2n-1}(\mathcal{A}_g)\right] \\ &= -c_n \int\limits_{\mathcal{M}_{2n-1}} \omega^0_{2n-1}(dgg^{-1}) - c_n \int\limits_{\mathcal{M}_{2n-2}} \alpha_{2n-2}(\mathcal{A}, g),\end{aligned} \tag{6.7}$$

where, for simplicity we omit the field strength forms from the arguments. Note that the dependence of the WZW action on the gauge field is given in terms of an integral over the $(2n-2)$-dimensional physical spacetime. In other words, unlike the Chern-Simons action, the WZW action is local and polynomial in the gauge fields. This is however not true for the dependence on the Goldstone modes, due to the presence of the higher dimensional integral in the second line of eq. (6.7). Using eqs. (B.12) and (B.13), together with the identity (6.7) we find in the four-dimensional case ($n = 3$)

$$\begin{aligned}\Gamma[\mathcal{A}, g]_{\rm WZW} = \frac{i}{240\pi^2} \int\limits_{\mathcal{M}_5} \mathrm{Tr}\,(dgg^{-1})^5 - \frac{i}{48\pi^2} \int\limits_{\mathcal{M}_4} \mathrm{Tr}\bigg[&(dgg^{-1})(\mathcal{A}\mathcal{F} + \mathcal{F}\mathcal{A} - \mathcal{A}^3) \\ &- \frac{1}{2}(dgg^{-1})\mathcal{A}(dgg^{-1})\mathcal{A} - \mathcal{A}(dgg^{-1})^3\bigg].\end{aligned} \tag{6.8}$$

As usual, the previous expression gives the WZW action for a right-handed gauge field $\mathcal{A}_R$ coupled to a set of Goldstone fields $g_R \equiv \exp[-i\xi^a_R t_a]$, and can be used also for left-handed fields, with a relative minus sign. When both types of fields are present, the total WZW action will be given by

$$\Gamma[\mathcal{A}_R, \mathcal{A}_L, g_R, g_L]_{\rm WZW} = \Gamma[\mathcal{A}_R, g_R]_{\rm WZW} - \Gamma[\mathcal{A}_L, g_L]_{\rm WZW}. \tag{6.9}$$

This description in terms of two sets of Goldstone modes is appropriate when the symmetry group $\mathcal{G}\times\mathcal{G}$ is completely broken. For applications to hadronic fluids, we are more interested

JHEP11(2018)076

in the case $\mathcal{G}\times\mathcal{G}\to\mathcal{G}$, where the symmetry is broken down to the diagonal subgroup of vector gauge transformations. In that case we have to halve the number of Goldstone bosons, in correspondence with the broken axial generators.

This can be accomplished by using the Chern-Simons form $\widetilde{\omega}^0_{2n-1}$ given in eq. (2.37) which preserves vector gauge transformations, instead of the left-right symmetric choice (2.34). Thus, we have

$$\begin{aligned}\Gamma[\mathcal{A}_{L,R},g_{L,R}]_{\rm WZW} &= c_n\int\limits_{\mathcal{M}_{2n-1}}\left[\widetilde{\omega}^0_{2n-1}(\mathcal{A}_R,\mathcal{A}_L)-T(g_R,g_L)\widetilde{\omega}^0_{2n-1}(\mathcal{A}_R,\mathcal{A}_L)\right]\\ &\equiv\widetilde{\Gamma}[\mathcal{A}_R,\mathcal{A}_L]_{\rm CS}-\widetilde{\Gamma}[\mathcal{A}_R^{g_R},\mathcal{A}_L^{g_L}]_{\rm CS},\end{aligned}\tag{6.10}$$

where we have introduced the notation

$$T(g)f(\mathcal{A})\equiv f(\mathcal{A}_g).\tag{6.11}$$

Now we have to take into account that the transformation (6.2) implies the following group composition law

$$T(g_1)T(g_2)=T(g_2g_1).\tag{6.12}$$

Remembering also that $\widetilde{\omega}^0_{2n-1}$ is invariant under finite vector gauge transformations $T(g,g)$, we can write

$$\begin{aligned}T(g_R,g_L)\widetilde{\omega}^0_{2n-1}(\mathcal{A}_R,\mathcal{A}_L)&=T(g_R,g_R)T(e,g_Lg_R^{-1})\widetilde{\omega}^0_{2n-1}(\mathcal{A}_R,\mathcal{A}_L)\\ &=T(e,U)\widetilde{\omega}^0_{2n-1}(\mathcal{A}_R,\mathcal{A}_L)\end{aligned}\tag{6.13}$$

where e represents the identity and

$$U=g_Lg_R^{-1}\equiv e^{2i\xi^a t_a},\tag{6.14}$$

is given in terms of a single set of Goldstone fields and transforms under $\mathcal{G}\times\mathcal{G}$ according to

$$U\to h_L^{-1}Uh_R.\tag{6.15}$$

With all these ingredients in mind, we find that the appropriate WZW action takes the form

$$\Gamma[\mathcal{A}_R,\mathcal{A}_L,U]_{\rm WZW}=\widetilde{\Gamma}[\mathcal{A}_R,\mathcal{A}_L]_{\rm CS}-\widetilde{\Gamma}[\mathcal{A}_R,\mathcal{A}_L^U]_{\rm CS},\tag{6.16}$$

where

$$\mathcal{A}_L^U=U^{-1}\mathcal{A}_LU+U^{-1}dU,\tag{6.17}$$

and $\widetilde{\Gamma}[\mathcal{A}_R,\mathcal{A}_L]_{\rm CS}$ is the integral of $\widetilde{\omega}^0_{2n-1}(\mathcal{A}_R,\mathcal{A}_L)$ over $\mathcal{M}_{2n-1}$, as shown in (6.10). Note that, by construction, this definition of the WZW action gives the correct anomaly in the Bardeen form.

We can obtain a more useful expression for the WZW action by using the property

$$\widetilde{\omega}^0_{2n-1}(\mathcal{A}_R,\mathcal{A}_L)=\omega^0_{2n-1}(\mathcal{A}_R,\mathcal{A}_L)+dS_{2n-2}(\mathcal{A}_R,\mathcal{A}_L)\tag{6.18}$$

JHEP11(2018)076

where S_{2n-2} is the Bardeen counterterm (see appendix B for explicit formulae). Using (6.16) together with (6.6) and (6.18) finally gives

$$\Gamma[\mathcal{A}_R,\mathcal{A}_L,U]_{\rm WZW} = c_n \int\limits_{\mathcal{M}_{2n-1}} \omega^0_{2n-1}(dUU^{-1}) \\ + c_n \int\limits_{\mathcal{M}_{2n-2}} \Big[\alpha_{2n-2}(\mathcal{A}_L,U) + S_{2n-2}(\mathcal{A}_R,\mathcal{A}_L) - S_{2n-2}(\mathcal{A}_R,\mathcal{A}_L^U)\Big]. \tag{6.19}$$

The explicit expression of $\Gamma_{\rm WZW}$ in four dimensions is rather long and will not be given here. It can be found, for instance, in refs. [20, 21, 43].

6.2 Gauge currents in general backgrounds

The formula for $\Gamma_{\rm WZW}$ given in (6.7) shows that, unlike the Chern-Simons action, the WZW effective action does *not* induce any bulk current. Instead, a general variation $\delta\mathcal{A} = B$ induces only a local consistent gauge current on the boundary spacetime $\mathcal{M}_{2n-2}$

$$\delta_B\Gamma[\mathcal{A},g]_{\rm WZW} = -c_n\delta_B \int\limits_{\mathcal{M}_{2n-2}} \alpha_{2n-2}(\mathcal{A},g) \\ = \int\limits_{\mathcal{M}_{2n-2}} {\rm Tr}\Big[B\mathcal{J}(\mathcal{A},g)_{\rm cons}\Big]. \tag{6.20}$$

This is a consistent current, as follows from the fact that adding to it the Bardeen-Zumino current yields a covariant current. Indeed, combining (6.1) with (3.6) gives

$$\delta_B\Gamma[\mathcal{A},g]_{\rm WZW} = \delta_B\Gamma[\mathcal{A}]_{\rm CS} - \delta_B\Gamma[\mathcal{A}_g]_{\rm CS} \\ = \int\limits_{\mathcal{M}_{2n-1}} \Big\{{\rm Tr}\Big[B\mathcal{J}(\mathcal{A})_{\rm bulk}\Big] - {\rm Tr}\Big[B_g\mathcal{J}(\mathcal{A}_g)_{\rm bulk}\Big]\Big\} \\ - \int\limits_{\mathcal{M}_{2n-2}} \Big\{{\rm Tr}\Big[B\mathcal{J}(\mathcal{A})_{\rm BZ}\Big] - {\rm Tr}\Big[B_g\mathcal{J}(\mathcal{A}_g)_{\rm BZ}\Big]\Big\}. \tag{6.21}$$

