Macroscopic Limits, Self-Organization and Stability in Systems with Singular Interactions Arising from Hydrodynamics and Life Sciences

Presented by

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Para mis padres, Fernando y Manoli,
y para el resto de mi familia.

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Pero no es ése el objetivo de estas líneas, pues el pasado solo es un punto de partida, uno de todos los muchos posibles. Lo más importante de éste, es que me ha dado la oportunidad de comenzar. Por eso, quiero realmente dedicar estas palabras como agradecimiento a todas aquellas personas que de alguna forma me han apoyado durante alguno de los pasos de éste, mi camino. Porque como dice el poema:

Caminante, son tus huellas
el camino y nada más;
caminante, no hay camino,
se hace camino al andar.

Al andar se hace el camino,
y al volver la vista atrás
se ve la senda que nunca
se ha de volver a pisar.

Caminante no hay camino
sino estelas en la mar.

Antonio Machado.
Proverbios y cantares.

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Dissertation Summary

This dissertation is centered around the analysis of non-linear partial differential equations that arise from models in physics, mathematical biology, social sciences and neuroscience. Specifically, we address a particular family of models that have been coined in the literature with the name of "collective dynamics models". The main idea is that from basic rules stating how a system of particles interact, the population often has the ability to self-organize collectively as a unique entity and it amounts to different emergent phenomena depending on the particular context., e.g., swarming, flocking, schooling, synchronization, etc.

Although these models appear in completely different settings, what make them so special from a mathematical point of view is the fact that their structural resemblance allow us to tackle them with common abstract mathematical tools. Indeed many relevant improvements and mathematical methods have emerged from this interface as we tackle the different effects that we can encounter (e.g., kinetic theory, stochastic equations, mean field limits, propagation of chaos, hydrodynamic limits, potential theory, optimal transport, etc).

We will draw our attention specially to kinetic collective dynamics models where interactions are mediated by singular kernels. This adds new analytical challenges that cannot be solved with standard methods and require novel mathematical tools. In particular, some of the problems that we will study in this thesis are: well-posedness, stability, scaling limits connecting different scales of description (microscopic, mesoscopic and macroscopic), asymptotics of solutions, etc.

Although equations arising in fluid mechanics can be considered as a special case of collective dynamics, we will treat them as a complementary topic. Namely, most of the collective dynamics models have well-defined associated macroscopic versions governed by a system of conservation laws similar to Euler or Navier–Stokes equations. This suggests that the study of fluid mechanics can shed light on the understanding of collective dynamics. In this thesis, we focus on the study of stability of a particular class of stationary solutions to the Euler equation that are called Beltrami fields. As it will be clarified later, such solutions have proved extremely relevant in the Lagrangian theory of turbulence.

List of works contained in this thesis

- Chapter 1: article [4] in collaboration with Giacomo Albi, Nicolla Bellomo, Luisa Fermo, Seung-Yeal Ha, Jeongho Kim and Juan Soler, to appear in *Mathematical Models and Methods in Applied Sciences*.

- Chapter 2: article [255], in collaboration with Juan Soler, published in *Mathematical Models and Methods in Applied Sciences*.

- Chapter 3: article [241], in collaboration with Jinyeong Park and Juan Soler, submitted for publication.
- Chapter 4: article [254], submitted for publication.
- Chapter 5: article [222], in collaboration with Javier Morales, submitted for publication.
- Chapter 6: article [118], in collaboration with Alberto Enciso and Juan Soler, published in *Communications in Mathematical Physics*.

Further works in preparation

- *Hydrodynamic limits of the thermomechanical Cucker–Smale model with fast and slow temperature relaxation*, in collaboration with Jeongho Kim and Juan Soler.
- *Modeling of morphogen transportation along moving cytonemes in Droshophila melanogaster*, in collaboration with Adrián Aguirre-Tamaral, Manuel Cambón, Isabel Guerrero and Juan Soler.

How to read this thesis

Each chapter of this thesis is self-contained and can be read separately. Indeed, the main concepts and equations that each part refers to are introduced in the corresponding chapter. However, we suggest the reader to follow the logical order in which the results are presented since some relations are drawn between the different parts as we read these pages.

For clarity of the presentation, we have decided to divide the contents into distinguished parts, that the reader may want to swap if necessary. Specifically, in Chapter 1 we review the preceding literature and we state the main problems of this thesis. In Part I (Chapter (2)) we focus on the study of the Cucker–Smale model of flocking. Part II (Chapters 3–5) contains several different contributions to the study of the Kuramoto model of coupled oscillators and some related versions. Part III (Chapter 6) centers on the analysis of existence and stability results in fluid mechanics and Part IV (Chapter 7) collects a list of further works in preparation that have emerged as a consequence of this thesis. We end that part with some conclusions and perspectives for future work.

For an easier readability, we recall the main necessary notation to be used throughout this dissertation in Section *Conventions and notation*. Also, to alleviate the presentation of the results, we include further important concepts and tools in Appendices A–H. Some of the results therein are classical whilst some others are original contributions of the author that will come to play along the reading.

Summary of the thesis

In the sequel, we outline the contents of the chapters that defines this thesis.

In Chapter 1 we introduce the state of the art for the main subjects within this thesis. Accordingly, we split it into two sections. The first one addresses the more recent contributions on the study of ODE and PDE-based models for collective dynamics in life sciences at each of their scaly of description (microscopic, mesoscopic and macroscopic), with additional emphasis on two prototype models: the Cucker–Smale model and the Kuramoto model. The second section focuses on the current studies of stability of stationary solutions in fluid mechanics, with special attention to the problem of existence of linked and knotted vortex structures of complicated topology in hydrodynamics. In this chapter, we state the main problems of this thesis and we compare them with preceding literature.

In Chapter 2 we derive some hyperbolic hydrodynamic limit of vanishing inertia type for the kinetic Cucker–Smale model towards singular weights. This rigorously justifies the macroscopic system

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) = 0, \\ \phi_0 * (\rho u) - (\phi_0 * \rho) u - u - \nabla \psi = 0, \\ \rho(0, \cdot) = \rho^0, \end{cases}$$

where $\rho = \rho(t, x)$ and $u = u(t, x)$ respectively represent the density and velocity field of the population, $F = -\nabla \psi$ is some external force and $\phi_0(r) = r^{-2\alpha}$ is a singular influence function. Such system can be regarded as a law of conservation of mass coupled with an implicit integral equation for the velocity field, that is governed by a commutator of singular integrals. First, we address the asymptotic limit for different appropriate scalings of the associated kinetic equation. This produces a method to construct weak measure-valued solutions to the above macroscopic system for $\alpha \in (0, \frac{1}{2})$. Second, we also introduce a local-in-time existence results in higher regularity spaces. Let us remark that this is one of the first results in the literature concerning hydrodynamic limits of the Cucker–Smale model with singular influence function, see also [126, 127, 186].

In Chapter 3 we introduce a new version of the Kuramoto model [193, 196] of N coupled oscillators with applications to neuronal synchronization. Specifically, we substitute uniform weights by phase-dependent weights that exhibit singularities at configurations with same phase and takes the following form

$$\begin{cases} \dot{\theta}_i = \Omega_i + \dfrac{K}{N} \displaystyle\sum_{j=1}^{N} h(\theta_j - \theta_i), \\ \theta_i(0) = \theta_{i,0}, \end{cases} \quad \text{where } h(\theta) := \frac{\sin \theta}{|\theta|_o^{2\alpha}},$$

for $i = 1, \ldots, N$ and $\alpha \in (0, 1)$. We justify such a model from first principles via a singular fast-learning approximation on a learning rule of weights that is governed by a plasticity function verifying Hebb's rule [166]. Such many-particle system shares some similarities with the Cucker–Smale model with singular influence function, see [226, 244, 245]. We elaborate on them along the Chapter. First, we derive a suitable well-posedness theory of global solutions after eventual collisions. The presence of singularities represents a clear obstruction to apply the classical theory and we decide to tackle the problem via the concept of Filippov solutions. We remark here that the interaction kernel is merely Hölder continuous and it represents severe problems with regards to uniqueness. To solve this fateful lack of regularity, we introduce a one-sided Lipschitz property, that arises from the structure of the kernel and is sufficient to derive uniqueness. Second, we explore emergence of synchronization. Interestingly, new phenomena is observed since oscillators have the ability of synchronizing in finite time. We provide sufficient conditions on the system's initial configuration that guarantee the global phase-synchronization in finite time. Indeed, we derive useful characterizations of the clustering and sticking phenomena of oscillators of each of the different regimes of singularity in the model, that open the scope in the paradigm of synchronization phenomena.

In Chapter 4 we present the associated kinetic counterpart of the above agent-based model in Chapter 3. Such equation consists in a Vlasov-type kinetic equation with nonlocal interactions described in terms of the interaction kernel h, i.e.,

$$\begin{cases} \dfrac{\partial f}{\partial t} + \dfrac{\partial}{\partial \theta}\left(\Omega f - K(h * \rho) f\right) = 0, \\ f(t,\theta,\Omega) = f(t,\theta+2\pi,\Omega), \\ f(0,\cdot,\cdot) = f_0, \end{cases}$$

where $f = f(t,\theta,\Omega)$ is the distribution function and $\rho = \rho(t,\theta)$ is the phase density. We remark that the interaction kernel is not Lipschitz and its study is a nontrivial matter. The contents are sorted according to different aspects of the macroscopic system. First, we propose a well-posedness theory of global weak measure-valued solutions that remains valid after collisions, for all the values $\alpha \in (0, \frac{1}{2}]$. We conduct our proof based on the existence of Filippov flows, see [251]. Second, we address the stability (or Dobrushin-type) estimates of the kinetic equation with respect to two different transportation distances: the quadratic Wasserstein distance and a new fiberwise quadratic Wasserstein distance that we propose here and has proved specially well adapted to the nonlinear problem. Again, the one-sided Lipschitz property of the kernel h is the heart of the matter in these inequalities. We apply such stability results to derive two important consequences. On the one hand, we obtain uniqueness of the above Cauchy problem for general initial data. On the other hand, we quantify the mean field limit as $N \to \infty$ of the particle system in the above Chapter towards the kinetic equation. Finally, we combine the preceding study of synchronization for the agent-based system with the above stability estimates to extend an analogue finite time phase synchronization of measure-valued solutions under certain assumptions of the support of the initial data. The case $\alpha \in (\frac{1}{2}, 1)$ is treated separately using the ideas in the above Chapter 2. We remark that a close approximation was conducted in [64, 67] for the aggregation equation in the Euclidean space. However, we do not use any gradient flow structure of our system.

In Chapter 5 we return to the original Kuramoto model, that corresponds to $h = \sin$. More specifically, we consider its kinetic counterpart, also called Kuramoto–Sakaguchi equation:

$$\begin{cases} \dfrac{\partial f}{\partial t} + \dfrac{\partial}{\partial \theta}\left(\Omega f - K(\sin * \rho) f\right) = 0, \\ f(t,\theta,\Omega) = f(t,\theta+2\pi,\Omega), \\ f(0,\cdot,\cdot) = f_0. \end{cases}$$

The goal of this part is to analyze the long-time asymptotics of the problem for general initial data $f_0 \in C^1(\mathbb{T} \times \mathbb{R})$. To the best of our knowledge, this is the first result in the literature where general initial data are considered (in contrast to preceding results with phase supports confined to the half circle) and explicit rates of convergence are computed. Historically, the main obstructions to apply classical theories are of two types. On the one hand, the heterogeneity introduced by the variable Ω implies that Kuramoto–Sakaguchi equation is not a Wasserstein gradient flow over the space of probability measures (in the terms of Otto calculus [237]). On the other hand, for identical oscillators the variable Ω can be neglected and the equation becomes a formal gradient flow but, unfortunately, the energy functional does not satisfy the necessary convexity assumptions. Our proof joins two different components. Firstly, we quantify a finite time that the system takes to enter a regime in which mass concentrates exponentially fast in a neighborhood of the average phase. To do so, we derive a system of differential inequalities for some useful quantities of the system that quantifies four different principles: soft entropy production, instability of equilibria with antipodal mass, emergence of attractor sets of characteristics and an accurate control on sliding L^2 norms along sets that evolve along the flow. Such a result extends to Kuramoto–Sakaguchi the heuristics in [104] for the Boltzmann equation. Secondly, once the system has entered into such a concentration regime, we derive generalized log-Sobolev and Talagrand inequalities (see [238] for the Fokker–Planck equation) for an appropriately defined dissipation of the system and the fiberwise distance in the above Chapter that allow concluding the convergence of the system to the global equilibrium.

In Chapter 6 we move to a different topic in the setting of fluid dynamics. Specifically, we face the study of stationary solutions of the incompressible Euler equation for ideal fluids in thee dimensions and we focus on a particularly interesting class of stationary solutions, that is, (generalized) Beltrami fields

$$\begin{cases} \operatorname{curl} u = fu, \\ \operatorname{div} u = 0, \end{cases}$$

where u represents the velocity field in $\mathbb{R}^3$ and f is any function that we call the proportionality factor. Their relevance arises in the Lagrangian theory of turbulence since they have important implications due to Arnold's structure theorem. Roughly speaking, it says that any incompressible fluid has to be either laminar (in the sense that the full space is smoothly foliated by curves coming from a first integral of the velocity field) or a Beltrami field. They have recently gained attention due to [115, 116], where they were used to solve the ancient Kelvin's conjecture of knotted and linked vortex structures in incompressible fluids. Specifically, the authors constructed (strong) Beltrami field with constant factor $f = \lambda$ that realize any arbitrary and topologically complicated ensemble of linked and knotted vortex lines and tubes. However, as depicted in [117] Beltrami fields are rare since f must verify some very specific geometrical constraints. Then, it stand to reason that it is hard to perturb a factor f so that the above constraints are met and new non-trivial Beltrami fields can be constructed. Our goal here is to address such lack of full stability. Namely, we propose two partial stability results that provide very specific ways to perturb f so that they lie within the constraints found in [117] and they have associated non-trivial generalized Beltrami fields arbitrarily close to the initial one. In the first one, only constant factors $f = \lambda$ can be perturbed in the complementary of any arbitrarily small ball. The second one applies to general factors in small enough balls around non-stagnation points of the fluid. Interestingly, the new Beltrami fields that we construct exhibit complicated vortex structures like in the preceding literature.

The last Chapter 7 addresses additional related works in progress that have emerged from this thesis:

• In Section 7.1 we extend the results in Chapter 2 to derive rigorous hyperbolic hydrodynamic limits of vanishing inertia type for some versions of the kinetic thermomechanical Cucker–Smale model [153, 161], whose discrete version takes the following form

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ \dfrac{dv_i}{dt} = \dfrac{K_v}{N} \displaystyle\sum_{j=1}^{N} \phi(|x_i - x_j|) \left(\dfrac{v_j}{T_j} - \dfrac{v_i}{T_i} \right), \\ \dfrac{dT_i}{dt} = \dfrac{K_T}{N} \displaystyle\sum_{j=1}^{N} \zeta(|x_i - x_j|) \left(\dfrac{1}{T_i} - \dfrac{1}{T_j} \right), \end{cases}$$

for any $i = 1, \ldots, N$. Here, the new variable T_i represents the "temperature" of the i-th particle and it is regarded as the internal energy of each particle. Under suitable scaling, we deduce two possible macroscopic limits that we call the slow and fast temperature relaxation regimes.

• In Section 7.2 we propose a mathematical model to explain some important process in developmental biology. Specifically, we address the study of cell-comunication mechanisms in *Drosophila melanogaster* which take place during the particular signalling pathway that is mediated by *Hh* morphogen and its target gene *Ci* (responsible for the correct formation of wins, among others). Very recently, experimental research has elucidated the basis of such a mechanism, see [34, 141]. Nowadays, the most likely hypothesis is that it is mediated by signalling filipodia (or cytonemes) that emanate from cell membranes. More specifically, proteins do no spread randomly throughout the full extra-cellular matrix, but its propagation is confined along such specific paths. Such filament-like structures are dynamic and both *Hh* emitting and receiving cells extend their cytonemes, that grow, retract and orient in an ordered manner so that after they connect in a sort of synapse, proteins are transferred from cell to cell. This microscopic dynamics generates the observed concentration gradient of morphogen which moves from emitting cells to receiving cells. In this section, we derive a mathematical model that joins three different components. First, we design a model based on first principles for the mechanics of each cytoneme, that is regarded as a mathematical curve with ability to grow and retract in a plastic but non-elastic way. Second, we describe the orientation potential that acts on each cytoneme and is responsible for the guidance of filopodia to the appropriate contact sites. Experimental evidence has shown that the involved proteins in such a mechanisms are *Ihog, Dally* and *Dlp*. Finally, we introduce a model for the morphogen transference after synapses that describes the propagation of proteins along moving cytonemes via flux-limited mechanisms on one-dimensional manifolds.

Resumen en castellano

Esta tesis se centra en el análisis de ecuaciones en derivadas parciales no lineales que surgen en modelos de física, biología matemática, ciencias sociales o neurociencia. En particular, estudiamos una familia de modelos, que en la literatura reciben el nombre de "modelos de dinámica colectiva". La idea fundamental es que partiendo de reglas sencillas que describen las interacciones entre partículas del sistema, la población tiene a menudo capacidad de organizarse colectivamente como si de un único ente se tratara. Esto lleva a diferentes tipos de fenómenos emergentes que dependerán del contexto en concreto, por ejemplo, "swarming", "flocking", "schooling", sincronización, etc.

A pesar de que estos modelos pueden aparecen en escenarios completamente diferentes, lo que los hace tan especiales desde el punto de vista matemático es que su parecido estructural nos permite atacarlos por medio de herramientas matemáticas abstractas comunes. De hecho, varios desarrollos importantes de las matemáticas han surgido de esta interfaz a medida que uno se enfrenta a los diversos componentes que estos modelos conllevan (por ejemplo, teoría cinética, ecuaciones estocásticas, límites de campo medio, propagación de caos, límites hidrodinámicos, teoría del potencial, transporte óptimo, etc).

En esta tesis, prestaremos especialmente atención a modelos cinéticos de dinámica colectiva donde las interacciones vienen descritas por medio de núcleos singulares. Esto genera nuevos retos matemáticos que no se pueden resolver mediante los métodos estándar y requieren el desarrollo de herramientas matemáticas novedosas. En particular, algunos de los problemas que abordaremos en esta tesis son: buen planteamiento, estabilidad, limites de escala conectando los diferentes niveles de descripción de los modelos (microscópico, mesoscópico y macrocópico), comportamiento asintótico de las soluciones, etc.

A pesar de que las ecuaciones de la mecánica de fluidos pueden verse como casos especiales de dinámica colectiva, trataremos estas últimos como un tema complementario en esta tesis. Concretamente, la mayoría de modelos de dinámica colectiva tienen versiones macroscópicas bien definidas que obedecen sistemas de leyes de conservación similares a las ecuaciones de Euler y Navier–Stokes. Esto sugiere que el estudio de mecánica de fluidos puede arrojar luz en la comprensión de dinámicas colectivas más generales. En esta tesis nos enfocaremos en el estudio de estabilidad de una clase de soluciones estacionarias especiales de las ecuaciones de Euler que reciben el nombre de campos de Beltrami. Como veremos más adelante, estas soluciones son de gran relevancia en la teoría lagrangiana de la turbulencia.

Lista de artículos de la tesis

- Capítulo 1: artículo [4] en colaboracón con Giacomo Albi, Nicolla Bellomo, Luisa Fermo, Seung-Yeal Ha, Jeongho Kim y Juan Soler, aceptado para publicación en *Mathematical Models and Methods in Applied Sciences*.

- Capítulo 2: artículo [255], en colaboración con Juan Soler, publicado en *Mathematical Models and Methods in Applied Sciences*.
- Capítulo 3: artículo [241], en colaboración con Jinyeong Park y Juan Soler, sometido para publicación.
- Capítulo 4: artículo [254], sometido a publicación.
- Capítulo 5: artículo [222], en colaboración con Javier Morales, sometido a publicación.
- Capítulo 6: artículo [118], en colaboración con Alberto Enciso y Juan Soler, publicado en *Communications in Mathematical Physics*.

Otros trabajos en preparación

- *Hydrodynamic limits of the thermomechanical Cucker–Smale model with fast and slow temperature relaxation*, en colaboración con Jeongho Kim y Juan Soler.
- *Modeling of morphogen transportation along moving cytonemes in Droshophila melanogaster*, en colaboración con Adrián Aguirre-Tamaral, Manuel Cambón, Isabel Guerrero y Juan Soler.

Cómo leer esta tesis

Cada capítulo de la tesis es autocontenido y puede ser leído de forma separada. De hecho, los conceptos principales y ecuaciones a los que hace referencia cada parte están presentados en los correspondientes capítulos. Sin embargo, sugerimos que el lector siga el orden lógico en el que se presentan los resultados dado que a lo largo de la lectura mostramos algunas relaciones entre las diferentes partes de esta tesis.

Para mayor claridad de la presentación, hemos decidido dividir los contenidos en diferentes partes, que el lector puede intercambiar si es necesario. Concretamente, en Capítulo 1 mostramos un repaso de la literatura previa y del estado del arte acerca de los problemas principales de la tesis. En Parte I (Capítulo (2)) nos centramos en el estudio del modelo de Cucker–Smale. Parte II (Capítulos 3–5) contiene varias contribuciones al estudio del modelo de Kuramoto para osciladores acoplados y algunas variantes relacionadas. Parte III (Capítulo 6) se enfoca en el análisis de resultados de existencia y estabilidad en mecánica de fluidos y finalmente, Parte IV (Capítulo 7) recoge una breve lista de otros trabajos en proceso de redacción que han surgido como consecuencia de esta tesis. Concluimos dicha parte con algunas conclusiones y trabajos futuros.

Para una mejor lectura, recordamos la notación necesaria a lo largo de esta tesis en Sección *Conventions and notation*. También, para aligerar la presentación de los resultados, incluimos el resto de conceptos y herramientas necesarias en Apéndices A–H. Algunos de los resultados ahí presentados son clásicos, mientras que otros son aportaciones originales del autor que serán necesarios en ciertos puntos a lo largo de la lectura.

Resumen de la tesis

A continuación, presentamos un breve resumen acerca de los contenidos de los diferentes capítulos que conforman esta tesis.

En Capítulo 1 presentamos el estado del arte acerca de los principales temas tratados en esta tesis. Dividimos este capítulo en dos secciones. La primera sección aborda las contribuciones más recientes al estudio de modelos de EDOs y EDPs para dinámica colectiva en ciencias de la vida, atendiendo a las diferentes escalas de descripcción (microscópica, mesoscópcia y macroscópica). Hacemos especial enfasis en dos modelos particulares: Cucker–Smale y Kuramoto. La segunda sección se centra en estudios actuales de propiedades de estabilidad de soluciones estacionarias en mecánica de fluidos, con especial atención al problema de existencia de estructura de vorticidad enlazadas y anudadas con topología compleja. En este capítulo, presentamos los principales problemas de la tesis y comparamos con literatura anterior.

En Capítulo 2 deduciremos el límite hidrodinámico singular de inercia pequeña para el modelo cinético de Cucker–Smale hacia pesos singulares. El objetivo es deducir rigurosamente el sistema macroscópico siguiente

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) = 0, \\ \phi_0 * (\rho u) - (\phi_0 * \rho) u - u - \nabla \psi = 0, \\ \rho(0,\cdot) = \rho^0, \end{cases}$$

donde $\rho = \rho(t,x)$ y $u = u(t,x)$ represetan respectivamente la densidad y campo de velocidades de la población, $F = -\nabla\psi$ es una fuerza externa y $\phi_0(r) = r^{-2\alpha}$ es una función de influencia con singularidades. Dicho sistema se puede observar como una ley de conservación de la masa acoplada con una ecuación integral implícita para el campo de velocidades, que está determinada por un conmutador de integrales singulares. Primero, estudiamos el límite asintótico asociado a distintos escalados de la ecuación cinética. Esto genera un método para construir soluciones débiles en sentido de medidas del sistema macroscópico anterior para valores del parámetro $\alpha \in (0, \frac{1}{2})$. En segundo lugar, mostramos resultados de existencia local de soluciones en espacios con mayor regularidad. Hacemos énfasis en el hecho de que el resultado presentado es uno de los primeros en la literatura en cuanto a límites hidrodinámicos singulares del modelo de Cucker-Smale con función de influencia singular, véase también [126, 127, 186].

En Capítulo 3 presentamos una nueva versión del modelo de Kuramoto [193, 196] de N osciladores acoplados con aplicaciones a sincronización neuronal. Concretamente, sustituimos los pesos uniformes por pesos que dependen de las fases y que presentan singularidad para configuraciones con la misma fase. Dicho modelo toma la siguiente forma

$$\begin{cases} \dot{\theta}_i = \Omega_i + \dfrac{K}{N} \displaystyle\sum_{j=1}^{N} h(\theta_j - \theta_i), \\ \theta_i(0) = \theta_{i,0}, \end{cases} \quad \text{donde } h(\theta) := \frac{\sin\theta}{|\theta|_o^{2\alpha}},$$

para $i = 1, \ldots, N$ y $\alpha \in (0,1)$. Justificamos dicho modelo desde primeros principios via una aproximación de aprendizaje rápido sobre una regla de aprendizaje para los pesos descrita en términos de una función de pasticidad verificando la regla de Hebb [166]. Dicho sistema guarda cierto parecido con el modelo de Cucker–Smale con función de influencia singular, véase [226, 244, 245]. Presentamos más detalles acerca de esta relación a lo largo del capítulo. En primer lugar, mostramos resultados de existencia y unicidad de soluciones globales, válidos incluso después de una posible colisión. La presencia de singularidades obviamente induce un serio problema para aplicar la teoría clásica. Decidimos abordar el problema mediante el concepto de solución en sentido de Filippov. Como quedará claro más adelante, el núcleo de interacción no es mas que Hölder continuo, lo cual añade problemas de unicidad al problema. Para resolver esta fatídica falta de regularidad, presentamos una propiedad de Lipschitz lateral, que surge de la estructura del núcleo y será suficiente para deducir la unicidad. En segundo lugar, analizamos la emergencia de sincronización y encontraremos nuevas dinámicas como consecuencia de que los osciladores tienen capacidad de sincronizarse en tiempo finito. Damos condiciones suficientes en la configuración inicial del sistema que garantizan la sincronización global de las fases en tiempo finito. De hecho, probaremos varias caracterizaciones interesantes sobre la formación de grupos y agregación en tiempo finito para los osciladores en los diferentes regimenes de singularidad del modelo, lo cual abre las fronteras en el paradigma del fenḿenos de sincronización.

En Capítulo 4 presentamos la contraparte cineética asociada al sistema discreto anterior del Capítulo 3. Dicha ecuación consiste en una ecuacón de tipo Vlasov con interacciones no locales descritas en términos del núcleo de interacción h, es decir,

$$\begin{cases} \dfrac{\partial f}{\partial t} + \dfrac{\partial}{\partial \theta}\left(\Omega f - K(h*\rho)f\right) = 0, \\ f(t,\theta,\Omega) = f(t,\theta+2\pi,\Omega), \\ f(0,\cdot,\cdot) = f_0, \end{cases}$$

donde $f = f(t,\theta,\Omega)$ es la función de distribucón y $\rho = \rho(t,\theta)$ es la densidad probabilidad de las fases. Nótese que el núcleo de interacción no es Lipschitz y por tanto el estudio del sistema cinético no es claro. Presentamos los siguientes resultados atendiendo a distintos aspectos del sistema macroscópico. En primer lugar, proponemos una teoría de existencia de soluciones débiles en sentido de las medidas que permanece siendo aplicable tras posibles colisiones en cualquier rango del parámetro $\alpha \in (0, \frac{1}{2}]$. Abordamos dicho resultado de la mano de la existencia de flujos de Filippov, véase [251]. En segundo lugar, estudiamos desigualdades de estabilidad (o de tipo Dobrushin) para la ecuación cinética con respecto a dos distancias de transporte diferentes en el espacio de probabilidades: la distancia cuadrática clásica de Wasserstein y una versión en fibras, que presentamos aquí y ha acabado estando especialmente bien adaptada al problema no lineal. De nuevo, la condición de Lipschitz lateral será esencial en esta parte. Aplicamos dichos resultados de estabilidad para concluir dos consecuencias importantes. Por un lado, obtenemos unidad del problema de Cauchy anterior para cualquier valor del dato inicial. En segundo lugar, cuantificamos el límite de campo medio cuando $N \to \infty$ del sistema de partículas del Capítulo anterior hacia la ecuación cinética. Finalmente, combinamos el estudio anterior de sincronización en el modelo discreto con los resultados de estabilidad para recuperar la sincronización en fase en tiempo finito en soluciones en sentido de las medidas con ciertas hipótesis acerca del diámetro inicial del soporte. El caso $\alpha \in (\frac{1}{2}, 1)$ será abordado de forma separada usando ideas similares a las obtenidas en el anterior Capítulo 2. Los resultados de este capítulo siguen un acercamiento similar al desarrollado en [64, 67] para la ecuación de agregación en el espacio Euclideo. Sin embargo, en este Capítulo no serán necesarias estructuras de tipo flujo gradiente de este modelo.

En Capítulo 5 regresamos al modelo original de Kuramoto, que se corresponde con el caso $h = \sin$. Más concretamente, consideramos su versión cinética que recibe el nombre de ecuación de Kuramoto–Sakaguchi en la literatura:

$$\begin{cases} \dfrac{\partial f}{\partial t} + \dfrac{\partial}{\partial \theta}\left(\Omega f - K(\sin*\rho)f\right) = 0, \\ f(t,\theta,\Omega) = f(t,\theta+2\pi,\Omega), \\ f(0,\cdot,\cdot) = f_0. \end{cases}$$

El objetico de esta parte es analizar la dinámica asintótica de las soluciones cuando $t \to \infty$ para datos iniciales generales $f_0 \in C^1(\mathbb{T} \times \mathbb{R})$. Hasta donde sabemos, éste es el primer resultado en la literatura donde consideramos datos inciales generales (al contrario de resultados previos donde el soporte de las fases se encuentra encerrado en un semicírculo) y computamos tasas de convergencia explícitas. Historicamente, los principales obstáculos para aplicar las teorías clásicas son de dos tipos. Por un lado, la heterogeneidad del sistema impuesta por la variable Ω implica que la ecuación de Kuramoto–Sakacuchi no es un flujo gradiente de Wasserstein sobre el espacio de medidas de probabilidad (en los términos del cálculo de Otto [237]). Por otro lado, para osciladores idénticos la variable Ω desaparece y ahora la ecuación sí obedece un flujo gradiente formal. Sin embargo, el funcional de energía no verifica las desigualdades de convexidad necesarias. Nuestra prueba reúne dos ideas principales. En primer lugar, cuantificamos el tiempo finito transiente que el sistema tarda en entrar en un régimen en el que la masa se concentra exponencialmente rápido en un entorno de la fase media del sistema. Para demostrar esto, obtenemos un sistema de desigualdades diferenciales para algunas magnitudes clave del sistema que cuantifican cuatro principios fundamentales: producción de entropía, inestabilidad de equilibrios con masa en la antípoda de la fase media, emergencia de conjuntos atractores de curvas características y un control preciso de norma L^2 de las soluciones sobre conjuntos móviles mediante el flujo del sistema. Dichos resultados extienden al modelo de Kuramoto–Sakaguchi la heurísitica desarrollada en [104] para la ecuación de Boltzmann. En segundo lugar, una vez que el sistema entra en el régimen de concentración, demostramos versiones adaptadas a nuestro problema de las desigualdades logarítmica de Sobolev y de Talagrand (véase [238] para la ecuación de Fokker–Planck) para una disipación apropriada del sistema y la distancia de transporte en fibras del Capítulo anterior. Estas desigualdades permiten finalmente concluir la convergencia del sistema hacia el equilibrio global.

En Capítulo 6 cambiamos a un tema diferente en el amibente de la mecánica de fluidos. Concretamente, afrontamos el estudio de soluciones estacionarias de las ecuaciones de Euler para fluidos ideales en tres dimensiones y nos centramos en la clase particular de soluciones estacionarias descrita por los campos de Beltrami (generalizados)

$$\begin{cases} \operatorname{curl} u = fu, \\ \operatorname{div} u = 0, \end{cases}$$

donde u representa el campo de velocidades en $\mathbb{R}^3$ y f es una función general que recibe el nombre de factor de proporcionalidad. Su relevancia nace en la teoría lagrangiana de la turbulencia y, más concretamente, en sus implicaciones de acuerdo al teorema de estructura de Arnold. A grosso modo, dicho resultado determina que cualquier fluido incompresible tiene que ser o bien laminar (en el sentido de que el espacio completo está foliado de curvas de nivel regulares que proceden de una integral primera del campo de velocidades) o bien un campo de Beltrami. En particular, dichos campos han ganado popularidad debido a [115, 116], donde los autores los usaron para resolver la antigua conjetura de Kelvin acerca de la existencia de estructuras de vorticidad anudadas y enlazadas en fluidos incompresibles. Concretamente, los autores construyeron campos de Beltrami (fuertes) con factor de proporcionalidad constante $f = \lambda$ que alcanzan cualquier conjunto arbitrario de líneas y tubos de vorticidad anudados y enlazados con topolog'ia arbitrariamente compleja. Sin embargo, como se mostró en [117], los campos de Beltrami son escasos en el sentido de que f debe verificar unas restricciones geométricas muy específicas. Entonces, parace claro que no es en general sencillo perturbar un factor de proporcionalidad f de manera que las restricciones anteriores se sigan verificando y dicho factor admita nuevos campos de Beltrami no triviales. Nuestro objetivo en este capítulo es abordar esta falta de estabilidad global. De hecho, introducimos dos resultados de estabilidad parcial que proporcionan formas concretas de perturbar f de modo que se verifiquen las condiciones encontradas en [117] y admita campos de Beltrami generalizados no triviales tan cercanos al inicial como se desee. En el primer resultado, solo abordaremos el caso de factores de proporcionalidad constantes $f = \lambda$, que pueden perturbarse en el comprementario de una bola arbitrariamente pequeña. El segundo aplica a factores de proporcionalidad generales en bolas suficientemente pequeñas entorno de puntos no críticos del campo de velocidades. Curiosamente, los nuevos campos de Beltrami presentan el mismo tipo de estructuras de vorticidad complejas de la literatura previa.

En el último Capítulo 7 presenta otros trabajos en proceso de redacción que han surgido de esta tesis:

• En la Sección 7.1 extendemos los resultados del Capítulo 2 para deducir de forma rigurosa el límite hidrodinámico riguroso de tipo inercia pequeña de algunas versiones del modelo cinético de Cucker–Smale termodinámico [153, 161], cuya versión discreta tiene la siguiente forma

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ \dfrac{dv_i}{dt} = \dfrac{K_v}{N} \displaystyle\sum_{j=1}^{N} \phi(|x_i - x_j|) \left(\dfrac{v_j}{T_j} - \dfrac{v_i}{T_i} \right), \\ \dfrac{dT_i}{dt} = \dfrac{K_T}{N} \displaystyle\sum_{j=1}^{N} \zeta(|x_i - x_j|) \left(\dfrac{1}{T_i} - \dfrac{1}{T_j} \right), \end{cases}$$

para cada $i = 1, \ldots, N$. Aquí, la nueva variable T_i representa la "temperatura" de la i-ésima particula y puede verse como la energía interna de cada partícula. Bajo un escalado adecuado, deducimos dos posibles límites macroscópicos que representatán respectívamente dos regímenes diferentes de relajació de la temperatura: rápida o lenta.

• En Sección 7.2 proponemos un modelo matemático para explicar algunos procesos importantes en biología del desarrollo. Concretamente, abordamos el estudio de los mecanismos de comunicación celular en *Drosophila melanogaster* que tienen lugar durante el concreto mecanismo de señalización mediado por el morfógeno *Hh* y su gen diana *Ci* (responsable de la formación correcta de las alas, entre otras cosas). Recientemente, algunos resultados experimentales han dado luz sobre las bases de dicho mecanismo de comunicación, véase [34, 141]. Actualmente, la hipótesis más acertada es que dicho proceso está mediado por filipodios de señalización (o citonemas) que crecen de las membranas de las células. Más especificamente, las proteinas no pueden difundirse aleatoramente por toda la membrana extracelular, sino que su propagación está subordinada a dichos caminos específicos. Dichos filamentos son estructuras dinámicas y tanto células productoras de *Hh* como células receptoras extienden sus filamentos, los cuales crecen, se retraen y se orientan de forma ordenada y, tras un contacto en forma de sinapsis, las proteinas son transferidas de célula a célula. Esta dinámica a nivel microscópico es la razón de la formación del gradiente de concentración de morfógenos que experimentalmente se observa y se propaga de células emisoras a células receptoras. En esta sección proponemos un modelo matemático que reune tres componentes diferentes. En primer lugar, diseñamos un modelo basado en primeros principios para la mecánica de cada citonema, los cuales serán interpretados como curvas matemáticas con habilidad de crecer y contraerse de forma plástica, pero no elástica. En segundo lugar, describimos los potenciales de orientación que actúan sobre cada citonema y son responsables de su guía hacia los sitios de contacto. Los resultados experimentales muestran que las proteínas involucradas en dicho mechanismo son *Ihog, Dally* y *Dlp*. Finalmente, introducimos un modelo para la transferencia de morfógeno tras la sinapsis, que describe la propagación de proteinas a lo largo de estas estructuras móviles mediante mecanismos de flujo limitado sobre variedades unidimensionales.

Conventions and notation

Notation for sets and structures

Given a set X and a subset $A \subseteq X$, we denote the complement of A in X by $X \setminus A$ or simply A^c when X is clear from the context. Also χ_A denotes the characteristic function of A, that is $\chi_A(x) = 1$ if $x \in A$, and $\chi_A(x) = 0$ if $x \in X \setminus A$.

For any set X, 2^X denotes its power set, that is, the family of subsets of X.

For any family of sets $X_1, \ldots, X_n$ we consider the product space $X_1 \times \cdots \times X_n$ and for any family of indices $i_1, \ldots, i_m \in \{1, \ldots, n\}$ we define the projection onto the $(i_1, \ldots, i_m)$-components by

$$\begin{array}{rccc} \pi_{(x_{i_1}, \ldots, x_{i_m})}: & X_1 \times \cdots X_n & \longrightarrow & X_{i_1} \times \cdots X_{i_m}, \\ & (x_1, \ldots, x_n) & \longmapsto & (x_{i_1}, \ldots, x_{i_m}). \end{array}$$

In particular, for the product space $\mathbb{T} \times \mathbb{R}$ we denote the projections

$$\begin{array}{rlcl} \pi_z & : \mathbb{T} \times \mathbb{R} & \longrightarrow & \mathbb{T}, \\ & (z, \Omega) & \longmapsto & z, \end{array} \qquad \begin{array}{rlcl} \pi_\Omega & : \mathbb{T} \times \mathbb{R} & \longrightarrow & \mathbb{R}, \\ & (z, \Omega) & \longmapsto & \Omega. \end{array} \tag{N.1}$$

Given a Banach space X and a subset $A \subseteq X$, $\mathrm{co}(A)$ denotes its convex hull , that is, the smallest convex subset of X containing A. The closed convex hull is denoted by $\overline{\mathrm{co}}(A)$ and represents the smallest closed convex subset that contains A.

$\mathbb{N}$ is the set of positive integers, that is, $\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{R}$ is the real line and $\mathbb{C}$ is the complex plane. For any complex number $z \in \mathbb{C}$ we denote its real and imaginary parts by $\Re z$ and $\Im z$ respectively. The Euclidean space of dimension $n \in \mathbb{N}$ is denoted by $\mathbb{R}^n$ and we define the Euclidean scalar product and the Euclidean norm by

$$v \cdot w = \sum_{i=1}^{n} v_i w_i \ \text{ and } \ |v| = (v \cdot v)^{1/2},$$

for any $v, w \in \mathbb{R}^n$. To avoid confusion, we shall sometimes use the notation $\langle v, w \rangle$ for the scalar product. If $n = 3$, $v \times w$ represents the cross product of v and w. The open ball of $\mathbb{R}^n$ centered at $x \in \mathbb{R}^n$ with radius $r > 0$ is $B_r(x)$ and $\bar{B}_r(x)$ represents the closed ball.

$\mathbb{N}^n$ represents the set of multi-indices and for any $\gamma = (\gamma_1, \ldots, \gamma_n)$ and any $x \in \mathbb{R}^d$ we denote the monomial

$$x^\gamma := x_1^{\gamma_1} \cdots x_n^{\gamma_n}.$$

Also, we denote the size of the multi-index by

$$|\gamma| = \gamma_1 + \cdots + \gamma_n.$$

And the factorial of γ is determined by

$$\gamma! = \gamma_1! \cdots \gamma_n!.$$

$\mathcal{M}_{n,m}(\mathbb{R})$ is the space of real $n \times m$ matrices, $\mathcal{M}_n(\mathbb{R})$ is the space of real squared matrices and I_n represents the identity matrix (we omit n when it is clear). $\mathrm{Skew}_n(\mathbb{R})$ is the subset of squared skewsymmetric real matrices. For any $A \in \mathcal{M}_n(\mathbb{R})$, we denote its transpose by $A^\top$, the symmetric part by $\mathrm{Sym}\, A = \frac{1}{2}(A + A^\top)$, its trace is represented by $\mathrm{Tr}\, A$ and its determinant is $\det A$. For any $A, B \in \mathcal{M}_n(\mathbb{R}^d)$ we define the Frobenius scalar product by $\langle A, B\rangle_F$ or $A : B$ and it reads

$$A : B = \mathrm{Tr}(A^\top B) = \sum_{i,j=1}^{n} a_{ij} b_{ij}.$$

For any $A \in \mathcal{M}_{n_1,m_1}(\mathbb{R})$ and $B \in \mathcal{M}_{n_2,m_2}(\mathbb{R})$, we define the Kronecker product blockwise by

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1m_1}B \\ a_{21}B & a_{22}B & \cdots & a_{2m_1}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n_1 1}B & a_{n_1 2}B & \cdots & a_{n_1 m_1}B \end{pmatrix}$$

Along this thesis, we shall set $\xi \in C_c^\infty([0,\infty))$ to be any non-increasing cut-off function verifying the following properties

1. $0 \leq \xi \leq 1$,
2. $\xi|_{[0,1]} \equiv 1$ and $\xi|_{[2,\infty]} \equiv 0$,
3. $\int_0^\infty \xi'(r)\, dr = -1$.

Also, we will use the following scaled cut-off functions for every $\varepsilon > 0$

$$\xi_\varepsilon(r) := \xi(\varepsilon^{-1} r),\ r \in [0, +\infty). \tag{N.2}$$

Real calculus

Given a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, we denote its gradient vector by ∇f and its Laplacian by Δf. In addition, for a multi-index $\gamma \in \mathbb{N}^n$ of size k, we denote the partial derivative of order γ by

$$D^\gamma f := \frac{\partial^k f}{\partial x^\gamma} = \frac{\partial^k f}{\partial x_1^{\gamma_1} \cdots x_n^{\gamma_n}}.$$

Given any scalar function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, any vector field $\Phi : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ and any multi-index $\gamma \in \mathbb{N}^m$, we shall systematically use the following expression for the chain rule (see [206]):

$$D^\gamma (f \circ \Phi)(x) = \gamma! \sum_{(l,\beta,\delta) \in \mathcal{D}(\gamma)} (D^\delta f)(\Phi(x)) \prod_{r=1}^{l} \frac{1}{\delta_r!} \left(\frac{1}{\beta_r!} D^{\beta_r} \Phi(x) \right)^{\delta_r}. \tag{N.3}$$

for every $x \in \mathbb{R}^m$. Here, $\mathcal{D}(\gamma)$ will represent the set of decompositions of γ as follows

$$\gamma = \sum_{r=1}^{l} |\delta_r|\beta_r,$$

where $l \in \mathbb{N}$, $\delta_r \in \mathbb{N}^n, \beta_r \in \mathbb{N}^m$ for $r = 1, \ldots, l$ are multi-indices such that for every $r = 1, \ldots, l-1$ there exists some $i_r \in \{1, \ldots, m\}$ such that $(\beta_r)_i = (\beta_{r+1})_i$ for every $i \neq i_r$ and $(\beta_r)_{i_r} < (\beta_{r+1})_{i_r}$ and we denote $\delta := \sum_{r=1}^{l} \delta_r$.

For any vector field $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $\operatorname{Jac} F$ is the Jacobian matrix and $\operatorname{div} F$ is the divergence. If $n = 3$, $\operatorname{curl} F$ denotes the rotational of the vector field. For any vector fields $F, G : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, we define

$$(G \cdot \nabla)F = \operatorname{Jac} F \cdot G.$$

Also, the divergence of its Kronecker product $F \otimes G$ can be computed blockwise as follows

$$\operatorname{div}(F \otimes G) = (\operatorname{div}(F_1 G), \ldots, \operatorname{div}(F_n G)) = (G \cdot \nabla)F + \operatorname{div}(G)F.$$

When $f = f(t)$ is any function defined on an interval of the real line (representing time), we will denote its time-derivative in any of the following ways

$$f'(t), \quad \frac{df}{dt}, \text{ or } \dot{f}(t).$$

For any set X and any couple of scalar functions $f, g : X \longrightarrow \mathbb{R}$, we shall say that $f \lesssim g$ if there is a universal constant C such that

$$f(x) \leq Cg(x), \text{ for all } x \in X.$$

When used repeatedly along an argument, the symbol $\lesssim$ may refer to possibly different values of the universal constant C that can change from line to line.

Calculus on Riemannian manifolds

Given any complete (finite-dimensional) Riemannian manifold $(M, \langle \cdot, \cdot \rangle)$, $T_x M$ represents the tangent space of M at $x \in M$, $\langle \cdot, \cdot \rangle_x : T_x M \to \mathbb{R}$ is the metric at x and TM is the tangent bundle of M. For any curve $\gamma : [0, 1] \longrightarrow M$, the parallel transport along γ from $\gamma(s_2)$ to $\gamma(s_2)$ is

$$\tau[\gamma]_{s_1}^{s_2} : T_{\gamma(s_1)} M \longrightarrow T_{\gamma(s_2)} M.$$

Also, $\exp_x : T_x M \longrightarrow M$ is the exponential map at $x \in M$ and $\operatorname{cut}(x)$ stands for its cut locus at x. The injectivity radius at x is represented by $\iota(x)$ and $\mathbb{B}_r(x) = \exp_x(B_r(0))$ is the geodesic ball centered at x with radius $r < \iota(x)$.

The space of smooth tangent fields along M is denoted by $\mathfrak{X}(M)$ and the space of 1-forms is $\Omega^1(M)$. Both can be identified via the musical isomorphisms $\flat$ and $\sharp$ as follows

$$V^\flat(W) = \langle V, W \rangle \text{ and } \left\langle \omega^\sharp, V \right\rangle = \omega(V),$$

for any $V, W \in \mathfrak{X}(M)$ and $\omega \in \Omega^1(M)$. For any function $f : M \longrightarrow \mathbb{R}$ its differential is $df \in \Omega^1(M)$ and its Riemannian gradient is $\nabla f = df^\sharp$.

Also, we denote the Riemannian distance between any couple of points $x, y \in M$ by

$$d(x,y) := \inf \left\{ \mathcal{L}[\gamma] := \int_0^1 |\gamma'(s)|\, ds : \gamma \in C^1([0,1], M),\ \gamma(0) = x, \gamma(1) = y \right\}. \tag{N.4}$$

By Hopf–Rinow's theorem [111], such infimum is attained at some minimizing geodesic on M. We refer to Appendix G for the particular case of compact hypersurfaces S of the Euclidean space $\mathbb{R}^3$, where the operators ∇_S, div_S and curl_S are introduced. We provide there a list of classical identities to bear in mind throughout the thesis and, specially, in Chapter 6.

Measure theory

We will denote measure spaces by $(E, \mathcal{F}, \mu)$ for some set E endowed with a σ-algebra $\mathcal{F}$ and a positive measure μ. Integrals with respect to such measure will be denoted by

$$\int_E f\, d\mu \quad \text{or} \quad \int_E f(x)\, d_x\mu,$$

if it is necessary to emphasize the variable $x \in E$. If μ is a probability measure, then $(E, \mathcal{F}, \mu)$ is called a probability space. We shall mostly deal with particular measure spaces where $E = \mathcal{X}$ is some Polish space, that is, a separable complete metric space, and $\mathcal{F} = \mathcal{B}(\mathcal{X})$ is the Borel σ-algebra of $\mathcal{X}$.

If $\mathcal{X} = \mathbb{R}^d$, the Lebesgue measure will be denoted $\mathcal{L}^d$ or simply dx and integrals are written by $\int_{\mathbb{R}^d} f(x)\, dx$. If $\mathcal{X} = M$ is some manifold, its Riemannian measure is denoted dS and integrals with respect to dS read $\int_M f(x)\, d_xS$.

Consider two Polish spaces $\mathcal{X}$ and $\mathcal{Y}$:

1. For any measurable map $T : \mathcal{X} \longrightarrow \mathcal{Y}$ and any finite measure μ on $\mathcal{X}$, we define the push-forwards measure $T_{\#}\mu$ on $\mathcal{Y}$ by the formula

$$\int_{\mathcal{Y}} \varphi\, d(T_{\#}\mu) = \int_{\mathcal{X}} \varphi \circ T\, d\mu,$$

   for any bounded continuous test function $\varphi : \mathcal{X} \longrightarrow \mathbb{R}$.

2. For any probability measure μ in the product space $\mathcal{X} \times \mathcal{Y}$, we denote the y-marginal by $\nu := \pi_{y\#}\mu$ and we define the family of conditional probabilities or disintegrations $\{\mu(\cdot|y)\}_{y \in \mathcal{Y}}$ on $\mathcal{Y}$ though the formula

$$\int_{\mathcal{X}\times\mathcal{Y}} \varphi(x,y)\, d\mu = \int_{\mathcal{Y}} \left(\int_{\mathcal{X}} \varphi(x,y)\, d_x\mu(\cdot|y) \right) d_y\nu,$$

   for any bounded continuous test function $\varphi : \mathcal{X} \times \mathcal{Y} \longrightarrow \mathbb{R}$.

Function spaces, measure spaces and optimal transport

We provide the list of the main function spaces to be used:

Name	Notation
Lebesgue	$L^p(\mathbb{R}^n)$
Weak Lebesgue	$L^{p,\infty}(\mathbb{R}^n)$
Bounded mean oscillation	$BMO(\mathbb{R}^n)$
Sobolev	$W^{k,p}(\mathbb{R}^n)$
Negative Sobolev	$W^{-k,p'}(\mathbb{R}^n)$
Bounded-continuous	$C_b(\mathbb{R}^n)$
Continuous that vanish at infinity	$C_0(\mathbb{R}^n)$
Continuous with compact support	$C_c(\mathbb{R}^n)$
Smooth with compact support	$C_c^\infty(\mathbb{R}^n)$
Bounded-continuous derivatives	$C^k(\mathbb{R}^n)$
Inhomogeneous Hölder	$C^{0,\alpha}(\mathbb{R}^n)$
Higher order inhomogeneous Hölder	$C^{k,\alpha}(\mathbb{R}^n)$

Here, $k \in \mathbb{N}$, $1 \leq p \leq \infty$, $0 < \alpha \leq 1$ and we denote $p' = p/(p-1)$ the conjugate exponent. We consider the classical Banach-space structure in the spaces L^p, BMO, $W^{k,p}$, $W^{-k,p'}$, C_b, C_0, C_c, C^k, $C^{k,\alpha}$. The space $L^{p,\infty}$ will be regarded as a quasi-normed space and C_c^∞ is endowed with the classical locally convex topology. For any such function space X, we will denote its norm (or quasinorm) by $\|\cdot\|_X$. To avoid confusion, we set the following norm in the inhomogeneous Hölder spaces $C^{k,\alpha}$ of higher order:

$$\|f\|_{C^{k,\alpha}(\mathbb{R}^n)} := \sum_{|\gamma|\leq k} \|D^\gamma f\|_{C_b(\mathbb{R}^n)} + \sum_{|\gamma|=k} [D^\gamma f]_{\alpha,\mathbb{R}^d}, \tag{N.5}$$

where $[\,\cdot\,]_{\alpha,\mathbb{R}^d}$ represent the homogeneous α-Hölder seminorm in $\mathbb{R}^d$.

The base space $\mathbb{R}^n$ can be replaced either by any open subset of $\mathbb{R}^n$ or any general Riemannian manifold $(M, \langle\cdot,\cdot\rangle)$. Indeed, if $\Omega \subseteq \mathbb{R}^n$ is any open subset, we shall distinguish between the notation $C^k(\Omega)$ and $C^k(\overline{\Omega})$, the latter meaning that all the derivatives up to order k can be continuously extended to the closure $\overline{\Omega}$. Similar comments apply to the Hölder spaces $C^{k,\alpha}(\Omega)$ and $C^{k,\alpha}(\overline{\Omega})$.

For any of the above regularity classes $\mathcal{C}$, the vector-valued counterparts of such function spaces will be denoted by $\mathcal{C}(\mathbb{R}^n,\mathbb{R}^m)$ (e.g., $L^p(\mathbb{R}^n,\mathbb{R}^m)$, $C^{k,\alpha}(\mathbb{R}^n,\mathbb{R}^m)$, etc) and they are considered component-wise.

For any Banach space X and any $T > 0$, we will considered the following Banach-valued functions spaces.

Name	Notation
Lebesgue–Bochner	$L^p(0,T;X)$
Weak Lebesgue–Bochner	$L^p_w(0,T;X^*)$
Sobolev–Bochner	$W^{1,p}(0,T;X)$
Weak Sobolev–Bochner	$W^{1,p}_w(0,T;X^*)$

Given any Polish space $\mathcal{X}$, that is, a complete separable metric space, we denote the following spaces of finite measures and probabilities:

Name	Notation
Finite Radon measures	$\mathcal{M}(\mathcal{X})$
Positive finite Radon measures	$\mathcal{M}^+(\mathcal{X})$
Probability measures	$\mathcal{P}(\mathcal{X})$
Wasserstein space	$\mathcal{P}_p(\mathcal{X})$

Again, we consider exponents $1 \leq p \leq \infty$. Unless otherwise stated, we will endow $\mathcal{M}(\mathcal{X})$ (as a consequence also $\mathcal{P}(\mathcal{X})$) with the weak-* topology associated with the standard duality pair with the space $C_c(\mathcal{X})$. Sometimes, we will also consider the narrow topology on such spaces, that is, the weak topology associated with the duality pair with $C_b(\mathcal{X})$. The Wasserstein space $\mathcal{P}_p(\mathcal{X})$ of p-th order will be endowed with the Wasserstein transportation distance W_p of p-th order. Finally, d_{BL} stands for the bounded-Lipschitz distance on $\mathcal{P}(\mathcal{X})$.

Along the thesis we will also define two adaptations of the quadratic Wasserstein distance W_2 in the product space $\mathcal{X} = \mathbb{T} \times \mathbb{R}$, that we respectively call the fiberwise quadratic Wasserstein distance $W_{2,g}$ and the scaled quadratic Wasserstein distance SW_2.

Potential theory

For any dimension $d \in \mathbb{N}$, I_β represents the fractional integral of order β. Specifically, we will set

$$(I_\beta f)(x) = \frac{1}{|x|^{d-\beta}} * f = \int_{\mathbb{R}^d} \frac{1}{|x-y|^{d-\beta}} f(y)\, dy, \qquad x \in \mathbb{R}^d,$$

for any measurable function $f : \mathbb{R}^d \longrightarrow \mathbb{R}$. Notice that, for simplicity, we will forget about the constant appearing in front of the Riesz potential $|x|^{-(d-\beta)}$.

In general, given a smooth open subset $D \subseteq \mathbb{R}^d$ and any kernel $K = K(x,z)$ with $x \in D$ and $z \in \mathbb{R}^d \setminus \{0\}$, the generalized volume potential generated by any density $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ is

$$(\mathcal{N}_K f)(x) = \int_{\mathbb{R}^d} K(x, x-y) f(y)\, dy, \qquad x \in D.$$

In particular, if $d = 3$ and we can consider as kernel $K(x,z)$ the fundamental solution of Helmoltz equation in $\mathbb{R}^3$, i.e.,

$$K(x,z) = \Gamma_\lambda(z) := \frac{e^{i\lambda|z|}}{4\pi|z|}.$$

In such particular case, we denote the generalized volume potential associated with Helmholtz equation generated by the density f in the simpler way:

$$\mathcal{N}_\lambda f := \mathcal{N}_{\Gamma_\lambda} f.$$

If f is supported on G we denote the inner and outer generalized volume potentials by

$$\mathcal{N}_\lambda^- f = (\mathcal{N}_\lambda f)|_G \quad \text{and} \quad \mathcal{N}_\lambda^+ f = (\mathcal{N}_\lambda f)|_\Omega .$$

Similarly, we denote the inner and outer single layer potentials associated with Helmholtz equation and generated by a density $f : S \longrightarrow \mathbb{R}$ as follows

$$(\mathcal{S}_\lambda f^-)(x) := \int_S \Gamma_\lambda(x-y)\, f(y)\, d_y S, \qquad x \in G,$$

$$(\mathcal{S}_\lambda f^+)(x) := \int_S \Gamma_\lambda(x-y)\, f(y)\, d_y S, \qquad x \in \Omega.$$

The case $\lambda = 0$ represents the Newtonian case and we call of the above potentials the Newtonian volume and single layer potentials.

Specific notation for many-particles systems and kinetic equations

We shall denote the spacial dimension by $d \in \mathbb{N}$. The main physical variables that will be considered are listed as follows:

Variable	Notation
Time	$t \geq 0$
Position	$x \in \mathbb{R}^d$
Velocity	$v \in \mathbb{R}^d$
Phase	$\theta \in \mathbb{R}$
Frequency	$\Omega \in \mathbb{R}$
Coupling strength	$K \geq 0$

For any phase $\theta \in \mathbb{R}$, we denote its representative modulo 2π within the interval $(-\pi, \pi]$ by $\bar{\theta}$.

In many-particles systems, N will represents the amount of particles. As $N \to \infty$, we represent the system by its distribution function f. In particular, if the physical variables are t, x, v, say, the distribution function $f = f(t, x, v)$ will depend upon such variables and it represent the probability of finding particles at time t, position x and velocity v. Associated with the distribution function we can set some relevant macroscopic quantities by coarse graining the variable v, namely,

Quantity	Notation
Local density	$\rho(t,x) = \int_{\mathbb{R}^d} f(t,x,v)\, dv$
Current	$j(t,x) = \int_{\mathbb{R}^d} v f(t,x,v)\, dv$
Velocity field	$u(t,x) = \frac{j(t,x)}{\rho(t,x)}$
Stress tensor	$\mathcal{S}(t,x) = \int_{\mathbb{R}^d} v \otimes v f(t,x,v)\, dv$
Kinetic energy	$E(t,x) = \frac{1}{2} \operatorname{Tr}(\mathcal{S}(t,x))$

Contents

CHAPTER 1

Introduction

How does water flow along a river? Why do fireflies synchronize their flashing? How does ice melts into water? Why do birds form flocks? How do cells duplicate? Why do aircrafts flight?...

These are just a few of the many questions that mankind has tried to answer driven by our natural curiosity about the world around us. These questions obviously do not have simple answers and address completely different type of phenomena that can range from physics, to biology, ecology, astrophysics, engineering or even social sciences. Since the origin of Science, one of the more successful methods to find a common convincing answer is through ”mathematical modeling”.

In a mathematical model, one tries to describe a system by using mathematical tools and an appropriate language: algebra, ODEs, PDEs, statistics, mean games, etc. Both tools and language depend upon the problem itself. Indeed, there are many choices that have to be made to formulate a model. In particular, we have to decide whether the model is deterministic or stochastic, quantitative or qualitative, analytical or numerical, discrete or continuous and also we must select the appropriate level of description, that is, either microscopic or macroscopic. If one wishes that the model is meaningful, two main aspects have to be respected. First, our model has to be supported by some empirical evidence or a priori information of the system under consideration. Such information will be translated into the mathematical hypothesis upon which we build our model. Second, the model should be able to provide predictions, which is the realm of mathematical modeling in Science.

Among all the above choices, which ones are the “best ones”? A first answer might be to stick to reality as close as possible. Despite the fact that additional complexity produces more realistic models, an excess of complexity often yields equations that are hard to tackle and analyze from a mathematical point of view. Sometimes this translates into computational problems that one is not able to solve with the current technology. A completely opposite (but also outdated) answer to such a criterion, might be to assume *Ockham's razor principle*:

Pluralitas non est ponenda sine necessitate (William of Ockham, 1287–1347),

whose meaning in our context gives precedence to simplicity and claims that of two competing theories, the simpler explanation of a phenomenon is the right one. This criterion proves difficult to apply in practice because it is hard to elucidate which one is the simpler model after we account for all the necessary a priori information about the system. Also, it is not clear that

(a) Murmuration of starlings at Gretna, Scotland.

(b) School of fish in Cabo Pulmo, Mexico.

Figure 1.1: Examples of collective dynamics, taken from the websites https://www.bbc.com/news and https://blog.nationalgeographic.org.

"the simplest model" must be the suitable one. Then, the final usual agreement is a balanced compromise between complexity and mathematical manageability of the model in terms of a careful dimensional reduction of the involved variables and parameters so that we reduce complexity, but we do not lose the most relevant features.

Historically, simple models have been studied first as representative of the more complex ones that one can propose to describe any physical or biological phenomenon. After such representatives are understood, we can then face more complex and realistic models explaining the complicatedness of structures and patterns around us. An even more captivating reality is that despite the fact that these simpler models may arise from completely different phenomena, some of them implicitly share a common abstract structure due to their mathematical resemblance. Then, we can think of such models as a connected family with an active feedback between them, so that by studying one, we may understand some properties of the others.

The ultimate mathematical interest of such groups of models is that it has historically given rise to strong mathematical advances as we try to analyze the different components. Indeed, very important areas of mathematics have emerged and grown as we try to solve those problems (e.g., kinetic theory, stochastic equations, potential theory, optimal transport, paradifferential calculus, harmonic analysis, etc).

In this thesis, we focus on the mathematical analysis of kinetic PDE models that arise from problems in physics, mathematical biology, ecology and neuroscience. They consists in deterministic evolutionary equations that explain how a biological or physical system, which departs from some initial configuration, evolves in time subordinated to specific laws. More specifically, we address the family of "collective dynamics models". Such equations describe a biological (or ecological) population of individuals that interact through a given set of rules. The main objective of such models is to show that the chosen interaction rules are robust enough to explain the formation of a self-organized dynamics at larger scales. This is often called collective behavior (or motion) and allows describing some complicated patterns and structures that we observe in nature, e.g., swarming of bacteria, flock of birds or schooling of fish, see Figure 1.1.

Along this thesis, we will pay special attention to systems of agents (particles, organisms, etc) that are subject to singular interactions. This adds new analytical challenges that require deriving novel mathematical theories and improving the preceding tools in the literature. In particular, we are interested in the following general problems, among others,

- Well posedness and stability properties of solutions.
- Scaling limits connecting different levels of description.
- Asymptotic behavior and emergent phenomena.

As a very specific example, we also address the study of some equations arising in fluid mechanics. Notice that although they are apparently of completely different nature, they are actually strongly connected via the second item above. Specifically, most of the macroscopic collective dynamic models are described in terms of conservation laws of Euler or Navier–Stokes type, what suggests that its study may shed some light on the general understanding of collective dynamics.

In the following sections, we review the state of the art of the main topics that this dissertation deals with. We also introduce the main problems of this thesis and related literature. The contents are arranged as follows:

Contents

1.1 Collective behavior in life sciences

In this section, we introduce a short review of collective behavior models in the literature. Since literature is huge, we do not aim at a comprehensive review. Instead, we will restrict our discussion to some models arising from biology, ecology and neuroscience, that belong to the area of *soft active matter* (see [210]), and that encompass the core of this thesis.

When designing a collective dynamics model, the description of the involved inter-particle interactions may not be clear at first glance. Indeed, they strongly depend on the particular phenomenon and on the specific nature of the population. Hence, we might require an interdisciplinary point of view in order to elucidate which are the fundamental variables and laws governing the social relations between agents. Specifically, a real population is actually a complex system and, as such, it involves plenty of physical, social, biological and cognitive variables. However, we still can give light to the problem if we restrict to part of the full complex dynamics by disregarding secondary variables and just looking at the effect of the main variables that that affect a given feature of the population. Although it clearly stops being a universal description that works for every the scenario, in this way, we can obtain a simpler and more manageable model that still recasts the most important features of the population, e.g., emergence of *global collective behavior.*

The interest on collective dynamics models has notably raised during the recent years. From the applied side, this is specially interesting since simple rules governing pairwise interactions between agents, leads to global emergent behavior of the total population as a whole. As it can be seen for instance in [24, 58, 64, 74, 128, 154, 157, 210, 241, 254, 255, 275, 287, 288] and references therein, collective dynamics models have proved relevant in several areas of *soft active*. In addition, those models are also important from a theoretical point of view since they can be regarded as a rich source of problems in mathematics. Indeed, as mentioned before, many recent strong techniques in mathematics have arisen as a consequence of such models, what highlights that the exchange of ideas between the applied and theoretical communities is undoubtedly positive.

1.1.1 Agent-based models of first and second order

In this part we shall focus on a few collective dynamics models that obey a similar structure and have been analyzed in the literature during the last years. We shall divide them into two categories: first order models and second order models, with regards to the amount of time-derivatives that are necessary for their description. Its relation will be clarified later on.

First, we describe first order models. At the agent-based level they consist of a system of N first order coupled ODEs, one for each agent. Although this thesis mainly centers on their kinetic description together with its macroscopic limit, we highlight in this part the microscopic description from which the mesoscopic one inherits a large part of its properties. Specifically, assume that each subject is located at a given position $x_i = x_i(t)$. Then, all these first order systems take the following form

$$\left\{\begin{array}{l}\dfrac{dx_i}{dt} = \nu_i + F_e(x_i) + \dfrac{K}{N}\displaystyle\sum_{j=1}^N m_j F(x_i, x_j),\\ x_i(0) = x_{i,0},\end{array}\right. \tag{1.1.1}$$

for $i = 1, \ldots, N$, where x_i represent positions in $\mathbb{R}^d$ (or $\mathbb{T}^d$, for periodic domains). Nevertheless, x_i will not necessarily restrict to positions, but can also represent any other physical, social, state or internal variable of agents. The parameter m_i is the mass of the i-th agent and $\nu_i \in \mathbb{R}$ are often called *natural velocities* and stand for heterogeneities in the population in the form of a biased tendency of individuals to follow a certain velocity in the absence of the remaining ones. Also, $F_e = F_e(x)$ represents an external force acting on the system and the more relevant terms $F = F(x, x')$ describes the interaction kernel governing the force that a field agent at x' exerts on a subject at x. Here, K is called the coupling strength and determines how strong inter-particle interactions are. Rigorously speaking, F should not me called force since there is no acceleration term in (1.1.1). Actually, F describes how each individual updates its instantaneous velocity as an effect of the remaining ones. To start, let us now list some of the main collective dynamics models of first order type that embed into the above formulation (1.1.1). For simplicity, we shall neglect the external force F_e and will focuse on the communication kernel $F(x, x')$ part between agents.

1. If $d = 1$, $x_i \equiv \theta_i \in \mathbb{R}$ are phases along the unite circle and we set

   $$F(x, x') = \sin(x' - x),\ \nu_i \equiv \Omega_i \ \text{ and } \ m_i = 1,$$

   then (1.1.1) reduces to the Kuramoto model for coupled oscillators

   $$\left\{\begin{array}{l}\dfrac{d\theta_i}{dt} = \Omega_i + \dfrac{K}{N}\displaystyle\sum_{j=1}^N \sin(\theta_j - \theta_i),\\ \theta_i(0) = \theta_{i,0}.\end{array}\right. \tag{1.1.2}$$

   This is a classical model that was proposed by Kuramoto as a prototype system exhibiting emergence of *synchronization* [114, 145, 147, 157, 202]. This model was originally proposed to describe chemical oscillations, but has been later applied to many other phenomena. Here, the heterogeneities Ω_i represent the natural frequency of each oscillator.

An extremely interestin setting appears when we try to apply Kuramoto model in neuroscience. In such case θ_i represent the phases of the neuronal signals, and $\dot{\theta}_i$ represents the firing frequencies of neurons in the brain. In this setting, the Kuramoto model consists in a first approach towards a mathematical description of *neuronal synchronization* [11, 45, 209, 248, 261, 297, 305, 306], that is known to rule many cognitive process of the brain that are activated when a specific group of neurons fire together forming a cluster. Of course, this model can be made more realistic by adding coupling weights governing the plasticity of connections via learning mechanisms [92, 159, 233, 247, 272], inertia terms and delays in time [76, 77, 78, 79], noise or many other features like singular couplings (see [241, 254] and Chapters 3 and 4 of this thesis). We will review some of this associated models later on.

2. If $x_i \in \mathbb{R}^d$ represent positions and we set

$$\nu_i \equiv 0, \ m_i = 1, \ \text{ and } \ F(x, x') = -\nabla_x W(x - x'),$$

then (1.1.1) agrees with the *aggregation equation* with a given potential function W

$$\left\{ \begin{array}{l} \dfrac{dx_i}{dt} = -\dfrac{K}{N} \displaystyle\sum_{j=1}^{N} \nabla_x W(x_i - x_j), \\ x_i(0) = x_{i,0}, \end{array} \right. \tag{1.1.3}$$

This is probably one of the best known models in this family. It represents *swarming* of a population of bacteria or other entities, that try to aggregate and from a unique group or cluster. Depending on the nature of potential W, one might include both attractive and repulsive interactions, then enriching the dynamics. This is one of the reason why this model has specially pulled the attention of the scientific community during the last years, indeed it has the ability to generate dynamics converging to equilibria that exhibit relevant patterns in biological contexts, see [37, 50, 49, 64, 67, 120, 121, 221, 220, 287, 288].

3. If $d = 2$, we neglect heterogeneities $\nu_i = 0$ and we set

$$m_i \equiv \omega_i \in \mathbb{R}, \ \text{ and } \ F(x, x') = \frac{(x - x')^\perp}{2\pi |x - x'|^2},$$

(where $(z_1, z_2)^\perp = (-z_2, z_1)$ for $z = (z_1, z_2) \in \mathbb{R}^2$), then (1.1.1) is the N vortex system [231]

$$\left\{ \begin{array}{l} \dfrac{dx_i}{dt} = \dfrac{1}{N} \displaystyle\sum_{\substack{1 \leq j \leq N \\ j \neq i}} \omega_j \dfrac{(x_i - x_j)^\perp}{2\pi |x_i - x_j|^2} \\ x_i(0) = x_{i,0}. \end{array} \right. \tag{1.1.4}$$

Here, the values ω_i denote the strength of vortices. This model has been widely studied in the fluid mechanics community. Specially, its associated PDE macroscopic model agrees with the 2*D Euler equation in vorticity form* for an perfect incompressible fluid. When, white noise is also added, we recover the well known Navier–Stockes system for viscous fluids. In this sense, the equations in fluid mechanics can be regarded as a particular case of collective dynamics where the emergent phenomenon is the rotation of individuals, also see [287, 288]. In next Section 1.2 we provide a more detailed introduction to the equations of fluid mechanics, that will be useful along Chapter 6 in this thesis.

Before we talk about the macroscopic counterparts of the fist order agent-based model (1.1.1), let us link the above system (1.1.1) with the classical well known second order models arising form Newton's second law. Specifically, consider the following second order deterministic model

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ \dfrac{dv_i}{dt} = -\dfrac{1}{\tau} v_i + F_e(x_i) + \dfrac{K}{N} \displaystyle\sum_{j=1}^{N} m_j F(x_i, x_j), \\ x_i(0) = x_{i,0},\ v_i(0) = v_{i,0}, \end{cases} \tag{1.1.5}$$

where we have included a friction term with the environment with associate relaxation time τ. Notice that in the *overdamped limit or Smoluchowski dynamics* (see [258]) damping dominates inertia and, consequently, the inertia term vanishes and make all the second order dependence of the system disappear. Then, the first order system (1.1.1) arises naturally from (1.1.5). This arguments can be made rigorous for Lipschitz-continuous forces via Tikhonov's theorem [183]. See [120, 121, 125, 126, 142, 255, 255] for some recent advances in this line both for smooth and singular kernels at the microscopic and macroscopic levels. Also see Chapter 2 and Section 7.1 of Chapter 7.

To complete this overview of agent-based models, let us mention that the forcing term in (1.1.5) has been chosen to depend only on positions. However, we can also consider velocity dependence of forces and, by Newton's second law, we obtain the following more general form of second order models:

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ \dfrac{dv_i}{dt} = -\dfrac{1}{\tau} v_i + F_e(x_i) + \dfrac{K}{N} \displaystyle\sum_{j=1}^{N} m_j F(x_i, v_i, x_j, v_j), \\ x_i(0) = x_{i,0},\ v_i(0) = v_{i,0}. \end{cases} \tag{1.1.6}$$

This family of second order models contains a specially interesting case in collective dynamics. Specifically, if we neglect external forces and friction, and we set

$$m_i = 1 \text{ and } F(x, x', v, v') = \phi(|x - x'|)(v' - v),$$

then, we arrive at the following model

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ \dfrac{dv_i}{dt} = \dfrac{K}{N} \displaystyle\sum_{j=1}^{N} \phi(|x_i - x_j|)(v_j - v_i), \\ x_i(0) = x_{i,0},\ v_i(0) = v_{i,0}. \end{cases} \tag{1.1.7}$$

This is the Cucker–Smale model with influence function described by the radial function ϕ. This model was introduced in [90, 91] as a prototype collective dynamics model where the emergent phenomenon is called *flocking* and represents the asymptotic alignment of all agents' velocities towards the average value.

Of course, such models (1.1.1)-(1.1.6) give rise to a large family of variants when, instead of smooth, interaction kernels are singular at the origin or, instead of isotropic, they include some more realistic anisotropy. Also, the addition of some noise that distorts the deterministic dynamics can be relevant. In this thesis we will be mostly interested in non-smooth forces. Notice that such lack of smoothness is expected for real-life systems. Also, the lack of smoothnes yield to substantially new phenomena of models like it is the case with *finite-time clustering*. Specifically, such "sticky" behavior of particles leads to the formation of distinguished groups in finite time with an eventual global collapse into a final unique cluster (or in several ones), see for example [241, 254] and Chapters 3 and 4, where such phenomenon is addressed for a singular version of (1.1.2) with singular weights. Here, we will be mostly interested in system (1.1.2) and (1.1.7), but some vague ideas and references will be provided later for the readers' convenience, specially regarding (1.1.3) and some anisotropic versions.

1.1.2 Mean field limit and propagation of chaos

In this section, we review the main techniques to derive the large crowd limit, or mean field limit, as $N \to$ in the collective dynamics models (1.1.1) or (1.1.6). We shall focus mainly in the former type of systems, but similar ideas apply to second order models as well.

There are two classical approaches in the literature: the *empirical measure* approach and the *BBGKY hierarchy* approach. The second method is stronger and harder to apply, but it has proved a strong method as it is intimately related to propagation of chaos in many particle systems. As it will be seen, the later has to do with a control of the fall-off of inter-particle correlations as the amount of agents N becomes large. For more accurate descriptions, see [163, 164, 176, 177, 178, 179, 181, 216, 217, 230, 281]. Let us assume that agents are all identical and mass are normalized to 1. Since the natural velocities in the discrete model (1.1.1) are constant parameters, we can equivalently restate the system as follows

$$\begin{cases} \dfrac{dx_i}{dt} = \nu_i + F_e(x_i) + \dfrac{K}{N}\displaystyle\sum_{j=1}^N F(x_i, x_j), \\ \dfrac{d\nu_i}{dt} = 0, \\ (x_i(0), \nu_i(0)) = (x_{i,0}, \nu_{i,0} \equiv \nu_i). \end{cases}$$

Then, both x_i an ν_i are regarded as variables, although the dynamics of ν_i is trivial.

On the one hand, for the first *empirical measures approach* let us recover the sequence of empirical measure of such N agents

$$\mu^N(t, x, \nu) := \frac{1}{N}\sum_{i=1}^N \delta_{x_i(t)}(x)\delta_{\nu_i(t))}(\nu), \tag{1.1.8}$$

for every $t \geq 0$. Then, it it is clear for Lipschitz forces that μ^N solves the following Vlasov equation in the sense of distributions

$$\frac{\partial \mu^N}{\partial t} + \operatorname{div}_x \left[\left(\nu + F_e(x) + K \int_{\mathbb{R}^d \times \mathbb{R}^d} F(x, x')\, d_{(x',\nu')}\mu_t^N \right) \mu_t^N \right] = 0. \tag{1.1.9}$$

We now take the initial data such that μ_0^N weakly-* converges towards the initial probability distribution $f_0 \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ by virtue of some law of large numbers, e.g. [291]. Then, our goal is to show that compactness is propagated uniformly in compact intervals of time and we get

$$\mu^N \to f \text{ in } C([0, T], \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d) - \text{narrow}),$$

for any $T > 0$. In such case, we may pass to the limit in (1.1.9) to recover the Vlasov equation

$$\frac{\partial f}{\partial t} + \operatorname{div}_x \left[\left(\nu + F_e(x) + K \int_{\mathbb{R}^d \times \mathbb{R}^d} F(x, x')\, d_{(x',\nu')} f_t \right) f_t \right] = 0. \tag{1.1.10}$$

Such ideas are classical and have been long used for $W^{1,\infty}$ kernels, see [230] for more precise details. Indeed, the following *estability estimate* or *Dobrushin-type inequality* (see [112]) with respect to the bounded-Lipschitz distance d_{BL} on the space of probability measures holds true for any two measure-valued solutions $f, g \in C([0,+\infty), \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}) - \text{narrow})$

$$d_{BL}(f_t, g_t) \leq e^{t(\|F_e\|_{W^{1,\infty}} + K\|F\|_{W^{1,\infty}})} d_{BL}(f_0, g_0),\ t \geq 0. \tag{1.1.11}$$

It is straightforward to check that in general, the minimal error to approximate any $f_0 \in \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d)$ by a sequence of empirical measures in d_{BL} is of order $N^{-\frac{1}{2d}}$. That provides an explicit estimate quantifying how the discrete system (1.1.1) can be approximated by the kinetic equation (1.1.10) as $N \to \infty$, namely,

$$d_{BL}(\mu_t^N, f_t) \lesssim e^{Ct} N^{-\frac{1}{2d}}, \tag{1.1.12}$$

for any $N \in \mathbb{N}$. Indeed, under such Lipschitz condition the characteristic flow is well defined and bi-Lipschitz, thus guaranteeing that absolutely continuous initial data f_0 propagates the same L^1 integrability for all times. Then, there is no way that Dirac masses can emerge from smooth initial data. Of course, the lack of Lipschitz-continuity breaks most of the above arguments down. This will be part of our goal in Chapter 4 of this thesis.

On the other hand, the *BBGKY approach* departs from the hierarchy of Liouville equations for the joint laws $f^N = f^N(t, x_1, \ldots, x_N, \nu_1, \ldots, \nu_N) \in \mathcal{P}_{sym}(\mathbb{R}^{dN} \times \mathbb{R}^{dN})$, that assumes the form

$$\frac{\partial f^N}{\partial t} + \sum_{i=1}^N \operatorname{div}_{x_i}\left[\left(\nu_i + F_e(x_i) + \frac{K}{N}\sum_{j=1}^N F(x_i, x_j)\right) f^N\right] = 0. \tag{1.1.13}$$

Notice that since $f_0^N \in \mathcal{P}_{sym}(\mathbb{R}^{dN} \times \mathbb{R}^{dN})$, in the sense that interchanging i-th and j-th positions and natural velocities let the measure invariant, then the same continues happening for all times by virtue of the properties of the system. Define the projection onto the first $k \in \{1, \ldots, N\}$ variables,

$$\begin{array}{rccc} \pi^{k,N}: & \mathbb{R}^{dN} \times \mathbb{R}^{dN} & \longrightarrow & \mathbb{R}^{dk} \times \mathbb{R}^{dk}, \\ & (X^N, \nu^N) & \longmapsto & (X^{k,N}, \nu^{k,N}), \end{array}$$

where for any $(X^N = (x_1, \ldots, x_N), \nu^N = (\nu_1, \ldots, \nu_N)) \in \mathbb{R}^{dN} \times \mathbb{R}^{dN}$ we are are denoting

$$X^{k,N} := (x_1, \ldots, x_k) \text{ and } \nu^{k,N} := (\nu_1, \ldots, \nu_k).$$

Consider the marginal measures $f_t^{k,N} := \pi_\#^{k,N}(f_t^N) \in \mathcal{P}_{sym}(\mathbb{R}^{dk} \times \mathbb{R}^{dk})$. Thanks to the assumed symmetry in the system, integration in (1.1.13) yields

$$\frac{\partial f^{k,N}}{\partial t} + \sum_{i=1}^k \operatorname{div}_{x_i}\left[\left(\nu_i + F_e(x_i) + \frac{K}{N}\sum_{j=1}^k F(x_i, x_j)\right) f^{k,N} \right.$$
$$\left. + K\frac{N-k}{N}\int_{\mathbb{R}^d \times \mathbb{R}^d} F(x_i, x_{k+1})\, d_{(x_{k+1}, \nu_{k+1})} f^{k+1,N}\right] = 0. \tag{1.1.14}$$

Via a diagonal argument we can obtain weak limits of an appropriate subsequence

$$f^{k,\infty} := \text{weak} * - \lim_{N \to \infty} f^{k,N}.$$

Although (1.1.14) is not closed for fixed N, some method to close it in the limit $N \to \infty$ is the *propagation of chaos*. Specifically, denote $f := f^{1,\infty}$ for simplicity and assume that all the initial values are tensorized, that is $f_0^{k,\infty} = f_0^{\otimes^k}$. Then, propagation of chaos means that such tensorization remains true for all times, i.e.,

$$f_t^{k,\infty} = f_t^{\otimes^k}, \quad \text{for all} \;\; t \geq 0.$$

Such property can be rigorously derived for Lipschitz kernels and symmetric interactions, like it is the case in this discussion. Then, we can pass to the limit in (1.1.14) as $N \to \infty$ for $k = 1$ and recover the Vlasov equation (1.1.10) for the limiting marginal $f^{1,\infty}$. Indeed, one can try to recover a similar Dobrushin inequality (1.1.12) between $f^{1,N}$ and $f^{1,\infty}$ with respect to some transportation distance. Indeed, such a bound has been studied for Wasserstein distances in [164, 216, 217] and others references. However, in [177, 178, 179] a different strategy has been followed to quantify chaoticity for a large class of models with non-smooth forces. Nevertheless, the presence of noise is required to guarantee existence of entropy solutions of (1.1.13). In other case, for the deterministic situation without noise some condition close to Lipschitz is required. Assume for the moment that the heterogeneities ν_i are neglected and we focus on forces of the type $F(x, x') = F(x - x')$. Then, for any entropy solution $f_t^N = f^N(x_1, \ldots, x_N)$ to (1.1.13) and any solution $f_t = f(x_1, \ldots, x_N)$ to (1.1.10), one can measure their "distance" in the sense of entropy by defining the following scaled entropy functionals

$$\mathcal{H}_N(f_t^N | f_t^{\otimes^N}) := \frac{1}{N} \int_{\mathbb{R}^{dN}} f_t^N \log\left(\frac{f_t^N}{f_t^{\otimes N}}\right) dx_1 \ldots dx_N.$$

Indeed, the following quantitative estimate can be obtained:

Theorem 1.1.1. *[179, Theorem 2] Assume that* $\operatorname{div} F_e \in L^\infty(\mathbb{T}^d)$, $\operatorname{div} F \in L^\infty(\mathbb{T}^d)$ *and that either* $F \in L^\infty(\mathbb{T}^d)$ *or for* $d \geq 2$, F *is an odd kernel and* $|x|F \in L^\infty(\mathbb{T}^d)$. *Then, there exists a constant* $M > 0$ *depending on* K, f_0 *and* $\|\operatorname{div} F_e\|_{L^\infty(\mathbb{T}^d)}$ *such that*

$$\mathcal{H}_N(f_t^N | f_t^{\otimes^N}) \leq e^{M\|F\|_\infty t} \left(\mathcal{H}_N(f_0^N | f_0^{\otimes^N}) + \frac{1}{N}\right),$$

for every $N \in \mathbb{N}$ *and* $t \geq 0$. *Here, we set the following norm of the kernel*

$$\|F\|_\infty := \begin{cases} \|F\|_{L^\infty(\mathbb{T}^d)} + \|\operatorname{div} F\|_{L^\infty(\mathbb{T}^d)}, & \textit{if } F \in L^\infty(\mathbb{T}^d), \\[2ex] \||x|F\|_{L^\infty(\mathbb{T}^d)} + \|\operatorname{div} F\|_{L^\infty(\mathbb{T}^d)}, & \textit{if } d \geq 2,\ F \textit{ is odd and } |x|F \in L^\infty(\mathbb{T}^d). \end{cases}$$

Notice that for every $k \leq N$ one has the relation

$$\mathcal{H}_k(f_t^{k,N} | f_t^{\otimes^k}) \leq \mathcal{H}_N(f_t^N | f_t^{\otimes^N}),$$

for all $t \geq 0$. Then, Theorem 1.1.1 amounts to quantitative estimates of propagation of chaos of system (1.1.1). Indeed, from it we can recover the standard propagation of chaos in L^1 and Wasserstein distances by virtue of *Csiszár–Kullback–Pinsker* and *Talagrand inequalities* respectively

$$\|f_t^{k,N} - f_t^{\otimes^k}\|_{L^1(\mathbb{T}^{dk})} \leq \sqrt{2k\mathcal{H}_k(f_t^{k,N} | f_t^{\otimes^k})},$$
$$W_p(f_t^{k,N}, f_t^{\otimes^k}) \leq C(f_t, p)\left(k\mathcal{H}_k(f_t^{k,N} | f_t^{\otimes^k})\right)^{1/2p},$$

for any $p \geq 1$, $N \geq k$ and $t \geq 0$.

In the following, we will come back to Kuramoto model (1.1.2) and to Cucker–Smale model (1.1.7). In the Lipschitz case, all the above theory works and we have well defined mesoscopic models coming from the agent-based descriptions. In addition, we shall review some singularly weighted versions (see [226, 241, 244, 245, 254, 255]) that will be the core of Chapters 2, 3 and 4 of the thesis. Notice that in such singular version models, $\operatorname{div} F$ is not bounded anymore and the above theorem 1.1.1 is not applicable in the deterministic case without noise.

1.1.3 The Cucker–Smale flocking model

Flocking models have been deeply studied in the context of starlings. For instance, [170] shows a study of the social behavior of tens of thousands of European Starlings (*Sturnus vulgaris*) in Rome, that analyzes the rules that guide them into an ordered motion culminating with the formation of a flock. Specifically, the population of starlings propagates and modifies its shape through the amazing aerial maneuvers taking place in winter in the period before migration. Such study, focuses on the internal natural laws and on explaining how they lead to the formation of the convoluted *patterns* in flocking that we observe in the nature, see Figure 1.1a. Computer simulation of complex systems is also a key ingredient to test whether the basic rules of organization yields the observed patterns. To such end, the first computer simulation (*Boids*) of flocking dynamics was pioneered by C. W. Reynodls [257].

The original model of flocking proposed by Reynolds relied in three different basic rules: *separation*, *alignment* and *cohesion*. Roughly speaking, there must exist some type of repulsive short range interaction (separation) that prevents two individuals in the crowd from getting too close. In the same way, some sort of attractive long range interaction (cohesion) should be considered in order for the population to form clusters that eventually flock. Finally, an intermediate third kind of interaction (alignment) should be borne to describe actual mechanism governing flocking collective behavior.

Later, partially influenced by the works of T. Vicsek [295], the Cucker–Smale model was proposed in [90, 91] by F. Cucker and S. Smale in the form that we introduced before in Equation (1.1.7). Notice that in such collective dynamics model, agent's velocities suffer a relaxation towards the values of their neigbour in an averaged way mediated by the influence function ϕ. Notice that natural way of choosing ϕ is as a decreasing function of distance so that the larger the interparticle distance, the weaker the influence. Indeed, the classical choice by Cucker and Smale is the truncated inverse power law

$$\phi(r) = \frac{1}{(1+r^2)^{\beta/2}}, \quad \text{for } r \geq 0, \tag{1.1.15}$$

where the exponent $\beta > 0$ describes the fall-off of influence between agents. By construction, we expect that all agent's velocity aligns asymptotically if influence is globally large enough. This gives rise to the following definitions that we shall use throughout the thesis.

Definition 1.1.2 (Alignment and flocking). *Let $(x_1, v_1), \ldots, (x_N, v_N)$ solve the Cucker–Smale model* (1.1.7)*. We will say that*

1. *The i-th and j-th agents collide at $t = t^*$ if*

$$x_i(t^*) = x_j(t^*).$$

2. *The i-th and j-th agents stick at $t = t^*$ if*

$$x_i(t) = x_j(t), \text{ for all } t \geq t^*.$$

3. *The system aligns asymptotically if*

$$\lim_{t\to\infty} \max_{i,j} |v_i(t) - v_j(t)| = 0.$$

4. *The system flocks asymptotically if it aligns asymptotically and, additionally,*

$$\sup_{t\geq 0} \max_{i,j} |x_i(t) - x_j(t)| < \infty.$$

Notice that when the system is well posed and it has unique solutions, it is clear that the sticking condition in the second item amounts to the condition $x_i(t^*) = x_j(t^*)$ and $v_i(t^*) = v_j(t^*)$. Also notice that only when the asymptotic alignment takes place fast enough we obtain a uniform bound of the diameter of the system. This can be regarded as the cohesion property by Reynolds and we readily recover the formation of a flock. In other case, the group may spread out while aligning their velocities. Finally, since the average velocity of agents is a conserved quantity of the system, that is,

$$\frac{d}{dt}\frac{1}{N}\sum_{i=1}^{N} v_i(t) = 0,$$

for all $t \geq 0$, then the asymptotic velocity of the population turns out to be precisely such average value.

Regular influence function

Regarding the classical Lipschitz influence function (1.1.15), the first results on the emergence of collective motion where derived in [90, 162, 158]. Indeed, if $\beta \leq 1$, then (1.1.7)-(1.1.15) exhibits unconditional flocking behavior with exponential fall-off of the velocity diameter. Otherwise for $\beta > 1$ flocking is conditional and is only valid for appropriate initial configurations. Using the techniques in Section 1.1.2, we readily recover the rigorous mean field limit in the Lipschitz case (1.1.15). Indeed, such ideas were conducted in [162, 158] to derive the Vlasov equation for the distribution function $f = f(t, x, v)$ as $N \to \infty$

$$\begin{cases} \dfrac{\partial f}{\partial t} + v \cdot \nabla_x f = \operatorname{div}_v(Q_{CS}(f, f)), \quad (x, v) \in \mathbb{R}^{2d},\ t > 0, \\ Q_{CS}(f, f)(t, x, v) := \displaystyle\int_{\mathbb{R}^{2d}} \phi(|x - y|)(v - w) f(t, x, v) f(t, y, w)\, dy\, dw. \end{cases} \tag{1.1.16}$$

See [51] for the analysis of well-posedness and stability of solutions to such system in a measure-valued setting using optimal transport tools that remind the ones by R. Dobrushin and H. Neunzert in [112, 230]. The macroscopic counterpart of (1.1.16) in the Lipschitz case (1.1.15) is determined by the Euler-alignment equation, that takes the form of the following system of conservation laws for density and velocity field of the population

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) = 0, \\ \dfrac{\partial u}{\partial t} + (u \cdot \nabla) u = \phi * (\rho u) - (\phi * \rho) u. \end{cases} \tag{1.1.17}$$

Such a system has been deeply analyzed. In particular, in [63, 282] the authors showed that global regularity is characterized by a explicit threshold on the initial data (ρ_0, u_0). Specifically, they discriminated initial data enjoying blow-up from data enjoying global regularity depending on the properties of a "magic quantity" in the $1D$ setting

$$G = \partial_x u + \phi * \rho.$$

Indeed, it was shown that smooth solutions must flock, for an appropriate definition of flocking that extends the microscopic one in Definition 1.1.2. A possible way to formally justify (1.1.17) is by imposing the monokinetic ansatz

$$f(t,x,v) = \rho(t,x)\delta_{u(t,x)}(v),$$

on the kinetic equation (1.1.16). The rigorous justification via hyperbolic hydrodynamic limits was recently derived in [127]. Some other attempts where an extra pressure term appears in the second balance equation (1.1.17) were addressed in [186] and related literature. However, in all the approaches to date, extra non-local damping terms have been added to the kinetic equation (1.1.16) to drive the solution towards the above monokinetic ansatz. Deriving the hydrodynamic limit in the absence of those artificial terms is still an open problem. We skip this topic here, and will return to it in Section 1.1.6, where further hydrodynamic limits are reviewed, in particular the one that we propose in Chapter 2.

In the above classical choice (1.1.15), influence is bounded by 1 for any couple of agents that are located at arbitrarily close positions. This suggests that the relaxation forces of the alignment term may not be strong enough for nearby agents to avoid collisions if their linear momenta are opposite and sufficiently large. Indeed, as discussed above in the macroscopic equation (1.1.17), this was materialized in [282] in terms of the description of a critical threshold on initial data (ρ_0, u_0) that discriminates solutions with blow up from global smooth solutions. Obviously, the repulsion property by Reynolds was formulated precisely in order for agents not to collide, bearing in mind its applications to the formation of flocks. Nevertheless, the above collisional property may be of interest in other situations physical or biological situations.

Singular influence function

With all the above intuition in mind, a plausible way to remove collisions from the dynamics is to design a more appropriate influence function ϕ with larger influence between nearby particles. This is the origin of the following singular choice

$$\phi(r) = \frac{1}{r^\beta}, \quad \text{for } r > 0, \tag{1.1.18}$$

where $\beta > 1$. Indeed, although tails behave in a similar way to the Lipschitz case (1.1.15), influence is arbitrarily large between agents separated by small distances. We here distinguish between two different regimes with regards to the singularity: the *weakly singular* case $\beta \in (0,1)$ and the *strongly singular* case $\beta \in [1,\infty)$. Formally, the same arguments as in [158] can be used to show that for $\beta \in (0,1]$, we can recover again unconditional flocking motion of the system, but it is not necessarily true for $\beta > 1$, where we expect conditional flocking for certain initial configuration. However, a fateful complication is that despite ϕ being singular, we still cannot assure that influence is strong enough to guarantee non-collision between agents. Consequently, the presence of a singular term in the system causes sever problems to derive a general well-posedness theory of the model that remains valid after potential collisions.

In the strongly singular case, particular initial conditions where found in [3] so that collisions do not take place along the dynamics and solutions remain classic and smooth for all times. Later, such a work was improved in [62], where the authors showed that in the strongly singular regime $\beta \geq 1$, collisions do not appear for any non-collisional initial data. Then, the above heuristics turns out to be true in the more singular regime. Nevertheless, it is easy to construct initial configurations so that solutions overcome eventual collisions in the weakly singular case, see [244, 245]. Indeed, in such papers, the authors proposed a well-posedness theory of piecewise weak solutions to (1.1.7)-(1.1.18) that accounts for the eventual emergence of collisions and sticking between agents (see Definition 1.1.2 above). In next Section 1.1.4, analogue results will be derived for a singularly weighted version of the Kuramoto model that become the core of Chapters 3 and 4. Indeed, in Section 1.1.5 some relations will be drawn between those singular collective dynamics models. We then postpone such discussion to forthcoming sections and, instead, we comment on the main implications of the above non-collision and collision character of the strongly singular and weakly singular cases..

Regarding the mean field limit, little in known due to the difficulties imposed by the singularity of ϕ. In a recent paper [152] a probabilistic approach to the derivation of the mean field limit was addressed for singular influence function with $\beta < d-1$. However, the main restriction of such result is that it is only valid when ϕ in (1.1.18) is replaced by an appropriate N-dependent cut-off near the origin that blows up when the number of particles N tends to infinity. The general result for the original kernel ϕ is still open.

Similarly, regarding the well-posedness of the kinetic equation, little is known as well. Indeed, only in the particular range of the weakly singular regime $\beta \in (0, \frac{1}{2})$ has been treated, mostly supported by the information at the microcopic scale provided in the above results in [244, 245]. Notice that the expected emergence of Dirac mases lead us to focus on measure-valued solutions. Specifically, in [226] the authors showed the following existence and (partial) uniqueness result of measure-valued solutions.

Theorem 1.1.3. *[226, Theorem 3.1]. Let us consider $0 < \beta < \frac{1}{2}$. For any compactly supported initial data $f_0 \in \mathcal{P}(\mathbb{R}^{2d})$ and any $T > 0$,* (1.1.16)-(1.1.18) *admits at least one weak measure-valued solution $f \in L^\infty(0,T;\mathcal{M}(\mathbb{R}^{2d}))$ with $\partial_t f \in L^p(0,T;C^1_c(\mathbb{R}^{2d})^*)$ for some $p > 1$. Moreover, if f_0 is of the form*

$$f_0(x,v) := \frac{1}{N}\sum_{i=1}^N \delta_{x_i(0)}(x)\delta_{v_i(0)}(v),$$

then f remains atomic of the form

$$f(t,x,v) := \frac{1}{N}\sum_{i=1}^N \delta_{x_i(t)}(x)\delta_{v_i(t)}(v).$$

for all $t \in (0,T]$. In particular, it is a unique measure-valued solution to (1.1.16)-(1.1.18) *(weak atomic uniqueness).*

Although this is not the main topic of this thesis, let us finally mention the main litterature concerning the Euler-alignment system with singular influence function (1.1.17)-(1.1.18). On the one hand, the above collision avoidance in the strongly singular case $\beta \geq 1$ is the cornerstone to derive a well-posedness. Indeed, for such regime in [110, 189, 275, 276, 277] the authors used that the non-local term of the right-hand side of the second equation in (1.1.17) turns out to be a (weighted) fractional Laplacian. Then, the case $d = 1$ is reminiscent of the Burger equation with fractional Laplacian. Thus, global regularity follows from similar regularity methods. For $d = 2$ similar result are obtained in [165] and the general d-dimensional case is still open. On the other hand, in the weakly singular case $\beta \in (0,1)$ one expects the eventual blow up of solutions depending on some threshold, like it was the case for regular influence function (1.1.15). Specifically, in the recent paper [283] some special conditions were found so that blow-up is guaranteed in (1.1.17)-(1.1.18).

However, to date, the derivation of singular Euler-alignment system via rigorous hydrodynamic limits has not been solved yet. We refer to Section (1.1.6) and Chapter 2 for some partial answers. We also refer to the recent review [213] for more details about the Cucker–Smale model with singular weights.

1.1.4 The Kuramoto model of coupled oscillators

The Kuramoto model with mean field coupling (1.1.2) (see [196, 195]) has been extensively studied during the last years as a first a approach to synchronization of agents. Their eventual applications are well known and many of them are addressed in the review [1] and also in [209, 248]. Although Kuramoto initially proposed it for synchronization in chemical reactions, it is a captivating cooperative phenomena that is also observed in biological, physical, and social systems and it has attracted the interest of scientists for centuries. Such mechanism governs the synchronization of flashing of fireflies [48], chorusing of crickets, beating of cardiac cells, metabolic synchrony in yeast cell suspension, etc. Here, we are mainly interested in the above-mentioned application on synchronization of the frequencies of synaptic firing of neurons in the brain. In particular, it allows explaining phase transitions from disordered to ordered states at a critical coupling strength, that is one of the main features of this model and will be slightly addressed later for the readers' convenience. For some applications to the human connective network and how the realistic connectome maps that are available in the literature affect the emergence of synchronization, see [297] and references therein. Those ideas exploit that the human connectome turns out to be organized in modula (characterized by a much larger intra than inter connectivity) structured in a hierarchical nested fashion across many scales, affecting to the neural dynamics [261, 305, 306].

Synchronization in the agent-based model

From a mathematical point of view, there have been important contribution in the analysis of phase and frequency synchronization in the system. The reader may want to look in [114, 145, 147, 157, 202] and references therein. Here we will sketch some of the most relevant results in the study of synchronization that will play a role in this thesis. Before we state them, let us formulate the corresponding Definition 1.1.2 in the particular setting of the Kuramoto model.

Definition 1.1.4 (Phase and frequency synchronization). *Let $\theta_1, \ldots, \theta_N$ solve the Kuramoto model* (1.1.2) *and define the phase and frequency diameters of the system in $\mathbb{R}^N$ as follows*

$$D(\Theta(t)) := \max\{\overline{\theta_i(t) - \theta_j(t)} : \, i, j = 1, \ldots, N\},$$
$$D(\dot{\Theta}(t)) := \max\{\dot{\theta}_i(t) - \dot{\theta}_j(t) : \, i, j = 1, \ldots, N\},$$

for all $t \geq 0$, where bars denote the representative modulo 2π inside the $(-\pi, \pi]$. Then, we say that

1. *The i-th and j-th oscillators collide at $t = t^*$ if*

$$\bar{\theta}(t^*) = \bar{\theta}(t^*).$$

2. *The i-th and j-th oscillators stick at $t = t^*$ if*

$$\bar{\theta}_i(t) = \bar{\theta}_j(t), \;\; \textit{for all} \;\; t \geq t^*.$$

3. *There is asymptotic phase synchronization (PS) if*

$$\lim_{t\to\infty} D(\Theta(t)) = 0.$$

4. *There is asymptotic frequency synchronization (FS) if*

$$\lim_{t\to 0} D(\dot{\Theta}(t)) = 0.$$

5. *The system converges towards a phase-locked state (PLS) if there are $\theta_{ij}^\infty \in \mathbb{R}$ such that*

$$\lim_{t\to 0} \theta_i(t) - \theta_j(t) = \theta_{ij}^\infty.$$

Notice that asymptotic phase synchronization only can take place for identical oscillators $\Omega_1 = \ldots = \Omega_N$. In addition, if there is PS then there also is FS. Also, notice that if there is PLS then there also is FS. The reverse is not necessarily true unless the fall-off of the frequencies diameter is fast enough. In particular, when the heterogeneity disappear and all the agents are identical (e.g. $\Omega_i = 0$), there is complete phase synchronization.

Theorem 1.1.5. *[145, Theorem 3.1] Let $\Theta(t) = (\theta_1(t), \ldots, \theta_N(t))$ be a smooth solution to* (1.1.2) *with $\Omega_i = 0$ and assume that $D(\Theta_0) < \pi$. Then, we have an asymptotic complete phase synchronization. Specifically,*

$$e^{-Kt} D(\Theta_0) \leq D(\Theta(t)) \leq e^{-K\frac{\sin(D(\Theta(0)))}{D(\Theta_0)}t} D(\Theta_0),\ t \geq 0.$$

Although some other approaches had been explored, most of them produced N-dependent rates that do not fit the mean field scaling and, to our best knowledge, this is the first result that avoids that issue. Of course, complete frequency synchronization of identical oscillators is the clear from the above result with a new exponential rate on the diameter. Regarding non-identical oscillators, one cannot expect global phase synchronization. However, one still expects frequency synchronization when the coupling strength K is large enough compared with the size of Ω_i. Indeed, when the decay rate is fast enough it implies emergence of phase-locked states. Those equilibria are characterized by the fact that the inter-particle distances remain constant while rotating in the unit circle. There are several approaches to that we will shortly sketch.

On the one hand, the first approach is based on an uniform bound of the phase diameter under appropriate conditions, that can be used to achieve an explicit exponential decay of the frequency diameter of the system. Specifically,

Theorem 1.1.6. *[145, Theorem 3.3] Assume that*

$$\frac{1}{N}\sum_{i=1}^N \Omega_i = 0,\ D(\Omega) > 0 \ \text{and}\ K > D(\Omega),$$

and let $\Theta(t) = (\theta_1(t), \ldots, \theta_N(t))$ be a smooth solution to (1.1.2) *such that $D(\Theta_0) < D^\infty$ and $\theta_{i,0} \neq \theta_{j,0}$ for every $i \neq j$, where $D^\infty \in (0, \frac{\pi}{2})$ is the unique root of*

$$\frac{D(\Omega)}{K} = \sin x,\ x \in \left(0, \frac{\pi}{2}\right).$$

Then, the phase diameter keeps bounded by D^∞ for all times and, in addition, we have asymptotic complete frequency synchronization

$$D(\dot{\Theta}(t)) \leq D(\dot{\Theta}_0) e^{-K\cos(D^\infty)t},\ t \geq 0.$$

Some other improvements are given in [114] and later in [147], where emergence of phase locked states was proved for any initial configuration. The main restriction in such extensions is the lack of estimate for the frequency decay, that in the general case is still an open problem.

The second approach exploits the gradient system structure of (1.1.2). Specifically, notice that for the potential energy

$$V(\Theta) = -\sum_{i=1}^{N} \theta_i \Omega_i + \frac{K}{2N} \sum_{i,j=1}^{N} (1 - \cos(\theta_i - \theta_j)), \quad \Theta = (\theta_1, \ldots, \theta_N) \in \mathbb{R}^N, \tag{1.1.19}$$

System (1.1.2) is nothing but the gradient flow of V, i.e.,

$$\dot{\Theta(t)} = -\nabla V(\Theta(t)). \tag{1.1.20}$$

Indeed, V is an analytic potential, what in particular implies Łojasiewicz gradient inequality. Namely, for every $\Theta^* \in \mathbb{R}^N$ there exist $\gamma \in \left[\frac{1}{2}, 1\right)$, $L > 0$ and a ball $B_R(\Theta)$ centered at Θ^* such that

$$|V(\Theta) - V(\Theta^*)|^{1-\gamma} \leq C|\nabla V(\Theta)|^2, \text{ for all } \Theta \in B_R(\Theta^*).$$

That can be used to prove that whenever one has a bounded trajectory $\Theta = \Theta(t)$ in $\mathbb{R}^N$, frequency has to converge to zero as $t \to \infty$ and a phase-locked state emerges, see [157]. However, the explicit rate is not given since it is known to depend on the explicit Łojasiewicz exponent γ of the phase-locked state θ'. See [202] where some decay rates have been given in particular cases.

Partial synchronization results in the kinetic model

Regarding the macroscopic model, notice that according to the above part, Neuzert's techniques [230] yields the rigorous mean field limit of (1.1.2), thanks to the regularity of the kernel $F(x, y) = \sin(y - x)$. Indeed, the Vlasov equation agrees with the well known Kuramoto-Sakaguchi equation

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial \theta} \left[\left(\Omega + K \int_{\mathbb{T}\times\mathbb{R}} \sin(\theta' - \theta) f(t, \theta', \Omega') \, d\theta' \, d\Omega' \right) f \right] = 0. \tag{1.1.21}$$

Such idea was first proposed in [198], where L^1 solutions where obtained. However, a more recent approach in [58] also address measure-valued solutions and a contractivity estimate was given in a sort of Dobrushin-type inequality 1.1.11 (see [112, 230]) with negative exponential decay. The authors used such information to transfer the above dynamical properties of agent-based system to the macroscopic system.

On the one hand, the mean field limit allows transferring complete phase synchronization in the identical case.

Theorem 1.1.7. *[58, Lemma 4.1, Theorem 4.1] Suppose $f_0 \in \mathcal{P}(\mathbb{T} \times \mathbb{R})$ such that all the oscillators are identical, i.e., $g = \delta_0(\Omega)$ where $g = (\pi_\Omega)_\# f_0$ is the Ω-marginal, or distribution of natural frequencies. Assume that*

$$\int_{[0,2\pi)\times\mathbb{R}} \theta f_0(\theta, \Omega) \, d\theta \, d\Omega = \pi, \quad D_\theta(f_0) < \pi \text{ and } K > 0.$$

Then, the measure-valued solution f to (1.1.21) *issued at f_0 satisfies*

$$D_\theta(f_t) \leq D_\theta(f_0) e^{-K \frac{\sin(D_\theta(f_0))}{D_\theta(f_0)} t},$$

for $t \geq 0$. *In particular,*

$$\lim_{t\to\infty} d_{BL}(f_t, f_\infty) = 0,$$

where the equilibrium reads $f_\infty = \delta_\pi(\theta) \otimes \delta_0(\Omega)$.

Here, $D_\theta(f_t) = \operatorname{diam}(\operatorname{supp}_\theta f_t)$ is the diameter of the θ-support of f_t, that is, $\operatorname{supp}_\theta(f_t) = \pi_\theta(\operatorname{supp} f_t)$, where π_θ stands for the projection onto the variable θ. The Ω-support of f_t and $D_\Omega(f_t)$ can be similarly defined.

Regarding complete frequency synchronization of non-identical oscillators, the necessary result is the aforementioned contractivity estimate with tespec to to an appropriate transportation distance $\widetilde{W}_p$ on the space of probability measures.

Theorem 1.1.8. *[58, Lemma 5.1] Suppose that two initial measures* $f_0, \widetilde{f}_0 \in \mathcal{P}(\mathbb{T}\times\mathbb{R})$ *and* $K > 0$ *satisfy the following constraints:*

1. $0 < D_\theta(\widetilde{f}_0) \leq D_\theta(f_0) < \pi$.
2. $\displaystyle\int_{[0,2\pi)\times\mathbb{R}} \theta f_0(\theta,\Omega)\, d\theta\, d\Omega = \int_{[0,2\pi)\times\mathbb{R}} \theta \widetilde{f}_0(\theta,\Omega)\, d\theta\, d\Omega = \pi.$
3. $\displaystyle K > D_\Omega(f_0)\max\left\{\frac{1}{\sin(D_\theta(f_0))}, \frac{1}{\sin(D_\theta(\widetilde{f}_0))}\right\}.$

Let $f, \widetilde{f}$ *be two measure-valued solutions to* (1.1.21) *corresponding to the initial data* f_0 *and* $\widetilde{f}_0$ *respectively. Then, there exists* $t_0 > 0$ *and some* $D^\infty \in (0, \frac{\pi}{2})$ *such that*

$$\widetilde{W}_p(f_t, \widetilde{f}_t) \leq e^{-\frac{2K\cos D^\infty}{\pi}(t-t_0)} \widetilde{W}_p(f_{t_0}, \widetilde{f}_{t_0}),$$

for every $t \geq t_0$ *and* $1 \leq p \leq \infty$.

Using such result frequency synchronization of non-identical oscillators takes place.

Corollary 1.1.9. *[58, Lemma 5.1] There exists a unique stationary state* f_∞ *in the set of probability measures fulfilling the properties in Theorem 1.1.8 such that for any other* $f_0 \in \mathcal{P}(\mathbb{T}\times\mathbb{R})$ *in such set, the solution* f *of* (1.1.21) *issued at* f_0 *verifies*

$$\widetilde{W}_p(f_t, f_\infty) \leq e^{-\frac{2K\cos D^\infty}{\pi}(t-t_0)} \widetilde{W}_p(f_{t_0}, f_\infty),$$

for any $t \geq t_0$.

In the above result, the transportation distance $\widetilde{W}_p$ proposed by the authors is not the usual Wasserstein distance in $\mathbb{T}\times\mathbb{R}$ as one might expect. Such a distance is constructed for $f, \widetilde{f} \in \mathcal{P}(\mathbb{T}\times\mathbb{R})$ through the following procedure. First, we unwrap the circle $\mathbb{T}$ into $[0, 2\pi)$ by opening at the point $z = 1 \in \mathbb{T}$. Then, we compute their associated cumulative distribution functions with respect to θ as follows

$$F(\theta,\Omega) := \int_0^\theta f(\theta',\Omega)\, d\theta', \quad \theta \in [0,2\pi),$$
$$\widetilde{F}(\theta,\Omega) := \int_0^\theta \widetilde{f}(\theta',\Omega)\, d\theta', \quad \eta \in [0,2\pi).$$

for any $\Omega \in \mathbb{R}$. Now, we compute their associated pseudoinverses

$$\phi(t,\eta,\Omega) := \inf\{\theta \in [0,2\pi) : \; F(t,\theta,\Omega) > \eta\}, \quad \eta \in [0, g(\Omega)],$$
$$\widetilde{\phi}(t,\eta,\Omega) := \inf\{\theta \in [0,2\pi) : \; \widetilde{F}(t,\theta,\Omega) > \eta\}, \quad \eta \in [0, \widetilde{g}(\Omega)],$$

Only when f and $\widetilde{f}$ have a common distribution of natural frequencies $g = \widetilde{g}$, we can finally compute the following "transportation distance"

$$\widetilde{W_p}(f,\widetilde{f}) := \left(\int_{\mathbb{R}} \|\phi(\cdot,\Omega) - \widetilde{\phi}(\cdot,\Omega)\|^p_{L^p(0,g(\Omega))} d\Omega\right)^{1/p}. \tag{1.1.22}$$

From our point of view, there are some implicit difficulties and restrictions to apply the above transportation distance $\widetilde{W_p}$. On the one hand, notice that it implicitly requires that we can fix the variable Ω and regard $f(\cdot,\Omega)$ and $\widetilde{f}(\cdot,\Omega)$ as functions only depending on θ. Of course, this can be done for absolutely continuous measures, but it is not fully clear for abstract measures. In addition, we implicitly need that both f and $\widetilde{f}$ have the same distribution of natural frequencies. Since f_t and f_∞ in Corollary 1.1.9 satisfy such hypothesis, then $\widetilde{W_p}$ can be applied in such case. However, it is not clear that we can apply it in contractivity Theorem 1.1.8 unless we restrict the class of initial data to those with common distribution g of natural frequencies. In the next part we will introduced a new fiberwise transportation distance $W_{2,g}$ that is related to $\widetilde{W_2}$ but is constructed in terms of the real Wasserstein in $\mathbb{T}$. From its definition, it will become clear at first glance that it applies to abstract probability measures not necessarily absolutely continuous. Indeed, such a metric will prove specially useful in Chapters 4 and 5 of this thesis.

In most of the above results, the cornerstone is the asumption on the phase diameter. Then, such results are nothing but a transference towards the kinetic equations of the preceding results at the discrete level. However, as explored in [147] and other papers, there is a classical quantity that simplifies the understanding of the synchronization dynamics, namely, the order parameter. For the continuous case (the reader can easily adapt it to the agent-based system), the order parameters $R = R(t)$ and $\phi = \phi(t)$ are given by the relation

$$R(t)e^{i\phi(t)} = \int_{\mathbb{T}\times\mathbb{R}} e^{i\theta}\, d_{(\theta,\Omega)} f_t, \; t \geq 0. \tag{1.1.23}$$

The parameter R is a measure of coherence in the system, ranging from disordered states with $R = 0$ to globally phase synchronized states with $R = 1$. In fact, such parameters allow restating the Kuramoto–Sakaguchi equation (1.1.21) as follows

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial \theta}\left(\Omega - KR\sin(\theta-\phi)f\right) = 0.$$

Therefore, we may want to analyze the dynamics of (1.1.21) in terms of the above macroscopic order parameters (1.1.23). This approach has been addressed in particular in [24, 154]. The aim in such ideas is to get rid of the diameter assumption in preceding results and, it has been successfully achieved in certain cases. Let us comment on the above improvements of such idea.

On the one hand, the identical case $g = \delta_0$ is much simpler since (1.1.21) can be restated in terms of the phase density $\rho_t = (\pi_\theta)_\# f_t$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \theta}(-KR\sin(\theta-\phi)\rho) = 0. \tag{1.1.24}$$

In addition, straightforward computations show that R is non-decreasing and, as shown in [24], $\phi(t)$ converges to an asymptotic value. Also, we can compute the stationary states that read as follows

$$\rho_\infty(\theta) = \sigma\delta_{\phi_\infty}(\theta) + (1-\sigma)\delta_{\phi_\infty+\pi}(\theta), \tag{1.1.25}$$

for some parameter $\sigma \in [\frac{1}{2}, 1)$. With all these ingredients, the following result holds true.

Theorem 1.1.10. *[24, Theorem 3.4] Consider $\rho_0 \in \mathcal{P}(\mathbb{T})$ and consider the measure-valued solution ρ to* (1.1.24). *Then, there exists $\phi^* \in \mathbb{R}$ and $\sigma \in [\frac{1}{2}, 1)$ such that*

$$\rho(t) \to \rho_\infty \ \ \textit{narrow in} \ \ \mathcal{P}(\mathbb{T}),$$

as $t \to \infty$, where ρ_∞ is defined in (1.1.25). *In addition, if ρ_0 is non atomic then $\sigma = 1$, i.e., there is complete phase synchronization.*

Such result also has also been proved in [154, Theorem 3.1], showing concentration of mass around $\phi(t)$ with exponential fall-off of the L^2 norm around $\phi(t) + \pi$.

On the other hand, regarding non-identical oscillators the distribution of natural frequencies $g = (\pi_\Omega)_\# f$ plays a role. Specifically, for compactly supported g and large enough K compared to its support, there are stationary solutions with corresponding order parameters ϕ_∞, R_∞ that play an analogue role to the above two-delta functions

$$f_\infty(\theta, \Omega) = g^+(\Omega)\delta_{\vartheta^+(\Omega)}(\theta) + g^-(\Omega)\delta_{\vartheta^-(\Omega)}(\theta), \tag{1.1.26}$$

where $g = g^+ + g^-$ and each Ω-dependent phase $\vartheta^\pm(\Omega)$ take the following form

$$\vartheta^+(\Omega) = \phi_\infty + \arcsin\left(\frac{\Omega}{KR_\infty}\right),$$
$$\vartheta^-(\Omega) = \phi_\infty + \pi - \arcsin\left(\frac{\Omega}{KR_\infty}\right).$$

In addition, the order parameter R_∞ has to verify the consistency relation

$$KR_\infty^2 = \int_{\mathbb{R}} \sqrt{K^2R_\infty^2 - \Omega^2}\, d_\Omega(g^+ - g^-). \tag{1.1.27}$$

However, in such case, the results in [24] are just conditional. They do not yield any rigorous convergence of the order parameters but only characterize the possible equilibria.

Proposition 1.1.11. *[24, Proposition 4.1] Consider $f_0 \in \mathcal{P}(\mathbb{T}\times\mathbb{R})$ non-atomic and let f_t be the solution to* (1.1.21) *issued at f_0. Assume that* $\operatorname{supp} g \subseteq [-KR_\infty, KR_\infty]$ *and that $R(t) \to R_\infty$ and $\phi(t) \to \phi_\infty$ as $t \to \infty$. Then,*

$$f(t) \overset{*}{\rightharpoonup} f_\infty \ \textit{weakly-}* \ \ \textit{in} \ \ \mathcal{P}(\mathbb{T}\times\mathbb{R}),$$

as $t \to \infty$, where f_∞ is given by (1.1.26) *with $g^- \equiv 0$.*

Mass concentration and full synchronization in the kinetic model

To the best of our knowledge, the first unconditional result in the non-identical case was analyzed in [154]. In that result, emergence of phase concentration for non identical oscillators was detected independently on the size of the diameter $D_\theta(f_0)$ of the initial configuration as long as $R_0 := R(0) > 0$ and K is larger that a large enough critical value depending on R_0 and the size of $\operatorname{supp} g$.

Theorem 1.1.12. *[154, Theorem 3.3] Let f be a classical solution to* (1.1.21) *with $R_0 > 0$ and assume that the distribution of natural frequencies has compact support* $\operatorname{supp} g \subseteq [-W, W]$*. Then, for large enough K compared to $\frac{1}{R_0}$ and W*

$$\liminf_{t\to\infty} R(t) \geq R_\infty := 1 + \frac{W}{K} - \sqrt{\frac{W^2}{K^2} + 4\frac{W}{K}},$$

and

$$\lim_{t\to\infty} \left\| f_t \chi_{(\mathbb{T}\setminus L_\infty(t))\times\mathbb{R}} \right\| = 0.$$

Here $L_\infty(t)$ is the interval centered at $\phi(t)$ with constant width larger, but arbitrarily close to

$$\arccos\left(\sqrt{1 - \left[\frac{W}{K}\frac{(1+R_\infty)}{R_\infty^2} + \frac{1-R_\infty}{R_\infty}\right]^2}\right).$$

Notice that as $\frac{K}{W} \to \infty$, the width can be made arbitrarily small and R_∞ tends to one.

The lower bound of the order parameter was essential and is the first result in this line. Also, it is reminiscent of *practical synchronization* at the agent-based level. That is, K has to be large enough, in order for the order parameter R to oscillate arbitrarily close to 1. On the one hand, notice that the above result is not strong enough to derive the convergence of the system towards an equilibrium of the family (1.1.26). At least, it is consistent with Proposition 1.1.11 in the sense that antipodal mass is ruled out by virtue of the above asymptotic convergence of mass towards the interval L_∞. On the other hand, another negative point of such result is that it does not quantifies the rate of convergence of mass to the neighborhood L_∞ of the order parameter.

In Chapter 5 we will provide the full answer to the above questions. Firstly, we quantify a finite time that the system takes to enter a regime in which mass concentrates exponentially fast around $\phi(t)$ by using the information provided by some system of differential inequalities that quantifies four well described principles:

- Soft entropy production
- Instability of equilibria with antipodal mass.
- Emergence of attractor sets of characteristics.
- Accurate control on sliding L^2 norms along sets that evolve along the flow.

Specifically, for an appropriately defined dissipation of the system (recall that non-identical Kuramoto–Sakaguchi is not a Wasserstein gradient flow), we will perform a subdivision that splits the dynamics into subintervals with dissipation below and above some critical threshold. Fortunatelly, the abovementioned system of differential inequalities will be enough to provide a sharp control on each regime so that we can extend to Kuramoto–Sakaguchi the ideas developed by L. Desvilletes and C. Villani in [104] for the Boltzmann equation and quantify mass concentration.

Lemma 1.1.13 (*Corollary 5.2.3 in Chapter 5*)**.** *Let f_0 be contained in $C^1(\mathbb{T}\times\mathbb{R})$ and let g be compactly supported in $[-W, W]$ and centered at $\Omega = 0$, that is,*

$$\int_{\mathbb{R}} \Omega g(\Omega)\, d\Omega = 0. \tag{1.1.28}$$

Consider the unique global-in-time classical solution $f = f(t,\theta,\omega)$ to (1.1.21) *and suppose that $\beta = \pi/3$. Then, there exists a universal constant C such that if*

$$\frac{W}{K} \leq CR_0^3,$$

then we can find a time T_0 with the property that

$$T_0 \lesssim \frac{1}{KR_0^2}\log\left(1 + W^{1/2}\|f_0\|_{L^2} + \frac{1}{R_0}\right),$$

and

$$R(t) \geq \frac{3}{5} \quad \textit{and} \quad \rho\big(\mathbb{T}\backslash L_\beta^+(t)\big) \leq e^{-\frac{1}{20}K(t-T_0)},$$

for every t in $[T_0,\infty)$.

Secondly, we derive generalized log-Sobolev and Talagrand inequalities (see the work [238] by F. Otto and C. Villani for the Fokker–Planck equation) that relate the rate of change of the aforementioned fiberwise transportation distance $W_{2,g}$ and the appropriate concept of dissipation of the system

$$\mathcal{I}[f_t] = \int_{\mathbb{T}\times\mathbb{R}} (\Omega - KR\sin(\theta-\phi))^2 f\, d\theta\, d\Omega.$$

Lemma 1.1.14 (*Lemma 5.3.6 in Chapter 5*)**.** *Assume that f_0 is contained in $C^1(\mathbb{T}\times\mathbb{R})$ and let g be compactly supported in $[-W,W]$ and centeted at Ω, i.e.,* (1.1.28)*. Consider the unique global-in-time classical solution $f = f(t,\theta,\omega)$ to* (1.1.21)*. Then,*

$$\frac{d}{ds}\frac{1}{2}W_{2,g}(f_t,f_s)^2 \leq \mathcal{I}[f]^{\frac{1}{2}} W_{2,g}(f_t,f_s),$$

for every $t \geq 0$ and almost every $s \geq 0$.

Using the above Lemma 1.1.14 after the concentration regime quantified by Lemma 1.1.13 we conclude the converge of the system towards the global equilibrium:

Theorem 1.1.15 (Theorem 5.1.2 of Chapter 5)**.** *Under the assumptions in 1.1.13, we obtain that*

$$W_{2,g}(f_t,f_\infty) \lesssim e^{-\frac{1}{40}K(t-T_0)},$$

for every t in $[T_0,\infty)$. Here f_∞ is the unique global equilibrium of the Kuramoto-Sakaguchi equation up to phase rotations.

For an easier readability, we advance here the definition of our fiberwise transportation distance $W_{2,g}$. Given $f,\widetilde{f} \in \mathcal{P}(\mathbb{T}\times\mathbb{R})$ with same distribution of natural fequencies $g := (\pi_\theta)_\# f = (\pi_\theta)_\# \widetilde{f}$, we first denote their family of conditional probabilities with respect to Ω by $\{f(\cdot|\Omega)\}_{\Omega\in\mathbb{R}}$ and $\{\widetilde{f}(\cdot|\Omega)\}_{\Omega\in\mathbb{R}}$ and we define the fiberwise distance as follows:

$$W_{2,g}(f,\widetilde{f}) := \left(\int_{\mathbb{R}} W_2\big(f(\cdot|\Omega),\widetilde{f}(\cdot|\Omega)\big)^2\, d_\Omega g\right)^{1/2}. \tag{1.1.29}$$

For the sake of clarity, we have provided an Appendix F, that contains an outline of the main required methods from optimal transport theory that will be used in this thesis. In particular, we introduce such fiberwise distance (F.4.2) in Appendix F and we draw some relations with the classical Wasserstein distances. Such a tool is a key point for both Chapters 4 and 5. We restrict our discussion here to clarify the main relations betweem $W_{2,g}$ in (1.1.29) and $\widetilde{W}_p$ in (1.1.22).

Notice that, as mentioned before, $W_{2,g}$ involves the quadratic Wasserstein distance W_2 on $\mathbb{T}$. However, using the notation in the definition (1.1.22) of $\widetilde{W}_p$, we obtain

$$\|\phi(\cdot,\Omega)-\phi(\cdot,\Omega)\|_{L^p(0,g(\Omega))}^p=\widehat{W}_p(f(\cdot|\Omega),\widetilde{f}(\cdot|\Omega))^p g(\Omega),$$

for any $\Omega\in\mathbb{R}$, where $\widehat{W}_p$ is the Wasserstein distance on the interval $[0,2\pi)$ (for such an identity in dimension one we refer to the textbooks [9, 296], see e.g., [268, Proposition 2.17]). Then, we recover the general inequality

$$W_{2,g}(f,\widetilde{f})^2=\int_{\mathbb{R}}W_2(f(\cdot|\Omega),\widetilde{f}(\cdot|\Omega))^2\,d_\Omega g\leq\int_{\mathbb{R}}\widehat{W}_2(f(\cdot|\Omega),\widetilde{f}(\cdot|\Omega))^2\,d_\Omega g=\widetilde{W}_2(f,\widetilde{f})^2,$$

where we have used the fact that transportation distance on $[0,2\pi)$ is in generally larger than on $\mathbb{T}$ and. Indeed, the latter transportation distance is infimum of all the possible transportation distances of the different intervals that we can unwrap $\mathbb{T}$ into. Also, observe that we do not need to make sense for $f(\cdot,\Omega)$ since our construction is described in terms of conditional probabilities $f(\cdot|\Omega)$, that are well defined for abstract probability measures.

The reason to define such a distance will become apparent later, but we here emphasize that such distance is extremely well conditioned to recover stability estimates (or Dobrushin inequalities) for the Kuramoto–Sakaguchi equation. Also, as mentioned before, it proves specially well adapted to derive log-Sobolev and Talagrand-type inequalities (see [238] for the Fokker–Planck equation).

Phase transition and Landau damping

As mentioned before, one of the most interesting features of the Kuramoto model in the mean field limit is the presence of a phase transition at a given critical coupling strength from disordered to ordered states. This was initially conjectured by Kuramoto, and was later rigorously obtained by several authors by analizing the bifurcation diagram, see [56]

Theorem 1.1.16. *[74, Theorems 1.1-1.3] Assume that $g=g(\Omega)$ is the Gaussian distribution or a rational function which is even, unimodal and bounded. Consider the Kuramoto transition point $K_c:=\frac{2}{\pi g(0)}$ and let $f_{inc}(\theta,\Omega)=\frac{g(\Omega)}{2\pi}$ be the incoherent state. Then, the following results hold true:*

1. *(**Instability of the incoherent state**) If $K>K_c$, then f_{inc} is linearly unstable.*
2. *(**Local stability of the coherent state**) If $0<K<K_c$, there exists $\delta>0$ such that if f_0 has distribution of natural frequencies equals g, i.e., $(\pi_\Omega)_\# f_0=g$ and*

$$\left|\int_{\mathbb{T}\times\mathbb{R}}e^{in\theta}\,d_{(\theta,\Omega)}f_0\right|<\delta,\ \text{for all}\ n\in\mathbb{N},\tag{1.1.30}$$

   then, $R(t)$ decays to zero exponentially fast.

3. *(**Bifurcation**) There exist $\varepsilon,\delta>0$ such that if $K_c<K<K_c+\varepsilon$ and f_0 fulfils (1.1.30) then*

$$R(t)=\sqrt{\frac{-16}{\pi K_c^4 g''(0)}}\sqrt{K-K_c}+\mathcal{O}(K-K_c)\ \text{as}\ t\to\infty.$$

Similar results were also obtained in [25, 106, 108]. Notice that the second item can be regarded as *Landau damping* in the vicinity of the incoherent state. Such a phenomenon was first observed in Vlasov equation, see [225] for a comprehensive aproach to the nonlinear version of the problem. Third result actually states that a pitchfork bifurcation arises after $K = K_c$, thus generating stable inhomogenous *partially locked states*. In relation with it, Landau damping towards those partially locked states was introduced in [107, 109].

The Kuramoto model with singular couplings

The above Kuramoto–Sakaguchi equation is still subject of deep study due to the complicatedness of the dynamics and its applications in many areas of Science. In particular, recall that in [261, 297, 305, 306] such model has been applied to model neuronal synchronization. Each node represents neurons in a specific area of the brain and the firing frequencies evolve through the coupled system (1.1.2) (or (1.1.21) for macroscopic description consisting of many nodes). However, uniform coupling weights between neurons are unrealistic in general. Specifically, connections should change with time and adapt to the dynamics itself:

$$\begin{cases} \dfrac{d\theta_i}{dt} = \Omega_i + \dfrac{K}{N}\displaystyle\sum_{j=1}^{N} a_{ij}\sin(\theta_j - \theta_i), \\ \theta_i(0) = \theta_{i,0}, \end{cases} \tag{1.1.31}$$

that is, $a_{ij} = a_{ij}(t)$ are time-evolving and coupled with the dynamics of phases. This is called *plasticity* and can be modelled via a *learning rule* [92, 159, 233, 247, 272], for instance

$$\frac{da_{ij}}{dt} = \eta(\Gamma(\theta_i - \theta_j) - a_{ij}). \tag{1.1.32}$$

The function $\Gamma = \Gamma(\theta)$ is called plasticity function and $\eta > 0$ determines the learning parameter. According to the neuroscientist D. O. Hebb [166], *any two cells or systems of cells that are repeatedly active at the same time will tend to become associated, so that activity in one facilitates activity in the other*. In our setting, it means that Γ must achieve a maximum at the origin so that neurons with close phases become associated and increase their coupling weights.

In the above reference, the choice $\Gamma(\theta) = \cos(\theta)$ was proposed as a particular prototype of *Hebbian learning*. However, since cosine is not positive everywhere, (unrealistic) negative coupling weights might eventually arise, although they end up disappearing after a finite time. Our goal in Chapter 3 is to propose the following modification of the plasticity function

$$\Gamma(\theta) := \frac{\sigma^{2\alpha}}{\left(\sigma^2 + |\theta|_o^2\right)^{\alpha}}, \tag{1.1.33}$$

where $\sigma \in (0, \pi)$, and $|\theta|_o$ is the geodesic distance of $e^{i\theta}$ to 1 along the unit circle, that is

$$|\theta|_o := |\bar{\theta}| \quad \text{for} \quad \bar{\theta} \equiv \theta \bmod 2\pi, \quad \bar{\theta} \in (-\pi, \pi].$$

In this way, only positive values of the weights a_{ij} arise in the dynamics of (1.1.32). More specifically, using a fast learning singular limit of the plasticity rule, we shall reduce the problem to the following collective dynamics agent-based model

$$\begin{cases} \dfrac{d\theta_i}{dt} = \Omega_i + \dfrac{K}{N}\displaystyle\sum_{j=1}^{N} h(\theta_j - \theta_i), \\ \theta_i(0) = \theta_{i,0}, \end{cases} . \tag{1.1.34}$$

where the interaction kernel reads

$$h(\theta) := \frac{\sin\theta}{|\theta|_o^{2\alpha}}, \quad \theta \in \mathbb{R}. \tag{1.1.35}$$

Notice that this is a sort of Kuramoto–Daido model with three regimes of singularity: subcritical for $\alpha \in (0, \frac{1}{2})$, critical for $\alpha = \frac{1}{2}$ and supercritical for $\alpha \in (\frac{1}{2}, 1)$. The presence of singularity is relevant as it opens the scope in the paradigm of the Kuramoto model.

Let us advance the main results that we shall address later in Chapter 3 with regards to such a model. First, we prove the well-posedness of global-in-time absolutely continuous solutions of the agent-based system. In in the critical and supercritical regime, where the kernel is discontinuous, solutions are considered in the sense of Filippov. Recall that Filippov solutions are nothing but solutions to the differential inclusion into the Filippov set-valued map of the system. In the most singular cases $\alpha \in (\frac{1}{2}, 1]$ we will see that the Filippov set-valued map at some point $\Theta \in \mathbb{R}^N$ consists of the values $(\omega_1, \ldots, \omega_N) \in \mathbb{R}^N$ parameterized by

$$\omega_i = \Omega_i + \frac{K}{N} \sum_{\substack{1\leq j\leq N \\ \bar{\theta}_j \neq \bar{\theta}_i}} h(\theta_j - \theta_i) + \frac{K}{N} \sum_{\substack{1\leq j\leq N \\ \bar{\theta}_j = \bar{\theta}_i}} y_{ij}, \tag{1.1.36}$$

for some skew-symmetric matrix $Y = (y_{ij})_{1\leq i,j\leq N}$ with general items in $\mathbb{R}$ if $\alpha \in (\frac{1}{2}, 1)$ or items in $[-1, 1]$ if $\alpha = \frac{1}{2}$. Although one-sided uniqueness will follow for $\alpha \in (0, \frac{1}{2}]$ (because the Filippov set-valued map (1.1.36) is one-sided Lipschitz), it is not clear yet for the supercritical regime $\alpha \in (\frac{1}{2}, 1)$. Indeed, we will propose two different methods to obtain solutions: rigorous limit towards singular kernel and a continuation criterion of classical solutions after collisions. Checking whether they agree is an open problem yet that we have only solved positively for two oscillators. From the point of view of the emergence of collective motion,we emphasize the main novelties that this new model introduces: *finite-time sticking* and *clustering into groups*.

Theorem 1.1.17 (*Theorem 3.3.6 in Chapter 3*)**.** *Consider $\Theta = (\theta_1, \ldots, \theta_N)$, the global-in-time classical solution to* (1.1.34)-(1.1.35) *for $\alpha \in (0, \frac{1}{2})$. Assume that two oscillators collide at t^*, i.e., $\bar{\theta}_i(t^*) = \bar{\theta}_j(t^*)$ for some $i \neq j$. Then, the following two statements are equivalent:*

1. *θ_i and θ_j stick together for all $t \geq t^*$.*
2. *Their natural frequencies agree, i.e., $\Omega_i = \Omega_j$.*

Recall Definition 1.1.4 for the concept of sticking of oscillators in this thesis that, in particular, we use in the above results. In the critical regime, some richer phenomena takes place.

Theorem 1.1.18 (*Corollary 3.3.14 in Chapter 3*)**.** *Consider $\Theta = (\theta_1, \ldots, \theta_N)$ the global-in-time Filippov solution to* (1.1.34)-(1.1.35) *for $\alpha = \frac{1}{2}$. Assume that t^* is some collision time and fix any formed cluster with indices in the set $E \subseteq \{1, \ldots, N\}$ and size $\#E = n$. Then, the following two statements are equivalent:*

1. *The n oscillators in the cluster E stick all together after $t = t^*$.*
2. *The next condition takes place*

$$\left| \frac{1}{n} \sum_{i\in E} \Omega_i - \frac{1}{m} \sum_{i\in I} \Omega_i \right| \leq \frac{K}{N}(n - m), \tag{1.1.37}$$

for every $1 \leq m \leq n$ and every $I \subseteq E$ such that $\#I = m$.

Notice that those results imply that oscillators with different natural frequencies are still allowed to stick in finite time after a collision takes place as long as condition (1.1.37) holds true. This is a conditional result, since we have not show yet that finite-time collision can take place. However, explicit sufficient conditions for global phase synchronization in finite time were also obtained in [241] for identical oscillators ($\Omega_i = 0$) initially confined to the half circle both in the subcritical and critical regime. It is an analogue to the above asymptotic complete phase synchronization of identical oscillators in Theorem 1.1.5.

Theorem 1.1.19 (*Theorem 3.5.4 of Chapter 3*). *Let $\Theta = (\theta_1, \cdots, \theta_N)$ be the classical solution to* (1.1.34)-(1.1.35) *with $\alpha \in \left(0, \frac{1}{2}\right)$ for identical oscillators ($\Omega_i = 0$). Assume that the initial configuration Θ_0 is confined in a half circle, i.e., $0 < D(\Theta_0) < \pi$. Then, there is complete phase synchronization at a finite time not larger than T_c, where*

$$T_c = \frac{D(\Theta_0)^{1-2\alpha}}{2\alpha K h(D(\Theta_0))}.$$

Theorem 1.1.20 (*Theorem 3.5.15 of Chapter 3*). *Let $\Theta = (\theta_1, \cdots, \theta_N)$ be the Filippov solution to* (1.1.34)-(1.1.35) *with $\alpha = \frac{1}{2}$ for identical oscillators ($\Omega_i = 0$). Assume that the initial configuration Θ_0 is confined in a half circle, i.e., $0 < D(\Theta_0) < \pi$. Then, there is complete phase synchronization in a finite time not larger than T_c, where*

$$T_c = \frac{D(\Theta_0)}{K h(D(\Theta_0))}.$$

Indeed, if the Filippov solutions obtained through the above two methods agree in the supercritical case then the latter result remains true in such more singular regime as well, see Remark 3.5.16. We refer to the corresponding chapter for more results and open problems regarding the dynamics of system (1.1.34)-(1.1.35) (e.g., non identical case, emergence of PLS, etc).

Let us move to the contents of Chapter 4 regarding the derivation and dynamics of the macroscopic counterpart of the above collective dynamics model (1.1.34)-(1.1.35). First of all, let us emphasize that the kernel h in (1.1.35) is no longer Lipschitz-continuous and the above techniques by R. Dobrushin and H. Neunzert [112, 230] that were used in for the Kuramoto–Sakaguchi equation [58, 198] do not work for these more singular regimes. Also, the approach in Theorem 1.1.1 does not yield any result since the divergence (derivative in 1D) of the coupling force is not bounded anymore in any of the regimes $\alpha \in (0, 1)$. In this chapter we will introduced a new approach to deal with this sort of kernels for $\alpha \in (0, \frac{1}{2}]$. Specifically, we shall construct weak measure-valued solutions (in the sense of the Filippov flow for $\alpha \in \frac{1}{2}$) to the kinetic singular Kuramoto model. Here, we will first recall the subcritical and critical regimes whilst the supercritical case will be sketched later.

On the one hand, it is clear that (formally) the kinetic singular Kuramoto model reads

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial \theta}\left[\left(\Omega + K \int_{\mathbb{T}\times\mathbb{R}} h(\theta' - \theta) f(t, \theta', \Omega')\, d\theta'\, d\Omega'\right) f\right] = 0,$$

whete $f = f(t, \theta, \Omega)$ is the distribution function of oscillators for all $t \geq 0$ and $(\theta, \Omega) \in \mathbb{T} \times \mathbb{R}$. Notice that we are assuming periodic boundary conditions with respect to the variable θ. Such a model equivalently takes the form of a nonlinear transport equation along the manifold $\mathbb{T}\times\mathbb{R}$, namely,

$$\begin{cases} \dfrac{\partial f}{\partial t} + \operatorname{div}(\mathcal{V}[f]f) = 0, \ (z, \Omega) \in \mathbb{T} \times \mathbb{R}, \\ f(0) = f_0, \end{cases} \tag{1.1.38}$$

where the divergence is considered along $\mathbb{T}\times\mathbb{R}$ and the non-linear transport field reads

$$\begin{aligned}\mathcal{V}[f](z,\Omega) &:= (\mathcal{P}[f](z,\Omega)iz, 0),\\ \mathcal{P}[f](z,\Omega) &:= \Omega + K\int_{\mathbb{T}\setminus\{z\}}\int_{\mathbb{R}} h(\theta'-\theta)\, d_{(\theta',\Omega')}f,\end{aligned} \tag{1.1.39}$$

for any $(z = e^{i\theta},\Omega)\in\mathbb{T}\times\mathbb{R}$. Obviously, the transport field only makes sense for $\alpha\in(0,\frac{1}{2}]$ because f_t is merely measure-valued. Indeed, the integral is intentionally considered off $\{z\}$ to avoid concentration issues for $\alpha = \frac{1}{2}$. Notice that it is totally consistent with the microscopic dynamics as $y_{ii} = 0$ in (1.1.36). Indeed, such definition does not make any sense for $\alpha\in(\frac{1}{2},1)$ unless f_t enjoys some extra integrability, that we do not expect to propagate due to concentration phenomena at the microscopic scale.

We will prove the existence and sided-uniqueness of a classical flow for $\alpha\in(0,\frac{1}{2})$ (respectively Filippov flow for $\alpha=\frac{1}{2}$) of the transport field $\mathcal{V}[f]$ due to the fact that the transport field is continuous with linear growth at infinity for $\alpha\in(0,\frac{1}{2})$ (respectively, it is locally bounded with linear growth at infinity for $\alpha=\frac{1}{2}$) and it is one-sided Lipschitz continuous. In addition, we will show that the mean-field limit approach works although Theorem 1.1.1 does not apply. Specifically, we prove that the empirical measures supported on classical (respectively Filippov) solutions to (1.1.34) are measure-valued solutions to (1.1.34) that converge to the unique weak measure-valued solution (respectively, solution in the sense of the Filippov flow) to (1.1.38) as $N\to\infty$. Indeed, we will derive a similar Dobrushin-type estimate to (1.1.11).

Theorem 1.1.21 (*Theorems 4.4.6 and 4.6.31 in Chapter 4*)**.** *Consider $\alpha\in(0,\frac{1}{2}]$, $K>0$ and two time-dependent probability measures $f,\widetilde{f}\in AC_{loc}([0,\infty), C_c^\infty(\mathbb{T}\times\mathbb{R})^* - weak*)$, solving* (1.1.38) *weakly in the sense of measures (respectively in the sense of Filippov flow) with associated initial data $f_0,\widetilde{f}_0\in\mathcal{P}_2(\mathbb{T}\times\mathbb{R})$. Then,*

$$W_2(f_t,\widetilde{f}_t)\le e^{\left(\frac{1}{2}+2KL_0\right)t}W_2(f_0,\widetilde{f}_0),$$

for every $t\ge 0$, where L_0 is the one-sided Lipschitz constant of $-h$.

Here, $\mathcal{P}_2(\mathbb{T}\times\mathbb{R})$ represents the metric space of probability measures on $\mathbb{T}\times\mathbb{R}$ with finite second order moment endowed with the standard 2-Wasserstein distance W_2 in the product space $\mathbb{T}\times\mathbb{R}$.

Such stability results imply two important consequences. On the one hand, it allows obtaining a quantitative mean field limit when $\widetilde{f}=\mu^N$ and the initial empirical measures approximate the initial datum f_0, i.e.,

$$\lim_{N\to\infty} W_2(\mu_0^N, f_0) = 0.$$

On the other hand, notice that uniqueness follows by simply choosing $f_0=\widetilde{f}_0\in\mathcal{P}_2(\mathbb{T}\times\mathbb{R})$.

In the above consequence, the extra tightness imposed by $\mathcal{P}_2(\mathbb{T}\times\mathbb{R})$ is technically assumed in order that W_2 makes sense. Nevertheless, we can still obtain uniqueness for general probability measures $f_0\in\mathcal{P}(\mathbb{T}\times\mathbb{R})$ by virtue of a similar Dobrushin inequality for the fiberwise distance $W_{2,g}$ that we introduced before.

Theorem 1.1.22 (*Theorems 4.4.2 and 4.6.28 in Chapter 4*)**.** *Consider $\alpha\in(0,\frac{1}{2}]$, $K>0$ and two time-dependent probability measures $f,\widetilde{f}\in AC_{loc}([0,\infty), C_c^\infty(\mathbb{T}\times\mathbb{R})^* - weak*)$, solving* (1.1.38) *weakly (respectively in the sense of Filippov flow) with initial data $f_0,\widetilde{f}_0\in\mathcal{P}(\mathbb{T}\times\mathbb{R})$. Consider the distribution of natural frequencies $g=(\pi_\Omega)_\# f_0$ and $\widetilde{g}=(\pi_\Omega)_\#\widetilde{f}_0$. If both distributions of natural frequencies agree $g=\widetilde{g}$, then*

$$W_{2,g}(f_t,\widetilde{f}_t)\le W_{2,g}(f_0,\widetilde{f}_0)e^{2KL_0t},$$

for every $t \geq 0$, where L_0 is the one-sided Lipschitz constant of $-h$.

In addition, the mean-field limit allows transferring Theorems 1.1.19 and 1.1.20 towards the macroscopic equation (1.1.38).

Theorem 1.1.23 (*Theorem 4.5.8 in Chapter 4*)**.** *Set $\alpha \in (0, \frac{1}{2})$ and consider any initial datum $f_0 \in \mathcal{P}(\mathbb{T} \times \mathbb{R})$ with identical distribution of natural frequencies, namely, $g = (\pi_\Omega)_\# f_0 = \delta_0$. Let f be the unique global-in-time weak measure-valued solution to* (1.1.38) *issued at f_0 and assume that $0 < D_\theta(f_0) < \pi$. Then,*

$$f(t) = f_\infty \text{ for all } t \geq T_c,$$

where

$$T_c = \frac{D_\theta(f_0)^{1-2\alpha}}{2\alpha K h(D_\theta(f_0))},$$

and the equilibrium f_∞ is given by the monopole $f_\infty := \delta_{z_{av}(0)}(z) \otimes \delta_0(\Omega)$ and z_{av} is the average phase of the oscillators.

Theorem 1.1.24 (*Theorem 4.6.33 in Chapter 4*)**.** *Set $\alpha = \frac{1}{2}$ and consider any initial datum $f_0 \in \mathcal{P}(\mathbb{T} \times \mathbb{R})$ with identical distribution of natural frequencies, namely, $g = (\pi_\Omega)_\# f_0 = \delta_0$. Let f be the unique global-in-time measure-valued solution to* (1.1.38) *in the sense of the Filippov flow issued at f_0 and assume that $0 < D_\theta(f_0) < \pi$. Then,*

$$f(t) = f_\infty \text{ for all } t \geq T_c,$$

where

$$T_c = \frac{D_\theta(f_0)}{K h(D_\theta(f_0))},$$

and the equilibrium f_∞ is given by the monopole $f_\infty := \delta_{z_{av}(0)}(z) \otimes \delta_0(\Omega)$ and z_{av} is the average phase of the oscillators.

1.1.5 From Kuramoto to Cucker–Smale

We devote this part to state some main relations between the two main collective dynamics model in this thesis: Cucker–Smale (1.1.7) and Kuramoto (1.1.2). Indeed, such relation does not necessarily restrict to smooth influence functions, but it will also remain valid for the singular versions that we have introduced in the preceding parts. In general, consider a Kuramoto–Daido model as follows

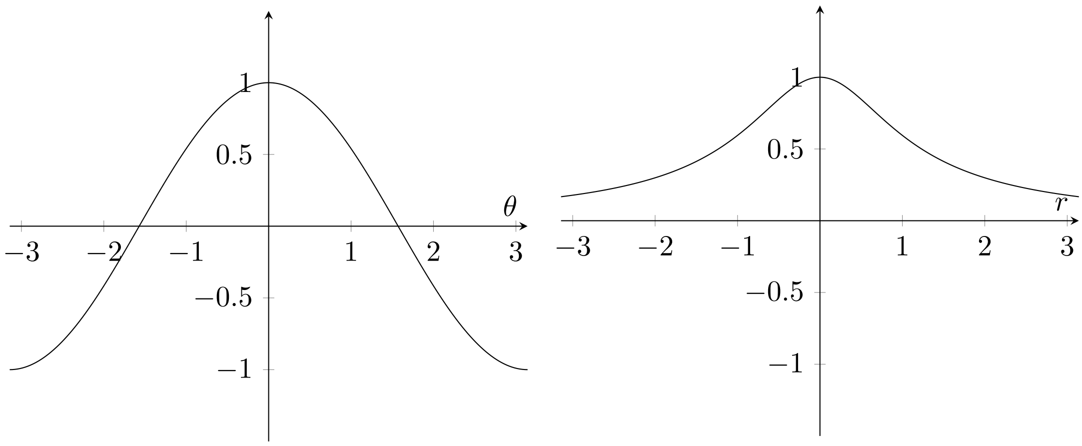
$$\begin{cases} \dot{\theta}_i = \Omega_i + \dfrac{K}{N}\displaystyle\sum_{j=1}^N h(\theta_j - \theta_i), \\ \theta_i(0) = \theta_{i,0}, \end{cases} \tag{1.1.40}$$

for an abstract periodic force h. If it is smooth, differentiation implies

$$\begin{cases} \dot{\theta}_i = \omega_i, \\ \dot{\omega}_i = \dfrac{K}{N}\displaystyle\sum_{j=1}^N \phi(\theta_j - \theta_i)(\omega_j - \omega_i), \\ (\theta_i(0), \omega_i(0)) = (\theta_{i,0}, \omega_{i,0}), \end{cases} \tag{1.1.41}$$

where $\phi := h'$. Notice that the initial and natural frequencies are related through the rule

$$\omega_{i,0} = \Omega_i + \frac{K}{N}\sum_{j=1}^N \phi(\theta_{j,0} - \theta_{i,0}).$$



(a) $phi(\theta) = \cos\theta$ (b) $\phi(r) = \frac{1}{(1+r^2)^{3/4}}$

Figure 1.2: On the left, influence function $\phi(\theta) = h'(\theta) = \cos\theta$ associated with the classical Kuramoto model. On the right, smooth infuence function (1.1.15) of the Cucker–Smale model

This amounts to say that Kuramoto–Daido model (1.1.40) agrees with Cucker–Smale model (1.1.41) for well-prepared initial data. In other words, the Kuramoto–Daido model (1.1.40) implicitly describes the evolution of swarms.

In particular, when $h(\theta) = \sin\theta$, one obtains that the classical Kuramoto model agrees with the Cucker–Smale model with an influence function $\phi(\theta) = \cos\theta$. This can be used to understand frequency synchronization as a flocking phenomenon. Observe indeed that alignment and frequency synchronization in Definitions 1.1.2 and 1.1.4 correspond each others. Then, the above relation allows transferring techniques between both models in 1D. In particular, see [150] where it was introduced and used to derive an alternative kinetic description for synchronization of oscillators

$$\frac{\partial F}{\partial t} + \omega\frac{\partial F}{\partial \theta} + \frac{\partial}{\partial \omega}\left[K\left(\int_{\mathbb{T}\times\mathbb{R}} \cos(\theta' - \theta)(\omega' - \omega)F(t, \theta', \Omega', \omega')\, d\theta'\, d\Omega'\, d\omega'\right) F\right] = 0.$$

However, let us notice that by doing so ϕ is not necessarily positive everywhere and this introduces a main difference in (1.1.41) in the form of “anti-alignment” between oscillators that are separated by distances close to π, see Figure 1.2.

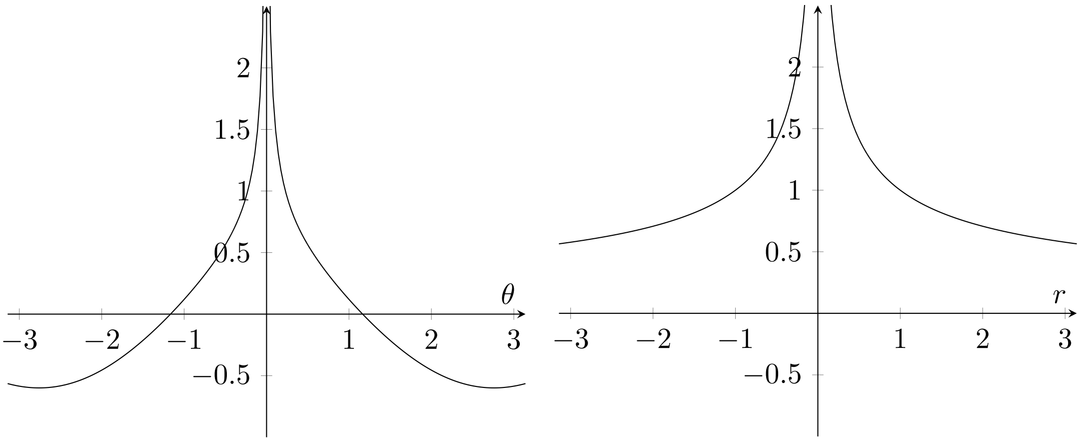
Regarding the singular cases, in the subcritical regime frequencies enjoy a minimum regularity required to describe the augmented second order equation. Namely,

Theorem 1.1.25 (*Remark 3.4.5 in Chapter 3*)**.** *Consider a classical solution $\Theta = (\theta_1, \ldots, \theta_N)$ to* (1.1.34) *with $\alpha \in (0, \frac{1}{2})$. Then, the frequencies verify $\dot{\theta}_i \in W^{1,p}([T_{k-1}, \tau])$, for $1 \leq p < \frac{1}{2\alpha}$, every $k \in \mathbb{N}$ and every $\tau \in (T_{k-1}, T_k)$. In addition, they verify the following equation in weak sense*

$$\ddot{\theta}_i = \frac{K}{N} \sum_{j \notin S_i(T_{k-1})} h'(\theta_j - \theta_i)(\dot{\theta}_j - \dot{\theta}_i), \tag{1.1.42}$$

for all $t \in [T_{k-1}, \tau]$. Here, $\{T_k\}_{k\in N}$ are the new collision times after some oscillators have stick together and $S_i(T_{k-1})$ means the set of indices j of oscillators that stick with the i-th one at $t = T_{k-1}$.

In other words, the singular Kuramoto model (1.1.34)-(1.1.35) in the weakly singular regime $\alpha \in (0, \frac{1}{2})$ is reminiscent of the Cucker–Smale model with singular influence function (1.1.7)-(1.1.18) for the choice of exponents $\beta = 2\alpha$. Here, we call the attentions of readers as this choice



(a) $\phi(\theta) = h'(\theta)$ (b) $\phi(r) = \frac{1}{r^{1/2}}$

Figure 1.3: On the left, influence function $\phi = \psi'$ in (1.1.35) associated with the singular Kuramoto model. On the right, weakly singular infuence function (1.1.18) of the Cucker–Smale model

of exponents will be propagated along the whole thesis. To unify notation, we will not use parameter β but parameter 2α here on.

Notice that the main difference between the (singular) Kuramoto and (singular) Cucker–Smale models is the periodicity assumption along with the fact that for the Kuramoto model, the influence function is not always positive. Indeed, as mentioned before, ϕ also attains negative values near $\theta = \pi$, see Figure 1.3. This means that Kuramoto oscillators with far apart phases are pushed away from the flock. Nevertheless, as shown in [24, 154, 222], the periodicity conditions recover the unique flock when the natural frequencies agree.

1.1.6 Hydrodynamic limits and other asymptotic limits

In this subsection we will review some hyperbolic hydrodynamic limits that have been analyzed in the literature for the kinetic Cucker–Smale model (1.1.16).

- In the first part, we will focus on Lipschitz influence functions ϕ, i.e. (1.1.15), which corresponds to the classical Cucker–Smale model. As mentioned below, the goal is to rigorously derive the Euler-alignment model (1.1.17) when the influence function is smooth.

- Later, since the full hydrodynamic limit in the singular regimes is still a hard open problem, we will introduce a particular macroscopic approximation. Specifically we shall propose a singular hyperbolic hydrodynamic limit of vanishing inertia type for the weakly singular case $\alpha \in (0, \frac{1}{2}]$, that covers the contents in Chapter 2 of this thesis. Such a method yields a reduced first order fluid model where inertia in the balance equation of momentum has been neglected in the flavour of the *overdamped limit* or *Smoluchowski dynamics* for the second order system with inertia and friction that we introduced in (1.1.5).

- Finally, due to its relation to this last case, we will show that a similar approach can be done to derive weak measure-valued solutions of the kinetic singular Kuramoto model (1.1.38) in the supercritical regime $\alpha \in (\frac{1}{2}, 1)$. We will sketch a similar singular hyperbolic hydrodynamic limit of vanishing inertia type on an augmented Kuramoto-type model with inertia and regularized weights.

Hydrodynamic limits for Lipschitz influence function

To derive macroscopic hydrodynamic models of the kinetic model of Cucker–Smale with classical Lipschitz interactions (1.1.15), new terms were introduced in [186]. Specifically, the following hyperbolic scaled model was proposed

$$\begin{cases} \dfrac{\partial f_\varepsilon}{\partial t} + v \cdot \nabla_x f_\varepsilon = \operatorname{div}_v(Q_{CS}(f_\varepsilon, f_\varepsilon)) + \dfrac{1}{\varepsilon} \operatorname{div}_v(\nabla_v f_\varepsilon + (v - u_\varepsilon) f_\varepsilon), \\ Q_{CS}(f_\varepsilon, f_\varepsilon)(t,x,v) := K \displaystyle\int_{\mathbb{R}^{2d}} \phi(|x - x'|)(v - v') f(t,x,v) f(t,x',v')\, dx'\, dv'. \end{cases} \tag{1.1.43}$$

Notice that such model includes velocity noise (through a Fokker–Planck term) and a *local alignment effect* of the velocity towards the mean velocity field

$$u_\varepsilon(t,x) = \frac{\int_{\mathbb{R}^d} v f_\varepsilon(t,x,v)\, dv}{\int_{\mathbb{R}^d} f_\varepsilon(t,x,v)\, dv}.$$

The hyperbolic scaling sets a regime with large noise and strong local alignment but weak nonlocal alignment of Cucker–Smale type. Notice that such local alignment term can be regarded as linear damping towards the macroscopic velocity field and provides no effect on the balance equation of momentum by virtue of its cancellations

$$\int_{\mathbb{R}^d} v \operatorname{div}_v((v - u_\varepsilon) f_\varepsilon)\, dv = -\int_{\mathbb{R}^d} (v - u_\varepsilon) f_\varepsilon\, dv = 0.$$

This local alignment term $(v - u)f$ was introduced in [223] as the singular limit $\phi \to \delta_0$ of in the Mostch–Tadmor nonlinear alignment term

$$Q_{MT}(f)(t,x,v) = K \frac{\int_{\mathbb{R}^{2d}} \phi(|x - x'|)(v - v') f(t,x,v) f(t,x',v')\, dx'\, dv'}{\int_{\mathbb{R}^{2d}} \phi(|x - x'|) f(t,x,v')\, dx'\, dv'}. \tag{1.1.44}$$

The main idea in (1.1.44) is to normalized the pairwise interactions $\phi(|x_i - x_j|)$ between agents in terms of a relative influence. Of course, it breaks the symmetry of the initial Cucker–Smale model, what in particular causes severe problems to recover such kinetic model as mean field limit of the corresponding agent-based description.

When $\varepsilon \to 0$, relative entropy methods were used in [186] to obtain the hydrodynamic limit of (1.1.43), that takes the form

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \operatorname{div}_x(\rho\, u) = 0, \\ \dfrac{\partial}{\partial t}(\rho\, u) + \operatorname{div}_x(\rho\, u \otimes u) = -\nabla_x \rho + \phi * (\rho u)\rho - (\phi * \rho)\rho u. \end{cases}$$

Note that such models maintain nonlocal alignment effects but does not include any local damping, as it disappeared in the limit. Indeed, the strong local alignment in (1.1.43) was only introduced as an extra term that helps the system reach the hydrodynamic regime. Unfortunately, notice that an extra pressure term $-\nabla_x \rho$ has appeared as a consequence of the velocity noise in the Fokker–Planck term of the right hand side of (1.1.43).

In relation with such scaling, the method was very recently improved in [127] to remove the velocity term noise. Specifically, the following system was considered

$$\begin{cases} \dfrac{\partial f_\varepsilon}{\partial t} + v \cdot \nabla_x f_\varepsilon = \operatorname{div}_v(Q_{CS}(f_\varepsilon, f_\varepsilon)) + \dfrac{1}{\varepsilon} \operatorname{div}_v((v - u_\varepsilon) f_\varepsilon), \\ Q_{CS}(f_\varepsilon, f_\varepsilon)(t,x,v) := K \displaystyle\int_{\mathbb{R}^{2d}} \phi(|x - x'|)(v - v') f(t,x,v) f(t,x',v')\, dx'\, dv'. \end{cases} \tag{1.1.45}$$

Again, similar relative methods allow recovering the well known pressureless Euler-alignment model

$$\left\{\begin{array}{l} \dfrac{\partial\rho}{\partial t}+\operatorname{div}_x(\rho\, u)=0,\\ \dfrac{\partial}{\partial t}(\rho\, u)+\operatorname{div}_x(\rho\, u\otimes u)=\phi*(\rho u)\rho-(\phi*\rho)\rho u. \end{array}\right. \tag{1.1.46}$$

where the above pressure term in the right hand side of the momentum equation has disappeared. This is precisely the abovementioned macroscopic system (1.1.17) that arises as the monokinetic antsatz of the Cucker–Smale model. However, its rigorous derivation has not been proved yet without the help of some extra damping or strong local alignment terms.

Another close approach was given in [185]. The velocity noise and nonlocal Cucker–Smale alignment term in (1.1.43) were neglected but the strong local alignment was kept and linear damping was also added to the system

$$\frac{\partial f_\varepsilon}{\partial t}+v\cdot\nabla_x f_\varepsilon=\lambda\operatorname{div}_v(vf_\varepsilon)+\frac{1}{\varepsilon}\operatorname{div}_v((v-u_\varepsilon)f_\varepsilon).$$

In this case, a similar analysis provides the limiting system

$$\left\{\begin{array}{l} \dfrac{\partial\rho}{\partial t}+\operatorname{div}_x(\rho\, u)=0,\\ \dfrac{\partial}{\partial t}(\rho\, u)+\operatorname{div}_x\cdot(\rho\, u\otimes u)=-\lambda\rho u. \end{array}\right.$$

Again, the strong local alignment is lost in the macroscopic system, but a linear damping has been recovered in the limit. This represents the compressible Euler equations with velocity damping.

In the same line, when agents are driven by a fluid, the following coupled system with fluids has been considered in [61]

$$\frac{\partial f_\varepsilon}{\partial t}+v\cdot\nabla_x f_\varepsilon=\operatorname{div}_v((v-U_\varepsilon)f_\varepsilon)+\frac{1}{\varepsilon}\operatorname{div}_v(\nabla_v f_\varepsilon+(v-u_\varepsilon)f_\varepsilon),$$

where U_ε is the velocity field of the fluid, which evolves according to the incompressible Navier–Stokes system, i.e.,

$$\left\{\begin{array}{l} \dfrac{\partial U_\varepsilon}{\partial t}+(U_\varepsilon\cdot\nabla_x)U_\varepsilon=-\nabla_x p_\varepsilon+\nu\Delta U_\varepsilon+(u_\varepsilon-U_\varepsilon)\rho_\varepsilon,\\ \operatorname{div}_x U_\varepsilon=0, \end{array}\right.$$

where $\nu\geq 0$ is the viscosity and p_ε stands for the pressure of the fluid. In such paper, an entropy method in the spirit of [186] was derived to pass to the limit $\varepsilon\to 0$ and the following limiting macroscopic system was obtained

$$\left\{\begin{array}{l} \dfrac{\partial\rho}{\partial t}+\operatorname{div}_x(\rho\, u)=0,\\ \dfrac{\partial}{\partial t}(\rho\, u)+\operatorname{div}_x(\rho\, u\otimes u)=-\nabla_x\rho+(U-u)\rho, \end{array}\right.$$

coupled with the limiting Navier–Stokes system

$$\left\{\begin{array}{l} \dfrac{\partial U}{\partial t}+(U\cdot\nabla_x)U=-\nabla_x p+\nu\Delta U+\rho(u-U),\\ \operatorname{div}_x U=0. \end{array}\right.$$

For the readers' convenience, let us mention another alternative to the above hydrodynamic limits in wich the scaling lead to a vanishing inertia effect on the macroscopic limit, thus reducing the second order dynamics to the Smoluchoski first order dynamics. In particular, this line has been developed in [22, 142, 232] for the Vlasov–Poisson–Fokker–Planck system, that give rise to the aggregation equation with Newtonian interactions, i.e.,

$$\begin{cases} \partial_t \rho + \operatorname{div}_x(\rho\, u) = 0, \\ u = -\nabla_x \varphi, \end{cases}$$

where potential $\varphi = \varphi(t,x)$ can be recovered from the density ρ through the Poisson equation

$$\Delta\varphi = \theta\rho,\ x \in \mathbb{R}^d,$$

and $\theta = 1$ or $\theta = -1$ respectively represent the attractive of repulsive character of the Newtonian interactions. In the parabolic case, the limiting system changes the velocity field from $u = -\nabla_x\varphi$ to $u = -\nabla_x\varphi + \frac{\nabla_x\rho}{\rho}$, that includes viscosity on the continuity equation for ρ.

Hydrodynamic singular limits of vanishing inertia type

Mimicking the preceding ideas, in Chapter 2 we will consider the kinetic singular Cucker–Smale model, with linear damping, velocity noise (Fokker-Planck term) and the effect of an external force $-\nabla_x\psi$. A dimensionless analysis will be proposed in Appendix 2.A of that chapter, leading to the following scaled system

$$\begin{cases} \dfrac{\partial f_\varepsilon}{\partial t} + v\cdot\nabla_x f_\varepsilon = \dfrac{1}{\varepsilon}\operatorname{div}_v(Q_\varepsilon(f_\varepsilon)f_\varepsilon) + \dfrac{1}{\varepsilon}\nabla_x\psi_\varepsilon\cdot\nabla_v f_\varepsilon + \dfrac{1}{\varepsilon}\operatorname{div}_v(v\, f_\varepsilon + \nabla_v f_\varepsilon), \\ Q_\varepsilon(f_\varepsilon, f_\varepsilon)(t,x,v) := \displaystyle\int_{\mathbb{R}^{2d}} \phi_\varepsilon(|x-x'|)(v-v')f_\varepsilon(t,x,v)f_\varepsilon(t,x',v')\,dx'\,dv'. \end{cases} \tag{1.1.47}$$

Here, the singular influence function (1.1.18), with $\beta = 2\alpha$, has been regularized as follows

$$\phi_\varepsilon(r) = \frac{1}{(\varepsilon^2 + c_\alpha r^2)^\alpha},\ r > 0, \tag{1.1.48}$$

for some $\alpha \in (0, \frac{1}{2}]$ and a α-dependent coefficient c_α. Notice that the scaled kernel (1.1.48) converges towards the singular one (1.1.18) as $\varepsilon \to 0$. Given a sufficiently regular initial data $f_\varepsilon(0) = f_\varepsilon(0,x,v)$ and the corresponding smooth solution $f_\varepsilon(t) = f_\varepsilon(t,x,v)$ to the regularized system (1.1.47), one can associate the macroscopic quantities:

$$\begin{aligned} \text{Density:}\quad & \rho_\varepsilon(t,x) := \int_{\mathbb{R}^d} f_\varepsilon(t,x,v)\,dv, \\ \text{Current:}\quad & j_\varepsilon(t,x) := \int_{\mathbb{R}^d} v\, f_\varepsilon(t,x,v)\,dv, \\ \text{Velocity field:}\quad & u_\varepsilon(t,x) := \frac{j_\varepsilon(t,x)}{\rho_\varepsilon(t,x)}, \\ \text{Stress tensor:}\quad & \mathcal{S}_\varepsilon(t,x) := \int_{\mathbb{R}^d} v\otimes v\, f_\varepsilon(t,x,v)\,dv, \end{aligned}$$

which verifies the following system of conservation laws

$$\frac{\partial\rho_\varepsilon}{\partial t} + \operatorname{div}_x j_\varepsilon = 0, \tag{1.1.49}$$

$$\varepsilon \frac{\partial j_\varepsilon}{\partial t} + \varepsilon \operatorname{div}_v \mathcal{S}_\varepsilon + \rho_\varepsilon \nabla_x \psi_\varepsilon + j_\varepsilon + (\phi_\varepsilon * \rho_\varepsilon)\, j_\varepsilon - (\phi_\varepsilon * j_\varepsilon)\, \rho_\varepsilon = 0. \tag{1.1.50}$$

Our goal is to find appropriate compactness on ρ_ε, j_ε and $\mathcal{S}_\varepsilon$ so that we can pass to the limit in the system (1.1.49)-(1.1.50). Specifically, we will show that, in particular, we get

$$\begin{aligned} \rho_\varepsilon &\overset{*}{\rightharpoonup} \rho, \quad \text{in } L^\infty_w(0,T;\mathcal{M}(\mathbb{R}^d)),\\ j_\varepsilon &\overset{*}{\rightharpoonup} j, \quad \text{in } L^2_w(0,T;\mathcal{M}(\mathbb{R}^d))^d. \end{aligned}$$

as $\varepsilon \to 0$ for an appropriate subsequence. In addition we achieve a useful dissipation estimate of the system that takes the following form

$$\int_0^T \int_{\mathbb{R}^{4d}} \phi_\varepsilon(|x-x'|)|v-v'|^2 f_\varepsilon(t,x,v) f_\varepsilon(t,x',v')\, dx\, dx'\, dv\, dv'\, dt \\ \leq C\left(d, T, \|\nabla\psi_\varepsilon\|_{L^2(0,T;L^\infty(\mathbb{R}^d))}, \||v|^2 f_\varepsilon(0)\|_{L^1(\mathbb{R}^{2d})}\right). \tag{1.1.51}$$

for every $\varepsilon > 0$. Passing to the limit in all the linear terms of (1.1.49)-(1.1.50) is clear. However, the main problem is the non-linear term in (1.1.50). To achieve this goal it is necessary to take into account the kindness of the commutator that defines the nonlinear term, the symmetry of the influence function, the range of values $\alpha \in (0, \frac{1}{2})$, as well as additional properties of convergence in time for ρ, see [255] for the details. Indeed, the above properties allow us to identify the limit

$$\rho_\varepsilon \otimes j_\varepsilon \overset{*}{\rightharpoonup} \rho \otimes j \quad \text{in } L^2_w(0,T;\mathcal{M}(\mathbb{R}^{2d}))^d,$$

that in turns, allows identifying the limit in (1.1.50) in the sense of distributions for $0 < \beta \leq 1$ and obtaining:

Theorem 1.1.26 (*Theorem 2.2.9 in Chapter 2*)**.** *Let f^0_ε verify the hypothesis*

$$\left\{\begin{array}{l} f^0_\varepsilon = f^0_\varepsilon(x,v) \geq 0 \ \text{ and } \ f^0_\varepsilon \in C^\infty_c(\mathbb{R}^d \times \mathbb{R}^d),\\ \|f^0_\varepsilon\|_{L^1(\mathbb{R}^{2d})} = 1 \ \text{ and } \rho^0_\varepsilon \overset{*}{\rightharpoonup} \rho^0 \text{ in } \mathcal{M}(\mathbb{R}^d),\\ \||x| f^0_\varepsilon\|_{L^1(\mathbb{R}^{2d})} \leq M_0 \ \text{ and } \ \||v|^2 f^0_\varepsilon\|_{L^1(\mathbb{R}^{2d})} \leq E_0, \end{array}\right.$$

for every $\varepsilon > 0$ and some ε-independent constants $M_0, E_0 > 0$. Also assume that the external $-\nabla_x\psi_\varepsilon$ forces satisfy appropriate mild assumptions. Let $f_\varepsilon = f_\varepsilon(t,x,v)$ be the smooth solutions to (1.1.47) *with $\alpha \in (0, \frac{1}{2}]$. Then, ρ_ε and j_ε converge in a weak sense to some finite Radon measure ρ and j that solve the Cauchy problem associated with the following Euler-type system in the distributional sense*

$$\left\{\begin{array}{ll} \dfrac{\partial \rho}{\partial t} + \operatorname{div}_x j = 0, & x \in \mathbb{R}^d,\ t \in [0,T),\\ \rho\,\nabla_x V + j = (\phi * j)\rho - (\phi * \rho) j, & x \in \mathbb{R}^d,\ t \in [0,T)\\ \rho(0,\cdot) = \rho^0, & x \in \mathbb{R}^d. \end{array}\right. \tag{1.1.52}$$

The endpoint case $\beta = 1$ is more involved due to the fact that test functions cannot cancel the full singularity of the kernel. This can be compared with the 2D Euler equations in vorticity formulation

$$\left\{\begin{array}{ll} \dfrac{\partial \omega}{\partial t} + u \cdot \nabla_x \omega = 0, & x \in \mathbb{R}^2,\ t > 0,\\ u = K_{BS} * \omega, & x \in \mathbb{R}^2,\ t > 0, \end{array}\right. \tag{1.1.53}$$

where K_{BS} is the so called Biot–Savart kernel, that reads

$$K_{BS}(x) = \frac{x^\perp}{2\pi|x|^2},\ x \in \mathbb{R}^2.$$

This is the mean field equation associated with the N vortex problem (1.1.4). In this context, a well known bound of vorticity in some logarithmic Morrey space is all we need to guarantee the absence of concentrations on the diagonal and to pass to the limit. Notice that in 2D Euler (1.1.53) the Biot–Savart kernel K_{BS} is odd. However, the Riesz-type ϕ in (1.1.18) for the weakly singular Cucker–Smale model is even and does not admit similar cancellations. Fortunately, the extra estimate for the dissipation (1.1.51) gives rise to the required the non-concentration estimate that allows the kinetic nonlinear term to be bounded for $\alpha = \frac{1}{2}$. This finally allows obtaining a measure-valued solution to the asymptotic system also in the endpoint case.

Hydrodynamic limits in the singular Kuramoto model

As mentioned before, similar hydrodynamic limits of vanishing inertia type have been considered in recent literature for related systems like the Vlasov–Poisson–Fokker–Planck equation, the aggregation equation, the alignment-aggregation system and some other anisotropic versions of the aggregation equation, see [120, 121, 125, 126, 142] and last Subsection 1.1.7. Before ending this part, we will sketch the idea for a different suitable system where one can apply such a method. This covers the derivation of existence of measure-valued solutions to the kinetic singular Kuramoto model (1.1.38) in the last supercritical regime of singularity $\alpha \in (\frac{1}{2}, 1)$. This is the content of Section 4.7. The cornerstone is again the cancellation property of the nonlinear term. We show that indeed, such cancellation works in the case of identical oscillators, i.e., $g = \delta_0$.

Let us briefly introduce the idea, that will be addressed later in Chapter 4 of this thesis. Specifically, one can consider the next scaled kinetic equation for the distribution function $F_\varepsilon = F_\varepsilon(t, \theta, \omega)$ at time t with phase $\theta \in \mathbb{T}$ and frequency $\omega \in \mathbb{R}$:

$$\frac{\partial F_\varepsilon}{\partial t} + \omega \frac{\partial F^\varepsilon}{\partial t} + \frac{1}{\varepsilon}\frac{\partial}{\partial \omega}\left[K\left(\int_{\mathbb{T}\times\mathbb{R}} h_\varepsilon(\theta' - \theta) F_\varepsilon(t, d\theta', d\omega')\right) F_\varepsilon\right] = \frac{1}{\varepsilon}\frac{\partial}{\partial \omega}\left(\omega F_\varepsilon + \frac{\partial F_\varepsilon}{\partial \omega}\right). \quad (1.1.54)$$

This is nothing but the Vlasov–McKean kinetic equation associated with the stochastic agent-based model

$$\begin{cases} d\theta_i = \omega_i \, dt, \\ \varepsilon d\omega_i = \dfrac{K}{N}\displaystyle\sum_{j=1}^{N} h_\varepsilon(\theta_j - \theta_i)\, dt - \omega_i \, dt + \sqrt{2\varepsilon}\, dW_t^i, \\ \theta_i(0) = \theta_{i,0}, \ \omega_i(0) = \omega_{i,0}. \end{cases} \quad (1.1.55)$$

Such second order system is a Kuramoto–Daido model with identical oscillators, regularized kernel h_ε, endowed with inertia inertia, white noise W_t^i and frequency damping. The inertia term and noise have been scaled so that they disappear as $\varepsilon \searrow 0$ while the scaled regularized kernel reads

$$h_\varepsilon(\theta) := \frac{\sin\theta}{(\varepsilon^2 + |\theta|_o^2)^\alpha},$$

and converges towards the singular kernel h in (1.1.35). Notice that the formal limit $\varepsilon \searrow 0$ in (1.1.55) recovers the singular first order system (1.1.34). Then, we expect that the hydrodynamic limit in (1.1.54) can be closed and yields rigorous weak solutions to (1.1.38). Starting with smooth initial data F_ε^0, the above system (1.1.55) produces smooth solutions due to the regularizing effect of the diffusion and the regularized kernels h_ε. Again, in [254] the following ω moments were considered as the analogues of those in the above subsection:

$$\text{Phase density:} \quad \rho_\varepsilon(t, \theta) := \int_{\mathbb{R}} F_\varepsilon(t, \theta, \omega)\, d\omega,$$

$$\text{Phase current:} \quad j_\varepsilon(t,\theta) := \int_{\mathbb{R}} \omega F_\varepsilon(t,\theta,\omega)\,d\omega,$$
$$\text{Phase stress:} \quad \mathcal{S}_\varepsilon(t,\theta) := \int_{\mathbb{R}} \omega^2 F_\varepsilon(t,\theta,\omega)\,d\omega.$$

The corresponding conservation laws read

$$\begin{aligned} \frac{\partial \rho_\varepsilon}{\partial t} + \frac{\partial j_\varepsilon}{\partial \theta} &= 0,\\ \varepsilon\,\frac{\partial j_\varepsilon}{\partial t} + \varepsilon\,\frac{\partial \mathcal{S}_\varepsilon}{\partial t} + j_\varepsilon + K(h_\varepsilon * \rho_\varepsilon)\rho_\varepsilon &= 0. \end{aligned}$$

Through a similar compactness method, we recover the following result, that is actually valid for any value $\alpha \in (0,1)$ of the singularity.

Theorem 1.1.27 (*Theorem 4.7.12 in Chapter 4*). *For any $\alpha \in (0,1)$, fix initial data*

$$\begin{cases} F_\varepsilon^0 = F_\varepsilon^0(\theta,\omega) \geq 0 \ \ and \ \ F_\varepsilon^0 \in C_c^\infty(\mathbb{T}\times\mathbb{R}),\\ \|F_\varepsilon^0\|_{L^1(\mathbb{T}\times\mathbb{R})} = 1 \ \ and \ \ \rho_\varepsilon^0 \overset{*}{\rightharpoonup} \rho^0 \ \ in \ \ \mathcal{M}(\mathbb{T}),\\ \frac{1}{2}\|\omega^2 F_\varepsilon^0\|_{L^1(\mathbb{T}\times\mathbb{R})} \leq E_0. \end{cases}$$

Consider the strong solution $F_\varepsilon = F_\varepsilon(t,\theta,\omega)$ to (1.1.54) *issued at F_ε^0. Then, for every $T > 0$ there is a limiting measure $\rho \in C([0,T],\mathcal{P}(\mathbb{T}))$ of ρ_ε, that verifies the kinetic singular Kuramoto model for identical oscillators in the sense of distributions, namely,*

$$\begin{cases} \dfrac{\partial \rho}{\partial t} - \dfrac{\partial}{\partial \theta}((h*\rho)\rho) = 0,\\ \rho(0) = \rho^0. \end{cases}$$

The above result provides global existence in the supercritical regime, but as for the microscopic scale, uniqueness is only guaranteed for $\alpha \in (0,\frac{1}{2}]$, the case $\alpha \in (\frac{1}{2},1)$ being an open problem. Similarly, emergence of phase synchronization is not guaranteed for these sort of (very) weak measure-valued solutions because the existence technique is not supported by the mean field limit approach this time.

1.1.7 Other models in collective dynamics

Apart from the preceding models, that have been exhibited as prototype of first and second order agent-based models where one can study its kinetic and macroscopic counterparts, there are a few more that have been proposed and analyzed in the literature. Although we will not enter into details, we will mention some of them and their main features.

Related to the last technique in Subsection 1.1.4, the classical Kuramoto model with inertia has been analyzed at the microscopic and kinetic scales in [76, 77, 78, 79]:

$$\frac{\partial f}{\partial t} + \omega\frac{\partial f}{\partial \theta} + \frac{\partial}{\partial \omega}\left[K\left((\Omega-\omega)f + K\int_{\mathbb{T}\times\mathbb{R}\times\mathbb{R}} \sin(\theta'-\theta)f\,d\theta'\,d\Omega'\,d\omega'\right)f\right] = 0, \tag{1.1.56}$$

where $f = f(t,\theta,\Omega,\omega)$ is the distribution of identical oscillators at time t, phase $\theta \in \mathbb{T}$, frequency $\omega \in \mathbb{R}$ and natural frequency $\Omega \in \mathbb{R}$. The dynamics introduces a transient regime due to inertia, that can be used to model certain physical situations. Nevertheless, the final dynamics essentially agrees with the starting model without inertia, as depicted in the above references. Although a hydrodynamic has not been proposed yet, the same vanishing inertia limit in Theorem 1.1.27 can be achieved mutatis mutandis, thus recovering the Kuramoto–Sakaguchi equation (1.1.21) with identical oscillators.

Another interesting swarming model in $\mathbb{R}^3$ arises when one consider constant speed. Then, only positions and orientations play a role. This is known as the *Couzin–Vicsek model* that in its kinetic version reads

$$\frac{\partial f}{\partial t} + c\,\omega \cdot \nabla_x f + \nabla_\omega \cdot [\nu(I - \omega \otimes \omega)\Omega(t,x,\omega) - D\nabla_\omega f] = 0, \tag{1.1.57}$$

where $f = f(t,x,\omega)$ is the distribution of particles at time t, position $x \in \mathbb{R}^3$ and orientation $\omega \in \mathbb{S}^2$. Here $c > 0$ is the constant speed of particles, $D > 0$ is the strength of the orientation noise and $\Omega = \Omega(t,x,\omega)$ is a normalized momentum vector $\Omega = \frac{J}{|J|}$ and

$$J(t,x,\omega) := \int_{\mathbb{R}^3\times\mathbb{S}^2} K_R(|x-x'|)\omega' f\,dx'\,d\omega'.$$

K_R is the "observation kernel" and typically stands for the characteristic function of the ball centered at the origin with radius R but one can consider general kernels modeling the fact that the influence of the particles falls off with distance. The global case $R \to \infty$ is classically considered, that is, J is the global momentum when all particles are taken into account. The model was rigorously derived via the mean-field limit approach in the discrete Couzin–Vicsek model [101] and some numerical simulations were obtained in [135]. The strong non-linearity, that gives rise to degenerate terms when the momentum vanishes, has proved a strong obstruction and makes well posedness a hard issue. Under the a priory assumption of positivity of momentum [136] shows well posedness of solutions in the full space-inhomogeneous case. The only unconditional results have been obtained for the space-homogeneous case, see [128, 184]. Hydrodynamic limits towards macroscopic limits have been derived in [101, 133]. Other corrections of (1.1.57) that smooth the momentum term have been studied in [40, 98].

Regarding the aggregation equation (1.1.3), there is a huge literature, see [37, 49, 57, 221, 220, 287, 288] and related references. Indeed, similar estimates to those in Theorem 1.1.21 have been proved for such family of *gradient-flow systems*. When W is λ-convex for some $\lambda \geq 0$, the same estimate was derived in the Euclidean space $\mathbb{R}^d$ for the associated kinetic equation [64, 67]

$$\frac{\partial \rho}{\partial t} + \operatorname{div}_x\left(-(\nabla_x W * \rho)\rho\right) = 0, \tag{1.1.58}$$

where $\rho = \rho(t,x)$ is the probability density of particles. The main differences between (1.1.38) and (1.1.58) are: the absence of heterogeneities ν_i in (1.1.58), the Wasserstein gradient-flow structure of (1.1.58) with potential

$$\mathcal{W}[\rho(t)] := \int_{\mathbb{R}^d\times\mathbb{R}^d} W(x-x')\rho(t,dx)\rho(t,dx'),$$

and the underlying Euclidean space $\mathbb{R}^d$ compared to the manifold $\mathbb{T}\times\mathbb{R}$ in (1.1.38).

The same ideas as in Subsection 1.1.6 were analyzed in [57] in order to derive a hydrodynamic limit of vanishing-inertia-type (or large friction) of the second-order kinetic aggregation equation towards the first-order aggregation equation (1.1.58). Also, explicit convergence rates were measured in Wasserstein distances, adding the above-mentioned strong local alignment, but not noise (to avoid pressure terms). To such end, the same scaling as above was considered to make inertia small.