Then, using that both B and $\mathcal{J}(\mathcal{A})_{\rm bulk}$ transform covariantly

$$B_g = g^{-1}Bg, \\ \mathcal{J}(\mathcal{A}_g)_{\rm bulk} = g^{-1}\mathcal{J}(\mathcal{A})_{\rm bulk}\, g, \tag{6.22}$$

we see that the first integral in the right-hand side of (6.21) vanishes, yielding

$$\delta_B\Gamma[\mathcal{A},g]_{\rm WZW} = \int\limits_{\mathcal{M}_{2n-2}} {\rm Tr}\Big\{B\Big[g\mathcal{J}(\mathcal{A}_g)_{\rm BZ}g^{-1} - \mathcal{J}(\mathcal{A})_{\rm BZ}\Big]\Big\}. \tag{6.23}$$

Given the gauge invariance of $\mathcal{A}_g$, the first term in the right-hand side can be identified with the covariant gauge current, and we conclude that

$$\mathcal{J}(\mathcal{A},g)_{\rm cons} = g\mathcal{J}(\mathcal{A}_g)_{\rm BZ}g^{-1} - \mathcal{J}(\mathcal{A})_{\rm BZ}, \\ \mathcal{J}(\mathcal{A},g)_{\rm cov} = g\mathcal{J}(\mathcal{A}_g)_{\rm BZ}g^{-1} = \mathcal{J}(\mathcal{A}+dgg^{-1})_{\rm BZ}. \tag{6.24}$$

This remarkable connection between the gauge currents and the Bardeen-Zumino current provides the most efficient computational method in the presence of spontaneous symmetry breaking, bypassing the need to use the WZW action [23, 24, 26, 44]. Indeed, using (3.14) we easily find the covariant current in four dimensions

$$\mathcal{J}(\mathcal{A},g)_{\rm cov} = \frac{i}{24\pi^2}\left[\mathcal{F}(\mathcal{A}+dgg^{-1}) + (\mathcal{A}+dgg^{-1})\mathcal{F} - \frac{1}{2}(\mathcal{A}+dgg^{-1})^3\right]. \tag{6.25}$$

This is valid for a right-handed fermion, while as usual the left-handed current differs just by a relative minus sign.

This can be easily extended to the phenomenologically interesting case $\mathcal{G}\times\mathcal{G}\to\mathcal{G}$, where the symmetry is broken down to the diagonal subgroup. Basically, one just have to make the replacements $\mathcal{A}\to(\mathcal{A}_R,\mathcal{A}_L)$ and $g\to(e,U)$ in the previous relations, obtaining

$$\begin{aligned}\mathcal{J}^R(\mathcal{A}_R,\mathcal{A}_L,U)_{\rm cov} &= \mathcal{J}^R(\mathcal{A}_R,\mathcal{A}_L^U)_{\rm BZ}\\ \mathcal{J}^L(\mathcal{A}_R,\mathcal{A}_L,U)_{\rm cov} &= U\mathcal{J}^L(\mathcal{A}_R,\mathcal{A}_L^U)_{\rm BZ}U^{-1}\,,\end{aligned} \tag{6.26}$$

where the Bardeen-Zumino currents can be computed from the relation [cf. (3.8)]

$$c_n\ell\,\widetilde{\omega}^0_{2n-1}(\mathcal{A}_R,\mathcal{A}_L) = -\mathrm{Tr}\left(B_R\mathcal{J}^R_{\rm BZ} + B_L\mathcal{J}^L_{\rm BZ}\right). \tag{6.27}$$

In four dimensions, the Bardeen-Zumino currents read

$$\begin{aligned}\mathcal{J}^R_{\rm BZ}(\mathcal{A}_L,\mathcal{A}_R) &= -\frac{i}{24\pi^2}\left[(\mathcal{A}_L-\mathcal{A}_R)\left(\mathcal{F}_R+\frac{1}{2}\mathcal{F}_L\right)\right.\\ &\qquad\left.+\left(\mathcal{F}_R+\frac{1}{2}\mathcal{F}_L\right)(\mathcal{A}_L-\mathcal{A}_R) - \frac{1}{2}(\mathcal{A}_L-\mathcal{A}_R)^3\right],\\ \mathcal{J}^L_{\rm BZ}(\mathcal{A}_L,\mathcal{A}_R) &= -\frac{i}{24\pi^2}\left[(\mathcal{A}_L-\mathcal{A}_R)\left(\mathcal{F}_L+\frac{1}{2}\mathcal{F}_R\right)\right.\\ &\qquad\left.+\left(\mathcal{F}_L+\frac{1}{2}\mathcal{F}_R\right)(\mathcal{A}_L-\mathcal{A}_R) - \frac{1}{2}(\mathcal{A}_L-\mathcal{A}_R)^3\right].\end{aligned} \tag{6.28}$$

Now, using (6.26) we can obtain the following expressions for the covariant current in the presence of Goldstone modes

$$\begin{aligned}\mathcal{J}^R(\mathcal{A}_R,\mathcal{A}_L,U)_{\rm cov} &= -\frac{i}{24\pi^2}\left[\left(U^{-1}\mathcal{A}_LU-\mathcal{A}_R+U_R\right)\left(\mathcal{F}_R+\frac{1}{2}U^{-1}\mathcal{F}_LU\right)\right.\\ &\quad\left.+\left(\mathcal{F}_R+\frac{1}{2}U^{-1}\mathcal{F}_LU\right)\left(U^{-1}\mathcal{A}_LU-\mathcal{A}_R+U_R\right)-\frac{1}{2}\left(U^{-1}\mathcal{A}_LU-\mathcal{A}_R+U_R\right)^3\right],\\ \mathcal{J}^L(\mathcal{A}_R,\mathcal{A}_L,U)_{\rm cov} &= -\frac{i}{24\pi^2}\left[\left(\mathcal{A}_L-U\mathcal{A}_RU^{-1}+U_L\right)\left(\mathcal{F}_L+\frac{1}{2}U\mathcal{F}_RU^{-1}\right)\right.\\ &\quad\left.+\left(\mathcal{F}_L+\frac{1}{2}U\mathcal{F}_RU^{-1}\right)\left(\mathcal{A}_L-U\mathcal{A}_RU^{-1}+U_L\right)-\frac{1}{2}\left(\mathcal{A}_L-U\mathcal{A}_RU^{-1}+U_L\right)^3\right],\end{aligned} \tag{6.29}$$

where we have defined the adjoint fields

$$\begin{aligned}U_R &\equiv U^{-1}dU,\\ U_L &\equiv dUU^{-1}.\end{aligned} \tag{6.30}$$

JHEP11(2018)076

From these results, we can obtain as well the vector and axial gauge currents as functions of $\mathcal{V}$ and $\mathcal{A}$, using

$$\begin{aligned}\mathcal{J}^V(\mathcal{V},\mathcal{A},U)_{\rm cov} &= \mathcal{J}^R(\mathcal{V}+\mathcal{A},\mathcal{V}-\mathcal{A},U)_{\rm cov}+\mathcal{J}^L(\mathcal{V}+\mathcal{A},\mathcal{V}-\mathcal{A},U)_{\rm cov},\\ \mathcal{J}^A(\mathcal{V},\mathcal{A},U)_{\rm cov} &= \mathcal{J}^R(\mathcal{V}+\mathcal{A},\mathcal{V}-\mathcal{A},U)_{\rm cov}-\mathcal{J}^L(\mathcal{V}+\mathcal{A},\mathcal{V}-\mathcal{A},U)_{\rm cov},\end{aligned} \tag{6.31}$$

but the explicit expressions resulting from the substitution of eq. (6.29) are rather cumbersome.