Related to the aggregation equations, several more realistic variants have been proposed in order to include anisotropies in the interactions. For instance, in [120, 121] the force $F(x,x') = \nabla_x W(x-x')$ in (1.1.3) is replaced by a velocity-dependent force

$$F(x,v,x',v') := \nabla_x W(x-x')\eta\left(\frac{x-x'}{|x-x'|}\cdot\frac{v}{|v|}\right).$$

The function η is considered a cut-off function at the origin so that if fades the effect of the aggregation force when the velocity of the particle and the director vector of such particle with respect to any other test particle are not sufficiently aligned. That anisotropic term arises as a sort of "cone of vision" that has been included in many other settings. In such paper, the anysotropies are velocity induced. However, there are some other related models where no velocity dependence appears. Specifically, in [50] the authors proposed a $2D$ model where the force reads

$$\begin{aligned} F(x,x') : &= F_A(x-x',T(x)) + F_R(x-x') \\ &= f_A(|x-x'|)T(x)(x-x') + f_R(|x-x'|)(x-x'). \end{aligned}$$

Then, there is an isotropic repulsive part and an anisotropic part dependent on the tensor

$$T(x) = \chi s(x)\otimes s(x) + (1-\chi)l(x)\otimes l(x),$$

for $\chi \in [0,1]$, being $\{s(x), l(x)\}$ an orthonormal frame of $\mathbb{R}^2$. The constant χ is regarded as the anisotropy parameter, being $\chi = \frac{1}{2}$ the isotropic case. Such model was proved useful to describe the formation of fingerprints and becomes a generalization of the Kücken–Champod model [194].

1.2 Stability and vortex structures in fluid mechanics

In this section, we provide a brief overview of the mathematical context of some recent results in fluid mechanics where Chapter 6 takes place. Specifically, we will first introduce the Euler equations of incompressible inviscid fluids in three dimensions. Later, we shall present a particular class of stationary solutions that have proved relevant in the literature and will become the main tool in our result. These are the Beltrami fields, that have deep implications on the understanding of Lagrangian theory of turbulence in hydrodynamics. Later, we review the old conjecture by L. Kelvin on the existence of linked and knotted vortex structures in incompressible fluids and we recall the recent poof by A. Enciso and D. Peralta Salas that, following the intuition of V. Arnold, is solved in terms of Beltrami fields. Finally, we state some properties of Beltrami fields regarding their (lack) of stability and we introduce our main contribution to the topic, that will be later developed in Chapter 6 of this thesis.

1.2.1 The Euler equations for incompressible fluids

The Euler equations represent a system of conservation laws describing the dynamics of an inviscid fluid that fills the whole space $\mathbb{R}^3$. They were first proposed by L. Euler [119] in the form of a coupled system for the continuity equation of the density and the balance equation of linear momentum. Throughout this thesis we are interested in a simpler case of inviscid fluids, often called perfect fluids. These are ideal, homogeneous and incompressible fluids, that can be represented in terms of the following system of nonlinear partial differential equations

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla) u = -\nabla p, & t > 0,\ x \in \mathbb{R}^3, \\ \operatorname{div} u = 0, & t > 0,\ x \in \mathbb{R}^3, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^3. \end{cases} \tag{1.2.1}$$

Here $u = u(t,x)$ stands for the velocity field at time $t \geq 0$ and position $x \in \mathbb{R}^3$ and determines the velocities of particles located at such points. Also, $p = p(t,x)$ is a scalar function that is often called the pressure.

On the one hand, since the fluid is homogeneous we have normalized density to one for simplicity. Then, the continuity equation is trivial and we can ignore it. On the other hand the first equation in (1.2.1) governs the evolution of linear momentum. It can be easily deduced from Newton's second law if we assume that the fluid is ideal. This means that the only forces exerted over any volume $D \subseteq \mathbb{R}^3$ within the fluid are normal to the boundary ∂D and proportional to p. Finally, the second equation in (1.2.1) represents the incompressibility condition and characterizes fluids so that the subdomain D keeps constant volume when we let it flow along the streamlines. See [5] for an easy derivation of the Euler equations and their viscous counterpart (i.e., the Navier–Stokes equations).

The Euler equations (1.2.1) have long been studied during the history. As a consequence, extremely important mathematics have emerged when we try to solve some of the involved problems. In particular, we mention that the question of global-in-time well posedness vs. blow-up is a extremely hard open problem with relevant implications in sciences. Indeed, the analogue version of such problem for the Navier–Stokes equation has been considered one of the problems of the millennium by Clay Mathematics Institute. Its resolution (either in an affirmative or negative way) entails a US$ 1 million prize to the discoverer, as a recognition of a strong advance in mathematics. We recall that, to date, only one of the seven millenium problems has been solved; namely the Poincaré's conjecture by G. Perelman in 2013.

The Euler equation (1.2.1) consists of four variables u_1, u_2, u_3, p and four equations (momentum equation and incompressibility condition), what suggests that they should be well posed. Notice that, by taking divergence on the first equation (1.2.1) and using the incompressibility condition, we achieve the following Poisson equation for pressure

$$-\Delta p = \operatorname{div}((u \cdot \nabla) u),$$

that suggests that pressure dynamics is subordinated to that of velocity. A different way to state it is in terms of the vorticity formulation of (1.2.1), that is obtained by applying curl operator to it, namely

$$\begin{cases} \dfrac{\partial \omega}{\partial t} + (u \cdot \nabla) \omega = (\omega \cdot \nabla) u, & t > 0,\ x \in \mathbb{R}^3, \\ \operatorname{div} u = 0, & t > 0,\ x \in \mathbb{R}^3, \\ \operatorname{curl} u = \omega, & t > 0,\ x \in \mathbb{R}^3, \\ \omega(0,x) = \omega_0(x), & x \in \mathbb{R}^3. \end{cases} \tag{1.2.2}$$

The vector field $\omega = \omega(t,x)$ is called the vorticity field and represents the tendency of the fluid to rotate around $x \in \mathbb{R}^3$. Observe that, in doing so, pressure has dissapeared from the dynamics. Once ω is known, we observe that u is subordinated to ω in terms of the div-curl problem in (1.2.2). It is well known that we can solve it in terms of the Biot–Savart law that describes a explicity formula for u in terms of ω

$$u(t,x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} \times \omega(t,y)\, dy,$$

for any $t > 0$ and $x \in \mathbb{R}^3$, see [5].

1.2.2 Stationary solutions and Beltrami fields

Along this thesis we are interested in the particular case of stationary solutions $u = u(x)$, $p = p(x)$ of the Euler equations (1.2.1). Indeed, notice that they must verify the simpler equations

$$\begin{cases} (u \cdot \nabla)u = -\nabla p, & x \in \mathbb{R}^3, \\ \operatorname{div} u = 0, & x \in \mathbb{R}^3. \end{cases}$$

For any given stationary solution (u, p) to such a problem, let us observe that the so called Bernoulli function is a first integral of the velocity field u. Specifically, if we set

$$B(x) := \frac{1}{2}|u|^2 + p,$$

then $u \cdot \nabla B = 0$. This is called Bernoulli law and has long been used in hydrodynamics and engineering. Using it, we can check that stationary solutions solved the following equivalent system that is often called, Bernoulli formulation

$$\begin{cases} u \times \omega = \nabla B, & x \in \mathbb{R}^3, \\ \operatorname{div} u = 0, & x \in \mathbb{R}^3, \\ \operatorname{curl} u = \omega, & x \in \mathbb{R}^3. \end{cases} \tag{1.2.3}$$

Notice that condition $u \cdot \nabla B$ determines that B is constant along stream lines (that is, integral lines of u). However, B must not be necessarily constant globally. Indeex, the latter assumption describes a specific subclass of stationary solutions with the condition $\omega \times u = 0$. That is, u and ω must be collinear. This gives rise to the following definition.

Definition 1.2.1 (Beltrami fields). *Let $u : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be any vector field. We say that u is a (generalized) Beltrami field if there exists a scalar function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ such that*

$$\begin{cases} \operatorname{curl} u = fu, & x \in \mathbb{R}^3, \\ \operatorname{div} u = 0, & x \in \mathbb{R}^3. \end{cases} \tag{1.2.4}$$

Moreover, when $f = \lambda$ is constant globally, we will say that u is a strong Beltrami field.

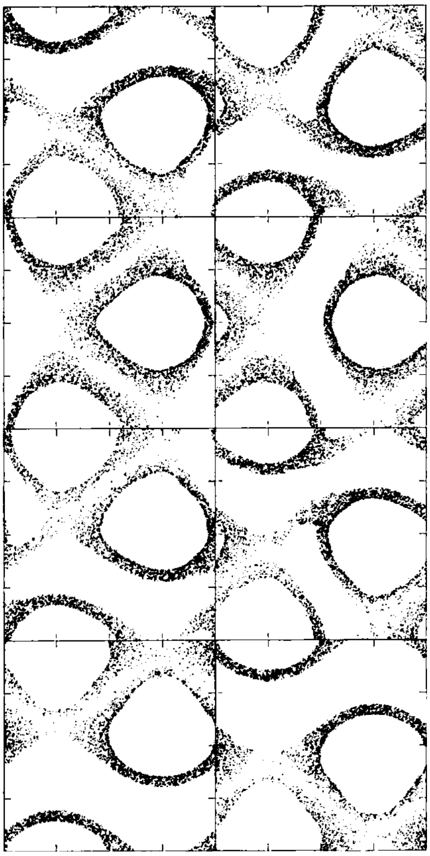
By construction, (generalized) Beltrami fields are particular stationary solutions of the Euler equations. Observe that if we compute the divergence of the first equation in (1.2.4), then we achieve the condition $u \cdot \nabla f$. Again, this means that f is a new first integral of u. This condition introduces some rigidity on the problem that will be discussed later.

1.2.3 Turbulence and Arnold's structure theorem

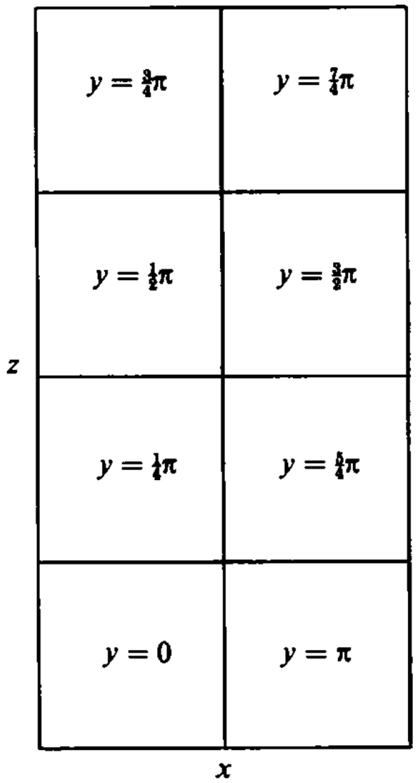
As a particular class of Beltrami fields, let us mention the so called ABC flows [208], that are periodic strong Beltrami fields determined by the expression $u = (u_1, u_2, u_3)$, where

$$\begin{aligned} u_1(x_1, x_2, x_3) &:= A \sin x_3 + C \cos x_2, \\ u_2(x_1, x_2, x_3) &:= B \sin x_1 + A \cos x_3, \\ u_3(x_1, x_2, x_3) &:= C \sin x_2 + B \cos x_1, \end{aligned}$$

for every $(x_1, x_2, x_3) \in \mathbb{R}^3$ and $A, B, C \in \mathbb{R}$ are constants. They were named after V. Arnold, E. Beltrami and S. Childress, but many studies are originally devoted to I. S. Gromeka, see [144].



(a) Poincaré sections of a single streamline of the ABC flow



(b) Arrangement of the Poincaré sections of the ABC flow

Figure 1.4: Pressence of chaotic and ordered areas for streamlines of ABC flows. Pictures have been taken from [113].

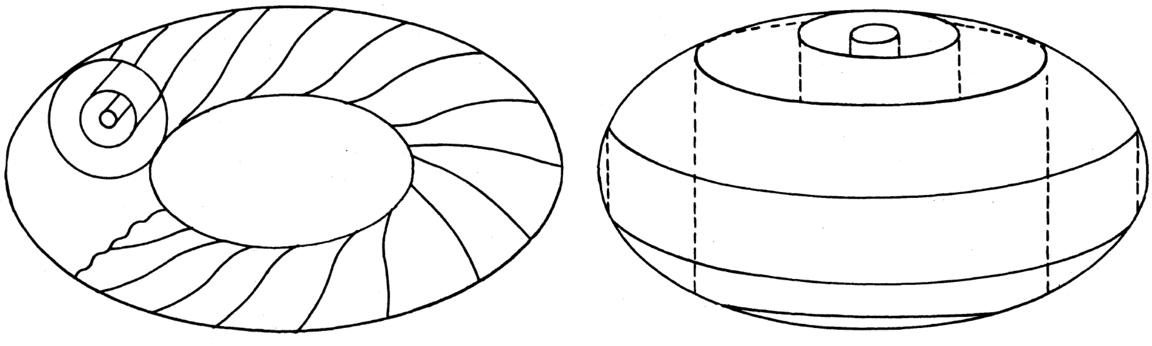
Here, we shall remark an interesting feature of ABC fields, as representative of more general Beltrami fields, see [113]. In such paper the authors found that ABC flows can have very complicated Lagrangian structure. Specifically, streamlines can behave in very convoluted ways. In particular, the authors showed that for the particular choice of parameters

$$A^2 = 1, \; B^2 = \frac{2}{3} \text{ and } C^2 = \frac{1}{3},$$

often called Hénon choice, the streamlines of the corresponding ABC flows exhibit both ordered regions (with KAM invariant surfaces) and chaotic regions (containing the unstable stagnation points), see Figure 1.4. As a consequence, starting at arbitrarily close points, the corresponding streamlines of the fluid may push particles towards completely different places or even fill a complete region of the fluid. This active enhancement of transport due to chaos in the dynamical system of streamlines is often called Lagrangian turbulence, to be distinguished from the usual Eulerian dynamics that emerges via bifurcation of structures as viscosity is decreased in the solution of the Navier–Stokes equation.

The above ideas suggest that Beltrami fields are good candidates of stationary perfect fluids with complicated vortex structures. Indeed, such heuristics were strengthened later after the so called Arnold's structure theorem in fluid mechanics:

Theorem 1.2.2. *[12, Théorème 7] Let $D \subseteq \mathbb{R}^3$ be a compact connected domain and assume that the*



(a) Subdomain fibered by invariant tori. (b) Subdomain fibered by invariant cylinders.

Figure 1.5: Scheme of the two posible subdomains of D that determines the internal structure of the stream lines of incompressible fluids where ω and u are not everywhere collinear. Pictures have been taken from [12].

velocity field $u : \overline{D} \longrightarrow \mathbb{R}^3$ *is a stationary solution of the Euler equation* (1.2.2) *in* D *so that* $u \cdot \eta = 0$ *on* ∂D*. Assume that* D*,* ∂D *and* u *are all analytic and that the vorticity field* ω *and the velocity field* u *of the fluid are not everywhere collinear in* D*, that is,*

$$\omega \times u \not\equiv 0 \ \text{ in } \ D.$$

Then, there exists an analytic compact subset $K \subseteq D$ *of codimension* 1 *or larger so that* $D \setminus K$ *can be split into finitely many connected subdomains* $D_1, \ldots, D_n$*, each of them being of one of the following two types:*

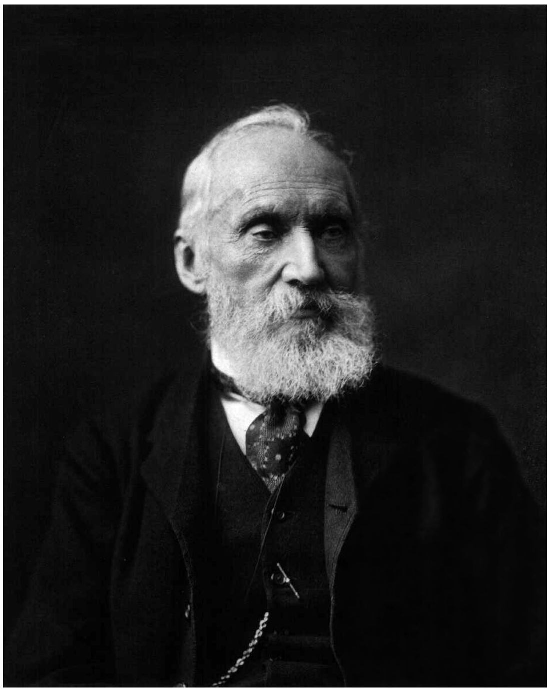
1. *If* $\partial D_i \cap \partial D = \emptyset$*, then* D_i *is fibered by invariant tori under* u*. On each torus, the flow of* u *is conjugate to linear a flow (either rational or irrational), so that, in particular, streamlines along each torus are either all dense or all periodic, see Figure 1.5a.*

2. *If* $\partial D_i \cap \partial D \neq \emptyset$*, then* D_i *is fibered by invariant cylinders under* u *whose boundary lie on* ∂D*. In addition, the streamlines of* u *on each cylinder are all closed curves, see Figure 1.5b.*

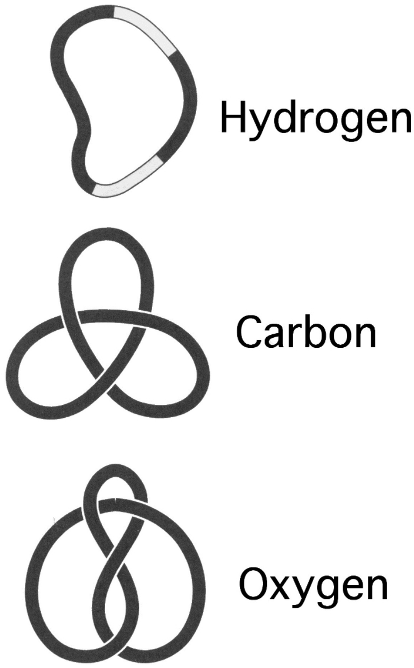
In particular, notice that only when velocity and vorticity are parallel everywhere we can expect existence of streamlines and stream tubes with more complicated structures than simple tori or cylinders in Theorem 1.2.2. This suggests that Beltrami fields are well conditioned to exhibit complicated vortex structures.

1.2.4 The Kelvin conjecture

The interest on the search for complicated vortex structures in fluid mechanics dates back to William Thomson, also known by Lord Kelvin, in 1875. After the seminal work [169] where H. Helmholtz showed that vortices in a perfect fluid are stable objects that exert long range forces according to the Biot–Savart (the same law describing magnetic forces), Lord Kelvin imagined that vortices should be the minimal perpetual constituents of matter. More specifically, he proposed a new atomic theory in which matter is regarded as a perfect fluid (aether) and atoms are identified with linked and knotted vortex filaments within the fluid. Indeed, Lord Kelvin aimed at describing all the possible states of matter in the universe (atoms) in terms of different complicated linked and knotted vortices, see Figure 1.6.



(a) Lord Kelvin



(b) Kelvin atoms

Figure 1.6: Some examples of atoms according to the atomic theory of vortices by Lord Kelvin. Pictures are taken from Wikipedia

For two decades, such atomic theory was popularly accepted in the scientific community due to the fact that it gathered several solid features that no other model had combined before. Unfortunately, such an atomic theory did not survived the test of time after the discovery of the electron by J. J. Thomson in the 1890s. However, it gave rise to an important conjecture, later popularized by V. Arnold and K. Moffat in 1960s':

For any ensemble of stream lines and tubes, that can be linked and knotted in an arbitrary way, there exists and incompressible fluid that realizes it?

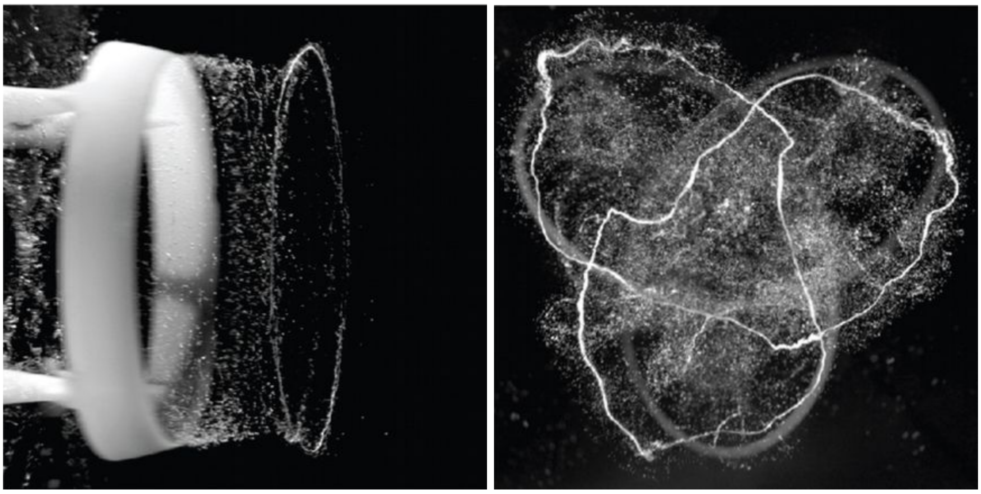
As discussed before, the abovementioned results in [12, 113] suggest that such a conjecture by L. Kelvin must be true and Beltrami fields are the good candidates for the solution.

On the one hand, from the experimental point of view a recent affirmative answer was drawn in [190]. Specifically, the authors printed 3D hydrofoils and used them to accelerate in a tank of water at Reynold number Re $\in (10^4, 10^5)$ compared to the size of the hydrofoil (close to inviscid). Then, tiny buoyant gas bubbles were used as indicators of regions with high vorticity and using high speed cameras they obtained single rings, trefoil knots and a couple of linked rings, see Figure 1.7.

The rigorous mathematical proof of such a conjecture was later derived in [115, 116] by A. Enciso and D. Peralta-Salas using strong Beltrami fields and takes the following form:

Theorem 1.2.3. *[116, Theorem 1.1] Consider any ensemble of disjoint linked and knotted closed curves $\gamma_1, \dots, \gamma_n$ in $\mathbb{R}^3$. Then, there exists a constant $\varepsilon_0 > 0$, depending on the topology of the curves, so that for any $\varepsilon \in (0, \varepsilon_0)$ and for the thin tubes*

$$\mathcal{T}_\varepsilon(\gamma_i) := \{x \in \mathbb{R}^3 : \operatorname{dist}(x, \gamma_i) \leq \varepsilon\}, \quad i = 1, \dots, n,$$



(a) Vortex ring (b) Trefoil vortex

Figure 1.7: Experimental vortices generated in a laboratory. Pictures are taken from [190].

there exists a diffemorphism $\Phi : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ that is arbitrarily closed to the identity in any C^m norm and there exists a strong Beltrami field $u : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ with constant factor $\lambda = \varepsilon^3$ so that $\Phi(\mathcal{T}_\varepsilon(\gamma_1)), \ldots, \Phi(\mathcal{T}_\varepsilon(\gamma_n))$ are vortex tubes of u. In addition, the decay of u and its derivatives is optimal, namely,

$$|D^\alpha u(x)| \leq \frac{C_\alpha}{|x|},\ x \in \mathbb{R}^3,$$

for every $\alpha \in \mathbb{N}^3$ and the following properties hold true inside each vortex tube $\Phi(\mathcal{T}_\varepsilon(\gamma_i))$:

1. *In the interior of $\Phi(\mathcal{T}_\varepsilon(\gamma_i))$ there are uncountably many nested tori invariant under the Beltrami field u. On each of these invariant tori, the field u is ergodic.*

2. *The set of invariant tori has positive Lebesgue measure in a small neighborhood of the boundary $\partial\Phi(\mathcal{T}_\varepsilon(\gamma_i))$.*

3. *In the region bounded by any pair of these invariant tori there are infinitely many closed vortex lines, not necessarily of the same knot type as the curve γ_i.*

4. *$\Phi(\gamma_i)$ is a closed vortex line of u.*

The above result joins three different components. First, the authors derived a local existence results of strong Beltrami fields inside each thin tubes. Second they developed a KAM-type theorem for Beltrami fields in generic thin tubes. Finally, they proved a Runge-type approximation theorem of the above local field in tubes by a global Beltrami field tending to zero at infinity.

As it will be of interest later, we recall two main important properties of the Beltrami fields with knotted and linked structures that were obtained in Theorem 1.2.3:

1. On the one hand, the fall-off of the above Beltrami fields is optimal as depicted in the Liouville-type theorem in [227].

2. On the other hand, the above fields are structurally stable. This means that, if $v : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is a divergence-free field that is close enough to u in C^m for large enough m. Then, there exists another diffemorphism $\Psi : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ that is arbitrarily close to the identity so that $\Psi(\mathcal{T}_\varepsilon(\gamma_i))$ are invariant tubes of v.

1.2.5 Obstructions to the existence of generalized Beltrami fields

In the above Theorem 1.2.3, the authors constructed non-trivial strong Beltrami fields, that is Beltrami fields with non-constant factor. Regarding generalized Beltrami fields, the existence theory is less abundant, probably due to the rigidity imposed by the aforementioned first integral condition $u{\cdot}\nabla f = 0$ of the proportionality factor f. Here, we review the main non-existence results of Beltrami fields with generic non-constant factor that were proved in [117]. Most of these results are of purely local nature.

The first obstruction to the existence of solutions to the Beltrami equation (1.2.4) is the following:

Theorem 1.2.4. *[117, Theorem 1.1] Let $D \subseteq \mathbb{R}^3$ be a domain and take any nonconstant proportionality factor $f \in C^{6,\alpha}(D)$. Suppose that the vector field u satisfies* (1.2.4) *in D with proportionality factor f. Then, there is a sixth order nonlinear partial differential operator $P \neq 0$, which can be computed explicitly, such that $u \equiv 0$ unless $P[f] \equiv 0$ in D. In particular, there exists an open set $\mathcal{O} \subseteq C^{k,\alpha}(D)$ with $k \geq 6$ and infinite codimension so that $u \equiv 0$ for all $f \in \mathcal{O}$.*

It should be noticed that Theorem 1.2.4 is of a purely local nature, as it provides obstructions for the existence of nontrivial Beltrami fields in any open set and most proportionality factors. A less powerful but more conceivable obstruction is that if f has a regular level set homeomorphic to the sphere, then (1.2.4) does not have nontrivial solutions.

Theorem 1.2.5. *Let $D \subseteq \mathbb{R}^3$ be a domain and take any nonconstant proportionality factor $f \in C^{2,\alpha}(D)$. Assume that a level set $f^{-1}(c)$, for some $c \in \mathbb{R}$, has a connected components in D that is homeomorphic to the sphere. Then, for any u solving* (1.2.4) *in D, $u \equiv 0$.*

There is a very useful direct consequence of the above theorem. Specifically, notice that whenever f has a strict local extrema or it is radially symmetric, then there is a level set of f that is homeomorphic to the sphere. In such cases, there is no non-trivial Beltrami field associated to such a non-constant factor. This is related to the classical theorem of Cowling on the nonexistence of poloidal Beltrami fields with nonconstant factor and axial symmetry [15].

The proof of these theorems is based on formulating the Beltrami equation (1.2.4) as a constrained evolution problem. Indeed, one can show that (1.2.4) is locally equivalent to the assertion that there is a time-dependent 1-form $\beta(t) \in \Omega^1(\Sigma)$ on a surface $\Sigma \subseteq D$ that satisfies the following equations

$$\frac{\partial \beta}{\partial t} = T(t)\beta, \qquad x \in \Sigma, \tag{1.2.5}$$

$$d\beta = 0, \qquad x \in \Sigma. \tag{1.2.6}$$

Here, $T(t)$ is a time-dependent tensor field that depends on f and the exterior differential d is computed with respect to the coordinates on the surface Σ, which, in turn, is a regular level set of f. It should be stressed that this formulation depends strongly on the choice of coordinates.

On the one hand (1.2.5) is an evolution equation for β governed by the tensor field T. Such an equation is not generally compatible with the constraint (1.2.6). The corresponding compatibility conditions translate into equations of β and T, that translate into a condition involving f and its derivatives. In Theorems 1.2.4 and 1.2.5 we have presented the first two of these compatibility conditions, but in fact the method of proof yields a whole hierarchy of explicitly computable obstructions to the existence of solutions. Computing a family that only contains all the possible independent obstructions is an interesting open problem.

1.2.6 Partial stability of generalized Beltrami fields

According to the above Theorems 1.2.4 and 1.2.5, we do not expect full stability results of Beltrami fields. Specifically, according the second part of Theorem (1.2.4), there are complete open subsets in $\mathcal{O} \subseteq C^{k,\alpha}(D)$ with infinite codimension containing non-constant factors that admit only trivial Beltrami fields. Hence, perturbations of proportionality factors cannot be conducted in an arbitrary way, but they have to be subordinated to the implicit set of obstructions imposed by the constrained evolution equation (1.2.5)-(1.2.6). In Chapter 6, we will introduce a very specific method to achieve perturbations of factors so that they admit non-trivial Beltrami fields. Our results are of two types and take the following forms.

On the one hand we introduce an almost global perturbation method of strong Beltrami fields in the complement of arbitrarily small balls.

Theorem 1.2.6 (Theorem 6.4.7 in Chapter 6). *Let $G \subseteq \mathbb{R}^3$ be an open domain that is homeomorphic to a sphere and consider $\lambda \in \mathbb{R} \setminus \{0\}$ so that it is not a Dirichlet eigenvalue of Laplace operator in G. Assume that $u_0 \in C^{k+1,\alpha}(\mathbb{R}^3 \setminus G, \mathbb{R}^3)$ is a strong Beltrami field with factor λ in the exterior domain, that is,*

$$\begin{cases} \operatorname{curl} u_0 = \lambda u_0, & x \in \mathbb{R}^3 \setminus G, \\ \operatorname{div} v_0 = 0, & x \in \mathbb{R}^3 \setminus G, \end{cases}$$

so that it admits an invariant cylinder $\mathcal{T}_0$ whose endpoints are both supported on ∂G. In addition, suppose that u_0 points outwards at some of the two tops surfaces Σ of the cylinder. Then, for every $\varepsilon > 0$ there exists $\delta > 0$ so that for any $\varphi_0 \in C_c^{k+1,\alpha}(\Sigma)$ with $\|\varphi_0\|_{C^{k+1,\alpha}(\Sigma)} < \delta$, there is another vector field $u \in C^{k+1,\alpha}(\mathbb{R}^3 \setminus G, \mathbb{R}^3)$ with $\|u - u_0\|_{C^{k+1,\alpha}(\mathbb{R}^3 \setminus G)} < \varepsilon$ so that it solves

$$\begin{cases} \operatorname{curl} u = (\lambda + \varphi) u, & x \in \mathbb{R}^3 \setminus G, \\ \operatorname{div} u = 0, & x \in \mathbb{R}^3 \setminus G, \\ u \cdot \eta = u_0 \cdot \eta, & x \in \Sigma, \\ |u(x)| \lesssim |x|^{-1}, & x \in \mathbb{R}^3 \setminus G. \end{cases}$$

In addition, $\varphi \in C^{k,\alpha}(\mathbb{R}^3 \setminus G)$ has compact support contained within a similar invariant cylinder $\mathcal{T}$ of u whose endpoints are supported on $\partial\Omega$, and it propagates the values of the prescription φ_0 along the stream lines that foliate the tube $\mathcal{T}$.

On the other hand, we derive a perturbation method of generalized Beltrami fields in small enough domains around a non-stagnation point of the original field.

Theorem 1.2.7 (Theorem 6.6.3 in Chapter 6). *Let $G \subseteq \mathbb{R}^3$ be an open domain and $u_0 \in C^{k+1,\alpha}(G, \mathbb{R}^3)$ be a generalized Beltrami field with factor $f_0 \in C^{k,\alpha}(G)$, that is,*

$$\begin{cases} \operatorname{curl} u_0 = f_0 u_0, & x \in G, \\ \operatorname{div} v_0 = 0, & x \in G. \end{cases}$$

Fix some none-stagnation point $x_0 \in G$ of u_0 (i.e., $u_0(x_0) \neq 0$), a small enough radius $R > 0$ and any surface $\Sigma_R \subseteq B_R(x_0)$ at which u_0 points inwards. Then, for any $\varepsilon > 0$ there exists some $\delta > 0$ so that for every $\varphi_0 \in C^{k+1,\alpha}(\Sigma_R)$ with $\|\varphi^0\|_{C^{k+1,\alpha}(\Sigma_R)} < \delta$ there exists another vector field $u \in C^{k+1,\alpha}(\overline{B}_R(x_0), \mathbb{R}^3)$ with $\|u - u_0\|_{C^{k+1,\alpha}(B_R(x_0))} < \varepsilon$ so that it solves

$$\begin{cases} \operatorname{curl} u = (f_0 + \varphi) u, & x \in B_R(x_0), \\ \operatorname{div} u = 0, & x \in B_R(x_0), \\ u \cdot \eta = u_0 \cdot \eta, & x \in \Sigma_R. \end{cases}$$

In addition, φ propagates the values of the prescription φ_0 along the streamlines of u, that determine a flow conjugate to a linear one across x_0.

We emphasize that, in our results, perturbations are not general but they have a very specific fiberwise structure along invariant cylinders of u that is compatible with the obstructions imposed by (1.2.5)-(1.2.6). Also, let us notice that the perturbed fields can be chosen arbitrarily close to the initial one by taking the perturbation φ_0 small enough. This, along with the structural stability of the Beltrami fields in Theorem 1.2.3 guarantee that the new fields can also exhibit complicated linked and knotted stream lines and tubes.

Part I

The Cucker-Smale model and some singular versions

CHAPTER 2

Singular hyperbolic limits of the kinetic Cucker–Smale model

Contents

2.1 Introduction

The aim of this chapter is to rigorously derive and analyze the following system of equations of Euler-type with singular commutators that arises as hydrodynamic hyperbolic limit of the kinetic Cucker–Smale model

$$\left\{\begin{array}{ll} \dfrac{\partial\rho}{\partial t}+\operatorname{div}(\rho u)=0, & t\geq 0,\ x\in\mathbb{R}^d \\ \phi_0*(\rho u)-(\phi_0*\rho)u-\mu u-\nabla\psi=0, & t\geq 0,\ x\in\mathbb{R}^d\,. \end{array}\right. \tag{2.1.1}$$

Here, $\rho = \rho(t,x)$ and $u = u(t,x)$ respectively represent the density and velocity field of the population at time t and position x. Likewise, d stands for the space dimension, being $d = 2$ and $d = 3$ the most physically meaningful cases. Also, throughout this chapter we will assume that ρ is normalized as a probability density, that is,

$$\int_{\mathbb{R}^d} \rho(t,x)\,dx = 1, \text{ for all } t \geq 0.$$

Notice that since the continuity equation in (2.1.1) conserves the initial mass, the above can be assumed without loss of generality by just restricting to initial data $\rho^0 \in \mathcal{P}(\mathbb{R}^d)$ in the space of probability measures. Notice that the main complexity in the above model arises from the coupling between velocity and density via the second equation in (2.1.1). Specifically, it consists of an integral equation given by a commutator of weakly singular integrals that involves linear friction with the medium (velocity damping) with constant coefficient $\mu \geq 0$, along with the effect of some external conservative force $F = -\nabla\psi$ described through a potential $\psi = \psi(t,x)$. Regarding the singular kernel in the commutator, it takes the form

$$\phi_0(r) = \frac{1}{c_\alpha^\alpha}\frac{1}{r^{2\alpha}}, \tag{2.1.2}$$

for any $r > 0$, some constant $c_\alpha > 0$ to be described later and any $\alpha > 0$, that measures the fall-off of the interactions between agents separated by large distances. Consequently, the commutator can be written in the following way

$$\begin{aligned} \phi_0 * (\rho u) - (\phi_0 * \rho)u &= -\frac{1}{c_\alpha^\alpha}[u, I_{d-2\alpha}]\rho \\ &= -\frac{1}{c_\alpha^\alpha}\int_{\mathbb{R}^d} \frac{u(t,x)-u(t,y)}{|x-y|^{2\alpha}}\rho(t,y)\,dy, \end{aligned} \tag{2.1.3}$$

where $I_\beta = (-\Delta)^{-\beta/2}$ denotes the Riesz potential or fractional integral operator of order $\beta \in (0,d)$ generated by a scalar measurable function f on $\mathbb{R}^d$ and $[u, I_\beta]$ stands for the commutator of I_β itself with the multiplication operator by u. For the sake of simplicity, we shall forget about the standard constant arising from the fundamental solution of the fractional Laplacian $(-\Delta)^{\beta/2}$ and we simply write

$$(I_\beta f)(x) = \left(\frac{1}{|\cdot|^{d-\beta}} * f\right)(x) = \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-\beta}}\,dy,$$

for each $x \in \mathbb{R}^d$, see [278, Chapter V, Section 1] for more details.

As depicted in Chapter 1, the ultimate goal of connecting microscopic and macroscopic scales intends to keep the basic features of the interactions between agents, that eventually is responsible for the self-organization of the complex system. This is a currently challenging issue in different areas of Science that steer towards the attainment of more complex rules beyond the Newtonian-type laws and it constitutes a particularly active area in the biological description of emergent processes in complex populations. Depending on the biological context, some of the emerging dynamics are called *swarming*, *schooling*, *flocking* or *synchronization*. For different models in this context we mention [1, 17, 21, 42, 58, 88, 92, 99, 100, 101, 170, 145, 195, 196, 210, 224, 257, 295], the references therein along with other works that have been mentioned in the introductory Chapter 1. In the same spirit, but within the framework of social and cellular interactions, we can also mention [19, 18, 20, 43, 140, 168, 210, 235, 240] among others. From a biological viewpoint, the main interest is to explore the convoluted set of rules stating how agents in an ecosystem self-organize. Such *local* organization or *cooperation* between the individuals in neighboring areas is expected to lead towards some kind of *global emergent behavior*, as a complex mechanism that brings all the agents of the population together and leads to dynamics driven by the collective motion of the rest of individuals. In other words, individuals in the ecosystem are self-driven by a self-generated collective motion that, in turns, such collective motion itself is determined by the local interactions between individuals. This is a captivating phenomenon that can be observed in some typical examples in nature like the formation of flocks of birds, schools of fish or a swarm of bacteria.

Regarding our system (2.1.1), we will see that such equation inherits the main insights of the microscopic system from which it is a macroscopic approximation, that is, the so-called Cucker–Smale model

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, & t \geq 0, \\ \dfrac{dv_i}{dt} = \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} \phi(|x_i - x_j|)(v_j - v_i), & t \geq 0. \end{cases} \tag{2.1.4}$$

Such system was proposed in [90, 91] as an adequate model for the alignment of velocities of N agents in a weighted way that depends on the influence function $\phi = \phi(|x-y|)$. More specifically, we shall require slightly modifying the original model (2.1.4) to account for velocity linear damping and noise along with the effect of some external conservative force $F = -\nabla\psi$. Then, we will show that (2.1.1) collects the nonlocal alignment effects of velocities of the population in a regime where inertia is negligible compared with friction and the forcing term $F = -\nabla\psi$. This is typically called the *overdamped Langevin dynamics*, *Brownian dynamics* or *Smoluchowski equation* in Physical Statistics, see [258, p. 257], and has proved a good simplification of the original full dynamics for certain bio-physical systems [210]. Although in this chapter $\psi(t,x)$ will be regarded as a external potential, it might also be considered internal (i.e., self-generated). In a sense, external potentials amount to simpler linear terms that shall help us focus on the most relevant nonlinear alignment terms. However, we believe that many of the techniques that will be exhibited in this chapter can be adapted to cover other self-generated potentials, for instance, those described in terms of more general nonlocal nonlinearities of the type $\psi = W * \rho$, for appropriate interaction potentials $W = W(x)$, or more general forces $F = K * \rho$, for adequate interaction kernels $K = K(x)$. Notice that such choices open our scope and lead to connecting system (2.1.1)-(2.1.2) with other systems in the literature. In particular, one recovers swarming effects modelled by aggregation-type terms [26, 37, 49, 64, 66, 67, 120, 126, 221] or more general terms [287, 288] inspired in Fluid Mechanics, the Euler equation and the closely related gSQG model [70, 97].

Notice that (2.1.1) can be compared with the recently proposed *pressureless Euler-alignment equation* arising in the setting of Cucker–Smale dynamics of flocking

$$\begin{cases} \dfrac{\partial\rho}{\partial t} + \operatorname{div}(\rho u) = 0, \\ \dfrac{\partial}{\partial t}(\rho u) + \operatorname{div}(\rho u \otimes u) = -\displaystyle\int_{\mathbb{R}^d} \phi(|x-y|)(u(t,x) - u(t,y))\rho(t,x)\rho(t,y)\,dy, \end{cases} \tag{2.1.5}$$

see [63, 165, 275, 276, 277, 282, 282] for some properties of such model. Specifically, when inertia effect (i.e., acceleration) is neglected in such hydrodynamic nonlocal alignment model but linear friction and some force $F = -\nabla\psi$ are assumed to act on the system, then we are formally led to the above general system (2.1.1).

Unfortunately, the rigorous derivation of the Euler system with nonlocal alignment effects (2.1.5) has been mainly achieved from mesoscopic models via formal arguments through a method of closure of the hierarchy of velocity moments based on the choice of a monokinetic distribution of particles [63, 66, 80, 162, 224, 282]. Very recently, several attempts to derive such model rigorously have been addessed via hydrodynamic limits in [127, 186]. Both results require introducing new "damping terms" in the associated kinetic equation. Specifically, the auhtors introduce the so called *local alignment force*, that can be understood as damping towards the mean velocity. An alternative derivation of such term can be obtained from the Mostch–Tadmor kinetic alignment term [223] when the influence function is approximated by a Dirac delta function. In [186], the authors also require introduce velocity noise in order for some relative-entropy method to work. Unfortunately, in the hyperbolic strong local alignment limit such velocity noise amounts to an artificial presure in the macroscopic equations. Later, the same relative-entropy method was achieved in [127] in the absence of velocity noise, leading to the above pressureless Euler-alignment system (2.1.5) in the hyperbolic strong local alignment limit. See Chapter 1 for some more details with regards to such literature on hydrodynamic limits towards Euler-alignment type models. We remark that such extra "damping term" has proved necessary in order to push the kinetic distribution of agents towards the monokinetic distribution in the hyperbolic limit. Indeed, an analogue result where the non-local alignment is the only responsible for the monokinetic distribution to emerge in the hyperbolic limit is still missing in the absence of local alignment terms.

Regarding regularity of interactions, we remark here that both [127, 186] assume Lipschitz influence functions $\phi = \phi(r)$. To our best knowledge, no rigorous derivation has been achieved for less regular influence functions. Specifically, the underlying choice of ϕ in those results is the classical one by Cucker and Smale [90, 91]

$$\phi(r) = \frac{1}{(1+r^2)^\alpha}, \tag{2.1.6}$$

for any $r > 0$. Naturally, the lack of singularity at $r = 0$ simplifies the mathematical handling of the kernel and nonlinear terms appearing in the Euler equation with nonlocal alignment. In addition, note that regular ϕ aims at preventing distant interparticle interactions. In order to augment the local interactions, a possible solution might be the one considered in [223] (assuming asymmetric interactions) or the model proposed in [244] (with singular interactions). From a mathematical point of view, very little has been obtained for the singular case (2.1.2) for values $\alpha \in (0, \frac{1}{2})$. In [60] local in-time well posedness of the sigular kinetic model has been proved. In [244, 245] the existence of piecewise weak solutions to the particle system along with existence and uniqueness of $W^{1,1}$ strong solutions in the restricted range $\alpha \in (0, \frac{1}{4})$ has been analyzed and the mean field limit has been explored in [226]. See also the review paper [213] for a comprehensive exposition of all the results up to date concerning the Cucker–Smale model with singular influence function at any level of description (microscopic, kinetic and macroscopic). The interest on singular interactions in not purely accidental, but actually this new paradigm leads to substantially new dynamics. Specifically, [244] exhibits some examples where (2.1.2) with $\alpha \in (0, \frac{1}{2})$ amounts to finite-time collisions and sticking of particles. See Section 3 and 4 for a detailed analysis of the sticky behavior of the related Kuramoto model with singular coupling weights.

In this chapter, we are interested in rigorously obtaining measure-valued solutions to (2.1.1)-(2.1.2) through a singular limit in the classical choice of the influence function devoted to Cucker and Smale. It will be done via a hydrodynamic singular limit of vanishing-inertia type on an appropriately chosen hyperbolic-type scaling for the kinetic Cucker–Smale system of flocking with linear damping and velocity noise, namely

$$\frac{\partial f_\varepsilon}{\partial t} + v \cdot \nabla_x f_\varepsilon - \frac{1}{\varepsilon}\nabla_x \psi_\varepsilon \cdot \nabla_v f_\varepsilon = \frac{1}{\varepsilon}\operatorname{div}_v \left(f_\varepsilon v + \nabla_v f_\varepsilon + Q_{CS}^{\phi_\varepsilon}(f_\varepsilon, f_\varepsilon) \right), \tag{2.1.7}$$

where the scaled kinetic Cucker–Smale alignment operator reads

$$Q_{CS}^{\phi_\varepsilon}(f_\varepsilon, f_\varepsilon)(t,x,v) := f_\varepsilon(t,x,v)\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\phi_\varepsilon(|x-y|)(v-w)f_\varepsilon(t,y,w)\,dy\,dw.$$

The interaction potential will be scaled as follows

$$\phi_\varepsilon(r) = \frac{1}{(\varepsilon^2 + c_\alpha r^2)^\alpha}. \tag{2.1.8}$$

As it is apparent now, the formal limit $\varepsilon \searrow 0$ yields singular interactions modeled by the Riesz kernel ϕ_0 of order $d-2\alpha$ in (2.1.2). In a general context, we can see (2.1.7)-(2.1.8) as a way to obtain an approximate sequence of solutions whose macroscopic quantities converge towards a measure-valued solution of (2.1.1)-(2.1.2) in a weak sense. Also, see [57, 126, 175] for recent related results on such vanishing inertia type limits or Smoluchowski limits. Finally, not only will we focus on the above-mentioned hyperbolic scaling, but we will also inspect appropriately chosen intermediate scalings where the velocity diffusion effects (that are inherited from the randomness in the microscopic equations) do not dominate and disappear in the limit. See [22, 232, 252] for a comprehensive study of the intermediate, hyperbolic and parabolic limits in the Vlasov–Poisson–Fokker–Planck system.

As it will be observed in the chapter, the main difficulties both to pass to the limit and to analyze the limiting hydrodynamic system obviously rely on handling with commutators. Specifically, the commutator in the hydrodynamic limit reads

$$(\phi_\varepsilon * j_\varepsilon)\rho_\varepsilon - (\phi_\varepsilon * \rho_\varepsilon)j_\varepsilon = \int_{\mathbb{R}^d}\frac{1}{(\varepsilon^2 + c_\alpha|x-y|^2)^\alpha}\left(\rho_\varepsilon(t,x)j_\varepsilon(t,y) - \rho_\varepsilon(t,y)j_\varepsilon(t,x)\right)\,dy,$$

where ρ_ε and j_ε stand for the density and current of particles associated with the particles distribution function f_ε, i.e.,

$$\rho_\varepsilon(t,x) := \int_{\mathbb{R}^d} f_\varepsilon(t,x,v)\,dv, \qquad j_\varepsilon(t,x) := \int_{\mathbb{R}^d} v f_\varepsilon(t,x,v)\,dv.$$

Then, the main result of this chapter reads as follows (also see Section 2.2 for a more detailed statement).

Theorem 2.1.1. *Under appropriate hypothesis on the initial data f_ε^0 and the external forces $-\nabla\psi_\varepsilon$, let f_ε be the smooth solutions to (2.1.7) with $\alpha \in (0, \frac{1}{2}]$. Then, the macroscopic quantities ρ_ε and j_ε converge in a weak sense to some probability measure ρ and some finite Radon measure j that solve the Cauchy problem associated with the following Euler-type system in the distributional sense*

$$\left\{\begin{array}{ll} \partial_t\rho + \operatorname{div} j = 0, & x\in\mathbb{R}^d,\ t\in[0,T),\\ (\phi_0 * j)\rho - (\phi_0*\rho)j - j - \rho\,\nabla\psi = 0, & x\in\mathbb{R}^d,\ t\in[0,T),\\ \rho(0,\cdot) = \rho^0, & x\in\mathbb{R}^d. \end{array}\right. \tag{2.1.9}$$

Naturally, we have some difficulties when trying to give some sense to the corresponding commutator when $\varepsilon \searrow 0$ since we do not expect that ρ_ε and j_ε converge stronger than in the weak-star sense of finite Radon measures. This is due to the fact that the distribution function and some of its functional moments with respect to velocity will be only bounded in L^1, independently of ε. Like it is the case for other related systems with antisymmetric interactions, one can solve such issue for singularities that lie in the range $\alpha \in (0, \frac{1}{2})$ via a symmetrization trick by just considering the above term in weak form and by cancelling the singularity through the use of appropriate test functions. Nevertheless, the same cannot be directly inferred for the endpoint case $\alpha = \frac{1}{2}$ (compare with the *2D Euler equations in vorticity formulation* [269]), that indeed agrees with the critical value of the parameter α over which one cannot expect unconditional flocking in the Cucker–Smale system (see [65, 162]). Again, comparing with the 2D Euler system in vorticity formulation, recall that some sort of logarithmic Morrey space estimate for the sequence of approximated vorticities can be obtained in order to ensure that the system do not exhibit concentration and to pass to the limit (see [103, 207, 293]). In our case, our scaled systems do not enjoy any additional estimate with respect to Morrey-type norms. Fortunately, we shall obtain an alternative uniform-in-ε bound on the *dissipation of the kinetic energy due to alignment interactions*, i.e.,

$$\int_0^T \int_{\mathbb{R}^{4d}} |v-w|^2 \phi_\varepsilon(|x-y|) f_\varepsilon(t,x,v) f_\varepsilon(t,y,w)\, dx\, dy\, dv\, dw\, dt \leq C_T,$$

for all $\varepsilon > 0$. Such bound will help us prove that the singular limit $\varepsilon \to 0$ involves no concentration and we can pass to the limit in the symmetrized version of the nonlinear term for all the values of the parameter $\alpha \in (0, \frac{d}{2})$, in the same spirit as in [269] or the 2D-case in [232]. Nevertheless, it will only prove useful for the restricted range $\alpha \in (0, \frac{1}{2}]$ because the absence of further L^p-integrability for ρ and j causes severe difficulties to identify the limiting nonlinear term in weak form. Interestingly, we will show that the nonlinear term is uniformly bounded with respect to ε though for a slightly larger range of the parameter, namely, $\alpha \in (0, 1)$, see Remark 2.2.19. It suggests that the convergence result might be extended to such case but, making such argument rigorous is a difficult task that we will not explore here because some convergence result for the particle distribution function f_ε towards f is missing and, more precisely, the extension of the above bound of the dissipation of kinetic energy to the limiting distribution seems unclear.

The dissipation of the kinetic energy due to alignment interactions also appeared in some preceding results [61, 127, 185, 186] as a dissipation term of a "generalized entropy" that is composed of the Boltzmann entropy, the kinetic energy and the second order moment with respect to position. The symmetries of the aforementioned local alignment forces along with the boundedness of the influence function (2.1.6) near $r = 0$ and an appropriate choice of the scaling in the kinetic equation ensured the presence of such entropy inequality in [127, 186]. However, as it will be made apparent later, we cannot ensure that the vanishing intertia type scaling in our system (2.1.7) do enjoy such sort of entropy results. It stands to reason that it is due to the singular scaling for ϕ_ε and the corresponding lack of further L^p norms for the macroscopic density $\rho = \rho(t,x)$. Then, in this chapter we shall forget about entropy methods and will simply resort on compactness methods, that have proven useful to deal with singular interactions. We must warn here about the fact that in our hyperbolic and intermediate scalings, the presence of the velocity linear damping term $\mathrm{div}_v(f_\varepsilon v)$ in the Fokker–Planck operator is essential to derive some of the only possible estimates for the current of particles that can be obtained. One might think that it perfectly fits the fact that in [127, 186] extra damping terms are required. However, such idea is not completely correct. Indeed, [126] explores the same type of vanishing inertia type limit in a system with regular influence function but no extra damping term was required by the authors to achieve the convergence towards the monokinetic distribution. In our case, the damping term is needed in order to neutralize the negative effect of working with purely singular interactions.

Once Theorem 2.1.1 is proved, we can formally cancel a factor ρ in the limiting equation (2.1.9) by considering the relation $j = \rho\, u$ linking current to velocity field of agents. Then, we are formally led to Equation (2.1.1) and the commutator in (2.1.3). This is somehow surprising because such commutator has not clear sense for $\rho \in L^1(\mathbb{R}^d)$. For such limiting Equation (2.1.1), that couples the velocity field and the density function, we shall explore the existence of strong solutions with higher regularity for the Cauchy problem associated with the system (2.1.1)-(2.1.2). To this end, one has to obtain accurate bounds for the commutator operator $[u, I_{d-2\alpha}]$ of a Riesz potential and the multiplication operator by u. The study of such bounds in L^p spaces is classical in harmonic analysis, see [73, 82, 89, 243]. In particular, it is known, by the *Hardy–Littlewood–Sobolev inequality*, that the *fractional integral* or *Riesz potential* of order $d-2\alpha$ is a bounded linear operator between the next spaces

$$I_{d-2\alpha} : L^p(\mathbb{R}^d) \longrightarrow L^q(\mathbb{R}^d),$$

where $1 < p < q < \infty$ and $1/p - (d-2\alpha)/d = 1/q$. The boundedness when $p = 1$ (then $q = d/2\alpha$) is not true in the $L^p - L^q$ sense but in the $L^p - L^{q,\infty}$ way, where $L^{q,\infty}(\mathbb{R}^d)$ stands for the weak Lebesgue space or Lorentz spaces. For further details, we refer to [278, Theorem 1.2.1] and the discussion in Appendix C. Regarding the commutator, one recovers the boundedness of the next linear operator

$$[u, I_{d-2\alpha}] : L^p(\mathbb{R}^d) \longrightarrow L^q(\mathbb{R}^d, \mathbb{R}^d),$$

if, and only if, $u \in BMO(\mathbb{R}^d, \mathbb{R}^d)$, see [73]. Nevertheless, the endpoint case $p = 1$ cannot be recovered in the $L^p - L^{q,\infty}$ sense [89], which is the natural framework according to the lack of estimates for ρ. The main difficulty when dealing with the Equation (2.1.1) is that it is not clear whether the operator $u \longmapsto [u, I_{d-2\alpha}]\rho$ is bounded from some Banach space to itself. Note that it is a key property to be checked in order for the classical existence techniques involving fixed-point theorems to work. We will show that this is the case for some well chosen Banach space of functions of Lipschitz type enjoying some summability properties. Namely, such normed space will be denoted by $W^{k,p,q}$, where

$$W^{k,p,q} := W^{k-1,p}(\mathbb{R}^d) \cap W^{k-1,q}(\mathbb{R}^d) \cap W^{k,\infty}(\mathbb{R}^d), \tag{2.1.10}$$

that is a Banach space when endowed with the norm

$$\|f\|_{W^{k,p,q}(\mathbb{R}^d)} := \|f\|_{W^{k-1,p}(\mathbb{R}^d)} + \|f\|_{W^{k-1,q}(\mathbb{R}^d)} + \|f\|_{W^{k,\infty}(\mathbb{R}^d)}.$$

Such commutator estimates constitute the cornerstone to construct our strong solutions to (2.1.1) via fixed point arguments. This will be the purpose of our second main result in this chapter.

Theorem 2.1.2. *Let us consider $\alpha \in (0, \frac{d}{2})$, $1 \leq p_1 < p_2 \leq \infty$ and $k \in \mathbb{N}$ such that*

$$\frac{1}{p_2} < 1 - \frac{2\alpha}{d} < \frac{1}{p_1} \text{ and } kp_1 > \frac{d}{2\alpha}.$$

Then, for any $R > 0$ there exists $\delta_R > 0$ depending on R such that (2.1.1) admits one, and only one solution $u \in L^1(0,T; W^{1,kp_1,kp_2}(\mathbb{R}^d, \mathbb{R}^d))$, $\rho \in L^\infty(0,T; L^{p_1}(\mathbb{R}^d) \cap L^{p_2}(\mathbb{R}^d))$ with

$$\|u\|_{L^1(0,T;W^{1,kp_1,kp_2}(\mathbb{R}^d,\mathbb{R}^d))} \leq R,$$

provided that the forcing term $F = -\nabla\psi$ and the initial datum ρ^0 fulfil the "smallness" conditions

$$\|\nabla\psi\|_{L^1(0,T;W^{1,kp_1,kp_2}(\mathbb{R}^d,\mathbb{R}^d))} < R \text{ and } \|\rho^0\|_{W^{k,p_1,p_2}(\mathbb{R}^d)} < \delta_R.$$

Such result then entails that the friction term tends to slow down the particles up to a state of absence of any motion unless some forcing term $F = -\nabla\psi$ assumed to act on the system. On the one hand, notice that the velocity damping is natural and it can be justifyied by the conditions of the medium. On the other hand, the presence of a force F can also be justified in certain biological settings like in the *soaring flight* of some flocks of big birds like pelicans or seagulls. Soaring birds avoids flapping in the presence of a thermal, since they can use wind currents to self-propel. See [234, Chapter 5] for a comprehensive classification of the kinds and mechanisms of soaring flights along with some pictures (e.g., slope, thermal, gust, frontal, wave and dynamic soaring). Another reference where such biological feature is considered (i.e., some kind of "external" force is assumed to propel birds) is the coupled macroscopic–mesoscopic system proposed in [61] consisting in a Vlasov–Fokker–Planck equation with local alignment effects describing the motion of birds in the core of a fluid obeying the Navier–Stokes equation of Fluid Mechanics. See also [16] for a similar analysis of global existence of strong solutions of such coupled systems.

The rest of the chapter is organized as follows. In Section 2.2 and 2.4 we shall focus on the relationship between the mesoscopic and macroscopic scales of descriptions via hydrodynamic limits. Specifically, in Section 2.2 we will concentrate on a hyperbolic and singular limit of the kinetic Cucker–Smale model whilst Section 2.4 shows that such techniques can be adapted to other relevant hydrodynamic limits of intermediate type. We will also address the frictionless case with appropriate scaling along with a hydrodynamic limit linking the Rayleigh–Helmholtz type friction terms with the standard linear damping. Section 2.3 presents the existence and uniqueness of global in time solutions of higher regularity for the limiting macrocopic systems arising from the preceding hydrodynamic and singular limits. To conclude, we derive in Appendix 2.A the preceding scalings of the kinetic equation through a nondimensionalization procedure and an appropriate choice of the dimensionless parameters.

2.2 Hyperbolic and singular hydrodynamic limit

In this section we shall focus on a hyperbolic scaling of the kinetic Cucker–Smale model with friction and diffusion effects in the velocity variable, see (2.1.7). Recall that the main highlight in this scaling is that the regular influence functions ϕ_ε converge towards the singular one ϕ_0 as $\varepsilon \to 0$, see (2.1.2) and (2.1.8). In Appendix 2.A such kinetic Cucker–Smale model was recovered from agent-based descriptions via mean field limits. Moreover, a dimensional study of the physical constants of the model has been conducted, thus producing the following hyperbolic scaling

$$\frac{\partial f_\varepsilon}{\partial t} + v\cdot\nabla_x f_\varepsilon - \frac{1}{\varepsilon}\nabla_x\psi_\varepsilon\cdot\nabla_v f_\varepsilon = \frac{1}{\varepsilon}\operatorname{div}_v\left(f_\varepsilon v + \nabla_v f_\varepsilon + Q_{CS}^{\phi_\varepsilon}(f_\varepsilon, f_\varepsilon)\right), \tag{2.2.1}$$

for the probability distribution $f_\varepsilon(t,\cdot,\cdot)\in\mathcal{P}(\mathbb{R}^d\times\mathbb{R}^d)$ under appropriate assumptions on the scaling of the scaled mean thermal velocity and mean free path, along with the scaled effective range and strength of the interactions. See Appendix 2.A for further details. Although we will be interested in (2.2.1) most of the time, we will show that our technique also yields some insight into the understanding of the rigorous hydrodynamic limit in an intermediate hyperbolic scaling for the frictional case and a hyperbolic scaling for the frictionless case that we have also introduced in the Appendix 2.A. This will be postponed to Section 2.4.

2.2.1 The Maxwell–Boltzmann distribution

For simplicity, in this chapter we will not address the rigorous convergence of f_ε, but we shall simply state some heuristics about the shape of the limiting distribution of f_ε as $\varepsilon \to 0$ in terms of its associated macroscopic quantities ρ and j, that are governed by (2.1.9) of this chapter. The reader might find this discussion useful to gain some intuition on the asymptotics that we have proposed for this model. We refer to [126, 175] for some rigorous results in such line regarding vanishing inertia limits for regular interaction kernels.

Theorem 2.2.1. *Let f_ε solve the scaled system* (2.2.1) *and consider any weak-* limit in the sense of distributions $f(t,x,v) = (weak\,*) - \lim_{\varepsilon\to 0} f_\varepsilon(t,x,v)$. Then, f formally takes the form of a Maxwell–Boltzmann distribution, namely,*

$$f(t,x,v) = \frac{\rho(t,x)}{(2\pi k_B T(t,x))^{d/2}} e^{-\frac{|v-u(t,x)|^2}{2k_B T(t,x)}}.$$

Here ρ and u respectively represent the macroscopic probability density and velocity field associated with f, that are governed by (2.1.1)*, and $T(t,x)$ is the thermodynamic temperature, i.e.,*

$$\rho := \int_{\mathbb{R}^d} f\,dv,$$
$$j := \rho\, u := \int_{\mathbb{R}^d} v f\,dv,$$
$$T := \frac{1}{k_B(1+\phi * \rho)},$$

where k_B stands for Boltzmann's constant.

In order to properly derive such formula for f, we shall require the following result.

Lemma 2.2.2. *Consider any $A \in \mathbb{R}^d$ and $B \in \mathbb{R}^+$, and define the Gaussian function*

$$G(v) = e^{A\cdot v - B|v|^2},\ v \in \mathbb{R}^d.$$

Then, we obtain that

$$\int_{\mathbb{R}} G(v)\,dv = \left(\frac{\pi}{B}\right)^{d/2} e^{\frac{|A|^2}{4B}}.$$

Proof of Theorem 2.2.1. Taking limits formally when $\varepsilon \to 0$ in (2.2.1), we can seek for f as solution to the following equation

$$\nabla_v f = (-v - (\phi * \rho)v + \phi * j - \nabla_x \psi)\, f.$$

By integration we obtain that f can be expressed as follows

$$f(t,x,v) = \exp\left(-\frac{1}{2}(1+\phi*\rho)|v|^2 + (\phi * j - \nabla_x\psi)\cdot v\right) F(t,x), \tag{2.2.2}$$

for some factor v-independent $F = F(t,x)$ that we will try to identify in terms of ρ and j. Define the coefficients

$$A := \phi * j - \nabla_x \psi \ \text{ and } \ B := \frac{1}{2}(1+\phi*\rho).$$

Then, by virtue of Lemma 2.2.2, we can integrate with respect to v in (2.2.2) to achieve the following relation

$$\rho = \left(\frac{2\pi}{1+\phi*\rho}\right)^{d/2} \exp\left(\frac{|\phi*j-\nabla_x\psi|^2}{2(1+\phi*\rho)}\right) F.$$

Notice that it relates F with ρ and j. Hence, substituting in (2.2.2) implies that

$$\begin{aligned} f(t,x,v) &= \rho\left(\frac{1+\phi*\rho}{2\pi}\right)^{d/2} \exp\left(-\frac{|(1+\phi*\rho)v-(\phi*j-\nabla_x\psi)|^2}{2(1+\phi*\rho)}\right) \\ &= \rho\left(\frac{1+\phi*\rho}{2\pi}\right)^{d/2} \exp\left(-\frac{(1+\phi*\rho)}{2}|v-u|^2\right), \end{aligned} \tag{2.2.3}$$

where the macroscopic velocity field u can be recovered from the relation $j=\rho u$ and we have used the second equation in (2.1.9) to simplify terms. Then, the proof follows by definition of the thermodynamic temperature. □

2.2.2 Hierarchy of moments

Before going into the heart of the matter, let us introduce some notation that will be used along the chapter and show the first equations of the hierarchy of velocity moments. We will focus on the next ones that correspond to the first forth velocity moments

$$\begin{aligned} \text{Density:} \quad & \rho_\varepsilon(t,x) := \int_{\mathbb{R}^d} f_\varepsilon(t,x,v)\,dv, \\ \text{Current:} \quad & j_\varepsilon(t,x) := \int_{\mathbb{R}^d} v\, f_\varepsilon(t,x,v)\,dv, \\ \text{Velocity field:} \quad & u_\varepsilon(t,x) := \frac{j_\varepsilon(t,x)}{\rho_\varepsilon(t,x)}, \\ \text{Stress tensor:} \quad & \mathcal{S}_\varepsilon(t,x) := \int_{\mathbb{R}^d} v\otimes v\, f_\varepsilon(t,x,v)\,dv, \\ \text{Stress flux tensor:} \quad & \mathcal{T}_\varepsilon(t,x) := \int_{\mathbb{R}^d} (v\otimes v)\otimes v f_\varepsilon(t,x,v)\,dv. \end{aligned}$$

First of all, let us note that the average kinetic energy consists of the macroscopic kinetic energy together with the internal energy and it can be obtained as a contraction of the stress tensor. Specifically, take the trace of the stress tensor to obtain

$$\begin{aligned} E_\varepsilon(t,x) := \frac{1}{2}\operatorname{Tr}(\mathcal{S}_\varepsilon(t,x)) &= \frac{1}{2}\int_{\mathbb{R}^d} |v|^2 f_\varepsilon(t,x,v)\,dv \\ &= \frac{1}{2}\rho_\varepsilon + \frac{1}{2}\int_{\mathbb{R}^d} |v-u_\varepsilon(t,x)|^2 f_\varepsilon(t,x,v)\,dv \\ &=: E_\varepsilon^{kin}(t,x) + E_\varepsilon^{int}(t,x). \end{aligned}$$

Another classical kinetic quantity is the well known energy flux that can be obtained as a new contraction of the stress flux tensor. Namely,

$$Q_\varepsilon(t,x) := \frac{1}{2}\operatorname{Tr}(\mathcal{T}_\varepsilon(t,x)) = \frac{1}{2}\int_{\mathbb{R}^d} v|v|^2 f_\varepsilon(t,x,v)\,dv.$$

Note that (2.2.1) can be restated in the following form that closes the nonlinear alignment operator in terms of some of the above macroscopic quantities

$$\varepsilon\frac{\partial f_\varepsilon}{\partial t}+\varepsilon v\cdot\nabla_x f_\varepsilon-\nabla_x\psi_\varepsilon\cdot\nabla_v f_\varepsilon=\operatorname{div}_v\left(f_\varepsilon v+\varepsilon\nabla_v f_\varepsilon+f_\varepsilon\phi_\varepsilon*\rho_\varepsilon v-f_\varepsilon\phi_\varepsilon*j_\varepsilon\right). \tag{2.2.4}$$

Also, notice that we have moved the parameter ε to the transport part in the left hand side of the equation in order to clarify that (2.2.4) indeed consists in a vanishing-inertia type scaling. Here on we will need to assume that the initial data f_ε^0 satisfy the next hypothesis

$$\left\{\begin{array}{l} f_\varepsilon^0=f_\varepsilon^0(x,v)\geq 0\ \text{ and }\ f_\varepsilon^0\in C_c^\infty(\mathbb{R}^d\times\mathbb{R}^d),\\ \|f_\varepsilon^0\|_{L^1(\mathbb{R}^{2d})}=1\ \text{ and } \rho_\varepsilon^0\overset{*}{\rightharpoonup}\rho^0 \text{ in } \mathcal{M}(\mathbb{R}^d),\\ \||x|f_\varepsilon^0\|_{L^1(\mathbb{R}^{2d})}\leq M_0\ \text{ and }\ \||v|^2f_\varepsilon^0\|_{L^1(\mathbb{R}^{2d})}\leq E_0, \end{array}\right. \tag{2.2.5}$$

for every $\varepsilon>0$, where $M_0,E_0>0$ are constants that do not depend on $\varepsilon>0$. Regarding the external forces $-\nabla\psi_\varepsilon$, we will assume that

$$\left\{\begin{array}{l} \psi_\varepsilon\in L^2(0,T;W^{1,\infty}(\mathbb{R}^d)) \text{ and } \psi\in L^2(0,T;W^{1,\infty}(\mathbb{R}^d)),\\ \nabla\psi_\varepsilon(t,\cdot)\in C_b(\mathbb{R}^d,\mathbb{R}^d) \text{ and } \nabla\psi(t,\cdot)\in C_b(\mathbb{R}^d,\mathbb{R}^d),\ \text{a.e. } t\in[0,T),\\ \|\nabla\psi_\varepsilon\|_{L^2(0,T;L^\infty(\mathbb{R}^d,\mathbb{R}^d))}\leq F_0\ \text{ and }\ \nabla\psi_\varepsilon\to\nabla\psi\ \text{ in } L^1(0,T;C_b(\mathbb{R}^d,\mathbb{R}^d)), \end{array}\right. \tag{2.2.6}$$

for every $\varepsilon>0$ and some $F_0>0$ not depending on ε.

Remark 2.2.3. *Notice that the uniform tightness condition in the third line of* (2.2.5) *allows guaranteeing that the weak * limit ρ^0 does not lose mass at infinity and, consequently $\rho^0\in\mathcal{P}(\mathbb{R}^d)$ and*

$$\rho_\varepsilon^0\to\rho^0\ \textit{ narrow in }\ \mathcal{M}(\mathbb{R}^d),$$

thanks to Prokhorov's compactness theorem. Indeed, we can readily see via standard arguments [296] that the limit preserves the same finite first order moment, i.e., $\rho^0\in\mathcal{P}_1(\mathbb{R}^d)$ and, in addition, the narrow convergence amounts to

$$\rho_\varepsilon^0\to\rho^0\ \textit{ in }\ \mathcal{P}_1(\mathbb{R}^d)-W_1.$$

However, in this chapter we will forget about Wasserstein spaces for simplicity. See Chapter 4, where a more sharper result is shown in Wasserstein spaces for the closely related singular Kuramoto model.

Under such assumptions (2.2.6)-(2.2.5) there exists a unique global in time strong solution to (2.2.4) such that $f_\varepsilon(0,\cdot,\cdot)=f_\varepsilon^0$ for every $\varepsilon>0$, see [44, 68, 162]. Respectively multiplying Equation (2.2.4) by 1, v and $v\otimes v$ (Kronecker product) and integrating by parts with respect to v, we obtain the next hierarchy of moment equations:

- **Mass conservation**

$$\frac{\partial\rho_\varepsilon}{\partial t}+\operatorname{div}_x j_\varepsilon=0. \tag{2.2.7}$$

- **Law of balance of current**

$$\varepsilon\frac{\partial j_\varepsilon}{\partial t}+\varepsilon\operatorname{div}_x\mathcal{S}_\varepsilon+\rho_\varepsilon\nabla_x\psi_\varepsilon+\left(1+\phi_\varepsilon*\rho_\varepsilon\right)j_\varepsilon-\left(\phi_\varepsilon*j_\varepsilon\right)\rho_\varepsilon=0. \tag{2.2.8}$$

- **Law of balance of stress**

$$\varepsilon\frac{\partial\mathcal{S}_\varepsilon}{\partial t}+\varepsilon\operatorname{div}_x\mathcal{T}_\varepsilon+2\operatorname{Sym}(j_\varepsilon\otimes\nabla_x\psi_\varepsilon)+2\left(\left(1+\phi_\varepsilon*\rho_\varepsilon\right)\mathcal{S}_\varepsilon-\rho_\varepsilon I\right)-2\operatorname{Sym}((\phi_\varepsilon*j_\varepsilon)\otimes j_\varepsilon)=0. \tag{2.2.9}$$

Here, $\mathrm{Sym}(M)$ denotes the symmetric part of a square matrix M, i.e., $\mathrm{Sym}(M) := \frac{1}{2}(M + M^{\mathsf{T}})$. Notice that we can take the spectral norm of the stress tensor and notice that it can be fully controlled by the average kinetic energy

$$|\mathcal{S}_\varepsilon(t,x)| \leq 2E_\varepsilon(t,x),\ \ t \geq 0,\ x \in \mathbb{R}^d.$$

Consequently, despite the fact that the stress tensor does contain all the information about the average kinetic energy of the system of particles, one might be interested in a priori control for the latter. Indeed, a explicit balance law for the average kinetic energy can be readily deduced from (2.2.9) by taking traces

$$\varepsilon\frac{\partial E_\varepsilon}{\partial t} + \varepsilon\,\mathrm{div}_x\, Q_\varepsilon + j_\varepsilon \cdot \nabla_x \psi_\varepsilon + 2\left((1+\phi_\varepsilon * \rho_\varepsilon)E_\varepsilon - \frac{d}{2}\rho_\varepsilon\right) - (\phi_\varepsilon * j_\varepsilon)\cdot j_\varepsilon = 0. \tag{2.2.10}$$

As it often happens in most of the kinetic equations, the hierarchy of velocity moments is not a closed system. Nevertheless, as it will be shown later, our hyperbolic hydrodynamic limit will close the system (2.2.7)–(2.2.8) when $\varepsilon \searrow 0$ as stated in Equation (2.1.1).

2.2.3 A priori bounds

Let us show first some useful bounds for the local interaction kernel ϕ_ε.

Lemma 2.2.4. *Let us fix any nonnegative number ε and any exponent $\alpha \in (0, 1/2)$. Then, the next estimates hold for every $r > 0$:*

1. $\phi_\varepsilon(r) \leq \dfrac{1}{c_\alpha^\alpha r^{2\alpha}}$,
2. $\phi_\varepsilon(r) \leq \dfrac{1}{\varepsilon^{2\alpha}}$,
3. $|\phi_\varepsilon(r) - \phi_0(r)| \leq C_\alpha \dfrac{\varepsilon^{1-2\alpha}}{r}$,

where $C_\alpha > 0$ is some constant depending on α but not depending neither on ε nor on r.

Proof. The first two estimates follows from the definition of ϕ_ε. Then, let us focus on the last property. The fundamental lemma of calculus leads to the next expression

$$\phi_\varepsilon(r) - \phi_0(r) = \int_0^\varepsilon \frac{d}{d\tau}\left[\frac{1}{(\tau^2 + c_\alpha r^2)^\alpha}\right] d\tau = -\alpha \int_0^\varepsilon \frac{2\tau}{(\tau^2 + c_\alpha r^2)^{\alpha+1}}\, d\tau.$$

Young's inequality for real numbers allows obtaining the next bound

$$2\tau = \frac{1}{\sqrt{c_\alpha} r} 2\tau\sqrt{c_\alpha} r \leq \frac{1}{\sqrt{c_\alpha} r}(\tau^2 + c_\alpha r^2).$$

Then, taking absolute values on the preceding identity we arrive at

$$\begin{aligned}|\phi_\varepsilon(r) - \phi_0(r)| &\leq \frac{\alpha}{\sqrt{c_\alpha}}\frac{1}{r}\int_0^\varepsilon \frac{1}{(\tau^2 + c_\alpha r^2)^\alpha}\, d\tau \\ &\leq \frac{\alpha}{\sqrt{c_\alpha}}\frac{1}{r}\int_0^\varepsilon \tau^{-2\alpha}\, d\tau = \frac{\alpha}{(1-2\alpha)\sqrt{c_\alpha}}\frac{\varepsilon^{1-2\alpha}}{r}.\end{aligned}$$

□

Let us now discuss some a priori bounds for the density function ρ_ε and the current j_ε.

Proposition 2.2.5. *Let the initial distribution functions f_ε^0 verify (2.2.5) and the external forces $-\nabla\psi_\varepsilon$ fulfill (2.2.6). Consider the strong global in time solution f_ε to (2.2.4) with initial data f_ε^0 and fix any nonnegative integer k. Then, the k-th order moments in x and v of f_ε obey the following equations,*

$$\begin{aligned}\varepsilon\frac{d}{dt}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|v|^k f_\varepsilon\,dx\,dv =& -k\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|v|^{k-2}v\cdot\nabla_x\psi_\varepsilon f_\varepsilon\,dx\,dv\\ &-k\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|v|^k f_\varepsilon\,dx\,dv + k(d+k-2)\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|v|^{k-2}f_\varepsilon\,dx\,dv\\ &-k\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|v|^{k-1}\left((\phi_\varepsilon*\rho_\varepsilon)|v|-(\phi_\varepsilon*j_\varepsilon)\cdot\frac{v}{|v|}\right)f_\varepsilon\,dx\,dv, \qquad (2.2.11)\\ \frac{d}{dt}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|x|^k f_\varepsilon\,dx\,dv =&\quad k\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|x|^{k-2}x\cdot v f_\varepsilon\,dx\,dv. \qquad (2.2.12)\end{aligned}$$

Note that in particular we recover the conservation of mass, namely,

$$\rho_\varepsilon(t,\cdot)\in\mathcal{P}(\mathbb{R}^d),\ \textit{ for all }\ t\geq 0,\ \varepsilon>0. \qquad (2.2.13)$$

Corollary 2.2.6. *Under the hypothesis in Proposition 2.2.5, we obtain that*

$$\begin{aligned}&\left\||v|^k f_\varepsilon\right\|_{L^1(0,T;L^1(\mathbb{R}^{2d}))}\\ &+\frac{1}{2}\int_0^T\int_{\mathbb{R}^{4d}}(|v|^{k-2}v-|w|^{k-2}w)\cdot(v-w)\phi_\varepsilon(|x-y|)f_\varepsilon(t,x,v)f_\varepsilon(t,y,w)\,dx\,dy\,dv\,dw\,dt\\ &\quad\leq\frac{\varepsilon}{k}\||v|^k f_\varepsilon^0\|_{L^1(\mathbb{R}^{2d})}+(d+k-2)\||v|^{k-2}f_\varepsilon\|_{L^1(0,T;L^1(\mathbb{R}^{2d}))}+\left\||v|^{k-1}f_\varepsilon\nabla_x\psi_\varepsilon\right\|_{L^1(0,T;L^1(\mathbb{R}^{2d}))},\\ &\||x|^k f_\varepsilon\|_{L^\infty(0,T;L^1(\mathbb{R}^d))}^{\frac{1}{k}}\leq\||x|^k f_\varepsilon^0\|_{L^1(\mathbb{R}^d)}^{\frac{1}{k}}+\int_0^T\left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|v|^k f_\varepsilon\,dx\,dv\right)^{\frac{1}{k}}dt.\end{aligned}$$

Proof. Regarding the first inequality, integrate Equation (2.2.11) with respect to time and neglect the term arising from the Barrow rule associated with the endpoint $t=T$ of the time interval. On the other hand, regarding the second inequality notice that we can use Hölder's inequality in the RHS of (2.2.12) and arrive at

$$\frac{d}{dt}\left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|x|^k f_\varepsilon\,dx\,dv\right)^{\frac{1}{k}}\leq\left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|v|^k f_\varepsilon\,dx\,dv\right)^{\frac{1}{k}}.$$

Then, integrating with respect to time yields the claimed inequality. □

Remark 2.2.7. *Note that the estimate in Corollary 2.2.6 leads to a real estimate of the k-the order velocity moment as long as the second term in the LHS is nonnegative. In turns, such term can be lower bounded through simple computations involving the Cauchy–Schwartz inequality as follows*

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^{4d}}(|v|^{k-1}-|w|^{k-1})(|v|-|w|)\phi_\varepsilon(|x-y|)f_\varepsilon(t,x,v)f_\varepsilon(t,y,w)\,dx\,dy\,dv\,dw\\ &\qquad\leq\frac{1}{2}\int_0^T\int_{\mathbb{R}^{4d}}(|v|^{k-2}v-|w|^{k-2}w)\cdot(v-w)f_\varepsilon(t,x,v)f_\varepsilon(t,y,w)\,dx\,dy\,dv\,dw\,dt.\end{aligned}$$

Note that when $k\geq 1$, then

$$(|v|^{k-1}-|w|^{k-1})(|v|-|w|)\geq 0\qquad v,w\in\mathbb{R}^d.$$

Consequently, for each $k \geq 1$ the k-th order moment can be bounded as follows

$$\left\| |v|^k f_\varepsilon \right\|_{L^1(0,T;L^1(\mathbb{R}^{2d}))} \leq \frac{\varepsilon}{k} \| |v|^k f_\varepsilon^0 \|_{L^1(\mathbb{R}^{2d})} + (d+k-2) \| |v|^{k-2} f_\varepsilon \|_{L^1(0,T;L^1(\mathbb{R}^{2d}))} + \left\| |v|^{k-1} f_\varepsilon \nabla_x \psi_\varepsilon \right\|_{L^1(0,T;L^1(\mathbb{R}^{2d}))}, \tag{2.2.14}$$

and the next term can be bounded similarly

$$\frac{1}{2} \int_0^T \int_{\mathbb{R}^{4d}} (|v|^{k-2} v - |w|^{k-2} w) \cdot (v-w) \phi_\varepsilon(|x-y|) f_\varepsilon(t,x,v) f_\varepsilon(t,y,w)\, dx\, dt\, dv\, dw\, dt \leq \frac{\varepsilon}{k} \| |v|^k f_\varepsilon^0 \|_{L^1(\mathbb{R}^{2d})} + (d+k-2) \| |v|^{k-2} f_\varepsilon \|_{L^1(0,T;L^1(\mathbb{R}^{2d}))} + \left\| |v|^{k-1} f_\varepsilon \nabla_x \psi_\varepsilon \right\|_{L^1(0,T;L^1(\mathbb{R}^{2d}))}. \tag{2.2.15}$$

In particular, the case $k=2$ yields the next bounds of the average kinetic energy, the current of particles and the *dissipation of kinetic energy due to alignment interaction between particles*.

Corollary 2.2.8. *Under the hypothesis in Proposition 2.2.5,*

$$\| |x| f_\varepsilon \|_{L^\infty(0,T;L^1(\mathbb{R}^{2d}))} \leq M_0 + T^{1/2} \| |v| f_\varepsilon \|^{1/2}_{L^2(0,T;L^1(\mathbb{R}^d))}, \tag{2.2.16}$$

$$\| |v| f_\varepsilon \|_{L^2(0,T;L^1(\mathbb{R}^{2d}))} \leq \left\| |v|^2 f_\varepsilon \right\|^{1/2}_{L^1(0,T;L^1(\mathbb{R}^{2d}))}, \tag{2.2.17}$$

$$\left\| |v|^2 f_\varepsilon \right\|_{L^1(0,T;L^1(\mathbb{R}^{2d}))} \leq 2\varepsilon E_0 + 2dT + F_0^2. \tag{2.2.18}$$

In addition, the next estimate holds

$$\int_0^T \int_{\mathbb{R}^{4d}} \phi_\varepsilon(|x-y|) |v-w|^2 f_\varepsilon(t,x,v) f_\varepsilon(t,y,w)\, dx\, dy\, dv\, dw\, dt \leq 2\varepsilon E_0 + 2dT + F_0^2. \tag{2.2.19}$$

Proof. By Corollary 2.2.6, the next bound follows for $k=2$

$$\begin{aligned} 2\| |v|^2 f_\varepsilon \|_{L^1(0,T;L^1(\mathbb{R}^{2d}))} &+ \int_0^T \int_{\mathbb{R}^{4d}} |v-w|^2 \phi_\varepsilon(|x-y|) f_\varepsilon(t,x,v) f_\varepsilon(t,y,w)\, dx\, dt\, dv\, dw\, dt \\ &\leq \varepsilon \| |v|^2 f_\varepsilon^0 \|_{L^1(\mathbb{R}^{2d})} + 2dT \| f_\varepsilon^0 \|_{L^1(\mathbb{R}^{2d})} + 2\, \| |v| f_\varepsilon \nabla_x \psi_\varepsilon \|_{L^1(0,T;L^1(\mathbb{R}^{2d}))} \\ &\leq \varepsilon \| |v|^2 f_\varepsilon^0 \|_{L^1(\mathbb{R}^{2d})} + 2dT \| f_\varepsilon^0 \|_{L^1(\mathbb{R}^{2d})} + 2 \| |v| f_\varepsilon \|_{L^2(0,T;L^1(\mathbb{R}^{2d}))} \| \nabla \psi_\varepsilon \|_{L^2(0,T;L^\infty(\mathbb{R}^{2d}))}. \end{aligned}$$

The Cauchy–Schwartz inequality leads to

$$\| |v| f_\varepsilon \|_{L^2(0,T;L^1(\mathbb{R}^{2d}))} \leq \left(\| f_\varepsilon^0 \|_{L^1(\mathbb{R}^{2d})} \right)^{1/2} \left(\left\| |v|^2 f_\varepsilon \right\|_{L^1(0,T;L^1(\mathbb{R}^{2d}))} \right)^{1/2},$$

and Young's inequality for real numbers yields

$$2\, \| |v| f_\varepsilon \|_{L^2(0,T;L^1(\mathbb{R}^{2d}))} \, \| \nabla_x \psi_\varepsilon \|_{L^2(0,T;L^\infty(\mathbb{R}^d))} \leq \| \nabla_x \psi_\varepsilon \|^2_{L^2(0,T;L^\infty(\mathbb{R}^{2d}))} \| f_\varepsilon^0 \|_{L^1(\mathbb{R}^{2d})} + \left\| |v|^2 f_\varepsilon \right\|_{L^1(0,T;L^1(\mathbb{R}^{2d}))},$$

and this ends the proof of the velocity moments and dissipation of kinetic energy due to alignment interactions. Finally, for $k=1$ we can control the position moments in terms of the velocity moments via the second relation in Corollary 2.2.6, namely

$$\begin{aligned} \| |x| f_\varepsilon \|_{L^\infty(0,T;L^1(\mathbb{R}^d))} &\leq \| |x| f_\varepsilon^0 \|_{L^1(\mathbb{R}^d)} + \| |v| f_\varepsilon \|_{L^1(0,T;L^1(\mathbb{R}^d))} \\ &\leq \| |x| f_\varepsilon^0 \|_{L^1(\mathbb{R}^d)} + T^{1/2} \| |v| f_\varepsilon \|^{1/2}_{L^2(0,T;L^1(\mathbb{R}^d))}, \end{aligned}$$

where the Cauchy–Scwarz inequality has been used in the last inequality. □

2.2.4 Passing to the limit

In this subsection, we will concentrate on the limiting procedure underlying the proof of Theorem 2.1.1. For the sake of completeness, let us specify the content and technical hypothesis that we will need in the proof.

Theorem 2.2.9. *Let f_ε^0 and $\nabla\psi_\varepsilon$ satisfy hypothesis (2.2.5)-(2.2.6) and consider a sequence f_ε of smooth solutions to (2.1.7) with $\alpha \in (0, \frac{1}{2}]$. Then, the macroscopic quantities ρ_ε and j_ε satisfy*

$$\begin{aligned}&\rho_\varepsilon \to \rho, \quad \textit{in } C([0,T]; \mathcal{M}(\mathbb{R}^d) - \textit{narrow}),\\ &j_\varepsilon \overset{*}{\rightharpoonup} j, \quad \textit{in } L^2_w(0,T; \mathcal{M}(\mathbb{R}^d))^d,\end{aligned}$$

when $\varepsilon \searrow 0$, for some probability measure ρ, some finite Radon measure j and some subsequences of $\{\rho_\varepsilon\}_{\varepsilon>0}$ and $\{j_\varepsilon\}_{\varepsilon>0}$ that we denote in the same way. In addition (ρ, j) is a weak measure-valued solution to the Cauchy problem associated with the following Euler-type system

$$\begin{cases}\partial_t \rho + \operatorname{div} j = 0, & x \in \mathbb{R}^d,\ t \in [0,T),\\ \rho\,\nabla\psi + j = (\phi_0 * j)\rho - (\phi_0 * \rho)j, & x \in \mathbb{R}^d,\ t \in [0,T),\\ \rho(0,\cdot) = \rho^0, & x \in \mathbb{R}^d.\end{cases} \tag{2.2.20}$$

We refer to Appendix A for a summarized presentation of *weak-* Lebesgue Bochner spaces* $L^p_w(0,T;X^*)$ for a Banach space X, their comparison with classical *Lebesgue-Bochner spaces* $L^p(0,T;X^*)$ along with their duality properties and how it can be used to obtain weak-* compactness of bounded sequences.

Remark 2.2.10. *Notice that according to* (2.2.16)*, our assumptions* (2.2.5) *imply that ρ_ε have finite first order moment uniformly in $\varepsilon > 0$ and $t \in [0,T]$. Such uniform tightness can be used to substantially improve the convergence result of densities. Specifically, elaborating a little bit on the arguments in this section, we can show that*

$$\rho_\varepsilon \to \rho \ \textit{ in } \ C([0,T], \mathcal{P}_1(\mathbb{R}^d) - W_1),$$

where $(\mathcal{P}_1(\mathbb{R}^d), W_1)$ stands for the Wasserstein space of probability measures with a finite first order moment, endowed with the associated Wasserstein distance. As discussed in Remark 2.2.3, for simplicity, we will avoid those technical details along this chapter. However, the reader may want to see Chapter 4, where those arguments are rigorously justified in a closely related model, namely, the Kuramoto model with singular coupling weights.

This subsection will be divided into three distinguished parts. The first step will collect the necessary compactness properties of the sequences $\{\rho_\varepsilon\}_{\varepsilon>0}$ and $\{j_\varepsilon\}_{\varepsilon>0}$ that can be inferred from the preceding part of the section, whilst the second step will try to give some insight into what the dissipation of kinetic energy due to alignment interactions provide to our system. The last step will show how to pass to the limit in all the terms in the weak formulation. Before entering into details, let indeed write (2.2.7)-(2.2.8) in weak form as follows

$$\int_0^T \int_{\mathbb{R}^d} \partial_t \varphi\, \rho_\varepsilon \,dt\,dx + \int_0^T \int_{\mathbb{R}^d} \nabla_x \varphi \cdot j_\varepsilon \,dt\,dx = -\int_{\mathbb{R}^d} \varphi(0,x)\rho_\varepsilon^0(x)\,dx, \tag{2.2.21}$$

for any $\varphi \in C_c^\infty([0,T) \times \mathbb{R}^d)$,

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^d} \rho_\varepsilon \nabla_x \psi_\varepsilon \cdot \varphi \,dt\,dx + \int_0^T \int_{\mathbb{R}^d} j_\varepsilon \cdot \varphi\,dt\,dx - \int_0^T \int_{\mathbb{R}^d} ((\phi_\varepsilon * j_\varepsilon)\rho_\varepsilon - (\phi_\varepsilon * \rho_\varepsilon)j_\varepsilon)\cdot \varphi\,dt\,dx\\ = \varepsilon \int_{\mathbb{R}^d} \varphi(0,x)\cdot j_\varepsilon^0(x)\,dx + \varepsilon \int_0^T \int_{\mathbb{R}^d} \partial_t \varphi \cdot j_\varepsilon\,dt\,dx + \varepsilon \int_0^T \int_{\mathbb{R}^d} \operatorname{Jac}_x \varphi : \mathcal{S}_\varepsilon\,dt\,dx,\end{aligned} \tag{2.2.22}$$

for any $\varphi \in C_c^\infty([0,T)\times\mathbb{R}^d,\mathbb{R}^d)$.

First step: Compactness

Let the initial distribution functions f_ε^0 verify (2.2.5), the external forces $-\nabla\psi_\varepsilon$ fulfil (2.2.6) and consider the strong global in time solution f_ε to (2.2.4) with initial data f_ε^0 and $\alpha\in(0,\frac{d}{2})$. Recall that since $\mathcal{M}(\mathbb{R}^d)$ fails the Radon–Nikodym property, it is difficult to identify $L^\infty(0,T;\mathcal{M}(\mathbb{R}^d))$ and $L^2(0,T;\mathcal{M}(\mathbb{R}^d))$ as dual spaces. Instead, we propose using the weak-* Lebesgue Bochner spaces that provide the duality representations

$$L_w^\infty(0,T;\mathcal{M}(\mathbb{R}^d))\equiv L^1(0,T;C_0(\mathbb{R}^d))^* \text{ and } L_w^2(0,T;\mathcal{M}(\mathbb{R}^d))\equiv L^2(0,T;C_0(\mathbb{R}^d))^*,$$

see Appendix A for further details. Now, observe that by Proposition 2.2.5, Corollary 2.2.8 and the Banach–Alaoglu theorem one has

$$\rho_\varepsilon \overset{*}{\rightharpoonup} \rho, \quad \text{in } L_w^\infty(0,T;\mathcal{M}(\mathbb{R}^d)), \tag{2.2.23}$$

$$j_\varepsilon \overset{*}{\rightharpoonup} j, \quad \text{in } L_w^2(0,T;\mathcal{M}(\mathbb{R}^d))^d, \tag{2.2.24}$$

when $\varepsilon\searrow 0$. Consequently, it is straightforward to pass to the limit in each term of the continuity equation in weak form (2.2.21) and obtain

$$\frac{\partial\rho}{\partial t}+\operatorname{div} j=0,$$

in the distributional sense, with initial datum $\rho(0,\cdot)=\rho^0$. Regarding the equation of balance of linear momentum in weak form (2.2.22), passing to the limit in all the linear terms is straightforward. The only term that requires some more elaboration is the nonlinear part

$$\int_0^T\int_{\mathbb{R}^d}((\phi_\varepsilon * j_\varepsilon)\rho_\varepsilon-(\phi_\varepsilon*\rho_\varepsilon)j_\varepsilon)\cdot\varphi\,dx\,dt$$
$$=\int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\phi_\varepsilon(|x-y|)\,(\rho_\varepsilon(t,x)j_\varepsilon(t,y)-\rho_\varepsilon(t,y)j_\varepsilon(t,x))\cdot\varphi(t,x)\,dx\,dy\,dt.$$

In order to show that such nonlinear term makes sense, let us use the kindness of the commutator therein along with the symmetries of the influence function. An easy change of variables that interchanges x with y then yields the next expression

$$\int_0^T\int_{\mathbb{R}^d}((\phi_\varepsilon * j_\varepsilon)\rho_\varepsilon-(\phi_\varepsilon*\rho_\varepsilon)j_\varepsilon)\cdot\varphi\,dx\,dt$$
$$=\frac{1}{2}\int_0^T\int_{\mathbb{R}^{2d}}H_\varphi^{\alpha,\varepsilon}(t,x,y)\cdot(\rho_\varepsilon(t,x)j_\varepsilon(t,y)-\rho_\varepsilon(t,y)j_\varepsilon(t,x))\,dx\,dy\,dt, \tag{2.2.25}$$

where the integral kernel $H_\varphi^{\alpha,\varepsilon}$ takes the form

$$H_\varphi^{\alpha,\varepsilon}(t,x,y):=\phi_\varepsilon(|x-y|)(\varphi(t,x)-\varphi(t,y))=\frac{\varphi(t,x)-\varphi(t,y)}{(\varepsilon^2+c_\alpha|x-y|^2)^\alpha},$$

for any test vector field $\varphi\in C_c^\infty([0,T)\times\mathbb{R}^d,\mathbb{R}^d)$. Combining the mean value theorem along with Lemma 2.2.4 we arrive at:

Lemma 2.2.11. *Let us fix $\varepsilon>0$, $\alpha\in(0,\frac{1}{2})$ and a test function $\varphi\in C_c^\infty([0,T)\times\mathbb{R}^d)$. Then, the next estimates hold for every couple $(x,y)\in\mathbb{R}^{2d}\setminus\Delta$ and $t\in[0,T)$:*

1. $\left|H_{\varphi}^{\alpha,\varepsilon}(t,x,y)\right| \leq \frac{1}{c_{\alpha}^{\alpha}}\|\nabla_x\varphi\|_{C([0,T)\times\mathbb{R}^d)}|x-y|^{1-2\alpha}$,

2. $|H_{\varphi}^{\alpha,\varepsilon}(t,x,y)| \leq \frac{1}{\varepsilon^{2\alpha}}\|\nabla_x\varphi\|_{C([0,T)\times\mathbb{R}^d)}|x-y|$,

3. $|H_{\varphi}^{\alpha,\varepsilon}(t,x,y) - H_{\varphi}^{\alpha,0}(t,x,y)| \leq \varepsilon^{1-2\alpha}C_{\alpha}\|\nabla_x\varphi\|_{C([0,T)\times\mathbb{R}^d)}$,

where Δ stands for the diagonal of $\mathbb{R}^{2d}$ i.e., $\Delta := \{(x,y)\in\mathbb{R}^{2d} : x = y\}$.

Let us first note some difference between the cases $\alpha \in (0,\frac{1}{2})$ and $\alpha \geq \frac{1}{2}$. In the former case, not only does $H_{\varphi}^{\alpha,\varepsilon}$ belong to $C([0,T],C_0(\mathbb{R}^d,\mathbb{R}^d))$ for every positive ε but it also belongs to such space when $\varepsilon = 0$. This is no longer true when $\alpha = \frac{1}{2}$ because despite the fact that the continuity is granted for positive ε, the corresponding kernel with $\varepsilon = 0$ takes the form

$$H_{\varphi}^{\frac{1}{2},0}(t,x,y) := \frac{\varphi(t,x)-\varphi(t,y)}{\sqrt{3}|x-y|},$$

and, although bounded, it clearly loses the continuity at the diagonal points $x = y$ regardless of the modulus of continuity of $\varphi(t,x)$. This is a common restriction in many other classical systems such as the 2D Euler equations in vorticity formulation (see e.g. [269]) or the 2D Vlasov–Poisson–Fokker–Planck system (see e.g. [232]). Similarly, in the latter case $\alpha > \frac{1}{2}$ the kernel is discontinuous along the diagonal points and it is not necessarily bounded at $x = y$ since it can even blow up. Hence, one cannot expect to pass to the limit for $\alpha > \frac{1}{2}$, but at most for $\alpha \in (0,\frac{1}{2}]$.

In this part we will focus on $\alpha \in (0,\frac{1}{2})$, where the continuity along the diagonal points is granted. In such case, Lemma 2.2.11 shows that

$$\lim_{\varepsilon\to 0}\|H_{\varphi}^{\alpha,\varepsilon} - H_{\varphi}^{\alpha,0}\|_{C([0,T],C_0(\mathbb{R}^d,\mathbb{R}^d))} = 0. \tag{2.2.26}$$

The next step will be to show that,

$$\rho_\varepsilon \otimes j_\varepsilon \overset{*}{\rightharpoonup} \rho\otimes j \ \text{ in } L^2_w(0,T;\mathcal{M}(\mathbb{R}^{2d}))^d,$$

which does not directly follows just from (2.2.23)-(2.2.24). In general some sort of time equicontinuity in certain space is needed (see [232, 252, 269]).

Theorem 2.2.12. *Let the initial distribution functions f_ε^0 verify (2.2.5), the external forces $-\nabla\psi_\varepsilon$ fulfill (2.2.6) and consider the strong global in time solution f_ε to (2.2.4) with initial data f_ε^0 and $\alpha \in (0,\frac{d}{2})$. Then, the following convergence takes place*

$$\rho_\varepsilon \to \rho \ \textit{in}\ C([0,T];\mathcal{M}(\mathbb{R}^d) - \textit{narrow}),$$

that is to say,

$$\lim_{\varepsilon\to 0}\sup_{t\in[0,T]}\left|\int_{\mathbb{R}^d}\phi(x)d_x(\rho_\varepsilon(t,\cdot)-\rho(t,\cdot))\right| = 0,$$

for every continuous test function $\phi \in C_b(\mathbb{R}^d)$.

Proof. Consider any test function $\varphi(t,x) = \eta(t)\phi(x)$, where $\eta \in C_c^\infty(0,T)$ and $\phi \in C_c^\infty(\mathbb{R}^d)$. Notice that (2.2.7) gives rise to the following equation in weak form

$$\int_0^T \frac{\partial\eta}{\partial t}(t)\left(\int_{\mathbb{R}^d}\rho_\varepsilon(t,x)\phi(x)\,dx\right)dt = \int_0^T \eta(t)\left(\int_{\mathbb{R}^d} j_\varepsilon(t,x)\cdot\nabla_x\phi(x)\,dx\right)dt.$$

Since by Corollary 2.2.8 the scaled current is bounded in $L^2(0,T;L^1(\mathbb{R}^d))$, then

$$\left|\int_0^T \frac{\partial\eta}{\partial t}(t)\left(\int_{\mathbb{R}^d}\rho_\varepsilon(t,x)\phi(x)\,dx\right)\,dt\right| \leq \|\eta\|_{L^2(0,T)}\|j_\varepsilon\|_{L^2(0,T;L^1(\mathbb{R}^d))}\|\phi\|_{W^{1,\infty}(\mathbb{R}^d)}$$
$$\leq (2\varepsilon E_0+2dT+F_0^2)^{1/2}\|\eta\|_{L^2(0,T)}\|\phi\|_{W^{1,\infty}(\mathbb{R}^d)}.$$

A well know characterization of Sobolev space leads to

$$\left\|\int_{\mathbb{R}^d}\rho_\varepsilon(\cdot,x)\phi(x)\,dx\right\|_{H^1(0,T)} \leq \left(T^{1/2}+(2\varepsilon E_0+2dT+F_0^2)^{1/2}\right)\|\phi\|_{W^{1,\infty}(\mathbb{R}^d)}.$$

In particular, Sobolev's embedding implies

$$\left|\int_{\mathbb{R}^d}(\rho_\varepsilon(t_1,x)-\rho_\varepsilon(t_2,x))\phi(x)\,dx\right| \leq \left(T^{1/2}+(2\varepsilon E_0+2dT+F_0^2)^{1/2}\right)\|\phi\|_{W^{1,\infty}(\mathbb{R}^d)}|t_1-t_2|^{1/2},$$

for every $t_1,t_2\in[0,T)$. Consequently, the next bound holds

$$\|\rho_\varepsilon\|_{C^{0,\frac{1}{2}}([0,T];W^{-1,1}(\mathbb{R}^d))} \leq T^{1/2}+(2\varepsilon E_0+2dT+F_0^2)^{1/2}.$$

Notice that the boundedness of $\{\rho_\varepsilon\}_{\varepsilon>0}$ in $C([0,T];W^{-1,1}(\mathbb{R}^d))$ also follows from the analogous bound in $L^\infty(0,T;L^1(\mathbb{R}^d))$ (see Proposition 2.2.5) and the chain of continuous inclusions

$$L^1(\mathbb{R}^d)\hookrightarrow\mathcal{M}(\mathbb{R}^d)\hookrightarrow W^{-1,1}(\mathbb{R}^d).$$

Then, the weak-* form of the Ascoli–Arzelà theorem yields the convergence in $C([0,T];W^{-1,1}(\mathbb{R}^d)-\text{weak}\,*)$, see Appendix B. Finally, a straightforward argument by density of $W^{1,\infty}(\mathbb{R}^d)$ in $C_0(\mathbb{R}^d)$ along with the boundedness of ρ_ε in $L^\infty(0,T;L^1(\mathbb{R}^d))$ and the uniform tightness (2.2.16) allows claiming that

$$\rho_\varepsilon\to\rho\quad\text{in } C([0,T];\mathcal{M}(\mathbb{R}^d)-\text{narrow}).$$

□

Let us conclude our assertion with the next result.

Corollary 2.2.13. *Let the initial distribution functions f_ε^0 verify (2.2.5), the external forces $-\nabla\psi_\varepsilon$ fulfil (2.2.6) and consider the strong global in time solution f_ε to (2.2.4) with initial data f_ε^0 and $\alpha\in(0,\frac{d}{2})$. Then, we obtain that*

$$\rho_\varepsilon\otimes j_\varepsilon\overset{*}{\rightharpoonup}\rho\otimes j\ \text{ in }\ L_w^2(0,T;\mathcal{M}(\mathbb{R}^{2d}))^d.$$

Proof. Notice that Theorem A.0.11 in Appendix A about Riesz representation for duals in Lebesgue–Bochner spaces in terms of weak-* Lebesgue–Bochner spaces leads to the next identification

$$L_w^2(0,T;\mathcal{M}(\mathbb{R}^{2d}))\equiv L^2(0,T;C_0(\mathbb{R}^{2d}))^*.$$

Let us then consider a test function $\varphi\in L^2(0,T;C_0(\mathbb{R}^{2d}))$. By density, one can assume that φ takes the form

$$\varphi(t,x,y):=\eta(t)\phi(x)\psi(y),$$

for $\eta\in L^2(0,T)$ and $\phi,\psi\in C_0(\mathbb{R}^d)$. The objective is to show that

$$I_{i,\varepsilon}:=\int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\varphi(t,x,y)\left(d_x(\rho_\varepsilon(t,\cdot))d_y\left(j_\varepsilon^i(t,\cdot)\right)-d_x(\rho(t,\cdot))d_y(j^i(t,\cdot))\right)\,dt\to 0,$$

as $\varepsilon \to 0$. The superscript $i \in \{1,\ldots,N\}$ stands for each component of the above current vectors. $I_{i,\varepsilon}$ can be split as follows

$$I_{i,\varepsilon} = I^1_{i,\varepsilon} + I^2_{i,\varepsilon} + I^3_{i,\varepsilon},$$

where each term reads

$$\begin{aligned}
I^1_{i,\varepsilon} &:= \int_0^T \eta(t)\left(\int_{\mathbb{R}^d}\phi(x)d_x(\rho_\varepsilon(t,\cdot)-\rho(t,\cdot))\right)\left(\int_{\mathbb{R}^d}\psi(y)d_y\left(j^i_\varepsilon(t,\cdot)-j^i(t,\cdot)\right)\right)dt,\\
I^2_{i,\varepsilon} &:= \int_0^T \eta(t)\left(\int_{\mathbb{R}^d}\phi(x)d_x(\rho_\varepsilon(t,\cdot)-\rho(t,\cdot))\right)\left(\int_{\mathbb{R}^d}\psi(y)d_y\left(j^i(t,\cdot)\right)\right)dt,\\
I^3_{i,\varepsilon} &:= \int_0^T \eta(t)\left(\int_{\mathbb{R}^d}\phi(x)d_x(\rho(t,\cdot))\right)\left(\int_{\mathbb{R}^d}\psi(y)d_y\left(j^i_\varepsilon(t,\cdot)-j^i(t,\cdot)\right)\right)dt.
\end{aligned}$$

We will restrict to the first term, $I^1_{i,\varepsilon}$, since the reasoning in the remaining two terms is similar. Let us define the sequence of scalar functions given by

$$R_\varepsilon(t) := \int_{\mathbb{R}^d}\phi(x)d_x\left(\rho_\varepsilon(t,\cdot)-\rho(t,\cdot)\right),\qquad \eta_\varepsilon(t) := \eta(t)R_\varepsilon(t).$$

By Theorem 2.2.12 one has that

$$\begin{aligned}
R_\varepsilon &\to 0 \quad \text{in } L^\infty(0,T),\\
\eta_\varepsilon &\to 0 \quad \text{in } L^2(0,T).
\end{aligned}$$

Then, $I^1_{i,\varepsilon}$ can be restated as

$$I^1_{i,\varepsilon} = \int_0^T\int_{\mathbb{R}^d}\eta_\varepsilon(t)\psi(y)\,d_y\left(j^i_\varepsilon(t,\cdot)-j^i(t,\cdot)\right)\,dt,$$

that converges to zero as $\varepsilon \to 0$ because the test functions $\eta_\varepsilon \otimes \psi$ strongly converges to zero in $L^2(0,T;C_0(\mathbb{R}^d))$ and $j_\varepsilon - j$ also converges to zero weakly-* in $L^2_w(0,T;\mathcal{M}(\mathbb{R}^d))^d$ by (2.2.24). □

Second step: Non-concentration

Recall that the limiting kernels $H^{\alpha,\varepsilon}_\varphi$ are continuous except at most at the diagonal points $x=y$. Then, in order for the nonlinear term in weak form

$$\frac{1}{2}\int_0^T\int_{\mathbb{R}^{2d}} H^{\alpha,\varepsilon}_\varphi(t,x,y)\cdot(\rho_\varepsilon(t,x)j_\varepsilon(t,y)-\rho_\varepsilon(t,y)j_\varepsilon(t,x))\,dx\,dy\,dt,$$

to pass to the limit, we have to ensure that the limiting terms $\rho\otimes j - j\otimes\rho$ do not concentrate on the set of points of discontinuity of $H^{\alpha,0}_\varepsilon$. This is the content of the next result, that holds for α belonging to the whole range $(0,\frac{d}{2})$.

Lemma 2.2.14. *Let the initial distribution functions f^0_ε verify (2.2.5), the external forces $-\nabla\psi_\varepsilon$ fulfil (2.2.6) and consider the strong global in time solution f_ε to (2.2.4) with initial data f^0_ε and $\alpha \in (0,\frac{d}{2})$. Then*

$$\liminf_{R,\varepsilon\to 0}|\rho_\varepsilon(t,\cdot)\otimes j_\varepsilon(t,\cdot) - j_\varepsilon(t,\cdot)\otimes\rho_\varepsilon(t,\cdot)|(\Omega_R) = 0,\ \ \textit{a.e. } t\in[0,T).$$

where Ω_R denotes the augmented diagonal of $\mathbb{R}^d\times\mathbb{R}^d$ with radius $R>0$, i.e.,

$$\Omega_R := \{(x,y)\in\mathbb{R}^{2d} :\ |x-y|<R\}.$$

Proof. Consider any $\varepsilon > 0$, fix any radius $R > 0$ and note that

$$\begin{aligned}
&|\rho_\varepsilon(t,\cdot) \otimes j_\varepsilon(t,\cdot) - j_\varepsilon(t,\cdot) \otimes \rho_\varepsilon(t,\cdot)|(\Omega_R) \\
&\quad = \iint_{|x-y|<R} |\rho_\varepsilon(t,x) \otimes j_\varepsilon(t,y) - j_\varepsilon(t,x) \otimes \rho_\varepsilon(t,y)| \, dx \, dy \\
&\quad \leq (\varepsilon^2 + R^2)^{\alpha/2} \int_{\mathbb{R}^{4d}} |v-w| \phi_\varepsilon(|x-y|)^{1/2} f_\varepsilon(t,x,v) f_\varepsilon(t,y,w) \, dx \, dy \, dv \, dw.
\end{aligned}$$

Then, the Cauchy–Schwartz inequality along with the estimate (2.2.19) in Corollary 2.2.8 yield

$$\left(\int_0^T |\rho_\varepsilon(t,\cdot) \otimes j_\varepsilon(t,\cdot) - j_\varepsilon(t,\cdot) \otimes \rho_\varepsilon(t,\cdot)|(\Omega_R)^2 \, dt \right)^{1/2} \leq (\varepsilon^2 + R^2)^{\alpha/2} (2\varepsilon E_0 + (2dT + F_0^2))^{1/2},$$

and consequently, taking limits when ε and R become zero

$$\liminf_{\varepsilon, R \to 0} \left(\int_0^T |\rho_\varepsilon(t,\cdot) \otimes j_\varepsilon(t,\cdot) - j_\varepsilon(t,\cdot) \otimes \rho_\varepsilon(t,\cdot)|(\Omega_R)^2 \, dt \right)^{1/2} = 0.$$

Then, the proof is done by virtue of Fatou's lemma. □

Now, let us recall the next result that may be used to show how to pass to the limit on a sequence of finite Radon measures (without distinguished sign) that do not exhibit concentrations on a set, against a bounded function that is discontinuous on such set (at most). It is a well known result for positive measures and it is the cornerstone in many other frameworks (e.g. 2D Euler with signed vortex sheet initial data [103, 269] or the 2D Vlasov–Poisson–Fokker–Planck system [232]). The proof for positive measures can be found in [250, Lemma 2.1], [269] or [270, Theorems 62-63, Chapter IV].

Proposition 2.2.15. *Let $\{\mu_\varepsilon\}_{\varepsilon>0} \subseteq \mathcal{M}(\mathbb{R}^d)$ be a sequence of general finite Radon measures and assume:*

1. *$\mu_\varepsilon \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\mathbb{R}^d)$ as $\varepsilon \to 0$, for some $\mu \in \mathcal{M}(\mathbb{R}^d)$,*
2. *There exists some closed set $C \subseteq \mathbb{R}^d$ such that*

$$\liminf_{\varepsilon, R \to 0} |\mu_\varepsilon|(\Omega_R) = 0,$$

where Ω_R stands for the augmented C with radius $R > 0$, i.e.,

$$\Omega_R := C + B_R(0) = \{x \in \mathbb{R}^d : \operatorname{dist}(x, C) < R\}.$$

Therefore, we obtain that

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} \varphi \, d\mu_\varepsilon = \int_{\mathbb{R}^d} \varphi \, d\mu, \tag{2.2.27}$$

for every measurable and bounded function $\varphi : \mathbb{R}^d \longrightarrow \mathbb{R}$ whose discontinuity points lie in C and such that φ falls off at infinity.

Proof. Let us decompose μ_ε into its positive and negative part, according to Hahn's decomposition theorem, i.e., $\mu_\varepsilon = \mu_\varepsilon^+ - \mu_\varepsilon^-$. We can assume, without loss of generality, that

$$\mu_\varepsilon^\pm \overset{*}{\rightharpoonup} \mu^\pm \text{ in } \mathcal{M}(\mathbb{R}^d),$$

where $\mu^\pm \in \mathcal{M}(\mathbb{R}^d)$ do not necessarily agree with the Jordan decomposition of $\mu = \mu^+ - \mu^-$. By hypothesis, it is clear that

$$\liminf_{\varepsilon,R\to 0} \mu_\varepsilon^\pm(\Omega_R) = 0.$$

As a consequence of the weak-* lower semicontinuity of the total variation norm, one obtains

$$\mu^\pm(\Omega_R) \leq \liminf_{\varepsilon\to 0} \mu_\varepsilon^\pm(\Omega_R).$$

By monotonicity $\mu^\pm(C) = 0$ and consequently, one can then apply Lemma 2.1 in [250] to the positive measures μ_ε^+ and μ_ε^- to end the proof of the theorem. □

Remark 2.2.16. *An important fact to be remarked is that, as stated in [250], the above result is not true if one only assumes that $|\mu|(C) = 0$. Indeed, some example is*

$$\mu_\varepsilon = \delta_\varepsilon - \delta_{-\varepsilon}, \;\; \mu = 0, \;\; C = \{0\}.$$

Obviously, $|\mu|(\{0\}) = 0$ but fails (2.2.27) (take φ any cutoff function at infinity of the sign function). Notice that there is no contradiction since the non-concentration property in the above result also fails

$$\liminf_{\varepsilon,R\to 0} |\mu_\varepsilon|(-R,R) = \liminf_{\varepsilon,R\to 0} (\delta_\varepsilon + \delta_{-\varepsilon})(-R,R) = 2 \neq 0.$$

Third step: Convergence

To end this section, let us show that the above results allow us to take limits in the symmetrized expression (2.2.25) of the nonlinear term in (2.2.22) as $\varepsilon \to 0$ for the restricted range of the parameter $\alpha \in (0, \frac{1}{2}]$. For $\alpha \in (0, \frac{1}{2})$, it is a direct consequence of the Corollary 2.2.13 and the uniform convergence (2.2.26). The case $\alpha = \frac{1}{2}$ requires a special analysis since (2.2.26) does not hold. However, the estimate (2.2.19) of the dissipation of kinetic energy due to alignment interactions will suffice to reinforce the lack of uniformity in the convergence of the approximate kernels $H_\varphi^{\alpha,\varepsilon}$.

Corollary 2.2.17. *Let the initial distribution functions f_ε^0 verify (2.2.5), the external forces $-\nabla\psi_\varepsilon$ fulfil (2.2.6) and consider the strong global in time solution f_ε to (2.2.4) with initial data f_ε^0 and $\alpha \in (0, 1/2]$. Then, we can pass to the limit in* (2.2.25) *as $\varepsilon \to 0$. Specifically,*

$$\lim_{\varepsilon\to 0} \frac{1}{2}\int_0^T \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} H_\varphi^{\alpha,\varepsilon}(\rho_\varepsilon \otimes j_\varepsilon - j_\varepsilon \otimes \rho_\varepsilon)\,dx\,dy\,dt = \frac{1}{2}\int_0^T \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} H_\varphi^{\alpha,0}(\rho \otimes j - j \otimes \rho)\,dx\,dy\,dt,$$

for every test function $\varphi \in C_c^\infty([0,T) \times \mathbb{R}^d, \mathbb{R}^d)$.

Proof. Since the proof for $\alpha \in (0, \frac{1}{2})$ is a direct consequence of Corollary 2.2.13 and the uniform convergence (2.2.26), then we restrict to the endpoint critical case $\alpha = \frac{1}{2}$. To this end, let us show that

$$I_\varepsilon := \frac{1}{2}\int_0^T \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left[H_\varphi^{\frac{1}{2},\varepsilon}(\rho_\varepsilon \otimes j_\varepsilon - j_\varepsilon \otimes \rho_\varepsilon) - H_\varphi^{\frac{1}{2},0}(\rho \otimes j - j \otimes \rho)\right] dx\,dy\,dt,$$

becomes zero when $\varepsilon \to 0$. For simplicity, we decompose I_ε into $I_\varepsilon = II_\varepsilon + III_\varepsilon$, where

$$II_\varepsilon := \frac{1}{2}\int_0^T \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (H_\varphi^{\frac{1}{2},\varepsilon} - H_\varphi^{\frac{1}{2},0})(\rho_\varepsilon \otimes j_\varepsilon - j_\varepsilon \otimes \rho_\varepsilon)\,dx\,dy\,dt,$$

$$III_\varepsilon := \frac{1}{2}\int_0^T \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} H_\varphi^{\frac{1}{2},0}\left[(\rho_\varepsilon\otimes j_\varepsilon - j_\varepsilon\otimes\rho_\varepsilon) - (\rho\otimes j - j\otimes\rho)\right]\,dx\,dy\,dt.$$

First, since $H_\varphi^{\frac{1}{2},0}$ is discontinuous at most at the diagonal Δ of $\mathbb{R}^d\times\mathbb{R}^d$, then Lemma 2.2.14 and Proposition 2.2.15 show that $III_\varepsilon\to 0$ when $\varepsilon\to 0$. Regarding II_ε, note that

$$|\phi_\varepsilon(r)-\phi_0(r)| = \frac{(\varepsilon^2+3r^2)^{1/2}-\sqrt{3}r}{\sqrt{3}r(\varepsilon^2+3r^2)^{1/2}} \le \frac{\varepsilon^{1/2}}{\sqrt{3}r}\phi_\varepsilon(r)^{1/2},\ \ r>0.$$

Consequently, there exists come constant $C>0$ depending on the test function $\varphi\in C_c^\infty([0,T)\times\mathbb{R}^d,\mathbb{R}^d)$ such that

$$|II_\varepsilon| \le C\varepsilon^{1/2}\int_0^T\int_{\mathbb{R}^{4d}} |v-w|\phi_\varepsilon(|x-y|)^{1/2} f_\varepsilon(t,x,v)f_\varepsilon(t,y,w)\,dx\,dy\,dv\,dw\,dt,$$

and by the Cauchy–Schwartz inequality and estimate (2.2.19)

$$|II_\varepsilon| \le CT^{1/2}(2\varepsilon E_0 + (2dT+F_0^2))^{1/2}\varepsilon^{1/2},$$

that obviously implies $II_\varepsilon\to 0$ when $\varepsilon\to 0$. □

Remark 2.2.18. 1. *For the sake of clarity, let us come back to the equation of balance of current* (2.2.22) *and comment why the hypothesis that we considered in* (2.2.6) *regarding the forcing term* $-\nabla_x\psi_\varepsilon$ *allows passing to the limit in the last term*

$$I_\varepsilon := \int_0^T\int_{\mathbb{R}^d} \rho_\varepsilon\nabla_x\psi_\varepsilon\cdot\varphi\,dx\,dt,\ \ \textit{for any}\ \ \varphi\in C_c^\infty([0,T)\times\mathbb{R}^d,\mathbb{R}^d),$$

as $\varepsilon\to 0$. *Specifically, notice that the assumption in* (2.2.6) *implies that* $\nabla_x\psi_\varepsilon\cdot\varphi\to\nabla_x\psi\cdot\varphi$ *in* $L^1(0,T;C_0(\mathbb{R}^d))$. *Indeed, for it to happen we do not really need that* $\nabla_x\psi_\varepsilon\to\nabla_x\psi$ *in* $L^1(0,T;C_b(\mathbb{R}^d,\mathbb{R}^d))$, *like in* (2.2.6), *but simply that*

$$\nabla_x\psi_\varepsilon\to\nabla_x\psi\ \ \textit{in}\ \ L^1(0,T;C(K)),\ \ \textit{for any}\ K\subset\subset\mathbb{R}^d.$$

This, along with the fact that $\rho_\varepsilon\overset{*}{\rightharpoonup}\rho$ *in* $L_w^\infty(0,T;\mathcal{M}(\mathbb{R}^d))$ *allow us to pass to the limit in* I_ε *as* $\varepsilon\to 0$ *and identify the limit in terms of* ρ *and* $\nabla_x\psi$.

2. *The assumption* $\nabla_x\psi(t,\cdot)\in C_b(\mathbb{R}^d,\mathbb{R}^d)$ *might seem very imposing at first glance, since generally forcing terms might not be even bounded. Recall that this requirement arose in the proof of Corollary 2.2.8 in order to achieve the bound*

$$\begin{aligned}\||v|f_\varepsilon\,\nabla_x\psi_\varepsilon\|_{L^1(0,T;L^1(\mathbb{R}^d))} &\le \||v|f_\varepsilon\|_{L^2(0,T;L^1(\mathbb{R}^d))}\|\nabla\psi_\varepsilon\|_{L^2(0,T;L^\infty(\mathbb{R}^d))}\\ &\le \||v|^2 f_\varepsilon\|_{L^1(0,T;L^1(\mathbb{R}^d))}^{1/2}\|\nabla\psi_\varepsilon\|_{L^2(0,T;L^\infty(\mathbb{R}^d))}.\end{aligned}$$

Indeed, as mentioned in the Introduction, one might have considered self-generated forcing terms $-\nabla_x\psi_\varepsilon$, *e.g.,* $\Delta\psi_\varepsilon=\theta\rho_\varepsilon$ *in the Vlasov–Poisson system (here* $\theta=\pm 1$ *suggests the attractive or repulsive character of the interactions). It can be restated as follows*

$$-\nabla\psi_\varepsilon = \theta(\nabla\Gamma_d)*\rho_\varepsilon,$$

where Γ_d *stands for the fundamental solution of* $-\Delta$ *in* $\mathbb{R}^d$. *Those nonlinearities have to be studied separately on each case like our nonlinear conmutator. Indeed, when the density only enjoys* L^1

bounds, then $-\nabla\psi_\varepsilon$ can be shown to be bounded just in the $d=1$ case. The case $d=2$ is harder and resembles our case $\alpha=\frac{1}{2}$, that required a non-concentration argument of the limiting density, see [232]. For general d, relative entropy methods (that do not work in our case) have been used in [142]. Consequently, despite having derived a method to deal with bounded continuous forces, each nonlocal force given by a potential $\psi_\varepsilon = W * \rho_\varepsilon$ and some interaction potential W needs to be treated separately through similar ideas to those developed here and in the related papers [22, 142, 232, 252]. Also, kernels appearing in other systems like the aggregation equation and fluid models like the gSQG or other Euler-type equations can be approached via this techniques, see [26, 70, 97, 287, 288].

To end this section, let us show, as already discussed in the Introduction, that the nonlinear term is bounded with respect to ε, not only for the above range $\alpha\in(0,\frac{1}{2}]$, but also for a larger range $\alpha\in(0,1)$.

Remark 2.2.19. *Consider $\alpha\in(0,1)$, $\delta>0$ and fix any cut-off function $\eta\in C_c^\infty([0,+\infty))$ such that $0\leq\eta\leq 1$, $\eta(r)=1$ for $r\in[0,1]$ and $\eta(r)=0$ for $r\in[2,+\infty)$. Define the associated dilation $\eta_\delta\in C_c^\infty([0,+\infty))$ of such cut-off function as follows*

$$\eta_\delta(r) := \eta\left(\frac{r}{\delta}\right),\ \ r>0.$$

Then, the nonlinear term in weak sense can be split as follows for every $\varepsilon>0$,

$$\int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} H_\varphi^{\alpha,\varepsilon}(\rho_\varepsilon\otimes j_\varepsilon - j_\varepsilon\otimes\rho_\varepsilon)\,dx\,dy\,dt = I_\varepsilon + II_\varepsilon,$$

where each term reads

$$\begin{aligned}
I_\varepsilon &:= \int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} H_{\varphi,+}^{\alpha,\varepsilon,\delta}(\rho_\varepsilon\otimes j_\varepsilon - j_\varepsilon\otimes\rho_\varepsilon)\,dx\,dy\,dt,\\
II_\varepsilon &:= \int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} H_{\varphi,-}^{\alpha,\varepsilon,\delta}(\rho_\varepsilon\otimes j_\varepsilon - j_\varepsilon\otimes\rho_\varepsilon)\,dx\,dy\,dt.
\end{aligned}$$

Here, φ is any test function in $C_c^\infty([0,T)\times\mathbb{R}^d)$ and the kernels read

$$\begin{aligned}
H_{\varphi,+}^{\alpha,\varepsilon,\delta}(t,x,y) &:= H_\varphi^{\alpha,\varepsilon}(t,x,y)\eta_\delta(|x-y|),\\
H_{\varphi,-}^{\alpha,\varepsilon,\delta}(t,x,y) &:= H_\varphi^{\alpha,\varepsilon}(t,x,y)\left(1-\eta_\delta(|x-y|)\right).
\end{aligned}$$

By virtue of the Cauchy–Schwartz inequality and Corollary 2.2.8, we obtain that

$$\begin{aligned}
|I_\varepsilon| \leq & \left(\int_0^T\int_{\mathbb{R}^{4d}}\phi_\varepsilon(|x-y|)|v-w|^2 f_\varepsilon(t,x,v)f_\varepsilon(t,y,w)\,dx\,dy\,dv\,dw\,dt\right)^{1/2}\\
&\times\left(\int_0^T\int_{|x-y|<2\delta}\phi_\varepsilon(|x-y|)|\varphi(t,x)-\varphi(t,y)|^2\rho_\varepsilon(t,x)\rho_\varepsilon(t,y)\,dx\,dy\,dt\right)^{1/2}\\
\leq & \frac{(2\delta)^{1-\alpha}}{c_\alpha^{\alpha/2}}T^{1/2}\left(2\varepsilon E_0 + \left(2dT+F_0^2\right)\right)^{1/2}\|\varphi\|_{C_0([0,T),C^1(\mathbb{R}^d))},\\
|II_\varepsilon| \leq & \int_0^T\int_{|x-y|\geq\delta}\phi_\varepsilon(|x-y|)\,|\varphi(t,x)-\varphi(t,y)|\,|\rho_\varepsilon(t,x)j_\varepsilon(t,y)-j_\varepsilon(t,x)\rho_\varepsilon(t,y)|\,dx\,dy\,dt\\
\leq & \frac{2}{c_\alpha^\alpha\delta^{2\alpha}}T^{1/2}(2\varepsilon E_0+(2dT+F_0^2))^{1/2}\|\varphi\|_{C_0([0,T),C_0(\mathbb{R}^d))}.
\end{aligned}$$

These bounds suggest that it might be possible to pass to the limit $\varepsilon \to 0$ for larger values of α than those $\alpha \in (0, \frac{1}{2}]$ considered in Theorem 2.2.9, namely, for $\alpha \in (\frac{1}{2}, 1)$, since the nonlinear terms are bounded with respect to ε for each fixed test function φ.

However, it is not clear whether the dissipation of kinetic energy due to interactions makes sense in the limit $\varepsilon \to 0$ since we have not compactness properties (in time) for the distribution function f_ε to pass to the limit in estimate (2.2.19). Let us specifically remark where passing to the limit is unclear. First, notice that a similar result to that in Lemma 2.2.4 entails

$$|\phi_\varepsilon(r) - \phi_0(r)| \leq C\frac{\varepsilon}{r^{2\alpha+1}},$$

for every $r > 0$. In particular,

$$\lim_{\varepsilon\to 0} \|H^{\alpha,\varepsilon,\delta}_{\varphi,-} - H^{\alpha,0,\delta}_{\varphi,-}\|_{C([0,T],C_0(\mathbb{R}^d,\mathbb{R}^d))} = 0,$$

and one can show, using similar ideas to those in Corollary 2.2.17, that

$$\int_0^T \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} H^{\alpha,\varepsilon,\delta}_{\varphi,-}(\rho_\varepsilon \otimes j_\varepsilon - j_\varepsilon \otimes \rho_\varepsilon)\,dx\,dy\,dt \longrightarrow \int_0^T \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} H^{\alpha,0,\delta}_{\varphi,-}(\rho \otimes j - j \otimes \rho)\,dx\,dy\,dt.$$

However, it is not clear how to give some meaning to

$$\int_0^T \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} H^{\alpha,0,\delta}_{\varphi,+}(\rho \otimes j - j \otimes \rho)\,dx\,dy\,dt,$$

since the kernel there needs not be neither bounded at the diagonal points.

2.3 Analysis of the limiting equations

In this section we shall focus on analyzing the macroscopic system arising from the singular and hydrodynamic limit of the preceding section. For simplicity, recall the velocity field u can be recovered from the law $j = \rho\, u$ and the limiting system can be restated formally as

$$\left\{\begin{array}{ll} \partial_t\rho + \operatorname{div}(\rho u) = 0, & x \in \mathbb{R}^d,\ t \geq 0,\\ \rho(0,x) = \rho^0(x), & x \in \mathbb{R}^d,\\ u = \phi_0 * (\rho u) - (\phi_0 * \rho)u - \nabla\psi, & x \in \mathbb{R}^d,\ t \geq 0. \end{array}\right. \tag{2.3.1}$$

Notice that such system consists of $d+1$ unknowns (ρ and $u = (u_1 \ldots, u_d)$), $d+1$ equations (a scalar conservation law for ρ and a vector-valued implicit equation for u) and a Cauchy datum at $t = 0$. Then, we expect that such macroscopic system enjoys some sort of well-posedness in appropriate functional settings to be explored in the sequel. First, we will revisit the existence theory of weak solutions in L^p for scalar conservation laws in conservative form driven by $W^{1,\infty}$ velocity fields. Second, we will analyze the properties of the above commutator

$$\phi_0 * (\rho u) - (\phi_0 * \rho)u = -\frac{1}{c^\alpha_\alpha}[M_u, I_{d-2\alpha}]\rho = -\int_{\mathbb{R}^d} \phi_0(|x-y|)(u(t,x) - u(t,y))\rho(t,y)\,dy,$$

for $u(t,\cdot) \in W^{1,\infty}(\mathbb{R}^d,\mathbb{R}^d)$ and $\rho(t,\cdot) \in L^p(\mathbb{R}^d)$. Finally, we put both properties together about scalar conservation laws and the commutator to obtain a solution to (2.3.1) via a fixed point method.

Remark 2.3.1. *Here we call the attention of the reader about a relevant issue to be borne in mind along this section. Recall that by construction ρ was set as a probability density. Nevertheless, we shall relax this normalization assumption here in order to first account for wellposedness results under small initial data assumptions. At first glance, such condition might be thought as a contradiction with the normalization assumption on $\rho \in \mathcal{P}(\mathbb{R}^d)$. However, we shall later show that this is not the case when appropriate exponents are chosen, thus recovering the case of probability densities. Unfortunately, such smallness assumption restricts our initial data to a class of flat enough densities.*

2.3.1 Linear transport equations in conservative form with Lipschitz transport field

The following result summarizes the existence and uniqueness results of weak solutions in L^p spaces when the transport field belongs to $L^1(0,T;W^{1,\infty}(\mathbb{R}^d,\mathbb{R}^d))$.

Theorem 2.3.2. *Set a velocity field $u \in L^1(0,T;W^{1,\infty}(\mathbb{R}^d,\mathbb{R}^d))$ and consider the next Cauchy problem for the density $\rho = \rho(t,x)$*

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, & x \in \mathbb{R}^d,\ t \geq 0, \\ \rho(0,x) = \rho^0(x), & x \in \mathbb{R}^d, \end{cases} \tag{2.3.2}$$

where the initial data $\rho^0 \geq 0$ is taken in $L^p(\mathbb{R}^d)$ for some $1 \leq p \leq \infty$. Then there exists one and only one weak solution ρ to (2.3.2) such that $\rho \in L^\infty(0,T;L^p(\mathbb{R}^d))$. Indeed, when $1 < p \leq \infty$ then

$$\|\rho(t,\cdot)\|_{L^p(\mathbb{R}^d)} \leq \exp\left(\frac{1}{p'}\|u\|_{L^1(0,T;W^{1,\infty}(\mathbb{R}^d))}\right)\|\rho^0\|_{L^p(\mathbb{R}^d)},$$

for any $t \in [0,T)$, where p' stands for the conjugated exponent of p i.e., $p' = \frac{p}{p-1}$. When $p = 1$ we recover the conservation of the total mass, namely,

$$\|\rho(t,\cdot)\|_{L^1(\mathbb{R}^d)} = \|\rho^0\|_{L^1(\mathbb{R}^d)},$$

for any $t \in [0,T)$.

Although the proof follows from standard arguments (see [7, Chapter 1]), we exhibit a short sketch of the proof in Appendix 2.B for the sake of completeness.

2.3.2 Commutator estimates and existence results for the limiting system

In this section, we explore the regularity properties of the commutator appearing in the limiting equation (2.3.1). This is the content of the next result.

Theorem 2.3.3. *Consider $1 \leq p_1 < p_2 \leq \infty$ such that*

$$\frac{1}{p_2} < 1 - \frac{2\alpha}{d} < \frac{1}{p_1}.$$

Then, there exists some positive constant $C = C(p_1,p_2,\alpha,d)$ such that

$$\begin{aligned}&\|([u, I_{d-2\alpha}]\rho)\|_{L^1(0,T;W^{1,\infty}(\mathbb{R}^d))} \\ &\quad \leq C\|u\|_{L^1(0,T;W^{1,\infty}(\mathbb{R}^d))}\|\rho\|_{L^\infty(0,T;L^{p_1}(\mathbb{R}^d))}^{(1/p_2'-\frac{2\alpha}{d})/(1/p_1-1/p_2)}\|\rho\|_{L^\infty(0,T;L^{p_2}(\mathbb{R}^d))}^{(\frac{2\alpha}{d}-1/p_1')/(1/p_1-1/p_2)},\end{aligned}$$

for every $\rho \in L^\infty(0,T;L^{p_1}(\mathbb{R}^d)) \cap L^\infty(0,T;L^{p_2}(\mathbb{R}^d))$ and every $u \in L^1(0,T;W^{1,\infty}(\mathbb{R}^d))$.

Proof. Write the commutator of the multiplier u with the Riesz potential in a more explicit way as follows

$$([u, I_{d-2\alpha}]\rho)(t,x) = \int_{\mathbb{R}^d} \phi_0(|x-y|)(u(t,x)-u(t,y))\rho(t,y)\,dy.$$

On the one hand, taking L^∞ norms with respect to space yields

$$\|([u, I_{d-2\alpha}]\rho)(t,\cdot)\|_{L^\infty(\mathbb{R}^d)} \leq 2\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^d)}\|\phi_0 * \rho(t,\cdot)\|_{L^\infty(\mathbb{R}^d)}.$$

The assumptions on ρ together with the L^∞ bounds provided by *Hardy–Littlewood–Sobolev's theorem* of fractional integrals lead to

$$\begin{aligned}&\|([u, I_{d-2\alpha}]\rho)\|_{L^1(0,T;L^\infty(\mathbb{R}^d))}\\ &\qquad \leq C\|u\|_{L^1(0,T;L^\infty(\mathbb{R}^d))}\|\rho\|_{L^\infty(0,T;L^{p_1}(\mathbb{R}^d))}^{(1/p_2'-\frac{2\alpha}{d})/(1/p_1-1/p_2)}\|\rho\|_{L^\infty(0,T;L^{p_2}(\mathbb{R}^d))}^{(\frac{2\alpha}{d}-1/p_1')/(1/p_1-1/p_2)},\end{aligned}$$

see Appendix C. Now, taking derivatives with respect to space in the commutator, we have

$$\frac{\partial}{\partial x_i}[u, I_{d-2\alpha}]\rho = [u, I_{d-(2\alpha+1)}]\rho + \frac{\partial u}{\partial x_i}\phi_0 * \rho.$$

Regarding the first term, one can cancel the extra degree of singularity thanks to the Lipschitz continuity with respect to space of u, leading to a similar estimate

$$\begin{aligned}&\|([u, I_{d-(2\alpha+1)}]\rho)\|_{L^1(0,T;L^\infty(\mathbb{R}^d))}\\ &\qquad \leq C\|u\|_{L^1(0,T;W^{1,\infty}(\mathbb{R}^d))}\|\rho\|_{L^\infty(0,T;L^{p_1}(\mathbb{R}^d))}^{(1/p_2'-\frac{2\alpha}{d})/(1/p_1-1/p_2)}\|\rho\|_{L^\infty(0,T;L^{p_2}(\mathbb{R}^d))}^{(\frac{2\alpha}{d}-1/p_1')/(1/p_1-1/p_2)}.\end{aligned}$$

The second term is easier to estimate in the same manner as follows

$$\begin{aligned}&\left\|\frac{\partial u}{\partial x_i}\phi_0 * \rho\right\|_{L^1(0,T;L^\infty(\mathbb{R}^d))}\\ &\qquad \leq C\|u\|_{L^1(0,T;W^{1,\infty}(\mathbb{R}^d))}\|\rho\|_{L^\infty(0,T;L^{p_1}(\mathbb{R}^d))}^{(1/p_2'-\frac{2\alpha}{d})/(1/p_1-1/p_2)}\|\rho\|_{L^\infty(0,T;L^{p_2}(\mathbb{R}^d))}^{(\frac{2\alpha}{d}-1/p_1')/(1/p_1-1/p_2)}.\end{aligned}$$

Arranging all the terms within the same bound ends the proof. □

Apart from the preceding regularity results, that leads to $W^{1,\infty}$ estimates of the commutator, we can indeed obtain some extra integrability by virtue of *Hardy–Littlewood–Sobolev's theorem* of fractional integrals, see Appendix C. Note that such integrability cannot be directly inferred from the $W^{1,\infty}$ regularity via *Sobolev's embedding theorem*.

Theorem 2.3.4. *Consider* $1 \leq p_1 < p_2 \leq \infty$ *and* $1 \leq s < \infty$ *such that*

$$\frac{1}{p_2} < 1 - \frac{2\alpha}{d} < \frac{1}{p_1} \quad \text{and} \quad s > \frac{d}{2\alpha}.$$

Then, there exists some positive constant $C = C(p_1, p_2, s, \alpha, d)$ *such that*

$$\begin{aligned}&\|([u, I_{d-2\alpha}]\rho)\|_{L^1(0,T;L^s(\mathbb{R}^d))}\\ &\qquad \leq C\|u\|_{L^1(0,T;L^s(\mathbb{R}^d))}\|\rho\|_{L^\infty(0,T;L^{p_1}(\mathbb{R}^d))}^{(1/p_2'-\frac{2\alpha}{d})/(1/p_1-1/p_2)}\|\rho\|_{L^\infty(0,T;L^{p_2}(\mathbb{R}^d))}^{(\frac{2\alpha}{d}-1/p_1')/(1/p_1-1/p_2)},\end{aligned}$$

for every $\rho \in L^\infty(0,T;L^{p_1}(\mathbb{R}^d)) \cap L^\infty(0,T;L^{p_2}(\mathbb{R}^d))$ *and every* $u \in L^1(0,T;L^s(\mathbb{R}^d))$.

Proof. First, split the commutator into two parts as follows

$$[u, I_{d-2\alpha}]\rho = F + G,$$

where the functions F and G take the form

$$F = (\phi_0 * \rho)u, \quad \text{and} \quad G = \phi_0 * (\rho u).$$

Note that one can apply the above reasoning to estimate F in any Lebesgue space as follows

$$\|F\|_{L^1(0,T;L^s(\mathbb{R}^d))} \leq C\|u\|_{L^1(0,T;L^s(\mathbb{R}^d))}\|\rho\|_{L^\infty(0,T;L^{p_1}(\mathbb{R}^d))}^{(1/p_2'-\frac{2\alpha}{d})/(1/p_1-1/p_2)}\|\rho\|_{L^\infty(0,T;L^{p_2}(\mathbb{R}^d))}^{(\frac{2\alpha}{d}-1/p_1')/(1/p_1-1/p_2)}.$$

Regarding the second term, define the exponent

$$p := \frac{d}{d-2\alpha},$$

and note that our hypothesis leads to $p_1 < p < p_2$. Then, the interpolation inequality of the Lebesgue spaces shows that $\rho \in L^\infty(0,T;L^p(\mathbb{R}^d))$ and

$$\|\rho\|_{L^\infty(0,T;L^p(\mathbb{R}^d))} \leq \|\rho\|_{L^\infty(0,T;L^{p_1}(\mathbb{R}^d))}^{\theta}\|\rho\|_{L^\infty(0,T;L^{p_2}(\mathbb{R}^d))}^{1-\theta},$$

for some exponent $\theta \in (0,1)$ given by

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{(1-\theta)}{p_2}.$$

By inspection, it is straightforward to check that

$$\theta = \frac{\frac{1}{p_2'} - \frac{2\alpha}{d}}{\frac{1}{p_1} - \frac{1}{p_2}}.$$

Hence,

$$\|\rho\|_{L^\infty(0,T;L^p(\mathbb{R}^d))} \leq \|\rho\|_{L^\infty(0,T;L^{p_1}(\mathbb{R}^d))}^{(1/p_2'-\frac{2\alpha}{d})/(1/p_1-1/p_2)}\|\rho\|_{L^\infty(0,T;L^{p_2}(\mathbb{R}^d))}^{(\frac{2\alpha}{N}-1/p_1')/(1/p_1-1/p_2)}.$$

Now, the Hölder inequality shows that $\rho\, u \in L^1(0,T;L^r(\mathbb{R}^d))$, where

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{s}.$$

Since we are assuming $s > d/2\alpha$, then $r > 1$. Moreover, the next identity holds

$$\frac{1}{s} = \frac{1}{r} + \frac{2\alpha}{d} - 1.$$

Consequently, the Hardy–Littlewood–Sobolev inequality (see Appendix C) entails

$$\begin{aligned}\|G\|_{L^1(0,T;L^s(\mathbb{R}^d))} &\leq C\|\rho u\|_{L^1(0,T;L^r(\mathbb{R}^d))}\\ &\leq C\|u\|_{L^1(0,T;L^s(\mathbb{R}^d))}\|\rho\|_{L^\infty(0,T;L^p(\mathbb{R}^d))}\\ &\leq C\|u\|_{L^1(0,T;L^s(\mathbb{R}^d))}\|\rho\|_{L^\infty(0,T;L^{p_1}(\mathbb{R}^d))}^{(1/p_2'-\frac{2\alpha}{d})/(1/p_1-1/p_2)}\|\rho\|_{L^\infty(0,T;L^{p_2}(\mathbb{R}^d))}^{(\frac{2\alpha}{d}-1/p_1')/(1/p_1-1/p_2)},\end{aligned}$$

and this end the proof of the theorem. ◻

In the next part, some particular choices of s will be needed. Specifically, s will equal some entire multiples of the upper and lower exponents p_1 and p_2 of ρ.

Corollary 2.3.5. *Let α be any exponent in $(0, \frac{d}{2})$ and $1 \leq p_1 < p_2 \leq \infty$ such that*

$$\frac{1}{p_2} < 1 - \frac{2\alpha}{d} < \frac{1}{p_1}.$$

Consider any $k \in \mathbb{N}$ so that $kp_1 > \frac{d}{2\alpha}$. Then, there exists $C = C(p_1, p_2, s, \alpha, d) > 0$ such that

$$\begin{aligned}&\|([u, I_{d-2\alpha}]\rho)\|_{L^1(0,T;L^{kp_i}(\mathbb{R}^d))}\\ &\qquad \leq C\|u\|_{L^1(0,T;L^{kp_i}(\mathbb{R}^d))}\|\rho\|_{L^\infty(0,T;L^{p_1}(\mathbb{R}^d))}^{(1/p_2'-\frac{2\alpha}{d})/(1/p_1-1/p_2)}\|\rho\|_{L^\infty(0,T;L^{p_2}(\mathbb{R}^d))}^{(\frac{2\alpha}{d}-1/p_1')/(1/p_1-1/p_2)},\end{aligned}$$

for every $i \in \{1,2\}$ and each $\rho \in L^\infty(0,T;L^{p_1}(\mathbb{R}^d)) \cap L^\infty(0,T;L^{p_2}(\mathbb{R}^d))$, $u \in L^1(0,T;L^{kp_1}(\mathbb{R}^d)) \cap L^1(0,T;L^{kp_2}(\mathbb{R}^d))$.

All the above results amounts to the necessary tools that we will need to construct a solution to (2.3.1) by the Banach contraction principle in the space W^{1,kp_1,kp_2} (see (2.1.10) for the definition of such Banach space). This is the content of Theorem 2.1.2 in the Introduction that we prove next.

Proof of Theorem 2.1.2. Let us define two operators $\mathcal{D}$ and $\mathcal{C}$ given by:

$$\begin{aligned}\mathcal{D}[u] &:= \left.\frac{\rho^0(y)}{J^u(t;0,y)}\right|_{y=X^u(0;t,\cdot)},\\ \mathcal{C}[\rho,u] &:= -\frac{1}{c_\alpha^\alpha}[u, I_{d-2\alpha}]\rho,\end{aligned}$$

for every $u \in L^1(0,T;W^{1,kp_1,kp_2}(\mathbb{R}^d,\mathbb{R}^d))$ and $\rho \in L^\infty(0,T;L^{p_1}(\mathbb{R}^d)\cap L^{p_2}(\mathbb{R}^d))$. Here, $X^u(t;t_0,x_0)$ and $J^u(t;t_0,x_0)$ stand for the forward flow of u and its associated Jacobian determinant. Notice that so defined, $\mathcal{D}[u]$ is the solution to the Cauchy problem obtained by means of Theorem 2.3.2 and $\mathcal{C}[\rho,u]$ is the above commutator of weakly singular integrals. With this notation, the existence and uniqueness of solution to (2.3.1) amounts to finding a solution to

$$\mathcal{C}[\mathcal{D}[u],u] - \nabla\psi = u, \quad u \in L^1(0,T;W^{1,kp_1,kp_2}(\mathbb{R}^d,\mathbb{R}^d)).$$

Recall the definition of W^{1,kp_1,kp_2} in (2.1.10). Naturally, it can be restated as a fixed point equation

$$\Phi[u] = u, \quad u \in L^1(0,T;W^{1,kp_1,kp_2}(\mathbb{R}^d,\mathbb{R}^d)),$$

for the new operator $\Phi[u] := \mathcal{C}[\mathcal{D}[u],u] - \nabla\psi$. First, notice that such fixed point problem is well posed because

$$\Phi(L^1(0,T;W^{1,kp_1,kp_2})) \subseteq L^1(0,T;W^{1,kp_1,kp_2}),$$

as a consequence of Theorem 2.3.2 and Corollary 2.3.5. Indeed,

$$\begin{aligned}\|\mathcal{D}[u]\|_{L^\infty(0,T;L^{p_i}(\mathbb{R}^d))} &\leq \exp\left(\frac{1}{p_i'}\|u\|_{L^1(0,T;W^{1,\infty}(\mathbb{R}^d))}\right)\|\rho^0\|_{L^{p_i}(\mathbb{R}^d)},\\ \|\mathcal{C}[\rho,u]\|_{L^1(0,T;W^{1,kp_1,kp_2}(\mathbb{R}^d,\mathbb{R}^d))} &\leq C\|u\|_{L^1(0,T;W^{1,kp_1,kp_2}(\mathbb{R}^d,\mathbb{R}^d))}\\ &\quad\times\|\rho\|_{L^\infty(0,T;L^{p_1}(\mathbb{R}^d))}^{(1/p_2'-\frac{2\alpha}{d})/(1/p_1-1/p_2)}\|\rho\|_{L^\infty(0,T;L^{p_2}(\mathbb{R}^d))}^{(\frac{2\alpha}{d}-1/p_1')/(1/p_1-1/p_2)}.\end{aligned}$$

Let us show that it is indeed a contraction. Set $u_1, u_2 \in L^1(0,T;W^{1,kp_1,kp_2}(\mathbb{R}^d,\mathbb{R}^d))$ and split

$$\Phi[u_1]-\Phi[u_2]=\mathcal{C}[\mathcal{D}[u_1],u_1-u_2]+\mathcal{C}[\mathcal{D}[u_1]-\mathcal{D}[u_2],u_2]=:U+V.$$

The first term can be bounded as follows

$$\begin{aligned}\|U\|_{L^1(0,T;W^{1,kp_1,kp_2}(\mathbb{R}^d,\mathbb{R}^d))}&\leq C\|u_1-u_2\|_{L^1(0,T;W^{1,kp_1,kp_2}(\mathbb{R}^d,\mathbb{R}^d))}\\ &\times\exp\left(\frac{2\alpha}{d}\|u_1\|_{L^1(0,T;W^{1,\infty}(\mathbb{R}^d))}\right)\|\rho^0\|_{L^{p_1}(\mathbb{R}^d)}^{(1/p_2'-\frac{2\alpha}{d})/(1/p_1-1/p_2)}\|\rho^0\|_{L^{p_2}(\mathbb{R}^d)}^{(\frac{2\alpha}{d}-1/p_1')/(1/p_1-1/p_2)}.\end{aligned}$$

Regarding the second term one arrives at the next slightly different estimate

$$\begin{aligned}\|V\|_{L^1(0,T;W^{1,kp_1,kp_2}(\mathbb{R}^d,\mathbb{R}^d))}&\leq C\|u_2\|_{L^1(0,T;W^{1,kp_1,kp_2}(\mathbb{R}^d,\mathbb{R}^d))}\\ &\times\|\mathcal{D}[u_1]-\mathcal{D}[u_2]\|_{L^\infty(0,T;L^{p_1}(\mathbb{R}^d))}^{(1/p_2'-\frac{2\alpha}{d})/(1/p_1-1/p_2)}\|\mathcal{D}[u_1]-\mathcal{D}[u_2]\|_{L^\infty(0,T;L^{p_2}(\mathbb{R}^d))}^{(\frac{2\alpha}{d}-1/p_1')/(1/p_1-1/p_2)}.\end{aligned}$$

The next step is to estimate $\mathcal{D}[u_1]-\mathcal{D}[u_2]$. To this end, the L^{kp_i} estimate of the commutator in Theorem 2.3.3 and Corollary 2.3.5 will be essential. As it will be checked, it will not directly follow from the $W^{1,\infty}$ estimate in Theorem 2.3.3. Since we are assuming smooth ρ^0, there is no problem in considering the pull-back pointwise. Thus,

$$\begin{aligned}&\mathcal{D}[u_1](t,x)-\mathcal{D}[u_2](t,x)\\ &=\rho^0(X^{u_1}(0;t,x))(J^{u_1}(0;t,x)-J^{u_2}(0;t,x))+(\rho^0(X^{u_1}(0;t,x))-\rho^0(X^{u_2}(0;t,x)))J^{u_2}(0;t,x).\end{aligned}$$

On the one hand, the first term can easily bounded by virtue of a straightforward change of variables, the Jacobi–Louville formula and the mean value theorem

$$\begin{aligned}&\|\rho^0(X^{u_1}(0;t,\cdot))(J^{u_1}(0;t,\cdot)-J^{u_2}(0;t,\cdot))\|_{L^{p_i}(\mathbb{R}^d)}\\ &\quad\leq\|\rho^0(X^{u_1}(0;t,\cdot))\|_{L^{p_i}(\mathbb{R}^d)}\|J^{u_1}(0;t,\cdot)-J^{u_2}(0;t,\cdot)\|_{L^\infty(\mathbb{R}^d)}\\ &\quad\leq\|\rho^0\|_{L^{p_i}(\mathbb{R}^d)}\exp\left(\frac{1}{p_i}\|u_1\|_{L^1(0,T;W^{1,\infty}(\mathbb{R}^d))}\right)\\ &\qquad\times\exp\left(\|u_1\|_{L^1(0,T;W^{1,\infty}(\mathbb{R}^d))}+\|u_2\|_{L^1(0,T;W^{1,\infty}(\mathbb{R}^d))}\right)\|u_1-u_2\|_{L^1(0,T;W^{1,\infty}(\mathbb{R}^d))}.\end{aligned}$$

On the other hand, the second term has to be studied separately as follows. First, the Jacobian determinant can be bounded in L^∞ as usual through (2.B.2). Second, the difference of the evaluation of ρ^0 along the flow of u_1 and u_2 can be split by means of the integral remainder version of Taylor's theorem, specifically

$$\begin{aligned}&\rho^0(X^{u_1}(0;t,x))-\rho^0(X^{u_2}(0;t,x))\\ &=\sum_{0<|\gamma|\leq k-1}\frac{1}{|\gamma|!}D^\gamma\rho^0(X^{u_2}(0;t,x))\,(X^{u_1}(0;t,x)-X^{u_2}(0;t,x))^\gamma\\ &\quad+\sum_{|\gamma|=k}\left(\int_0^1\frac{(1-\theta)^k}{k!}D^\gamma\rho^0(X^{u_2}(0;t,x)+\theta(X^{u_1}(0;t,x)-X^{u_2}(0;t,x)))\,d\theta\right)\\ &\qquad\times(X^{u_1}(0;t,x)-X^{u_2}(0;t,x))^\gamma.\end{aligned}$$

As a consequence, one can obtain the next bound

$$\|(\rho^0(X^{u_1}(0;t,\cdot))-\rho^0(X^{u_2}(0;t,\cdot)))J^{u_2}(0;t,\cdot)\|_{L^{p_i}(\mathbb{R}^d)}$$

$$\leq \sum_{0<|\gamma|\leq k-1} \frac{1}{|\gamma|!} \|D^\gamma \rho^0\|_{L^{p_i}(\mathbb{R}^d)} \exp\left(\frac{1}{p_i}\|u_2\|_{L^1(0;T;W^{1,\infty}(\mathbb{R}^d))}\right) \|X^{u_1}(0;t,\cdot) - X^{u_2}(0;t,\cdot)\|^{|\gamma|}_{L^\infty(\mathbb{R}^d)}$$
$$+ \frac{1}{(k+1)!} \sum_{|\gamma|=k} \|D^\gamma \rho^0\|_{L^\infty(\mathbb{R}^d)} \exp\left(\frac{1}{p_i}\|u_2\|_{L^1(0;T;W^{1,\infty}(\mathbb{R}^d))}\right) \|X^{u_1}(0;t,\cdot) - X^{u_2}(0;t,\cdot)\|^{k}_{L^{kp_i}(\mathbb{R}^d)}$$

Since $u_i(t,\cdot)$ belongs both to $W^{1,\infty}(\mathbb{R}^d)$ and L^{kp_i}, then a straightforward application of Gronwall's Lemma yields the next upper bounds

$$\begin{aligned}
&\|X^{u_1}(0;t,\cdot)-X^{u_2}(0;t,\cdot)\|_{L^\infty(\mathbb{R}^d)} \\
&\qquad \leq \exp\left(\|u_1\|_{L^1(0,T;W^{1,\infty}(\mathbb{R}^d))}\right) \|u_1 - u_2\|_{L^1(0,T;L^\infty(\mathbb{R}^d))}, \\
&\|X^{u_1}(0;t,\cdot)-X^{u_2}(0;t,\cdot)\|_{L^{kp_i}(\mathbb{R}^d)} \\
&\qquad \leq \exp\left(\frac{1}{kp_i}\|u_1\|_{L^1(0,T;W^{1,\infty}(\mathbb{R}^d))}\right) \exp(\|u_2\|_{L^1(0,T;W^{1,\infty}(\mathbb{R}^d))})\|u_1 - u_2\|_{L^1(0,T;L^{kp_i}(\mathbb{R}^d))}.
\end{aligned}$$

To sum up, there exists some separately increasing function $\kappa_2 : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+$ that does not depend on u_1, u_2 or ρ^0 such that

$$\begin{aligned}
&\|\Phi[u_1] - \Phi[u_2]\|_{L^1(0,T;W^{1,kp_1,kp_2}(\mathbb{R}^d,\mathbb{R}^d))} \leq \|\rho^0\|_{W^{k,p_1,p_2}(\mathbb{R}^d)} \\
&\times \kappa_2(\|u_1\|_{L^1(0,T;W^{1,kp_1,kp_2}(\mathbb{R}^d,\mathbb{R}^d))}, \|u_2\|_{L^1(0,T;W^{1,kp_1,kp_2}(\mathbb{R}^d,\mathbb{R}^d))})\|u_1 - u_2\|_{L^1(0,T;W^{1,kp_1,kp_2}(\mathbb{R}^d,\mathbb{R}^d))}.
\end{aligned}$$

Similarly, one obtains the next estimate of $\Phi[u]$

$$\begin{aligned}
&\|\Phi[u]\|_{L^1(0,T;W^{1,kp_1,kp_2}(\mathbb{R}^d,\mathbb{R}^d))} \\
&\quad \leq \|\rho^0\|_{L^{p_1}(\mathbb{R}^d)}^{(1/p_2'-\frac{2\alpha}{d})/(1/p_1-1/p_2)} \|\rho^0\|_{L^{p_2}(\mathbb{R}^d)}^{(\frac{2\alpha}{d}-1/p_1')/(1/p_1-1/p_2)} \kappa_1(\|u\|_{L^1(0,T;W^{1,kp_1,kp_2}(\mathbb{R}^d,\mathbb{R}^d))}) \\
&\qquad + \|\nabla\psi\|_{L^1(0,T;W^{1,kp_1,kp_2}(\mathbb{R}^d,\mathbb{R}^d))},
\end{aligned}$$

for some increasing function $\kappa_1 : \mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+$ which does not depend on u or ρ^0. Consider any radius $R > \|\nabla\psi\|_{L^1(0,T;W^{1,kp_1,kp_2}(\mathbb{R}^d,\mathbb{R}^d))}$ and define the unit ball of $L^1(0,T;W^{1,kp_1,kp_2}(\mathbb{R}^d,\mathbb{R}^d))$ centered at the origin

$$\mathcal{B}_R := \{u \in L^1(0,T;W^{1,kp_1,kp_2}(\mathbb{R}^d,\mathbb{R}^d)) : \|u\|_{L^1(0,T;W^{1,kp_1,kp_2}(\mathbb{R}^d,\mathbb{R}^d))} \leq R\}.$$

Assume that ρ^0 is "small enough" so that

$$\begin{aligned}
\|\rho^0\|_{L^{p_1}(\mathbb{R}^d)}^{(1/p_2'-\frac{2\alpha}{d})/(1/p_1-1/p_2)} \|\rho^0\|_{L^{p_2}(\mathbb{R}^d)}^{(\frac{2\alpha}{d}-1/p_1')/(1/p_1-1/p_2)} \kappa_1(R) &\leq R - \|\nabla\psi\|_{L^1(0,T;W^{1,kp_1,kp_2}(\mathbb{R}^d,\mathbb{R}^d))}, \\
\|\rho^0\|_{W^{k,p_1,p_2}(\mathbb{R}^d)}\kappa_2(R,R) &< 1.
\end{aligned} \tag{2.3.3}$$

Then, $\Phi(\mathcal{B}_R) \subseteq \mathcal{B}_R$, Φ is a contraction for the norm $\|\cdot\|_{L^1(0,T;W^{1,kp_1,kp_2}(\mathbb{R}^d,\mathbb{R}^d))}$ and the Banach contraction principle shows the existence of a unique solution u to (2.3.1) in $\mathcal{B}_R$. □

Remark 2.3.6. *1. Notice that since T is arbitrary, then the above theorem yields a global existence result for velocities fields $u \in L^1(0,\infty;W^{1,kp_1,kp_2}(\mathbb{R}^d))^d$. Nevertheless, initial data have to be considered small enough so that condition* (2.3.3) *fulfils. At first glance, it seems that* (2.3.3) *might contradict the normalization assumption $\rho^0 \in \mathcal{P}(\mathbb{R}^d)$ because $\kappa_1, \kappa_2 \geq 1$ by construction.*

2. *Recall that global existence results for small initial data usually amount to local-in-time existence results for general initial data in many systems of conservation laws. However, the implicit equation for u does not involve any time derivative and, consequently, those ideas cannot be easily implemented. Indeed, a classical way to do so is to replace $u \in L^1(0,T;W^{1,kp_1,kp_2}(\mathbb{R}^d))^d$ with $u \in L^q(0,T;W^{1,kp_1,kp_2}(\mathbb{R}^d))^d$ for some $q > 1$, e.g., $q = \infty$. By doing so, we can generalize Theorems 2.3.3 and 2.3.4, specifically*

$$\|\mathcal{D}[u]\|_{L^\infty(0,T;L^{p_i}(\mathbb{R}^d))} \leq \exp\left(\frac{T^{1/q'}}{p_i'}\|u\|_{L^q(0,T;W^{1,\infty}(\mathbb{R}^d))}\right)\|\rho^0\|_{L^{p_i}(\mathbb{R}^d)},$$

$$\|\mathcal{C}[\rho,u]\|_{L^q(0,T;W^{1,kp_1,kp_2}(\mathbb{R}^d,\mathbb{R}^d))} \leq C\|u\|_{L^q(0,T;W^{1,kp_1,kp_2}(\mathbb{R}^d,\mathbb{R}^d))} \times \|\rho\|_{L^\infty(0,T;L^{p_1}(\mathbb{R}^d))}^{(1/p_2'-\frac{2\alpha}{N})/(1/p_1-1/p_2)}\|\rho\|_{L^\infty(0,T;L^{p_2}(\mathbb{R}^d))}^{(\frac{2\alpha}{N}-1/p_1')/(1/p_1-1/p_2)}.$$

Then, a similar procedure like in the above proof yields a local-in-time existence result for (2.3.1) with $L^q(0,T;W^{1,kp_1,kp_2}(\mathbb{R}^d))$ velocity fields. Unfortunately, the smallness assumption on ρ^0 still cannot be removed due to the fact that $\kappa_1,\kappa_2 \geq 1$.

3. *Recall that the smallness assumption explicitly reads*

$$\|\rho^0\|_{W^{k,\infty}(\mathbb{R}^d)} + \|\rho^0\|_{W^{k-1,p_1}(\mathbb{R}^d)} + \|\rho^0\|_{W^{k-1,p_2}(\mathbb{R}^d)} < \delta_R.$$

Notice that if we set $p_1 = 1$, then we indeed obtain the explicit constraint $\|\rho^0\|_{L^1(\mathbb{R}^d)} < \delta_R$, that clearly contradicts the normalization assumptions $\rho^0 \in \mathcal{P}(\mathbb{R}^d)$. However, we can set $p_1 > 1$. In this case, Sobolev's embedding theorem characterizes in a sharp way the continuous embedding of the involved Sobolev spaces $W^{k-1,p_1}(\mathbb{R}^d)$ and $W^{k-1,p_2}(\mathbb{R}^d)$ into further $L^r(\mathbb{R}^d)$ spaces. This sharp result guarantees that $r > p_1$ and, consequently, such eventual gain of L^r-integrability does not prevent us from choosing any $\rho^0 \in \mathcal{P}(\mathbb{R}^d)$ as long as it fulfils the corresponding "smallness" (flatness) assumption.

2.4 Other relevant hydrodynamic limits

This section focuses on showing that the techniques in Section 2.2 remain valid for other relevant scalings of the system. Specifically, regarding the friction case we will analyze an intermediate scaling (see Equation (2.A.5) in Appendix 2.A). Note that this scaled system includes the velocity diffusion at a lower order of ε. Hence, we cannot expect the system to converge towards a Maxwellian as it was expected in the hyperbolic scaling in the preceding section. Nevertheless, such scaling in the diffusion term is compulsory in order to get some estimate on the scaled current (coming from the friction term) that allows passing to the limit $\varepsilon \to 0$. In the frictionless case, we will introduce another hyperbolic scaling (see Equation (2.A.6) in Appendix 2.A) where the velocity diffusion is again of a low order of ε and similar a priori bounds for the current can be obtained from the inertial terms, not from the friction term. Finally, we will address the well known Rayleigh–Helmholtz friction, that has been considered of great help in the modeling of flocking, swarming and self-propelling phenomena through the recent years [66, 259]. Specifically, we shall introduce a new hydrodynamic limit where not only hyperbolic, intermediate scalings and singular influence functions can be considered, but also the Rayleigh–Helmholtz-type damping can be assumed as an approximation of the classical linear friction when $\varepsilon \to 0$.

2.4.1 Hydrodynamic limit with intermediate scalings in the damping case

The intermediate hyperbolic scaling (2.A.5) was derived in 2.A and takes the form

$$\varepsilon^{1+\gamma}\frac{\partial f_\varepsilon}{\partial t}+\varepsilon v\cdot\nabla_x f_\varepsilon-\varepsilon^\gamma\nabla_x\psi_\varepsilon\cdot\nabla_v f_\varepsilon=\operatorname{div}_v\left(f_\varepsilon v+\varepsilon^{2\gamma}\nabla_v f_\varepsilon+Q_{CS}^{\phi_\varepsilon}(f_\varepsilon,f_\varepsilon)\right),\tag{2.4.1}$$

for some parameter $\gamma\in[0,1]$. Note that when $\gamma=0$, it agrees with the hyperbolic scaling in Section 2.2 and the choice $\gamma=1$ reminds us of a parabolic scaling. In this case, the hierarchy of velocity moments now takes the form:

• **Mass conservation**

$$\frac{\partial\rho_\varepsilon}{\partial t}+\operatorname{div}_x\left(\frac{j_\varepsilon}{\varepsilon^\gamma}\right)=0.\tag{2.4.2}$$

• **Law of balance of current**

$$\varepsilon^{1+\gamma}\frac{\partial j_\varepsilon}{\partial t}+\varepsilon\operatorname{div}_x\mathcal{S}_\varepsilon+\varepsilon^\gamma\rho_\varepsilon\nabla_x\psi_\varepsilon+(1+\phi_\varepsilon*\rho_\varepsilon)\,j_\varepsilon-(\phi_\varepsilon*j_\varepsilon)\,\rho_\varepsilon=0.\tag{2.4.3}$$

• **Law of balance of stress**

$$\varepsilon^{1+\gamma}\frac{\partial\mathcal{S}_\varepsilon}{\partial t}+\varepsilon\operatorname{div}_x\mathcal{T}_\varepsilon+2\varepsilon^\gamma\operatorname{Sym}(j_\varepsilon\otimes\nabla_x\psi_\varepsilon)+2\left((1+\phi_\varepsilon*\rho_\varepsilon)\,\mathcal{S}_\varepsilon-\varepsilon^{2\gamma}\rho_\varepsilon I\right)-2\operatorname{Sym}((\phi_\varepsilon*j_\varepsilon)\otimes j_\varepsilon)=0.\tag{2.4.4}$$

Thanks to our choice for the scaling of the velocity diffusion term, one arrives at an analogue of Corollary 2.2.6.

Corollary 2.4.1. *Let the initial distribution functions f_ε^0 verify (2.2.5), the external forces $-\nabla\psi_\varepsilon$ fulfil (2.2.6) and consider the strong global in time solution f_ε to (2.4.1) with initial data f_ε^0 and $\alpha\in(0,\frac{d}{2})$. Then, for any nonnegative integer k the next bound holds true*

$$\begin{aligned}
&k\left\||v|^k\frac{f_\varepsilon}{\varepsilon^{2\gamma}}\right\|_{L^1(0,T;L^1(\mathbb{R}^{2d}))}\\
&+\frac{k}{2}\frac{1}{\varepsilon^{2\gamma}}\int_0^T\int_{\mathbb{R}^{4d}}(|v|^{k-2}v-|w|^{k-2}w)\cdot(v-w)\phi_\varepsilon(|x-y|)f_\varepsilon(t,x,v)f_\varepsilon(t,y,w)\,dx\,dy\,dv\,dw\,dt\\
&\leq\varepsilon^{1-\gamma}\||v|^kf_\varepsilon(0)\|_{L^1(\mathbb{R}^{2d})}+k(d+k-2)\||v|^{k-2}f_\varepsilon\|_{L^1(0,T;L^1(\mathbb{R}^{2d}))}+k\left\||v|^{k-1}\frac{f_\varepsilon}{\varepsilon^\gamma}\nabla_x\psi_\varepsilon\right\|_{L^1(0,T;L^1(\mathbb{R}^{2d}))},
\end{aligned}$$

$$\||x|^kf_\varepsilon\|_{L^\infty(0,T;L^1(\mathbb{R}^d))}^{\frac{1}{k}}\leq\||x|^kf_\varepsilon\|_{L^1(\mathbb{R}^d)}^{\frac{1}{k}}+\int_0^T\left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|x|^k}{\varepsilon^{k\gamma}}f_\varepsilon\,dx\,dv\right)^{\frac{1}{k}}\,dt.$$

Again, the choice $k=2$ and $k=1$ on each inequality leads to the next estimates for the scaled first and second order velocity moments.

Corollary 2.4.2. *Let the initial distribution functions f_ε^0 verify (2.2.5), the external forces $-\nabla\psi_\varepsilon$ fulfil (2.2.6) and consider the strong global in time solution f_ε to (2.4.1) with initial data f_ε^0 and $\alpha\in(0,\frac{d}{2})$. Then, we obtain*

$$\begin{aligned}
\||x|f_\varepsilon\|_{L^\infty(0,T;L^1(\mathbb{R}^d))}&\leq M_0+T^{1/2}\left(2\varepsilon^{1-\gamma}E_0+\left(2dT+F_0^2\right)\right)^{1/2},\\
\left\||v|\frac{f_\varepsilon}{\varepsilon^\gamma}\right\|_{L^2(0,T;L^1(\mathbb{R}^{2d}))}&\leq\left(2\varepsilon^{1-\gamma}E_0+\left(2dT+F_0^2\right)\right)^{1/2},\\
\left\||v|^2\frac{f_\varepsilon}{\varepsilon^{2\gamma}}\right\|_{L^1(0,T;L^1(\mathbb{R}^{2d}))}&\leq2\varepsilon^{1-\gamma}E_0+\left(2dT+F_0^2\right).
\end{aligned}$$

In addition, the next bound also holds

$$\frac{1}{\varepsilon^{2\gamma}}\int_0^T\int_{\mathbb{R}^{4d}}\phi_\varepsilon(|x-y|)|v-w|^2 f_\varepsilon(t,x,v)f_\varepsilon(t,y,w)\,dx\,dy\,dv\,dw\,dt\leq 2\varepsilon^{1-\gamma}E_0+\left(2dT+F_0^2\right).$$

By virtue of the preceding Corollary 2.4.2, we can pass to the limit again in the linear terms of the weak form of Equation (2.4.3). The only term that is not completely clear is the nonlinear term again. To identify the limit as $\varepsilon\to 0$, we proceed as in Section 2.2. Indeed, the above a priori estimates in Corollary 2.4.2 and the ideas in Appendix A imply

$$\begin{aligned}\rho_\varepsilon &\overset{*}{\rightharpoonup}\rho, \quad \text{in } L^\infty_w(0,T;\mathcal{M}(\mathbb{R}^d)),\\ \frac{j_\varepsilon}{\varepsilon^\gamma} &\overset{*}{\rightharpoonup} j, \quad \text{in } L^2_w(0,T;\mathcal{M}(\mathbb{R}^d))^d.\end{aligned}$$

Again, the continuity equation provides us with some extra compactness with respect to time and the ideas in Appendix B (see also Theorem 2.2.12) amount to

$$\rho_\varepsilon\to\rho, \quad \text{in } C([0,T];\mathcal{M}(\mathbb{R}^d)-\text{narrow}),$$

that ensure the convergence of the tensor product of both measures

$$\rho_\varepsilon\otimes\frac{j_\varepsilon}{\varepsilon^\gamma}\overset{*}{\rightharpoonup}\rho\otimes j, \ \text{ in } L^2_w(0,T;\mathcal{M}(\mathbb{R}^{2d}))^d.$$

Consequently, we recover an analogous convergence result to Corollary 2.2.17 for any exponent $\alpha\in(0,\frac{1}{2}]$ and the same limiting macroscopic system (2.2.20), hence (2.1.1) in the Introduction.

Remark 2.4.3. *Let us recall here that the intermediate scaling in Equation (2.4.1) with $\gamma\in(0,1]$ that we have proposed here is different from the one that was considered in [22, 252] for the the Vlassov–Poisson–Fokker–Planck system. Specifically, in* (2.4.1)*, the velocity diffusion is of lower order of ε and each term in the Fokker–Planck differential operator is scaled in a different way, see Appendix 2.A. This implies that the parabolic case $\gamma=1$ in [252] does not enjoy any bound of the scaled average kinetic energy like in Corollary 2.4.2 for our scaled system. Consequently, in the case $\gamma=1$ in [252] the term $\operatorname{div}_x\mathcal{S}_\varepsilon$ does not have any scaling factor of ε in front and the authors could not show converge towards zero of such term. Indeed, the authors showed in [252] that, in the sense of distributions,*

$$\operatorname{div}_x\mathcal{S}_\varepsilon\rightharpoonup\nabla_x\rho, \ \text{ in } \mathcal{D}^*((0,T)\times\mathbb{R}^d).$$

As a consequence, an extra diffusive term appears in the continuity equation for such scaling of the Vlasov–Poisson–Fokker–Planck system, but it does no longer happen in our particular parabolic-type scaling (2.4.1) of the kinetic Cucker–Smale model.

To conclude, we formally recover an analogue to Theorem 2.2.1 to exhibit the expected shape for the limiting distribution f.

Theorem 2.4.4. *Let f_ε solve the scaled system* (2.4.1)*. Then, f_ε asymptotically behaves as follows*

$$f_\varepsilon\sim\frac{\rho_\varepsilon}{(2\pi k_BT_\varepsilon)^{d/2}}\frac{1}{\varepsilon^{\gamma d}}\exp\left(-\frac{\left|(1+\phi_\varepsilon*\rho_\varepsilon)\frac{v}{\varepsilon^\gamma}-\phi_\varepsilon*\frac{j_\varepsilon}{\varepsilon^\gamma}+\nabla_x\psi_\varepsilon\right|^2}{2k_BT_\varepsilon}\right),$$

as $\varepsilon\to 0$, where $T_\varepsilon:=\frac{1}{k_B(1+\phi_\varepsilon*\rho_\varepsilon)}$ is the thermodynamic temperature of f_ε and k_B stands for Boltzmann's constant. In addition, consider a weak- limit in the sense of distributions $f(t,x,v)=$ weak $*-\lim_{\varepsilon\to 0}f(t,x,v)$, then f agrees with the monokinetic distribution*

$$f(t,x,v)=\frac{\rho(t,x)}{(2\pi k_BT(t,x))^{d/2}}\delta_{u(t,x)}(v),$$

where ρ and u evolve according to (2.1.1).

2.4.2 Hydrodynamic limit in the frictionless case

In the frictionless case, the same technique cannot provide us with the desired a priori estimates since the friction term is missing in the system. Consequently, one has to rely on estimates that arise from the inertial terms of the system. To this end, the appropriate choice of the scaling is some hyperbolic scale where the velocity diffusion and external force are of low order of ε. Specifically, we will consider the scaled system (2.A.6) in Appendix 2.A, i.e.,

$$\frac{\partial f_\varepsilon}{\partial t} + v \cdot \nabla_x f_\varepsilon - \nabla_x \psi_\varepsilon \cdot \nabla_v f_\varepsilon = \operatorname{div}_v(\nabla_v f_\varepsilon) + \frac{1}{\varepsilon} \operatorname{div}_v(f_\varepsilon(\phi_\varepsilon * \rho_\varepsilon)v - f_\varepsilon \phi_\varepsilon * j_\varepsilon). \tag{2.4.5}$$

The associated hierarchy of velocity moments then takes the form:

• **Mass conservation**

$$\frac{\partial \rho_\varepsilon}{\partial t} + \operatorname{div}_x j_\varepsilon = 0. \tag{2.4.6}$$

• **Law of balance of current**

$$\varepsilon \frac{\partial j_\varepsilon}{\partial t} + \varepsilon \operatorname{div}_x \mathcal{S}_\varepsilon + \varepsilon \rho_\varepsilon \nabla_x \psi_\varepsilon + (\phi_\varepsilon * \rho_\varepsilon)\, j_\varepsilon - (\phi_\varepsilon * j_\varepsilon)\, \rho_\varepsilon = 0. \tag{2.4.7}$$

• **Law of balance of stress**

$$\varepsilon \frac{\partial \mathcal{S}_\varepsilon}{\partial t} + \varepsilon \operatorname{div}_x \mathcal{T}_\varepsilon + 2\varepsilon \operatorname{Sym}(j_\varepsilon \otimes \nabla_x \psi_\varepsilon) + 2\left((\phi_\varepsilon * \rho_\varepsilon)\, \mathcal{S}_\varepsilon - \varepsilon \rho_\varepsilon I\right) - 2\operatorname{Sym}((\phi_\varepsilon * j_\varepsilon) \otimes j_\varepsilon) = 0. \tag{2.4.8}$$

First, let us obtain some estimates for the moments of f_ε in the same spirit as Proposition 2.2.5. Straightforward computations that are identical to that in the above-mentioned propositions yield the next analogue.

Proposition 2.4.5. *Let the initial distribution functions f_ε^0 verify (2.2.5) and consider the strong global in time solution f_ε to (2.4.1) with initial data f_ε^0 and $\alpha \in (0, \frac{d}{2})$. Consider any $k \in \mathbb{N}$. Then,*

$$\begin{aligned}
\varepsilon \frac{d}{dt} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |v|^k f_\varepsilon \, dx \, dv = & - k\varepsilon \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |v|^{k-2} v \cdot \nabla_x \psi_\varepsilon f_\varepsilon \, dx \, dv \\
& + k(d+k-2)\varepsilon \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |v|^{k-2} f_\varepsilon \, dx \, dv \\
& - k \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |v|^{k-1} \left((\phi_\varepsilon * \rho_\varepsilon)|v| - (\phi_\varepsilon * j_\varepsilon) \cdot \frac{v}{|v|} \right) f_\varepsilon \, dx \, dv \\
\frac{d}{dt} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |x|^k f_\varepsilon \, dx \, dv = & \quad k \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |x|^{k-2} x \cdot v f_\varepsilon \, dx \, dv.
\end{aligned}$$

Note that in particular we recover the conservation of mass, namely,

$$\rho_\varepsilon(t, \cdot) \in \mathcal{P}(\mathbb{R}^d) \text{ for all } t \geq 0, \ \varepsilon > 0.$$

As a direct consequence, taking $k = 2$ and $k = 1$ respectively entails the next analogue of Corollary 2.2.6.

Corollary 2.4.6. *Let the initial distribution functions f_ε^0 verify (2.2.5) and the forcing terms $-\nabla \psi_\varepsilon$ be merely bounded in $L^2(0, T; L^\infty(\mathbb{R}^d))^d$. Consider the strong global in time solution f_ε to (2.4.1) with initial data f_ε^0 and $\alpha \in (0, \frac{d}{2})$. Then,*

$$\| |x| f_\varepsilon \|_{L^\infty(0,T; L^1(\mathbb{R}^d))} \leq M_0 + T \left(4E_0 + 4dT + 4F_0^2 \right)^{1/2},$$

$$\| |v| f_\varepsilon \|_{L^\infty(0,T;L^1(\mathbb{R}^{2d}))} \leq \left(4E_0 + 4dT + 4F_0^2\right)^{1/2},$$
$$\frac{1}{2} \| |v|^2 f_\varepsilon \|_{L^\infty(0,T;L^1(\mathbb{R}^{2d}))} \leq 2E_0 + 2dT + 2F_0^2.$$

In addition, the next bound also holds

$$\frac{1}{\varepsilon^{2\alpha}} \int_0^T \int_{\mathbb{R}^{4d}} \phi_\varepsilon(|x-y|)|v-w|^2 f_\varepsilon(t,x,v) f_\varepsilon(t,y,w)\, dx\, dy\, dv\, dw\, dt$$
$$\leq 2\varepsilon^{1-2\alpha} E_0 + \varepsilon^{1-2\alpha}\left(2dT + 2F_0^2\right).$$

Again, the preceding estimates can be arranged to show that, up to subsequence, we get

$$\rho_\varepsilon \otimes j_\varepsilon \overset{*}{\rightharpoonup} \rho \otimes j \ \text{ in } \ L^2_w(0,T;\mathcal{M}(\mathbb{R}^{2d}))^d.$$

The same reasoning as in Section 2.2, where one distinguishes again the regimes $\alpha \in (0, \frac{1}{2})$ and $\alpha = \frac{1}{2}$ because of the concentration issues, yields the next result.

Corollary 2.4.7. *Let f_ε^0 and $\nabla\psi_\varepsilon$ satisfy hypothesis (2.2.5), the forcing terms $-\nabla\psi_\varepsilon$ be merely bounded in $L^2(0,T;L^\infty(\mathbb{R}^d))^d$, and consider a sequence f_ε of smooth solutions to (2.1.7) with $\alpha \in (0, \frac{1}{2}]$. Then, the macroscopic quantities ρ_ε and j_ε satisfy*

$$\rho_\varepsilon \to \rho, \quad \text{in } C([0,T];\mathcal{M}(\mathbb{R}^d) - \text{narrow}),$$
$$j_\varepsilon \overset{*}{\rightharpoonup} j, \quad \text{in } L^\infty_w(0,T;\mathcal{M}(\mathbb{R}^d))^d,$$

when $\varepsilon \searrow 0$, for some probability measure ρ, some finite Radon measure j and some subsequences of $\{\rho_\varepsilon\}_{\varepsilon>0}$ and $\{j_\varepsilon\}_{\varepsilon>0}$ that we denote in the same way. In addition (ρ, j) is a weak measure-valued solution to the Cauchy problem associated with the following Euler-type system in the sense of distribution

$$\left\{\begin{array}{ll} \partial_t \rho + \operatorname{div} j = 0, & x \in \mathbb{R}^d,\ t \in [0,T), \\ 0 = (\phi_0 * j)\rho - (\phi_0 * \rho) j, & x \in \mathbb{R}^d,\ t \in [0,T), \\ \rho(0,\cdot) = \rho^0, & x \in \mathbb{R}^d. \end{array}\right. \tag{2.4.9}$$

Remark 2.4.8. *In this particular case, the last estimate in Corollary 2.4.6 provides an improved bound that allows quantifying convergence to zero of the commutator term, namely, strong convergence (not only in the sense of distributions) for the range $\alpha \in (0, \frac{1}{2})$. Specifically, notice that the Cauchy–Schwartz inequality, the obvious inequality $\phi_\varepsilon \leq \frac{1}{\varepsilon^\alpha}\phi_\varepsilon^{1/2}$ and Corollary 2.4.6 entail*

$$\|(\phi_\varepsilon * \rho_\varepsilon) j_\varepsilon - (\phi_\varepsilon * j_\varepsilon)\rho_\varepsilon\|_{L^2(0,T;L^1(\mathbb{R}^{2d}))}$$
$$\leq \left(\frac{1}{\varepsilon^{2\alpha}} \int_0^T \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \phi_\varepsilon(|x-y|)|v-w|^2 f_\varepsilon(t,x,v) f_\varepsilon(t,y,w)\, dx\, dy\, dv\, dw\, dt\right)^{1/2} \|f_\varepsilon^0\|_{L^1(\mathbb{R}^{2d})}$$
$$\leq \varepsilon^{\frac{1}{2}-\alpha}\left(2E_0 + 2dT + 2F_0^2\right)^{1/2}.$$

In particular, since $\alpha < \frac{1}{2}$ then

$$\lim_{\varepsilon \to 0} \|(\phi_\varepsilon * \rho_\varepsilon) j_\varepsilon - (\phi_\varepsilon * j_\varepsilon)\rho_\varepsilon\|_{L^2(0,T;L^1(\mathbb{R}^{2d}))} = 0.$$

2.4.3 Rayleigh–Helmholtz friction towards linear friction

As it has be shown in Section 2.3 in the analysis of the limiting equation (2.1.1), the classical friction in the Fokker–Planck differential operator prevents the individuals from the desired self-propelled behavior since it aims at reducing the particles' velocity to zero. Then, classical linear friction tends to halt the dynamics of the individuals unless we include an external force given by a potential $\psi = \psi(t, x)$ (see Section 2.2 and Subsection 2.4.1) or we neglect friction effects (see Subsection 2.4.2). Depending on what are we modeling it might (or not) make sense. Actually, it is the nature of the environment where agents live that determines the kind of friction to be considered. For instance, assume that individuals live inside some viscous material so that its velocity decreases proportionally to the velocity itself at a constant rate μ. In such case, the microscopic Langevin equation reads

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, & t \geq 0, \\ \dfrac{dv_i}{dt} = -\mu v + \dfrac{1}{n} \displaystyle\sum_{j \neq i} \phi(|x_i - x_j|)(v_j - v_i) + \sqrt{2D}\xi_i(t), & t \geq 0, \end{cases}$$

and the kinetic description takes the form

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \mathrm{div}_v(\mu v f + D\nabla_v f + Q^{\phi}_{CS}(f, f)),$$

which is the kinetic model that we have focused on so far. Naturally, such velocity damping comes imposed by the medium. However, the individuals might slow down its velocity according to any v-dependent friction coefficient $\mu = \mu(v)$. A particularly interesting choice by its consequences in the modeling of self-propelled behavior is the *Rayleigh–Helmholtz friction*, see [66, 259]. It arose from the theory of sound developed by Rayleigh and Helmholtz and takes the form

$$\mu(v) := \beta|v|^2 - \delta.$$

Note that, it consists of a decrease term, $-\beta|v|^2 v$, in the spirit of the classical linear damping and an increase term, $+\delta v$, that can be understood as an intrinsic self-propulsion of individuals to surpass the medium natural friction. Such competition leads to a natural asymptotic velocity $\sqrt{\delta/\beta}$. To understand it, let us forget about the interactions and stochastic effects and restrict ourselves to one single particle and one spacial dimension. This gives rise to the next first order scalar ODEs

$$\frac{dx}{dt} = -\mu v, \qquad \frac{dv}{dt} = (\delta - \beta v^2)v.$$

On the one hand, the former linear damping has only one velocity equilibrium, namely $v = 0$, that is asymptotically stable. On the second hand, the latter Rayleigh–Helmholtz friction enjoys three different equilibria, namely,

$$v = -\sqrt{\delta/\beta}, \quad v = 0 \text{ and } v = \sqrt{\delta/\beta}.$$

Here, $v = 0$ is unstable whilst the remaining two equilibria are asymptotically stable. Consequently, the velocities evolve as depicted in Figure 2.1. Note that when the initial velocity is under the threshold one, the self-propulsion term helps to achieve such asymptotic velocity. Similarly, when the initial velocity is over the threshold one, the medium friction slow down agent's velocity until it relaxes towards the asymptotic one. In particular, note that whenever $\delta \neq 0$ the asymptotic velocity of the particle is no longer zero, which is a much more realistic property to model flocking behavior of birds under the influence of some friction arising from

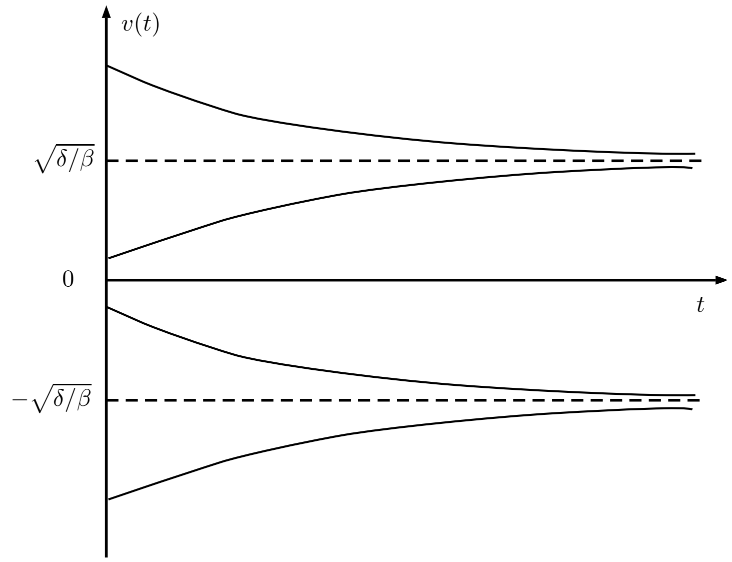


Figure 2.1: Velocity tendency under the Rayleigh–Helmholtz friction equation

the medium. Hence, why don't we study the same kind of hydrodynamic limit as above when one considers alignment interactions along with Rayleigh–Helmholtz-type friction terms, e.g.,

$$\varepsilon^{1+\gamma}\frac{\partial f_\varepsilon}{\partial t}+\varepsilon v\cdot\nabla_x f-\varepsilon^\gamma\nabla_x\psi_\varepsilon\cdot\nabla_v f_\varepsilon=\operatorname{div}_v((\beta|v|^2-\delta)vf+\varepsilon^{2\gamma}\nabla_v f_\varepsilon+Q_{CS}^{\phi_\varepsilon}(f_\varepsilon,f_\varepsilon))?$$

The reason becomes apparent when we compute the hierarchy of moments and try to obtain the same type of estimates as in the preceding sections. Specifically, it is clear that one can obtain bounds for velocity moments of high enough order; however, it is not clear at all how the hydrodynamic limit might help on closing the hierarchy of macroscopic equations of moments since in the equation of any velocity moment involves twice higher order velocity moment (that cannot be shown to converge to zero, but only to remain bounded).