6.3 Stationary backgrounds and the WZW partition function

The partition function with spontaneously broken symmetry can be readily obtained from the corresponding one in the absence of Goldstone bosons by applying (6.1) in a time-independent background

$$W[\mathcal{A}_0,\boldsymbol{A},g]_{\rm WZW} = i\Gamma[\mathcal{A}_0,\boldsymbol{A},g]_{\rm WZW} = i\Gamma[\mathcal{A}_0,\boldsymbol{A}]_{\rm CS} - i\Gamma[\mathcal{A}_{0g},\boldsymbol{A}_g]_{\rm CS}. \tag{6.32}$$

Then, using (4.19), we find that the invariant part cancels and the partition function can be written only in terms of the local anomalous part as

$$W[\mathcal{A}_0,\boldsymbol{A},g]_{\rm WZW} = W[\mathcal{A}_0,\boldsymbol{A}]_{\rm anom} - W[\mathcal{A}_{0g},\boldsymbol{A}_g]_{\rm anom}, \tag{6.33}$$

where, to avoid a cumbersome notation here and in the following, we have omitted the dependence on the metric function a_i. Noting that $\mathcal{A}_0$ transforms covariantly as an adjoint field under time-independent gauge transformations, $\mathcal{A}_{0g}=g^{-1}\mathcal{A}_0 g$, and using the cyclic property of the trace, the formula for the partition function can be simplified to

$$W[\mathcal{A}_0,\boldsymbol{A},g]_{\rm WZW} = W[\mathcal{A}_0,\boldsymbol{A}]_{\rm anom} - W[\mathcal{A}_0,\boldsymbol{A}+dgg^{-1}]_{\rm anom}. \tag{6.34}$$

Using the explicit form of $W[\mathcal{A}_0,\boldsymbol{A}]_{\rm anom}$ given in (4.22), yields the partition function in four dimensions

$$\begin{aligned}W[\mathcal{A}_0,\boldsymbol{A},g]_{\rm WZW} = \frac{i}{24\pi^2 T_0}\int_{S^3}\mathrm{Tr}\,\bigg\{\mathcal{A}_0\bigg[&\boldsymbol{F}dgg^{-1}+dgg^{-1}\boldsymbol{F}+\frac{1}{2}\boldsymbol{A}^3\\ &-\frac{1}{2}\Big(\boldsymbol{A}+dgg^{-1}\Big)^3\bigg]+\mathcal{A}_0^2\,dgg^{-1}da\bigg\}.\end{aligned} \tag{6.35}$$

The result (6.33) can be obviously extended to the case where the gauge group is spontaneously broken to its diagonal vector subgroup, $\mathcal{G}\times\mathcal{G}\to\mathcal{G}$. In this case, using the partition function in the Bardeen form, the WZW action reads

$$W[\mathcal{V}_0,\mathcal{A}_0,\boldsymbol{V},\boldsymbol{A},U]_{\rm WZW} = W[\mathcal{V}_0,\mathcal{A}_0,\boldsymbol{V},\boldsymbol{A}]_{\rm anom} - W[\mathcal{V}_0^U,\mathcal{A}_0^U,\boldsymbol{V}^U,\boldsymbol{A}^U]_{\rm anom}, \tag{6.36}$$

where $\mathcal{V}_0$ and $\mathcal{A}_0$ transform under time-independent gauge transformations as

$$\begin{aligned}\mathcal{V}_0^U &= \frac{1}{2}(\mathcal{V}_0+\mathcal{A}_0)+\frac{1}{2}U^{-1}(\mathcal{V}_0-\mathcal{A}_0)U,\\ \mathcal{A}_0^U &= \frac{1}{2}(\mathcal{V}_0+\mathcal{A}_0)-\frac{1}{2}U^{-1}(\mathcal{V}_0-\mathcal{A}_0)U,\end{aligned} \tag{6.37}$$

JHEP11(2018)076

with identical transformations for $\boldsymbol{F}_V$ and $\boldsymbol{F}_A$. On the other hand, the transformation of $\boldsymbol{V}$ and $\boldsymbol{A}$ picks up an extra non-homogeneous term

$$\begin{aligned}\boldsymbol{V}^U &= \frac{1}{2}(\boldsymbol{V}+\boldsymbol{A}) + \frac{1}{2}U^{-1}(\boldsymbol{V}-\boldsymbol{A})U + \frac{1}{2}U^{-1}dU, \\ \boldsymbol{A}^U &= \frac{1}{2}(\boldsymbol{V}+\boldsymbol{A}) - \frac{1}{2}U^{-1}(\boldsymbol{V}-\boldsymbol{A})U - \frac{1}{2}U^{-1}dU.\end{aligned} \tag{6.38}$$

Using the transformation laws (6.37) and (6.38) on (4.28) yields the partition function $W[\mathcal{V}_0,\mathcal{A}_0,\boldsymbol{V},\boldsymbol{A},U]_{\rm WZW}$. Although the computation is straightforward the result is rather long and will not be reproduced here.

To illustrate this technique, we consider again the exampled discussed in section 5.3, two-flavor QCD in four dimensions, where now the global symmetry group $\mathrm{U}(2)_L\times\mathrm{U}(2)_R$ is broken down to its vector subgroup $\mathrm{U}(2)_V$. The WZW action can be computed using the prescription (6.36) with the anomalous piece of the effective action obtained in eq. (4.31). In doing this, we have to keep in mind that the $\mathrm{U}(1)_A$ factor is broken at the nonperturbative level, so there is no Goldstone mode associated with $\mathrm{U}(1)_A$ and the field U only has components on the $\mathrm{SU}(2)_A$ factor. Again, the external fields are taken to lie in the Cartan subalgebra in the field configuration defined in eq. (5.17), with the corresponding field strengths given in eq. (5.18). Using these expressions, we find from (4.31) that $W[\mathcal{V}_0,\mathcal{A}_0,\boldsymbol{V},\boldsymbol{A}]_{\rm anom}=0$ for $\boldsymbol{A}=0$. On the other hand, to compute the second term in (6.36), we need eq. (6.37), together with

$$\begin{aligned}\boldsymbol{A}^U &= \frac{1}{2}V_3\Big(t_3 - U^{-1}t_3U\Big) - \frac{1}{2}U^{-1}dU, \\ \boldsymbol{F}_V^U &= t_0 dV_0 + \frac{1}{2}dV_3\Big(t_3 + U^{-1}t_3U\Big), \\ \boldsymbol{F}_A^U &= \frac{1}{2}dV_3\Big(t_3 - U^{-1}t_3U\Big).\end{aligned} \tag{6.39}$$

Notice that, from the first and last equations, we find

$$\begin{aligned}\mathrm{Tr}\,\boldsymbol{A}^U &= -\frac{1}{2}\mathrm{Tr}\left(U^{-1}dU\right) = 0, \\ \mathrm{Tr}\,\boldsymbol{F}_A^U &= 0,\end{aligned} \tag{6.40}$$

since U only takes value on SU(2). We have then

$$\begin{aligned}W[\mathcal{V}_0,\mathcal{A}_0,\boldsymbol{V},\boldsymbol{A},U]^{\mathrm{U}(2)\times\mathrm{U}(2)}_{\rm WZW} = \frac{i}{4\pi^2T_0}\int_{S^3}\Big\{&(\mathrm{Tr}\,\mathcal{V}_0^U)\,\mathrm{Tr}\left[\boldsymbol{F}_V^U\boldsymbol{A}^U - \frac{2}{3}(\boldsymbol{A}^U)^3\right] \\ &+ (\mathrm{Tr}\,\boldsymbol{F}_V^U)\,\mathrm{Tr}\,(\mathcal{V}_0^U\boldsymbol{A}^U) + \frac{1}{3}(\mathrm{Tr}\,\mathcal{A}_0^U)\,\mathrm{Tr}\,(\boldsymbol{F}_A^U\boldsymbol{A}^U) \\ &+ da(\mathrm{Tr}\,\mathcal{V}_0^U)\,\mathrm{Tr}\,(\mathcal{V}_0^U\boldsymbol{A}^U) + \frac{1}{3}da(\mathrm{Tr}\,\mathcal{A}_0^U)\,\mathrm{Tr}\,(\mathcal{A}_0^U\boldsymbol{A}^U)\Big\},\end{aligned} \tag{6.41}$$

JHEP11(2018)076

which upon substitution renders the result

$$\begin{aligned}W[\mathcal{V}_0,\mathcal{A}_0,\boldsymbol{V},\boldsymbol{A},U]^{\mathrm{U}(2)\times\mathrm{U}(2)}_{\mathrm{WZW}} = \frac{i}{8\pi^2 T_0}\int\limits_{S^3}\bigg\{&\frac{1}{2}\mathcal{V}_{00}V_3\mathrm{Tr}\Big[t_3 d(U_R+U_L)\Big]+\frac{1}{6}\mathcal{V}_{00}\mathrm{Tr}\left(U_L^3\right)\\&-\frac{1}{2}\Big(\mathcal{V}_{00}dV_3+\mathcal{V}_{03}dV_0+da\mathcal{V}_{00}\mathcal{V}_{03}\Big)\mathrm{Tr}\Big[t_3\big(U_R+U_L\big)\Big]\\&-\frac{1}{6}\mathcal{A}_{00}\Big(dV_3+da\mathcal{V}_{03}\Big)\mathrm{Tr}\Big[t_3(U_R-U_L)\Big]\\&+\frac{1}{3}\mathcal{A}_{00}\Big(dV_3V_3+da\mathcal{V}_{03}V_3\Big)\mathrm{Tr}\Big[t_3\Big(t_3-U^{-1}t_3U\Big)\Big]\bigg\},\end{aligned} \tag{6.42}$$

where we have used U_R, U_L defined in eq. (6.30). This equation is one of the main results of our work and illustrates the power of the differential geometry methods introduced.