Remark 2.4.9. *For completeness of the above exposition, we recall here a related nonlinear damping (that will not be studied here). Specifically, we can recover the same sort of behavior and self-propulsion effects with a first order nonlinear friction term like*

$$-\mu(v)v=-\gamma_0(|v|-v_0)\frac{v}{|v|},$$

for $\gamma_0, v_0>0$. This is known as the Schienbein–Gruler friction (see [259] and references therein, where its is proposed as a linearization of the above-mentioned Rayleigh–Helmholtz friction in the theory of active Brownian motions). It also arises in the modeling of certain type of cell of granulocyte type. As it is apparent, the velocity v_0 is the asymptotic one that the system tends to achieve. However, from a mathematical point of view, such nonlinear friction term behaves worse due to the obvious discontinuity at $v=0$. From a dynamical point of view, this friction term can also be compared with the classical linear friction in the limit $|v|\to+\infty$.

In this subsection we will compare the first two types of friction (classical and Rayleigh–Helmholtz) by considering a family of intermediate terms that, in the limit $\varepsilon\to 0$, approach the classical friction case. Such intermediate frictions take the form

$$\mu(v)=\beta|v|^k-\delta,$$

for $0 \leq k \leq 2$ and $\delta, \beta \geq 0$. Notice that $k = 0$ and $\delta = 0$ yields the classical linear friction whilst $k = 2$ and $\delta > 0$ represents Rayleigh–Helmholtz. To summarize, the kind of kinetic equations that we will deal with takes the form

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f - \nabla_x \psi \cdot \nabla_v f = \operatorname{div}_v((\beta|v|^k - \delta)vf + D\nabla_v f + Q_{CS}^{\phi}(f, f)).$$

First note that the asymptotic velocity of the friction term is now $(\delta/\beta)^{1/k}$ and it converges towards zero in the limit $\delta \to 0$ and $k \to 0$. A dimensional analysis like in Appendix 2.A allows introducing the next intermediate scaling of the system

$$\varepsilon^{1+\gamma}\partial_t f_\varepsilon + \varepsilon v \cdot \nabla_x f_\varepsilon - \varepsilon^\gamma \nabla_x \psi_\varepsilon \cdot \nabla_v f_\varepsilon = \operatorname{div}_v \left((|v|^{k(\varepsilon)} - \delta(\varepsilon))v f_\varepsilon + \varepsilon^{2\gamma} \nabla_v f_\varepsilon + Q_{CS}^{\phi_\varepsilon}(f_\varepsilon, f_\varepsilon) \right), \tag{2.4.10}$$

for some functions $k = k(\varepsilon)$ and $\delta = \delta(\varepsilon)$ such that $k(\varepsilon) \searrow 0$ and $\delta(\varepsilon) \searrow 0$ when $\varepsilon \searrow 0$. In the formal limit $\varepsilon \to 0$, we expect to recover the same hydrodynamic limit as with a fixed classical friction, i.e., Equation (2.1.1). As in the previous case, we refer to [44, 68, 162] for the combination of ideas to deal with the analysis of existence for solutions to this system. Let us sketch how does the hierarchy of velocity moments looks like and how can we obtain similar estimates to rigorously passing to the limit.

Hierarchy of moments

In addition to the velocity moments $\rho_\varepsilon, j_\varepsilon, \mathcal{S}_\varepsilon, \mathcal{T}_\varepsilon$, let us define

$$q_\varepsilon^{k(\varepsilon)+1} := \int_{\mathbb{R}^d} |v|^{k(\varepsilon)} v f_\varepsilon \, dv,$$
$$\mathcal{Q}_\varepsilon^{k(\varepsilon)+2} := \int_{\mathbb{R}^d} |v|^{k(\varepsilon)} v \otimes v f_\varepsilon \, dv.$$

Then, the first three velocity moments read as follows:

- **Mass conservation**

$$\frac{\partial \rho_\varepsilon}{\partial t} + \operatorname{div}_x \left(\frac{j_\varepsilon}{\varepsilon^\gamma} \right) = 0. \tag{2.4.11}$$

- **Law of balance of current**

$$\varepsilon^{1+\gamma} \frac{\partial j_\varepsilon}{\partial t} + \varepsilon \operatorname{div}_x \mathcal{S}_\varepsilon + \varepsilon^\gamma \rho_\varepsilon \nabla_x \psi_\varepsilon + (q_\varepsilon^{k(\varepsilon)+1} - \delta(\varepsilon) j_\varepsilon) + (\phi_\varepsilon * \rho_\varepsilon) j_\varepsilon - (\phi_\varepsilon * j_\varepsilon) \rho_\varepsilon = 0. \tag{2.4.12}$$

- **Law of balance of stress**

$$\begin{aligned} \varepsilon^{1+\gamma} \frac{\partial \mathcal{S}_\varepsilon}{\partial t} &+ \varepsilon \operatorname{div}_x \mathcal{T}_\varepsilon + 2\varepsilon^\gamma \operatorname{Sym}(j_\varepsilon \otimes \nabla_x \psi_\varepsilon) + 2(\mathcal{Q}_\varepsilon^{k(\varepsilon)+2} - \delta(\varepsilon)\mathcal{S}_\varepsilon) \\ &+ 2\left((\phi_\varepsilon * \rho_\varepsilon) \mathcal{S}_\varepsilon - \varepsilon^{2\gamma} \rho_\varepsilon I \right) - 2 \operatorname{Sym}((\phi_\varepsilon * j_\varepsilon) \otimes j_\varepsilon) = 0. \end{aligned} \tag{2.4.13}$$

A priori bounds

We will need some hypothesis on the coefficients $k(\varepsilon)$ and $\delta(\varepsilon)$ in order to obtain appropriate a priori bounds. On the one hand, in the hyperbolic case, i.e., $\gamma = 0$ we will assume that

$$k(\varepsilon) = o(1) \quad \text{and} \quad \delta(\varepsilon) = o(1) \qquad \text{when} \quad \varepsilon \to 0. \tag{2.4.14}$$

On the other hand in the purely intermediate or parabolic cases $\gamma \in (0,1]$ we will assume

$$k(\varepsilon) = O(\varepsilon^{2\gamma}),\ \ k'(\varepsilon) = O(\varepsilon^{-(1-\gamma)}) \ \text{ and } \ \delta(\varepsilon) = o(1) \quad \text{when } \ \varepsilon \to 0. \tag{2.4.15}$$

For the sake of simplicity, we first present the next lemma that will help us to control the velocity moments of order 1, 2 and $k(\varepsilon)+1$ in terms of that of order $k(\varepsilon)+2$.

Lemma 2.4.10. *Let the initial distribution functions f_ε^0 verify (2.2.5), the external forces $-\nabla\psi_\varepsilon$ fulfill (2.2.6) and consider the strong global in time solution f_ε to (2.4.10) with initial data f_ε^0 and $\alpha \in (0, \frac{d}{2})$. Let us also define the exponents*

$$p_\varepsilon := 1 + \frac{k(\varepsilon)}{2},\quad q_\varepsilon := 2 + k(\varepsilon),\quad r_\varepsilon := 2 - \frac{k(\varepsilon)}{1+k(\varepsilon)}$$

Then, the next estimates hold true

$$\begin{aligned}
\||v|^2 f_\varepsilon\|_{L^{p_\varepsilon}(0,T;L^1(\mathbb{R}^{2d}))} &\leq \||v|^{k(\varepsilon)+2} f_\varepsilon\|_{L^1(0,T;L^1(\mathbb{R}^{2d}))}^{1/p_\varepsilon},\\
\||v| f_\varepsilon\|_{L^{q_\varepsilon}(0,T;L^1(\mathbb{R}^{2d}))} &\leq \||v|^{k(\varepsilon)+2} f_\varepsilon\|_{L^1(0,T;L^1(\mathbb{R}^{2d}))}^{1/q_\varepsilon},\\
\||v|^{k(\varepsilon)+1} f_\varepsilon\|_{L^{r_\varepsilon}(0,T;L^1(\mathbb{R}^{2d}))} &\leq \||v|^{k(\varepsilon)+2} f_\varepsilon\|_{L^1(0,T;L^1(\mathbb{R}^{2d}))}^{1/r_\varepsilon}.
\end{aligned}$$

In particular, Hölder's inequality together with Young's inequality for real numbers imply

$$\begin{aligned}
\||v|^2 f_\varepsilon\|_{L^1(0,T;L^1(\mathbb{R}^{2d}))} &\leq \||v|^{k(\varepsilon)+2} f_\varepsilon\|_{L^1(0,T;L^1(\mathbb{R}^{2d}))}^{1/p_\varepsilon} T^{1/p_\varepsilon'}\\
&\leq \frac{2}{2+k(\varepsilon)}\||v|^{k(\varepsilon)+2} f_\varepsilon\|_{L^1(0,T;L^1(\mathbb{R}^{2d}))} + \frac{k(\varepsilon)}{2+k(\varepsilon)}T,\\
\||v| f_\varepsilon\|_{L^2(0,T;L^1(\mathbb{R}^{2d}))} &\leq \||v|^{k(\varepsilon)+2} f_\varepsilon\|_{L^1(0,T;L^1(\mathbb{R}^{2d}))}^{1/q_\varepsilon} T^{\frac{k(\varepsilon)}{2(k(\varepsilon)+2)}}\\
&= \left(\||v|^{k(\varepsilon)+2} f_\varepsilon\|_{L^1(0,T;L^1(\mathbb{R}^{2d}))}^{1/p_\varepsilon} T^{1/p_\varepsilon'}\right)^{1/2}\\
&\leq \left(\frac{2}{2+k(\varepsilon)}\||v|^{k(\varepsilon)+2} f_\varepsilon\|_{L^1(0,T;L^1(\mathbb{R}^{2d}))} + \frac{k(\varepsilon)}{2+k(\varepsilon)}T\right)^{1/2}.
\end{aligned}$$

Remark 2.4.11. *Note that we have not considered neither a L^1 norm nor a L^2 norm with respect to time for the last term $\||v|^{k(\varepsilon)+1} f_\varepsilon\|_{L^{r_\varepsilon}(0,T;L^1(\mathbb{R}^{2N}))}$. The reason is twofold.*

1. *First, the exponent $r_\varepsilon \nearrow 2$ as $\varepsilon \searrow 0$. Consequently, the natural L^2 norm cannot be achieved for each fixed ε but in the limit $\varepsilon \to 0$.*

2. *Second, one could have obtained a L^1 norm with respect to time in the same spirit as in the second order velocity moment since $r_\varepsilon > 1$ for small enough ε. However, if we do so, then the coefficient in the Young inequality that comes before $\|f_\varepsilon^0\|_{L^1(\mathbb{R}^{2d})}$ would be $1/(2+k(\varepsilon))$. Although it is obviously bounded with respect to ε, it does not vanish in the limit $\varepsilon \to 0$ and it is a serious obstruction in order to obtain bounds for the needed scaled velocity moments.*

Proposition 2.4.12. *Let the initial distribution functions f_ε^0 verify (2.2.5), the external forces $-\nabla\psi_\varepsilon$ fulfil (2.2.6) and consider the strong global in time solution f_ε to (2.4.10) with initial data f_ε^0 and $\alpha \in (0, \frac{d}{2})$. Then,*

$$\frac{1}{\varepsilon^{2\gamma}}\int_0^T\int_{\mathbb{R}^{2d}} |v|^{k(\varepsilon)+2} f_\varepsilon\,dx\,dv\,dt + \frac{1}{\varepsilon^{2\gamma}}\int_0^T\int_{\mathbb{R}^{4d}} \phi_\varepsilon(|x-y|)|v-w|^2 f_\varepsilon(t,x,v) f_\varepsilon(t,y,w)\,dx\,dy\,dv\,dw$$

$$\leq \frac{\delta(\varepsilon)}{\varepsilon^{2\gamma}} \int_0^T \int_{\mathbb{R}^{2d}} |v|^2 f_\varepsilon \, dx \, dv \, dt + \varepsilon^{1-\gamma} \int_{\mathbb{R}^{2d}} |v|^2 f_\varepsilon(0) \, dx \, dv$$
$$- \frac{1}{\varepsilon^\gamma} \int_0^T \int_{\mathbb{R}^{2d}} v \cdot \nabla_x \psi_\varepsilon \, f_\varepsilon \, dx \, dv \, dt + T \int_{\mathbb{R}^{2d}} f_\varepsilon^0 \, dx \, dv.$$

Now, one can apply apply the estimates of the L^1 and L^2 norms of the second and first order velocity moments above to obtain the next inequality.

Corollary 2.4.13. *Under the assumptions of Proposition 2.4.12, the next property holds*

$$\left[1 - \left(\delta(\varepsilon) + \frac{1}{2}\right) \frac{2}{2+k(\varepsilon)}\right] \frac{1}{\varepsilon^{2\gamma}} \| |v|^{k(\varepsilon)+2} f_\varepsilon \|_{L^1(0,T;L^1(\mathbb{R}^{2d}))}$$
$$+ \frac{1}{\varepsilon^{2\gamma}} \int_0^T \int_{\mathbb{R}^{4d}} \phi_\varepsilon(|x-y|) |v-w|^2 f_\varepsilon(t,x,v) f_\varepsilon(t,y,w) \, dx \, dy \, dv \, dw$$
$$\leq 2\varepsilon^{1-\gamma} E_0 + \left[\frac{1}{\varepsilon^{2\gamma}} \left(\alpha(\varepsilon) + \frac{1}{2}\right) \frac{k(\varepsilon)}{2+k(\varepsilon)} + 1\right] T + \frac{1}{2} F_0^2.$$

For ε small enough one can obtain a lower estimate of the first factor as follows

$$\left[1 - \left(\delta(\varepsilon) + \frac{1}{2}\right) \frac{2}{2+k(\varepsilon)}\right] \geq \frac{1}{4}.$$

Consequently, the preceding results yield the next list of estimates for the scaled velocity moments under consideration.

Corollary 2.4.14. *Let the initial distribution functions f_ε^0 verify (2.2.5), the external forces $-\nabla \psi_\varepsilon$ fulfill (2.2.6), consider the strong global in time solution f_ε to (2.4.10) with initial data f_ε^0 and $\alpha \in (0, d/2)$ and assume the hypothesis (2.4.14)–(2.4.15). Then, there exists some constant $C > 0$ that does not depend on ε such that*

$$\frac{1}{\varepsilon^{2\gamma}} \| |v|^{k(\varepsilon)+2} f_\varepsilon \|_{L^1(0,T;L^1(\mathbb{R}^{2d}))} \leq C,$$
$$\frac{1}{\varepsilon^{\gamma}} \| |v|^{k(\varepsilon)+1} f_\varepsilon \|_{L^{r_\varepsilon}(0,T;L^1(\mathbb{R}^{2d})} \leq C,$$
$$\frac{1}{\varepsilon^{2\gamma}} \| |v|^2 f_\varepsilon \|_{L^1(0,T;L^1(\mathbb{R}^{2d}))} \leq C,$$
$$\frac{1}{\varepsilon^{\gamma}} \| |v| f_\varepsilon \|_{L^2(0,T;L^1(\mathbb{R}^{2d}))} \leq C,$$
$$\| |x| f_\varepsilon \|_{L^\infty}(0,T; L^1(\mathbb{R}^d)) \leq C,$$

and

$$\frac{1}{\varepsilon^{2\gamma}} \int_0^T \int_{\mathbb{R}^{4d}} \phi_\varepsilon(|x-y|) |v-w|^2 f_\varepsilon(t,x,v) f_\varepsilon(t,y,w) \, dx \, dy \, dv \, dw \leq C,$$

where r_ε agrees with the exponent in Lemma 2.4.10.

Proof. All the estimates obviously follows from Lemma 2.4.10 and Corollary 2.4.13. Let us just sketch the proof of the second one which is less apparent according to the moment relations in Lemma 2.4.10. Such result shows that

$$\frac{1}{\varepsilon^\gamma} \| |v|^{k(\varepsilon)+1} f_\varepsilon \|_{L^{r_\varepsilon}(0,T;L^1(\mathbb{R}^{2d})} \leq \frac{\varepsilon^{2\gamma/r_\varepsilon}}{\varepsilon^\gamma} \left(\frac{1}{\varepsilon^{2\gamma}} \| |v|^{k(\varepsilon)+2} f_\varepsilon \|_{L^1(0,T;L^1(\mathbb{R}^{2d})} \right)^{1/r_\varepsilon}.$$

Now, one can apply the Young inequality for real numbers once more to obtain the next bound

$$\frac{1}{\varepsilon^\gamma}\||v|^{k(\varepsilon)+1}f_\varepsilon\|_{L^{r_\varepsilon}(0,T;L^1(\mathbb{R}^{2N})}\leq \varepsilon^{\gamma\frac{k(\varepsilon)}{1+k(\varepsilon)}}\left\{\frac{1+k(\varepsilon)}{2+k(\varepsilon)}\frac{1}{\varepsilon^{2\gamma}}\||v|^{k(\varepsilon)+2}f_\varepsilon\|_{L^1(0,T;L^1(\mathbb{R}^{2N}))}+\frac{1}{2+k(\varepsilon)}\right\}.$$

Note now that

$$\varepsilon^{\gamma\frac{k(\varepsilon)}{1+k(\varepsilon)}}=\exp\left(\frac{\gamma}{1+k(\varepsilon)}k(\varepsilon)\log\varepsilon\right)\to 1\ \text{ when } \varepsilon\to 0,$$

because we are assuming $k=O(\varepsilon^{2\gamma})$. In particular, note that one cannot say that such scaled momentum converges to zero as $\varepsilon\to 0$ because the above factor, although bounded, does not converges to zero. □

The above results show that we again can consider weak-* limits for ρ_ε and $\frac{j_\varepsilon}{\varepsilon^\gamma}$ by virtue of Appendix A to obtain

$$\rho_\varepsilon \overset{*}{\rightharpoonup} \rho \ \text{ in } \ L^\infty_w(0,T;\mathcal{M}(\mathbb{R}^d)),$$
$$\frac{j_\varepsilon}{\varepsilon^\gamma} \overset{*}{\rightharpoonup} j \ \text{ in } \ L^2_w(0,T;\mathcal{M}(\mathbb{R}^d))^d,$$

thus leading to the limiting continuity equation in the sense of distributions

$$\frac{\partial\rho}{\partial t}+\operatorname{div} j=0.$$

Indeed, one can repeat the same idea as in the preceding sections to ensure (thanks to the continuity equation (2.4.11) and Appendix B) that

$$\rho_\varepsilon\to\rho \ \text{ in } \ C([0,T],\mathcal{M}(\mathbb{R}^d)-\text{narrow}).$$

In particular, $\rho(0,\cdot)=\rho^0$. Recall that the balance law for the scaled current now takes the form

$$\varepsilon^{1+\gamma}\frac{\partial}{\partial t}\left(\frac{j_\varepsilon}{\varepsilon^\gamma}\right)+\varepsilon^{1-\gamma}\operatorname{div}_x\mathcal{S}_\varepsilon+\rho_\varepsilon\nabla_x\psi_\varepsilon-\left(\frac{1}{\varepsilon^\gamma}q_\varepsilon^{k(\varepsilon)+1}-\delta(\varepsilon)\frac{j_\varepsilon}{\varepsilon^\gamma}\right)+(\phi_\varepsilon*\rho_\varepsilon)\frac{j_\varepsilon}{\varepsilon^\gamma}-\left(\phi_\varepsilon*\frac{j_\varepsilon}{\varepsilon^\gamma}\right)\rho_\varepsilon=0,$$

or, in weak form,

$$\begin{aligned}&-\varepsilon^{1+\gamma}\int_0^T\int_{\mathbb{R}^d}\frac{\partial\varphi}{\partial t}\cdot\frac{j_\varepsilon}{\varepsilon^\gamma}\,dx\,dt-\varepsilon^{1-\gamma}\int_0^T\int_{\mathbb{R}^d}\operatorname{Jac}\varphi:\mathcal{S}_\varepsilon\,dx\,dt\\&\quad+\int_0^T\int_{\mathbb{R}^d}\rho_\varepsilon\nabla_x\psi_\varepsilon\cdot\varphi\,dx\,dt-\int_0^T\int_{\mathbb{R}^d}\frac{q_\varepsilon^{k(\varepsilon)+1}}{\varepsilon^\gamma}\cdot\varphi\,dx\,dt+\delta(\varepsilon)\int_0^T\int_{\mathbb{R}^d}\frac{j_\varepsilon}{\varepsilon^\gamma}\cdot\varphi\,dx\,dt\\&\qquad-\frac{1}{2}\int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}H_\varphi^{\alpha,\varepsilon}(t,x,y)\cdot\left(\rho_\varepsilon(t,x)\frac{j_\varepsilon}{\varepsilon^\gamma}(t,y)-\rho_\varepsilon(t,y)\frac{j_\varepsilon}{\varepsilon^\gamma}(t,y)\right)dx\,dy\,dt=0,\end{aligned}$$

for any $\varphi\in C_c^\infty([0,T];C_0^1(\mathbb{R}^d,\mathbb{R}^d))$. The same arguments as before guarantee that we can pass to the limit in each term (including the commutator) by virtue of Corollary 2.4.14. Recall that it holds for all the parameters in the range $\alpha\in(0,\frac{1}{2}]$. Again, the endpoint case $\alpha=\frac{1}{2}$ has to be considered separately because of the concentration issues of the term $\rho\otimes j-j\otimes\rho$ and the only term that requires a special analysis is the forth term one, i.e.,

$$\int_0^T\int_{\mathbb{R}^d}\frac{q_\varepsilon^{k(\varepsilon)+1}}{\varepsilon^\gamma}\cdot\varphi\,dx\,dt.$$

Moreover, the first, second and fifth terms vanish when $\varepsilon\to 0$. The next result shows that one can indeed pass to the limit in the forth term and identify it in terms of the limit of $\frac{j_\varepsilon}{\varepsilon^\gamma}$, i.e., j.

Theorem 2.4.15. *Let the initial distribution functions f_ε^0 verify (2.2.5), the external forces $-\nabla\psi_\varepsilon$ fulfil (2.2.6), consider the strong global in time solution f_ε to (2.4.10) with initial data f_ε^0 and $\alpha \in (0, \frac{d}{2})$ and assume the hypothesis (2.4.14)–(2.4.15). Then,*

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^\gamma} \left\| ||v|^{k(\varepsilon)} - 1|\,|v| f_\varepsilon \right\|_{L^p(0,T;L^1(\mathbb{R}^{2d}))} = 0,$$

for every exponent $1 \leq p < 2$.

Proof. We start by restating the difference $|v|^{k(\varepsilon)} - 1$ though the integral version of the mean value theorem as follows

$$|v|^{k(\varepsilon)} - 1 = \int_0^{k(\varepsilon)} \frac{d}{d\tau}|v|^\tau\,d\tau = \int_0^{k(\varepsilon)} \log|v|\,|v|^\tau\,d\tau.$$

As a consequence,

$$\frac{1}{\varepsilon^\gamma}\int_{\mathbb{R}^{2d}} ||v|^{k(\varepsilon)} - 1|\,|v| f_\varepsilon\,dx\,dv \leq \frac{1}{\varepsilon^\gamma}\int_0^{k(\varepsilon)}\int_{\mathbb{R}^{2d}} |\log|v||\,|v|^{\tau+1} f_\varepsilon\,dx\,dv\,d\tau.$$

Then, all our efforts must be conducted to bound logarithmic velocity moments of the particle distribution. This will be done by controlling them in terms of the already known bounds for the velocity moments in Corollary 2.4.14. Then, let us recall that for any couple of positive exponent a, b, the asymptotic behavior of the logarithm near $r = 0$ and $r = \infty$ is explicitly given by

$$|\log r| \leq \begin{cases} \frac{1}{ae} r^{-a}, & r \in (0,1], \\ \frac{1}{be} r^b, & r \in [1, +\infty). \end{cases}$$

In a more compact (although less sharp) way

$$|\log r|\, r^{\tau+1} \leq \frac{1}{ae} r^{\tau+1-a} + \frac{1}{be} r^{\tau+1+b},$$

for every $\tau \in (0, k(\varepsilon))$ and every $a, b > 0$. Fix now any couple of exponents $m \in [0,1)$ and $n \in (0,1]$. Note that when ε is small enough, then $0 < k(\varepsilon) < n$ and this allows choosing $a = a(\tau)$ and $b = b(\tau)$ as follows

$$a = (1-m) + \tau, \qquad b = n - \tau.$$

Such choice amounts to

$$|\log r| r^{\tau+1} \leq \frac{1}{((1-m)+\tau)e} r^m + \frac{1}{(n-\tau)e} r^{1+n}, \tag{2.4.16}$$

for every $r > 0$. Thus, our integral can be split into two terms

$$\frac{1}{\varepsilon^\gamma}\int_{\mathbb{R}^{2d}} ||v|^{k(\varepsilon)} - 1|\,|v| f_\varepsilon\,dx\,dv \leq F_\varepsilon(t) + G_\varepsilon(t),$$

where

$$F_\varepsilon(t) := \frac{1}{\varepsilon^\gamma}\int_0^{k(\varepsilon)} \frac{1}{((1-m)+\tau)e}\int_{\mathbb{R}^{2d}} |v|^m f_\varepsilon\,dx\,dv\,d\tau,$$

$$
\begin{aligned}
&= \frac{1}{\varepsilon^\gamma}\frac{1}{e}\log\left(1+\frac{1}{1-m}k(\varepsilon)\right)\||v|^m f_\varepsilon\|_{L^1(\mathbb{R}^{2d})}\\
G_\varepsilon(t) &:= \frac{1}{\varepsilon^\gamma}\int_0^{k(\varepsilon)}\frac{1}{(n-\tau)e}\int_{\mathbb{R}^{2d}}|v|^{1+n}f_\varepsilon\,dx\,dv\,d\tau,\\
&= -\frac{1}{\varepsilon^\gamma}\frac{1}{e}\log\left(1-\frac{1}{n}k(\varepsilon)\right)\||v|^{1+n}f_\varepsilon\|_{L^1(\mathbb{R}^{2d})}.
\end{aligned}
$$

Regarding the first term, let us use once more the Hölder inequality to obtain

$$
\|F_\varepsilon\|_{L^{2/m}(0,T)} \leq \frac{1}{e}\frac{\log\left(1+\frac{1}{1-m}k(\varepsilon)\right)}{\varepsilon^{\gamma(1-m)}}\left(\frac{1}{\varepsilon^\gamma}\||v|f_\varepsilon\|_{L^1(0,T;L^1(\mathbb{R}^{2d}))}\right)^m\|f_\varepsilon^0\|_{L^1(\mathbb{R}^{2d})}^{1-m}.
$$

In this case, Corollary 2.4.14, our choice of $k(\varepsilon)$ and L'Hôpital's rule show that

$$
\lim_{\varepsilon\to 0}\|F_\varepsilon\|_{L^{2/m}(0,T)} = 0.
$$

Note that the endpoint case $m=0$ is also allowed, giving rise to L^∞ norms with respect to times. Regarding the second term,

$$
\|G_\varepsilon\|_{L^{\frac{2}{1+n}}(0,T)} \leq -\frac{1}{e}\log\left(1-\frac{1}{n}k(\varepsilon)\right)\varepsilon^{\gamma n}\left(\frac{1}{\varepsilon^{2\gamma}}\||v|^2 f_\varepsilon\|_{L^1(0,T;L^1(\mathbb{R}^{2d}))}\right)^{\frac{1+n}{2}}\|f_\varepsilon^0\|_{L^1(\mathbb{R}^{2d})}^{\frac{1-n}{2}}.
$$

The same reasoning as above also shows that

$$
\lim_{\varepsilon\to 0}\|G_\varepsilon\|_{L^{\frac{2}{1+n}}(0,T)} = 0.
$$

Since both exponents are ordered, namely

$$
\frac{2}{1+n} < \frac{2}{m},
$$

then, one can conclude that

$$
\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^\gamma}\left\|\left||v|^{k(\varepsilon)}-1\right||v|f_\varepsilon\right\|_{L^{\frac{2}{1+n}}(0,T;L^1(\mathbb{R}^{2d}))} = 0,
$$

for every $n\in(0,1]$. Since $n=0$ is not allowed in the splitting (2.4.16) (to obtain convergent τ-integrals), then we arrive at the claimed convergence result for each exponent $1\leq p<2$. □

Corollary 2.4.16. *Let the initial distribution functions f_ε^0 verify (2.2.5), the external forces $-\nabla\psi_\varepsilon$ fulfil (2.2.6), consider the strong global in time solution f_ε to (2.4.10) with initial data f_ε^0 and $\alpha\in(0,\frac{d}{2})$ and assume the hypothesis (2.4.14)–(2.4.15). Then,*

$$
\frac{1}{\varepsilon^\gamma}q_\varepsilon^{k(\varepsilon)+1}\overset{*}{\rightharpoonup} j
$$

when $\varepsilon\to 0$ both in $L^p_w(0,T;\mathcal{M}(\mathbb{R}^d))$, for every $1<p<2$, and in $\mathcal{M}([0,T]\times\mathbb{R}^d)$.

Proof. For simplicity, we restrict to the case $1<p<2$, although the other case can be proved through similar arguments. First, recall that

$$
\frac{j_\varepsilon}{\varepsilon^\gamma}\overset{*}{\rightharpoonup} j \text{ in } L^2_w(0,T;\mathcal{M}(\mathbb{R}^d)).
$$

Since we are assuming $p < 2$, then

$$\frac{j_\varepsilon}{\varepsilon^\gamma} \overset{*}{\rightharpoonup} j \text{ in } L^p_w(0,T;\mathcal{M}(\mathbb{R}^d)).$$

By virtue of Theorem 2.4.15 one has that $\frac{1}{\varepsilon^\gamma} q_\varepsilon^{k(\varepsilon)+1}$ is bounded in $L^p(0,T;L^1(\mathbb{R}^{2N}))$. Then, some common subsequence (that we do not distinguish from the original for the sake of simplicity) weakly-* converges to some limit $q \in L^p_w(0,T;\mathcal{M}(\mathbb{R}^d))$, i.e.,

$$\frac{1}{\varepsilon^\gamma} q_\varepsilon^{k(\varepsilon)+1} \overset{*}{\rightharpoonup} q \text{ in } L^p_w(0,T;\mathcal{M}(\mathbb{R}^d)).$$

Hence,

$$\frac{q_\varepsilon^{k(\varepsilon)+1}}{\varepsilon^\gamma} - \frac{j_\varepsilon}{\varepsilon^\gamma} \overset{*}{\rightharpoonup} q - j \text{ in } L^p_w(0,T;\mathcal{M}(\mathbb{R}^d)).$$

Then, the weak-star lower semicontinuity of dual norms yields the estimate

$$\begin{aligned}
\|q-j\|_{L^p_w(0,T;\mathcal{M}(\mathbb{R}^d))} &\leq \liminf_{\varepsilon\to 0} \left\| \frac{q_\varepsilon^{k(\varepsilon)+1}}{\varepsilon^\gamma} - \frac{j_\varepsilon}{\varepsilon^\gamma} \right\|_{L^p_w(0,T;L^1(\mathbb{R}^{2N}))} \\
&= \liminf_{\varepsilon\to 0} \left\| \frac{q_\varepsilon^{k(\varepsilon)+1}}{\varepsilon^\gamma} - \frac{j_\varepsilon}{\varepsilon^\gamma} \right\|_{L^p(0,T;L^1(\mathbb{R}^{2N}))} \\
&\leq \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^\gamma} \left\| ||v|^{k(\varepsilon)+1} - 1|\,|v| f_\varepsilon \right\|_{L^p(0,T;L^1(\mathbb{R}^{2N}))} = 0,
\end{aligned}$$

where in the second line we have used that $\frac{q_\varepsilon^{k(\varepsilon)+1}}{\varepsilon^\gamma} - \frac{j_\varepsilon}{\varepsilon^\gamma}$ is actually strongly measurable along with the discussions in A to identify the norms of both the Lebesgue–Bochner spaces and their weak-* version. Hence, we conclude that $q \approx j$ and this ends the proof. □

Appendices

2.A The agent-based system, its mean field limit and scalings

In this Appendix we will introduce the kinetic Cucker–Smale model of interest with all its physical constants. Later we shall develop a nondimensional analysis and we will identify some dimensionless parameters that will be useful to justify the proposed scalings of hyperbolic and semi-hyperbolic type in this chapter. To start, we consider a system of N interacting particles particles subject to Cucker–Smale type of interactions, for any $N \in \mathbb{N}$. We shall assume that particles are influenced by some external force $F = -\nabla\psi$ described in terms of a potential function $\psi = \psi(t,x)$. In addition, we shall assume that particles might be affected by a *thermal bath*; specifically, velocities experiment linear damping and white noise due to interaction with such bath. This amounts to a coupled system of stochastic ODEs of Langevin-type:

$$\left\{\begin{aligned}
dx_i &= v_i\,dt,\\
m\,dv_i &= \frac{m}{N}\sum_{j=1}^N \phi^{K,\alpha,\sigma}(|x_i - x_j|)(v_j - v_i)\,dt - \nabla_x\psi(t,x_i)\,dt - \frac{m}{\tau} v_i\,dt + m\sqrt{\frac{2\mu}{\tau^2}}\,dW_t^i,
\end{aligned}\right. \tag{2.A.1}$$

for every $i = 1,\ldots,N$, where W_t^i are N independent standard Wiener processes (Brownian motions) in $\mathbb{R}^d$ and the stochastic term in last equation is understood in Itô sense. Here m stands

Parameter	Description
m	Mass of each particle
τ	Relaxation time under the thermal bath
$\sqrt{\mu}$	Mean thermal velocity
σ	Range of effective interactions
K	Coupling strength

Table 2.1: Physical parameters of the model.

for the mass of particles, that we assume is equal for each of them for the sake of simplicity. The influence function $\phi^{K,\alpha,\sigma} = \phi^{K,\alpha,\sigma}(r)$ represents agents connectivity. As it is usual, we will sort such connectivity decreasingly depending on inter-particle distances so that the larger the distances, the lower the interactions. Inspired in the choice by F. Cucker and S. Smale [90, 91] we shall set the influence function as follows

$$\phi^{K,\alpha,\sigma}(r) = K a^{\alpha,\sigma}(r) := K \frac{\sigma^{2\alpha}}{(\sigma^2 + c_\alpha |x|^2)^\alpha}, \tag{2.A.2}$$

where K is called the coupling strength with units times^{-1} and $a^{\alpha,\sigma}(r)$ is dimensionless. Here $\alpha > 0$ controls the the fall-off of connectivity with the interparticle distance, $c_\alpha = \zeta^{-1/\alpha} - 1$ for any fixed value $\zeta \in (0,1)$ (e.g., $\zeta = \frac{1}{2}$) and σ has spacial units and represents the range of effective interactions, namely,

$$r \leq \sigma \implies a^{\alpha,\sigma}(r) \leq \zeta = \frac{1}{2}.$$

In addition, notice that $a^{\sigma,\alpha}(r) \in (0,1]$ for every $r \geq 0$ and consequently K controls the maximum strength of interactions. Finally, τ represents the *relaxation time* of each particle on the thermal bath whilst $\sqrt{\mu}$ is the *mean thermal velocity* and defines the typical velocity of the thermal motion of particles. Sometimes, the *mean free path* $\sqrt{\mu}\tau$ is considered instead. Such magnitude consists in the average distance that particles must travels between collisions with other moving particles. All that terminology has been borrowed from the kinetic theory of gases. Indeed, those concepts appear in Boltzmann equation to characterize the dynamics of collision between particles within gas, but can be extended to related descriptions, like Fokker–Planck-type equations. See [71, Section 2.10] and [258, Section 10.1] for further details about those physical magnitudes along with the approximation of Boltzmann equation by Fokker–Planck-type equations. To summarize, the main parameters are given in Table 2.1

In order to derive the mesoscopic descriptions we refer to the discussion in Section 1.1.2 of the introductory Chapter 1 about the mean-field limit and propagation of chaos. Also, we recall the bilbiography [163, 164, 176, 177, 178, 179, 181, 216, 217, 230, 281]. Specifically, we can consider the BBGKY hierarchy of Liouville-type equations associated with (2.A.1). Indeed, the deterministic joint laws $f^N = f^N(t, x_1, \ldots, x_N, v_1, \ldots, v_N)$ are governed by

$$\frac{\partial f^N}{\partial t} + \sum_{i=1}^N v_i \cdot \nabla_{x_i} f^N - \frac{1}{m} \sum_{i=1}^N \nabla_x \psi(t, x_i) \cdot \nabla_{v_i} f^N$$
$$= \sum_{i=1}^N \operatorname{div}_{v_i} \left(\frac{1}{\tau} v_i f^N + \frac{\mu}{\tau} \nabla_{v_i} f^N + \frac{K}{N} \sum_{j=1}^N a^{\alpha,\sigma}(|x_i - x_j|)(v_i - v_j) f^N \right).$$

Notice that the symmetry of the interactions guarantee that if $f_0^N \in \mathcal{P}_{sym}(\mathbb{R}^{dN} \times \mathbb{R}^{dN})$ is a symmetric probability density initially, then the same continues happening for all times, i.e., $f_t^N \in \mathcal{P}_{sym}(\mathbb{R}^{dN} \times \mathbb{R}^{dN})$ for $t \geq 0$. This is precisely what allows writing a closed hierarchy for the marginal probabilities $f^{k,N} = f^{k,N}(t, x_1, \ldots, x_k, v_1, \ldots, v_k)$

$$\frac{\partial f^{k,N}}{\partial t} + \sum_{i=1}^{k} v_i \cdot \nabla_{x_i} f^{k,N} - \frac{1}{m} \sum_{i=1}^{k} \nabla_x \psi(t, x_i) \cdot \nabla_{v_i} f^{k,N} = \sum_{i=1}^{k} \operatorname{div}_{v_i} \left(\frac{1}{\tau} v_i f^N + \frac{\mu}{\tau} \nabla_{v_i} f^N \right)$$
$$+ K \frac{(N-k)}{N} \sum_{i=1}^{k} \operatorname{div}_{v_i} \left(\sum_{j=1}^{N} \int_{\mathbb{R}^{2d}} a^{\alpha,\sigma}(|x_i - x_{k+1}|)(v_i - v_{k+1}) f^{k+1,N} \, dx_{k+1} \, dv_{k+1} \right),$$

for $1 \leq k < N$. Take limits $f^{k,\infty} = \lim_{N\to\infty} f^{k,N}$ and use *propagation of chaos*, (that usually takes place in this type of systems, see references above), meaning that

$$f_0^{2,\infty} = f_0^{1,\infty} \otimes f_0^{1,\infty} \implies f_t^{2,\infty} = f_t^{1,\infty} \otimes f_t^{1,\infty}, \ t \geq 0.$$

Then, we can write down the limiting equation for the marginal $f^{1,\infty} \equiv f$, that agrees with the kinetic Cucker–Smale model for the probability distribution of particles $f = f(t, x, v)$

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f - \frac{1}{m} \nabla_x \psi \cdot \nabla_v f = L_{FP}(f) + K Q_{CS}^{\alpha,\sigma}(f, f), \tag{2.A.3}$$

where $L_{FP}(f)$ is the classical Fokker–Planck operator, that is reminiscent of the thermal bath, and $Q_{CS}^{\alpha,\sigma}(f, f)$ is the Cucker–Smale integro-differential operator associated to the influence function $\phi^{\alpha,\sigma}$, that is contains all the information about alignment interactions, specifically

$$L_{FP}(f) := \operatorname{div}_v \left(\frac{1}{\tau} v f + \frac{\mu}{\tau} \nabla_v f \right),$$
$$Q_{CS}^{\alpha,\sigma}(f) := f(t, x, v) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} a^{\alpha,\sigma}(|x - y|)(v - w) f(t, y, w) \, dy \, dw.$$

In the sequel, we shall introduce a nondimensional analysis of the above model (2.A.3), see [252, Appendix] for a comprehensive dimensional analysis of the Vlasov–Poisson–Fokker–Planck system. Namely, consider some characteristic units for time, space, velocity and typical value of the potential

$$\bar{t} = \frac{t}{T}, \quad \overline{x} = \frac{x}{R}, \quad \overline{v} = \frac{v}{V}, \quad \overline{\psi}(\bar{t}, \overline{x}) = \frac{\psi(t, x)}{\psi_0}.$$

Straightforward computations on the original model lead to the next dimensionless model, where we have removed bars for the sake of simplicity in the notation

$$\frac{\partial f}{\partial t} + \frac{VT}{R} v \cdot \nabla_x f - \frac{1}{m} \frac{\psi_0 T}{RV} \nabla_x \psi \cdot \nabla_v f = \operatorname{div}_v \left(\frac{T}{\tau} f v + \frac{T}{\tau} \frac{\mu}{V^2} \nabla_v f + V^d R^d T K Q_{CS}^{\alpha,\sigma/R}(f, f) \right).$$

Now, let us introduce a scaling where the characteristic units and the physical constants are linked thought the next formulas:

$$\frac{R}{T} = \frac{1}{m} \frac{\tau}{R} \psi_0, \quad V^d R^d = 1.$$

The above choices are considered in order for the quotient of the characteristic length over the characteristic time to agree with the drift velocity associated with the potential $\psi(t, x)$ and for

f to preserve the normalization property. Indeed, notice that the only constraint in the rescaling is that f has to preserve the total probability 1. Finally, consider the next dimensionless parameters for the thermal mean velocity, the thermal mean free path, the scaled range of the effective interactions and the scaled coupling strength

$$\lambda := \frac{\sqrt{\mu}}{R/T}, \quad \beta := \frac{\sqrt{\mu}\tau}{R}, \quad \nu := \frac{\sqrt{\mu}}{V}, \quad \delta := \frac{\sigma}{R}, \quad \kappa := \tau K.$$

Consequently, the corresponding dimensionless model reads

$$\frac{\partial f}{\partial t} + \frac{\lambda}{\nu} v \cdot \nabla_x f - \frac{\nu}{\beta}\nabla_x \psi \cdot \nabla_v f = \frac{\lambda}{\beta}\operatorname{div}_v \left(fv + \nu^2 \nabla_v f + \kappa\, Q_{CS}^{\alpha,\delta}(f,f)\right).$$

In this chapter, we are interested in "singular" scalings of the system depending on a parameter $\varepsilon \to 0$. To illustrate the idea, let us look at the corresponding influence function of the dimensionless model

$$\kappa a^{\alpha,\delta}(r) = \frac{\kappa \sigma^{2\alpha}}{(\sigma^2 + c_\alpha r^2)^\alpha}.$$

Then, we are interested in taking $\sigma \to 0$ and $\kappa\sigma^{2\alpha} = 1$ as $\varepsilon \to 0$, so that a singular influence kernel arises. Notice that such singular limit amounts to small range of effective interactions compared with the typical size of the system and large coupling strength compared to the inverse relaxation time of the thermal bath. We shall propose the following different scaling that are compatible with such idea:

1. First, a hyperbolic scaling can be obtained by choosing the next order of the parameters:

   $$\lambda = 1, \quad \beta = \varepsilon, \quad \nu = 1, \quad \delta = \varepsilon, \quad \kappa = \varepsilon^{-2\alpha}.$$

   Note that, in particular, we are assuming the characteristic velocity of the system to agree with the mean thermal velocity in the thermal bath. In this case the system takes the form

   $$\varepsilon \frac{\partial f_\varepsilon}{\partial t} + \varepsilon v \cdot \nabla_x f_\varepsilon - \nabla_x \psi_\varepsilon \cdot \nabla_v f_\varepsilon = \operatorname{div}_v \left(f_\varepsilon v + \nabla_v f_\varepsilon + Q_{CS}^{\phi_\varepsilon}(f_\varepsilon, f_\varepsilon)\right). \tag{2.A.4}$$

2. Second, one can suppose that the characteristic velocity of the system is much larger that the mean thermal velocity. An appropriate choice of the scaling is

   $$\lambda = 1, \quad \beta = \varepsilon^{1+\gamma}, \quad \nu = \varepsilon^\gamma, \quad \delta = \varepsilon, \quad \kappa = \varepsilon^{-2\alpha},$$

   for some parameter $\gamma \in [0,1]$. We shall call such scaling, an intermediate hyperbolic scaling and it reads

   $$\varepsilon^{1+\gamma} \frac{\partial f_\varepsilon}{\partial t} + \varepsilon v \cdot \nabla_x f_\varepsilon - \varepsilon^\gamma \nabla_x \psi_\varepsilon \cdot \nabla_v f_\varepsilon = \operatorname{div}_v \left(f_\varepsilon v + \varepsilon^{2\gamma}\nabla_v f_\varepsilon + Q_{CS}^{\phi_\varepsilon}(f_\varepsilon, f_\varepsilon)\right). \tag{2.A.5}$$

3. Finally, one might be interested on neglecting friction effects. In this case, the factor fv disappears in the above equation. We are interested in a hyperbolic scaling that stands for the next choice of the dimensionless parameters

   $$\lambda = 1, \quad \beta = \varepsilon, \quad \nu = \varepsilon, \quad \delta = \varepsilon, \quad \kappa = \varepsilon^{-2\alpha}.$$

   Such system takes the form

   $$\varepsilon \frac{\partial f_\varepsilon}{\partial t} + \varepsilon v \cdot \nabla_x f_\varepsilon - \varepsilon \nabla_x \psi_\varepsilon \cdot \nabla_v f_\varepsilon = \operatorname{div}_v \left(\varepsilon \nabla_v f_\varepsilon + Q_{CS}^{\phi_\varepsilon}(f_\varepsilon, f_\varepsilon)\right). \tag{2.A.6}$$

In all the above scalings (2.A.4), (2.A.5) and (2.A.6) we have considered the scaled influence kernel ϕ_ε given as follows

$$\phi_\varepsilon(r) := \frac{1}{(\varepsilon^2 + c_\alpha r^2)^\alpha}, \quad r > 0.$$

When $\varepsilon \to 0$, we recover singular kernels of Newtonian type as opposed to the classical regular setting in the original works of F. Cucker and C. Smale [90, 91]. Along this chapter, our main goal is to take the rigorous limit as $\varepsilon \to 0$, that amounts to consider a coupled hydrodynamic and singular limit in the corresponding scaled kinetic model.

2.B Sketch of proof of Theorem 2.3.2

First, let us recall that $\rho \in L^1_{loc}([0,T) \times \mathbb{R}^d)$ is said to be a weak solution to (2.3.2) when

$$\int_0^T \int_{\mathbb{R}^d} \left(\frac{\partial \varphi}{\partial t} + u \cdot \nabla \varphi \right) \rho \, dx \, dt = - \int_{\mathbb{R}^d} \rho^0(x) \varphi(0,x) \, dx, \tag{2.B.1}$$

for every test function $\varphi \in C^1_c([0,T) \times \mathbb{R}^d)$.

Second, the assumed hypothesis on the transport field u yields the existence of an uniquely defined flux of u in the extended sense of Caratheodory (see [81, Theorem 1.1, Chapter 2]). Specifically, since the three Caratheodory conditions (continuity in x, measurability in t and boundedness both in t and x by a t-dependent integrable function) fulfill, then the characteristic system

$$\begin{cases} \dfrac{d}{dt} X(t; t_0, x_0) = u(t, X(t; t_0, x_0)), \quad t \in [0,T) \\ X(t_0; t_0, x_0) = x_0, \end{cases}$$

has a unique absolutely continuous global-in-time solution $X(t; t_0, x_0)$ for every $0 \le t_0 < T$ and $x_0 \in \mathbb{R}^d$. Notice that such ODE holds in the a.e. sense (recall that absolutely continuous functions are a.e. differentiable). The Lipschitz continuity of u does not only ensures the global definition by the Gronwall lemma but also its uniqueness. Indeed, $X(t; t_0, \cdot)$ is a bi-Lipschitz diffeomorphism for each t, t_0. Specifically, note that the Lipschitz continuity with respect to x shows that

$$\begin{aligned} &\frac{d}{dt} |X(t; t_0, x_1) - X(t; t_0, x_2)|^2 \\ &\quad = 2 \left(X(t; t_0, x_1) - X(t; t_0, x_2) \right) \cdot \left(u(t, X(t; t_0, x_1)) - u(t, X(t; t_0, x_2)) \right) \\ &\quad \le 2 \|u(t, \cdot)\|_{W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)} |X(t; t_0, x_1) - X(t; t_0, x_2)|^2, \end{aligned}$$

and consequently,

$$|X(t; t_0, x_1) - X(t; t_0, x_2)| \le \exp\left(\|u\|_{L^1(0,T; W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d))} \right) |x_1 - x_2|,$$

for every $0 \le t, t_0 < T$ and $x_1, x_2 \in \mathbb{R}^d$. In particular, one can define its Jacobian determinant a.e. because Lipschitz functions are a.e. differentiable by *Rademacher's theorem*

$$J(t; t_0, x_0) := \det \left(\mathrm{Jac}_{x_0} X(t; t_0, x_0) \right).$$

If u were smooth, then *Liouville's theorem* would hold, i.e.,

$$\frac{d}{dt} J(t; t_0, x_0) = \operatorname{div} u(t, X(t; t_0, x_0)) \, J(t; t_0, x_0),$$

and consequently,

$$J(t;t_0,x_0)=\exp\left(\int_{t_0}^t \operatorname{div} u(s,X(s;t_0,x_0))\,ds\right).$$

In particular, one would obtain the next upper and lower bound of J

$$\exp\left(-\|u\|_{L^1(0,T;W^{1,\infty})}\right)\leq J(t;t_0,x_0)\leq \exp\left(\|u\|_{L^1(0,T;W^{1,\infty})}\right). \tag{2.B.2}$$

Nevertheless, it is not compulsory to assume that u is smooth. Indeed, if $u\in L^1(0,T;W^{1,\infty}(\mathbb{R}^d,\mathbb{R}^d))$ then (2.B.2) also holds by a density argument (see [7, Section 2]).

The corresponding theory for classical solutions with smooth initial data ρ^0 provides the following candidate of weak solution

$$\rho(t,x):=\rho^0(X(0;t,x))\,J(0;t,x)=\left.\frac{\rho^0(y)}{J(t;0,y)}\right|_{y=X(0,t,x)}. \tag{2.B.3}$$

First of all, let us show that although evaluation of measurable functions like ρ^0 might not have sense at some points (because they are defined almost everywhere), the above definition makes sense and does not depend on the representative. To this end, note that $X(0;t,\cdot)$ has its Jacobian determinant lower and upper bounded by positive constants by virtue of (2.B.2) and, consequently, the theorem of change of variables shows that the flux $X(0;t,\cdot)$ preserves the negligible sets in $\mathbb{R}^d$. Thus, no matter the chosen representative of the measurable function, the above composition yields functions that agree almost everywhere (hence, representing same equivalence classes).

On the one hand, the claimed bound in $L^\infty(0,T;L^p(\mathbb{R}^d))$ when $1<p\leq\infty$ is clear by the change of variables theorem for bi-Lipschitz vector fields and the above bound (2.B.2). Similarly, the $L^\infty(0,T;L^1(\mathbb{R}^d))$ estimate is also apparent when $p=1$ through the same argument since the $L^1(\mathbb{R}^d)$ norm is indeed constant for all time. Let us now show that, so defined, ρ gives rise to a weak solution to (2.3.2). We will face first the case $1<p\leq\infty$. Fix any test function $\varphi\in C_c^1([0,T)\times\mathbb{R}^d)$ and notice that definition (2.B.3) implies

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^d}\left(\frac{\partial\varphi}{\partial t}+u\cdot\nabla\varphi\right)\rho\,dx\,dt&=\int_0^T\int_{\mathbb{R}^d}\left.\left(\frac{\partial\varphi}{\partial t}+u\cdot\nabla\varphi\right)\right|_{(t,X(t;0,x))}\rho^0(x)\,dx\\&=\int_0^T\int_{\mathbb{R}^d}\frac{d}{dt}\left[\varphi(t,X(t;0,x))\right]\rho^0(x)\,dx\\&=-\int_{\mathbb{R}^d}\varphi(0,x)\rho^0(x)\,dx,\end{aligned}$$

where we have used the fundamental lemma of calculus in the last equality to arrive at the weak formulation (2.B.1) for every $\varphi\in C_c(\mathbb{R}^d)$ (notice that locally absolutely continuous functions also verify such result). A similar argument also proves the weak formulation when $p=1$.

Finally, let us prove the uniqueness of weak solution in the sense of (2.B.1) under the stronger assumption that u is smooth. Consider two weak solutions $\rho_1(t,x)$ and $\rho_2(t,x)$ to the same Cauchy problem (2.3.2). Then, by linearity it is clear that $\rho=\rho_1-\rho_2$ solves (2.3.2) with $\rho^0\equiv 0$, i.e.,

$$\int_0^T\int_{\mathbb{R}^d}\left(\frac{\partial\varphi}{\partial t}+u\cdot\nabla\varphi\right)\rho\,dx\,dt=0,$$

for every test function $\varphi \in C_c^1([0,T), \mathbb{R}^d)$. Note that given any test function $\psi \in C_c^1((0,T) \times \mathbb{R}^d)$ the classical theory also allows solving the backwards Cauchy problem

$$\begin{cases} \partial_t \varphi + u \cdot \nabla \varphi = \psi, & t \in [0,T], \\ \varphi(T) = 0, \end{cases}$$

then leading to a test function $\varphi \in C^1([0,T), C^1(\mathbb{R}^d, \mathbb{R}^d))$. Since ψ is arbitrary, the fundamental lemma of the calculus of variations shows that $\rho \equiv 0$, i.e., $\rho_1 \equiv \rho_2$. When u is not smooth but only $L^1(0,T; W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d))$, a similar duality argument can be proved, leading to the uniqueness of weak solutions. Also, Wasserstein-type estimates prove uniqueness for Lipschitz kernels in an alternative way, without resorting on duality arguments or further smoothness assumptions on u. See Chapter 4 for further details.

Part II

The Kuramoto model and some singular versions

CHAPTER 3

The Kuramoto model with singular couplings: the agent based system

Contents

3.1 Introduction

Synchronization is the natural collective behavior arising from agents-based interactions described by periodic rules. These rhythmical motions can be easily observed in various biological complex systems such as flashing of fireflies, beating of cardiac cells, etc. Since Kuramoto proposed such mathematical model for coupled oscillators in [196, 195], synchronization has received a lot of attention and has been studied extensively in various disciplines from this point of view, see [1] for a comprehensive review on this topic. In the classical model by Kuramoto, the system of N coupled oscillators has an all-to-all coupling with uniform weights characterized by a constant coupling strength K

$$\dot{\theta}_i = \Omega_i + \frac{K}{N}\sum_{j=1}^{N} \sin(\theta_j - \theta_i), \tag{3.1.1}$$

for $i = 1, \ldots, N$, where θ_i' represent the phases and Ω_i's are the natural frequencies of each oscillator. The original system by Kuramoto is a paradigmatic model describing collective synchronization of a large population of coupled oscillators, that spontaneously synchronize to oscillate at a common frequency. Although Kuramoto initially proposed it for synchronization of chemical reactions, such many-body cooperative effect can be observed in many other examples in nature, see [1].

One of the most significant examples of synchronization appear in neurons. For some applications to *neuronal synchronization* and how the realistic human connectome maps that are available in the literature affect the emergence of synchronization, see [297] and references therein. Such ideas exploit neuronal connections in the brain turn out to be organized in moduli structured in a hierarchical nested fashion across many scales, and it affects the neural dynamics [262, 305, 306]. *Associative* or *Hebbian learning* [166] proposes an explanation for the adaptation of neurons in the brain during the learning process. Such mechanism is founded in the assumption that synchronous activation of cells (firing of neurons) leads to selectively pronounced increases in synaptic strength between those cells. The consequence is that the pattern of activity will become self-organized. In Hebb's words

> *Any two cells or systems of cells that are repeatedly active at the same time will tend to become associated, so that activity in one facilitates activity in the other*, D. O. Hebb. (3.1.2)

see [166, p. 70]. In neuroscience, this processes provide the neuronal basis of unsupervised learning of cognitive functions in neural networks and can explain the phenomena that arises in the development of the nervous system. However, uniform and time-constant coupling weights like in (3.1.1) are too restrictive to explain the complicatedness of the above phenomena. Thus, it is more interesting to consider a generalization of the Kuramoto model introduced in [10, 156, 159, 233, 247, 256, 272]

$$\dot{\theta}_i = \Omega_i + \frac{1}{N}\sum_{j=1}^{N} K_{ij}\sin(\theta_j - \theta_i) \tag{3.1.3}$$

$$\dot{a}_{ij} = \eta(\Gamma(\theta_j - \theta_i) - a_{ij}), \tag{3.1.4}$$

for $i = 1, \ldots, N$. In such model, each couple of oscillators θ_i and θ_j is equipped with a plastic (adaptive) coupling $K_{ij} = K_{ij}(t)$, that has its own dynamics depending on the phase configuration. Indeed, for convenience the couplings K_{ij} are often split into the dimensional and dimensionless parts as follows

$$K_{ij} = K a_{ij},$$

where $a_{ij} = a_{ij}(t) \in [0,1]$ measures the degree of connectivity between the i-th and j-th oscillators and K is the coupling strength and has frequency units. Also, recall similar considerations for the influence function $\phi = \phi(r)$ of the related Cucker–Smale model in Chapter 2. Equation (3.1.4) governing connectivity can be regarded as a dynamic learning rule for the coupling weights. Here, $\Gamma = \Gamma(\theta) \in [0,1]$ represents the plasticity function and η is called the learning rate. In a way, small values of η delay the adaptation of weights a_{ij} towards the value of $\Gamma(\theta_i - \theta)_j$ whilst large values of η accelerates such adaptation mechanism.

According to the choice of the plasticity function Γ, the dynamics of the system (3.1.3)-(3.1.4) shows various scenarios. In neural networks systems, Hebbian-type dynamics is often considered for the learning algorithm of couplings between oscillators. Specifically, from a qualitative point of view, *Hebb's learning rule* (3.1.2) amounts to saying that the couplings tend to increase when oscillators fire simultaneusly, that is, when phases get close to each other. Equivalently, for a mathematical point of view Hebb's rule sets $\Gamma = \Gamma(\theta)$ as a decreasing function of the phase distance θ that achieves the maximum value 1 at $\theta = 0$ (i.e., total connectivity when $\theta_i = \theta_j$). For example, in [159, 233, 272] the authors assumed the *Hebbian type* plasticity function $\Gamma(\theta) = \cos\theta$ so that attraction between near oscillators is reinforced, but also repulsive interaction arises between apart phases. On the other hand, some other processes follow *anti-Hebbian type* rules like $\Gamma(\theta) = |\sin\theta|$, that was considered in [159, 256]. In this case, synchronization emerges slowly due to the reduction of weight for nearby oscillators. Other types of adaptive rules are considered in [156, 247]. In this chapter, we shall consider a more realistic Hebbian-like plasticity function Γ that in particular solves the "problematic" eventual changes of sign in the aforementioned case $\Gamma(\theta) = \cos\theta$. Namely, we will propose

$$\Gamma(\theta) := \frac{\sigma^{2\alpha}}{\left(\sigma^2 + c_{\alpha,\zeta}|\theta|_o^2\right)^{\alpha}}, \tag{3.1.5}$$

where $\sigma \in (0,\pi)$, $\zeta \in (0,1]$ and $|\theta|_o := d_{\mathbb{T}}(e^{i\theta}, 1)$ is the (orthodromic) distance along $\mathbb{T}$, i.e.,

$$|\theta|_o := |\bar{\theta}| \quad \text{for} \quad \bar{\theta} \equiv \theta \bmod 2\pi, \quad \bar{\theta} \in (-\pi, \pi].$$

Here, the parameter $c_{\alpha,\zeta} := 1 - \zeta^{-1/\alpha}$ has been chosen so that whenever two phases θ_i and θ_j stay at orthodromic distance σ or larger, then the adaptive function Γ predicts a maximum degree of connectivity not larger than ζ between such oscillators. See also Chapter 2 for comparison with the influence function $\phi^{K,\alpha,\sigma}$ in (2.A.2) for the Cucker–Smale model.

Along this chapter we will not address the full dynamics of the system (3.1.3)-(3.1.4). Instead, we will consider the *fast learning regime* in order to reduce the learning rule (3.1.4) to a simpler instantaneous adaptation and start gaining some intuition about the model. Specifically, since the plasticity function Γ in (3.1.5) is Lipschitz-continuous, then we can apply the classical *Tikhonov theorem* [183] to system (3.1.3)-(3.1.4) and rigorously take the limit $\eta \to +\infty$ to arrive at the following simplified Kuramoto model with weighted coupling structure

$$\dot{\theta}_i = \Omega_i + \frac{K}{N}\sum_{j=1}^{N} \Gamma(\theta_j - \theta_i)\sin(\theta_j - \theta_i), \tag{3.1.6}$$

for $i = 1, \ldots, N$. This model will play a central role in this chapter. Notice that if either the parameter $\alpha = 0$ or $\zeta = 1$, then our plasticity function (3.1.5) becomes 1 everywhere and (3.1.6) reduces to the classical Kuramoto model (3.1.1). Hence, we will assume that $\alpha > 0$ and $\zeta \in (0,1)$ from now on. Our main purpose is to analyze (3.1.5)-(3.1.6) and compare it with the associated singular counterpart governed by the following singular plasticity function

$$\Gamma(\theta) := \frac{1}{c_{\alpha,\zeta}^{\alpha}|\theta|_o^{2\alpha}}. \tag{3.1.7}$$

Notice that plasticity function is actually a normalized quantity and by definition it cannot be unbounded like in (3.1.7). However, in the next section we shall derive this new singular dynamics from the regular one via a singular scaling of the parameters K and σ that is reminiscent of the one proposed in the preceding chapter. Notice that the singularly weighted system indeed differs from the regular one for several reasons. First, in the regular case (3.1.5) Γ is Lipschitz-continuous function and (3.1.6) becomes the Kuramoto model with regular weights depending on a different sine-type coupling force. Hence, well-posedness of global-in-time classical solutions is standard. Unfortunately, in the singular case (3.1.7) the system (3.1.6) has non-Lipschitz right hand sides due to the presence of a singular weight. Then, the Cauchy–Lipschitz theory cannot be used again to guarantee existence and uniqueness of global-in-time solutions. More interestingly, we also expect that such a difference plays a fundamental role at the dynamical level and the singularly weighted case introduces substantially new dynamics in the paradigm of synchronization. When dealing with singularities, we shall distinguish three different regimes $\alpha \in (0, \frac{1}{2})$, $\alpha = \frac{1}{2}$ and $\alpha \in (\frac{1}{2}, 1)$ that we respectively call the *subcritical*, *critical* and *supercritical* cases for simplicity.

The main results of this chapter are listed as follows. First, we study well-posedness of the singular weighted system. Depending on the value of α, the properties of the right hand side of (3.1.6) vary. Specifically, in the subcritical regime, we deal with systems of ODEs with Hölder-continuous right hand side while we face discontinuous right hand side of both bounded and unbounded type in the critical and supercritical cases respectively. This suggests that the appropriate concept of uniqueness that we should explore in these systems is not the standard one but just *one-sided forward uniqueness*. This allows the possibility that a group of oscillators might gather and collide in finite time to form a *cluster* of oscillators with same phase but larger mass that stay stuck together for all times. This is a phenomenon that was recently found in other relevant agent-based systems like Cucker–Smale model with weakly singular influence function (see [244, 245]) or the aggregation equation with mildly singularity potentials (see [27, 28, 29, 64, 200])

Our second result characterizes the explicit conditions for sticking in the subcritical and critical regimes. In the former case, we show that only clusters of oscillators with the same natural frequencies can stick together. Nevertheless, in the latter case, cluster of oscillators with different natural frequencies may stick together as long as such frequencies fulfill an appropriate condition. Regarding the supercritical case, the analogue sticking condition becomes trivial and we can show a continuation procedure of classical solutions after finite-time collisions. Namely, after a cluster is formed in finite time, the cluster keep stuck together no matter which are the natural frequencies of the involved oscillators.

The third result consists in showing that these singular weights are actually physically relevant. Specifically, we will show that the system (3.1.6)-(3.1.7) with singular weights can be recovered as a rigorous singular limit of the regular model (3.1.5)-(3.1.6). Again, the strategy will differ in each of the regimes. For the subcritical case, similar tools to those in [244, 245] for the singular Cucker–Smale model can be adapted. What is more, we can even obtain an analogue gain of extra $W^{1,1}$ piece-wise regularity of the frequencies of oscillators. For the critical and supercritical cases we cannot resort on the same ideas. Hence, we use the underlying gradient-flow structure to gain compactness of frequencies. Identifying the limit will be the heart of the matter in this part.

Our last result faces the emergence of synchronization in each regime of the parameter α. For identical oscillators, we show the emergence of complete phase synchronization in finite-time under appropriate assumptions on the initial diameter of phases. At least in the subcritical regime, where frequencies become more regular, we study the asymptotic emergence of complete frequency synchronization of non-identical oscillators. Also, we study the stability properties of collision-less phase-locked states in all the three regimes.

The techniques are firstly inspired by a combination of results for the classical Kuramoto model, but these techniques require of a new perspective allowing for singular interactions. For this purpose, we introduce a well-posedness result "à la Filippov" that is valid for systems of ODEs with discontinuous right-hand sides. Specifically, we will rely on the study of absolutely continuous solutions of the differential inclusions associated with the Filippov's set-valued map. The values of such map are convex polytopes that are bounded and unbounded in the critical and supercritical case respectively. Hence, the classical theory can be used in the former case whereas new ideas are developed for the latter case. Also, we prove some one-sided uniqueness results for non–Lipschitzian interactions that rely on the structure of interaction kernel near the points of loss of Lipschitz-continuity. For the stability of equilibria, Lyapunov's first method entails a similar scenario to that of the classical Kuramoto model in the critical and supercritial regime. On the other hand, the subcritical regime requires a center manifold approach that yields the stability of the corresponding equilibria. Regading the dynamics, it is interestingn that we can still reproduce some accurate control of the diameter of the system. Such control can be used to show finite-time and asymptotic synchronization for the identical and non-identical cases respectively. Unfortunately, the emergence of phased-locked states cannot be takled with the same ideas as in [157] for the Kuramoto model due to the fact that Łojasiewicz' gradient inequality [204] might not holds for non-analytic systems with gradient structure like this. Similarly, it is not clear whether the ideas in [147] can be conducted to recover emergence of phase-locked states for any initial data in the large coupling regime in these models. We do not address such ideas here, that will be object of study in future works. Finally, regarding the singular limit from regular coupling weights towards the singularly weighted cases, our main goal is to prove that solutions of the regularized system converge towards absolutely continuous trajectories that verify the differential inclusion into Filippov's map. The cornerstone of such results is the derivation of an appropriate H-representation (half-space representation) of such convex polytopes through convex analysis techniques. Then, the preceding gain of compactness of frequencies along with such geometric representation of the Filippov map will become the necessary tools for the singular limit to work in the critical and supercritical regimes.

The remaining section of this chapter are organized as follows. In Section 3.2, we present definitions, basic properties of the weighted Kuramoto model, the underlying gradient-flow structure, the passage from regular to singular plasticity function and the expected macroscopic equations. Since Chapter 4 is devoted to the associated kinetic equations, we skip the details here and focus on the agent-based system. In Section 3.3, we study the system with singular weights and we prove the well-posedness theory in each regime. In Section 3.4, we prove the rigorous singular limit in every regime and compare the model with previous results derived in other agent-based systems, in particular we compare with Cucker–Smale models. In Section 3.5, we show the emergence of synchronization for the singular weighted system. For the sake of clarity, we summarize in Appendix 3.A the main classical tools that have been used for the Kuramoto models and we apply them to show emergence of synchronization for the regular weighted system. Appendix 3.B shows the proofs of the H-representation of the Filippov set-valued map in the critical and supercritical cases. Finally, Appendix 3.C introduces the explicit characterization of the sticking conditions. For a brief summary of the basis of Filippov's existence theory that we will use in this chapter, we refer to Appendix D of the thesis.

3.2 Preliminaries

3.2.1 Basic properties and definitions

In this section, we study the basic properties of the weighted Kuramoto system and introduce some related results that will be useful in the following sections. For simplicity, let us denote the interaction kernel by $h(\theta) := \Gamma(\theta)\sin\theta$ (here Γ can be any even function, e.g., (3.1.5) or (3.1.7)). Then the system (3.1.6) can be expressed as

$$\dot{\theta}_i = \Omega_i + \frac{K}{N}\sum_{j=1}^{N} h(\theta_j - \theta_i). \tag{3.2.1}$$

For simplicity, we shall sometimes use vector notation in (3.2.1). We define the vector field $H(\Theta) = (H_1(\Theta), \dots, H_N(\Theta))$ whose components read

$$H_i(\Theta) = \Omega_i + \frac{K}{N}\sum_{j=1}^{N} h(\theta_j - \theta_i). \tag{3.2.2}$$

Then, (3.2.1) can be restated as

$$\dot{\Theta} = H(\Theta). \tag{3.2.3}$$

Since h is an odd function, by taking sums on both sides of (3.2.1), we have

$$\sum_{j=1}^{N}\dot{\theta}_i = \sum_{j=1}^{N}\Omega_i,$$

i.e., the average of frequencies is conserved. Thus, without loss of generality, we may assume that the average of natural frequencies is zero, $\frac{1}{N}\sum_{j=1}^{N}\Omega_i = 0$, in order to focus on the fluctuation from the constant average motion.

For the discussion in Section 3.4, we briefly introduce the second order augmentation of Kuramoto model, see [150]. By taking one more derivative on the system (3.2.1), we have the second order model

$$\begin{cases} \dot{\theta}_i = \omega_i, \\ \dot{\omega}_i = \dfrac{K}{N}\displaystyle\sum_{j=1}^{N} h'(\theta_j - \theta_i)(\omega_j - \omega_i). \end{cases} \tag{3.2.4}$$

For both systems (3.2.1) and (3.2.4) we have the following equivalence.

Theorem 3.2.1. *The Kuramoto model* (3.2.1) *is equivalent to an augmented Kuramoto model* (3.2.4) *in the following sense.*

1. *If $\Theta = (\theta_1, \dots, \theta_N)$ is a solution to* (3.2.1) *with initial data Θ_0, then $(\Theta, \omega := \dot{\Theta})$ is a solution to* (3.2.4) *with well-prepared initial data (Θ_0, ω_0):*

$$\omega_{i,0} := \Omega_i + \frac{\kappa}{N}\sum_{j=1}^{N} h(\theta_{j,0} - \theta_{i,0}).$$

2. *If (Θ, ω) is a solution to* (3.2.4) *with initial data (Θ_0, ω_0), then Θ is a solution to* (3.2.1) *with natural frequencies:*

$$\Omega_i := \omega_{i,0} - \frac{\kappa}{N}\sum_{j=1}^{N} h(\theta_{j,0} - \theta_{i,0}).$$

For the regular cases (3.1.5), the proof can be found in [150]. However, one has to take special care with the time regularity of solutions in the singular cases (3.1.7) before we take derivatives in (3.2.1). In that later case with $\alpha \in (0, \frac{1}{2})$, the type of solutions to be considered for (3.2.1) are absolutely continuous solutions, while for (3.2.4), solutions have to be taken in weak sense with C^1 and piecewise $W^{2,1}$ regularity (see [244] for this concept of solution for the discrete Cucker–Smale model with singular influence function). The well–posedness of both singular systems (3.2.1) and (3.2.4) will be established in Sections 3.3 and 3.4 (see Theorems 3.3.5, 3.3.12, 3.4.2, 3.4.4 and Remark 3.4.5) and comparisons with Cucker–Smale models with singular influence function will be given in Subsection 3.4.4.

For the sake of completeness, we recall the different definitions of synchronization, [145].

Definition 3.2.2. *Let $\Theta(t) = (\theta_1(t), \ldots, \theta_N(t))$ be the phase configuration of oscillators of which the dynamics is governed by the system* (3.1.6).

1. *The system shows the complete phase synchronization asymptotically if, and only if, the following condition holds:*
$$\lim_{t\to\infty} |\theta_i(t) - \theta_j(t)| = 0, \quad \textit{for all} \quad i \neq j.$$
2. *The system shows the complete frequency synchronization asymptotically if, and only if, the following condition holds:*
$$\lim_{t\to\infty} |\dot{\theta}_i(t) - \dot{\theta}_j(t)| = 0, \quad \textit{for all} \quad i \neq j.$$
3. *The system shows the emergence of a phase-locked state asymptotically if, and only if, there exist constants θ_{ij}^∞ such that*
$$\lim_{t\to\infty} |\theta_i(t) - \theta_j(t)| = \theta_{ij}^\infty, \quad \textit{for all} \quad i \neq j.$$

Analogue definitions of synchronization will be considered if, instead of asymptotically, the emergent dynamics takes place in some finite time T. In such case ∞ will be replaced by such finite time T in the above definitions.

We note that the complete phase-synchronization is a special case of phase-locked state. It is obvious that if the solution shows the emergence of phase-locked state, then it implies the complete frequency synchronization. However, the converse is valid when the frequency synchronization occurs fast, i.e., integrable decay of frequency differences.

3.2.2 Singular weighted model

In this part, we introduce the formal derivation of the Kuramoto model with singular weights as singular limit of the regular weighted model. We note that the regular weighted model is (3.2.1) with interaction kernel given by

$$h(\theta) := \frac{\sigma^{2\alpha} \sin\theta}{(\sigma^2 + c_{\alpha,\zeta}|\theta|_o^2)^\alpha}.$$

Recall that the degree of connectivity is smaller than ζ for interparticle distances larger than σ and α imposes the fall-off of the interactions. Consequently, σ measures the effective range of interactions. Similarly, the parameter K measures the maximum strength of interactions. Hence, one can propose the following scaling

$$\sigma = \mathcal{O}(\varepsilon), \quad K\sigma^{2\alpha} = \mathcal{O}(1), \quad \text{when } \varepsilon \to 0.$$

Or more specifically, using the change of variables

$$\sigma \to \varepsilon \text{ and } K \to K\varepsilon^{-2\alpha},$$

where ε is a dimensionless parameter, we arrive at the next scaled system

$$\dot{\theta}_i = \Omega_i + \frac{K}{N}\sum_{j=1}^{N} h_\varepsilon(\theta_j - \theta_i), \tag{3.2.5}$$

where the scaled interaction kernel now reads

$$h_\varepsilon(\theta) := \frac{\sin\theta}{(\varepsilon^2 + c_{\alpha,\zeta}|\theta|_o^2)^\alpha}. \tag{3.2.6}$$

If we formally take limits when $\varepsilon \to 0$, then we arrive at the desired singular weighted Kuramoto model, whose singular interaction kernel is

$$h(\theta) := \frac{\sin\theta}{c_{\alpha,\zeta}^\alpha |\theta|_o^{2\alpha}}.$$

All these arguments are heuristic. However they might become rigorous depending on the value of α. For a rigorous derivation of the singular limit in all the subcritical, critical and supercritical regimes, see Section 3.4.

3.2.3 Emergence of clusters: collision and sticking of oscillators

In this part we introduce some notation that will be used along the whole chapter. We will denote the set of pair-wise collisions of the i-th and j-th oscillators by

$$\mathcal{C}_{ij} := \{\Theta \in \mathbb{R}^N : \bar{\theta}_i = \bar{\theta}_j\},$$

where $\bar{\theta}$ denotes again the representative of θ in $(-\pi, \pi]$. Then, the set of collisions reads

$$\mathcal{C} := \bigcup_{i \neq j} \mathcal{C}_{ij} = \{\Theta \in \mathbb{R}^N : \exists\, i \neq j \text{ such that } \bar{\theta}_i = \bar{\theta}_j\}.$$

Consider any phase configuration of the N oscillators, i.e.,

$$\Theta = (\theta_1, \ldots, \theta_N) \in \mathbb{R}^N.$$

We will say that the i-th oscillator *collides* with j-th oscillator when $\Theta \in \mathcal{C}_{ij}$ and we will say that Θ is a *collision state* when $\Theta \in \mathcal{C}$. In order to manage with collisions, let us define the following binary relation

$$i \overset{\Theta}{\sim} j \text{ when } \Theta \in \mathcal{C}_{ij}.$$

Since it is an equivalence relation, we can denote its equivalence classes by

$$\mathcal{C}_i(\Theta) := \{j \in \{1, \ldots, N\} : i \overset{\Theta}{\sim} j\} = \{j \in \{1, \ldots, N\} : \Theta \in \mathcal{C}_{ij}\}. \tag{3.2.7}$$

As it is apparent from the definition, $\mathcal{C}_i(\Theta)$ is the set of *indices of collision* with the i-th oscillator. Then, Θ is a collision state when some of its equivalence classes is non-trivial. Consequently, each of the equivalence classes can be regarded as a *cluster of oscillators* with the same phase value. Let us denote by $\mathcal{E}(\Theta)$ the family of all the different equivalence classes (clusters). It is apparent that $\mathcal{E}(\Theta)$ determines a partition of $\{1,\ldots,N\}$, that we will call the *collisional type* of Θ. It represents the way oscillators are arranged into clusters. For simplicity of notation, we shall enumerate the equivalence classes

$$\mathcal{E}(\Theta)=\{E_1(\Theta),\ldots,E_{\kappa(\Theta)}(\Theta)\},$$

in such a way that the minimal representatives in each of them, i.e., $\iota_k(\Theta):=\min E_k(\Theta)$, are increasingly ordered, that is,

$$\iota_1<\iota_2<\cdots<\iota_\kappa,$$

where $\kappa(\Theta):=\#\mathcal{E}(\Theta)$ represents the total amount of different clusters for such phase configuration Θ. Finally, we shall denote $n_k(\Theta):=\#E_k(\Theta)$ to the size of the k-th cluster, that is the number of particles which form the k-th cluster, for each $k=1,\ldots,\kappa(\Theta)$.

Assume now that not only do we know some phase configuration at a particular time, but a whole absolutely continuous trajectory $t\mapsto\Theta(t)=(\theta_1(t),\ldots\theta_N(t))\in\mathbb{R}^N$ governing the dynamics of the N oscillators. It is clear that the collisional type might change from time to time subordinated to the dynamics of phases itself. Then, as long as it is clear from the context, we shall omit the dependence on the particular trajectory and will simplify the notation as follows

$$\mathcal{C}_i(t):=\mathcal{C}_i(\Theta(t)),\ \mathcal{E}(t):=\mathcal{E}(\Theta(t)),\ \kappa(t):=\kappa(\Theta(t)),\ n_k(t):=n_k(\Theta(t)).$$

Similarly, time may be omitted in our notation for simplicity. Apart form collisions into clusters, it is important to characterize when those clusters remain stuck together. If the i-th and j-th oscillators have collided at time t, we will say that they *stick together* (for all times) when

$$\bar{\theta}_i(s)=\bar{\theta}_j(s),\ \text{ for all } s\geq t.$$

Then, we can define the set of *indices of sticking* with the i-th oscillator by

$$S_i(t):=\{j\in\mathcal{C}_i(t):\ \bar{\theta}_i(s)=\bar{\theta}_j(s),\ \text{ for all } s\geq t\}. \tag{3.2.8}$$

In Section 3.3 we will introduce some results about *clustering* and *sticking behavior* of solutions to our singular weighted Kuramoto model corresponding to (3.2.5)-(3.2.6) with $\varepsilon=0$.

3.2.4 Gradient flow structure

In this part, let us remark that our system (3.2.1) can be equivalently turned into a gradient flow system:

$$\dot{\Theta}=-\nabla V(\Theta), \tag{3.2.9}$$

governed by a potential $V=V(\Theta)$ that is defined by

$$V(\Theta)=-\sum_{i=1}^N\Omega_i\theta_i+V_{int}(\Theta):=-\sum_{i=1}^N\Omega_i\theta_i+\frac{K}{2N}\sum_{i\neq j}W(\theta_j-\theta_i). \tag{3.2.10}$$

Here, W is the primitive function of h such that $W(0)=0$, i.e.,

$$W(\theta):=\int_0^\theta h(\theta')\,d\theta'. \tag{3.2.11}$$

The function W can be regarded as the interaction potential of binary interactions while V_{int} stands for the total interaction potential due to binary interactions. This approach is obviously formal and depends on the specific regularity of the plasticity function Γ. For instance, if we choose Γ to be analytic, then (3.2.1) can be regarded as a gradient flow system with analytic potential V. In such particular case, one can oversimplify the proof of emergence of synchronization like in the classical Kuramoto model, see [147]. Specifically, some boundedness property of the trajectory is all we need to ensure the exponential convergence towards a phase-locked state by virtue of the Łojasiewicz inequality for analytic functions. For the choices of plasticity function of interest in this chapter, i.e., either (3.1.5) or (3.1.7), analyticity is lacking and the same approach does not necessarily work. Nevertheless, we shall focus on values of the parameter α that belong to the range $\alpha \in (0,1)$ and produce potentials V globally a continuous and smooth off the set of collisions $\mathcal{C}$. Since in general we are missing either analyticity or convexity of V, the gradient flow structure will not be used much along this chapter, except in Subsections 3.4.2 and 3.4.3.

3.2.5 Kinetic formulation of the problem

In this part, let us formally introduce the expected kinetic models associated with (3.2.5). Also see the discussion in Section 1.1.2 of the introductory Chapter 1 about mean-field limit and propagation of chaos and the bibliography [163, 164, 176, 177, 178, 179, 181, 216, 217, 230, 281].

On the one-hand, for every $\varepsilon > 0$ the mean field limit for the distribution function of oscillators $f_\varepsilon = f_\varepsilon(t,\theta,\Omega)$ is governed by the following Vlasov equation with regular kernels

$$\frac{\partial f_\varepsilon}{\partial t} + \frac{\partial}{\partial \theta}\left[(\Omega - Kh_\varepsilon * \rho_\varepsilon f_\varepsilon\right] = 0, \;\; t \in \mathbb{R}_0^+,\, \theta \in [0,2\pi],\, \Omega \in \mathbb{R}, \tag{3.2.12}$$

where periodic boundary conditions in the variable θ are assumed. Here the macroscopic phase-density ρ_ε is nothing but

$$\rho_\varepsilon(t,\theta) := \int_{\mathbb{R}} f_\varepsilon \, d\Omega.$$

Similarly, when $\varepsilon = 0$ the corresponding mean field limit for the distribution function of oscillators for $f = f(t,\theta,\Omega)$ is subject to a Vlasov equation with singular kernels

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial \theta}\left[(\Omega - Kh *_\rho f\right] = 0, \;\; t \in \mathbb{R}_0^+,\, \theta \in [0,2\pi],\, \Omega \in \mathbb{R}, \tag{3.2.13}$$

with analogous periodic conditions in θ. The derivation of the mean field limit is more involved in this latter case and requires a sharper analys, see references [59, 163, 226] for related singular models like. Let us briefly recall the main formal idea supporting the above mean field limit through the *empirical measures approach*. Fix the following empirical measure as initial condition in (3.2.13)

$$\mu_0^N(\theta,\Omega) = \frac{1}{N}\sum_{i=1}^N \delta_{\theta_{i,0}^N}(\theta)\delta_{\Omega_i^N}(\Omega),$$

associated to some discrete initial configuration $\Theta_0^N = (\theta_{1,0}^N, \ldots, \theta_{N,0}^N)$. Because of the results in this chapter, the Filippov solution $\Theta^N(t) = (\theta_1^N(t), \ldots, \theta_N^N(t))$ to the singular discrete model allows considering the next measure-valued solution to (3.2.13)

$$\mu_t^N(\theta,\Omega) = \frac{1}{N}\sum_{i=1}^N \delta_{\theta_i^N(t)}(\theta)\delta_{\Omega_i^N}(\Omega).$$

The ultimate effort to be done is to show that the weak limit f of μ^N as $N \to \infty$ is another measure-valued solution in some generalized sense to the singular macroscopic system.

We postpone this topic to the forthcoming Chapter 4, where a comprehensive analysis of the singular macroscopic model (3.2.13) has been conducted. Also, see [226] for a close approach in the Cucker–Smale model with weakly singular influence kernel corresponding to the smaller range of parameters $\alpha \in (0, \frac{1}{4})$ of the subcritical regime. Analogue results in aggregation models and classical Kuramoto model has been studied in [59, 64, 67] and [58, 198] respectively.

3.3 Well–posedness of the singular weighted system

We now consider the Kuramoto model with singular coupling that we introduced in Section 3.2. For simplicity, we will forget about the constant $c = c_{\alpha,\zeta} = 1 - \zeta^{-1/\alpha}$ and the system gets simplified as follows

$$\dot{\theta}_i = \Omega_i + \frac{K}{N}\sum_{j=1}^{N} h(\theta_j - \theta_i), \tag{3.3.1}$$

for $i = 1, \ldots, N$, with coupling force given by

$$h(\theta) := \frac{\sin\theta}{|\theta|_o^{2\alpha}}. \tag{3.3.2}$$

Regarding the parameter α, it belongs to the interval $(0, 1)$ to allow for mild singularities. Note that the kernel is continuous for $\alpha \in (0, \frac{1}{2})$, it exhibits a jump discontinuity at $\theta \in 2\pi\mathbb{Z}$ for $\alpha = \frac{1}{2}$, and it shows essential discontinuities for $\alpha \in (\frac{1}{2}, 1)$, see Figure 3.1.

In this section, we shall focus on developing the well-posedness theory of such system (3.3.1)-(3.3.2) of coupled ODEs. Note that uniqueness is not trivial even in the subcritical case. Indeed, due to the choice of singular plasticity function, the coupling force (3.3.2) is not Lipschitz-continuous in any of the subcritical, critical and supercritical regimes. Thus, we must explore existence and uniqueness of (generalized) solutions to the singular weighted system before we proceed with the study of synchronization. Notice that the main obstruction appears after each collision time, where classical solutions stop existing. This contribution is then valuable as we are able to specify in which sense classical solutions can be extended after such collision times. For the following discussion, we recall the definition of the vector field $H = H(\Theta)$ in (3.2.2) that allows stating (3.3.1)-(3.3.2) in the vector form (3.2.3).

3.3.1 Well–posedness in the subcritical regime

In the subcritical case, namely $\alpha \in \left(0, \frac{1}{2}\right)$, the vector field $H = H(\Theta)$ in (3.2.2) is continuous. Therefore, it is a clear consequence of Peano's theorem that (3.3.1)-(3.3.2) has a local-in-time solution for every initial configuration $\Theta(0) = \Theta_0 \in \mathbb{R}^N$. Unfortunately, note that $h(\theta)$ has unbounded slope at the phase values $\theta \in 2\pi\mathbb{Z}$. Indeed, the modulus of continuity of h is strictly worst than Lipschitz as it is shown in this result.

Lemma 3.3.1. *Consider $\alpha \in (0, \frac{1}{2})$. Then, h is a $(1-2\alpha)$-Hölder continuous periodic function, namely,*

$$|h(\theta_1) - h(\theta_2)| \leq \cosh\pi\,|\theta_1 - \theta_2|_o^{1-2\alpha},$$

for every couple $\theta_1, \theta_2 \in \mathbb{R}$.

Proof. Taking appropriate representatives for the phases, we can assume that $\theta_1 - \theta_2 \in (-\pi, \pi]$ without loss of generality. To simplify the proof, we shall divide it into two different cases:

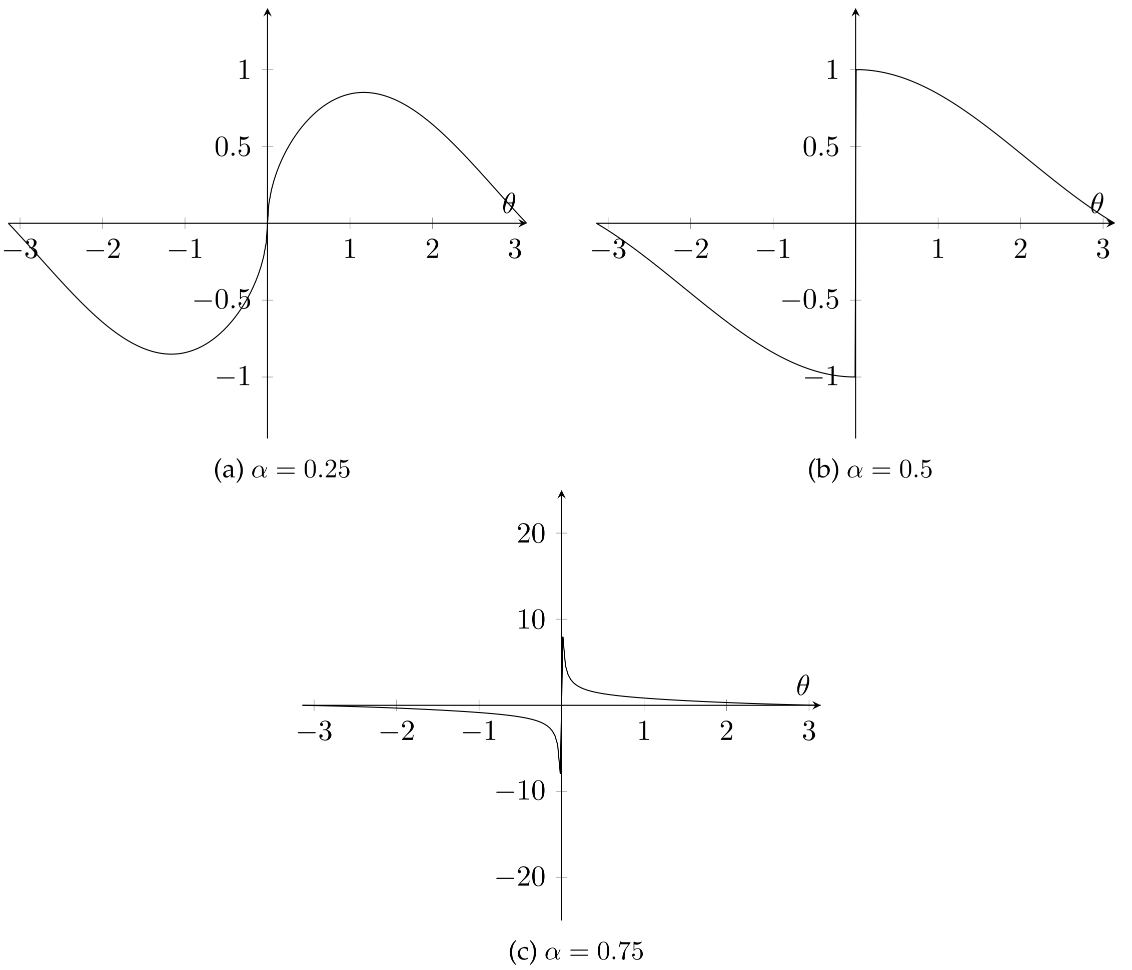


Figure 3.1: Plot of the interaction kernel $h = h(\theta)$ in (3.3.2) for the values $\alpha = 0.25$, $\alpha = 0.5$ and $\alpha = 0.75$, respectively.