It is instructive to compute the previous partition function for the particular case of the chiral-imbalanced electromagnetic background studied in section 5.3, this time on a flat background ($a_k = 0$). Here we just give the results, the full calculation being deferred to a future work [31]. Using the Pauli matrices, the unitary Goldstone matrix U can be parametrized as

$$U = \exp\left(i\sum_{a=1}^{3}\zeta_a\sigma_a\right), \tag{6.43}$$

where, in terms of the conventionally normalized pion fields $\{\pi^0,\ \pi^\pm\}$, we write

$$\sum_{a=1}^{3}\zeta_a\sigma_a = \frac{\sqrt{2}}{f_\pi}\begin{pmatrix}\frac{1}{\sqrt{2}}\pi^0 & \pi^+\\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0\end{pmatrix}, \tag{6.44}$$

with $f_\pi \approx 92\,\mathrm{MeV}$ the pion decay constant. Expanding the effective action (6.42) in powers of the pion fields, after a lengthy calculation we find the following expression for the partition function

$$\begin{aligned}T_0 W = \int d^3x\,\bigg[&\frac{e^2N_c}{12\pi^2 f_\pi}\mathbb{V}_0\,\partial_i\pi^0 B^i - \frac{ie\mu_5 N_c}{12\pi^2 f_\pi^2}\Big(\pi^-\partial_j\pi^+ - \pi^+\partial_j\pi^- - 2ie\pi^-\pi^+\mathbb{V}_j\Big)B^j\\&+\mathcal{O}(\pi^3)\bigg],\end{aligned} \tag{6.45}$$

where $B^i = \epsilon^{ijk}\partial_j\mathbb{V}_k$ is the magnetic field. The first term on the right-hand side of eq. (6.45), linear in the pion field, is in perfect agreement with the magnitude of the term in the effective Lagrangian giving the electromagnetic decay of the neutral pion, $\pi^0\to 2\gamma$

$$\mathscr{L}_{\mathrm{eff}} \supset \frac{e^2N_c}{96\pi^2 f_\pi}\pi^0\epsilon^{\mu\nu\alpha\beta}\mathbb{F}_{\mu\nu}\mathbb{F}_{\alpha\beta}, \tag{6.46}$$

where $\mathbb{F}_{\mu\nu} = \partial_\mu\mathbb{V}_\nu - \partial_\nu\mathbb{V}_\mu$ is the field strength associated with the electromagnetic potential $\mathbb{V}_\mu$. It also reproduces the result found in ref. [45]. In addition, the quadratic term in (6.45) also agrees with the known form of the parity-odd couplings obtained from the Wess-Zumino-Witten action in the presence of chiral imbalance (see, for example, refs. [20, 46]).

JHEP11(2018)076

6.4 Gauge currents and energy-momentum tensor in stationary backgrounds with spontaneous symmetry breaking

The application of this formalism to hydrodynamics requires the knowledge of the covariant gauge currents in stationary backgrounds [28]. One possible route is to compute the consistent currents from a general variation (5.3) of the partition function, which then can be covariantized through the addition of the (dimensionally reduced) Bardeen-Zumino current. A more efficient way is to perform the dimensional reduction of the covariant gauge currents, which in the presence of symmetry breaking admit explicit, local expressions [cf. eqs. (6.25) and (6.29) in four dimensions].

An efficient way to carry out the dimensional reduction of the currents is by writing

$$\int_{\mathcal{M}_{2n-2}} \mathrm{Tr}\left[B\mathcal{J}(\mathcal{A},\mathcal{F},g)\right] = \frac{1}{T_0}\int_{S^{2n-3}} \mathrm{Tr}\left[\mathcal{B}_0\mathcal{J}_0(\mathcal{A}_0,\boldsymbol{A},g) + \boldsymbol{B}\boldsymbol{J}(\mathcal{A}_0,\boldsymbol{A},g)\right], \tag{6.47}$$

and using the identity

$$\int_{\mathcal{M}_{2n-2}} f(B,\mathcal{A},\mathcal{F}) = \frac{1}{T_0}\int_{S^{2n-3}} \left(\mathcal{B}_0\frac{\delta}{\delta B} + \mathcal{A}_0\frac{\delta}{\delta\mathcal{A}} - \boldsymbol{D}\mathcal{A}_0\frac{\delta}{\delta\mathcal{F}}\right) f(B,\mathcal{A},\mathcal{F})\Big|_{\substack{\mathcal{A},B\to\boldsymbol{A},\boldsymbol{B}\\ \mathcal{F}\to\boldsymbol{F}+\mathcal{A}_0 da}}, \tag{6.48}$$

where $\boldsymbol{D}$ is the covariant derivative with respect to the connection $\boldsymbol{A}$ defined in eq. (4.10). This equation follows from the decompositions (4.7) and (5.3), together with (4.9), which reads

$$\mathcal{F} = \boldsymbol{F} + \mathcal{A}_0 da - \theta\boldsymbol{D}\mathcal{A}_0. \tag{6.49}$$

Notice that this identity introduces terms proportional to dx^0 that survive upon dimensional reduction on the thermal cycle, hence the functional derivative with respect to $\mathcal{F}$. Applying eq. (6.48) to (6.47) immediately yields

$$\mathcal{J}_0(\mathcal{A}_0,\boldsymbol{A},\boldsymbol{F},g) = \mathcal{J}(\mathcal{A},\mathcal{F},g)\Big|_{\substack{\mathcal{A}\to\boldsymbol{A}\\ \mathcal{F}\to\boldsymbol{F}+\mathcal{A}_0 da}}, \tag{6.50}$$

for the anomalous non-abelian charge density, and

$$\boldsymbol{J}(\mathcal{A}_0,\boldsymbol{A},\boldsymbol{F},g) = \left(-\mathcal{A}_0\frac{\delta}{\delta\mathcal{A}} + \boldsymbol{D}\mathcal{A}_0\frac{\delta}{\delta\mathcal{F}}\right)\mathcal{J}(\mathcal{A},\mathcal{F},g)\Big|_{\substack{\mathcal{A}\to\boldsymbol{A}\\ \mathcal{F}\to\boldsymbol{F}+\mathcal{A}_0 da}}, \tag{6.51}$$

for the anomalous current.

We finally study the four-dimensional case. Using eq. (6.25) for the covariant current gives

$$\begin{aligned}\mathcal{J}_0(\mathcal{A}_0,\boldsymbol{A},\boldsymbol{F},g)_{\mathrm{cov}} = \frac{i}{24\pi^2}\bigg[&(\boldsymbol{F}+\mathcal{A}_0 da)(\boldsymbol{A}+dgg^{-1})\\ &+(\boldsymbol{A}+dgg^{-1})(\boldsymbol{F}+\mathcal{A}_0 da) - \frac{1}{2}(\boldsymbol{A}+dgg^{-1})^3\bigg],\end{aligned}$$

JHEP11(2018)076

and

$$
\begin{aligned}
\boldsymbol{J}(\mathcal{A}_0,\boldsymbol{A},\boldsymbol{F},g)_{\rm cov} = \frac{i}{24\pi^2}\Big[&\boldsymbol{D}\mathcal{A}_0\Big(\boldsymbol{A}+dgg^{-1}\Big)-\Big(\boldsymbol{A}+dgg^{-1}\Big)\boldsymbol{D}\mathcal{A}_0 \\
&-\Big(\boldsymbol{F}+\mathcal{A}_0 da\Big)\mathcal{A}_0-\mathcal{A}_0\Big(\boldsymbol{F}+\mathcal{A}_0 da\Big)+\frac{1}{2}\mathcal{A}_0\Big(\boldsymbol{A}+dgg^{-1}\Big)^2 \\
&-\frac{1}{2}\Big(\boldsymbol{A}+dgg^{-1}\Big)\mathcal{A}_0\Big(\boldsymbol{A}+dgg^{-1}\Big)+\frac{1}{2}\Big(\boldsymbol{A}+dgg^{-1}\Big)^2\mathcal{A}_0\Big].
\end{aligned} \tag{6.52}
$$

As in many other instances along this paper, it should be clear that eqs. (6.50) and (6.51) can also be applied to a system with left and right (or vector and axial-vector) gauge currents, with obvious modifications.

Finally, it is easy to see that the anomalous energy-momentum tensor must vanish in a system with spontaneously broken symmetry described by the WZW action. The reason is that, as can be seen from eq. (5.11), the anomalous components ${T_0}^i$ of the energy-momentum tensor in stationary backgrounds are invariant under time-independent gauge transformations

$$
{T_0}^i(\mathcal{A}_0^g,\boldsymbol{A}_g)={T_0}^i(\mathcal{A}_0,\boldsymbol{A}). \tag{6.53}
$$

Thus, eq. (6.33) implies that the anomalous energy-momentum tensor is zero in the presence of spontaneous symmetry breaking. Indeed, computing its only potentially nonvanishing component, we find that the contribution of the gauge fields is exactly cancelled by that of the Goldstone modes

$$
{T_0}^i(\mathcal{A}_0,\boldsymbol{A},g)={T_0}^i(\mathcal{A}_0,\boldsymbol{A})-{T_0}^i(\mathcal{A}_0^g,\boldsymbol{A}_g)=0. \tag{6.54}
$$

7 Discussion and outlook

Differential geometry is indeed a powerful tool to address the issue of quantum field theory anomalies. In this work we have employed it to carry out a systematic construction of partition functions for nondissipative fluids in the presence of non-Abelian anomalies. We have provided explicit expressions for the anomalous contribution to the fluid partition function for generic theories in terms of functional derivatives of the corresponding Chern-Simons effective action. More importantly, our analysis can be also applied to theories with spontaneous symmetry breaking, in which case the starting point is the WZW effective action for Goldstone modes, which is straightforwardly constructed in terms of the corresponding partition functions.