• *Case 1*: $\theta_1 - \theta_2 \in [0, \pi]$. By the Taylor expansion of the sine function, we obtain

$$h(\theta_1) - h(\theta_2) = \frac{\sin\theta_1}{\theta_1^{2\alpha}} - \frac{\sin\theta_2}{\theta_2^{2\alpha}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(\theta_1^{2n+1-2\alpha} - \theta_2^{2n+1-2\alpha}).$$

Since $\alpha \in (0, \frac{1}{2})$, then $1 - 2\alpha \in (0, 1)$ and, consequently, for $n = 0$ we infer the next estimate

$$|\theta_1^{1-2\alpha} - \theta_2^{1-2\alpha}| \leq |\theta_1 - \theta_2|^{1-2\alpha}.$$

For any other $n \geq 1$ we apply the mean value theorem to obtain

$$\begin{aligned} |\theta_1^{2n+1-2\alpha} - \theta_2^{2n+1-2\alpha}| &\leq (2n+1-2\alpha)\pi^{2n-2\alpha}|\theta_1 - \theta_2| \\ &\leq (2n+1-2\alpha)\pi^{2n}|\theta_1 - \theta_2|^{1-2\alpha} \\ &\leq (2n+1)\pi^{2n}|\theta_1 - \theta_2|^{1-2\alpha}. \end{aligned}$$

Putting everything together, we achieve the desired estimate

$$|h(\theta_1) - h(\theta_2)| \leq \sum_{n=0}^{\infty} \frac{\pi^{2n}}{(2n)!}|\theta_1 - \theta_2|^{1-2\alpha} = \cosh\pi\,|\theta_1 - \theta_2|^{1-2\alpha}.$$

• *Case 2*: $\theta_1 - \theta_2 \in (-\pi, 0]$. By the antisymmetry of the kernel h with respect to the origin, we can reduce this case to the latter one, thus we omit the proof.

Finally, in order to end the proof notice that $\theta_1 - \theta_2 \in (-\pi, \pi]$ and, consequently, $|\theta_1 - \theta_2| = |\theta_1 - \theta_2|_o = d(z_1, z_2)$. □

Such loss of Lipschitz-continuity of $H = H(\Theta)$ causes some problems and, in particular, the classical Cauchy–Picard–Lindelöf theorem does not apply. Hence, the study of uniqueness requires an alternative approach that we discuss in the sequel. Roughly speaking the method is supported by the following fact: near the points of loss of Lipschitz-continuity our vector field can be locally split into the sum of a decreasing vector field and a Lipschitz-continuous vector field, then ensuring the local one-sided Lipschitz condition that is enough to obtain a forward one-sided uniqueness result.

Lemma 3.3.2. *Let $F : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ be a bounded and continuous vector field and assume that for every $x^* \in \mathbb{R}^N$ there exists some open neighborhood $V \subseteq \mathbb{R}^N$ and a positive constant M so that F verifies the one-sided Lipschitz condition in V*

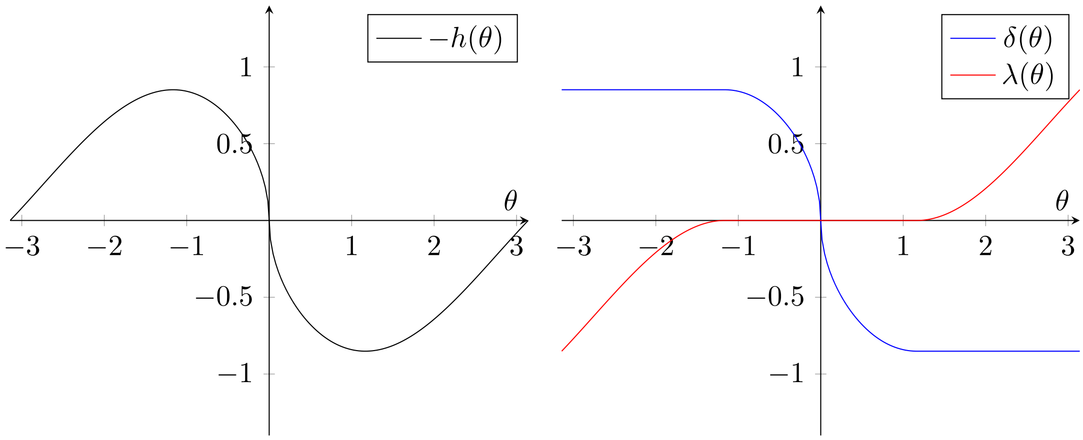
$$(F(x) - F(y)) \cdot (x - y) \leq M|x - y|^2,$$

for every couple $x, y \in V$. Then, the following initial value problem (IVP) associated with any initial configuration $x_0 \in \mathbb{R}^N$ enjoys one global-in-time solution, that is unique forward in time

$$\begin{cases} \dot{x} = F(x), & t \geq 0, \\ x(0) = x_0. \end{cases}$$

Since the proof is classical, we omit it here. Let us now apply such result to our case of interest. To do so, it is enough to introduce a decomposition of the vector field $H = H(\Theta)$ in the Kuramoto model (3.3.1)-(3.3.2). We first set the following split of the interaction function $h = h(\theta)$. First, consider $\overline{h}$ and $\tilde{\theta} \in \left(0, \frac{\pi}{2}\right)$ such that

$$\overline{h} := \max_{0<r<\pi} h(r) \quad \text{and} \quad 2\alpha\sin\tilde{\theta} = \tilde{\theta}\cos\tilde{\theta}.$$



(a) $-h(\theta)$

(b) $\delta(\theta)$ and $\lambda(\theta)$

Figure 3.2: Plot of the function $-h(\theta)$ and the functions $\delta(\theta)$ and $\lambda(\theta)$ in the decomposition for the value $\alpha = 0.25$.

Note that $\tilde{\theta}$ is uniquely defined as the value in $(0, \pi)$ where h attains its maximum. Second, define the couple of functions $\delta = \delta(\theta)$ and $\lambda = \lambda(\theta)$ in $(-\pi, \pi)$ as follows

$$\delta(\theta) := \begin{cases} \overline{h}, & \text{for } \theta \in (-\pi, -\tilde{\theta}), \\ -h(\theta), & \text{for } \theta \in [-\tilde{\theta}, \tilde{\theta}), \\ -\overline{h}, & \text{for } \theta \in [\tilde{\theta}, \pi), \end{cases}$$

$$\lambda(\theta) := \begin{cases} -\overline{h} - h(\theta), & \text{for } \theta \in (-\pi, -\tilde{\theta}), \\ 0, & \text{for } \theta \in [-\tilde{\theta}, \tilde{\theta}), \\ \overline{h} - h(\theta), & \text{for } \theta \in [\tilde{\theta}, \pi). \end{cases}$$

Notice that

$$-h(\theta) = \delta(\theta) + \lambda(\theta), \quad \text{for all } \theta \in (-\pi, \pi), \tag{3.3.3}$$

as depicted in Figure 3.2.

Remark 3.3.3. *Note that although $-h(\theta)$ is not a Lipschitz-continuous function because of the unbounded slope at $\theta \in 2\pi\mathbb{Z}$, we can locally decompose it around such values in terms of a decreasing function $\delta(\theta)$ and a Lipschitz-continuous function $\lambda(\theta)$.*

Finally, consider any value $\Theta^* = (\theta_1^*, \ldots, \theta_N^*) \in \mathbb{R}^N$ to locally decompose H around it. For $\Theta = (\theta_1, \ldots, \theta_N)$ in a small enough neighborhood $\mathcal{V}$ of Θ^* in $\mathbb{R}^N$, we set

$$\Delta(\Theta) := \frac{K}{N} \sum_{j \in \mathcal{C}_i(\Theta^*)} \delta\left(\overline{\theta_i - \theta_j}\right), \tag{3.3.4}$$

$$\Lambda(\Theta) := \Omega_i + \frac{K}{N} \sum_{j \in \mathcal{C}_i(\Theta^*)} \lambda\left(\overline{\theta_i - \theta_j}\right) - \frac{K}{N} \sum_{j \notin \mathcal{C}_i(\Theta^*)} h(\theta_i - \theta_j), \tag{3.3.5}$$

where we recall that $\mathcal{C}_i(\Theta^*)$ stands for the set of indices of collision with the i-th oscillator in the phase configuration Θ^*, see Subsection 3.2.3.

Proposition 3.3.4. *Let $\Theta^* = (\theta_1^*, \ldots, \theta_N^*) \in \mathbb{R}^N$, and define the vector fields*

$$\Delta : \mathcal{U} \longrightarrow \mathbb{R}^N, \quad \Lambda : \mathcal{U} \longrightarrow \mathbb{R}^N,$$

via the formulas (3.3.4)-(3.3.5), for a small enough neighborhood $\mathcal{U}$ of Θ^* in $\mathbb{R}^N$. Then,

1. *$H = \Delta + \Lambda$ in $\mathcal{U}$.*
2. *Δ is decreasing in $\mathcal{U}$.*
3. *Λ is Lipschitz-continuous in $\mathcal{U}$.*
4. *H is one-sided Lipschitz continuous in $\mathcal{U}$.*

Proof. The decomposition of H is clear by virtue of the decomposition (3.3.3) and the definitions (3.3.4)-(3.3.5). Let us then focus on the last three properties. Fist, consider $\Theta = (\theta_1, \ldots, \theta_N), \widetilde{\Theta} = (\widetilde{\theta}_1, \ldots, \widetilde{\theta}_N) \in \mathbb{R}^N$ in a small enough neighborhood of Θ^*. Without loss of generality, we will directly assume that $\theta_i - \theta_j$ and $\widetilde{\theta}_i - \widetilde{\theta}_j$ belong to $(-\pi, \pi]$. In other case, we just need to work with the representatives. On the one hand,

$$(\Delta(\Theta) - \Delta(\widetilde{\Theta})) \cdot (\Theta - \widetilde{\Theta}) = \frac{K}{N} \sum_{i=1}^N \sum_{j \in \mathcal{C}_i(\Theta^*)} (\delta(\theta_i - \theta_j) - \delta(\widetilde{\theta}_i - \widetilde{\theta}_j))(\theta_i - \widetilde{\theta}_i).$$

Changing the indices i and j we obtain

$$\begin{aligned}(\Delta(\Theta) - \Delta(\widetilde{\Theta})) \cdot (\Theta - \widetilde{\Theta}) &= \frac{K}{N} \sum_{j=1}^N \sum_{i \in \mathcal{C}_j(\Theta^*)} (\delta(\theta_j - \theta_i) - \delta(\widetilde{\theta}_j - \widetilde{\theta}_i))(\theta_j - \widetilde{\theta}_j) \\ &= -\frac{K}{N} \sum_{i=1}^N \sum_{j \in \mathcal{C}_i(\Theta^*)} (\delta(\theta_i - \theta_j) - \delta(\widetilde{\theta}_i - \widetilde{\theta}_j))(\theta_j - \widetilde{\theta}_j),\end{aligned}$$

where the properties of the sets $\mathcal{C}_i(\Theta^*)$ along with the antisymmetry of δ have been used in the last line. Taking the mean value of both expressions and using that δ is decreasing, we arrive at

$$(\Delta(\Theta) - \Delta(\widetilde{\Theta})) \cdot (\Theta - \widetilde{\Theta}) = \frac{K}{2N} \sum_{i=1}^N \sum_{j \in \mathcal{C}_i(\Theta^*)} (\delta(\theta_i - \theta_j) - \delta(\widetilde{\theta}_i - \widetilde{\theta}_j))((\theta_i - \theta_j) - (\widetilde{\theta}_i - \widetilde{\theta}_j)) \leq 0,$$

and, as a consequence, to the monotonicity of Δ. On the other hand,

$$|\Lambda_i(\Theta) - \Lambda_i(\widetilde{\Theta})| \leq \frac{K}{N} \sum_{j \in \mathcal{C}_i(\Theta^*)} |\lambda(\theta_i - \theta_j) - \lambda(\widetilde{\theta}_i - \widetilde{\theta}_j)| + \frac{K}{N} \sum_{j \notin \mathcal{C}_i(\Theta^*)} |h(\theta_i - \theta_j) - h(\widetilde{\theta}_i - \widetilde{\theta}_j)|.$$

Since λ is Lipschitz-continuous in $(-\pi, \pi)$ and h is locally Lipschitz-continuous in $(-\pi, \pi) \setminus \{0\}$, then there exists some constant $M = M(\mathcal{U})$ so that

$$|\Lambda_i(\Theta) - \Lambda_i(\widetilde{\Theta})| \leq \frac{KM}{N} \sum_{j=1}^N |(\theta_i - \theta_j) - (\widetilde{\theta}_i - \widetilde{\theta}_j)| \leq \frac{N+1}{N} KM |\Theta - \widetilde{\Theta}|,$$

for every index $i \in \{1, \ldots, N\}$, thus yielding the Lipschitz-continuity of Λ in $\mathcal{U}$. The last part is a simple consequence of all the other properties. Namely, consider $x, y \in \mathcal{U}$ and note that

$$(H(x) - H(y)) \cdot (x - y) = (\Delta(x) - \Delta(y)) \cdot (x - y) + (\Lambda(x) - \Lambda(y)) \cdot (x - y) \leq \frac{N+1}{N} KM |x - y|^2,$$

where we have used the above two properties along with the Cauchy–Schwartz inequality. □

Finally, putting together Lemma 3.3.2 and Proposition 3.3.4, one concludes the following well-posedness property.

Theorem 3.3.5. *There is one global-in-time strong solution to the system* (3.3.1)-(3.3.2)*, with* $\alpha \in (0, \frac{1}{2})$*, which is unique forwards in time, for any initial configuration.*

The next result is a simple consequence of the above well-posedness theorem and characterizes the eventual emergence of sticking in a cluster after a potential collision.

Theorem 3.3.6. *Consider* $\Theta = (\theta_1, \ldots, \theta_N)$*, the global-in-time solution in Theorem 3.3.5. Assume that two oscillators collide at* t^**, i.e.,* $\bar{\theta}_i(t^*) = \bar{\theta}_j(t^*) = \theta^*$ *for some* $i \neq j$*. Then, the following two statements are equivalent:*

1. θ_i *and* θ_j *stick together at* t^**.*

2. *Their natural frequencies agree, i.e.,*

$$\Omega_i = \Omega_j. \tag{3.3.6}$$

Proof. Without loss of generality, let us assume that $i = 1$, $j = 2$ and $\theta_1(t^*) = \theta_2(t^*) \in (-\pi, \pi]$. Assume that the two particles keep stuck together after time t^*. Then, looking at the first two equations in system (3.3.1)-(3.3.2) it is clear that $\Omega_1 = \Omega_2$. Conversely, let us assume that $\Omega_1 = \Omega_2 =: \Omega$ and consider the following system of $N-1$ coupled ODEs.

$$\dot{\vartheta} = \Omega + \frac{K}{N} \sum_{j=3}^{N} h(\vartheta_j - \vartheta),$$
$$\dot{\vartheta}_i = \Omega_i + \frac{2K}{N} h(\vartheta - \vartheta_i) + \frac{K}{N} \sum_{j=3}^{N} h(\vartheta_j - \vartheta_i), \;\; i = 3, \ldots, N,$$

with initial data given by

$$(\vartheta(t^*), \vartheta_3(t^*), \ldots, \vartheta_N(t^*)) = (\theta^*, \theta_3(t^*), \ldots, \theta_N(t^*)).$$

A similar technique to that in Theorem 3.3.5 clearly yields a global-in-time solution to such initial value problem. Hence, the following two trajectories in $\mathbb{R}^N$

$$t \mapsto (\theta_1(t), \theta_2(t), \theta_3(t), \ldots, \theta_N(t)),$$
$$t \mapsto (\vartheta(t), \vartheta(t), \vartheta_3(t), \ldots, \vartheta_N(t)),$$

are both solutions to (3.3.1)-(3.3.2) such that at $t = t^*$ they take the value

$$(\theta^*, \theta^*, \theta_3(t^*), \ldots, \theta_N(t^*)).$$

By uniqueness they agree and, in particular, $\theta_1(t) = \vartheta(t) = \theta_2(t)$ for all $t \geq t^*$. □

3.3.2 Well–posedness in the critical regime

In the critical case, i.e. $\alpha = \frac{1}{2}$, the vector field $H = H(\Theta)$ is no longer continuous and the Peano existence theorem does not work. Nevertheless, in such case H is still a measurable and essentially bounded vector field. Consequently, one can apply *Filippov's existence criterion*, see [14, 130]. Such method provides a criterion to ensure the existence of *Filippov-solutions* to (3.3.1)-(3.3.2), that is, absolutely continuous trajectories

$$t \in [0, +\infty) \longmapsto \Theta(t) = (\theta_1(t), \ldots, \theta_N(t)),$$

that solve the differential inclusion

$$\begin{cases} \dot{\Theta}(t) \in \mathcal{H}(\Theta(t)), & \text{a.e. } t \geq 0, \\ \Theta(0) = \Theta_0, \end{cases} \tag{3.3.7}$$

where $\mathcal{H} = \mathcal{H}(\Theta)$ is the *Filippov set-valued map* associated with $H = H(\Theta)$ for $\alpha = \frac{1}{2}$, i.e.,

$$\mathcal{H}(\Theta) = \bigcap_{\delta > 0} \bigcap_{|\mathcal{N}| N} \overline{\text{co}}(H(B_\delta(\Theta) \setminus \mathcal{N})),$$

for any $\Theta \in \mathbb{R}^N$, where $|\mathcal{N}|$ represents the Lebesgue measure of a measurable subset $\mathcal{N} \subseteq \mathbb{R}^N$ and $\overline{co}(A)$ represents the closed convex hull of a subset $A \subseteq \mathbb{R}^N$. See also Definition D.1.1 and the remaining results in Appendix D for further details about Filippov's theory that will be used later along this part. However, before Filippov existence theory, let us first explore a more explicit representation of $\mathcal{H} = \mathcal{H}(\Theta)$ to gain some intuition about the Filippov set-valued map of many-particle systems with discontinuous forces. We recall Subsection 3.2.3 about the collision equivalence relation and the necessary notation to deal with clusters of oscillators.

Proposition 3.3.7. *In the critical regime $\alpha = \frac{1}{2}$, the Filippov set-valued map $\mathcal{H} = \mathcal{H}(\Theta)$ associated with $H = H(\Theta)$ stands for the convex and compact polytope consisting of the points $(\omega_1, \ldots, \omega_N) \in \mathbb{R}^N$ such that*

$$\omega_i = \Omega_i + \frac{K}{N} \sum_{j \notin \mathcal{C}_i(\Theta)} h(\theta_j - \theta_i) + \frac{K}{N} \sum_{j \in \mathcal{C}_i(\Theta) \setminus \{i\}} y_{ij}, \quad \textit{for all } i = 1, \ldots, N,$$

for some $Y = (y_{ij})_{1 \leq i,j \leq N} \in \text{Skew}_N([-1, 1])$.

Here, $\text{Skew}_N([-1, 1])$ represents the space of skew-symmetric $N \times N$ matrices with items in the interval $[-1, 1]$. Since the proof is straightforward by Definition D.1.1 of the Filippov set-valued map, we omit it here.

Remark 3.3.8. *Notice that for every $(\omega_1, \ldots, \omega_N) \in \mathcal{H}(\Theta)$ the next property holds true*

$$\sum_{i=1}^N \omega_i = \sum_{i=1}^N \Omega_i.$$

In particular, every Filippov solution $(\theta_1, \ldots, \theta_N)$ to (3.3.1)-(3.3.2)*, in the case $\alpha = \frac{1}{2}$, verifies*

$$\sum_{i=1}^N \dot{\theta}_i(t) = \sum_{i=1}^N \Omega_i, \textit{ for a.e. } t \geq 0.$$

Hence, Filippov solutions in the critical case still preserve the average frequency like classical solutions do for the subcritical case or the original Kuramoto model.

Example 3.3.9. *In order to gain some intuition about those sets, let us exhibit some particular examples:*

1. *For every* $N \in \mathbb{N}$*, if* $\Theta \notin \mathcal{C}$*, then* $\mathcal{H}(\Theta) = \{H(\Theta)\}$*.*

2. *For* $N = 2$*, if* $\Theta = (\theta_1, \theta_2) \in \mathcal{C}_{12}$*, then* $\mathcal{H}(\Theta)$ *is the polytope consisting of points* $(\omega_1, \omega_2) \in \mathbb{R}^2$ *such that*
$$\omega_1 = \Omega_1 + \frac{K}{2} y_{12},$$
$$\omega_2 = \Omega_2 - \frac{K}{2} y_{12},$$
for some $y_{12} \in [-1, 1]$*.*

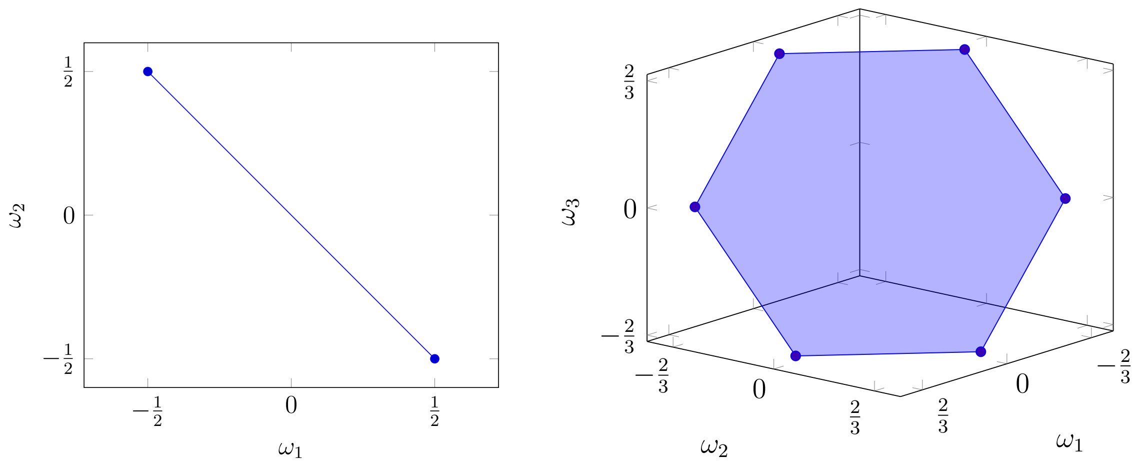
3. *For* $N = 3$*, if* $\Theta = (\theta_1, \theta_2, \theta_3) \in \mathcal{C}_{12} \setminus \mathcal{C}_{13}$*, then* $\mathcal{H}(\Theta)$ *is the polytope consisting of the points* $(\omega_1, \omega_2, \omega_3) \in \mathbb{R}^3$ *such that*
$$\omega_1 = \Omega_1 + \frac{K}{3} h(\theta_3 - \theta_1) + \frac{K}{3} y_{12},$$
$$\omega_2 = \Omega_2 + \frac{K}{3} h(\theta_3 - \theta_2) - \frac{K}{3} y_{12},$$
$$\omega_3 = \Omega_3 + \frac{K}{3} h(\theta_1 - \theta_3) + \frac{K}{3} h(\theta_2 - \theta_3),$$
for some $y_{12} \in [-1, 1]$*. That is a line segment, see Figure 3.3a.*

4. *For* $N = 3$*, if* $\Theta = (\theta_1, \theta_2, \theta_3) \in \mathcal{C}_{12} \cap \mathcal{C}_{13}$*, then* $\mathcal{H}(\Theta)$ *is the polytope consisting of the points* $(\omega_1, \omega_2, \omega_3) \in \mathbb{R}^3$ *such that*
$$\omega_1 = \Omega_1 + \frac{K}{3} y_{12} + \frac{K}{3} y_{13},$$
$$\omega_2 = \Omega_2 - \frac{K}{3} y_{12} + \frac{K}{3} y_{23},$$
$$\omega_3 = \Omega_3 - \frac{K}{3} y_{13} - \frac{K}{3} y_{23}.$$
for some $y_{12}, y_{13}, y_{23} \in [-1, 1]$*. This is a regular hexagon, see Figure 3.3b.*

Finally, let us apply Filippov theory in Appendix D to construct the unique Filippov solutions of our particular system (3.3.1)-(3.3.2) in the critical case $\alpha = \frac{1}{2}$. Notice first that global existence is guaranteed by Lemma D.1.5 thanks to the boundedness of the vector field H. However, uniqueness is less apparent. The way to go is similar to that in the preceding Subsection 3.3.1 and relies on a good decomposition of $-h$. Define the couple of function $\delta = \delta(\theta)$ and $\lambda = \lambda(\theta)$ in $(-\pi, \pi)$ as follows

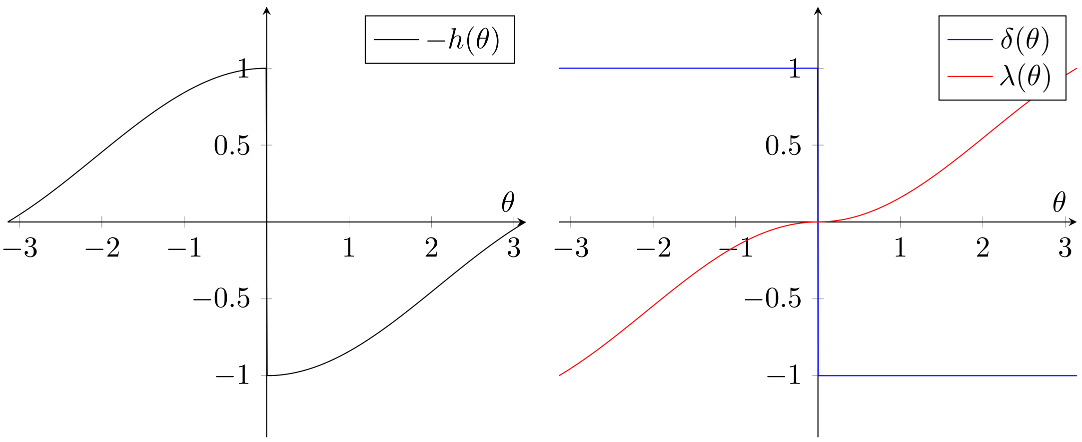
$$\delta(\theta) := \begin{cases} 1 & \text{for} \quad \theta \in (-\pi, 0), \\ -1, & \text{for} \quad \theta \in [0, \pi), \end{cases}$$
$$\lambda(\theta) := \begin{cases} -1 - h(\theta), & \text{for} \quad \theta \in (-\pi, 0), \\ 1 - h(\theta), & \text{for} \quad \theta \in [0, \pi). \end{cases}$$

Notice that
$$-h(\theta) = \delta(\theta) + \lambda(\theta), \quad \text{for all } \theta \in (-\pi, \pi), \tag{3.3.8}$$
as depicted in Figure 3.4.



(a) $N = 2$ (b) $N = 3$

Figure 3.3: Pictures of the Filippov set-valued map in the critical case at a total collision phase configuration. In Figure 3.3a, $N = 2$ and the polytope is a line segment joining $\left(\Omega_1 \pm \frac{K}{2}, \Omega_2 \mp \frac{K}{2}\right)$. In Figure 3.3b, $N = 3$ and the polytope is a regular hexagon with vertices $\left(\Omega_1 \pm \frac{2K}{3}, \Omega_2 \mp \frac{2K}{3}, \Omega_3\right)$, $\left(\Omega_1 \pm \frac{2K}{3}, \Omega_2, \Omega_3 \mp \frac{2K}{3}\right)$ and $\left(\Omega_1, \Omega_2 \pm \frac{2K}{3}, \Omega_3 \mp \frac{2K}{3}\right)$. For simplicity, the natural frequencies are set to zero and $K = 1$ in the figures.



(a) $-h(\theta)$ (b) $\delta(\theta)$ and $\lambda(\theta)$

Figure 3.4: Plot of the function $-h(\theta)$ and the functions $\delta(\theta)$ and $\lambda(\theta)$ in the decomposition for the value $\alpha = 0.5$.

Remark 3.3.10. *Note that although $-h(\theta)$ is a continuous function because of the jump discontinuities at $\theta \in 2\pi\mathbb{Z}$, one can locally decompose it around such values in terms of a decreasing function $\delta(\theta)$ and a Lipschitz-continuous function $\lambda(\theta)$.*

Finally, for every $\Theta^* = (\theta_1^*, \ldots, \theta_N^*) \in \mathbb{R}^N$ we locally decompose H around it as follows

$$\Delta_i(\Theta) := \frac{K}{N} \sum_{j \in \mathcal{C}_i(\Theta^*)} \delta\left(\overline{\theta_i - \theta_j}\right), \tag{3.3.9}$$

$$\Lambda_i(\Theta) := \Omega_i + \frac{K}{N} \sum_{j \in \mathcal{C}_i(\Theta^*)} \lambda\left(\overline{\theta_i - \theta_j}\right) - \frac{K}{N} \sum_{j \notin \mathcal{C}_i(\Theta^*)} h(\theta_i - \theta_j), \tag{3.3.10}$$

for every $i = 1, \ldots, N$, where the above functions are defined almost everywhere. Indeed, notice that since δ does not make sense at 0, thus Δ is not defined on $\mathbb{C}$, but it is a negligible set. Again, we recall that $\bar{\theta}$ is its representative modulo 2π in the interval $(-\pi, \pi]$, for any $\theta \in \mathbb{R}$.

Proposition 3.3.11. *Let $\Theta^* = (\theta_1^*, \ldots, \theta_N^*) \in \mathbb{R}^N$ and define the vector fields*

$$\Delta : \mathcal{U} \longrightarrow \mathbb{R}^N, \quad \Lambda : \mathcal{U} \longrightarrow \mathbb{R}^N,$$

via the formulas (3.3.9)-(3.3.10), for a small enough neighborhood $\mathcal{V}$ of Θ^* in $\mathbb{R}^N$. Then,

1. *$H = \Delta + \Lambda$ in $\mathcal{U}$.*
2. *Δ is decreasing in $\mathcal{U}$.*
3. *Λ is Lipschitz-continuous in $\mathcal{U}$.*
4. *H is one-sided Lipschitz continuous in $\mathcal{U}$.*

Proof. The proof is analogous to Proposition 3.3.4. □

Finally, putting Lemmas D.1.5 and D.1.7 in Appendix D and Proposition 3.3.11 together, we conclude the following well-posedness result of Filippov solutions for (3.3.1)-(3.3.2).

Theorem 3.3.12. *There is one global-in-time Filippov solution to the system* (3.3.1)-(3.3.2) *with $\alpha = \frac{1}{2}$ for any initial configuration, that is unique forwards in time.*

Again, we can characterize the eventual emergence of sticking of a cluster after a potential collision in a similar way as we did in Theorem 3.3.6. We require the following notation. For any $N \in \mathbb{N}$, each $1 \leq m \leq N$ and every permutation σ of $\{1, \ldots, N\}$ we define the following couple of $m \times m$ matrices:

$$M_m^\sigma(\Omega) := (\Omega_{\sigma_i} - \Omega_{\sigma_j})_{1 \leq i,j \leq m}, \qquad \mathbf{J}_m = (1)_{1 \leq i,j \leq m}, \tag{3.3.11}$$

i.e., $M_m^\sigma(\Omega)$ stands for the matrix of relative natural frequencies of the only m oscillators with indices $i = \sigma_1, \ldots, \sigma_m$ and $\mathbf{J}_m$ is a $m \times m$ matrix whose components are all equal to one.

Theorem 3.3.13. *Consider $\Theta = (\theta_1, \ldots, \theta_N)$ the global-in-time Filippov solution in Theorem 3.3.12. Assume that t^* is some collision time and fix any cluster $E_k(t^*) \equiv E_k$ with $k = 1, \ldots, \kappa(t^*)$. Then, the following two statements are equivalent:*

1. *The $n_k(t^*)$ oscillators in such cluster $E_k(t^*)$ stick all together at time t^*.*

2. *There exists a bijection* $\sigma:\{1\ldots,n_k\}\to E_k$ *and* $Y\in\mathrm{Skew}_{n_k}([-1,1])$ *such that*

$$M^\sigma_{n_k}(\Omega)=\frac{K}{N}\left(Y\cdot\mathbf{J}_{n_k}+\mathbf{J}_{n_k}\cdot Y\right). \tag{3.3.12}$$

Proof. Let us call $n:=n_k$ for simplicity and assume that the oscillators in such cluster agree precisely with the first n oscillators, i.e., $E_k=\{1,\ldots,n\}$. By continuity, let us take some small $\varepsilon>0$ such that $\bar\theta_j(t)\neq\bar\theta_i(t)$, for every $t\in[t^*,t^*+\varepsilon]$, any $i\in E_k$ and each $j\notin E_k$. First, let us assume that the former statement holds true. Without loss of generality we might assume that $\theta_1(t)=\cdots=\theta_n(t)$ for all $t\geq t^*$ and we define $\theta(t):=\theta_1(t)=\cdots=\theta_n(t)$ for all $t\geq t^*$. Then, looking at the explicit expression in Proposition 3.3.7 for the Filippov set-valued map $\mathcal{H}$, it is clear that the following identities hold true

$$\dot\theta_i=\Omega_i+\frac{K}{N}\sum_{j=n+1}^N h(\theta_j(t)-\theta(t))+\frac{K}{N}\sum_{j=1}^n y_{ij}(t),$$

for a.e. $t\in[t^*,t^*+\varepsilon]$ and every $i=1,\ldots,n$, where $y_{ij}\in L^\infty(t^*,t^*+\varepsilon)$ and $Y(t)=(y_{ij}(t))_{1\leq i,j\leq n}\in\mathrm{Skew}_n([-1,1])$ for almost all $t\in[t^*,t^*+\varepsilon]$. Since $\dot\theta_i=\dot\theta_j$ a.e., for every $i,j=1,\ldots,n$, then we obtain the next system of equations

$$\Omega_i-\Omega_j=-\frac{K}{N}\sum_{\substack{l=1\\l\neq i}}^n y_{il}(t)+\frac{K}{N}\sum_{\substack{l=1\\l\neq j}}^n y_{jl}(t),$$

for a.e. $t\in[t^*,t^*+\varepsilon]$. In particular, (3.3.12) holds. Conversely, let us assume that (3.3.12) is verified for some $Y\in\mathrm{Skew}_n([-1,1])$, then we have

$$\Omega_i+\frac{K}{N}\sum_{\substack{l=1\\l\neq i}}^n y_{il}=\Omega_j+\frac{K}{N}\sum_{\substack{l=1\\l\neq j}}^n y_{jl}=:\widehat\Omega.$$

Let us now consider the vector field

$$\widehat H^n=(\widehat H^n_0,\widehat H^n_{n+1},\ldots,\widehat H^n_N):\mathbb{R}^{N-n+1}\longrightarrow\mathbb{R}^{N-n+1}$$

given by the formulas

$$\widehat H^n_0(\vartheta,\vartheta_{n+1},\ldots,\vartheta_N)=\widehat\Omega+\frac{K}{N}\sum_{j=n+1}^N h(\vartheta_j-\vartheta),$$

$$\widehat H^n_i(\vartheta,\vartheta_{n+1},\ldots,\vartheta_N)=\Omega_i+\frac{nK}{N}h(\vartheta-\vartheta_i)+\frac{K}{N}\sum_{j=n+1}^N h(\vartheta_j-\vartheta_i),$$

for every $i=n+1,\ldots,N$. Also, consider its associated Filippov set-valued map $\widehat{\mathcal{H}}^n$ and the associated differential inclusion

$$(\dot\vartheta,\dot\vartheta_{n+1},\ldots,\dot\vartheta_N)\in\widehat{\mathcal{H}}^n(\vartheta,\vartheta_{n+1},\ldots,\vartheta_N),$$

with initial datum given by

$$(\vartheta(t^*),\vartheta_{n+1}(t^*),\ldots,\vartheta_N(t^*))=(\theta^*,\theta_{n+1}(t^*),\ldots,\theta_N(t^*)).$$

A similar well-posedness result to that in Theorem 3.3.12 shows that such IVP enjoys one global-in-time solution. In addition, by definition it is apparent that whenever we pick

$$(\omega, \omega_{n+1}, \ldots, \omega_N) \in \widehat{\mathcal{H}}^n(\vartheta, \vartheta_{n+1}, \ldots, \vartheta_N),$$

then, we obtain

$$\big(\underbrace{\omega, \ldots, \omega}_{n\,\text{pairs}}, \omega_{n+1}, \ldots, \omega_N\big) \in \mathcal{H}\big(\underbrace{\vartheta, \ldots, \vartheta}_{n\,\text{pairs}}, \vartheta_{n+1}, \vartheta_N\big).$$

Consequently, the following two trajectories in $\mathbb{R}^N$

$$\begin{aligned} t &\mapsto (\theta_1(t), \theta_2(t), \ldots, \theta_n(t), \theta_{n+1}(t), \ldots, \theta_N(t)), \\ t &\mapsto (\underbrace{\vartheta(t),\ \vartheta(t),\ \ldots,\ \vartheta(t)}_{n\,\text{pairs}}, \vartheta_{n+1}(t), \ldots, \vartheta_N(t)), \end{aligned}$$

are Filippov solutions to (3.3.1)-(3.3.2) such that they take the same value at $t = t^*$, namely,

$$\big(\underbrace{\theta^*, \ldots, \theta^*}_{n\,\text{pairs}}, \theta_{n+1}(t^*), \ldots, \theta_N(t^*)\big).$$

By uniqueness they agree and, in particular,

$$\theta_i(t) = \vartheta(t) \text{ for all } t \geq t^* \text{ and every } i = 1, \ldots, n.$$

□

The sticking condition (3.3.12) can be characterized in a much more explicit manner by convex analysis techniques supported by *Farkas' alternative*. See Appendix 3.C and, in particular, the characterization of condition (3.3.12) in Lemma 3.C.3. Such ideas can be arranged in the next result.

Corollary 3.3.14. *Under the same assumptions as in Theorem 3.3.13. The following two assertions are equivalent:*

1. *The n_k oscillators in the cluster E_k stick all together at time t^*.*
2. *We have*
$$\frac{1}{m}\sum_{i\in I}\Omega_i - \frac{1}{n_k}\sum_{i\in E_k}\Omega_i \in \left[-\frac{K}{N}(n_k - m), \frac{K}{N}(n_k - m)\right], \tag{3.3.13}$$
for every $1 \leq m \leq n_k$ and every $I \subseteq E_k$ such that $\#I = m$.

Remark 3.3.15.

- *Notice that in Theorem 3.3.13 and Corollary 3.3.14 we have characterized when the whole cluster E_k remains stuck together, but not when a subcluster of a given size instantaneously splits from the remaining oscillators of the cluster. The main problem to extend the above proof is that it is hard to quantify the way in which an oscillator splits from the subcluster. Specifically, it is possible that an oscillator departs from the cluster exhibiting a left accumulation of switches of state where it instantaneously splits and collides with the formed subcluster. This accumulating phenomenon will come to play several times along this chapter and will cause some problems throughout the chapter that we shall try to overcome.*

- *The above accumulating phenomenon is called left Zeno behavior in the literature. It appears in Filippov solutions of some systems like the reversed bouncing ball. For instance, in [130, p. 116] Filippov proposed a discontinuous first order system with solutions exhibiting Zeno behavior. In [130, Theorem 2.10.4], the same author considered absence of Zeno behavior as part of the sufficient conditions (but not necessary) guaranteeing forwards uniqueness. We skip the analysis of Zeno behavior here and will address it in a future work.*

3.3.3 Well–posedness in the supercritical regime

Recall that in the supercritical regime, i.e., $\alpha > \frac{1}{2}$, the vector field $H = H(\Theta)$ is not only discontinuous at the collision states but it is also unbounded near those points, see Figure 3.1. Thus, the classical theory for well-posedness cannot be applied neither and one might seek for a notion of generalized solutions in the same sense as in the critical case $\alpha = \frac{1}{2}$ (see Subsection 3.3.2). Again, a plausible strategy is to turn the differential equation (3.3.1)-(3.3.2) of interest into the augmented differential inclusion (3.3.7) associated to the Filippov set-valued map $\mathcal{H} = \mathcal{H}(\Theta)$. A similar analysis to that in Proposition 3.3.7 yields the following characterization of the Filippov set-valued map for the supercritical regime.

Proposition 3.3.16. *In the supercritical regime $\alpha > \frac{1}{2}$, the Filippov set-valued map $\mathcal{H} = \mathcal{H}(\Theta)$ associated with $H = H(\Theta)$ stands for the convex and unbounded polytope consisting of the points $(\omega_1, \ldots, \omega_N) \in \mathbb{R}^N$ such that*

$$\omega_i = \Omega_i + \frac{K}{N} \sum_{j \notin \mathcal{C}_i(\Theta)} h(\theta_j - \theta_i) + \frac{K}{N} \sum_{j \in \mathcal{C}_i(\Theta) \setminus \{i\}} y_{ij}, \text{ for all } i = 1, \ldots, N,$$

for some $Y = (y_{ij})_{1 \leq i,j \leq N} \in \mathrm{Skew}_N(\mathbb{R})$.

Now, notice that $\mathrm{Skew}_N(\mathbb{R})$ represents the set of $N \times N$ skew-symmetric matrices with items in the whole real line. The Filippov set-valued map then enjoys similar expressions in the critical and supercritical regimes except for a "slight" change. In the former case, the coefficients y_{ij} range in the interval $[-1, 1]$ whereas in the latter case they take values in the whole $\mathbb{R}$. Indeed, the same examples for $\alpha = \frac{1}{2}$ in Example 3.3.9 can be considered for $\alpha > \frac{1}{2}$. For instance, similar polytopes to those in Figure 3.3 are obtained at the total collision phase configurations when the corresponding polygon is replaced by its affine envelope. Those similarities ensure that any Filippov solution to (3.3.1)-(3.3.2) with $\alpha > \frac{1}{2}$ also conserve the average frequency as in Remark 3.3.8. What is more, since $\mathcal{H}(\Theta)$ is apparently non-empty, then Lemma D.1.3 in Appendix D shows that $\mathcal{H}$ takes values in the non-empty, closed and convex sets and it has closed graph in the set-valued sense. However, the unboundedness in y_{ij} entails a severe change of behavior. Specifically, it violates the local compactness of the minimal selection $m(\mathcal{H})$ and, as a consequence, the existence result in Lemma D.1.4 does not work. Such loss of compactness is fateful and implies that the supercritical regime $\alpha > \frac{1}{2}$ lies in the setting where all the "classical" assumptions ensuring global existence and one-sided uniqueness does not hold. The literature about the abstract analysis of unbounded differential inclusions is rare, see [172, 286]. In addition, all those results require some sort of relaxed set-valued Lipschitz condition and linear growth that do not hold in our particular problem. Nevertheless, we will show that in some cases we can still construct a Filippov solution which is unique under some conditions.

Remark 3.3.17. *Notice that we still do not know anything about uniqueness results in the supercritical case. However, the approach in Theorem 3.3.13 can still be used to obtain a partial answer. Namely, it might give a sufficient condition on the natural frequencies to ensure that after a collision of a classical* *solution, we can extend a Filippov solution with sticking of the formed cluster. We shall elaborate on this idea later after this remark and we skip it here. Instead, let us just focus on the study of the analogue necessary condition of sticking like in* (3.3.12). *Indeed, consider some Filippov solution* $\Theta = (\theta_1, \ldots, \theta_N)$ *to* (3.3.1)-(3.3.2) *with* $\alpha > \frac{1}{2}$ *and assume that it is defined in an interval* $[0, T)$ *and that* $t^* \in (0, T)$ *is some collision time. Then, we might fix a cluster* $E_k(t^*) \equiv E_k$ *and assume that the* $n_k(t^*) \equiv n_k$ *oscillators in such cluster stick all together at time* t^**. Hence, a similar proof to that of Theorem 3.3.13 yields the existence of some bijection* $\sigma : \{1, \ldots, n_k\} \longrightarrow E_k$ *and some* $Y \in \mathrm{Skew}_{n_k}(\mathbb{R})$ *such that the following equation fulfils*

$$M_m^\sigma(\Omega) = \frac{K}{N}\left(Y \cdot \mathbf{J}_{n_k} + \mathbf{J}_{n_k} \cdot Y\right). \tag{3.3.14}$$

Again, let us obtain a more explicit characterization of such condition. We can resort on similar ideas coming from Farkas' alternative, see Lemma 3.C.2 in Appendix 3.C. Such Lemma ensures that (3.3.14) *is perfectly equivalent to the condition* (3.C.2)*, i.e.,*

$$m_{ij} + m_{jk} + m_{ki} = 0,$$

for every $i, j, k = 1, \ldots, n_k$*, where* m_{ij} *denotes the* (i, j)*-th component of the matrix* $M_m^\sigma(\Omega)$*. Let us look into the particular structure of* $M_{n_k}^\sigma(\Omega)$ *to restate the above condition (see* (3.3.11)*)*

$$m_{ij} + m_{jk} + m_{ki} = (\Omega_{\sigma_i} - \Omega_{\sigma_j}) + (\Omega_{\sigma_j} - \Omega_{\sigma_k}) + (\Omega_{\sigma_k} - \Omega_{\sigma_i}).$$

Then, the necessary sticking condition is automatically satisfied for every given configuration of natural frequencies. This suggests that any classical solution in the supercritical case that stops at a collision state might always be continued as Filippov solution with sticking of the cluster without any constraint for the chosen natural frequencies.

Our next goal is to show that we can indeed prolong classical solutions by Filippov solutions in this way. To such end, we require some more accurate control of the behavior of such classical solutions at the maximal time of existence. This is the content of the next result where the (dissipative) gradient-flow structure in Subsection 3.2.4 comes into play.

Lemma 3.3.18. *Consider* $\Theta = (\theta_1, \ldots, \theta_N)$ *any classical solution to* (3.3.1)-(3.3.2) *with* $\alpha \in (\frac{1}{2}, 1)$ *that is defined in a finite maximal existence interval* $[0, t^*)$*. Then,*

1. *The solution does not blow up at* t^**, i.e.,*

$$\lim_{t \to t^*} |\Theta(t)| \neq \infty,$$

2. *The solution converges towards a collision state, i.e., there exists* $\Theta^* \in \mathcal{C}$ *such that*

$$\lim_{t \to t^*} \Theta(t) = \Theta^*.$$

In addition, the trajectory $t \mapsto \Theta(t)$ *remains absolutely continuous up to the collision time* $t = t^*$*; specifically,* $\dot{\Theta} \in L^2((0, t^*), \mathbb{R}^N)$*.*

Proof. We split the proof into three parts. The first part is devoted to show that the classical trajectories verify the following fundamental inequalities:

$$\frac{1}{2}\int_0^t |\dot{\Theta}(s)|^2\, ds \leq V_{int}(\Theta_0) + \frac{C_\Omega^2}{2} t, \tag{3.3.15}$$

$$|\Theta(t)| \leq |\Theta_0| + \int_0^t |\dot{\Theta}(s)|\,ds, \tag{3.3.16}$$

for every $t \in [0, t^*)$. Here, $V_{int}(\Theta)$ is the second term of the potential $V(\Theta)$ in (3.2.10) and we set the constant

$$C_\Omega := |(\Omega_1, \ldots, \Omega_N)| = \sqrt{\Omega_1^2 + \cdots \Omega_N^2}.$$

We shall show in the second step that such inequalities (3.3.15) and (3.3.16) infer the next ones

$$\frac{1}{2}\int_0^{t^*} |\dot{\Theta}(s)|^2\,ds \leq V_{int}(\Theta_0) + \frac{C_\Omega^2}{2}t^* < \infty, \tag{3.3.17}$$

$$\int_0^{t^*} |\dot{\Theta}(s)|\,ds \leq V_{int}(\Theta_0) + \frac{1 + C_\Omega^2}{2}t^* < \infty, \tag{3.3.18}$$

$$|\Theta(t)| \leq |\Theta_0| + V_{int}(\Theta_0) + \frac{1 + C_\Omega^2}{2}t^*, \tag{3.3.19}$$

for every $t \in [0, t^*)$. Finally, the third part will focus on proving the assertions in the statement of the Lemma via such fundamental inequalities (3.3.15)-(3.3.19).

• *Step 1:* Recall that in Section 3.2, the classical solution $t \longmapsto \Theta(t)$ of (3.3.1)-(3.3.2) equivalently solves a gradient flow system (3.2.9), i.e.,

$$\dot{\Theta}(t) = -\nabla V(\Theta(t)),$$

for all $t \in [0, t^*)$, where V is given in (3.2.10). Hence,

$$\frac{d}{dt}V(\Theta(t)) = \nabla V(\Theta(t)) \cdot \dot{\Theta}(t) = -|\dot{\Theta}(t)|^2,$$

for every $t \in [0, t^*)$. Taking integrals in time, we obtain

$$\int_0^t |\dot{\Theta}(s)|^2\,ds = V(\Theta_0) - V(\Theta(t)) = \sum_{i=1}^N \Omega_i(\theta_{i,0} - \theta_i(t)) + V_{int}(\Theta_0) - V_{int}(\Theta(t)), \tag{3.3.20}$$

for every $t \in [0, t^*)$. Recall that the function W in (3.2.11) involved in the potential (3.2.10) is a primitive function of h. Then, $W \geq 0$ as a consequence of the antisymmetry of h and our choice $W(0) = 0$ and, in particular, $V_{int} \geq 0$. This, together with the Cauchy–Schwarz inequality, yield

$$\int_0^t |\dot{\Theta}(s)|^2\,ds \leq C_\Omega \int_0^t |\dot{\Theta}(s)|\,ds + V_{int}(\Theta_0), \tag{3.3.21}$$

for every $t \in [0, t^*)$. Using Young's inequality in the first term of (3.3.21), we arrive at the first fundamental inequality (3.3.15). The second inequality (3.3.16) is standard, but let us sketch it for the sake of clarity

$$\frac{d}{dt}\frac{|\Theta|^2}{2} = \Theta \cdot \dot{\Theta} \leq |\Theta|\,|\dot{\Theta}|,$$

for all $t \in [0, t^*)$. Then, we arrive at

$$\frac{d}{dt}|\Theta(t)| \leq |\dot{\Theta}(t)|,$$

for every $t \in [0, t^*)$ and integrating with respect to time yields (3.3.16).

• *Step 2:* First, taking limits $t \to t^*$ in (3.3.15), we clearly obtain (3.3.17). Also, the finite length of the trajectory (3.3.18) holds true by virtue of the Cauchy–Schwarz inequality and Young's inequality both applied to the preceding one. Finally, inequalities (3.3.16) and (3.3.18) entail (3.3.19).

• *Step 3:* The classical trajectory $t \mapsto \Theta(t)$ is defined up to a finite maximal time t^*. Hence, classical results show that either it blows up at $t = t^*$ or there exists some sequence $\{t_n\}_{n\in\mathbb{N}} \nearrow t^*$ and some $\Theta^* \in \mathcal{C}$ such that $\{\Theta(t_n)\}_{n\in\mathbb{N}} \to \Theta^*$. Since the former option is prevented by (3.3.19), then the latter must hold true. Let us prove that the whole trajectory converges towards that collision state Θ^*. In other case, there exists another sequence $\{s_n\}_{n\in\mathbb{N}} \nearrow t^*$ and some $\varepsilon_0 > 0$ such that

$$|\Theta(s_n) - \Theta^*| \geq \varepsilon_0, \tag{3.3.22}$$

for all $n \in \mathbb{N}$. Without loss of generality we can assume that the sequences $\{t_n\}_{n\in\mathbb{N}}$ and $\{s_n\}_{n\in\mathbb{N}}$ are ordered as follows

$$t_1 < s_1 < t_2 < s_2 < \ldots$$

and that

$$|\Theta(t_n) - \Theta^*| \leq \frac{\varepsilon_0}{2^n}, \tag{3.3.23}$$

for every $n \in \mathbb{N}$. Thereby,

$$\begin{aligned}
|\Theta(t_n) - \Theta(s_n)| &\geq |\Theta(s_n) - \Theta^*| - |\Theta(t_n) - \Theta^*| \geq \varepsilon_0 - \frac{\varepsilon_0}{2^n}, \\
|\Theta(s_n) - \Theta(t_{n+1})| &\geq |\Theta(s_n) - \Theta^*| - |\Theta(t_{n+1}) - \Theta^*| \geq \varepsilon_0 - \frac{\varepsilon_0}{2^{n+1}},
\end{aligned}$$

for all $n \in \mathbb{N}$. Then, it is clear that

$$\begin{aligned}
\int_0^{t^*} |\dot{\Theta}(t)|\,dt &\geq \int_{t_1}^{t^*} |\dot{\Theta}(t)|\,dt \\
&= \sum_{n=1}^{\infty} \int_{t_n}^{s_n} |\dot{\Theta}(t)|\,dt + \sum_{n=1}^{\infty} \int_{s_n}^{t_{n+1}} |\dot{\Theta}(t)|\,dt \\
&\geq \sum_{n=1}^{\infty} |\Theta(t_n) - \Theta(s_n)| + \sum_{n=1}^{\infty} |\Theta(s_n) - \Theta(t_{n+1})| \\
&\geq \sum_{n=1}^{\infty} \varepsilon_0 \left(1 - \frac{1}{2^n}\right) + \sum_{n=1}^{\infty} \varepsilon_0 \left(1 - \frac{1}{2^{n+1}}\right) = \infty.
\end{aligned}$$

Thus, the trajectory would have infinite length and that contradicts (3.3.18). Hence, we find

$$\lim_{t\to t^*} \Theta(t) = \Theta^*.$$

□

Such conclusion shows that, as expected, it is plausible to continue classical solutions by Filippov solutions (hence absolutely continuous) after a possible collision. The explicit method of continuation is discussed in the following result.

Theorem 3.3.19. *Consider $\Theta = (\theta_1, \ldots, \theta_N)$ any classical solution to* (3.3.1)-(3.3.2) *with $\alpha \in \left(\frac{1}{2}, 1\right)$ that is defined in a finite maximal existence interval $[0, t^*)$ and, according to Lemma 3.3.18, let us consider the collision state $\Theta^* \in \mathcal{C}$ such that*

$$\lim_{t\to t^*} \Theta(t) = \Theta^*.$$

Then, there exists some $\varepsilon > 0$ so that the classical trajectory $t \mapsto \Theta(t)$ can be continued by a Filippov solution to (3.3.1)-(3.3.2) *in a short interval $[t^*, t^* + \varepsilon)$ in such a way that oscillators belonging to the same cluster of the collision state Θ^* remain all stuck together after t^*.*

Proof. Let E_k be the k-th cluster of oscillators with $n_k = \#E_k$ for $k = 1, \cdots, \kappa$. We consider a bijection $\sigma^k : \{1, \ldots, n_k\} \longrightarrow E_k$, for every $k = 1, \ldots, \kappa$. Since the necessary condition (3.3.14) is automatically satisfied as discussed in Remark 3.3.17, then there exists some matrix $Y^k \in \mathrm{Skew}_{n_k}(\mathbb{R})$ such that

$$\Omega_{\sigma_i^k} + \frac{K}{N} \sum_{\substack{l=1 \\ l \neq i}}^{n_k} y_{il}^k = \Omega_{\sigma_j^k} + \frac{K}{N} \sum_{\substack{l=1 \\ l \neq j}}^{n_k} y_{jl}^k =: \widehat{\Omega}_k, \tag{3.3.24}$$

for every couple of indices $i, j \in \{1, \ldots, n_k\}$. Let us define the following system of κ differential equations

$$\dot{\vartheta}_k = \widehat{H}_k(\vartheta_1, \ldots, \vartheta_k) := \widehat{\Omega}_k + \frac{K}{N} \sum_{\substack{m=1 \\ m \neq k}}^{\kappa} n_m h(\vartheta_m - \vartheta_k), \tag{3.3.25}$$

for $k = 1, \ldots, \kappa$, with initial data given by

$$(\vartheta_1(t^*), \ldots, \vartheta_\kappa(t^*)) = (\theta_{\iota_1}^*, \ldots, \theta_{\iota_\kappa}^*). \tag{3.3.26}$$

Since the initial datum is a non-collision state in a lower dimension space $\mathbb{R}^\kappa$ of phase configurations, then there exists a unique classical solution to such problem that is defined in a maximal existence interval $[t^*, t^{**})$ and such that if $t^{**} < \infty$, then $(\vartheta_1, \ldots, \vartheta_\kappa)$ converges towards a new collision state by virtue of Lemma 3.3.18 (merge of clusters). The same result ensures that

$$\begin{aligned} t \in [0, t^*) &\longmapsto (\theta_1(t), \ldots, \theta_N(t)), \\ t \in [t^*, t^{**}) &\longmapsto (\vartheta_1(t), \ldots, \vartheta_\kappa(t)), \end{aligned}$$

belong to $W^{1,2}((0, t^*), \mathbb{R}^N)$ and $W^{1,2}((t^*, t^{**}), \mathbb{R}^\kappa)$, respectively. Let us set the prolongation of $t \longmapsto \Theta(t)$ in $[t^*, t^{**})$ in such a way that

$$\theta_{\sigma_i^k}(t) := \vartheta_k(t), \ \forall t \in [t^*, t^{**}),$$

for every $i \in E_k$ and $k = 1, \ldots, \kappa$. Both trajectories glue in a $W^{1,2}$ way and it is clear, by virtue of the definition of $\widehat{H}^k$ in (3.3.25) and $\widehat{\Omega}_k$ in (3.3.24) along with the explicit expression of the Filippov map in Proposition 3.3.16, that $t \in [0, t^{**}) \longmapsto \Theta(t)$ becomes a Filippov solution to (3.3.1)-(3.3.2) in $[0, t^{**})$. □

Remark 3.3.20. *The above procedure can be repeated as many times as needed after each collision time of the corresponding classical solutions to the reduced systems* (3.3.25)-(3.3.26)*. Indeed, by Remark 3.3.17 the necessary condition* (3.3.14) *is automatically satisfied. Since there cannot be more than $N - 1$ different sticking times, we may apply Theorem 3.3.19 finitely many times to obtain global-in-time Filippov solutions to* (3.3.1)-(3.3.2) *in the supercritical case. However, one may wonder whether this global-in-time continuation procedure is unique or oscillators may also be allowed to split instantaneously after a collision. Although answering the general question for any number N of oscillators and any collision state is really convoluted, let us give some particular answer for the case $N = 2$:*

$$\dot{\theta}_1 = \Omega_1 + \frac{K}{2} h(\theta_2 - \theta_1), \tag{3.3.27}$$

$$\dot\theta_2 = \Omega_2 + \frac{K}{2} h(\theta_1 - \theta_2). \tag{3.3.28}$$

Consider the relative phase $\theta := \theta_2 - \theta_1$ *and relative natural frequency* $\Omega := \Omega_2 - \Omega_1$. *Then, the associated dynamics of a classical solution is governed by the next equation*

$$\dot\theta = \Omega - Kh(\theta),$$

in the maximal interval of existence $[0, t^*)$. *The dynamics of the above ODE is analyzed later in Proposition 3.5.2, from which we can infer that* $t^* = +\infty$ *if* $\theta(0) = \bar\theta$, *whereas* $t^* < +\infty$ *if* $\theta(0) \notin \{0, \bar\theta\}$. *Here,* $\bar\theta$ *stands for the unique (unstable) equilibrium of the system, see Proposition 3.5.2 in the subsequent Section 3.5. Without loss of generality, we will fix the initial relative phase so that* $\theta(0) \in (0, \bar\theta)$ *(the other cases are similar). Then, Lemma 3.3.18 guarantees that* $t = t^*$ *must be a collision time, i.e.,*

$$\lim_{t\to t^*} \theta(t) = 0.$$

1. *Let us assume by contradiction that there was another Filippov solution in* $[t^*, t^{**})$ *consisting of two particles that instantaneously split again after* $t = t^*$. *Such split can arise in only two different manners:*

   (a) *(Sharp split) There exists some small* $\varepsilon > 0$ *such that* $\theta(t) \neq 0$, *for every* $t \in (t^*, t^* + \varepsilon)$. *In such case, either* $\theta(t) > 0$, *for all* $t \in (t^*, t^* + \varepsilon)$, *or* $\theta(t) < 0$, *for all* $t \in (t^*, t^* + \varepsilon)$.

   (b) *(Zeno split) There exist a couple of sequences* $\{t_n\}_{n\in\mathbb{N}} \searrow t^*$ *and* $\{s_n\}_{n\in\mathbb{N}} \searrow t^*$ *such that* $\theta(s_n) = 0$ *but* $\theta(t_n) \neq 0$, *for every* $n \in \mathbb{N}$. *Recall Remark 3.3.15 for the left accumulations of switches or Zeno behavior and see Figure 3.5.*

   Replacing t^* *by a suitable time, it is apparent that the second type of split at* t^* *guarantees the first one at a (possibly) latter time. Let us then focus just on the fist case. Looking at the profile of* $\Omega - kh(\theta)$ *in Figure 3.6, we then would arrive at the following conclusion: either* $\dot\theta(t) < 0$ *and* $\theta(t) > 0$ *for all* $t \in (t^*, t^* + \varepsilon)$ *or* $\dot\theta(t) > 0$ *and* $\theta(t) < 0$ *for all* $t \in (t^*, t^* + \varepsilon)$. *In any case, we obtain a contradiction.*

2. *Hence, the only choice for the oscillators after the collision state is to stick together. Let us define the phase of the reduced system, see* (3.3.24)

   $$\widehat\Omega := \Omega_1 + y_{12} = \Omega_2 + y_{21},$$

   where $Y \in \mathrm{Skew}_2(\mathbb{R})$ *is any matrix verifying the necessary condition* (3.3.14). *Indeed, there just exists one such matrix* Y, *whose items read* $y_{12} = -y_{21} = \frac{\Omega_2 - \Omega_1}{2}$. *Then,* $\widehat\Omega = \frac{\Omega_1+\Omega_2}{2}$ *and the reduced system* (3.3.25) *looks like*

   $$\dot\vartheta = \widehat\Omega,\ t \in [t^*, \infty).$$

   Consequently, the only Filippov solution to (3.3.1)-(3.3.2) *evolves through* (3.3.27)-(3.3.28) *up to the collision time* t^*. *After it, both oscillators stick together and they move with constant frequency equals to the average natural frequency.*

For general N, *it is not clear whether (b) in the above first item can be reduced to (a) in a similar way. Namely, we cannot guarantee that along a whole time interval* $(t^*, t^*+\varepsilon)$ *all the formed subclusters splitting from the given cluster remain at positive distance or they actually merge and split instantaneously with eventual switches of collisional type in a similar way to Figure 3.6 in Zeno behavior. Also, studying the higher dimensional phase portrait in the same spirit as we have done for* $N = 2$ *is not easy and we shall address it in future works.*

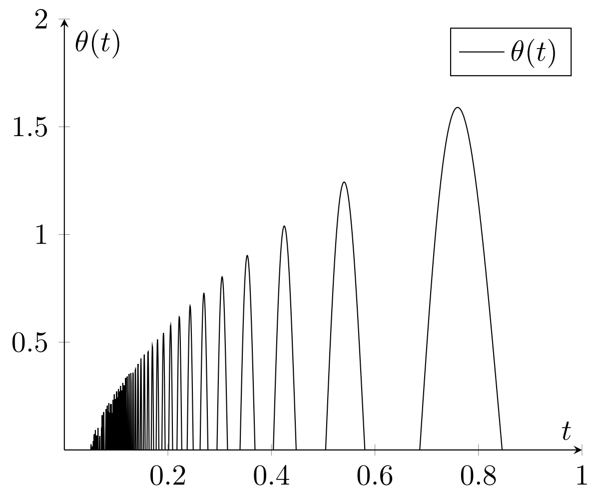


Figure 3.5: Left-Zeno behavior in the relative phase $\theta(t) = \theta_2(t) - \theta_1(t)$ of two oscillators

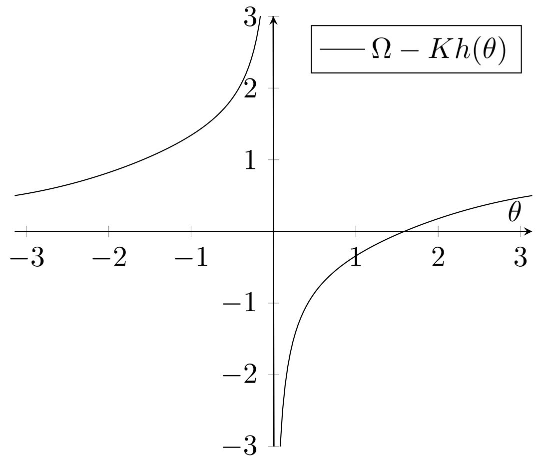


Figure 3.6: Profile of $\Omega - Kh(\theta)$ for $\Omega = 0.25$, $K = 1$ and $\alpha = 0.75$.

3.4 Rigorous limit towards singular weights

In the previous section, we studied the existence and one-sided uniqueness of absolutely continuous solutions to the singular weighted first order Kuramoto model in all the subcritical, critical and supercritical cases. Because of the continuity of the kernel for $\alpha \in \left(0, \frac{1}{2}\right)$, we can show that in that case the solutions are indeed C^1, although we cannot say the same neither for the critical case $\alpha = \frac{1}{2}$ nor for the supercritical case $\alpha \in \left(\frac{1}{2}, 1\right)$. Also, these results does not necessarily provide any extra regularity of the frequencies $\omega_i = \dot{\theta}_i$ for an augmented second order model to make sense.

Let us recall that in Subsection 3.2.2, the singular Kuramoto model was formally obtained as singular limit $\varepsilon \to 0$ of the scaled regular model (3.2.5)-(3.2.6). Notice that if apart form heuristically, we rigorously proved the limit $\varepsilon \to 0$, then we would achieve an alternative existence result for the singular models. In this section, we shall inspect to what extend such idea works and how many exponents we can obtain with such technique. In particular, we will recover the existence results in Section 3.3. Indeed, this technique will yield a gain of piecewise $W^{1,1}$ regularity of the frequencies ω_i in the subcritical case and will provide an equation for them in weak sense that will be discussed and related with similar models in Subsection 3.4.4. However, such idea fails for the more singular cases, where the compactness of frequencies is very weak. While the singular limit for the subcritical case is straightforward, we need to develop new ideas to tackle the limiting set-valued Filippov map in the critical and supercritical cases along with the loss of strong compactness of the frequencies in such cases.

3.4.1 Singular limit in the subcritical case and augmented flocking model

The following result provides a list of a priori estimates for the global-in-time classical solutions of the regularized system (3.2.5)-(3.2.6), for any $\varepsilon > 0$:

Lemma 3.4.1. *Let us consider any initial data $\Theta_0 = (\theta_{1,0}, \ldots, \theta_{N,0}) \in \mathbb{R}^N$ and set the unique global-in-time classical solution $\Theta^\varepsilon = (\theta_1^\varepsilon, \ldots, \theta_N^\varepsilon)$ to* (3.2.5)-(3.2.6) *in the subcritical case $\alpha \in \left(0, \frac{1}{2}\right)$, for every $\varepsilon > 0$. Then, there exists some non-negative constant C such that*

$$\begin{aligned} \|\dot{\Theta}^\varepsilon\|_{C^{0,1-2\alpha}([0,\infty),\mathbb{R}^N)} &\leq C, \\ \|\Theta^\varepsilon\|_{C^{1,1-2\alpha}([0,T],\mathbb{R}^N)} &\leq |\Theta_0| + CT, \end{aligned}$$

for every $T > 0$ and $\varepsilon > 0$. As a consequence, there exists some subsequence of $\{\Theta^\varepsilon\}_{\varepsilon>0}$, that we denote in the same way for simplicity, and some $\Theta \in C^1([0,+\infty),\mathbb{R}^N)$ such that $\dot{\Theta} \in C^{0,1-2\alpha}([0,\infty),\mathbb{R}^N)$, it verifies the same estimates as above and

$$\{\Theta^\varepsilon\}_{\varepsilon>0} \to \Theta \ \text{ in } C^1([0,T],\mathbb{R}^N),$$

for every $T > 0$.

Proof. All the properties directly follow from the first one along with the Ascoli–Arzelà theorem. Recall that there is some constant $M > 0$ such that

$$|h_\varepsilon(\theta)| \leq M \ \text{ and } \ |h_\varepsilon(\theta_1) - h_\varepsilon(\theta_2)| \leq M|\theta_1 - \theta_2|_o^{1-2\alpha},$$

for every θ, θ_1, $\theta_2 \in \mathbb{R}$ and every $\varepsilon > 0$, recall Lemma 3.3.1. Then, the first property is also a straightforward consequence of such uniform-in-ε boundedness and Hölder-continuity of the kernel. □

The following result holds true as a clear consequence of the uniform equicontinuity of the sequence h_ε along with the compactness of the sequence $\{\Theta^\varepsilon\}_{\varepsilon>0}$.

Theorem 3.4.2. *The limit function Θ of $\{\Theta^\varepsilon\}_{\varepsilon>0}$ in Lemma 3.4.1 is a classical global-in-time solution of the singular model* (3.3.1)-(3.3.2) *in the subcritical case $\alpha \in \left(0, \frac{1}{2}\right)$.*

Notice that we have arrived at a construction of classical global-in-time solutions of the singular problem with $0 < \alpha < \frac{1}{2}$ through two different techniques: Theorems 3.3.5 and 3.4.2. However, both techniques are actually very related since originally, the Filippov theory relies on a similar regularizing procedure. In what follows, we will see that such procedure provides us with extra a priori estimates for the "acceleration" (derivatives of frequencies). Also, such procedure will allow us to derive a "piecewise weak equation" for them. This is the rest of the content of this subsection.

Remark that a necessary and sufficient condition for two oscillators θ_i and θ_j that collide at some time to stick together is that $\Omega_i = \Omega_j$ by virtue of Theorem 3.3.6. In some sense, those two oscillators are identified in an unique cluster with bigger "mass". Then, we can quantify the times of "pure collisions" as follows. Starting with $T_0 = 0$, we define

$$T_k := \inf\{t > T_{k-1} : \exists\, i \text{ and } j \in S_i(T_{k-1})^c \text{ such that } \bar{\theta}_i(t) = \bar{\theta}_j(t)\}, \tag{3.4.1}$$

for every $k \in \mathbb{N}$. Recall the notation in Subsection 3.2.3, and see [245] for related notation in the discrete Cucker–Smale model with singular influence function. Then, taking derivatives in (3.2.5)-(3.2.6) we can obtain the next split

$$\begin{aligned}\ddot{\theta}_i^\varepsilon = & \frac{K}{N} \sum_{j \notin \mathcal{C}_i(T_{k-1})} h_\varepsilon'(\theta_j^\varepsilon - \theta_i^\varepsilon)(\dot{\theta}_j^\varepsilon - \dot{\theta}_i^\varepsilon) \\ & + \frac{K}{N} \sum_{j \in (\mathcal{C}_i \setminus S_i)(T_{k-1})} h_\varepsilon'(\theta_j^\varepsilon - \theta_i^\varepsilon)(\dot{\theta}_j^\varepsilon - \dot{\theta}_i^\varepsilon) \\ & + \frac{K}{N} \sum_{j \in S_i(T_{k-1})} h_\varepsilon'(\theta_j^\varepsilon - \theta_i^\varepsilon)(\dot{\theta}_j^\varepsilon - \dot{\theta}_i^\varepsilon),\end{aligned} \tag{3.4.2}$$

where $t \in [T_{k-1}, T_k)$. The idea is to show that we can pass to the limit in the above expressions in $L^1([T_{k-1}, \tau])$-weak, for every $k \in \mathbb{N}$ and for every $\tau \in (T_{k-1}, T_k)$. This is the content of the next theorem. Before going on, let us discuss the possible scenarios for the sequence $\{T_k\}_{k\in\mathbb{N}}$ and how can we cover the whole interval $[0, +\infty)$ with them in any case so that our dynamics can be reduced to each of them:

1. First, it might happen that there exists some $k_0 \in \mathbb{N}$ such that $T_{k_0+1} = +\infty$ (then, $T_k = +\infty$ for every $k > k_0$). In this case either all particles have stuck together in finite time or after some finite time there is no more collision. Then, we recover

$$[0, +\infty) = \bigcup_{0 \le k \le k_0 - 1} [T_k, T_{k+1}) \cup [T_{k_0}, +\infty),$$

and at each interval there are no further collisions.

2. Also it might happen that $\{T_k\}_{k\in\mathbb{N}}$ is infinite and unbounded, i.e., $T_k \nearrow +\infty$. Hence,

$$[0, +\infty) = \bigcup_{k \ge 0} [T_k, T_{k+1}),$$

and there is no collision in each interval.

3. Finally, it might also be the "odd" case that the sequence $\{T_k\}_{k\in\mathbb{N}}$ is infinite but bounded. In such case, there exists some $T^\infty \in \mathbb{R}^+$ with right Zeno behavior, i.e. $T_k \nearrow T^\infty$. Then, a straightforward argument involving the mean value theorem shows that T^∞ is a sticking point. Then we can split the dynamics up to time T^∞ through

$$[0, T^\infty) = \bigcup_{k \ge 0} [T_k, T_{k-1}).$$

Taking T^∞ as our initial time, we can repeat each of the steps 1, 2 and 3 above so that we can globally recover the whole dynamics. Notice that since there just can be $N - 1$ times of sticking, then there just can be $N - 1$ times like T^∞.

For simplicity in our arguments, we shall assume that we always lie in the second case, although the same result also apply to any other case. Before going to the heart of the result, let us summarize some good properties of the kernel h'_ε.

Lemma 3.4.3. *Consider any value $\alpha \in \left(0, \frac{1}{2}\right)$. Then, the following properties hold true:*

1. *Formula for the derivative:*
$$h'_\varepsilon(\theta) = \frac{1}{(\varepsilon^2 + c|\theta|_o^2)^\alpha}\left[\cos\theta - 2\alpha c \frac{\sin|\theta|_0}{|\theta|_0}\frac{|\theta|_0^2}{\varepsilon^2 + c|\theta|_o^2}\right].$$
2. *Upper bound by $L^1(\mathbb{T})$-function:*
$$|h'_\varepsilon(\theta)|, |h'(\theta)| \leq M\frac{1}{|\theta|_o^{2\alpha}}.$$
3. *Strong convergence in $L^1(\mathbb{T})$:*
$$h'_\varepsilon \to h' \ \text{ in } L^1(\mathbb{T}).$$
4. *Weighted Hölder-continuity:*
$$|h'_\varepsilon(\theta_1) - h'_\varepsilon(\theta_2)| \leq M\frac{|\theta_1 - \theta_2|_o^\beta}{\min\{|\theta_1|_o, |\theta_2|_o\}^\gamma},$$
for any couple of exponents $\beta, \gamma \in (0,1)$ such that $\gamma = 2\alpha + \beta$.
5. *Weighted convergence in $L^\infty(\mathbb{T})$:*
$$|h'_\varepsilon(\theta) - h'(\theta)| \leq M\frac{\varepsilon^{1-2\alpha}}{|\theta|_o}.$$

Proof. The first two results are straightforward and the third one is a clear consequence of the dominated convergence theorem. The fourth property follows from an obvious application of the mean value theorem and the fifth one is a standard property of mildly singular kernels (one can indeed show that $M = \alpha/\beta$) . □

Theorem 3.4.4. *For any initial datum $\Theta_0 \in \mathbb{R}^N$, consider Θ^ε, the classical global-in-time solution of* (3.2.5)-(3.2.6) *in the subcritical case $\alpha \in \left(0, \frac{1}{2}\right)$. Also, consider the limiting Θ in Theorem 3.4.2 and the collision times $\{T_k\}_{k\in\mathbb{N}}$ in* (3.4.1). *Then, the following properties hold true :*

1. *For every $i \in \{1, \ldots, N\}$ and $j \notin \mathcal{C}_i(T_{k-1})$*
$$h'_\varepsilon(\theta_j^\varepsilon - \theta_i^\varepsilon) \to h'(\theta_j - \theta_i), \quad \text{as} \quad \varepsilon \to 0, \quad \text{in} \quad C([T_{k-1}, \tau]).$$
2. *For every $i \in \{1, \ldots, N\}$ and $j \in \mathcal{C}_i(T_{k-1}) \setminus S_i(T_{k-1})$*
$$h'_\varepsilon(\theta_j^\varepsilon - \theta_i^\varepsilon) \rightharpoonup h'(\theta_j - \theta_i), \quad \text{as} \quad \varepsilon \to 0, \quad \text{in} \quad L^1([T_{k-1}, \tau]).$$
3. *For every $i \in \{1, \ldots, N\}$ and $j \in S_i(T_{k-1})$*
$$\frac{d}{dt}h_\varepsilon(\theta_j^\varepsilon - \theta_i^\varepsilon) \to 0, \quad \text{as} \quad \varepsilon \to 0, \quad \text{in} \quad W^{-1,\infty}([T_{k-1}, \tau]).$$

Proof. We split the proof in three steps.

• *Step 1:* In the first case, fix any $i \in \{1, \ldots, N\}$ and $j \notin \mathcal{C}_i(T_{k-1})$. There exists (by definition) some positive constant $\delta_0 = \delta_0(k, \tau) < \pi$ such that

$$|\theta_i(t) - \theta_j(t)|_o \geq \delta_0, \text{ for all } t \in [T_{k-1}, \tau].$$

Then, by the uniform convergence in Lemma 3.4.1 there exists some $\varepsilon_0 > 0$ such that

$$|\theta_i^\varepsilon(t) - \theta_j^\varepsilon(t)|_o \geq \frac{\delta_0}{2}, \text{ for all } t \in [T_{k-1}, \tau], \tag{3.4.3}$$

for every $\varepsilon \in (0, \varepsilon_0)$. Consequently, by crossing terms we have

$$\begin{aligned} &|h_\varepsilon'(\theta_j^\varepsilon(t) - \theta_i^\varepsilon(t)) - h'(\theta_j(t) - \theta_i(t))| \\ &\qquad \leq |h_\varepsilon'(\theta_j^\varepsilon(t) - \theta_i^\varepsilon(t)) - h'(\theta_j^\varepsilon(t) - \theta_i^\varepsilon(t))| + |h'(\theta_j^\varepsilon(t) - \theta_i^\varepsilon(t)) - h'(\theta_j(t) - \theta_i(t))|, \end{aligned}$$

for every $t \in [T_{k-1}, \tau]$. Hence, both two terms converge to zero uniformly in $[T_{k-1}, \tau]$, as $\varepsilon \to 0$. This is due to (3.4.3), the third property in Lemma 3.4.3, the uniform continuity of h' in compact sets away from $2\pi\mathbb{Z}$ and the uniform convergence of the phases in Lemma 3.4.1. This ends the proof of the first part.

• *Step 2*: In the second case, $i \in \{1, \ldots, N\}$ and $j \in \mathcal{C}_i(T_{k-1}) \setminus S_i(T_{k-1})$. Then,

$$\bar{\theta}_j(T_{k-1}) = \bar{\theta}_i(T_{k-1}) \text{ but } \dot{\theta}_j(T_{k-1}) \neq \dot{\theta}_i(T_{k-1}).$$

Thus, it is clear that we again have $|\theta_j(t) - \theta_i(t)|_o > 0$, for $t \in [\tau^*, \tau]$ and for every $\tau^* \in (T_{k-1}, \tau)$. This amounts to saying that the preceding argument again holds in $[\tau^*, \tau]$ and consequently,

$$h_\varepsilon'(\theta_j^\varepsilon - \theta_i^\varepsilon) \to h'(\theta_j - \theta_i) \text{ in } C([\tau^*, \tau]),$$

for every $\tau^* \in (T_{k-1}, \tau)$. Then, we just need to prove the weak convergence in some interval $[T_{k-1}, \tau^*]$. Let us set τ^*. Since $\dot{\theta}_j(T_{k-1}) \neq \dot{\theta}_i(T_{k-1})$, we can assume without loss of generality that $\delta_0 := \dot{\theta}_j(T_{k-1}) - \dot{\theta}_i(T_{k-1}) > 0$. By continuity of $\dot{\theta}_j$ and $\dot{\theta}_i$, there exists some small $\tau^* \in (T_{k-1}, \tau)$ such that

$$\dot{\theta}_i(t) - \dot{\theta}_j(t) \geq \frac{\delta_0}{2}, \text{ for all } t \in [T_{k-1}, \tau^*]. \tag{3.4.4}$$

Then, by the uniform convergence of the frequencies (see Lemma 3.4.1), we can take a small enough $\varepsilon_0 > 0$ such that if $\varepsilon \in (0, \varepsilon_0)$ then

$$\dot{\theta}_i^\varepsilon(t) - \dot{\theta}_j^\varepsilon(t) \geq \frac{\delta_0}{4}, \text{ for all } t \in [T_{k-1}, \tau^*]. \tag{3.4.5}$$

In particular, we have well defined inverses of $\theta_j - \theta_i$ and $\theta_j^\varepsilon - \theta_i^\varepsilon$ in $[T_{k-1}, \tau^*]$, for every $\varepsilon \in (0, \varepsilon_0)$. Indeed, the inverse function theorem states that:

$$((\theta_j - \theta_i)^{-1})' = \frac{1}{(\dot{\theta}_j - \dot{\theta}_i) \circ (\theta_j - \theta_i)^{-1}}, \tag{3.4.6}$$

and a similar statement holds for $\theta_j^\varepsilon - \theta_i^\varepsilon$. In order to show the weak convergence in $L^1([T_{k-1}, \tau^*])$, we equivalently claim that the following assertions are true:

1. Uniform-in-ε L^1 bound of $h'_\varepsilon(\theta_j^\varepsilon - \theta_i^\varepsilon)$ and $h(\theta_j - \theta_i)$ in $[T_{k-1}, \tau^*]$, i.e., there exists some constant $M > 0$ such that

$$\|h'_\varepsilon(\theta_j^\varepsilon - \theta_i^\varepsilon)\|_{L^1([T_{k-1},\tau^*])}, \|h'(\theta_j - \theta_i)\|_{L^1([T_{k-1},\tau^*])} \leq M,$$

for every $\varepsilon \in (0, \varepsilon_0)$.

2. Convergence of the mean values over finite intervals, i.e.,

$$\lim_{\varepsilon\to 0} \int_{T_{k-1}}^{\tau^{**}} (h'_\varepsilon(\theta_j^\varepsilon(t) - \theta_i^\varepsilon(t)) - h'(\theta_j(t) - \theta_i(t)))\, dt = 0,$$

for every $\tau^{**} \in (T_{k-1}, \tau^*)$.

Let us then prove such claim. Regarding the first assertion, we just focus on $h'_\varepsilon(\theta_j^\varepsilon - \theta_i^\varepsilon)$ (the other case is similar). Due to a simple change of variables $\theta = (\theta_j^\varepsilon - \theta_i^\varepsilon)(t)$ and (3.4.5)-(3.4.6)

$$\int_{T_{k-1}}^{\tau^{**}} |h'_\varepsilon(\theta_j^\varepsilon(t) - \theta_i^\varepsilon(t))|\, dt = \int_{\theta_j^\varepsilon(T_{k-1})-\theta_i^\varepsilon(T_{k-1})}^{\theta_j^\varepsilon(\tau^{**})-\theta_i^\varepsilon(\tau^{**})} \frac{|h'_\varepsilon(\theta)|d\theta}{(\dot\theta_j^\varepsilon - \dot\theta_i^\varepsilon)((\theta_j^\varepsilon - \theta_i^\varepsilon)^{-1}(\theta))} \leq \|h'_\varepsilon\|_{L^1(\mathbb{T})} \frac{4}{\delta_0}.$$

Then the assertion under consideration follows from the second item in Lemma 3.4.3. Regarding the second assertion we split into two terms

$$\int_{T_{k-1}}^{\tau^{**}} (h'_\varepsilon(\theta_j^\varepsilon - \theta_i^\varepsilon) - h'(\theta_j - \theta_i))\, dt = I_\varepsilon + II_\varepsilon,$$

where,

$$\begin{aligned} I_\varepsilon &:= \int_{T_{k-1}}^{\tau^{**}} (h'_\varepsilon(\theta_j^\varepsilon - \theta_i^\varepsilon) - h'(\theta_j^\varepsilon - \theta_i^\varepsilon))\, dt, \\ II_\varepsilon &:= \int_{T_{k-1}}^{\tau^{**}} (h'(\theta_j^\varepsilon - \theta_i^\varepsilon) - h'(\theta_j - \theta_i))\, dt. \end{aligned}$$

The same change of variables as above allows us restate I_ε in the following way

$$I_\varepsilon = \int_{\theta_j^\varepsilon(T_{k-1})-\theta_i^\varepsilon(T_{k-1})}^{\theta_j^\varepsilon(\tau^{**})-\theta_i^\varepsilon(\tau^{**})} (h'_\varepsilon(\theta) - h'(\theta)) \frac{d\theta}{(\dot\theta_j^\varepsilon - \dot\theta_i^\varepsilon)((\theta_j^\varepsilon - \theta_i^\varepsilon)^{-1}(\theta))}.$$

Then, estimate (3.4.5) along with the strong $L^1(\mathbb{T})$ convergence of the kernels in (3) of Lemma 3.4.3 shows that I_ε vanishes when $\varepsilon \to 0$:

$$|I_\varepsilon| \leq \frac{4}{\delta_0} \int_{\theta_j^\varepsilon(T_{k-1})-\theta_i^\varepsilon(T_{k-1})}^{\theta_j^\varepsilon(\tau^{**})-\theta_i^\varepsilon(\tau^{**})} |h'_\varepsilon(\theta) - h'(\theta)|d\theta = \frac{4}{\delta_0} \|h'_\varepsilon(\theta) - h'(\theta)\|_{L^1(\mathbb{T})} \to 0, \quad \text{as} \quad \varepsilon \to 0.$$

For the term II_ε, we use the forth item in Lemma 3.4.3 to show

$$\begin{aligned} |II_\varepsilon| &\leq M \int_{T_{k-1}}^{\tau^{**}} \frac{|(\theta_j^\varepsilon - \theta_j) - (\theta_i^\varepsilon - \theta_i)|_o^\beta}{\min\{|\theta_j^\varepsilon - \theta_i^\varepsilon|_o, |\theta_j - \theta_i|_o\}^\gamma}\, dt \\ &\leq 2^\beta M \|\Theta^\varepsilon - \Theta\|_{C([T_{k-1},\tau^{**}],\mathbb{R}^N)}^\beta \int_{T_{k-1}}^{\tau^{**}} \frac{1}{\min\{|\theta_j^\varepsilon - \theta_i^\varepsilon|_o, |\theta_j - \theta_i|_o\}^\gamma}\, dt \end{aligned}$$

$$\leq 2^\beta M\|\Theta^\varepsilon-\Theta\|^\beta_{C([T_{k-1},\tau^{**}],\mathbb{R}^N)}\int_{T_{k-1}}^{\tau^{**}}\max\left\{\frac{1}{|\theta_j^\varepsilon-\theta_i^\varepsilon|_o^\gamma},\frac{1}{|\theta_j-\theta_i|_o^\gamma}\right\}\,dt$$
$$\leq 2^\beta M\|\Theta^\varepsilon-\Theta\|^\beta_{C([T_{k-1},\tau^{**}],\mathbb{R}^N)}\int_{T_{k-1}}^{\tau^{**}}\left(\frac{1}{|\theta_j^\varepsilon-\theta_i^\varepsilon|_o^\gamma}+\frac{1}{|\theta_j-\theta_i|_o^\gamma}\right)\,dt.$$

Then, a new change of variables along with the equations (3.4.5)-(3.4.6) and the local integrability in one dimension of an inverse power of order γ entail the existence of a non-negative constant C that does not depend on ε such that

$$|II_\varepsilon|\leq C\|\Theta^\varepsilon-\Theta\|_{C([T_{k-1},\tau^{**}],\mathbb{R}^N)}.$$

Then, the second step follows from the uniform convergence of the phases in Lemma 3.4.1.

• *Step 3:* In the third case, consider $i\in\{1,\ldots,N\}$ and $j\in S_i(T_{k-1})$. By the uniqueness in Theorem 3.3.5, we can ensure that $\theta_j(t)=\theta_i(t)$ for all $t\geq T_{k-1}$. Then, the uniform convergence of the kernels h_ε along with the uniform convergence of the phases in Lemma 3.4.1 shows that

$$h_\varepsilon(\theta_j^\varepsilon-\theta_i^\varepsilon)\to 0 \text{ in } C([T_{k-1},\tau]),$$

and then, the result holds true by definition of the norm in $W^{-1,\infty}([T_{k-1},\tau])$. □

Remark 3.4.5. *The preceding results show that the unique global-in-time solution* Θ *to the problem* (3.3.1)-(3.3.2) *with* $\alpha\in(0,\frac{1}{2})$ *in Theorem 3.3.5, satisfies that* $\theta_i\in C^{1,1-2\alpha}([0,\infty),\mathbb{R}^N)$ *and frequencies* $\dot\theta_i$ *enjoy higher higher regularity. Indeed, they are piece-wise* $W^{1,1}$ *in the sense that* $\dot\theta_i\in W^{1,1}([T_{k-1},\tau])$*, for every* $k\in\mathbb{N}$ *and every* $\tau\in(T_{k-1},T_k)$*. In addition, they verify the following equation in weak sense*

$$\ddot\theta_i=\frac{K}{N}\sum_{j\notin S_i(T_{k-1})}h'(\theta_j-\theta_i)(\dot\theta_j-\dot\theta_i), \tag{3.4.7}$$

in $[T_{k-1},\tau]$*. Throughout the proof of the above result we have just used the local integrability in one dimension of any inverse power of order smaller than* 1*. However, one might have tried to use that such inverse powers actually belong to* L^p_{loc} *in order to show that in Steps 2 the convergence take place in* $L^p([T_{k-1},\tau])$*-weak for any* $1\leq p<\frac{1}{2\alpha}$*. In this way, the gain of regularity is in reality higher, namely* $\dot\theta_i\in W^{1,p}([T_{k-1},\tau])$*, for every* $1\leq p<\frac{1}{2\alpha}$*.*

In the following, we shall discuss the corresponding singular limit in the critical and supercritical case. Since the Filippov set-valued map is relatively simpler in that latter case, we will start with that supercritical case. Later, we will adapt the ideas therein to show a parallel result in the critical regime.

3.4.2 Singular limit in the supercritical case

Using a similar vector notation to that in (3.2.3) for the singular weighted model, our regularized system (3.2.5)-(3.2.6) can be restated as

$$\left\{\begin{array}{l}\dot\Theta^\varepsilon=H^\varepsilon(\Theta^\varepsilon),\\ \Theta^\varepsilon(0)=\Theta_0,\end{array}\right.$$

where the components of the vector field H^ε read

$$H_i^\varepsilon(\Theta)=\Omega_i+\frac{K}{N}\sum_{j\neq i}h_\varepsilon(\theta_j-\theta_i),$$

for every $\Theta \in \mathbb{R}^N$ and every $i \in \{1, \ldots, N\}$. Then, one can mimic the ideas in Section 3.2 to show that the regularized system can also be written as a gradient flow

$$\begin{cases} \dot{\Theta}^\varepsilon = -\nabla V^\varepsilon(\Theta^\varepsilon), \\ \Theta^\varepsilon(0) = \Theta_0, \end{cases} \tag{3.4.8}$$

where the regularized potential now reads

$$V^\varepsilon(\Theta) := -\sum_{i=1}^N \Omega_i\theta_i + V_{int}^\varepsilon(\Theta) := -\sum_{i=1}^N \Omega_i\theta_i + \frac{K}{2N}\sum_{i\neq j} W_\varepsilon(\theta_i - \theta_j), \tag{3.4.9}$$

for every $\Theta \in \mathbb{R}^N$. Again, W_ε is the anti-derivative of h_ε such that $W_\varepsilon(0) = 0$, i.e.,

$$W_\varepsilon(\theta) := \int_0^\theta h_\varepsilon(\theta')\, d\theta'.$$

Also, it is clear that $W_\varepsilon \geq 0$ in the supercritical case, for every $\varepsilon > 0$. Then, the following result holds true.

Lemma 3.4.6. *In the supercritical case $\alpha \in (\frac{1}{2}, 1)$, consider the unique global-in-time classical solution Θ^ε to the regularized system* (3.4.8). *Then,*

$$\frac{1}{2}\int_0^t |\dot{\Theta}^\varepsilon(s)|^2\, ds \leq \frac{C_\Omega^2}{2} t + V_{int}(\Theta_0),$$

for every $t > 0$ and every $\varepsilon > 0$, where $C_\Omega := |(\Omega_1, \ldots, \Omega_N)| = \sqrt{\Omega_1^2 + \cdots + \Omega_N^2}$.