We have also studied in detail the gauge currents and energy-momentum tensor induced by the anomaly, giving operational expressions for these quantities. The covariant current, being determined solely from the nonlocal invariant piece of the partition function [18], is therefore independent of any local counterterms modifying the anomalous part. This in particular means that its form does not depend on using either the symmetric or the Bardeen form of the anomaly. In the case of the energy-momentum tensor, we have shown that the requirement of KK invariance of the low energy fields forces a nonvanishing anomaly-induced contribution. The situation is quite different for theories with spontaneous symmetry breaking. In this case, the Bardeen-Zumino current fully determines both the consistent and covariant currents. These can be computed either from the variation of the partition function or, alternatively, from the dimensional reduction of the Bardeen-Zumino current. In the case of the induced energy-momentum tensor, we find a vanishing result due to the cancellation between the contribution of gauge fields and Goldstone modes.

There are a number of issues that can be effectively addressed using the methods described in this work. For example, the anomaly-induced equilibrium partition functions obtained here only include the effect of chiral anomalies. This is clear from the fact that our effective actions only contain first derivatives of the metric functions, whereas gravitational anomalies depend on the curvature two-form which includes second-order derivatives. A next step is to incorporate the effect of gravitational and mixed gauge-gravitational anomalies into the formalism, thus generalizing existing analysis in the literature (e.g., [17, 18]).

This can be done following the same strategy used in this work for chiral anomalies: implementing dimensional reduction on the Chern-Simons form associated with the appropriate anomaly polynomial, which now includes both the contributions of the gauge fields and the background curvature [47, 48]. Using homotopy techniques similar to the ones employed in this paper, it is possible to write general formulae for the Chern-Simons form in any dimension, including also the contribution of the background curvature. This opens the way to give general prescriptions for the construction of equilibrium partition functions for fluids including the effects of gravitational anomalies. This problem will be addressed in detail elsewhere.

Finally, in this article we have confined our attention to the application of differential geometry methods to give a general prescription for the construction of nondissipative partition functions. These can be applied to the study of a variety of hydrodynamic systems with non-Abelian anomalies. A natural task now is to exploit these techniques to study the constitutive relations, as well as the corresponding anomalous transport coefficients, of a variety of systems of physical interest, including hadronic fluids relevant for the physics of heavy ion collisions [49–52]. These issues will be addressed in a forthcoming publication [31].

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JHEP11(2018)076

A The generalized transgression formula

The aim of this appendix is to summarize basic aspects of the generalized transgression formula introduced in ref. [32], which has been used at various points throughout the paper. Let us consider a family of connections $\mathcal{A}_t$ depending on a set of $p+2$ continuous parameters $t \equiv (t_0, \ldots, t_{p+1})$, satisfying the constraint

$$\sum_{r=0}^{p+1} t_r = 1, \tag{A.1}$$

and taking values in a domain T. We denote by ℓ_t the substitution operator replacing the standard exterior differential d by the differential in parameter space d_t

$$\ell_t \equiv d_t \frac{\partial}{\partial(d)}, \tag{A.2}$$

where the odd differential operator d_t is defined by

$$d_t \equiv \sum_{r=0}^{p+1} dt_r \frac{\partial}{\partial t_r}. \tag{A.3}$$

It is important to notice that the operator ℓ_t is even, since it replaces one exterior differential by another. We consider now a generic polynomial $\mathscr{Q}$ depending on $\{\mathcal{A}_t, \mathcal{F}_t, d_t\mathcal{A}_t, d_t\mathcal{F}_t\}$, of degree q in d_t. Then, the following generalized transgression formula holds [32]

$$\int_{\partial T} \frac{\ell_t^p}{p!}\mathscr{Q} = \int_T \frac{\ell_t^{p+1}}{(p+1)!} d\mathscr{Q} + (-1)^{p+q} d \int_T \frac{\ell_t^{p+1}}{(p+1)!}\mathscr{Q}. \tag{A.4}$$

For illustration, we apply this formula to the family of connections interpolating between $\mathcal{A}$ and $\mathcal{B}$

$$\mathcal{A}_t = t\mathcal{A} + (1-t)\mathcal{B}, \tag{A.5}$$

with $0 \leq t \leq 1$. The action of the operator ℓ_t on the connection $\mathcal{A}_t$ and the associated field strength $\mathcal{F}_t = d\mathcal{A}_t + \mathcal{A}_t^2$ is given by

$$\begin{aligned} &\ell_t \mathcal{A}_t = 0, \\ &\ell_t \mathcal{F}_t = d_t \mathcal{A}_t = dt\big(\mathcal{A} - \mathcal{B}\big). \end{aligned} \tag{A.6}$$

Let us take the polynomial defined by the Chern-Simons form associated with $\mathcal{A}_t$

$$\mathscr{Q} = \omega^0_{2n-1}(\mathcal{A}_t), \tag{A.7}$$

which is of degree $q = 0$ in d_t. Since our family of connections depends on a single independent parameter ($p = 0$), the generalized transgression formula (A.4) in this case renders eq. (3.1)

$$\begin{aligned} \omega^0_{2n-1}(\mathcal{A}) - \omega^0_{2n-1}(\mathcal{B}) &= \int_0^1 \ell_t d\omega^0_{2n-1}(\mathcal{A}_t) + d\int_0^1 \ell_t \omega^0_{2n-1}(\mathcal{A}_t) \\ &= \int_0^1 \ell_t \operatorname{Tr} \mathcal{F}_t^n + d\int_0^1 \ell_t \omega^0_{2n-1}(\mathcal{A}_t), \end{aligned} \tag{A.8}$$

where we have used that $d\omega^0_{2n-1}(\mathcal{A}_t) = \mathrm{Tr}\,\mathcal{F}^n_t$.

Particularizing this expression to $\mathcal{A} = \mathcal{A}_0\theta + \boldsymbol{A}$ and $\mathcal{B} = \boldsymbol{A}$, we obtain eq. (4.16). To compute the first term, we just notice that

$$\begin{aligned}\ell_t d\omega^0_{2n-1} &= \ell_t \mathrm{Tr}\,\mathcal{F}^n_t = n\mathrm{Tr}\,(\mathcal{F}^{n-1}_t \ell_t \mathcal{F}_t)\\ &= n\mathrm{Tr}\,(\mathcal{F}^{n-1}_t d_t\mathcal{A}_t) = dt\, n\,\mathrm{Tr}\,(\mathcal{F}^{n-1}_t \mathcal{A}_0\theta),\end{aligned} \tag{A.9}$$

which, together with

$$\mathcal{F}_t = \boldsymbol{F} + t\mathcal{A}_0 da + t(\boldsymbol{D}\mathcal{A}_0)\theta, \tag{A.10}$$

leads to eq. (4.17). Here again $\boldsymbol{D}$ denotes the covariant derivative with respect to the connection $\boldsymbol{A}$ defined in (4.10). As for the second term in (A.8), we have to remember that ℓ_t acts on products through the Leibniz rule, replacing $\mathcal{F}_t$ by $d_t\mathcal{A}_t$. Thus, we can write

$$\ell_t\omega^0_{2n-1}(\mathcal{A}_t,\mathcal{F}_t) = (d_t\mathcal{A}_t)\frac{\delta}{\delta\mathcal{F}}\omega^0_{2n-1}(\mathcal{A},\mathcal{F})\Big|_{\substack{\mathcal{A}\to\boldsymbol{A}\\ \mathcal{F}\to\boldsymbol{F}+t\mathcal{A}_0 da+t(\boldsymbol{D}\mathcal{A}_0)\theta}}. \tag{A.11}$$

Taking finally into account that $d_t\mathcal{A}_t = dt\mathcal{A}_0\theta$, and that $\theta^2 = 0$, we retrieve eq. (4.18).

B Some explicit expressions

To make our presentation self-contained, in this appendix we list some relevant expressions refereed to at various points of the article.

A closed expression of the Bardeen counterterm. Taking the two-parameter family of connections

$$\mathcal{A}_t = t_1\mathcal{A}_R + t_2\mathcal{A}_L, \tag{B.1}$$

a simple explicit expression for the Bardeen counterterm can be written as [21]

$$S_{2n-2}(\mathcal{A}_{R,L},\mathcal{F}_{R,L}) = \frac{1}{2}n(n-1)\int_T \mathrm{Str}\left[(d_t\mathcal{A}_t)(d_t\mathcal{A}_t)\mathcal{F}^{n-2}_t\right], \tag{B.2}$$

where d_t is defined in eq. (A.3) and the integration domain T is bounded by the right triangle on the (t_1,t_2)-plane with vertices at $(0,0)$, $(1,0)$, and $(0,1)$. In writing this formula we have used a symmetrized trace defined as

$$\mathrm{Str}\left(\mathcal{O}_1,\ldots,\mathcal{O}_n\right) = \frac{1}{n!}\sum_{\sigma\in S_n}\mathrm{sgn}(\sigma)\mathrm{Tr}\left(\mathcal{O}_{\sigma(1)},\ldots,\mathcal{O}_{\sigma(n)}\right), \tag{B.3}$$

where $\mathrm{sgn}(\sigma)$ is the sign arising from the permutation of the forms inside the trace.

The Bardeen counterterm in four dimensions is obtained by setting $n=3$ in (B.2)

$$S_4(\mathcal{A}_{R,L},\mathcal{F}_{R,L}) = 3\int_T d^2t\,\mathrm{Str}\Big[(\mathcal{A}_R\mathcal{A}_L + \mathcal{A}_L\mathcal{A}_R)\mathcal{F}_t\Big], \tag{B.4}$$

where

$$\mathcal{F}_t = t_1\mathcal{F}_R + t_2\mathcal{F}_L + t_1(t_1-1)\mathcal{A}^2_R + t_2(t_2-1)\mathcal{A}^2_L + t_1t_2(\mathcal{A}_R\mathcal{A}_L + \mathcal{A}_L\mathcal{A}_R). \tag{B.5}$$

After evaluating the integral, the result is

$$S_4(\mathcal{A}_{R,L},\mathcal{F}_{R,L}) = \frac{1}{2}\mathrm{Tr}\Big[(\mathcal{A}_L\mathcal{A}_R+\mathcal{A}_R\mathcal{A}_L)(\mathcal{F}_R+\mathcal{F}_L) \\ +\mathcal{A}_R^3\mathcal{A}_L+\mathcal{A}_L^3\mathcal{A}_R+\frac{1}{2}\mathcal{A}_L\mathcal{A}_R\mathcal{A}_L\mathcal{A}_R\Big]. \tag{B.6}$$

Currents and energy-momentum tensor. We have seen that, upon the dimensional reduction carried out in section 4, the corresponding functionals depend on the KK-invariant gauge fields, $\mathcal{A}_0$ and $\boldsymbol{A}$, as well as on the field strength da of the Abelian KK field. Thus, upon a generic variation of the gauge field (5.3), we have the following definition of the consistent and covariant currents $\mathcal{J}_0$ and $\boldsymbol{J}$

$$\begin{aligned}\delta_B W[\mathcal{A}_0,\boldsymbol{A},da]_{\mathrm{inv}} &= \int_{S^{2n-3}} \mathrm{Tr}\left(\mathcal{B}_0\mathcal{J}_{0,\mathrm{cov}}+\boldsymbol{B}\boldsymbol{J}_{\mathrm{cov}}\right)+\text{bulk term},\\ \delta_B W[\mathcal{A}_0,\boldsymbol{A},da]_{\mathrm{anom}} &= \int_{S^{2n-3}} \mathrm{Tr}\left(\mathcal{B}_0\mathcal{J}_{0,\mathrm{cons}}+\boldsymbol{B}\boldsymbol{J}_{\mathrm{cons}}\right).\end{aligned} \tag{B.7}$$

The second equation is the variation of a local functional, already defined as an integral over the spatial manifold S^{2n-3}, so it only gives boundary contributions. Notice that $\mathcal{B}_0$ is a zero-form, whereas $\boldsymbol{B}$ is a one form. Thus, the currents in eq. (B.7) are respectively $(2n-3)$-forms ($\mathcal{J}_{0,\mathrm{cov}}$ and $\mathcal{J}_{0,\mathrm{cons}}$) and $(2n-4)$-forms ($\boldsymbol{J}_{\mathrm{cov}}$ and $\boldsymbol{J}_{\mathrm{cons}}$). The associated bona-fide currents are defined as the corresponding Hodge duals

$$\begin{aligned} j_{0,\mathrm{cov}} &= \star\mathcal{J}_{0,\mathrm{cov}},\\ \boldsymbol{j}_{\mathrm{cov}} &= \star\boldsymbol{J}_{\mathrm{cov}},\end{aligned} \tag{B.8}$$

and similarly for the consistent ones. Notice in particular that, whereas $\boldsymbol{j}_{\mathrm{cov}}$ and $\boldsymbol{j}_{\mathrm{cons}}$ are one-forms, $j_{0,\mathrm{cov}}$ and $j_{0,\mathrm{cons}}$ are zero-forms.

We can proceed along similar lines with the energy-momentum tensor. Using eq. (4.2), we have that the mixed components G_{0i} of the metric are

$$G_{0i}(\mathbf{x}) = -e^{2\sigma(\mathbf{x})}a_i(\mathbf{x}) \quad\Longrightarrow\quad \frac{\delta}{\delta G_{0i}} = -e^{-2\sigma}\frac{\delta}{\delta a_i}, \tag{B.9}$$

where the second identity has to be understood as acting on functionals independent of σ. When applying this expression to our effective actions, we have to keep in mind that the functionals do depend on a both explicitly, through da, and implicitly via the combination $\boldsymbol{A}=\boldsymbol{\mathcal{A}}-\mathcal{A}_0 a$ [cf. eq. (4.6)]. Lowering the zero index, we find the following expression for the mixed components of the energy-momentum tensor

$$\begin{aligned}\sqrt{-G}T_0{}^i &\equiv T_0 G_{00}\frac{\delta}{\delta G_{0i}}W[\mathcal{A}_0,\boldsymbol{A},da] = T_0\frac{\delta}{\delta a_i}W[\mathcal{A}_0,\boldsymbol{\mathcal{A}}-\mathcal{A}_0 a,da]\\ &= T_0\left[\left(\frac{\delta}{\delta a_i}\right)_{\boldsymbol{A}} - \mathcal{A}_0\frac{\delta}{\delta A_i}\right]W[\mathcal{A}_0,\boldsymbol{A},da],\end{aligned} \tag{B.10}$$

where $G\equiv -e^{2\sigma}\det g$ is the determinant of the stationary metric (4.2), T_0 is the equilibrium temperature, and in the second line the functional derivative with respect to a_i is taken

JHEP11(2018)076

at fixed $\boldsymbol{A}$. Applying this to the anomalous part of the partition function, we define the $(2n-4)$-form $\boldsymbol{T}$ from the linear variation of $W[\mathcal{A}_0, \boldsymbol{A}, da]_{\rm anom}$ with respect to a as

$$\delta W[\mathcal{A}_0, \boldsymbol{A}, da]_{\rm anom} = \int\limits_{S^{2n-3}} \delta a\, \boldsymbol{T}, \tag{B.11}$$

where $\boldsymbol{T}$ is dual to the one form $\sqrt{-G}T_{0i}dx^i = e^{\sigma}\sqrt{\det g}\, g_{ij}{T_0}^j dx^i$.

Gauge transformation of the Chern-Simons form. In order to construct the effective action for Goldstone bosons, it is necessary to have explicit expressions for the terms of the gauge transformation of the Chern-Simons form given in eq. (6.6). Using the homotopy formula given in (2.25), it is immediate to arrive at the following equation for the second term on the right-hand side

$$\omega^0_{2n-1}(dgg^{-1}, 0) = (-1)^{n+1}\frac{n!(n-1)!}{(2n-1)!}\mathrm{Tr}\left[(dgg^{-1})^{2n-1}\right]. \tag{B.12}$$

In the case of the third term, $\alpha_{2n-2}(\mathcal{A}, \mathcal{F}, g)$, a higher-order homotopy formula leads to [21]

$$\alpha_{2n-2}(\mathcal{A}, \mathcal{F}, g) = n(n-1)\int_T \mathrm{Str}\left[\mathcal{A}(dgg^{-1})\mathcal{F}_t^{n-2}\right], \tag{B.13}$$

where here $\mathcal{A}_t$ denotes the two-parameter family of connections

$$\mathcal{A}_t = t_1\mathcal{A} - t_2 dgg^{-1}, \tag{B.14}$$

and the integration domain T is the same right triangle as in (B.2).