The above result shows that $\{\Theta^\varepsilon\}_{\varepsilon>0}$ is bounded in $H^1((0,T), \mathbb{R}^N)$, for every $T > 0$. Then, there exists some subsequence that we denote in the same way so that $\{\Theta^\varepsilon\}_{\varepsilon>0}$ weakly converge to some $\Theta \in H^1_{loc}((0,\infty), \mathbb{R}^N)$ in $H^1((0,T), \mathbb{R}^N)$ for every $T > 0$. The Sobolev embedding and the definition of weak convergence ensure that

$$\begin{aligned} &\Theta^\varepsilon \to \Theta \text{ in } C([0,T], \mathbb{R}^N), \\ &\dot{\Theta}^\varepsilon \rightharpoonup \dot{\Theta} \text{ in } L^2((0,T), \mathbb{R}^N), \end{aligned}$$

for every $T > 0$. Before we obtain the desired convergence result of (3.4.8) towards a Filippov solution, let us introduce the following split of the frequencies:

$$\dot{\Theta}^\varepsilon(t) = x^\varepsilon(t) + y^\varepsilon(t), \tag{3.4.10}$$

where, componentwise, each term reads as follows

$$\begin{aligned} x_i^\varepsilon(t) &= \frac{K}{N}\sum_{j\notin \mathcal{C}_i(t)} (h_\varepsilon(\theta_j^\varepsilon(t) - \theta_i^\varepsilon(t)) - h(\theta_j(t) - \theta_i(t))), \\ y_i^\varepsilon(t) &= \frac{K}{N}\sum_{j\notin \mathcal{C}_i(t)} h(\theta_j(t) - \theta_i(t)) + \frac{K}{N}\sum_{j\in \mathcal{C}_i(t)} h_\varepsilon(\theta_j^\varepsilon(t) - \theta_i^\varepsilon(t)). \end{aligned}$$

Then, it is clear by definition that

$$\begin{aligned} &x^\varepsilon \to 0 \text{ in } C([0,T], \mathbb{R}^N), \\ &y^\varepsilon \rightharpoonup \dot{\Theta} \text{ in } L^2((0,T), \mathbb{R}^N), \end{aligned}$$

for every $T > 0$, and $y^\varepsilon(t) \in \mathcal{H}(\Theta(t))$, for every $t \geq 0$. As a consequence, we infer that Θ^ε becomes a Filippov approximate solution in the following sense:

$$\dot{\Theta}^\varepsilon(t) \in \mathcal{H}(\Theta(t)) + x^\varepsilon(t). \tag{3.4.11}$$

Remark 3.4.7. *Recall that $\mathcal{H}(\Theta(t))$ is a closed set, for every $t \geq 0$, see Proposition D.1.3. Consequently, in order to prove that the limiting $\Theta(t)$ yields a Filippov solution, it would be enough to show the almost everywhere convergence of the sequence $\{\dot{\Theta}^\varepsilon\}_{\varepsilon>0}$ towards $\dot{\Theta}$. Unfortunately, it is well known that weak convergence in L^2 is not enough for that purpose. Hence, we must deal only with such weak convergence.*

Before going to the heart of the matter, we need to exhibit another characterization of the Filippov set-valued map in terms of implicit equations. The next technical lemma will be used for that. For the sake of clarity, a proof has been provided in Lemma 3.B.2 of Appendix 3.B.

Lemma 3.4.8. *Consider $n \in \mathbb{N}$ and any vector $x \in \mathbb{R}^n$. Then, the following assertions are equivalent:*

1. *There exists some $Y \in \mathrm{Skew}_n(\mathbb{R})$ such that*
$$x = Y \cdot \mathbf{j}.$$
2. *The following implicit equation holds true*
$$x \cdot \mathbf{j} = 0,$$

where $\mathbf{j} = \underbrace{(1, \cdots, 1)}_{n\ pairs}$ stands for the vector of ones.

Hence, we are ready to obtain the above-mentioned characterization.

Proposition 3.4.9. *In the supercritical regime $\alpha > \frac{1}{2}$, the Filippov set-valued map $\mathcal{H} = \mathcal{H}(\Theta)$ associated with $H = H(\Theta)$ consists in the affine subspace of dimension $N - \kappa$ of points $(\omega_1, \ldots, \omega_N) \in \mathbb{R}^N$ obeying the following implicit equations (recall Subsection 3.2.3)*

$$\frac{1}{n_k} \sum_{i \in E_k} \omega_i = \frac{1}{n_k} \sum_{i \in E_k} \left(\Omega_i + \frac{K}{N} \sum_{j \notin \mathcal{C}_i} h(\theta_j - \theta_i) \right), \tag{3.4.12}$$

for every $k = 1, \ldots, \kappa$.

Proof. By Proposition 3.3.16, $\mathcal{H}(\Theta)$ consists of the set of points $(\omega_1, \ldots, \omega_N) \in \mathbb{R}^N$ such that for every $k = 1, \ldots, \kappa$ there exist a skew symmetric matrix $Y^k \in \mathrm{Skew}_{n_k}(\mathbb{R})$ and a bijection $\sigma^k : \{1, \ldots, n_k\} \longrightarrow E_k$ such that the following equations hold true

$$\omega_{\sigma_i^k} = \Omega_{\sigma_i^k} + \frac{K}{N} \sum_{\substack{m=1 \\ m \neq k}}^{\kappa} n_m h(\theta_{\iota_m} - \theta_{\iota_k}) + \frac{K}{N} \sum_{j=1}^{n_k} y_{ij}^k,$$

for every $i = 1, \ldots, n_k$. Then, the result follows by applying Lemma 3.4.8 to each of the above sets of n_k equations to the particular vectors $x^k \in \mathbb{R}^{n_k(\Theta)}$ with components:

$$x_i^k := \omega_{\sigma_i^k} - \Omega_{\sigma_i^k} - \frac{K}{N} \sum_{\substack{m=1 \\ m \neq k}}^{\kappa} n_m h(\theta_{\iota_m} - \theta_{\iota_k}),\ i = 1, \ldots, n_k,$$

when we equivalently restate it using the notation in Subsection 3.2.3. □

Remark 3.4.10. *Here we discuss why the same approach as in Subsection 3.4.1 to decompose the dynamics for $\alpha \in [\frac{1}{2}, 1)$ into subintervals (T_k, T_{k+1}) with same collisional type cannot be conducted:*

1. *Recall that in the subcritical case $\alpha \in (0, \frac{1}{2})$ in Subsection 3.4.1, any strong limit Θ already yielded a solution to the limiting system* (3.3.1)-(3.3.2). *Indeed, there just can be one and only one such strong limit by the one-sided uniqueness of the limiting system* (3.3.1)-(3.3.2) *Also, in that subcritical case one can find a nice split of the dynamics in a sequence of intervals where no collision happens. Thus, on every such interval, the kind of collisional state of our trajectory remains unchanged. Let us remember that the reason why that sequence fills the whole half line in the subcritical case relies on the following facts: first, by uniqueness we can characterize the sticking of oscillators and once they stick during some time they remain stuck for all times. In particular, only $N-1$ sticking times can exist. Second, when an accumulation of collisions takes place, it has to be at a sticking time. Hence, there just can be $N-1$ such accumulations of collisions, thus recovering the whole half line.*

2. *Unfortunately, for $\alpha \in (\frac{1}{2}, 1)$ or $\alpha = \frac{1}{2}$ we still do not know at this point whether any limit Θ becomes a Filippov solution to the limiting system* (3.3.1)-(3.3.2). *Thus, despite the fact that we have a clear characterizations of sticking of such solutions, we cannot apply them to any such limit Θ. In addition, the behavior of any H^1 weak limit can be very wild. Specifically, a possible scenario of a H^1_{loc} trajectory is that sticking might happen just for a short period of time and, after it, the cluster detaches. Also, "pure collisions" might accumulate at a non-sticking time exhibiting Zeno behavior (recall Remark 3.3.15 and Figure 3.5). Thereby, a split of the dynamics into countably many intervals (T_k, T_{k+1}) like in the above Subsection 3.4.1, where the collisional state remains unmodified, is not viable.*

As discussed in the above Remark, it is not clear how to achieve a split of the dynamics into countably many time intervals covering the whole half-line, each of them exhibiting unvaried collisional state. Then, we require the development of a new approach, where the above explicit H-representation of the Filippov set-valued map at any collision state will play a role. One of our main tools here will be the *Kuratowski–Ryll–Nardzewski measurable selection theorem*, see [197]. This result guarantees the existence of measurable selections of any set-valued *Effros-measurable map*. Sometimes, it is necessary to know how many of these single-valued measurable selections do we need in order to essentially have the whole set-valued map "represented" in an appropriate set sense. This is called *Castaign representation*, see [69, Theorem III.30], and it turns out that we only require countably many such measurable selectors to densely fill the range of the set-valued map. Such results will be directly applied to the critical case in the next Subsection 3.4.3. However, for the supercritical case, we shall further refinements of the above theorem to allow for integrable representations of the set-valued map. Specifically, the Effros-measurability has to be improved to some stronger integrability condition for set-valued maps. We refer to Appendix E of this thesis for a short introduction to the above results and the above-mentioned Corollaries to the integrable setting, see Lemma E.0.4 and Remark E.0.5.

Theorem 3.4.11. *Consider the classical solutions $\{\Theta^\varepsilon\}_{\varepsilon>0}$ to the regularized system* (3.4.8) *with $\alpha \in (\frac{1}{2}, 1)$ and any weak H^1_{loc} limit Θ. Then,*

$$\dot{\Theta}(t) \in \mathcal{H}(\Theta(t)) \ \textit{for a.e.} \ \ t \geq 0.$$

Proof. • *Step 1*: H-representation of the Filippov map.
By virtue of Proposition 3.4.9

$$\mathcal{H}(\Theta(t)) = \bigcap_{l=1}^{\kappa(t)} \mathcal{P}_l(t), \tag{3.4.13}$$

where each of the $\mathcal{P}_l(t)$ stands for the hyperplane $\mathcal{P}_l(t) := \{x \in \mathbb{R}^N : a_l(t) \cdot x = b_l(t)\}$. Here, the above vector and scalar functions $a_l(t)$ and $b_l(t)$ read as follows

$$a_l(t) := \frac{1}{n_l(t)} \sum_{i \in E_l(t)} e_i, \quad b_l(t) := \frac{1}{n_l(t)} \sum_{i \in E_l(t)} \left(\Omega_i + \frac{K}{N} \sum_{j \notin \mathcal{C}_i(t)} h(\theta_j - \theta_i) \right).$$

• *Step 2*: Castaing representation of coefficients.
Also, let us define $\mathcal{A} : \mathbb{R}_0^+ \longrightarrow 2^{\mathbb{R}^N}$ and $\mathcal{B} : \mathbb{R}_0^+ \longrightarrow 2^{\mathbb{R}}$ by

$$\mathcal{A}(t) := \{a_l(t) : l = 1, \ldots, \kappa(t)\},$$
$$\mathcal{B}(t) := \{b_l(t) : l = 1, \ldots, \kappa(t)\},$$

for any $t \geq 0$. It is clear that both maps take closed non-empty values and they are Effros-measurable. Then, Lemma E.0.3 in Appendix E allows obtaining a Castaing representation of both maps. Indeed, notice that $\mathcal{A}$ is strongly essentially bounded, see Remark E.0.5. Hence, there exists a sequence $\{A^n\}_{n \in \mathbb{N}} \subseteq L^\infty(0, +\infty)$ such that

$$\mathcal{A}(t) = \overline{\{A^n(t) : n \in \mathbb{N}\}},$$

for almost every $t \geq 0$. By the finiteness of $\mathcal{A}(t)$ we equivalently have

$$\{a_l(t) : l = 1, \ldots, \kappa(t)\} = \{A^n(t) : n \in \mathbb{N}\}, \tag{3.4.14}$$

for almost every $t \geq 0$. However, notice that it is not clear yet whether $\mathcal{B}$ is strongly locally integrable due to the fact that eventual switches of the collisional type for the limiting $\Theta(t)$ (thus on its coefficients $b_l(t)$) is expected.

• *Step 3*: Strong local integrability of $\mathcal{B}$.
Consider the regularized coefficients in the H-representation as follows

$$b_l^\varepsilon(t) := \frac{1}{n_l(t)} \sum_{i \in E_l(t)} \left(\Omega_i + \frac{K}{N} \sum_{j \notin \mathcal{C}_i(t)} h_\varepsilon(\theta_j^\varepsilon - \theta_i^\varepsilon) \right), \ l = 1, \ldots, \kappa(t).$$

We can associate a similar set-valued map $\mathcal{B}^\varepsilon : \mathbb{R}_0^+ \longrightarrow 2^{\mathbb{R}}$ defined by

$$\mathcal{B}^\varepsilon(t) = \{b_l^\varepsilon(t) : l = 1, \ldots, \kappa(t)\}.$$

Notice that, by definition, it is clear that

$$\lim_{\varepsilon \to 0} b_l^\varepsilon(t) = b_l(t),$$

for every $l = 1, \ldots, \kappa(t)$ since $j \notin \mathcal{C}_i(t)$ in their definitions and, at those $\theta_j(t) - \theta_i(t)$, the limiting kernel h is continuous. Since both $\mathcal{B}(t)$ and $\mathcal{B}^\varepsilon(t)$ consist of finitely many terms, we deduce that

$$|\mathcal{B}^\varepsilon(t)| \longrightarrow |\mathcal{B}(t)|, \ \text{ a.e. } t \in \mathbb{R}_0^+. \tag{3.4.15}$$

Then, Fatou's lemma on any finite time interval $[0, T] \subseteq \mathbb{R}_0^+$ with $T > 0$ entails

$$\int_0^T |\mathcal{B}|(t)\, dt \leq \liminf_{\varepsilon \to 0} \int_0^T |\mathcal{B}^\varepsilon|(t)\, dt. \tag{3.4.16}$$

By definition, it is clear that

$$\dot{\Theta}^\varepsilon(t)\cdot a_l(t) = \frac{1}{n_l(t)}\sum_{i\in E_l(t)}\left(\Omega_i + \frac{K}{N}\sum_{j=1}^N h_\varepsilon(\theta_j^\varepsilon - \theta_i^\varepsilon)\right) = b_l^\varepsilon(t),$$

where we have cancelled the terms with $j\in E_l(t)$ in the last step by the antisymmetry of h_ε. Then, our set-valued maps are strongly dominated as follows

$$|\mathcal{B}^\varepsilon|(t)\leq |\dot{\Theta}^\varepsilon(t)| \text{ a.e. } t\geq 0. \tag{3.4.17}$$

Putting (3.4.17) into 3.4.16 we obtain

$$\begin{aligned}\int_0^T |\mathcal{B}|(t)\,dt &= \int_0^T \liminf_{\varepsilon\to 0}|\mathcal{B}^\varepsilon|(t)\,dt \leq \liminf_{\varepsilon\to 0}\int_0^T |\dot{\Theta}^\varepsilon(t)|\,dt\\ &\leq T^{1/2}\liminf_{\varepsilon\to 0}\left(\int_0^T|\dot{\Theta}^\varepsilon(t)|^2\,dt\right)^{1/2} \leq T^{1/2}\left(C_\Omega^2 T + 2V_{int}(\Theta_0)\right)^{1/2}<\infty.\end{aligned}$$

Here, we have used the Cauchy–Schwarz inequality in the second step and the a priori bound in Lemma 3.4.6 in the last one. Then, Remark E.0.5 yields the existence of a Castaing representation $\{B^n\}_{n\in\mathbb{N}}\subseteq L^1_{loc}(0,+\infty)$ of the map $\mathcal{B}$. Again, we conclude that

$$\{b_l(t)\,:\, l=1,\ldots,\kappa(t)\} = \{B^n(t)\,:\, n\in\mathbb{N}\}, \tag{3.4.18}$$

for almost every $t\geq 0$.

• *Step 4*: Conclusion.
Since $y^\varepsilon(t)\in\mathcal{H}(\Theta(t))$, for every $\varepsilon>0$ and every $t\geq 0$, then the H-representation (3.4.13) along with the essentially bounded and locally integrable representations (3.4.14) and (3.4.18) yield the equations

$$A^n(t)\cdot y^\varepsilon(t) = B^n(t),\ n\in\mathbb{N},$$

for almost every $t\geq 0$. In particular,

$$\int_0^{+\infty} A^n(t)\cdot y^\varepsilon(t)\,\varphi(t)\,dt = \int_0^{+\infty} B^n(t)\,\varphi(t)\,dt,$$

for every $\varepsilon>0$, each $\varphi\in C_c(\mathbb{R}^+)$ and any $n\in\mathbb{N}$. Notice that the boundedness and local integrability of our selectors allows such expression to make sense. We can now use the weak convergence in L^2 of y^ε towards $\dot{\Theta}$ to obtain

$$\int_0^{+\infty} A^n(t)\cdot \dot{\Theta}(t)\,\varphi(t)\,dt = \int_0^{+\infty} B^n(t)\,\varphi(t)\,dt,$$

for every $\varphi\in C_c(\mathbb{R}^+)$ and each $n\in\mathbb{N}$. The fundamental lemma of calculus of variations along with the Castaing representations in (3.4.14) and (3.4.18) and the H-representation in (3.4.13) allow us to conclude the claimed result. □

3.4.3 Singular limit in the critical case

In this Subsection, we will address the singular limit of the regularized system (3.2.5)-(3.2.6) towards a Filippov solution to (3.3.1)-(3.3.2) in the critical regime $\alpha=\frac{1}{2}$. We will mostly apply a similar approach to that in the supercritical regime. Nevertheless, there are several novelties to be considered, that make the study slightly different. First, we will show that we actually enjoy a better $W^{1,\infty}$ a priori estimate, apart from the above H^1 bound in Lemma 3.4.6. Second, the explicit expression of the Filippov map in Proposition 3.4.9 in terms of intersection of hyperplanes will be adapted to this case.

Lemma 3.4.12. *In the critical regime $\alpha = \frac{1}{2}$, consider the unique global-in-time solution Θ^ε to the regularized system* (3.4.8). *Then,*

$$\|\dot{\Theta}^\varepsilon\|_{L^\infty((0,\infty),\mathbb{R}^N)} \leq C_\Omega + K,$$

for every $\varepsilon > 0$, where $C_\Omega := |(\Omega_1, \ldots, \Omega_N) = \sqrt{\Omega_1^2 + \cdots \Omega_N^2}$.

We omit the proof since it is a clear consequence of the boundedness of h in the critical case. As a consequence of the above Lemma 3.4.12, we infer the existence of a subsequence of $\{\Theta^\varepsilon\}_{\varepsilon>0}$ that we denote in the same way so that it weakly-* converges to some $\Theta \in W^{1,\infty}_{loc}((0,\infty);\mathbb{R}^N)$ in $W^{1,\infty}((0,T),\mathbb{R}^N)$, for every $T > 0$. In particular

$$\begin{aligned} &\Theta^\varepsilon \to \Theta \text{ in } C([0,T],\mathbb{R}^N),\\ &\dot{\Theta}^\varepsilon \overset{*}{\rightharpoonup} \dot{\Theta} \text{ in } L^\infty((0,T),\mathbb{R}^N), \end{aligned}$$

for every $T > 0$. In addition, the same split as in (3.4.10) can be considered and we obtain

$$\begin{aligned} &x^\varepsilon \to 0 \text{ in } C([0,T],\mathbb{R}^N),\\ &y^\varepsilon \overset{*}{\rightharpoonup} \dot{\Theta} \text{ in } L^\infty((0,T),\mathbb{R}^N), \end{aligned}$$

and $y^\varepsilon(t) \in \mathcal{H}(\Theta(t))$, for every $t \geq 0$ and $\varepsilon > 0$. Hence, Θ^ε becomes an approximate solution in the same sense as in (3.4.11). What is more, the same Remark 3.4.7 is in order. Then, again we cannot ensure pointwise convergence of $\dot{\Theta}^\varepsilon$. In order to obtain an analogue characterization of the Filippov map, we will need the next technical lemma.

Lemma 3.4.13. *Consider any $n \in \mathbb{N}$ and any vector $x \in \mathbb{R}^n$. Then, the following two assertions are equivalent:*

1. *There exists some $Y \in \mathrm{Skew}_n([-1,1])$ such that*
$$x = Y \cdot \mathbf{j}.$$
2. *We have*
$$\frac{1}{k}\sum_{i=1}^{k} x_{\sigma_i} \in [-(n-k),(n-k)],$$
for every permutation σ of $\{1,\ldots,n\}$ and any $k \in \mathbb{N}$.

For an easier readability, we postpone the proof to Appendix 3.B in this Chapter. The following result the becomes a straightforward consequence of Lemma 3.4.13 along with the explicit formula in Proposition 3.3.7.

Proposition 3.4.14. *In the critical regime $\alpha = \frac{1}{2}$, the Filippov set-valued map $\mathcal{H} = \mathcal{H}(\Theta)$ associated with $H = H(\Theta)$ is the compact and convex polytope of points $(\omega_1, \ldots, \omega_N) \in \mathbb{R}^N$ whose H-representation consist of the affine inequalities (recall Subsection 3.2.3)*

$$\frac{1}{m}\sum_{i\in I}\omega_i \in \frac{1}{m}\sum_{i\in I}\left(\Omega_i + \frac{K}{N}\sum_{j\notin\mathcal{C}_i} h(\theta_j - \theta_i)\right) + \left[-\frac{K}{N}(n_k - m), \frac{K}{N}(n_k - m)\right], \tag{3.4.19}$$

for every $k = 1, \ldots, \kappa$, and $I \subseteq E_k$ with $\#I = m$.

Then, we move to the main result, i.e., the convergence of the singular limit towards a Filippov solution to the critical system.

Theorem 3.4.15. *Consider the classical solutions* $\{\Theta^\varepsilon\}_{\varepsilon>0}$ *to the regularized system* (3.4.8) *with* $\alpha = \frac{1}{2}$ *and any weak-* limit* Θ *in* $W^{1,\infty}_{loc}$. *Then,*

$$\dot{\Theta}(t) \in \mathcal{H}(\Theta(t)) \text{ for a.e. } t \geq 0.$$

Proof. We mimic the proof of Theorem 3.4.11. Recall that by the above Proposition 3.4.14, an analogue H-representation to that in (3.4.13) holds. Specifically,

$$\mathcal{H}(\Theta(t)) = \bigcap_{l=1}^{\kappa(t)} \bigcap_{I\subseteq E_l} (\mathcal{S}^+_{l,I}(t) \cap \mathcal{S}^-_{l,I}(t)), \tag{3.4.20}$$

where the semi-spaces read

$$\mathcal{S}^+_{l,I}(t) := \{x \in \mathbb{R}^N : a_{l,I}(t)\cdot x \leq b^+_{l,I}(t)\},$$
$$\mathcal{S}^-_{l,I}(t) := \{x \in \mathbb{R}^N : a_{l,I}(t)\cdot x \geq b^-_{l,I}(t)\},$$

for every $I \subseteq E_l(t)$. We set

$$a_{l,I}(t) := \frac{1}{m}\sum_{i\in I} e_i,$$
$$b^\pm_{l,I}(t) := \frac{1}{m}\sum_{i\in I}\left(\Omega_i + \frac{K}{N}\sum_{j\notin \mathcal{C}_i(t)} h(\theta_j(t)-\theta_i(t))\right) \pm (n_l(t)-m),$$

where $m = \#I$. Now, the coefficients are clearly uniformly bounded. Then, a straightforward application of Remark E.0.5 leads to the existence of essentially bounded selectors for the coefficients. Namely, we can give an ordering such as

$$\{a_{l,I}(t) : l = 1,\ldots,\kappa(t),\, I \subseteq E_l(t)\} = \{A^n(t) : n \in \mathbb{N}\} \tag{3.4.21}$$
$$\{b^\pm_{l,I}(t) : l = 1,\ldots,\kappa(t),\, I \subseteq E_l(t)\} = \{B^{\pm,n}(t) : n \in \mathbb{N}\}, \tag{3.4.22}$$

for almost every $t \geq 0$. Recall that $y^\varepsilon(t) \in \mathcal{H}(\Theta(t))$, for every $\varepsilon > 0$ and every $t \geq 0$. Then, by virtue of (3.4.20), (3.4.21) and (3.4.22), we equivalently have

$$A^n(t)\cdot y^\varepsilon(t) \leq B^{+,n}(t) \text{ and } A^n(t)\cdot y^\varepsilon(t) \geq B^{-,n}(t),$$

for all $n \in \mathbb{N}$, each $\varepsilon > 0$ and almost every $t \geq 0$. In particular,

$$\int_0^{+\infty} A^n(t)\cdot y^\varepsilon(t)\,\varphi(t)\,dt \leq \int_0^{+\infty} B^{+,n}\,dt,$$
$$\int_0^{+\infty} A^n(t)\cdot y^\varepsilon(t)\,\varphi(t)\,dt \geq \int_0^{+\infty} B^{-,n}\,dt,$$

for all $n \in \mathbb{N}$, each $\varepsilon > 0$ and any non-negative $\varphi \in C_c(\mathbb{R}^+)$. Then, using the weak-* convergence in L^∞ we obtain that

$$\int_0^{+\infty} A^n(t)\cdot \dot{\Theta}(t)\,\varphi(t)\,dt \leq \int_0^{+\infty} B^{+,n}\,dt,$$
$$\int_0^{+\infty} A^n(t)\cdot \dot{\Theta}(t)\,\varphi(t)\,dt \geq \int_0^{+\infty} B^{-,n}\,dt,$$

for all $n \in \mathbb{N}$ and any non-negative $\varphi \in C_c(\mathbb{R}^+)$. Hence, the result follows from the fundamental lemma of calculus of variations along with the Castaing representations (3.4.21)-(3.4.22) and the H-representation (3.4.20). □

3.4.4 Comparison with previous results about singular weighted systems

In the previous parts of this Chapter, we studied the existence and one-sided uniqueness for the singular weighted first order Kuramoto model (3.3.1)-(3.3.2) in all the subcritical, critical and supercritical regimes. We now compare our result with previous research on the singular weighted Cucker–Smale model which is a second order system describing the flocking behavior of interacting particles. In order to set these relations, let us recall Section 3.2, where the first order Kuramoto model (3.2.1) was shown to be equivalent to its second order augmentation (3.2.4). On the one hand, this is clear for regular weights as studied in Theorem 3.2.1, see [150, 160]. What is more, it remains true in our case, which is characterized by singular weights. However, we must be specially careful with the time regularity in order for such heuristic arguments to become true. Let us focus on the subcritical regime, where the rigorous equivalence between (3.2.1) and (3.2.4) follows from Remark 3.3.20 by virtue of the one-sided uniqueness in both models. Indeed, in such subcritical case, the "influence function" of the augmented flocking-type model reads

$$h'(\theta) = \frac{1}{|\theta|_o^{2\alpha}} \left[\cos\theta - 2\alpha \frac{\sin|\theta|_o}{|\theta|_o}\right] \sim \frac{1-2\alpha}{|\theta|_o^{2\alpha}} \quad \text{near} \quad \theta \in 2\pi\mathbb{Z}, \tag{3.4.23}$$

which enjoys mild singularities of order $2\alpha < 1$ in the subcritical case. Such singular second order model (3.2.4)-(3.4.23) shares some similarities with the Cucker–Smale model with singular weights,

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = \dfrac{K}{N}\displaystyle\sum_{j=1}^{N} \psi(|x_j - x_i|)(v_j - v_i), \end{cases} \tag{3.4.24}$$

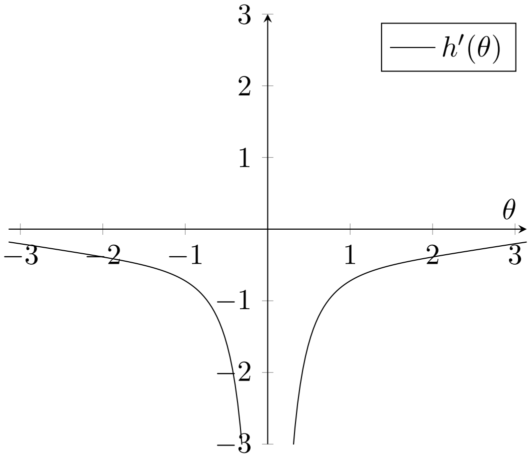
where the communication weight ψ is given by

$$\psi(r) := \frac{1}{r^\beta}, \tag{3.4.25}$$

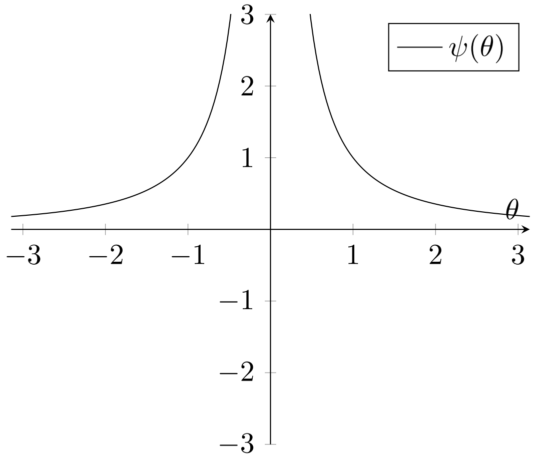
for $r > 0$ and $\beta > 0$. Although some results regarding the asymptotic behavior of such system have been established [158], the well–posedness theory has not been addressed until very recently in [244, 245] for the microscopic model and [60, 226, 255, 275, 276, 277] for some first and second order kinetic and macroscopic versions of the model. Regarding the microscopic system (3.4.24)-(3.4.25), the existence of global C^1 piece-wise weak $W^{2,1}$ solutions $(x_1, \ldots, x_N)$ has been established in [244] for $\beta \in (0, 1)$, which corresponds to $\alpha \in (0, \frac{1}{2})$ in our setting (see Theorem 3.3.5, Theorem 3.4.2 and Remark 3.4.5). Also, in the weakly singular regime $\beta \in (0, \frac{1}{2})$ (i.e., $\alpha \in (0, \frac{1}{4})$), the same author proved in [245] that the velocities $(v_1, \ldots, v_N)$ are indeed absolutely continuous. Consequently, the C^1 weak solutions $(x_1, \ldots, x_N)$ are actually $W^{2,1}_{loc}$ in such latter case. This latter property was proved through a differential inequality.

The method of proof is similar to ours in Section 3.4 and relies on a regularization process of the second order model near the collision times. In our case, we have obtained a similar regularization process of the first order model, entailing the corresponding regularization of the augmented second order model. Indeed, such method has not only proved successful in our subcritical case, but also in the critical and supercritical case. Also, we have obtained the well-posedness results in an alternative way based on the gain of continuity of the kernel in the first order model along with its particular structure near the points of loss of Lipschitz-continuity. Indeed, we have succeeded in introducing an analogue well–posedness theory in Filippov sense for the endpoint case $\alpha = \frac{1}{2}$ and the supercritical case $\alpha > \frac{1}{2}$.

Regarding the more singular cases $\beta \geq 1$ (i.e., $\alpha \geq \frac{1}{2}$), one can show that there exists some class of initial data for (3.4.24)-(3.4.25) such that one can avoid collisions and the solutions remain smooth for all times. Indeed, such solutions exhibit asymptotic flocking dynamics, see [3]. Very recently, it was shown in [62] that the loss of integrability of the kernel when $\beta \geq 1$ actually ensures the avoidance of collisions for general initial data. In such regime, the asymptotic flocking behavior is not guaranteed for any initial data. However, such ideas for (3.4.24)-(3.4.25) fails in our model (3.2.4)-(3.4.23) because the kernel h' with $\alpha \geq \frac{1}{2}$ does no longer behave like the communication weight ψ with $\beta \geq 1$. Specifically, ψ is always a positive and decreasing function whereas h' is negative and increasing (see Figure 3.7). Then, we do expect our solutions to exhibit finite time collisions as depicted in the results in next Section 3.5. This is the reason for the generalized theory in Filippov sense to come into play in the critical and supercritical cases.



(a) $h'(\theta)$



(b) $\psi(\theta)$

Figure 3.7: Comparison of the functions $h'(\theta)$ and $\psi(\theta)$ with $\alpha = 0.75$.

3.5 Synchronization of the singular weighted system

We now analyze the collective behavior in the system (3.3.1)-(3.3.2). We first consider the system of two interacting oscillators. We extend the argument to the N-oscillator system in succession.

3.5.1 Two oscillator case

In this part, we consider the dynamics of two oscillator. The system (3.3.1)-(3.3.2) for two oscillator becomes

$$\begin{aligned} \dot{\theta}_1 &= \Omega_1 + \frac{K}{2} \frac{\sin(\theta_2 - \theta_1)}{|\theta_2 - \theta_1|_o^{2\alpha}}, \\ \dot{\theta}_2 &= \Omega_2 + \frac{K}{2} \frac{\sin(\theta_1 - \theta_2)}{|\theta_1 - \theta_2|_o^{2\alpha}}. \end{aligned} \tag{3.5.1}$$

Recall that in the critical and supercritical cases we do expect collisions, see Subsections 3.3.2 an 3.3.3. Then, the above representation of the system is only valid before the first collision. After that, the right-hand side has to be replaced with the corresponding Filippov set-valued map. At this step, we shall focus on the dynamics before the first collision. Let us define the relative phase and natural frequency by $\theta := \theta_2 - \theta_1$ and $\Omega := \Omega_2 - \Omega_1$. Then, the system (3.5.1) can be rewritten into the following form:

$$\dot{\theta} = \Omega - K\frac{\sin\theta}{|\theta|_o^{2\alpha}}. \tag{3.5.2}$$

Proposition 3.5.1. *Let $\theta : [0,T) \to \mathbb{R}$ be a maximal classical solution to the differential equation* (3.5.2) *with $\alpha \in (0,1)$ such that the oscillators are identical, i.e., $\Omega = 0$, and initial datum $0 < |\theta_0| < \pi$. Then, the maximal time of existence T lies in the interval $[t_{min}, t_{max}]$, where*

$$t_{min} = \frac{|\theta_0|^{2\alpha}}{2K\alpha} \quad \textit{and} \quad t_{max} = \frac{|\theta_0|^{2\alpha+1}}{2K\alpha\sin|\theta_0|}.$$

In addition, the following lower and upper estimates

$$|\theta_0|^{2\alpha} - 2K\alpha t \leq |\theta|^{2\alpha} \leq |\theta_0|^{2\alpha} - 2K\alpha t\frac{sin|\theta_0|}{|\theta_0|}t$$

hold, for all $t \in [0,T)$ and $\lim_{t\to T}\theta(t) = 0$. Hence, two identical oscillators confined to the half-circle exhibit finite-time phase synchronization.

Proof. First of all, let us note that in the identical case $\pi + 2\pi\mathbb{Z}$ are equilibria of (3.5.2) where the interaction kernel is Lipschitz-continuous. Hence, the maximal solution θ cannot touch such values if initially started at θ_0. Thereby, $\theta(t) \in (-\pi,\pi)$ for every $t \in [0,T)$ and consequently, $|\theta(t)|_o = |\theta(t)|$ for $t \in [0,T)$. Let us now multiply by $(2\alpha+1)|\theta|^{2\alpha}\mathrm{sgn}(\theta)$ on both side to obtain

$$\frac{d}{dt}|\theta|^{2\alpha+1} = (2\alpha+1)|\theta|^{2\alpha}\mathrm{sgn}(\theta)\frac{d}{dt}\theta = -K(2\alpha+1)\sin\theta\,\mathrm{sgn}(\theta) = -K(2\alpha+1)\sin|\theta|.$$

Denote $y = |\theta|^{2\alpha+1}$, then the equation becomes

$$\frac{d}{dt}y = -K(2\alpha+1)\sin y^{\frac{1}{2\alpha+1}}. \tag{3.5.3}$$

We now consider upper and lower estimates for (3.5.3) separately.

• *Lower estimate:* Since $|y| \geq \sin|y|$, we have

$$\frac{d}{dt}y \geq -K(2\alpha+1)y^{\frac{1}{2\alpha+1}}.$$

By multiplying by $\frac{2\alpha}{2\alpha+1}y^{-\frac{1}{2\alpha+1}}$ on both sides, we obtain

$$\frac{d}{dt}y^{\frac{2\alpha}{2\alpha+1}} \geq -2K\alpha.$$

This yields

$$y^{\frac{2\alpha}{2\alpha+1}} \geq y_0^{\frac{2\alpha}{2\alpha+1}} - 2K\alpha t.$$

Thus, we have a lower estimate

$$|\theta|^{2\alpha} \geq |\theta_0|^{2\alpha} - 2K\alpha t \quad \text{for} \quad 0 \leq t < T.$$

In particular, the above lower estimate shows that

$$T \geq \frac{|\theta_0|^{2\alpha}}{2K\alpha} \equiv t_{min}.$$

• *Upper estimate:* As long as $0 \leq y < \pi^{2\alpha+1}$, the solution y is non-increasing, i.e., $\frac{d}{dt}y \leq 0$. Since the initial data θ_0 satisfies $|\theta_0| < \pi$, we have $y_0 < \pi^{2\alpha+1}$, thus $y(t) \leq y_0$, for $t > 0$. Hence, we have the following inequality

$$\sin y^{\frac{1}{2\alpha+1}} \geq \frac{\sin y_0^{\frac{1}{2\alpha+1}}}{y_0^{\frac{1}{2\alpha+1}}} y^{\frac{1}{2\alpha+1}}. \tag{3.5.4}$$

Applying (3.5.4) to (3.5.3), we find

$$\frac{d}{dt}y \leq -K(2\alpha+1)\frac{\sin y_0^{\frac{1}{2\alpha+1}}}{y_0^{\frac{1}{2\alpha+1}}} y^{\frac{1}{2\alpha+1}}.$$

Multiplying by $\frac{2\alpha}{2\alpha+1}y^{-\frac{1}{2\alpha+1}}$ on both sides, we obtain

$$\frac{d}{dt}y^{\frac{2\alpha}{2\alpha+1}} \leq -2K\alpha\frac{\sin y_0^{\frac{1}{2\alpha+1}}}{y_0^{\frac{1}{2\alpha+1}}},$$

which yields

$$y^{\frac{2\alpha}{2\alpha+1}} \leq y_0^{\frac{2\alpha}{2\alpha+1}} - 2K\alpha\frac{\sin y_0^{\frac{1}{2\alpha+1}}}{y_0^{\frac{1}{2\alpha+1}}}t.$$

This is equivalent to

$$|\theta|^{2\alpha} \leq |\theta_0|^{2\alpha} - 2K\alpha\frac{\sin|\theta_0|}{|\theta_0|}t \quad \text{for} \quad 0 \leq t < T.$$

Again, the upper estimate shows that

$$T \leq \frac{|\theta_0|^{2\alpha+1}}{2K\alpha\sin|\theta_0|} \equiv t_{max}.$$

□

Assume that the oscillators are non-identical $\Omega = \Omega_2 - \Omega_1 > 0$ and the system (3.5.1) has a phase-locked state $(\bar{\theta}_1, \bar{\theta}_2)$ satisfying $0 < \bar{\theta}_2 - \bar{\theta}_1 < \pi$. Then, the equation (3.5.2) has an equilibrium $\bar{\theta} = \bar{\theta}_2 - \bar{\theta}_1 \in (0, \pi)$ such that

$$\Omega - K\frac{\sin\bar{\theta}}{|\bar{\theta}|_o^{2\alpha}} = 0. \tag{3.5.5}$$

To guarantee the existence of such equilibrium, we need the following conditions for the coupling strength K:

$$\text{if} \quad \alpha < \frac{1}{2}, \quad \text{choose} \quad K \geq \frac{\Omega}{\bar{h}},$$

$$\text{if} \quad \alpha = \frac{1}{2}, \quad \text{choose} \quad K > \Omega,$$

where $\overline{h} := \max_{0<r<\pi} h(r)$. Note that the equilibrium exists for the case of $\alpha > \frac{1}{2}$ without any condition on the coupling $K > 0$. We now investigate the stabilities of the equilibria in each cases.

Proposition 3.5.2. *Let θ be a solution of* (3.5.2)*. We have the following stability results.*

1. *For $\alpha \geq \frac{1}{2}$, the equilibrium $\bar{\theta}$ is unstable. Furthermore, if the initial datum θ_0 satisfies*

$$\theta_0 \neq 0 \quad \textit{and} \quad \theta_0 \neq \bar{\theta},$$

then the solution θ reaches 0 *or* 2π *in finite time.*

2. *For $\alpha < \frac{1}{2}$, there are a stable equilibrium $\bar{\theta} \in (0, \tilde{\theta})$ and an unstable equilibrium $\bar{\theta}^* \in (\tilde{\theta}, \pi)$, where $\tilde{\theta} \in (0, \frac{\pi}{2})$ is the solution to $\tilde{\theta} = 2\alpha \tan \tilde{\theta}$. Moreover, if the initial datum θ_0 is located in $(-2\pi + \bar{\theta}^*, \bar{\theta}^*)$, the solution θ converges to $\bar{\theta}$ asymptotically.*

Proof. We linearize the equation (3.5.2) near $\bar{\theta}$ as

$$\dot{\theta} = -kh'(\bar{\theta})(\theta - \bar{\theta}) + R(\bar{\theta}).$$

When $\alpha \geq \frac{1}{2}$, we have $h'(\bar{\theta}) < 0$, for $\theta \in (0, \pi)$. Thus, the equilibrium $\bar{\theta}$ is unstable. For $\alpha < \frac{1}{2}$, if the equilibrium $\bar{\theta}$ is located in $(0, \tilde{\theta})$, we have $h'(\bar{\theta}) > 0$, i.e., it is stable. By similar argument, due to $h'(\bar{\theta}^*) < 0$, the equilibrium $\bar{\theta}^*$ located in $(\tilde{\theta}, \pi)$ is unstable. We now investigate the convergence of the solution.

• *Step 1:* Critical case $\alpha \geq \frac{1}{2}$.

∘ *Case 1 ($\theta_0 > \bar{\theta}$)*: Since the function h is decreasing in $(0, 2\pi)$, we have $h(\theta) < h(\bar{\theta})$, for $\theta \in (\bar{\theta}, 2\pi)$. Thus, we find

$$\dot{\theta} = \Omega - Kh(\theta) > \Omega - Kh(\bar{\theta}) = 0, \quad \text{for} \quad \theta \in (\bar{\theta}, 2\pi).$$

Moreover, due to the monotonic increase of θ, we obtain the lower estimate for the frequency:

$$\dot{\theta} = \Omega - Kh(\theta) > \Omega - Kh(\theta_0) > 0, \quad \text{for} \quad \theta \in (\bar{\theta}, 2\pi).$$

Hence, there exists a finite time $t_1 < \frac{2\pi - \theta_0}{\Omega - Kh(\theta_0)}$, for which the solution converges to 2π.

∘ *Case 2 ($\theta_0 < \bar{\theta}$)*: We can apply an analogous argument for this case. Since the function h is decreasing, we deduce $h(\theta) > h(\bar{\theta})$ for $\theta \in (0, \bar{\theta})$. Thus, we have

$$\dot{\theta} = \Omega - Kh(\theta) < \Omega - Kh(\bar{\theta}) = 0, \quad \text{for} \quad \theta \in (0, \bar{\theta}).$$

This monotonic decrease of phase yields the upper estimate for the frequency:

$$\dot{\theta} = \Omega - Kh(\theta) < \Omega - Kh(\theta_0) < 0, \quad \text{for} \quad \theta \in (0, \bar{\theta}).$$

So, there exists a finite time $t_2 < \frac{\theta_0}{|\Omega - Kh(\theta_0)|}$, for which the solution converge to zero.

• *Step 2*: Subcritical case $\alpha < \frac{1}{2}$.

We consider two different steps for the asymptotic convergence to the equilibrium:

∘ *Step 2a:* We first show the solution moves into the interval $(0, \tilde{\theta})$ in finite time when the initial datum θ_0 is located in $(-2\pi + \bar{\theta}^*, 0] \cup [\tilde{\theta}, \bar{\theta}^*)$. As long as the solution θ located in $[\tilde{\theta}, \bar{\theta}^*)$, we have $h(\theta) > h(\bar{\theta})$. Thus, the solution is non-increasing:

$$\dot{\theta} = \Omega - Kh(\theta) < \Omega - Kh(\bar{\theta}) = 0, \quad \text{for} \quad \theta \in [\tilde{\theta}, \bar{\theta}^*).$$

Moreover, the non-increase of solution $\theta(t) \leq \theta_0$ gives an upper bound of frequency:

$$\dot{\theta} = \Omega - Kh(\theta) < \Omega - Kh(\theta_0) < 0,$$

while θ is in $[\tilde{\theta}, \bar{\theta}^*)$. So, there exists a finite time $t_3 := \frac{\theta_0 - \tilde{\theta}}{|\Omega - Kh(\theta_0)|}$ such that the solution verifies $\theta(t) < \tilde{\theta}$ for $t > t_3$. Analogously, if the initial datum θ_0 is given in $(-2\pi + \bar{\theta}^*, 0]$, then we have $h(\theta) < h(\bar{\theta})$, the solution is non-decreasing:

$$\dot{\theta} = \Omega - Kh(\theta) > \Omega - Kh(\bar{\theta}) = 0,$$

and the frequency has a lower bound

$$\dot{\theta} = \Omega - Kh(\theta) > \Omega - Kh(\theta_0) > 0,$$

as long as $\theta \in (-2\pi + \bar{\theta}^*, 0]$. Thus, there exists a finite time $t_4 := \frac{|\theta_0|}{|\Omega - Kh(\theta_0)|}$ such that the solution verifies $\theta(t) > 0$, for $t > t_4$.

∘ *Step 2b:* We will show that the solution converges to the stable equilibrium $\bar{\theta}$ asymptotically, when the initial datum is in $(0, \tilde{\theta})$. Suppose the initial data is located in $(0, \bar{\theta})$. Then, the following inequality

$$\frac{h(\bar{\theta})}{\bar{\theta}}\theta < h(\theta) < h'(\bar{\theta})(\theta - \bar{\theta}) + h(\bar{\theta}),$$

holds for the function h. Thus, the solution satisfies the differential inequality

$$\Omega - K\big(h'(\bar{\theta})(\theta - \bar{\theta}) + h(\bar{\theta})\big) < \dot{\theta} < \Omega - \frac{Kh(\bar{\theta})}{\bar{\theta}}\theta.$$

By Grönwall's lemma, we obtain

$$\bar{\theta} - (\bar{\theta} - \theta_0)e^{-kh'(\bar{\theta})t} < \theta(t) < \bar{\theta} - (\bar{\theta} - \theta_0)e^{-\frac{Kh(\bar{\theta})}{\bar{\theta}}t}.$$

Similarly, if the initial datum θ_0 is in $(\bar{\theta}, \tilde{\theta})$, the function h satisfies

$$\frac{h(\tilde{\theta}) - h(\bar{\theta})}{\tilde{\theta} - \bar{\theta}}(\theta - \bar{\theta}) + h(\bar{\theta}) < h(\theta) < h'(\bar{\theta})(\theta - \bar{\theta}) + h(\bar{\theta}).$$

Then, we have the following differential inequality:

$$\Omega - K\big(h'(\bar{\theta})(\theta - \bar{\theta}) + h(\bar{\theta})\big) < \dot{\theta} < \Omega - K\left(\frac{h(\tilde{\theta}) - h(\bar{\theta})}{\tilde{\theta} - \bar{\theta}}(\theta - \bar{\theta}) + h(\bar{\theta})\right).$$

Hence, by Grönwall's lemma, we find

$$\bar{\theta} - (\theta_0 - \bar{\theta})e^{-Kh'(\bar{\theta})t} < \theta(t) < \bar{\theta} - (\theta_0 - \bar{\theta})e^{-K\frac{h(\tilde{\theta}) - h(\bar{\theta})}{\tilde{\theta} - \bar{\theta}}t}.$$

□

Remark 3.5.3. *In the subcritical case $\alpha \in \left(0, \frac{1}{2}\right)$, the emergence of phase-locked state for two non-identical oscillators occurs asymptotically (see Proposition 3.5.2), whereas the phase synchronization for two identical oscillators appears in finite time (see Proposition 3.5.1). However, in the critical and supercritical cases $\alpha \in \left[\frac{1}{2}, 1\right)$, phase synchronization always appears in finite time as depicted in the above-mentioned Propositions 3.5.2 and 3.5.1 as long as the initial phase configuration does not agree with the unstable phase-locked state $\overline{\theta}$. Namely, in the supercritical case both oscillators stick together into a unique cluster moving at constant frequency $\widehat{\Omega} = \frac{\Omega_1+\Omega_2}{2}$, independently on the chosen natural frequencies. However, in the critical case, the same only happens under the assumption $|\Omega_1 - \Omega_2| \leq K$. In other case, the formed cluster will instantaneously split.*

3.5.2 N-oscillator case

In this subsection, we consider the system of N interacting oscillators. We will first focus on the dynamics in the simpler subcritical case $\alpha \in \left(0, \frac{1}{2}\right)$, where solutions have proved to be classical, see Theorem 3.3.5. The reason to start with this case is that the right hand side of (3.3.1)-(3.3.2) can be considered in the single-valued sense for that case. The dynamics in the critical case $\alpha = \frac{1}{2}$ and some intuition about the dynamics in the supercritical regime $\alpha \in (\frac{1}{2}, 1)$ will be provided at the end of this Subsection.

Let $\Theta = (\theta_1, \ldots, \theta_N)$ be the solution to the system (3.3.1)-(3.3.2). We first study the phase synchronization for identical oscillators. Fist, let us set the indices M and m to satisfy

$$\theta_M(t) := \max\{\theta_1(t), \ldots, \theta_N(t)\} \text{ and } \theta_m(t) := \min\{\theta_1(t), \ldots, \theta_N(t)\}, \tag{3.5.6}$$

for each time $t \geq 0$. Then, we can define the diameter of phase to be

$$D(\Theta) := \theta_M - \theta_m. \tag{3.5.7}$$

Theorem 3.5.4. *Let $\Theta = (\theta_1, \cdots, \theta_N)$ be the solution to (3.3.1)-(3.3.2) with $\alpha \in \left(0, \frac{1}{2}\right)$ for identical oscillators ($\Omega_i = 0$), for $i = 1, \ldots, N$. Assume that the initial configuration Θ_0 is confined in a half circle, i.e., $0 < D(\Theta_0) < \pi$. Then, there is complete phase synchronization at a finite time not larger than T_c where*

$$T_c = \frac{D(\Theta_0)^{1-2\alpha}}{2\alpha K h(D(\Theta_0))}.$$

Proof. We consider the dynamics of phase diameter:

$$\frac{d}{dt}D(\Theta) = \frac{K}{N}\sum_{j=1}^{N}\Big(h(\theta_j - \theta_M) - h(\theta_j - \theta_m)\Big).$$

Since $h(\theta_j - \theta_M) < 0$ and $h(\theta_j - \theta_m) > 0$ as long as $D(\Theta) < \pi$, we have

$$\frac{d}{dt}D(\Theta) \leq 0 \quad \text{and} \quad D(\Theta(t)) \leq D(\Theta_0) < \pi, \quad \text{for} \quad t > 0.$$

Due to the contraction of phase, and the fact that $\theta \in (0, \pi) \mapsto \frac{h(\theta)}{\theta}$ is decreasing, we have

$$h(\theta_j - \theta_M) \leq \frac{h\big(D(\Theta_0)\big)}{D(\Theta_0)}(\theta_j - \theta_M) \quad \text{and} \quad h(\theta_j - \theta_m) \geq \frac{h\big(D(\Theta_0)\big)}{D(\Theta_0)}(\theta_j - \theta_m).$$

Thus, we attain the following differential inequality:

$$\frac{d}{dt}D(\Theta) \leq \frac{K}{N}\sum_{j=1}^{N}\Big(\frac{h\big(D(\Theta_0)\big)}{D(\Theta_0)}(\theta_j - \theta_M) - \frac{h\big(D(\Theta_0)\big)}{D(\Theta_0)}(\theta_j - \theta_m)\Big)$$

$$= \frac{K}{N}\frac{h\big(D(\Theta_0)\big)}{D(\Theta_0)}\sum_{j=1}^{N}\Big((\theta_j-\theta_M)-(\theta_j-\theta_m)\Big)$$
$$= -K\frac{h\big(D(\Theta_0)\big)}{D(\Theta_0)}D(\Theta).$$

By Grönwall's lemma, we obtain

$$D(\Theta) \le D(\Theta_0)e^{-K\frac{h(D(\Theta_0))}{D(\Theta_0)}t} \quad \text{for} \quad t \ge 0.$$

Notice that $h(\theta)$ behaves like $\theta^{1-2\alpha}$ near the origin. Indeed, it is easy to prove that for every $\theta_* \in (0,\pi)$

$$h(\theta) \ge \frac{h(\theta_*)}{\theta_*^{1-2\alpha}}\,\theta^{1-2\alpha},\ \forall\,\theta \in [0,\theta_*].$$

The main idea is to show that the mapping

$$\theta \longmapsto \frac{h(\theta)}{\theta^{1-2\alpha}},$$

is nonincreasing in $[0,\pi]$. Since the phase diameter $D(\Theta)$ is bounded above by $D(\Theta_0)$ we can take $\theta_* = D(\Theta_0)$ and apply the above lower estimate for h to attain the following estimate of the phase diameter

$$\begin{aligned}\frac{d}{dt}D(\Theta) &= \frac{K}{N}\sum_{j=1}^{N}\Big(h(\theta_j-\theta_M)-h(\theta_j-\theta_m)\Big)\\
&\le \frac{K}{N}\frac{h(D(\Theta_0))}{D(\Theta_0)}\sum_{j=1}^{N}\Big(-(\theta_M-\theta_j)^{1-2\alpha}-(\theta_j-\theta_m)^{1-2\alpha}\Big)\\
&\le -\frac{K}{N}\frac{h(D(\Theta_0))}{D(\Theta_0)}\sum_{j=1}^{N}((\theta_M-\theta_j)+(\theta_j-\theta_m))^{1-2\alpha}\\
&= -\frac{Kh(D(\Theta_0))}{D(\Theta_0)}D(\Theta)^{1-2\alpha},\end{aligned}$$

for every $t \ge 0$. In the last inequality we have used that $1-2\alpha \in (0,1)$ and, consequently,

$$(a+b)^{1-2\alpha} \le a^{1-2\alpha}+b^{1-2\alpha},$$

for every couple of nonnegative numbers $a,b \in \mathbb{R}$. Then, integrating the above differential inequality implies

$$D(\Theta(t)) \le \left(D(\Theta_0)^{2\alpha}-2\alpha K\frac{h(D(\Theta_0))}{D(\Theta_0)}t\right)^{\frac{1}{2\alpha}},$$

for all $t \ge 0$. This implies the convergence to zero at a finite time not larger than T_c. □

We now consider the system for non-identical oscillators. The next proposition yields the structure of phase-locked state of (3.3.1)-(3.3.2) for non-identical oscillators with mutually distinct natural frequencies in the subcritical regime.

Proposition 3.5.5. *Let* $\alpha \in \left(0,\frac{1}{2}\right)$ *and* $\bar\Theta = (\bar\theta_1,\cdots,\bar\theta_N)$ *be an equilibrium of the system* (3.3.1)-(3.3.2) *such that* $\max_{i,j}|\bar\theta_i-\bar\theta_j| < \tilde\theta$ *where* $\tilde\theta \in (0,\frac{\pi}{2})$ *is the solution to* $\tilde\theta = 2\alpha\tan\tilde\theta$. *Assume the natural frequencies satisfy the ordering* $\Omega_1 < \ldots < \Omega_N$. *Then, the phase-locked state* $\bar\Theta$ *verifies the ordering* $\bar\theta_1 < \ldots < \bar\theta_N$.

Proof. First, we show that the equilibria $\bar{\theta}_i$'s are mutually distinct, i.e.,

$$\bar{\theta}_i \neq \bar{\theta}_j \quad \text{for} \quad i \neq j.$$

Since $\bar{\Theta}$ is an equilibrium, it satisfies

$$\Omega_i + \frac{K}{N}\sum_{k\neq i} h(\bar{\theta}_k - \bar{\theta}_i) = 0, \tag{3.5.8}$$

for every $i = 1, \ldots, N$. If there existed two oscillators having the same equilibria $\bar{\theta}_i = \bar{\theta}_j$, then we would have

$$\frac{K}{N}\sum_{k\neq i} h(\bar{\theta}_k - \bar{\theta}_i) = \frac{K}{N}\sum_{k\neq j} h(\bar{\theta}_k - \bar{\theta}_j),$$

which contradicts with $\Omega_i \neq \Omega_j$. We now show the ordering property. From (3.5.8), we have

$$\begin{aligned}
\Omega_{i+1} - \Omega_i &= -\frac{K}{N}\sum_{j\neq i+1} h(\bar{\theta}_j - \bar{\theta}_{i+1}) + \frac{K}{N}\sum_{j\neq i} h(\bar{\theta}_j - \bar{\theta}_i) \\
&= \frac{K}{N}\Big(\sum_{j\neq i,i+1} h(\bar{\theta}_{i+1} - \bar{\theta}_j) - h(\bar{\theta}_i - \bar{\theta}_j)\Big) - \frac{K}{N}\big(h(\bar{\theta}_i - \bar{\theta}_{i+1}) - h(\bar{\theta}_{i+1} - \bar{\theta}_i)\big) \\
&= \frac{K}{N}\Big(\sum_{j\neq i,i+1} h(\bar{\theta}_{i+1} - \bar{\theta}_j) - h(\bar{\theta}_i - \bar{\theta}_j)\Big) + \frac{2K}{N} h(\bar{\theta}_{i+1} - \bar{\theta}_i) \\
&= \frac{K}{N}\sum_{j\neq i,i+1} c_{i,j}(\bar{\theta}_{i+1} - \bar{\theta}_i) + \frac{2K}{N} h(\bar{\theta}_{i+1} - \bar{\theta}_i),
\end{aligned}$$

where the coefficients $c_{i,j}$ read

$$c_{i,j} := \frac{h(\bar{\theta}_{i+1} - \bar{\theta}_j) - h(\bar{\theta}_i - \bar{\theta}_j)}{\bar{\theta}_{i+1} - \bar{\theta}_i}.$$

They are properly defined because all the equilibria are mutually distinct and they are positive because h is strictly increasing in $(-\tilde{\theta}, \tilde{\theta})$. Thus, the order $\Omega_{i+1} > \Omega_i$ yields the order of equilibria $\bar{\theta}_{i+1} > \bar{\theta}_i$. □

In the subcritical case, we can attain the uniform boundedness of phase differences under sufficiently large coupling strength.

Lemma 3.5.6. *Let Θ be the solution to* (3.3.1)-(3.3.2) *for $\alpha \in \left(0, \frac{1}{2}\right)$ and non-identical oscillators with initial data Θ_0, satisfying $D(\Theta_0) < D^\infty < \tilde{\theta}$. If the coupling strength K is sufficiently large such that*

$$K > \frac{D(\dot{\Theta}_0)}{h'(D^\infty)(D^\infty - D(\Theta_0))},$$

then, the phase diameter $D(\Theta)$ is uniformly bounded by D^∞:

$$D(\Theta(t)) < D^\infty, \quad \textit{for} \quad t \geq 0.$$

Proof. Suppose there exists a finite time $t^* > 0$ such that

$$t^* := \sup\{t : D(\Theta(s)) < D^\infty \quad \text{for} \quad 0 \leq s \leq t\} \quad \text{and} \quad D(\Theta(t^*)) = D^\infty.$$

We set indices F and S so that

$$\dot\theta_F := \max\{\dot\theta_1, \dots, \dot\theta_N\} \quad \text{and} \quad \dot\theta_S := \min\{\dot\theta_1, \dots, \dot\theta_N\},$$

for each time t. We define the frequency difference so that

$$D(\dot\Theta(t)) := \dot\theta_F - \dot\theta_S.$$

We note that

$$D(\dot\Theta(t)) - D(\dot\Theta_0) = \int_0^t \frac{d}{dt} D(\dot\Theta(s))\, ds. \tag{3.5.9}$$

By taking time derivative on $D(\dot\Theta)$, we obtain

$$\frac{d}{dt} D(\dot\Theta) = \frac{K}{N} \sum_{j=1}^N \Big(h'(\theta_j - \theta_F)(\dot\theta_j - \dot\theta_F) - h'(\theta_j - \theta_S)(\dot\theta_j - \dot\theta_m) \Big).$$

As long as $D(\Theta) < D^\infty$, we have

$$h'(\theta_j - \theta_i) \geq h'(D^\infty) > 0.$$

Thus, we get

$$\frac{d}{dt} D(\dot\Theta) \leq \frac{K}{N} \sum_{j=1}^N h'(D^\infty) \Big((\dot\theta_j - \dot\theta_F) - (\dot\theta_j - \dot\theta_S) \Big) = -K h'(D^\infty) D(\dot\Theta). \tag{3.5.10}$$

We combine (3.5.9) and (3.5.10) to obtain

$$D(\dot\Theta(t)) \leq D(\dot\Theta_0) - K h'(D^\infty) \int_0^t D(\dot\Theta(s))\, ds. \tag{3.5.11}$$

Setting $y(s) := \int_0^t D(\dot\Theta(s))\, ds$, we can rewrite (3.5.11) into

$$y'(t) \leq y'(0) - K h'(D^\infty) y(t).$$

Hence, we have

$$y(t) \leq \frac{y'(0)}{K h'(D^\infty)} (1 - e^{-K h'(D^\infty) t}) \leq \frac{y'(0)}{K h'(D^\infty)}.$$

Since $D(\Theta(t^*)) = D^\infty$ and $K > \frac{D(\dot\Theta_0)}{h'(D^\infty)(D^\infty - D(\Theta_0))}$, we get

$$\begin{aligned} D^\infty &= D(\Theta_0) + \int_0^{t^*} \frac{d}{ds} D(\Theta(s))\, ds \\ &\leq D(\Theta_0) + \int_0^{t^*} D(\dot\Theta(s))\, ds \\ &\leq D(\Theta_0) + \frac{D(\dot\Theta_0)}{K h'(D^\infty)} < D^\infty, \end{aligned}$$

which is a contradiction. Thus, we have the desired uniform bound for phase difference

$$D(\Theta(t)) < D^\infty, \quad \text{for} \quad t \geq 0.$$

□

Remark 3.5.7. *Note that, in the preceding proof, the solution $\Theta = \Theta(t)$ is C^1 but not necessarily C^2 because of the essential discontinuity of h'. Then, one cannot directly argue with two time derivatives in the computation of $\frac{d}{dt}D(\dot{\Theta})$. However, the preceding arguments can be made rigorous because the C^1 solution of* (3.3.1)-(3.3.2) *is a piece-wise $W^{2,1}$ solution of the augmented model* (3.2.4)-(3.4.23) *as discussed in Remark 3.4.5 in the preceding Section 4. Other possible approach is to directly show the Gröwall inequality* (3.5.11) *in integral form.*

In the following result, we show the collision avoidance when the oscillators are initially well-ordered.

Lemma 3.5.8. *Let Θ be the solution to* (3.3.1)-(3.3.2)*, with $\alpha \in \left(0, \frac{1}{2}\right)$, and initial data Θ_0 satisfying $D(\Theta_0) < D^\infty < \tilde{\theta}$. Assume the natural frequencies and the initial configuration satisfy the ordering $\Omega_1 < \cdots < \Omega_N$ and $\theta_{1,0} < \cdots < \theta_{N,0}$, respectively. We assume the coupling strength K is sufficiently large such that*

$$K > \frac{D(\dot{\Theta}_0)}{h'(D^\infty)(D^\infty - D(\Theta_0))}.$$

Then, there is no collision between oscillators, i.e.,

$$\theta_i(t) \neq \theta_j(t) \quad \text{for} \quad i \neq j, \quad t > 0.$$

Proof. From Lemma 3.5.6, we have a uniform bound of the phase diameter $D(\Theta(t)) < D^\infty$, for $t \geq 0$. Let ℓ be an index such that

$$\theta_{\ell+1}(t) - \theta_\ell(t) = \min_{j=1,\dots,N-1} \theta_{j+1}(t) - \theta_j(t),$$

for each time $t \geq 0$. For notationally simplicity, we set $\Delta := \theta_{\ell+1} - \theta_\ell$. Then, we have

$$\begin{aligned} \dot{\Delta} &= \Omega_{\ell+1} - \Omega_\ell + \frac{K}{N}\sum_{j=1}^{N}\Big(h(\theta_j - \theta_{\ell+1}) - h(\theta_j - \theta_\ell)\Big) \\ &\geq \Omega_\delta + \frac{K}{N}\sum_{j=1}^{N}\Big(h(\theta_j - \theta_{\ell+1}) - h(\theta_j - \theta_\ell)\Big), \end{aligned} \tag{3.5.12}$$

where $\Omega_\delta := \min_{j=1,\dots,N-1} \Omega_{j+1} - \Omega_j > 0$. We define the sets of indices such that

$$\mathcal{S}_1(\ell) := \{j : j < \ell\} \quad \text{and} \quad \mathcal{S}_2(\ell) := \{j : j > \ell + 1\}.$$

Note that $h(\theta)$ is convex increasing for $\theta \in (-\tilde{\theta}, 0)$ and is concave increasing for $\theta \in (0, \tilde{\theta})$. Thus, we have

$$\begin{aligned} 0 < h'(b) \leq \frac{h(b) - h(a)}{b - a} \leq h'(a) \quad &\text{for} \quad 0 \leq a < b \leq \tilde{\theta}, \\ 0 < h'(c) \leq \frac{h(d) - h(c)}{d - c} \leq h'(d) \quad &\text{for} \quad -\tilde{\theta} \leq c < d \leq 0. \end{aligned} \tag{3.5.13}$$

From (3.5.12) and (3.5.13), we obtain

$$\begin{aligned} \dot{\Delta} \geq \Omega_\delta &+ \frac{K}{N}\sum_{j\in\mathcal{S}_1(\ell)}\Big(h(\theta_j - \theta_{\ell+1}) - h(\theta_j - \theta_\ell)\Big) + \frac{K}{N}h(\theta_\ell - \theta_{\ell+1}) \\ &- \frac{K}{N}h(\theta_{\ell+1} - \theta_\ell) + \frac{K}{N}\sum_{j\in\mathcal{S}_2(\ell)}\Big(h(\theta_j - \theta_{\ell+1}) - h(\theta_j - \theta_\ell)\Big) \end{aligned}$$

$$\begin{aligned}
&\geq \Omega_\delta - \frac{K}{N}\sum_{j\in\mathcal{S}_1(\ell)} h'(\theta_j-\theta_\ell)\,\Delta - \frac{K}{N}\sum_{j\in\mathcal{S}_2(\ell)} h'(\theta_j-\theta_{\ell+1})\,\Delta - \frac{2K}{N}h(\Delta)\\
&\geq \Omega_\delta - \frac{K|\mathcal{S}_1(\ell)|}{N}h'(\Delta)\,\Delta - \frac{K|\mathcal{S}_2(\ell)|}{N}h'(\Delta)\,\Delta - \frac{2K}{N}h(\Delta)\\
&\geq \Omega_\delta - Kh'(\Delta)\,\Delta - \frac{2K}{N}h(\Delta)\\
&\geq \Omega_\delta - C\Delta^\gamma =: q(\Delta),
\end{aligned}$$

where we have used

$$h(\theta) \leq C_1\theta^\gamma \quad \text{and} \quad h'(\theta)\theta \leq C_2\theta^\gamma,$$

for $\theta \geq 0$ and $0 < \gamma < 1-2\alpha$ in the last inequality. Since $\lim_{\theta\to 0+} q(\theta) = \Omega_\delta > 0$ and $q(\theta)$ is continuous for $\theta > 0$, there exists a positive $\varepsilon > 0$ such that $q(\theta) > 0$, for $\theta \in (0,\varepsilon)$. Hence, the distance Δ has a positive lower bound. □

In the sequel, we study the stability of the phase-locked state for the system of non-identical oscillators. We use the center manifold theorem to investigate the stability of linearized system.

Lemma 3.5.9 (Center Manifold Theorem [56]). *Consider the system*

$$\begin{aligned}
\dot{x} &= Ax + f_A(x,y)\\
\dot{y} &= By + f_B(x,y)
\end{aligned} \tag{3.5.14}$$

where $x\in\mathbb{R}^n, y\in\mathbb{R}^m$ and A and B are constant matrices such that all the eigenvalues of A have zero real parts while all the eigenvalues of B have negative real parts. Assume that the functions f_A and f_B are C^2 with $f_A(0,0)=0, \nabla f_A(0,0)=0, f_B(0,0)=0, \nabla f_B(0,0)=0$. Then, we have the following results:

1. *There exists a center manifold for* (3.5.14), *$y=\phi(x), |x|<\delta$, where $\phi=\phi(x)$ is C^2. The flow on the center manifold is governed by the n-dimensional system:*

$$\dot{u} = Au + f_A(u,\phi(u)) \tag{3.5.15}$$

2. *Assume the zero solution of* (3.5.15) *is stable (respectively asymptotically stable/unstable). Then, the zero solution of* (3.5.14) *is stable (respectively asymptotically stable/unstable).*

Theorem 3.5.10. *Let $\bar\Theta := (\bar\theta_1,\cdots,\bar\theta_N)\notin\mathcal{C}$ be a collision-less equilibrium of* (3.3.1)-(3.3.2).

1. *If $\alpha\geq\frac{1}{2}$, then the phase-locked state $\bar\Theta$ is unstable.*
2. *If $\alpha<\frac{1}{2}$, then the phase-locked state $\bar\Theta$ is stable.*

Proof. • *Step 1*: Critical and supercritical regimes $\alpha\in[\frac{1}{2},1)$.

We first linearize the system (3.3.1)-(3.3.2):

$$\dot\Theta = A(\Theta-\bar\Theta) + R(\bar\Theta), \tag{3.5.16}$$

where the elements of matrix $A=(a_{ij})_{1\leq i,j,\leq N}$ are determined by

$$\begin{aligned}
a_{ij} &= \frac{\cos(\bar\theta_j-\bar\theta_i)}{|\bar\theta_j-\bar\theta_i|_o^{2\alpha}} - 2\alpha\frac{\sin|\bar\theta_j-\bar\theta_i|_o}{|\bar\theta_j-\bar\theta_i|_o^{2\alpha+1}} \quad \text{for} \quad i\neq j,\\
a_{ii} &= -\sum_{j\neq i} a_{ij}.
\end{aligned} \tag{3.5.17}$$

If $\alpha \geq \frac{1}{2}$, we find $a_{ij} < 0$, for $i \neq j$, and hence $a_{ii} > 0$, for $i = 1, \ldots, N$. This leads to a matrix A of Laplacian type and, consequently, all its eigenvalues are non-negative. Indeed, we can elaborate on this argument and notice that since the matrix A represents all-to-all connected network, then there exists a zero eigenvalue for which the multiplicity is one (amount of connected components of the network) and all the other eigenvalues are positive, which implies the unstability of the equilibrium.

• *Step 2*: Subcritical regime $\alpha \in (0, \frac{1}{2})$.

Since the equilibrium satisfies $\max_{i,j} |\bar{\theta}_i - \bar{\theta}_j| < \tilde{\theta}$ and $\bar{\theta}_i \neq \bar{\theta}_j$ for $i \neq j$, the elements of the matrix have signs so that $a_{ij} > 0$ for $i \neq j$ and $a_{ii} < 0$, for $i = 1, \ldots, N$. By similar argument as above, we can obtain that the eigenvalues of A are non-positive and there is a zero eigenvalue with multiplicity 1. Let $\lambda_1 = 0$ and $\lambda_2, \ldots, \lambda_N < 0$ be the eigenvalues for matrix A and let $v_1, \ldots, v_N$ be the corresponding left eigenvectors such that

$$v_i A = \lambda_i v_i \quad \text{for} \quad i = 1, \ldots, N.$$

We note that $v_1 = (1, \cdots, 1)$. We set the matrices P and D so that

$$P^{-1} := \begin{pmatrix} 1 & \cdots & 1 \\ & v_2 & \\ & \vdots & \\ & v_N & \end{pmatrix} \quad \text{and} \quad D := \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_N \end{pmatrix}.$$

Then, we can diagonalize the matrix A:

$$P^{-1} A P = D. \tag{3.5.18}$$

We change the variables from $\Theta = (\theta_1, \ldots, \theta_N)$ to $X = (x_1, \ldots, x_N)$ such that

$$X := P^{-1} \Theta. \tag{3.5.19}$$

Then, the system (3.5.16) can be transformed into the following form:

$$\dot{X} = D(X - \bar{X}) + \tilde{R}(X) \tag{3.5.20}$$

Let $\hat{x}_1 := (x_2, \ldots, x_N)$ and $\hat{D}$ be a minor matrix of D such that

$$\hat{D} := \begin{pmatrix} \lambda_2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_N \end{pmatrix}.$$

Then, we can rewrite the system (3.5.20) in the following form:

$$\begin{pmatrix} x_1 \\ \hat{x}_1 \end{pmatrix}' = \begin{pmatrix} 0 & 0 \\ 0 & \hat{D} \end{pmatrix} \begin{pmatrix} x_1 - \bar{x}_1 \\ \hat{x}_1 - \hat{\bar{x}}_1 \end{pmatrix} + \begin{pmatrix} \tilde{R}_1(x_1, \hat{x}_1) \\ \hat{\tilde{R}}_1(x_1, \hat{x}_1) \end{pmatrix}. \tag{3.5.21}$$

Consider the center manifold in Lemma 3.5.9, that can be written as follows

$$W_c := \{(x, y) \in \mathbb{R} \times \mathbb{R}^{N-1} : y = c(x) \quad \text{for} \quad |x| < \varepsilon, \quad \phi(\bar{x}_1) = 0, \quad D\phi(\bar{x}_1) = 0\},$$

and consider the equation

$$\dot{x}_1 = \tilde{R}_1(x_1, \phi(x_1)). \tag{3.5.22}$$

By the Center Manifold Theorem, the stability of (3.5.22) implies the stability of the system (3.5.21). Since the equality (3.5.19) yields $x_1 = \theta_1 + \cdots + \theta_N$ and we have

$$\dot{x}_1 = \sum_{i=1}^{N} \dot{\theta}_i = \sum_{i=1}^{N} \Omega_i = 0.$$

Thus, the right hand side $\tilde{R}_1 \equiv 0$ and the dynamics of (3.5.22) is stable. Therefore, the phase-locked state $\bar{\Theta}$ is stable for $\alpha < \frac{1}{2}$. □

Finally, we are ready to show the emergence of phase locked state for non-identical oscillators.

Theorem 3.5.11. *Let Θ be a solution to* (3.3.1)-(3.3.2) *with initial data Θ_0 satisfying $D(\Theta_0) < D^\infty < \tilde{\theta}$ for $\alpha \in \big(0, \frac{1}{2}\big)$. If the coupling strength is sufficiently large such that*

$$K > \frac{D(\dot{\Theta}_0)}{h'(D^\infty)(D^\infty - D(\Theta_0))},$$

then we can show the emergence of phase-locked state. Moreover, if each oscillator has distinct natural frequency, i.e., $\Omega_i \neq \Omega_j$ for $i \neq j$, then, the synchronization occurs asymptotically.

Proof. By applying Gronwall's lemma on (3.5.10), we have an exponential decay of upper estimate on the frequency diameter:

$$D(\dot{\Theta}(t)) \leq D(\dot{\Theta}_0) e^{-Kh'(D^\infty)t}.$$

This exponential decay implies the emergence of phase-locked state.

Now, assume that the oscillators have mutually distinct natural frequencies. Since Proposition 3.5.5 gives the structure of phase-locked state, the oscillators draw in descending order of natural frequencies in finite time. After this time, by Lemma 3.5.8, we have a positive lower bound $\varepsilon_\Delta > 0$ of distance between oscillators. Then, we have

$$\begin{aligned}\frac{d}{dt} D(\dot{\Theta}) &= \frac{K}{N} \sum_{j=1}^{N} \Big(h'(\theta_j - \theta_F)(\dot{\theta}_j - \dot{\theta}_F) - h'(\theta_j - \theta_S)(\dot{\theta}_j - \dot{\theta}_S) \Big) \\ &\geq \frac{K}{N} \sum_{j=1}^{N} \Big(h'(\varepsilon_\Delta)(\dot{\theta}_j - \dot{\theta}_F) - h'(\varepsilon_\Delta)(\dot{\theta}_j - \dot{\theta}_S) \Big) \\ &= -Kh'(\varepsilon_\Delta) D(\dot{\Theta}).\end{aligned}$$

By Grönwall's lemma, we have an lower estimate on the frequency diameter:

$$D(\dot{\Theta}(t)) \geq D(\dot{\Theta}_0) e^{-Kh'(\varepsilon_\Delta)t}.$$

□

Let us now explore the behavior of Filippov solutions to (3.3.1)-(3.3.2) (see Theorems 3.3.12 and 3.3.19) in the most singular cases $\alpha = \frac{1}{2}$ and $\alpha \in (\frac{1}{2}, 1)$. Looking at Remark 3.5.3 for the dynamics of 2 oscillators, we expect global synchronization in finite time for N oscillators. Specifically, in the supercritical case, the emerged global cluster is hoped to stay stuck independently on the chosen natural frequencies. In the critical case, the sticking conditions (3.3.13) are required for the cluster to remain stuck. To start with, let us prove the finite-time global phase synchronization of identical oscillators in the critical and supercritical cases. To that end, we need the following technical results.

Lemma 3.5.12. *Consider* $\alpha \in [\frac{1}{2}, 1)$*,* $\beta \in (0, 2\alpha]$ *and* $\theta_* \in (0, \pi)$ *and define the number*

$$c(\alpha, \beta) = \left(\frac{2\alpha - \beta}{\beta}\right)^{1/2}.$$

Then, the following lower bound for h_ε *holds true*

$$h_\varepsilon(\theta) \geq \frac{h_\varepsilon(\theta_*)}{\theta_*^\beta}\theta^\beta, \ \forall\, \theta \in [c(\alpha, \beta)\varepsilon, \theta_*],$$

for every $0 < \varepsilon < c(\alpha, \beta)^{-1}\theta_*$*.*

Proof. Define a scalar function

$$g_\varepsilon(\theta) := \frac{h_\varepsilon(\theta)}{\theta^\beta} = \frac{\frac{\sin\theta}{(\varepsilon^2+\theta^2)^\alpha}}{\theta^\beta}, \ \theta \in (0, \pi).$$

We claim that g_ε is nonincreasing in the interval $(c(\alpha, \beta)\varepsilon, \pi)$ for every $\varepsilon \in (0, c(\alpha, \beta)^{-1}\theta_*)$. Then, the result is apparent once monotonicity of g_ε is proved. Indeed, taking derivatives we have

$$\begin{aligned} g'_\varepsilon(\theta) &= \frac{1}{\theta^{\beta+1}(\varepsilon^2 + \theta^2)^\alpha}\left[\theta\cos\theta - \left(2\alpha\frac{\theta^2}{\varepsilon^2 + \theta^2} + \beta\right)\sin\theta\right] \\ &= \frac{1}{\theta^{\beta+1}(\varepsilon^2 + \theta^2)^\alpha}\left[\theta\cos\theta - \left(2\alpha + \frac{\beta\theta^2 - (2\alpha - \beta)\varepsilon^2}{\theta^2 + \varepsilon^2}\right)\sin\theta\right], \end{aligned}$$

for every $\theta \in (0, \frac{\pi}{2})$. Notice that $2\alpha \geq 1$ and $\beta \leq 2\alpha$. Then, by virtue of the definition of $c(\alpha, \beta)$ one checks that

$$\theta\cos\theta - \left(2\alpha + \frac{\beta\theta^2 - (2\alpha - \beta)\varepsilon^2}{\theta^2 + \varepsilon^2}\right)\sin\theta \leq \theta\cos\theta - \sin\theta \leq 0,$$

for every $\theta \in (c(\alpha, \beta)\varepsilon, \pi)$ and the monotonicity of g_ε becomes clear. □

Lemma 3.5.13. *Let* $\Theta = (\theta_1, \cdots, \theta_N)$ *be the solution to* (3.3.1)-(3.3.2) *with* $\alpha \in [\frac{1}{2}, 1)$ *for identical oscillators,* $\Omega_i = 0$*, for* $i = 1, \ldots, N$ *obtained in Theorems 3.4.11 and 3.4.15 as singular limits. Suppose the initial configuration* Θ_0 *is confined in a half circle, i.e.,* $0 < D(\Theta_0) < \pi$*. Then,*

$$\begin{aligned} &D(\Theta(t)) \leq D(\Theta_0)e^{-K\frac{h(D(\Theta_0))}{D(\Theta_0)}t}, && \text{if } \alpha = \tfrac{1}{2}, \\ &D(\Theta(t)) \leq \left(D(\Theta_0)^{1-2\alpha} + (2\alpha - 1)2^{2\alpha-1}K\frac{h(D(\Theta_0))}{D(\Theta_0)^{2\alpha}}t\right)^{-\frac{1}{2\alpha-1}}, && \text{if } \alpha \in (\tfrac{1}{2}, 1), \end{aligned}$$

for every $t \geq 0$*.*

Proof. The main idea is to handle the approximate sequence $\{\Theta^\varepsilon\}_{\varepsilon>0}$ obtained as solutions to the regularized system (3.4.8) and to take limits $\varepsilon \to 0$ in the phase diameter estimates. First, notice that by virtue of the assumed initial condition on the diameter one has that

$$\frac{d}{dt}D(\Theta^\varepsilon) \leq 0 \ \text{ and } \ D(\Theta^\varepsilon(t)) \leq D(\Theta_0) < \pi, \ \text{ for } \ t > 0.$$

Indeed, note that we can obtain an explicit decay rate for the diameter by mimicking the ideas in Theorem 3.5.4. Namely, choosing $\theta_* = D(\Theta_0)$ and $\beta = 2\alpha$ in Lemma 3.5.12, we notice that

$c(\alpha,\beta) = 0$. Consequently, the lower bound of the kernel h_ε is valid in the whole interval $[0, D(\Theta_0)]$. Then,

$$\begin{aligned}
\frac{d}{dt}D(\Theta^\varepsilon) &= \frac{K}{N}\sum_{j=1}^{N}(h_\varepsilon(\theta_j^\varepsilon-\theta_M^\varepsilon)-h_\varepsilon(\theta_j^\varepsilon-\theta_m^\varepsilon))\\
&\quad -\frac{K}{N}\sum_{j=1}^{N}(h_\varepsilon(\theta_M^\varepsilon-\theta_j^\varepsilon)+h_\varepsilon(\theta_j^\varepsilon-\theta_m^\varepsilon))\\
&\leq -\frac{K}{N}\frac{h_\varepsilon(D(\Theta_0))}{D(\Theta_0)^{2\alpha}}\sum_{j=1}^{N}((\theta_M^\varepsilon-\theta_j^\varepsilon)^{2\alpha}+(\theta_j^\varepsilon-\theta_m^\varepsilon)^{2\alpha})\\
&\leq -\frac{K}{N}\frac{h_\varepsilon(D(\Theta_0))}{D(\Theta_0)^{2\alpha}}2^{2\alpha-1}\sum_{j=1}^{N}((\theta_M^\varepsilon-\theta_j^\varepsilon)+(\theta_j^\varepsilon-\theta_m^\varepsilon))^{2\alpha}\\
&= -K\frac{h_\varepsilon(D(\Theta_0))}{D(\Theta_0)^{2\alpha}}2^{2\alpha-1}D(\Theta)^{2\alpha}.
\end{aligned}$$

Let us integrate the above differential inequality. We need to distinguish the cases $\alpha = \frac{1}{2}$ and $\alpha \in (\frac{1}{2}, 1)$:

$$\begin{aligned}
&D(\Theta^\varepsilon(t)) \leq D(\Theta_0)e^{-K\frac{h_\varepsilon(D(\Theta_0))}{D(\Theta_0)}}t, && \text{if } \alpha=\tfrac{1}{2},\\
&D(\Theta^\varepsilon(t)) \leq \left(D(\Theta_0)^{1-2\alpha}+(2\alpha-1)2^{2\alpha-1}K\frac{h_\varepsilon(D(\Theta_0))}{D(\Theta_0)^{2\alpha}}t\right)^{-\frac{1}{2\alpha-1}}, && \text{if } \alpha\in(\tfrac{1}{2},1),
\end{aligned}$$

for every $t \geq 0$. Recall that by virtue of Lemmas 3.4.6 and 3.4.12, we obtained $\Theta^\varepsilon \overset{*}{\rightharpoonup} \Theta$ in $H^1((0,T),\mathbb{R}^N)$. In particular, $\Theta^\varepsilon \to \Theta$ in $C([0,T],\mathbb{R}^N)$. Then, we can take the limit $\varepsilon \to 0$ in the above estimates to attain the desired result. □

Under the assumptions in the preceding Lemma 3.5.13 one obtains exponential decay of the diameter in the critical case and algebraic decay in the supercritical regime. However, a finite-time global synchronization is expected. This is the content of the following result.

Theorem 3.5.14. *Let $\Theta = (\theta_1, \cdots, \theta_N)$ be the solution to* (3.3.1)-(3.3.2) *with $\alpha \in [\frac{1}{2}, 1)$ for identical oscillators, $\Omega_i = 0$, for $i = 1, \ldots, N$ obtained in Theorems 3.4.11 and 3.4.15 as singular limits of the regularized solutions Θ^ε to* (3.4.8)*. Assume that the initial configuration Θ_0 is confined in a half circle, i.e., $0 < D(\Theta_0) < \pi$. Then, for every $\beta \in (0, 1)$ there exist two oscillators that collide at some time not larger than T_c^1, where*

$$T_c^1 = \frac{D(\Theta_0)}{(1-\beta)Kh(D(\Theta_0))}.$$

Proof. Let us assume the contrary. Then, by continuity there exists some $T > T_c^1$ so that there is no collision between oscillators along the time interval $[0, T]$. Again, by continuity there exists $\delta_T \in (0, D(\Theta_0))$ so that

$$|\theta_i(t) - \theta_j(t)| \geq \frac{\delta_T}{2},$$

for all $t \in [0, T]$ and every $i \neq j$. Since $\Theta^\varepsilon \to \Theta$ in $C([0,T],\mathbb{R}^N)$, then there exists $\varepsilon_0 > 0$ so that

$$|\theta_i^\varepsilon(t) - \theta_j^\varepsilon(t)| \geq \delta_T,$$

for all $t \in [0,T]$ and every $i \neq j$ and every $\varepsilon \in (0,\varepsilon_0)$. Take $\theta_* = D(\Theta_0)$ and consider a nonnegative

$$\varepsilon_1 < \min\{\varepsilon_0, c(\alpha,\beta)\theta_*^{-1}), c(\alpha,\beta)^{-1}\delta_T\}.$$

Then, it is clear that

$$|\theta_i^\varepsilon(t) - \theta_j^\varepsilon(t)| \in [c(\alpha,\beta)\varepsilon, \theta_*],$$

for every $t \in [0,T]$ any $\varepsilon \in (0,\varepsilon_1)$ and each $i \neq j$. Applying Lemma 3.5.12 we obtain

$$\begin{aligned}
\frac{d}{dt}D(\Theta^\varepsilon) &= \frac{K}{N}\sum_{j=1}^N (h_\varepsilon(\theta_j^\varepsilon - \theta_M^\varepsilon) - h_\varepsilon(\theta_j^\varepsilon - \theta_m^\varepsilon)) \\
&\quad - \frac{K}{N}\sum_{j=1}^N (h_\varepsilon(\theta_M^\varepsilon - \theta_j^\varepsilon) + h_\varepsilon(\theta_j^\varepsilon - \theta_m^\varepsilon)) \\
&\leq -\frac{K}{N}\frac{h_\varepsilon(D(\Theta_0))}{D(\Theta_0)^\beta}\sum_{j=1}^N ((\theta_M^\varepsilon - \theta_j^\varepsilon)^\beta + (\theta_j^\varepsilon - \theta_m^\varepsilon)^\beta) \\
&\leq -\frac{K}{N}\frac{h_\varepsilon(D(\Theta_0))}{D(\Theta_0)^\beta}\sum_{j=1}^N ((\theta_M^\varepsilon - \theta_j^\varepsilon) + (\theta_j^\varepsilon - \theta_m^\varepsilon))^\beta \\
&= -K\frac{h_\varepsilon(D(\Theta_0))}{D(\Theta_0)^\beta}D(\Theta)^\beta,
\end{aligned}$$

for every $t \in [0,T]$ and $\varepsilon \in (0,\varepsilon_1)$. Integrating the differential inequality yields

$$D(\Theta(t)^\varepsilon) \leq \left(D(\Theta_0)^{1-\beta} - (1-\beta)K\frac{h_\varepsilon(D(\Theta_0))}{D(\Theta_0)^\beta}t\right)^{\frac{1}{1-\beta}},$$

for every $t \in [0,T]$ and $\varepsilon \in (0,\varepsilon_1)$. Taking limits when $\varepsilon \to 0$ amounts to

$$D(\Theta(t)) \leq \left(D(\Theta_0)^{1-\beta} - (1-\beta)K\frac{h(D(\Theta_0))}{D(\Theta_0)^\beta}t\right)^{\frac{1}{1-\beta}},$$

for each $t \in [0,T]$. However, it clearly yields a contradiction with the fact that $T > T_c^1$ due to the definition of T_c^1. □

The above result leads to a time estimate for the first collision between a couple of oscillators in the critical and supercritical cases. However, such idea can be repeated and improved in the critical case to give a total collision in finite time. The key ideas will be the uniqueness in Theorem 3.3.12 or, more specifically, the characterization of sticking of oscillators in Corollary 3.3.14.