C Trace identities for U(2)

In applying our expressions to hadronic fluids, it is useful to consider the two-flavor QCD case where the chiral group is $\mathrm{U}(2)_L\times\mathrm{U}(2)_R$. Here we list some relevant trace identities for the non-semisimple group U(2) leading to some of the expressions found in this paper. Taking the four generators $\{t_0, t_i\}$ $(i = 1, 2, 3)$ defined in eq. (4.30), the properties of the Pauli matrices imply the identity

$$t_j t_k = \frac{1}{4}\delta_{jk}\mathbb{1} + \frac{i}{2}\epsilon_{jk\ell}t_\ell. \tag{C.1}$$

For arbitrary p- and q-forms ω_p and η_q in the adjoint representation of U(2), we can immediately write

$$\mathrm{Tr}\,(\omega_p\eta_q) = \frac{1}{2}(\mathrm{Tr}\,\omega_p)(\mathrm{Tr}\,\eta_q) + \mathrm{Tr}\,(\widehat{\omega}_p\widehat{\eta}_q), \tag{C.2}$$

where the hat indicates the components of the p- and q-forms over the SU(2) factor of U(2)=U(1)×SU(2), namely

$$\widehat{\omega}_p \equiv \omega_p - \frac{1}{2}(\mathrm{Tr}\,\omega_p)\mathbb{1}. \tag{C.3}$$

JHEP11(2018)076

Using eq. (C.1), we can find the corresponding identity for the trace of three p-, q-, and r-forms

$$\begin{aligned}\mathrm{Tr}\,(\omega_p\eta_q\xi_r) = &-\frac{1}{2}(\mathrm{Tr}\,\omega_p)(\mathrm{Tr}\,\eta_q)(\mathrm{Tr}\,\xi_r) + \frac{1}{2}\Big[(\mathrm{Tr}\,\omega_p)\mathrm{Tr}\,(\eta_q\xi_r) \\ &+ (-1)^{pq}(\mathrm{Tr}\,\eta_q)\mathrm{Tr}\,(\omega_p\xi_r) + \mathrm{Tr}\,(\omega_p\eta_q)(\mathrm{Tr}\,\xi_r)\Big] + \mathrm{Tr}\,(\widehat{\omega}_p\widehat{\eta}_q\widehat{\xi}_r),\end{aligned} \tag{C.4}$$

where once again hatted quantities lie on the SU(2) factor. A similar calculation can be carried out for a trace with four adjoint differential forms to give

$$\begin{aligned}\mathrm{Tr}\,(\omega_p\eta_q\xi_r\zeta_s) = &\frac{3}{8}(\mathrm{Tr}\,\omega_p)(\mathrm{Tr}\,\eta_q)(\mathrm{Tr}\,\xi_r)(\mathrm{Tr}\,\zeta_s) - \frac{3}{8}\Big[(\mathrm{Tr}\,\omega_p)(\mathrm{Tr}\,\eta_q)\mathrm{Tr}\,(\xi_r\zeta_s) \\ &+ (-1)^{rs}(\mathrm{Tr}\,\omega_p)\mathrm{Tr}\,(\eta_q\zeta_s)(\mathrm{Tr}\,\xi_r) + (\mathrm{Tr}\,\omega_p)\mathrm{Tr}\,(\eta_q\xi_r)(\mathrm{Tr}\,\zeta_s) \\ &+ (-1)^{pq+rs}(\mathrm{Tr}\,\eta_q)\mathrm{Tr}\,(\omega_p\zeta_s)(\mathrm{Tr}\,\xi_r) + (-1)^{pq}(\mathrm{Tr}\,\eta_q)\mathrm{Tr}\,(\omega_p\xi_r)(\mathrm{Tr}\,\zeta_s) \\ &+ \mathrm{Tr}\,(\omega_p\eta_q)(\mathrm{Tr}\,\xi_r)(\mathrm{Tr}\,\zeta_s)\Big] + \frac{1}{2}\Big[(\mathrm{Tr}\,\omega_p)\mathrm{Tr}\,(\eta_q\xi_r\zeta_s) \\ &+ (-1)^{pq}(\mathrm{Tr}\,\eta_q)\mathrm{Tr}\,(\omega_p\xi_r\zeta_s) + (-1)^{rs}\mathrm{Tr}\,(\omega_p\eta_q\zeta_s)(\mathrm{Tr}\,\xi_r) \\ &+ \mathrm{Tr}\,(\omega_p\eta_q\xi_r)(\mathrm{Tr}\,\zeta_s)\Big] + \mathrm{Tr}\,(\widehat{\omega}_p\widehat{\eta}_q\widehat{\xi}_r\widehat{\zeta}_s).\end{aligned} \tag{C.5}$$

As a further useful example, we compute the trace

$$\mathrm{Tr}\,\omega^5 = \mathrm{Tr}\,\left[\widehat{\omega} + \frac{1}{2}(\mathrm{Tr}\,\omega)\mathbb{1}\right]^5 = \mathrm{Tr}\,\widehat{\omega}^5, \tag{C.6}$$

with ω a one form. To write the last equality we have taken into account that $\mathrm{Tr}\,\omega$ is an "Abelian" one-form and therefore $(\mathrm{Tr}\,\omega)^n = 0$ for $n \geq 2$. Moreover, all terms linear in $\mathrm{Tr}\,\omega$ vanish, since $\mathrm{Tr}\,\widehat{\omega}^4 = 0$, which follows from the cyclicity property. Finally, applying the identity

$$\mathrm{Tr}\,(t_jt_kt_\ell t_mt_n) = \frac{i}{16}\Big(\delta_{jk}\epsilon_{\ell mn} + \delta_{\ell m}\epsilon_{jkn} + \delta_{kn}\epsilon_{j\ell m} + \delta_{jn}\epsilon_{k\ell m}\Big), \tag{C.7}$$

we see that, given that ω is a one-form,

$$\mathrm{Tr}\,\omega^5 = 0. \tag{C.8}$$

Finally, we evaluate the traces $\mathrm{Tr}\,(u\omega^3)$ and $\mathrm{Tr}\,(u\omega^4)$, with u and ω zero- and one-forms respectively. In the first case

$$\mathrm{Tr}\,(u\,\omega^3) = \mathrm{Tr}\,(u\,\widehat{\omega}^3) + \frac{1}{2}\mathrm{Tr}\,(u\,\widehat{\omega}^2)\mathrm{Tr}\,\omega, \tag{C.9}$$

whereas using

$$\mathrm{Tr}\,(t_it_jt_kt_\ell) = \frac{1}{8}\Big(\delta_{ij}\delta_{k\ell} - \delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}\Big), \tag{C.10}$$

we find that

$$\mathrm{Tr}\,(u\,\omega^3) = \frac{1}{2}(\mathrm{Tr}\,u)\mathrm{Tr}\,\widehat{\omega}^3 + \frac{1}{2}\mathrm{Tr}\,(u\,\widehat{\omega}^2)\mathrm{Tr}\,\omega. \tag{C.11}$$

As a final step, we can express the right-hand side of this equation in terms of hatless quantities. Using eq. (C.4), we find $\mathrm{Tr}\,\widehat{\omega}^3 = \mathrm{Tr}\,\omega^3$. In addition, the presence of $\mathrm{Tr}\,\omega$ in the

JHEP11(2018)076

second term of the right-hand side of (C.11) allows to remove the hat on the prefactor, to finally write

$$\mathrm{Tr}\,(u\,\omega^3) = \frac{1}{2}(\mathrm{Tr}\,u)\mathrm{Tr}\,\omega^3 + \frac{1}{2}\mathrm{Tr}\,(u\,\omega^2)\mathrm{Tr}\,\omega. \tag{C.12}$$

A similar computation leads to

$$\mathrm{Tr}\,(u\,\omega^4) = 0. \tag{C.13}$$

References

[1] B. Zumino, *Chiral anomalies and differential geometry*, in *Relativity, groups and topology*, Elsevier, The Netherlands, (1983).

[2] L. Álvarez-Gaumé, *An introduction to anomalies*, in *Fundamental problems of gauge field theory*, Plenum Press, U.S.A., (1985).

[3] R.A. Bertlmann, *Anomalies in quantum field theory*, Oxford University Press, Oxford, U.K., (1996).

[4] K. Fujikawa and H. Suzuki, *Path integrals and quantum anomalies*, Oxford University Press, Oxford, U.K., (2004).

[5] J.A. Harvey, *TASI* 2003 *lectures on anomalies*, hep-th/0509097 [INSPIRE].

[6] D.T. Son and P. Surowka, *Hydrodynamics with triangle anomalies*, *Phys. Rev. Lett.* **103** (2009) 191601 [arXiv:0906.5044] [INSPIRE].

[7] Y. Neiman and Y. Oz, *Relativistic hydrodynamics with general anomalous charges*, *JHEP* **03** (2011) 023 [arXiv:1011.5107] [INSPIRE].

[8] A.V. Sadofyev and M.V. Isachenkov, *The chiral magnetic effect in hydrodynamical approach*, *Phys. Lett.* **B 697** (2011) 404 [arXiv:1010.1550] [INSPIRE].

[9] V.P. Kirilin, A.V. Sadofyev and V.I. Zakharov, *Chiral vortical effect in superfluid*, *Phys. Rev.* **D 86** (2012) 025021 [arXiv:1203.6312] [INSPIRE].

[10] K. Fukushima, *Views of the chiral magnetic effect*, *Lect. Notes Phys.* **871** (2013) 241 [arXiv:1209.5064] [INSPIRE].

[11] V.I. Zakharov, *Chiral magnetic effect in hydrodynamic approximation*, in *Strongly interacting matter in magnetic fields*, D. Kharzeev, K. Landsteiner, A. Schmitt and H.-U. Yee eds., Springer Verlag, Germany, (2013), pg. 295 [arXiv:1210.2186] [INSPIRE].

[12] D.E. Kharzeev, J. Liao, S.A. Voloshin and G. Wang, *Chiral magnetic and vortical effects in high-energy nuclear collisions — a status report*, *Prog. Part. Nucl. Phys.* **88** (2016) 1 [arXiv:1511.04050] [INSPIRE].

[13] K. Landsteiner, *Notes on anomaly induced transport*, *Acta Phys. Polon.* **B 47** (2016) 2617 [arXiv:1610.04413] [INSPIRE].

[14] J. Gooth et al., *Experimental signatures of the mixed axial-gravitational anomaly in the Weyl semimetal NbP*, *Nature* **547** (2017) 324 [arXiv:1703.10682] [INSPIRE].

[15] N. Banerjee, J. Bhattacharya, S. Bhattacharyya, S. Jain, S. Minwalla and T. Sharma, *Constraints on fluid dynamics from equilibrium partition functions*, *JHEP* **09** (2012) 046 [arXiv:1203.3544] [INSPIRE].

[16] K. Jensen, M. Kaminski, P. Kovtun, R. Meyer, A. Ritz and A. Yarom, *Towards hydrodynamics without an entropy current*, *Phys. Rev. Lett.* **109** (2012) 101601 [arXiv:1203.3556] [INSPIRE].

[17] K. Jensen, R. Loganayagam and A. Yarom, *Thermodynamics, gravitational anomalies and cones*, *JHEP* **02** (2013) 088 [arXiv:1207.5824] [INSPIRE].

[18] K. Jensen, R. Loganayagam and A. Yarom, *Anomaly inflow and thermal equilibrium*, *JHEP* **05** (2014) 134 [arXiv:1310.7024] [INSPIRE].

[19] J. Wess and B. Zumino, *Consequences of anomalous Ward identities*, *Phys. Lett.* **B 37** (1971) 95 [INSPIRE].

[20] E. Witten, *Global aspects of current algebra*, *Nucl. Phys.* **B 223** (1983) 422 [INSPIRE].

[21] J.L. Mañes, *Differential geometric construction of the gauged Wess-Zumino action*, *Nucl. Phys.* **B 250** (1985) 369 [INSPIRE].

[22] F.M. Haehl, R. Loganayagam and M. Rangamani, *Effective actions for anomalous hydrodynamics*, *JHEP* **03** (2014) 034 [arXiv:1312.0610] [INSPIRE].

[23] S. Lin, *On the anomalous superfluid hydrodynamics*, *Nucl. Phys.* **A 873** (2012) 28 [arXiv:1104.5245] [INSPIRE].

[24] V.P. Nair, R. Ray and S. Roy, *Fluids, anomalies and the chiral magnetic effect: a group-theoretic formulation*, *Phys. Rev.* **D 86** (2012) 025012 [arXiv:1112.4022] [INSPIRE].

[25] S. Bhattacharyya, S. Jain, S. Minwalla and T. Sharma, *Constraints on superfluid hydrodynamics from equilibrium partition functions*, *JHEP* **01** (2013) 040 [arXiv:1206.6106] [INSPIRE].

[26] M. Lublinsky and I. Zahed, *Anomalous chiral superfluidity*, *Phys. Lett.* **B 684** (2010) 119 [arXiv:0910.1373] [INSPIRE].

[27] Y. Neiman and Y. Oz, *Anomalies in superfluids and a chiral electric effect*, *JHEP* **09** (2011) 011 [arXiv:1106.3576] [INSPIRE].

[28] S. Lin, *An anomalous hydrodynamics for chiral superfluid*, *Phys. Rev.* **D 85** (2012) 045015 [arXiv:1112.3215] [INSPIRE].

[29] C. Hoyos, B.S. Kim and Y. Oz, *Odd parity transport in non-Abelian superfluids from symmetry locking*, *JHEP* **10** (2014) 127 [arXiv:1404.7507] [INSPIRE].

[30] A. Jain, *Theory of non-Abelian superfluid dynamics*, *Phys. Rev.* **D 95** (2017) 121701 [arXiv:1610.05797] [INSPIRE].

[31] J.L. Mañes, E. Megías, M. Valle and M.Á. Vázquez-Mozo, *Anomalous currents of nuclear matter in the chiral limit*, to appear.

[32] J. Mañes, R. Stora and B. Zumino, *Algebraic study of chiral anomalies*, *Commun. Math. Phys.* **102** (1985) 157 [INSPIRE].

[33] M. Nakahara, *Geometry, topology and physics*, 2nd edition, Taylor & Francis, U.K., (2003).

[34] L. Álvarez-Gaumé and P.H. Ginsparg, *The topological meaning of non-Abelian anomalies*, *Nucl. Phys.* **B 243** (1984) 449 [INSPIRE].

JHEP11(2018)076

[35] W.A. Bardeen, *Anomalous Ward identities in spinor field theories*, *Phys. Rev.* **184** (1969) 1848 [INSPIRE].

[36] R. Kaiser, *Anomalies and WZW term of two flavor QCD*, *Phys. Rev.* **D 63** (2001) 076010 [hep-ph/0011377] [INSPIRE].

[37] W.A. Bardeen and B. Zumino, *Consistent and covariant anomalies in gauge and gravitational theories*, *Nucl. Phys.* **B 244** (1984) 421 [INSPIRE].

[38] C.G. Callan Jr. and J.A. Harvey, *Anomalies and fermion zero modes on strings and domain walls*, *Nucl. Phys.* **B 250** (1985) 427 [INSPIRE].

[39] H.E. Haber and H.A. Weldon, *Finite temperature symmetry breaking as Bose-Einstein condensation*, *Phys. Rev.* **D 25** (1982) 502 [INSPIRE].

[40] D. Yamada and L.G. Yaffe, *Phase diagram of $N = 4$ super-Yang-Mills theory with R-symmetry chemical potentials*, *JHEP* **09** (2006) 027 [hep-th/0602074] [INSPIRE].

[41] K. Fukushima, D.E. Kharzeev and H.J. Warringa, *The chiral magnetic effect*, *Phys. Rev.* **D 78** (2008) 074033 [arXiv:0808.3382] [INSPIRE].

[42] K. Landsteiner, E. Megías and F. Peña Benítez, *Anomalous transport from Kubo formulae*, *Lect. Notes Phys.* **871** (2013) 433 [arXiv:1207.5808] [INSPIRE].

[43] H. Kawai and S.-H. Henry Tye, *Chiral anomalies, effective Lagrangian and differential geometry*, *Phys. Lett.* **B 140** (1984) 403 [INSPIRE].

[44] K. Fukushima and K. Mameda, *Wess-Zumino-Witten action and photons from the chiral magnetic effect*, *Phys. Rev.* **D 86** (2012) 071501 [arXiv:1206.3128] [INSPIRE].

[45] D.T. Son and M.A. Stephanov, *Axial anomaly and magnetism of nuclear and quark matter*, *Phys. Rev.* **D 77** (2008) 014021 [arXiv:0710.1084] [INSPIRE].

[46] A. Andrianov, V. Andrianov and D. Espriu, *Chiral imbalance in QCD*, *EPJ Web Conf.* **138** (2017) 01007 [INSPIRE].

[47] L. Álvarez-Gaumé and E. Witten, *Gravitational anomalies*, *Nucl. Phys.* **B 234** (1984) 269 [INSPIRE].

[48] L. Álvarez-Gaumé and P.H. Ginsparg, *The structure of gauge and gravitational anomalies*, *Annals Phys.* **161** (1985) 423 [*Erratum ibid.* **171** (1986) 233] [INSPIRE].

[49] Y. Yin and J. Liao, *Hydrodynamics with chiral anomaly and charge separation in relativistic heavy ion collisions*, *Phys. Lett.* **B 756** (2016) 42 [arXiv:1504.06906] [INSPIRE].

[50] X.-G. Huang, *Electromagnetic fields and anomalous transports in heavy-ion collisions — a pedagogical review*, *Rept. Prog. Phys.* **79** (2016) 076302 [arXiv:1509.04073] [INSPIRE].

[51] Y. Sun, C.M. Ko and F. Li, *Anomalous transport model study of chiral magnetic effects in heavy ion collisions*, *Phys. Rev.* **C 94** (2016) 045204 [arXiv:1606.05627] [INSPIRE].

[52] S. Shi, Y. Jiang, E. Lilleskov and J. Liao, *Anomalous chiral transport in heavy ion collisions from anomalous-viscous fluid dynamics*, *Annals Phys.* **394** (2018) 50 [arXiv:1711.02496] [INSPIRE].

JHEP11(2018)076