Theorem 3.5.15. *Let $\Theta = (\theta_1, \cdots, \theta_N)$ be the solution to* (3.3.1)-(3.3.2) *with $\alpha = \frac{1}{2}$ for identical oscillators, $\Omega_i = 0$, for $i = 1, \ldots, N$. Assume that the initial configuration Θ_0 is confined in a half circle, i.e., $0 < D(\Theta_0) < \pi$. Then, there is complete phase synchronization in a finite time not larger than T_c, where*

$$T_c = \frac{D(\Theta_0)}{Kh(D(\Theta_0))}.$$

Proof. Let us assume the contrary, i.e., complete synchronization does not arises along $[0, T_c]$. By continuity there exists some $T > T_c$ so that it does not happen along $[0,T]$ neither. Recall that by virtue of Corollary 3.3.14, sticking of oscillators takes place in the critical case after any collision. Then, the collision classes $\mathcal{C}_i(t)$ and sticking classes $S_i(t)$ in Subsection 3.2.3 agree each other. Let us list the family of collision (sticking) classes, i.e., the various clusters at time t

$$\mathcal{E}(t) = \{\mathcal{C}_1(t), \ldots, \mathcal{C}_N(t)\} = \{E_1(t), \ldots, E_{\kappa(t)}\}.$$

As a consequence of the assumed hypothesis $\kappa(t)$, is nonincreasing with respect to t and bounded below by 2. Coming back to the initial configuration, we define i_M and i_m in such a way that

$$\max_{1\leq j\leq N} \theta_{j,0} = \theta_{i_M,0} \text{ and } \min_{1\leq j\leq N} \theta_{j,0} = \theta_{i_m,0}.$$

Since the regularized system (3.4.8) enjoys uniqueness in full sense, the oscillators θ_i^ε and θ_j^ε cannot cross. Similarly, by the Corollary 3.3.14, the oscillators θ_i and θ_j cannot cross neither unless they keep stuck together after that time. In any case, it is clear that

$$\begin{aligned} &\max_{1\leq j\leq N} \theta_j(t) = \theta_{i_M}(t), && \min_{1\leq j\leq N} \theta_j(t) = \theta_{i_m}(t),\\ &\max_{1\leq j\leq N} \theta_j^\varepsilon(t) = \theta_{i_M}^\varepsilon(t), && \min_{1\leq j\leq N} \theta_j^\varepsilon(t) = \theta_{i_m}^\varepsilon(t), \end{aligned}$$

for every $t \geq 0$ and any $\varepsilon > 0$. Then, we have

$$D(\Theta^\varepsilon(t)) = \theta_{i_M}^\varepsilon(t) - \theta_{i_m}^\varepsilon(t) \text{ and } D(\Theta(t)) = \theta_{i_M}(t) - \theta_{i_m}(t),$$

for every $t \geq 0$ and any $\varepsilon > 0$. Notice that all the above remarks ensure that for every $t \in [0, T]$

$$\begin{aligned} &\theta_j(t) - \theta_{i_m}(t) > 0, && \text{for all } j \in \mathcal{C}_{i_M}(t),\\ &\theta_{i_M}(t) - \theta_j(t) > 0, && \text{for all } j \in \mathcal{C}_{i_m}(t),\\ &\theta_{i_M}(t) - \theta_j(t) > 0, && \text{for all } j \notin \mathcal{C}_{i_M} \cup \mathcal{C}_{i_m}(t),\\ &\theta_j(t) - \theta_{i_m}(t) > 0, && \text{for all } j \notin \mathcal{C}_{i_M} \cup \mathcal{C}_{i_m}(t). \end{aligned}$$

Since $\Theta^\varepsilon \to \Theta$ in $C([0,T], \mathbb{R}^N)$, by continuity we can obtain $\varepsilon_0 > 0$ and $\delta_T > 0$ so that

$$\begin{aligned} &\theta_j^\varepsilon(t) - \theta_{i_m}^\varepsilon(t) > \delta_T, && \text{for all } j \in \mathcal{C}_{i_M}(t),\\ &\theta_{i_M}^\varepsilon(t) - \theta_j^\varepsilon(t) > \delta_T, && \text{for all } j \in \mathcal{C}_{i_m}(t),\\ &\theta_{i_M}^\varepsilon(t) - \theta_j^\varepsilon(t) > \delta_T, && \text{for all } j \notin \mathcal{C}_{i_M} \cup \mathcal{C}_{i_m}(t),\\ &\theta_j^\varepsilon(t) - \theta_{i_m}^\varepsilon(t) > \delta_T, && \text{for all } j \notin \mathcal{C}_{i_M} \cup \mathcal{C}_{i_m}(t). \end{aligned} \tag{3.5.23}$$

for every $t \in [0, T]$ and every $\varepsilon \in (0, \varepsilon_0)$. Take $\theta_* = D(\Theta_0)$, fix $\beta \in (0,1)$ and consider a nonnegative

$$\varepsilon_1 < \min\{\varepsilon_0, c(\alpha,\beta)\theta_*^{-1}), c(\alpha,\beta)^{-1}\delta_T\}.$$

Then, it is clear that

$$\begin{aligned} &\theta_j^\varepsilon(t) - \theta_{i_m}^\varepsilon(t) \in [c(\alpha,\beta)\varepsilon, \theta_*], && \text{for all } j \in \mathcal{C}_{i_M}(t),\\ &\theta_{i_M}^\varepsilon(t) - \theta_j^\varepsilon(t) \in [c(\alpha,\beta)\varepsilon, \theta_*], && \text{for all } j \in \mathcal{C}_{i_m}(t),\\ &\theta_{i_M}^\varepsilon(t) - \theta_j^\varepsilon(t) \in [c(\alpha,\beta)\varepsilon, \theta_*], && \text{for all } j \notin \mathcal{C}_{i_M} \cup \mathcal{C}_{i_m}(t),\\ &\theta_j^\varepsilon(t) - \theta_{i_m}^\varepsilon(t) \in [c(\alpha,\beta)\varepsilon, \theta_*], && \text{for all } j \notin \mathcal{C}_{i_M} \cup \mathcal{C}_{i_m}(t), \end{aligned} \tag{3.5.24}$$

for every $t \in [0, T]$ and any $\varepsilon \in (0, \varepsilon_1)$. Now, let us split as follows

$$\frac{d}{dt} D(\Theta^\varepsilon) = -\frac{K}{N} \sum_{j\in\mathcal{C}_{i_M}(t)} (h_\varepsilon(\theta_{i_M}^\varepsilon - \theta_j^\varepsilon) + h_\varepsilon(\theta_j^\varepsilon - \theta_{i_m}^\varepsilon))$$

$$
\begin{aligned}
&-\frac{K}{N}\sum_{j\in\mathcal{C}_{i_m}(t)}(h_\varepsilon(\theta^\varepsilon_{i_M}-\theta^\varepsilon_j)+h_\varepsilon(\theta^\varepsilon_j-\theta^\varepsilon_{i_m}))\\
&-\frac{K}{N}\sum_{j\notin\mathcal{C}_{i_M}(t)\cup\mathcal{C}_{i_m}(t)}(h_\varepsilon(\theta^\varepsilon_{i_M}-\theta^\varepsilon_j)+h_\varepsilon(\theta^\varepsilon_j-\theta^\varepsilon_m))\\
\leq&-\frac{K}{N}\sum_{j\in\mathcal{C}_{i_M}(t)}h_\varepsilon(\theta^\varepsilon_j-\theta^\varepsilon_{i_m})-\frac{K}{N}\sum_{j\in\mathcal{C}_{i_m}(t)}h_\varepsilon(\theta^\varepsilon_{i_M}-\theta^\varepsilon_j)\\
&-\frac{K}{N}\sum_{j\notin\mathcal{C}_{i_M}(t)\cup\mathcal{C}_{i_m}(t)}(h_\varepsilon(\theta^\varepsilon_{i_M}-\theta^\varepsilon_j)+h_\varepsilon(\theta^\varepsilon_j-\theta^\varepsilon_{i_m})),
\end{aligned}
$$

for every $t\in[0,T]$ and every $\varepsilon\in(0,\varepsilon_1)$. By virtue of Lemma 3.5.12 and the estimates in (3.5.24), the above chain of inequalities implies

$$
\begin{aligned}
\frac{d}{dt}D(\Theta^\varepsilon)\leq&-\frac{K}{N}\frac{h_\varepsilon(D(\Theta_0))}{D(\Theta_0)^\beta}\sum_{j\in\mathcal{C}_{i_M}(t)}(\theta^\varepsilon_j-\theta^\varepsilon_{i_m})^\beta-\frac{K}{N}\frac{h_\varepsilon(D(\Theta_0))}{D(\Theta_0)^\beta}\sum_{j\in\mathcal{C}_{i_m}(t)}(\theta^\varepsilon_{i_M}-\theta^\varepsilon_j)^\beta\\
&-\frac{K}{N}\frac{h_\varepsilon(D(\Theta_0))}{D(\Theta_0)^\beta}\sum_{j\notin\mathcal{C}_{i_M}(t)\cup\mathcal{C}_{i_m}(t)}((\theta^\varepsilon_{i_M}-\theta^\varepsilon_j)^\beta+(\theta^\varepsilon_j-\theta^\varepsilon_{i_m})^\beta).
\end{aligned}
$$

Let us integrate such differential inequality to obtain

$$
\begin{aligned}
D(\Theta^\varepsilon(t))\leq D(\Theta^0)&-\frac{K}{N}\frac{h_\varepsilon(D(\Theta_0))}{D(\Theta_0)^\beta}\int_0^t\sum_{j\in\mathcal{C}_{i_M}(s)}(\theta^\varepsilon_j(s)-\theta^\varepsilon_{i_m}(s))^\beta\,ds\\
&-\frac{K}{N}\frac{h_\varepsilon(D(\Theta_0))}{D(\Theta_0)^\beta}\int_0^t\sum_{j\in\mathcal{C}_{i_m}(s)}(\theta^\varepsilon_{i_M}(s)-\theta^\varepsilon_j(s))^\beta\,ds\\
&-\frac{K}{N}\frac{h_\varepsilon(D(\Theta_0))}{D(\Theta_0)^\beta}\int_0^t\sum_{j\notin\mathcal{C}_{i_M}(s)\cup\mathcal{C}_{i_m}(s)}((\theta^\varepsilon_{i_M}(s)-\theta^\varepsilon_j(s))^\beta+(\theta^\varepsilon_j(s)-\theta^\varepsilon_{i_m}(s))^\beta)\,ds,
\end{aligned}
$$

for every $t\in[0,T]$ and every $\varepsilon\in(0,\varepsilon_1)$. Taking limits as $\varepsilon\to 0$ we obtain

$$
\begin{aligned}
D(\Theta(t))\leq D(\Theta^0)&-\frac{K}{N}\frac{h(D(\Theta_0))}{D(\Theta_0)^\beta}\int_0^t\sum_{j\in\mathcal{C}_{i_M}(s)}(\theta_{i_M}(s)-\theta_{i_m}(s))^\beta\,ds\\
&-\frac{K}{N}\frac{h(D(\Theta_0))}{D(\Theta_0)^\beta}\int_0^t\sum_{j\in\mathcal{C}_{i_m}(s)}(\theta_{i_M}(s)-\theta_{i_m}(s))^\beta\,ds\\
&-\frac{K}{N}\frac{h(D(\Theta_0))}{D(\Theta_0)^\beta}\int_0^t\sum_{j\notin\mathcal{C}_{i_M}(s)\cup\mathcal{C}_{i_m}(s)}((\theta_{i_M}(s)-\theta_j(s))^\beta+(\theta_j(s)-\theta_{i_m}(s))^\beta)\,ds.
\end{aligned}
$$

for every $t\in[0,T]$. To sum up, we obtain,

$$D(\Theta(t))\leq D(\Theta_0)-K\frac{h(D(\Theta_0))}{D(\Theta_0)^\beta}\int_0^t D(\theta(s))^\beta\,ds.$$

Hence, we find

$$D(\Theta(t))\leq\left(D(\Theta_0)^{1-\beta}-(1-\beta)K\frac{h(D(\Theta_0))}{D(\Theta_0)^\beta}t\right)^{\frac{1}{1-\beta}},$$

for all $t \in [0, T]$. Then, it is clear that

$$T < \frac{D(\Theta_0)}{(1-\beta)Kh(D(\Theta_0))},$$

for all $\beta \in (0,1)$. Taking limits when $\beta \to 0$ shows that $T \leq T_c$ and this yields the contradiction. □

Remark 3.5.16. *Notice that Theorem 3.5.14 also works in the supercritical case. However, the same proof as in Theorem 3.5.15 is not valid to show finite-time complete phase synchronization of identical oscillators for $\alpha \in (\frac{1}{2}, 1)$. The reason is that at this point we cannot guarantee whether the Filippov solutions in Θ obtained as singular limit of the regularized solutions Θ^ε to system* (3.4.8) *in Theorem 3.4.11 agrees with the solution obtained in Remark 3.3.20 via the "sticking after collision" continuation procedure of classical solutions. However, if the limiting Θ obtained in Theorem 3.4.11 satisfies such "sticking after collision" property, we can mimic Theorem 3.5.15 to show that it exhibits complete phase synchronization at a finite time not larger than*

$$T_c = \frac{D(\Theta_0)}{Kh(D(\Theta_0))}.$$

Appendices

3.A Regular interactions

In this Appendix, we study the Kuramoto model with regular coupling weights:

$$\dot{\theta}_i = \Omega_i + \frac{K}{N}\sum_{j=1}^{N} \frac{\sigma^{2\alpha}}{(\sigma^2 + c|\theta_j - \theta_i|_o^2)^\alpha} \sin(\theta_j - \theta_i) \quad \text{for} \quad i = 1, \ldots, N, \tag{3.A.1}$$

where we denote $c \equiv c_{\alpha,\zeta} = 1 - \zeta^{-1/\alpha}$ for simplicity. Recall that such model comes from the choice (3.1.5) of Γ as the Hebbian plasticity function in (3.1.6). Since the right hand side of (3.A.1) is Lipschitz continuous, then the system (3.A.1) has a unique solution by Cauchy–Lipschitz theory in this case.

For positive σ, we get the following bounds for Γ:

$$\varepsilon_\sigma := \frac{\sigma^{2\alpha}}{(\sigma^2 + c\pi^2)^\alpha} \leq \Gamma(\theta) \leq 1, \qquad \Gamma(0) = \Gamma(2\pi) = 1.$$

Note that ε_σ converges to zero as $\sigma \to 0$. We will study the emergence of synchronization for identical and non-identical oscillators and, we will use the idea of [145] for the proof of synchronization.

3.A.1 Identical oscillators

Consider the Kuramoto model (3.A.1) for identical oscillators, which have the same natural frequency. Without loss of generality, we may assume $\Omega_i = 0$ for all $i = 1, \ldots, N$. The system (3.A.1) becomes as follows:

$$\dot{\theta}_i = \frac{K}{N}\sum_{j=1}^{N} \frac{\sigma^{2\alpha}}{(\sigma^2 + c|\theta_j - \theta_i|_o^2)^\alpha} \sin(\theta_j - \theta_i), \quad i = 1, \ldots, N. \tag{3.A.2}$$

We can show the complete phase synchronization asymptotically for (3.A.2) with a constraint on initial configuration. Let us recall the notation $\theta_M(t)$ and $\theta_m(t)$ in (3.5.6) for the indices of largest and shortest phases and $D(\Theta)$ for the phase diameter defined in (3.5.7).

Theorem 3.A.1. *Let* $\Theta = (\theta_1, \ldots, \theta_N)$ *be the solution to* (3.A.2). *Assume that the initial configuration is confined in a half circle, i.e.* $D(\Theta_0) < \pi$, *and the coupling strength* K *is positive. Then, the solution* Θ *shows the complete phase synchronization asymptotically:*

$$D(\Theta_0)e^{-Kt} \le D(\Theta) \le D(\Theta_0)e^{-\frac{K\Gamma(D(\Theta_0))\sin D(\Theta_0)}{D(\Theta_0)}t}.$$

Proof. We consider the dynamics of phase diameter

$$\frac{d}{dt}D(\Theta) = \frac{K}{N}\sum_{j=1}^{N}\Big(\Gamma(\theta_j - \theta_M)\sin(\theta_j - \theta_M) - \Gamma(\theta_j - \theta_m)\sin(\theta_j - \theta_m)\Big). \tag{3.A.3}$$

Since $\sin(\theta_j - \theta_M) \le 0$ and $\sin(\theta_j - \theta_m) \ge 0$, as long as $D(\Theta) \le \pi$, we have

$$\frac{d}{dt}D(\Theta) \le 0 \quad \text{and} \quad D(\Theta(t)) \le D(\Theta_0) < \pi \quad \text{for} \quad t > 0.$$

By this contraction of phase difference, we have

$$\sin(\theta_j - \theta_M) \le \frac{\sin D(\Theta_0)}{D(\Theta_0)}(\theta_j - \theta_M) \quad \text{and} \quad \sin(\theta_j - \theta_m) \ge \frac{\sin D(\Theta_0)}{D(\Theta_0)}(\theta_j - \theta_m). \tag{3.A.4}$$

On the other hand, we get

$$\varepsilon_\sigma < \Gamma(D(\Theta_0)) \le \Gamma(D(\Theta)) \le 1. \tag{3.A.5}$$

By applying (3.A.4) and (3.A.5) to (3.A.3), we attain the following differential inequality:

$$\begin{aligned}
\frac{d}{dt}D(\Theta) &\le \frac{K}{N}\sum_{j=1}^{N}\Big(\Gamma(\theta_j - \theta_M)\frac{\sin D(\Theta_0)}{D(\Theta_0)}(\theta_j - \theta_M) - \Gamma(\theta_j - \theta_m)\frac{\sin D(\Theta_0)}{D(\Theta_0)}(\theta_j - \theta_m)\Big) \\
&= -\frac{K}{N}\frac{\sin D(\Theta_0)}{D(\Theta_0)}\sum_{j=1}^{N}\Big(\Gamma(\theta_j - \theta_M)(\theta_M - \theta_j) + \Gamma(\theta_j - \theta_m)(\theta_j - \theta_m)\Big) \\
&\le -\frac{K}{N}\frac{\Gamma(D(\Theta_0))\sin D(\Theta_0)}{D(\Theta_0)}\sum_{j=1}^{N}\Big((\theta_M - \theta_j) + (\theta_j - \theta_m)\Big) \\
&= -\frac{K\Gamma(D(\Theta_0))\sin D(\Theta_0)}{D(\Theta_0)}D(\Theta).
\end{aligned}$$

Grönwall's lemma yields the desired upper estimate. Similarly, from (3.A.5) and $\sin x \le x$ for $0 \le x \le \pi$, we have

$$\frac{d}{dt}D(\Theta) \ge \frac{K}{N}\sum_{j=1}^{N}\Big((\theta_j - \theta_M) - (\theta_j - \theta_m)\Big) = -KD(\Theta),$$

which gives the lower estimate. □

3.A.2 Non–identical oscillators

We assume that the diameter of initial configuration is less than $D^\infty < \frac{\pi}{2}$. We first show that the diameter of phase is less than D^∞ for all time $t \geq 0$ for sufficiently large coupling strength K. Let us recall that for $\theta \in (-\pi, \pi)$ the plasticity function reads $\Gamma(\theta) = \frac{\sigma^{2\alpha}}{(\sigma^2 + c\theta^2)^\alpha}$. Then, we have

$$\Gamma'(\theta) = -\frac{2\sigma^{2\alpha}\alpha c\theta}{(\sigma^2 + c\theta^2)^{\alpha+1}},$$
$$\Gamma''(\theta) = -\frac{2\sigma^{2\alpha}\alpha c\big[\sigma^2 - (2\alpha+1)c\theta^2\big]}{(\sigma^2 + c\theta^2)^{\alpha+2}}.$$

If we set

$$\theta_\pm := \pm\frac{\sigma}{\sqrt{c(2\alpha+1)}},$$

then Γ' attains its global extrema on such points, namely

$$\Gamma'(\theta_-) = \max_{\theta\in(-\pi,\pi)} \Gamma'(\theta) > 0 \quad \text{and} \quad \Gamma'(\theta_+) = \min_{\theta\in(-\pi,\pi)} \Gamma'(\theta) < 0.$$

Indeed, we get

$$\Gamma'(\theta_-) = -\Gamma'(\theta_+) = \frac{2\alpha\sqrt{c}}{\sigma\sqrt{2\alpha+1}(1 + \frac{1}{2\alpha+1})^{\alpha+1}}.$$

We first show the boundedness of phase differences.

Lemma 3.A.2. *Assume that $D(\Theta_0) < D^\infty$, for some small $D^\infty < \frac{\pi}{2}$, and that the coupling strength is sufficiently large so that*

$$-\Gamma'(\theta_+) < \frac{\Gamma(D^\infty)}{\tan D^\infty} \quad \textit{and} \quad K > \frac{D(\dot\Theta_0)}{\Big[\Gamma'(\theta_+)\sin D^\infty + \Gamma(D^\infty)\cos D^\infty\Big](D^\infty - D(\Theta_0))}.$$

Then, we have

$$D(\Theta(t)) < D^\infty \quad \textit{for} \quad t \geq 0.$$

Proof. Assume that there exists a time for which $D(\Theta(t)) \geq D^\infty$. Then, due to the continuity

$$t^* := \sup\{t > 0 : D(\Theta(s)) < D^\infty \text{ for } 0 \leq s \leq t\},$$

is positive and finite and $D(\Theta(t^*)) = D^\infty$. We set indices F and S so that

$$\dot\theta_F(t) := \max\{\dot\theta_1(t), \ldots, \dot\theta_N(t)\} \quad \text{and} \quad \dot\theta_S(t) := \min\{\dot\theta_1(t), \ldots, \dot\theta_N(t)\},$$

for each time t and define the diameter of frequency so that

$$D(\dot\Theta(t)) := \dot\theta_F(t) - \dot\theta_S(t).$$

Then, we have

$$D(\dot\Theta(t)) - D(\dot\Theta_0) = \int_0^t \frac{d}{ds} D(\dot\Theta(s))\, ds. \tag{3.A.6}$$

By taking time derivative on $D(\dot{\Theta})$, we get

$$\begin{aligned}\frac{d}{dt}D(\dot{\Theta}) = &\frac{K}{N}\sum_{j=1}^{N}\Big[\Gamma'(\theta_j-\theta_F)\sin(\theta_j-\theta_F)+\Gamma(\theta_j-\theta_F)\cos(\theta_j-\theta_F)\Big](\dot{\theta}_j-\dot{\theta}_F)\\ &-\frac{K}{N}\sum_{j=1}^{N}\Big[\Gamma'(\theta_j-\theta_S)\sin(\theta_j-\theta_S)+\Gamma(\theta_j-\theta_S)\cos(\theta_j-\theta_S)\Big](\dot{\theta}_j-\dot{\theta}_S).\end{aligned} \tag{3.A.7}$$

Then, we get the following couple of upper and lower bounds

$$\Gamma'(\theta_+)\sin D^\infty \le \Gamma'(\theta_j-\theta_i)\sin(\theta_j-\theta_i)\le 0, \tag{3.A.8}$$

$$\Gamma(D^\infty)\cos D^\infty \le \Gamma(\theta_j-\theta_i)\cos(\theta_j-\theta_i)\le 1. \tag{3.A.9}$$

By applying (3.A.8) and (3.A.9) into (3.A.7), we deduce

$$\begin{aligned}\frac{d}{dt}D(\dot{\Theta}) &\le \frac{K}{N}\sum_{j=1}^{N}\Big[\Gamma'(\theta_+)\sin D^\infty+\Gamma(D^\infty)\cos D^\infty\Big]\Big((\dot{\theta}_j-\dot{\theta}_F)-(\dot{\theta}_j-\dot{\theta}_S)\Big)\\ &= -K\Big[\Gamma'(\theta_+)\sin D^\infty+\Gamma(D^\infty)\cos D^\infty\Big](\dot{\theta}_F-\dot{\theta}_S)\\ &\le -K\underbrace{\Big[\Gamma'(\theta_+)\sin D^\infty+\Gamma(D^\infty)\cos D^\infty\Big]}_{>0}D(\dot{\Theta}),\end{aligned} \tag{3.A.10}$$

for every $t\in[0,t^*]$. Combining (3.A.6) and (3.A.10), we obtain

$$D(\dot{\Theta}(t)) \le D(\dot{\Theta}_0) - K\Big[\Gamma'(\theta_+)\sin D^\infty+\Gamma(D^\infty)\cos D^\infty\Big]\int_0^t D(\dot{\Theta}(s))ds, \tag{3.A.11}$$

for every $t\in[0,t^*]$. Let us define $y(t) := \int_0^t D(\dot{\Theta}(s))ds$. Thus, the inequality (3.A.11) can be rewritten into

$$y'(t)\le y'(0)-Cy(t).$$

Here, $C := K\Big[\Gamma'(\theta_+)\sin D^\infty+\Gamma(D^\infty)\cos D^\infty\Big]$ and $t\in[0,t^*]$. Then, we find

$$y(t)\le\frac{y'(0)}{C}(1-e^{-Ct})\le\frac{y'(0)}{C},$$

for all $t\in[0,t^*]$. However, since $D(\Theta(t^*)) = D^\infty$, we get

$$\begin{aligned}D^\infty = D(\Theta_0)+\int_0^{t^*}\frac{d}{ds}D(\Theta(s))\,ds &\le D(\Theta_0)+\int_0^{t^*}D(\dot{\Theta}(s))\,ds\\ &\le D(\Theta_0)+y(t^*)\le D(\Theta_0)+\frac{y'(0)}{C}<D^\infty,\end{aligned}$$

when

$$K > \frac{D(\dot{\Theta}_0)}{\Big[\Gamma'(\theta_+)\sin D^\infty+\Gamma(D^\infty)\cos D^\infty\Big](D^\infty-D(\Theta_0))},$$

which yields to a contradiction. Thus, $D(\Theta(t)) < D^\infty$, for all $t\ge 0$. □

We are ready to prove the frequency synchronization for non-identical oscillators.

Theorem 3.A.3. *Assume that $D(\Theta_0) < D^\infty$, for some small $D^\infty < \frac{\pi}{2}$, and that the coupling strength is sufficiently large so that*

$$-\Gamma'(\theta_+) < \frac{\Gamma(D^\infty)}{\tan D^\infty} \quad \text{and} \quad K > \frac{D(\dot{\Theta}_0)}{\Big[\Gamma'(\theta_+)\sin D^\infty + \Gamma(D^\infty)\cos D^\infty\Big](D^\infty - D(\Theta_0))}.$$

Then, we deduce a complete frequency synchronization

$$D(\dot{\Theta}(0))e^{-Kt} \leq D(\dot{\Theta}(t)) \leq D(\dot{\Theta}(0))e^{-K\Big[\Gamma'(\theta_+)\sin D^\infty + \Gamma(D^\infty)\cos D^\infty\Big]t}.$$

Proof. From (3.A.7)-(3.A.10), we obtain

$$\frac{d}{dt}D(\dot{\Theta}) \leq -K\Big[\Gamma'(\theta_+)\sin D^\infty + \Gamma(D^\infty)\cos D^\infty\Big]D(\dot{\Theta}).$$

On the other hand, from (3.A.7)-(3.A.9), we have

$$\frac{d}{dt}D(\dot{\Theta}) \geq -KD(\dot{\Theta}).$$

By Gronwall's lemma, we achieve the exponential estimates for the frequency synchronization. □

Since the decay rate of the asymptotic frequency synchronization is exponential, then the solution Θ shows the emergence of a phase-locked state.

3.B H-representation of the Filippov set-valued maps

In this appendix, we exhibit the proofs of the technical Lemmas 3.4.8 and 3.4.13 that have been used in Chapter 3. Recall that such results were respectively applied in Propositions 3.4.9 and 3.4.14 in order to characterize explicitly some H-representation of the Filippov set-valued map in the supercritical and critical cases. We introduce some notation that will be used here on.

Definition 3.B.1. *Consider $n \in \mathbb{N}$. For every $i, j \in \{1, \ldots, n\}$ we define the linear operator*

$$\begin{array}{rccc} L_{ij}: & \operatorname{Skew}_n(\mathbb{R}) & \longrightarrow & \mathbb{R}, \\ & Y & \longmapsto & y_{ij}, \\ L_i: & \operatorname{Skew}_n(\mathbb{R}) & \longrightarrow & \mathbb{R}, \\ & Y & \longmapsto & \sum_{k=1}^n y_{ik}, \\ \mathcal{L}: & \operatorname{Skew}_n(\mathbb{R}) & \longrightarrow & \mathbb{R}^n, \\ & Y & \longmapsto & Y\cdot \mathbf{j}. \end{array}$$

By definition, the following relations hold true

$$L_i = \sum_{k=1}^n L_{ik} \text{ and } \mathcal{L} = (L_1, \ldots, L_n).$$

First, we give the simpler proof of Lemma 3.4.8:

Lemma 3.B.2. *Consider any $n \in \mathbb{N}$ and any vector $x \in \mathbb{R}^n$. Then, the following assertions are equivalent:*

1. *There exists some* $Y \in \mathrm{Skew}_n(\mathbb{R})$ *such that*

$$x = Y \cdot \mathbf{j}.$$

2. *The following implicit equation holds true*

$$x \cdot \mathbf{j} = 0,$$

where $\mathbf{j} = \underbrace{(1, \ldots, 1)}_{n \text{ pairs}}$ *stand for the vector of ones.*

Proof. Let us define the following linear operator

$$\begin{array}{rccc} \mathcal{L}: & \mathrm{Skew}_n(\mathbb{R}) & \longrightarrow & \mathbb{R}^n, \\ & Y & \longmapsto & Y \cdot \mathbf{j}. \end{array}$$

Then, the thesis of this lemma is equivalent to

$$\mathcal{L}(\mathrm{Skew}_n(\mathbb{R})) = \mathbf{j}^{\perp}. \tag{3.B.1}$$

On the one hand, it is clear that the inclusion $\subseteq$ in (3.B.1) fulfils by virtue of the properties of the skew symmetric matrices. On the other hand, let us define the matrices

$$E_{ij} := \frac{1}{2}(e_i \otimes e_j - e_j \otimes e_i), \tag{3.B.2}$$

for every $i \neq j$, where $\{e_i : i = 1, \ldots, N\}$ is the standard basis of $\mathbb{R}^n$ and $\otimes$ denotes the Kronecker product. Notice that

$$\mathcal{L}(E_{ij}) = \frac{1}{2}(e_i \otimes e_j - e_j \otimes e_i) \cdot \mathbf{j} = e_i - e_j.$$

Hence, $\{\mathcal{L}(E_{i,i+1}) : i = 1, \ldots, n-1\} = \{e_i - e_{i+1} : i = 1, \ldots, n-1\}$ consist of $n-1$ independent vectors. Consequently, $\mathcal{L}$ has rank larger or equal to $n-1$. Since $\mathbf{j}^{\perp}$ has rank equal to $n-1$ the full identity in (3.B.1) holds true. □

Now, we focus on the proof of Lemma 3.4.13. Our main tool in this part will be the *Farkas alternative* from convex analysis that we recall in the subsequent result.

Lemma 3.B.3 (Farkas alternative)**.** *Consider any finite-dimensional vector space V, some finite family of linear operators $T_1, \ldots, T_k : V \longrightarrow \mathbb{R}$ and $b = (b_1, \ldots, b_k) \in \mathbb{R}^k$. Then, exactly one of the following statements holds true:*

1. *There exists* $v \in V$ *such that*
$$T_i(v) \leq b_i, \;\; i = 1, \ldots, k.$$

2. *There exists* $q \in \mathbb{R}^n$ *with* $q_i \geq 0$ *for all* $i = 1, \ldots k$ *such that*

$$\sum_{i=1}^{k} q_i T_i \equiv 0 \;\; \text{and} \;\; q \cdot b < 0.$$

This result has several equivalent representations in the literature and it is sometimes called the *Theorem of Alternatives*. One clear reference where we can infer our version from can be found in [284, Lemma 2.54]. We are now ready to give a proof of Lemma 3.4.13.

Lemma 3.B.4. *Consider any $n \in \mathbb{N}$ and any vector $x \in \mathbb{R}^n$. Then, the following two assertions are equivalent:*

1. *There exists some $Y \in \mathrm{Skew}_n([-1,1])$ such that*
$$x = Y \cdot \mathbf{j}.$$
2. *There exists some $Y \in \mathrm{Skew}_n(\mathbb{R})$ such that*
$$L_{ij}(Y) \leq 1, \; L_i(Y) \leq x_i \text{ and } -L_i(Y) \leq -x_i.$$
3. *The following inequality*
$$\sum_{i,j=1}^{n} q_{ij} + \lambda_i x_i \geq 0, \tag{3.B.3}$$
holds, for every $Q \in \mathcal{M}_n(\mathbb{R}_0^+)$ and $\lambda \in \mathbb{R}^n$ such that $q_{ij} + \lambda_i = q_{ji} + \lambda_j$.
4. *We have that*
$$\frac{1}{k}\sum_{i=1}^{k} x_{\sigma_i} \in [-(n-k),(n-k)],$$
for every permutation σ of $\{1,\ldots,n\}$ and any $k \in \mathbb{N}$.

Proof. For the shake of simplicity in our arguments, we will split the proof into two parts. In the first part, we establish the equivalence of the first three assertions in the statement. The main tool to be used in such part is the above Lemma 3.B.3. In the second part, we will focus on the more convoluted equivalence between the first group of equivalent assertions in the above-mentioned step and the last assertion.

• *Step 1*: Equivalence of the first three assertions.

On the one hand, the first two assertion are perfectly equivalent by virtue of Definition 3.B.1. Then, our problem consists in a system of affine inequalities in the vector space $\mathrm{Skew}_n(\mathbb{R})$ of skew symmetric matrices. Hence, by Farkas alternative (see Lemma 3.B.3) such assertions amounts to saying that whenever q_{ij}, q_i^+, q_i^- are non-negative coefficients verifying
$$\sum_{i,j=1}^{n} q_{ij}L_{ij} + \sum_{i=1}^{n} q_i^+ L_i - \sum_{i=1}^{n} q_i^- L_i \equiv 0 \text{ in } \mathrm{Skew}_n(\mathbb{R}),$$
then
$$\sum_{i,j=1}^{n} q_{ij} + \sum_{i=1}^{n} q_i^+ x_i - \sum_{i=1}^{n} q_i^- x_i \geq 0.$$
Defining $\lambda_i = q_i^+ - q_i^-$, we can simplify an equivalent assertion: for every $Q \in \mathcal{M}_n(\mathbb{R}_0^+)$ and $\lambda \in \mathbb{R}^n$ such that
$$\sum_{i,j=1}^{n} q_{ij}L_{ij} + \sum_{i=1}^{n} \lambda_i L_i \equiv 0 \text{ in } \mathrm{Skew}_n(\mathbb{R}), \tag{3.B.4}$$
then
$$\sum_{i,j=1}^{n} q_{ij} + \sum_{i=1}^{n} \lambda_i x_i \geq 0.$$

Thus, the equivalence with the third assertion follows by evaluating the identity (3.B.4) on every matrix in the canonical basis of $\mathrm{Skew}_n(\mathbb{R})$, i.e.,

$$\{e_i \otimes e_j - e_j \otimes e_i : 1 \leq i < j \leq n\},$$

and noticing that we obtain the condition $q_{ij} + \lambda_i = q_{ji} + \lambda_j$ in such third assertion.

• *Step 2*: Equivalence with the last assertion.

On the one hand, let us assume that the first assertion is satisfied, i.e., $x = Y \cdot \mathbf{j}$ for some $Y \in \mathrm{Skew}_n([-1,1])$. Taking any permutation σ of $\{1,\ldots,n\}$ and any $1 \leq k \leq n$ we obtain

$$\sum_{i=1}^{k} x_{\sigma_i} = \sum_{i=1}^{k}\sum_{j=1}^{n} y_{\sigma_i\sigma_j} = \sum_{i=1}^{k}\sum_{j=1}^{k} y_{\sigma_i\sigma_j} + \sum_{i=1}^{k}\sum_{j=k+1}^{n} y_{\sigma_i\sigma_j}.$$

Since the first term becomes zero (by anti-symmetry) and the second term consists of $n(n-k)$ terms with values in $[-1,1]$, then

$$\sum_{i=1}^{k} x_{\sigma_i} \in [-k(n-k), k(n-k)].$$

Conversely, assume that the last assertion is true and let us prove (3.B.3) in the third assertion. Consider $Q \in \mathcal{M}_n(\mathbb{R}_0^+)$ and $\lambda \in \mathbb{R}^n$ such that

$$q_{ij} - q_{ji} = \lambda_j - \lambda_i. \tag{3.B.5}$$

Without loss of generality we will assume that $q_{ii} = 0$, for every $i = 1,\ldots,n$ (notice that in other case, (3.B.3) is even larger), and let us split

$$I := \sum_{i\neq j} q_{ij} + \sum_{i=1}^{n} \lambda_i x_i =: I_1 + I_2.$$

On the one hand, let us rewrite I_2 and notice that

$$I_2 = \sum_{i=1}^{n} \lambda_i x_i = \sum_{i=1}^{n} (\lambda_i - \lambda_j) x_i + \lambda_j \sum_{i=1}^{n} x_i,$$

for every $j = 1,\ldots,n$. Since the sum of all the x_i becomes zero by hypothesis, taking averages with respect to all the indices $j = 1,\ldots,n$ we obtain that

$$I_2 = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} (\lambda_i - \lambda_j) x_i.$$

Finally, changing the indices i with j and taking the average of both expressions we can equivalently write

$$I_2 = \frac{1}{2n}\sum_{i=1}^{n}\sum_{j=1}^{n} (\lambda_i - \lambda_j)(x_i - x_j) = \frac{1}{n}\sum_{i<j} (\lambda_j - \lambda_i)(x_j - x_i).$$

Thus, substituting (3.B.5) into I_2 and putting it together with I_1 we can rewrite

$$I = \sum_{i\neq j} q_{ij} + \frac{1}{2n}\sum_{i\neq j} (q_{ij} - q_{ji})(x_j - x_i) = \sum_{i\neq j} q_{ij}\left(1 + \frac{1}{n}(x_j - x_i)\right). \tag{3.B.6}$$

Let us consider a permutation σ of $\{1,\ldots,n\}$ so that we can order the coefficients λ_i in increasing way, i.e.,

$$\lambda_{\sigma_1} \leq \lambda_{\sigma_2} \leq \cdots \leq \lambda_{\sigma_n}. \tag{3.B.7}$$

Then,

$$\begin{aligned} I &= \sum_{i\neq j} q_{\sigma_i\sigma_j}\left(1+\frac{1}{n}(x_{\sigma_j}-x_{\sigma_i})\right)\\ &= \sum_{i<j}(q_{\sigma_i\sigma_j}-q_{\sigma_j\sigma_i})\left(1+\frac{1}{n}(x_{\sigma_j}-x_{\sigma_i})\right)+2\sum_{i<j} q_{\sigma_j\sigma_i} =: I_3+I_4. \end{aligned}$$

It is clear that I_4 is non-negative. Hence, we will focus on showing that so is I_3 too. By virtue of (3.B.5), it is easy to show that

$$q_{\sigma_i\sigma_j} - q_{\sigma_j}\sigma_i = \sum_{k=i}^{j-1}(q_{\sigma_k\sigma_{k+1}} - q_{\sigma_{k+1}\sigma_k}),$$

for every $i<j$. Thereby,

$$\begin{aligned} I_3 &= \sum_{i<j}\sum_{k=i}^{j-1}(q_{\sigma_k\sigma_{k+1}} - q_{\sigma_{k+1}\sigma_k})\left(1+\frac{1}{n}(x_{\sigma_j}-x_{\sigma_i})\right)\\ &= \sum_{k=1}^{n-1} a_k(q_{\sigma_k\sigma_{k+1}} - q_{\sigma_{k+1}\sigma_k}) = \sum_{k=1}^{n-1} a_k(\lambda_{\sigma_{k+1}}-\lambda_{\sigma_k}), \end{aligned} \tag{3.B.8}$$

where in the last step we have used (3.B.5) again and the coefficients a_k read

$$a_k := \sum_{\substack{i\leq k\\ j\geq k+1}}\left(1+\frac{1}{n}(x_{\sigma_j}-x_{\sigma_i})\right) = k(n-k)+\frac{k}{n}\sum_{j=k+1}^{n} x_{\sigma_j} - \frac{n-k}{n}\sum_{i=1}^{k} x_{\sigma_i}.$$

Bearing in mind that the sum of all the x_i vanishes by hypothesis, then

$$a_k = k(n-k) - \sum_{i=1}^{k} x_{\sigma_i}.$$

Then, $a_k \geq 0$ by hypothesis. Since we have chosen σ so that (3.B.7) takes place, then the result follows from the above expression (3.B.8) for I_3. □

3.C Characterizing the sticking conditions

Our purpose in this appendix is to characterize explicit conditions for the weights specifying the necessary and sufficient conditions for sticking of particles (3.3.13) and (3.3.14) in the Subsections (3.4.3) and (3.4.1) respectively. The first part is devoted to the latter condition for the supercritical case and the second part will focus on the former critical case.

Apart form the linear operators in Definition 3.B.1 we will need the following ones.

Definition 3.C.1. *Consider $n \in \mathbb{N}$. For every $i, j \in \{1, \ldots, n\}$ we define the linear operator*

$$\begin{array}{cccc} T_{ij}: & \mathrm{Skew}_n(\mathbb{R}) & \longrightarrow & \mathbb{R}, \\ & Y & \longmapsto & \sum_{k=1}^n (y_{ik} - y_{jk}). \end{array}$$

Notice that by definition we get the following relation with the operators in Definition 3.B.1

$$T_{ij} = \sum_{k=1}^{n} (L_{ik} - L_{jk}).$$

Then, the next result yields a characterization for the sticking condition (3.3.14) to hold.

Lemma 3.C.2. *Consider any $n \in \mathbb{N}$ and any matrix $M \in \mathrm{Skew}_n(\mathbb{R})$. Then, the following assertions are equivalent:*

1. *There exists some $Y \in \mathrm{Skew}_n(\mathbb{R})$ such that*

$$M = Y \cdot \mathbf{J} + \mathbf{J} \cdot Y.$$

2. *There exits some $Y \in Skew_n(\mathbb{R})$ such that*

$$T_{ij}(Y) \leq m_{ij} \text{ and } -T_{ij}(Y) \leq -m_{ij}.$$

3. *We have*

$$m_{1i} + m_{ij} + m_{j1} = 0, \tag{3.C.1}$$

for every $2 \leq i < j \leq n$.

4. *The equality*

$$m_{ij} + m_{jk} + m_{ki} = 0, \tag{3.C.2}$$

holds, for every $1 \leq i < j < k \leq n$.

Proof. First, it is clear that the first two assertions are equivalent. Second, let us briefly show that (3.C.1) and (3.C.2) are equivalent. On the one hand, it is clear that (3.C.1) is a particular case of (3.C.2). On the other hand, let us assume that (3.C.1) fulfills. Then, we have in particular the next three equations for $1 \leq i < j < k \leq n$

$$\begin{aligned} m_{1i} + m_{ij} + m_{j1} &= 0, \\ m_{1j} + m_{jk} + m_{k1} &= 0, \\ m_{1k} + m_{ki} + m_{i1} &= 0. \end{aligned}$$

Taking the sum of such equations we obtain (3.C.2) by virtue of the skew-symmetry of M. Hence, let us just concentrate on proving the equivalence between the second and third assertions. By Lemma 3.B.3, the second assertion amounts to saying that whenever $\Lambda \in \mathcal{M}_n(\mathbb{R})$ verifies

$$\sum_{i,j=1}^{n} \lambda_{ij} T_{ij} \equiv 0,$$

then the following condition fulfills

$$\sum_{i,j=1}^{n} \lambda_{ij} m_{ij} \geq 0.$$

Evaluating along the basis $e_i \otimes e_j - e_j \otimes e_i$ we equivalently write the former condition as

$$\sum_{k=1}^{n} [(\lambda_{ik} - \lambda_{ki}) - (\lambda_{jk} - \lambda_{kj})] = 0.$$

Hence, if we define $p_{ij} = \lambda_{ij} - \lambda_{ji}$ we can conclude that the second assertion of this Lemma is completely equivalent to the fact that whenever $P \in \mathrm{Skew}_n(\mathbb{R})$ verifies

$$\sum_{k=1}^{n} (p_{ik} - p_{jk}) = 0, \tag{3.C.3}$$

for every $i, j \in \{1, \ldots, n\}$, then

$$\sum_{i,j=1}^{n} p_{ij} m_{ij} \geq 0. \tag{3.C.4}$$

• *Step 1:* Characterizing condition (3.C.3).
Taking the vector x as follows

$$x = \left(\sum_{k=1}^{n} p_{jk} \right) \mathbf{j},$$

and applying Lemma 3.4.8 shows that those matrices $P \in \mathrm{Skew}_n(\mathbb{R})$ fulfilling (3.C.3) agree with the matrices that lie in the kernel of the operator $\mathcal{L} = (L_1, \ldots, L_n)$. Recall that by virtue of such result, $\mathcal{L}$ has rank equal to $n-1$. Since $\mathrm{Skew}_n(\mathbb{R})$ is a vector space with dimension $d_1 := n(n-1)/2$, then we know that

$$d_2 := \dim(\ker \mathcal{L}) = \frac{n(n-1)}{2} - (n-1) = \frac{(n-1)(n-2)}{2}.$$

Consider the following matrices

$$P_{ij} := E_{1i} + E_{ij} + E_{j1} = E_{1i} - E_{ij} + E_{ij}, \tag{3.C.5}$$

where E_{ij} are the skew symmetric matrices in (3.B.2). Then,

$$\mathcal{L}(P_{ij}) = \mathcal{L}(E_{1i}) + \mathcal{L}(E_{ij}) + \mathcal{L}(E_{j1}) = (e_1 - e_j) + (e_i - e_j) + (e_j - e_1) = 0.$$

Hence, the following subset

$$\mathcal{P} := \{P_{ij} : 2 \leq i < j \leq n\} \subseteq \ker \mathcal{L},$$

consists of $(n-1)(n-2)/2$ different elements, which we can be classified via the lexicographic order of multi-indices (i, j). Let us show that all of them are linearly independent, thus generating the whole kernel. We first consider the basis of skew-symmetric matrices

$$\mathcal{B} := \{E_{ij} : 1 \leq i < j \leq n\},$$

and, again, we can list them ordered with respect to the lexicographic order. Let us consider the matrix $\mathcal{M} \in \mathcal{M}_{d_2 \times d_1}(\mathbb{R})$ of coordinates of the elements in $\mathcal{P}$ with respect to the basis $\mathcal{B}$. Then, by the definition (3.C.5) one infers that the $d_2 \times d_2$ identity matrix appears as the submatrix of $\mathcal{M}$ consisting of all the d_2 rows but just the last d_2 columns. Hence, $\operatorname{rank} \mathcal{M} = d_2$ and, consequently,

$$\ker \mathcal{L} = \operatorname{span}(\mathcal{P}).$$

• *Step 2:* Characterize condition (3.C.4).
Such condition clearly amounts to show that

$$\sum_{i,j=1}^{n} p_{ij} m_{ij} = 0,$$

for every $P \in \mathcal{P}$. Taking $P = P_{ij}$ for $2 \leq i < j \leq n$ we get

$$\sum_{i,j=1}^{n} p_{ij} m_{ij} = \frac{1}{2}\left(m_{1i} - m_{i1} + m_{ij} - m_{ji} + m_{j1} - m_{1j}\right) = m_{1i} + m_{ij} + m_{j1},$$

and this concludes the full proof of our result. □

Finally, we focus on the sticking condition (3.3.12) in the critical case. The next result exhibits an explicit characterization that follows similar techniques to those in Lemma 3.B.4.

Lemma 3.C.3. *Consider any $n \in \mathbb{N}$ and any matrix $M \in \operatorname{Skew}_n(\mathbb{R})$. Then, the following assertions are equivalent:*

1. *There exists some $Y \in \operatorname{Skew}_n([-1,1])$ such that*
$$M = Y \cdot \mathbf{J} + \mathbf{J} \cdot Y.$$
2. *There exists some $Y \in \operatorname{Skew}_n(\mathbb{R})$ such that*
$$T_{ij}(Y) \leq m_{ij},\ -T_{ij}(Y) \leq -m_{ij} \ \text{ and } \ L_{ij}(Y) \leq 1.$$
3. *The following inequality*
$$\sum_{i,j=1}^{n} q_{ij} + \frac{1}{2}\sum_{i,j=1}^{n} p_{ij} m_{ij} \geq 0,$$
holds, for any $i,j = 1,\ldots,n$, and for every $P \in \operatorname{Skew}_n(\mathbb{R})$ and $Q \in \mathcal{M}_n(\mathbb{R}_0^+)$ such that $\sum_{k=1}^{n}(p_{ik} - p_{jk}) + q_{ij} - q_{ji} = 0$.
4. *The following two conditions fulfill*
   (a) *Condition* (3.C.2) *holds true.*
   (b) *We have that*
$$\sum_{i=1}^{m}\sum_{j=m+1}^{n} m_{\sigma_i \sigma_j} \in [-nm(n-m), nm(n-m)], \tag{3.C.6}$$
   for every permutation σ of $\{1,\ldots,n\}$ and any $1 \leq m \leq n$.

Proof. The assertions 1 and 2 are apparently equivalent due to the definition of the involved linear operators. Also, both properties 2 and 3 are equivalent by virtue of an application of Lemma 3.B.3 that is analogue to that in the proof of Lemma 3.B.4; hence, we skip the proof for simplicity. Thereby, we will only focus on the equivalence with the former assertion. First, let us assume that for some $Y \in \operatorname{Skew}_n([-1,1])$ the first assertion holds true, i.e.,

$$m_{ij} = \sum_{k=1}^{n}(y_{ik} - y_{jk}).$$

By Lemma 3.C.2 we arrive at (3.C.2). Moreover,

$$\begin{aligned}\sum_{i=1}^{m}\sum_{j=m+1}^{n} m_{\sigma_i\sigma_j} &= \sum_{i=1}^{m}\sum_{j=m+1}^{n}\sum_{k=1}^{n}(y_{\sigma_i\sigma_k}-y_{\sigma_j\sigma_k})\\ &= (n-m)\sum_{i=1}^{m}\sum_{k=m+1}^{n} y_{\sigma_i\sigma_k} - m\sum_{j=m+1}^{n}\sum_{k=1}^{m} y_{jk}\\ &= n\sum_{i=1}^{m}\sum_{k=m+1}^{n} y_{ik}.\end{aligned}$$

Since it is n times the sum of $m(n-m)$ numbers in $[-1,1]$, then the condition (3.C.6) is also satisfied. Conversely, let us assume that both (3.C.2) and (3.C.6) fulfill and take any $P \in \mathrm{Skew}_n(\mathbb{R})$ and $Q \in \mathcal{M}_n(\mathbb{R}_0^+)$ such that

$$\sum_{k=1}^{n}(p_{ik}-p_{jk}) + q_{ij} - q_{ji} = 0, \tag{3.C.7}$$

for any couple of indices $i, j = 1, \ldots, n$. Without loss of generality we can assume that $q_{ii} = 0$, for every $i = 1, \ldots, n$. Also, let us define the coefficients $\lambda_i := \sum_{k=1}^n p_{ik}$ and consider a permutation σ of $\{1, \ldots, n\}$ so that λ_{σ_i} are ordered in a non-decreasing way, i.e.,

$$\lambda_{\sigma_1} \le \lambda_{\sigma_2} \le \cdots \le \lambda_{\sigma_n}. \tag{3.C.8}$$

Let us split

$$I := \sum_{i,j=1}^{n} q_{\sigma_i\sigma_j} + \frac{1}{2}\sum_{i,j=1}^{n} p_{\sigma_i\sigma_j} m_{\sigma_i\sigma_j} =: I_1 + I_2.$$

Using (3.C.2) in the second term we can write

$$I_2 = \frac{1}{2}\sum_{i,j=1}^{n} p_{\sigma_i\sigma_j}(m_{\sigma_i\sigma_k} - m_{\sigma_j\sigma_k}),$$

for any $k = 1, \ldots, n$. Let us take the average with respect to k in the above expression

$$\begin{aligned} I_2 &= \frac{1}{2n}\sum_{i=1}^{n}\left(\sum_{k=1}^{n} m_{\sigma_i\sigma_k}\right)\lambda_{\sigma_i} + \frac{1}{2n}\sum_{j=1}^{n}\left(\sum_{k=1}^{n} m_{\sigma_j\sigma_k}\right)\lambda_{\sigma_j}\\ &= \frac{1}{n}\sum_{i=1}^{n}\left(\sum_{k=1}^{n} m_{\sigma_i\sigma_k}\right)\lambda_{\sigma_i} = \frac{1}{n}\sum_{i=1}^{n}\left(\sum_{k=1}^{n} m_{\sigma_i\sigma_k}\right)(\lambda_{\sigma_j} + q_{\sigma_j\sigma_i} - q_{\sigma_i\sigma_j}),\end{aligned}$$

for any $j = 1 \ldots, n$, where (3.C.7) has been used in the last step. Taking the average with respect to j we get to

$$\begin{aligned} I_2 &= \frac{1}{n^2}\sum_{i,j=1}^{n}\left(\sum_{k=1}^{n} m_{\sigma_i\sigma_k}\right)(q_{\sigma_j\sigma_i} - q_{\sigma_i\sigma_j})\\ &= \frac{1}{2n^2}\sum_{i,j=1}^{n}\left(\sum_{k=1}^{n}(m_{\sigma_i\sigma_k} - m_{\sigma_j\sigma_k})\right)(q_{\sigma_j\sigma_i} - q_{\sigma_i\sigma_j})\\ &= \frac{1}{n^2}\sum_{i<j}\left(\sum_{k=1}^{n}(m_{\sigma_i\sigma_k} - m_{\sigma_j\sigma_k})\right)(q_{\sigma_j\sigma_i} - q_{\sigma_i\sigma_j}).\end{aligned} \tag{3.C.9}$$

On the other hand

$$I_2 = \sum_{j>i} q_{\sigma_i \sigma_j} + \sum_{i<j} (q_{\sigma_i \sigma_j} - q_{\sigma_j \sigma_i}). \tag{3.C.10}$$

Putting (3.C.9)-(3.C.10) together we obtain

$$I = 2\sum_{j>i} q_{ij} + \sum_{i<j} \left(1 - \frac{1}{n^2}\sum_{k=1}^{n}(m_{\sigma_i \sigma_k} - m_{\sigma_j \sigma_k})\right)(q_{\sigma_i \sigma_j} - q_{\sigma_j \sigma_i}).$$

Finally, notice that for every $i < j$, the condition (3.C.7) entails

$$q_{\sigma_i \sigma_j} - q_{\sigma_j \sigma_i} = \sum_{m=i}^{j-1} (q_{\sigma_m \sigma_{m+1}} - q_{\sigma_{m+1} \sigma_m}),$$

and, consequently

$$I = 2\sum_{j>i} q_{ij} + \sum_{k=1}^{n} a_m (q_{\sigma_m \sigma_{m+1}} - q_{\sigma_{m+1} \sigma_m}),$$

where the coefficients read

$$\begin{aligned} a_m &= \sum_{i=1}^{m} \sum_{j=m+1}^{n} \left(1 - \frac{1}{n^2}\sum_{k=1}^{n}(m_{\sigma_i \sigma_k} - m_{\sigma_j \sigma_k})\right) \\ &= m(n-m) - \frac{1}{n}\sum_{i=1}^{m} \sum_{j=m+1}^{n} m_{\sigma_i \sigma_j}. \end{aligned}$$

Here, (3.C.2) has been used again in the last identity. Since a_m are all non-negative by (3.C.6) and λ_{σ_i} are ordered by (3.C.8), we can conclude that $I \geq 0$ and this ends the proof. $\square$

CHAPTER 4

The Kuramoto model with singular couplings: the kinetic equation

Contents

4.1 Introduction

In this chapter, we shall continue the analysis of the Kuramoto model with singular coupling weights that was introduced in Chapter 3 at the agent-based level. Indeed, the above system of N coupled oscillators in (3.3.1)-(3.3.2) was derived from a singular fast learning limit in the Kuramoto model with adaptive coupling weights towards a Hebbian-type plasticity function governing learning of weights. For the sake of easier readability, we recall the system under consideration:

$$\begin{cases} \dot{\theta}_i = \Omega_i + \dfrac{K}{N}\displaystyle\sum_{j=1}^{N} h(\theta_j - \theta_i), \\ \theta_i(0) = \theta_{i,0}. \end{cases} \tag{4.1.1}$$

We can think of such model to describe the evolution of the phases $\theta_i = \theta_i(t)$ of neuron signals located at specific areas in the brain. Hence, $\dot{\theta}_i = \dot{\theta}_i(t)$ represent the firing frequencies of neurons and Ω_i appears as an heterogeneity that play the role of a biased tendency of agents to move at their own frequency while being influenced by their neighbors via the coupling force h. Recall that if $h(\theta) = \sin\theta$, then we recover the classical Kuramoto model. In our study, the proposed periodic coupling force stands for the following non-smooth kernel

$$h(\theta) = \frac{\sin\theta}{|\theta|_o^{2\alpha}},\ \theta \in \mathbb{R}, \tag{4.1.2}$$

where $|\theta|_o$ is the Riemannian distance of $e^{i\theta}$ towards 1 along the unit circle, that is

$$|\theta|_o := d_{\mathbb{T}}(e^{i\theta}, 1) = |\bar{\theta}| \quad \text{for} \quad \bar{\theta} \equiv \theta \bmod 2\pi, \quad \bar{\theta} \in (-\pi, \pi].$$

For further details about the well-posedness and dynamical properties of solutions, see the preceding Chapter 3. For the sake of clarity, we provide here a very brief summary of the main properties of such system that we shall bear in mind in the sequel. Recall that for the parameter α there are three different regimes $\alpha \in (0, \frac{1}{2})$, $\alpha = \frac{1}{2}$ and $\alpha \in (\frac{1}{2}, 1)$ that we will respectively call the *subcritical*, *critical* and *supercritical* cases. In the subcritical case the coupling function is Hölder-continuos, in the critical case it is bounded but discontinuous, while the supercritical regime corresponds to an unbounded singular kernel. Therefore, the Cauchy–Lipschitz theory cannot guarantee existence and uniqueness of global-in-time solutions, then requiring the concept of *Filippov trajectories* [14, 130]. In particular, uniqueness of Filippov solutions was proven to hold forwards-in-time only. Although it might appear a deficiency on the model at first glance, it actually suggests a new relevant dynamics of oscillators: *finite-time sticking* and *clustering into groups*. Namely, after some phases eventually agree in finite time, there is a chance that they keep stuck together for all times or they instantaneously disassociate. The rule governing such behavior only depends upon certain specific conditions on the natural frequencies of the oscillators belonging to the formed cluster. Such sticky behavior suggests that small clusters can emerge in finite time ending up with the eventual global synchronizations of all the oscillators in a unique big group (*finite-time phase synchronization*) under certain initial assumptions. Also recall that such sticky dynamics was also found in related models like

Cucker–Smale model with weakly singular influence function (see [244, 245]) or the aggregation equation with mildly singularity potentials (see [27, 28, 29, 64, 200]).

Closely related to the aforementioned issues, one of the most classical problems in physics and mathematics is to achieve the perfect fit between such individual-based descriptions and the associated coarse-grained continuous versions, where systems are explained in an averaged way though a probabilistic distribution function. In kinetic theory, it is called *mean-field limit* and, when the amount of individuals is large enough, it yields accurate macroscopic descriptions of the system in terms of a kinetic Vlasov-type equation. As it is already known, such mean-field methods for particle systems give rise to plenty of relevant models in statistical physics and fluid mechanics like Boltzmann, Vlasov–Poisson, Euler or Navier–Stokes equations, that have become the motor of important advances in mathematics. In particular, one can try to extend the methods to the new class of non-Newtonian interactions coming from the *active soft matter* community, see [210]. When the coupling force between any two agents is Lipschitz, the mean-field limit can be recovered from standard methods [212, 230]. However, interactions are usually discontinuous or even singular for most of the real systems. In such cases, the mean-field approximation is not obvious and new methods are required, see [51, 134, 163, 158, 164, 176, 177, 178, 179, 181, 216, 217, 230, 281] and the discussion in the introductory Chapter 1.

This chapter is precisely devoted to such subject. Specifically, we shall focus on the rigorous derivation of the Vlasov-type equation with non-smooth interactions that arises as mean-field limit of the above agent-based model (4.1.1)-(4.1.2) as $N \to \infty$

$$\left\{ \begin{array}{l} \dfrac{\partial f}{\partial t} + \dfrac{\partial}{\partial \theta}\left(\Omega f - K(h*\rho) f\right) = 0, \\ f(0,\cdot,\cdot) = f_0. \end{array} \right. \tag{4.1.3}$$

Here, $f = f(t,\theta,\Omega)$ represents the probability distribution of finding an oscillator at time $t \geq 0$, with frequency $\theta \in (-\pi,\pi]$ and natural frequency $\Omega \in \mathbb{R}$, endowed with periodic boundary conditions for the variable θ. Due to the periodicity of phases, such kinetic equation will be identified in a natural way with a nonlinear transport equation along the Riemannian manifold $\mathbb{T}\times\mathbb{R}$. This changes the natural phase-space from the standard Euclidean space to a non-Euclidean ambient space. Specifically (4.1.3) can be restated as follows

$$\left\{ \begin{array}{l} \dfrac{\partial f}{\partial t} + \operatorname{div}_{(z,\Omega)}(\mathcal{V}[f] f) = 0, \\ f(0,\cdot,\cdot) = f_0, \end{array} \right.$$

for some tangent transport field $\mathcal{V}[f]$ to be defined later in Section 4.2. We remark here that we will not use any extra gradient-type structure of the system, as it was the case in previous literature for other models, see e.g. [64, 76, 77, 128, 154, 184]. Contrarily, we will just work with the non-smooth tangent transport field $\mathcal{V}[f]$ with any extra structure. Such point of view requires extending the well known analysis of a non-smooth transport field given in [8, 9] from the classical Euclidean setting to our new phase-space.

Recall that from the dynamic of the agent-based system, we also expect that global phase synchronization at finite time (emergence of Dirac masses) can also take place at the macroscopic scale. Then, we must deal with weak solutions that are merely measure-valued. Our approach will be supported by the methods of *Filippov characteristics* and is inspired in [67, 251]. Specifically, we shall try to give some sense to the Filippov flow of this non-smooth tangent transport field. To such end, we will study conditions on $\mathcal{V}[f]$ in the different regimes of α that allows constructing the associated characteristic flow globally-in-time. The uniqueness of the flow is again one-sided and will be guaranteed by the internal structure of the kernel h at points of loss of Lipschitz-continuity. In particular, we shall show that although non-smooth, the tangent vector field $\mathcal{V}[f]$ is one-sided Lipschitz-continuous in a rigorous sense to be specified later. Then, the associated characteristic system

$$\left\{\begin{array}{l}\frac{dX}{dt}(t;t_0,x_0)\in\mathcal{K}[\mathcal{V}[f_t]](X(t;t_0,x_0)),\\ X(t_0;t_0,x_0)=x_0,\end{array}\right.$$

enjoys global solutions in Filippov's sense that are unique forwards-in-time [14, 130] (recall the ideas in the above Chapter 3). Here $\mathcal{K}[\mathcal{V}[f]]$ stands for the Filippov set-valued tangent field associated with $\mathcal{V}[f]$ to be defined later in Section 4.6 as an extension of the classical Filippov set-valued map in Appendix D in the Euclidean space. This will be the first result, that becomes the cornerstone to construct measure-valued solutions in the subcritical and critical cases.

Our second result is the rigorous derivation of the mean field limit. Specifically, consider empirical measures supported on Filippov solutions to the discrete system (4.1.1)-(4.1.2) of N oscillators, that is,

$$\mu_t^N:=\frac{1}{N}\sum_{i=1}^N\delta_{z_i^N(t)}(z)\otimes\delta_{\Omega_i^N}(\Omega),$$

where $z_i^N(t):=e^{i\theta_i^N(t)}$. Then under appropriate assumptions coming from the law of large numbers, we shall show that $\mu^N\rightarrow f$ in $C([0,T],\mathcal{P}_1(\mathbb{T}\times\mathbb{R})-W_1)$ and the limiting f solves (4.1.3) in the *sense of Filippov flow*

$$X_f(t;0,\cdot)_{\#}f_0=f_t,\ t\geq 0,$$

where $X_f=X_f(t;0,z,\Omega)$ is the Filippov flow associated with $\mathcal{V}[f]$. Here, W_1 means the Rubinstein–Kantorovich distance on $\mathbb{T}\times\mathbb{R}$ and $\mathcal{P}_1$ stands for the probability measures with finite first order moment, see Appendix F for a brief summary of the main necessary tools inspired in optimal transport theory.

Indeed, this will become a special consequence of our third result, namely, stability with respect to initial data in Wasserstein-type distances. Specifically, we will introduce some *Dobrushin*-type estimate (see [112]) for any two measure-valued solutions $f^1=f_t^1(\theta,\Omega)$ and $f^2=f_t^2(\theta,\Omega)$ to (4.1.3) with respect to two different metrics on the set of probability measures: the standard 2-Wasserstein distance W_2 and a new version $W_{2,g}$, specially designed for this problem, that will be called *fiberwise Wasserstein distance*:

$$\begin{array}{ll}W_2(f_t^1,f_t^2)\leq e^{(\frac{1}{2}+2KL_0)t}W_2(f_0^1,f_0^2), & t\geq 0,\\ W_{2,g}(f_t^1,f_t^2)\leq e^{2KL_0t}W_{2,g}(f_0^1,f_0^2), & t\geq 0,\end{array}$$

for some constant L_0 to appear later. The latter one only works for solutions with the same distribution g of natural frequencies. In particular, it allows recovering uniqueness results for any general initial data. Our forth result is the study of asymptotic behavior of solutions. Namely, we will show that under appropriate conditions, the macroscopic system (4.1.3) enjoys finite-time global phase synchronization.

Unfortunately, notice that there is no way to extend any (generalized) characteristic method to the most singular regime $\alpha\in(\frac{1}{2},1)$ because $\mathcal{V}[f]$ lacks of sense. To cover such supercritical case, we shall develop an alternative method that is valid for all $\alpha\in(0,1)$ (at least for identical oscillators $g=\delta_0$). We shall augment the first order singular Kuramoto system into an auxiliary second order regularized system with inertia, frequency damping and diffusion, see

[76, 77, 78, 79]. Under an appropriate scaling depending on a parameter $\varepsilon \searrow 0$, the inertia and noise terms will vanish and singularity of the coupling function will be recovered. This defines a singular hyperbolic hydrodynamic limit of vanishing inertia type like in [142, 232] that resembles the one that we have proposed in Chapter 2 for the Cucker–Smale model with singular influence function. For the application of similar methods in other models both with regular and singular interaction kernels, see [120, 121, 125, 126]. Via a compactness method, the sequence of augmented regularized distribution functions $P^\varepsilon = P^\varepsilon(t,\theta,\omega,\Omega)$ can be shown to have bounded zeroth and first order frequency moments that weakly converge to a weak measure-valued solution of the macroscopic singular system. Like in Chapter 2, the main point here is an accurate a priori control on the hierarchy of frequency moments of the second order regularized kinetic description, that in particular, includes time-equicontinuity for the sequence of reduced regularized distribution functions $f^\varepsilon = f^\varepsilon(t,\theta,\Omega)$ in some negative Sobolev space, specifically

$$\|f^\varepsilon_{t_1} - f^\varepsilon_{t_2}\|_{W^{-1,1}(\mathbb{T}\times\mathbb{R})} \leq C|t_1 - t_2|^{1/2}.$$

Although in this chapter we will just illustrate the techniques in the particular setting of the singular Kuramoto model, we believe that most of the tools can be adapted to other type systems lacking gradients structure, whose internal variables belong, in a natural way, to more general Lie groups. In addition, interactions do not need to be necessarily smooth neither, but at least enjoy similar one-sided Lipschitz properties or sided Osgood-type moduli of continuity.

The remaining sections of this chapter are structured as follows. Sections 4.2, 4.3, 4.4 and 4.5 are devoted to the subcritical regime $\alpha \in (0, \frac{1}{2})$. Specifically, in Section 4.2 we will introduce the model (4.1.3) as a nonlinear transport equation along $\mathbb{T}\times\mathbb{R}$, some regularity properties of the tangent transport field $\mathcal{V}[f]$ will be derived and we will also revisit the main concepts of measure-valued solutions in the literature. In Section 4.3 we shall prove the existence of global-in-time weak measure-valued solutions to (4.1.3). The above-mentioned Dobrushin-type estimates will be discussed in Section 4.4, in particular their applications to obtain uniqueness results of weak measure-valued solutions and the mean-field limit from the particle system (4.1.1)-(4.1.2) towards (4.1.3). Finally, we will prove emergence of global phase synchronization in finite time for this subcritical regime under appropriate initial assumptions. Section 4.6 will focus on the critical case $\alpha = \frac{1}{2}$. Specifically, we will adapt the above methods to obtain measure-valued solutions in Filippov's sense and we will provide analogue results regarding stability, uniqueness, mean field limit and emergence of synchronization. In Section 4.7 we will explore the aforementioned singular hyperbolic limit of vanishing inertia type for the supercritical regime $\alpha \in (\frac{1}{2}, 1)$. The appendices are devoted to some technical observations that we will use throughout the chapter. In Appendix 4.A we will recall some notation about measures along $\mathbb{T}$ or periodic measures. Finally, Appendix 4.B recalls the differentiability properties of the squared Riemannian distance in a complete Riemannian manifold. In particular, we will introduce the concept of *one-sided upper Dini differentiability* that we will sometimes use to deal with quadratic Wasserstein distances along manifolds.

4.2 Tangent transport field along $\mathbb{T}\times\mathbb{R}$ and measure-valued solutions

In this section we will introduce some tools and notation for the model that we will use along the chapter with regards to the subcritical regime of the singularity $\alpha \in (0, \frac{1}{2})$. On the one hand, we will reformulate the macroscopic system as a genuine transport equation along a manifold and will focus on deriving properties of the tangent transport field. That will be the cornerstone to derive the well-posedness of global-in-time measure-valued solutions to the system with $\alpha\in(0,\frac{1}{2})$ in the forthcoming Sections 4.3 and 4.4. On the other hand, we will revisit different equivalent concepts of measure-valued solutions ranging from *weak measure-valued solutions* to *superposition solutions* and *solutions in the sense of the characteristic flow*. Although the regularity of the tangent transport field will vary (or even may not make sense at all) in the critical and supercritical regimes, we will later try to adapt this ideas to those more singular cases in Sections 4.6 and 4.7.

4.2.1 Formal derivation of the Vlasov equation

In this chapter we will not address the problem of propagation of chaos. For that issue, the reader may be interested in the following related literature [163, 164, 176, 177, 178, 181, 216, 217, 281]. Nevertheless, let us recall a formal derivation of (4.1.3) that may arise form the study of propagation of chaos in the system, also see the introductory Chapter 1. Specifically, since the natural frequencies in the discrete model (4.1.1)-(4.1.2) are constant parameters, we can equivalently state the system as follows

$$\begin{cases}\dot{\theta}_i=\Omega_i+\dfrac{K}{N}\displaystyle\sum_{j=1}^N h(\theta_j-\theta_i),\\ \dot{\Omega}_i=0,\\ \theta_i(0)=\theta_{i,0},\ \Omega_i(0)=\Omega_{i,0}\equiv\Omega_i.\end{cases}$$

Then, not only θ_i is regarded as a mechanical variable but also Ω_i is. Consequently, the joint laws $F_t^N=F_t^N(x_1,\ldots,x_N,\Omega_1,\ldots,\Omega_N)\in\mathcal{P}_{sym}(\mathbb{R}^N\times\mathbb{R}^N)$ obey the BBGKY hierarchy of Liouville equations

$$\frac{\partial F^N}{\partial t}+\sum_{i=1}^N\frac{\partial}{\partial\theta_i}\left(\left(\Omega_i+\frac{K}{N}\sum_{j=1}^N h(\theta_j-\theta_i)\right)F^N\right)=0. \tag{4.2.1}$$

Let us consider the projection map onto the first $k\in\{1,\ldots,N\}$ variables, that is,

$$\begin{array}{rccc}\pi^{k,N}: & \mathbb{R}^N\times\mathbb{R}^N & \longrightarrow & \mathbb{R}^k\times\mathbb{R}^k,\\ & (\Theta^N,\Omega^N) & \longmapsto & (\Theta^{k,N},\Omega^{k,N}),\end{array}$$

where we denote $\Theta^{k,N}:=(\theta_1,\ldots,\theta_k)$ and $\Omega^{k,N}:=(\Omega_1,\ldots,\Omega_k)$, for any $\Theta^N=(\theta_1,\ldots,\theta_N)\in\mathbb{R}^N$ and $\Omega^N=(\Omega_1,\ldots,\Omega_N)\in\mathbb{R}^N$. Then, we can consider the marginal measures $F_t^{k,N}:=\pi_{\#}^{k,N}(F_t^N)\in\mathcal{P}_{sym}(\mathbb{R}^k\times\mathbb{R}^k)$. Thanks to the assumed symmetry in the system, integration in (4.2.1) yields

$$\begin{split}\frac{\partial F^{k,N}}{\partial t}+\sum_{i=1}^k\frac{\partial}{\partial\theta_i}&\left(\left(\Omega_i+\frac{K}{N}\sum_{j=1}^k h(\theta_j-\theta_i)\right)F^{k,N}\right.\\ &\left.+K\frac{N-k}{N}\int_{\mathbb{R}\times\mathbb{R}}h(\theta_{k+1}-\theta_i)\,d_{(\theta_{k+1},\Omega_{k+1})}F^{k+1,N}\right)=0.\end{split} \tag{4.2.2}$$

Observe that the hierarchy is not necessarily closed yet. Via a diagonal argument we can obtain weak limits of an appropriate subsequence (that we denote in the same manner)

$$F^{k,\infty}:=\text{weak}*-\lim_{N\to\infty}F^{k,N}.$$

For simplicity of the notation, let us denote $F := F^{1,\infty}$. Let us assume that all the initial values are tensorized, that is $F_0^{k,\infty} = F_0^{\otimes^k}$. Then, *propagation of chaos* means that such tensorization remains true for all times, i.e.,

$$F_t^{k,\infty} = F_t^{\otimes^k}, \quad \text{for all } t \geq 0.$$

Conditionally under such property, we can pass to the limit as $N \to \infty$ in (4.2.2) for $F^{1,N}$ and close an equation for F as follows

$$\frac{\partial F}{\partial t} + \frac{\partial}{\partial \theta}\left(\left(\Omega + K\int_{\mathbb{R}^2} h(\theta' - \theta)\, d_{(\theta',\Omega')}F\right) F\right) = 0.$$

This is the classical Vlasov equation of the system. Finally, since the phases $\theta \in \mathbb{R}$ only make sense modulo 2π, we can consider the following map projecting each $\theta \in \mathbb{R}$ into its representative $\bar{\theta} \in (-\pi, \pi]$ modulo 2π

$$\begin{array}{rccc} \bar{\pi}: & \mathbb{R} & \longrightarrow & (-\pi,\pi], \\ & \theta & \longmapsto & \bar{\theta}. \end{array}$$

It clearly maps F_t into $f_t \in \mathcal{P}((-\pi,\pi] \times \mathbb{R})$ through $\bar{\pi}_{\#}F_t = f_t$, and $f = f_t$ obviously fulfils (4.1.3) in distributional sense. As previously stated, we are not interested in the propagation of chaos topic, but rather on the mean field limit via the empirical measure technique that will be described throughout the chapter.

4.2.2 Reformulating a nonlinear transport equation along a manifold

In the above part, the kinetic singular Kuramoto model (4.1.3) was formally derived. Solutions are regarded as periodic measures $f_t \in \mathcal{P}((-\pi,\pi] \times \mathbb{R})$ with respect to θ. It is clear that we can equivalently regard them as measures $f_t \in \mathcal{P}(\mathbb{T} \times \mathbb{R})$ by virtue of the identification of the interval $(-\pi,\pi]$ with the torus $\mathbb{T}$, see Theorem 4.A.10 in Appendix 4.A. Here on, we will think of both spaces as the same space and will change notation from one to another without any notice for simplicity of arguments. Before going further in our results, we will introduce some basic properties and notation with regards to the Riemannian manifold $\mathbb{T} \times \mathbb{R}$. In the first result we comment on the structure of the Riemannian distance, its tangent space and tangent vector fields.

Lemma 4.2.1. *Consider the Riemannian manifold $\mathbb{T} \times \mathbb{R}$ endowed with the standard metric. Then,*

1. *The Riemannian distance in $\mathbb{T} \times \mathbb{R}$ between any couple $(z_1,\Omega_1),(z_2,\Omega_2) \in \mathbb{T} \times \mathbb{R}$ is defined by*

$$d((z_1,\Omega_1),(z_2,\Omega_2)) = \left(d(z_1,z_2)^2 + (\Omega_1 - \Omega_2)^2\right)^{1/2} = \left(|\theta_1 - \theta_2|_o^2 + (\Omega_1 - \Omega_2)^2\right)^{1/2},$$

for any $\theta_1, \theta_2 \in \mathbb{R}$ such that $z_1 = e^{i\theta_1}$ and $z_2 = e^{i\theta_2}$. Here, $|\cdot|_o$ means the orthodromic distance in the unit torus, that is $|\theta|_o = |\bar{\theta}|$, where $\bar{\theta}$ is the representative modulo 2π of θ in $(-\pi,\pi]$.

2. *The tangent space at $(z,\Omega) \in \mathbb{T} \times \mathbb{R}$ reads*

$$T_{(z,\Omega)}(\mathbb{T} \times \mathbb{R}) = T_z\mathbb{T} \times T_\Omega\mathbb{R} = \{(p\, iz, q) : p, q \in \mathbb{R}\}.$$

As a consequence, the space $\mathfrak{X}\mathcal{C}(\mathbb{T} \times \mathbb{R})$ of $\mathcal{C}$-regular tangent vectors along $\mathbb{T} \times \mathbb{R}$, for any given regularity class $\mathcal{C}$, consist of the fields V with components

$$V_{(z,\Omega)} = (P(z,\Omega)\, iz, Q(z,\Omega)), \tag{4.2.3}$$

for a couple of scalar functions $P, Q : \mathbb{T} \times \mathbb{R} \longrightarrow \mathbb{R}$ with regularity in the class $\mathcal{C}$.

3. *Consider* $V \in \mathfrak{X}C^1(\mathbb{T} \times \mathbb{R})$ *given by* (4.2.3) *for some* $P, Q \in C^1(\mathbb{T} \times \mathbb{R})$. *Then,*

$$\operatorname{div} V = \frac{\partial P}{\partial \theta} + \frac{\partial Q}{\partial \Omega} \equiv iz\frac{\partial P}{\partial z} + \frac{\partial Q}{\partial \Omega}, \tag{4.2.4}$$

where we have used the identifications in the Appendix 4.A and the notation of the complex derivatives in Definition 4.A.2. The same formula (4.2.4) *holds true for distributional derivatives.*

In the second lemma, we introduce the geodesics and parallel transport in $\mathbb{T} \times \mathbb{R}$ that will be of interest in some upcoming results.

Lemma 4.2.2. *Let us set any point* $x = (z, \Omega) \in \mathbb{T} \times \mathbb{R}$, *where* $z = e^{i\theta}$ *for some* $\theta \in \mathbb{R}$, *and any tangent vector* $v = (p\, iz, q) \in T_{(z,\Omega)}\mathbb{T} \times \mathbb{R}$.

1. *Define the geodesic* $\gamma_{z,p}$ *of* $\mathbb{T}$ *issued at* z *in the direction* $p\, iz$, *i.e.,*

$$\gamma_{z,p}(s) := e^{i(\theta + ps)}.$$

Then, the geodesic $\widehat{\gamma}_{x,v}$ *of* $\mathbb{T} \times \mathbb{R}$ *issued at* x *in the direction* v *reads*

$$\widehat{\gamma}_{x,v}(s) = (\gamma_{z,p}(s), \Omega + qs) = (e^{i(\theta+ps)}, \Omega + qs).$$

In particular, the Riemannian exponential map reads

$$\widehat{\exp}_x(v) = (\exp_z(p\, iz), \Omega + q) = (e^{i(\theta+p)}, \Omega + q).$$

2. *Let* $\widehat{\gamma}_{x,v}$ *be the associated geodesic, and set* $s_1, s_2 \in \mathbb{R}$. *Then, the parallel transport from* $\widehat{\gamma}_{x,v}(s_1)$ *to* $\widehat{\gamma}_{x,v}(s_2)$ *along* $\widehat{\gamma}_{x,v}$ *is the linear isometry (see [111])*

$$\begin{array}{rccc} \tau[\widehat{\gamma}_{x,v}]_{s_1}^{s_2}: & T_{\widehat{\gamma}_{x,v}(s_1)}(\mathbb{T} \times \mathbb{R}) & \longrightarrow & T_{\widehat{\gamma}_{x,v}(s_2)}(\mathbb{T} \times \mathbb{R}), \\ & (p' i\gamma_{z,p}(s_1), q') & \longmapsto & (p' i\gamma_{z,p}(s_2), q'). \end{array}$$

Since the proofs are standard, we omit them.

Definition 4.2.3. *Consider* $\alpha \in (0, \frac{1}{2})$ *and* $K > 0$. *We (formally) define the function* $\mathcal{P}[\mu]$ *and the tangent vector field* $\mathcal{V}[\mu]$ *along the manifold* $\mathbb{T} \times \mathbb{R}$ *by*

$$\begin{aligned} \mathcal{P}[\mu](\theta, \Omega) &:= \Omega - K \int_{\mathbb{T}\times\mathbb{R}} h(\theta - \theta')\, d_{(\theta',\Omega')}\mu, \\ \mathcal{V}[\mu](z, \Omega) &:= (\mathcal{P}[\mu](z, \Omega)\, iz, 0), \end{aligned}$$

where $\mu \in \mathcal{M}(\mathbb{T} \times \mathbb{R})$ *is any finite Radon measure.*

Using such notation and (4.2.4) one can easily check that the kinetic singular Kuramoto model (4.1.3) can be restated as the following nonlinear transport equation in conservative form along the manifold $\mathbb{T} \times \mathbb{R}$:

$$\left\{ \begin{array}{l} \dfrac{\partial f}{\partial t} + \operatorname{div}_{(z,\Omega)}(\mathcal{V}[f]f) = 0, \\ f(0, \cdot, \cdot) = f^0. \end{array} \right. \tag{4.2.5}$$

Of course, now there is no need to impose explicit periodicity conditions because they are actually considered implicitly in the geometry of the space $\mathbb{T} \times \mathbb{R}$.

4.2.3 Properties of the transport field

Here on, most of our effort will be devoted to derive some one-sided modulus of continuity of $\mathcal{V}[f]$. In particular, such (weak) regularity will entail that the characteristic flow associated with $\mathcal{V}[f]$ is well–defined forwards-in-time, for any weak measure-valued solution to (4.4.11). This will be the cornerstone to show well-posedness of global-in-time weak measure valued solutions to (4.2.5) in Sections 4.3 and 4.4. Before moving to such regularity issues of the transport field $\mathcal{V}[f]$, let us set the appropriate spaces of time-dependent measures.

Definition 4.2.4. *Fix $T > 0$, then we will consider*

$$\begin{aligned}
\mathcal{C}_{\mathcal{M}}(0,T) &:= \left\{\mu \in L^\infty_w(0,T;\mathcal{M}(\mathbb{T}\times\mathbb{R})) : \, t \longmapsto \int_{\mathbb{T}\times\mathbb{R}} \varphi\, d\mu_t \in C([0,T]),\ \forall\, \varphi \in C_c(\mathbb{T}\times\mathbb{R})\right\},\\
\widetilde{\mathcal{C}}_{\mathcal{M}}(0,T) &:= \left\{\mu \in L^\infty_w(0,T;\mathcal{M}(\mathbb{T}\times\mathbb{R})) : \, t \longmapsto \int_{\mathbb{T}\times\mathbb{R}} \varphi\, d\mu_t \in C([0,T]),\ \forall\, \varphi \in C_b(\mathbb{T}\times\mathbb{R})\right\},\\
\mathcal{T}_{\mathcal{M}}(0,T) &:= \left\{\mu \in L^\infty_w(0,T;\mathcal{M}(\mathbb{T}\times\mathbb{R})) : \ \textit{the family}\ \ (\mu_t)_{t\in[0,T]}\ \ \textit{is uniformly tight}\right\},\\
\mathcal{AC}_{\mathcal{M}}(0,T) &:= \left\{\mu \in L^\infty_w(0,T;\mathcal{M}(\mathbb{T}\times\mathbb{R})) : \, t \longmapsto \int_{\mathbb{T}\times\mathbb{R}} \varphi\, d\mu_t \in AC(0,T),\ \forall\, \varphi \in C^\infty_c(\mathbb{T}\times\mathbb{R})\right\},
\end{aligned}$$

where the above L^∞_w denotes the weak- Lebesgue-Bochner space, see Appendix A. For simplicity of the notation, we will sometimes remove the dependence on T when it is clear.*

Some properties about the preceding spaces of measures are in order:

Proposition 4.2.5. *For the spaces in Definition 4.2.4, the following properties hold true:*

1. *The spaces $\mathcal{C}_{\mathcal{M}}$ and $\widetilde{\mathcal{C}}_{\mathcal{M}}$ can be represented as follows*

$$\begin{aligned}
\mathcal{C}_{\mathcal{M}} &= C([0,T],\mathcal{M}(\mathbb{T}\times\mathbb{R}) - \textit{weak}\,*),\\
\widetilde{\mathcal{C}}_{\mathcal{M}} &= C([0,T],\mathcal{M}(\mathbb{T}\times\mathbb{R}) - \textit{narrow}).
\end{aligned}$$

2. *In the above definition of $\mathcal{C}_{\mathcal{M}}$, test functions $\varphi \in C_c(\mathbb{T}\times\mathbb{R})$ can be replaced by any intermediate regularity class being dense in $C_0(\mathbb{T}\times\mathbb{R})$, e.g.,*

$$C^\infty_c(\mathbb{T}\times\mathbb{R}),\ C^k_c(\mathbb{T}\times\mathbb{R})\ \textit{and}\ \ W^{k,p}(\mathbb{T}\times\mathbb{R}),\ \ \textit{with}\ k\in\mathbb{N},\ p>2.$$

3. *The space $\widetilde{\mathcal{C}}_{\mathcal{M}}$ can be represented as follows*

$$\widetilde{\mathcal{C}}_{\mathcal{M}} = \mathcal{C}_{\mathcal{M}} \cap \mathcal{T}_{\mathcal{M}}.$$

4. *The next embeddings take place*

$$\mathcal{AC}_{\mathcal{M}} \subseteq \mathcal{C}_{\mathcal{M}}\ \ \textit{and}\ \ \mathcal{AC}_{\mathcal{M}} \cap \mathcal{T}_{\mathcal{M}} \subseteq \widetilde{\mathcal{C}}_{\mathcal{M}}.$$

5. *The next embedding takes place for any $1 \le p,q \le \infty$*

$$L^\infty_w(0,T;\mathcal{M}(\mathbb{T}\times\mathbb{R})) \cap W^{1,q}_w(0,T;W^{-1,p'}(\mathbb{T}\times\mathbb{R})) \subseteq \mathcal{AC}_{\mathcal{M}},$$

   where L^∞_w and $W^{1,p}_w$ denote the weak- Lebesgue and Sobolev Bochner-type spaces in Appendix A.*

Proof. The first assertion is clear and the second one is a straightforward density argument of the set of smooth and compactly supported functions $C_c^\infty(\mathbb{T}\times\mathbb{R})$ in $C_0(\mathbb{T}\times\mathbb{R})$. The third item is nothing but *Prokhorov's compactness theorem*, whilst the forth is clear by definition. The last claim is clear as Sobolev regularity in one dimensions implies absolute continuity. □

Let us now recall the concept of weak measure-valued solution to (4.2.5).

Definition 4.2.6. *We will say that $f\in\mathcal{C}_{\mathcal{M}}$ is a weak measure-valued solution to* (4.2.5) *when*

$$\int_0^T\int_{\mathbb{T}\times\mathbb{R}}\frac{\partial\varphi}{\partial t}\,d_{(z,\Omega)}f_t\,dt+\int_0^T\int_{\mathbb{T}\times\mathbb{R}}\left\langle\mathcal{V}[f_t],\nabla_{(z,\Omega)}\varphi\right\rangle\,d_{(z,\Omega)}f_t\,dt=-\int_{\mathbb{T}\times\mathbb{R}}\varphi(0,z,\Omega)d_{(z,\Omega)}f^0,$$

for every $\varphi\in C_c^1([0,T)\times\mathbb{T}\times\mathbb{R})$.

Notice that the nonlinear term in Definition 4.2.6 makes sense for any $f\in\mathcal{C}_{\mathcal{M}}$, no matter whether f also belongs to $\widetilde{C}_{\mathcal{M}}$ or $\mathcal{AC}_{\mathcal{M}}$. However, as it is the case for many other models, solutions end up being more regular in time than simply $\mathcal{C}_{\mathcal{M}}$. In such case, we can restate the above weak formulation for $f\in\mathcal{AC}_{\mathcal{M}}$ as follows.

Proposition 4.2.7. *Consider $\alpha\in(0,1)$, $K>0$ and fix $f\in\mathcal{AC}_{\mathcal{M}}$. Then, the following two statement are equivalent:*

1. *f is a weak measure-valued solution to* (4.2.5) *in the sense of Definition 4.2.6.*
2. *The following identity holds*

$$\frac{d}{dt}\int_{\mathbb{T}\times\mathbb{R}}\phi\,d_{(z,\Omega)}f_t=\int_{\mathbb{T}\times\mathbb{R}}\left\langle\mathcal{V}[f_t],\nabla_{(z,\Omega)}\phi\right\rangle\,d_{(z,\Omega)}f_t,\tag{4.2.6}$$

   for a.e. $t\in[0,T]$ and $f(0,\cdot,\cdot)=f^0$, for every $\phi\in C_c^\infty(\mathbb{T}\times\mathbb{R})$.

Proof. First, let us assume that f solves (4.2.5) in the sense of Definition 4.2.6. Take any $\phi\in C_c^\infty(\mathbb{T}\times\mathbb{R})$, consider $\eta\in C_c^\infty(0,T)$ and define the test function

$$\varphi(t,z,\Omega):=\eta(t)\phi(z,\Omega).$$

Since $\varphi\in C_c^\infty((0,T)\times\mathbb{T}\times\mathbb{R})$, then Definition 4.2.6 entails

$$\int_0^T\eta'(t)\left(\int_{\mathbb{T}\times\mathbb{R}}\phi\,d_{(z,\Omega)}f_t\right)\,dt+\int_0^T\eta(t)\left(\int_{\mathbb{T}\times\mathbb{R}}\left\langle\mathcal{V}[f_t],\nabla_{(z,\Omega)}\phi\right\rangle\,d_{(z,\Omega)}f_t\right)\,dt=0.$$

By the arbitrariness of η, we can identify the weak derivative of $t\mapsto\int_{\mathbb{T}\times\mathbb{R}}\phi\,d_{(z,\Omega)}f_t$ (that exists since $f\in\mathcal{AC}_{\mathcal{M}}$) as the right-hand side in (4.2.6) and it concludes the first part. Conversely, let us assume that f is a solution in the sense of Equation (4.2.6) and consider a test function $\varphi\in C_c^\infty([0,T)\times\mathbb{T}\times\mathbb{R})$. By density, we can assume that it has separate variables, i.e.,

$$\varphi(t,z,\Omega)=\eta(t)\phi(z,\Omega),$$

for $\eta\in C_c^\infty([0,T))$ and $\phi\in C_c^\infty(\mathbb{T}\times\mathbb{R})$. Consider any non-increasing cut-off functions $\xi_\varepsilon\in C_c^\infty([0,\infty))$ like in (N.2) for any $\varepsilon>0$ and consider the smooth approximate test functions

$$\eta_\varepsilon(t):=(1-\xi_\varepsilon(t))\eta(t).$$

It is clear that $\eta_\varepsilon \in C_c^\infty(0,T)$ and, consequently, witting (4.2.6) in weak form amounts to

$$\int_0^T \eta_\varepsilon'(t)\left(\int_{\mathbb{T}\times\mathbb{R}} \phi\, d_{(z,\Omega)} f_t\right) dt + \int_0^T \eta_\varepsilon(t)\left(\int_{\mathbb{T}\times\mathbb{R}} \left\langle \mathcal{V}[f_t], \nabla_{(z,\Omega)}\phi\right\rangle\, d_{(z,\Omega)} f_t\right) dt = 0,$$

i.e., expanding the derivatives of η_ε,

$$\begin{aligned}\int_0^T (1-\xi_\varepsilon(t))\eta'(t)\left(\int_{\mathbb{T}\times\mathbb{R}} \phi\, d_{(z,\Omega)} f_t\right) dt \\ + \int_0^T (1-\xi_\varepsilon(t))\eta(t)\left(\int_{\mathbb{T}\times\mathbb{R}} \left\langle \mathcal{V}[f_t], \nabla_{(z,\Omega)}\phi\right\rangle\, d_{(z,\Omega)} f_t\right) dt \\ = \int_0^T \xi_\varepsilon'(t)\eta(t)\left(\int_{\mathbb{T}\times\mathbb{R}} \phi\, d_{(z,\Omega)} f_t\right) dt.\end{aligned}$$

Notice that $1-\xi_\varepsilon \to 1$ in $C([0,T])$ and $\xi_\varepsilon' \overset{*}{\rightharpoonup} -\delta_0$ in $\mathcal{M}([0,T))$ as $\varepsilon \to 0$. Then, taking limits as $\varepsilon \to 0$ in the above identities yields to the weak formulation in Definition 4.2.6. ☐

Notice that for general $f \in \mathcal{C}_\mathcal{M}$ (also for $f \in \widetilde{\mathcal{C}}_\mathcal{M}$) we do not expect $\mathcal{V}[f]$ to be fully Lipschitz-continuous since h was proved to be barely Hölder-continuous, see Lemma 3.3.1 in Chapter 3. Obviously, this causes severe problems with regards to the standard theory. Before introducing sharper regularity properties for the tangent transport field, let us comment on the basic properties that we can infer from such uniform continuity of the kernel h.

Theorem 4.2.8. *Consider $\alpha \in (0, \frac{1}{2})$, $K > 0$ and set $\mu \in \widetilde{\mathcal{C}}_\mathcal{M}$. Then,*

$$\frac{\mathcal{P}[\mu]}{1+|\Omega|} \in C_b([0,T]\times\mathbb{T}\times\mathbb{R}).$$

In addition, there exists $C > 0$, that does not depend on μ, such that

$$|\mathcal{P}[\mu_t](z_1,\Omega_1) - \mathcal{P}[\mu_t](z_2,\Omega_2)| \leq |\Omega_1 - \Omega_2| + CK\|\mu\|_{L^\infty(0,T;\mathcal{M}(\mathbb{T}\times\mathbb{R}))} d(z_1,z_2)^{1-2\alpha},$$

for every $t \in [0,T]$ and $(z_1,\Omega_1),(z_2,\Omega_2) \in \mathbb{T}\times\mathbb{R}$.

Proof. First, let us show the second property. Fix $t \in [0,T]$ and $(z_1,\Omega_1),(z_2,\Omega_2) \in \mathbb{T}\times\mathbb{R}$ and notice that

$$\mathcal{P}[\mu_t](z_1,\Omega_1) - \mathcal{P}[\mu_t](z_2,\Omega_2) = \Omega_1 - \Omega_2 - K\int_{\mathbb{T}\times\mathbb{R}} (h(z_1\overline{z'}) - h(z_2\overline{z'}))d_{(z',\Omega')}\mu_t.$$

Then, the triangle inequality together with Lemma 3.3.1 in the preceding Chapter 3 imply

$$|\mathcal{P}[\mu_t](z_1,\Omega_1) - \mathcal{P}[\mu_t](z_2,\Omega_2)| \leq |\Omega_1 - \Omega_2| + K\cosh\pi \int_{\mathbb{T}\times\mathbb{R}} d(z_1\overline{z'}, z_2\overline{z'})^{1-2\alpha} d_{(z',\Omega')}\mu_T.$$

Since the Riemannian distance along the torus $\mathbb{T}$ is translation invariant, then $d(z_1\overline{z'}, z_2\overline{z'}) = d(z_1,z_2)$, for every $z' \in \mathbb{T}$, thus yielding

$$|\mathcal{P}[\mu_t](z_1,\Omega_1) - \mathcal{P}[\mu_t](z_2,\Omega_2)| \leq |\Omega_1 - \Omega_2| + K\cosh\pi\|\mu\|_{L^\infty_w(0,T;\mathcal{M}(\mathbb{T}\times\mathbb{R}))} d(z_1,z_2)^{1-2\alpha}.$$

Second, let us prove the full continuity in all the variables. Consider $t \in [0,T]$ and $(z,\Omega) \in \mathbb{T}\times\mathbb{R}$ ant let $\{t_n\}_{n\in\mathbb{N}} \subseteq [0,T]$ and $\{(z_n,\Omega_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{T}\times\mathbb{R}$ such that $t_n \to t$ and $(z_n,\Omega_n) \to (z,\Omega)$. We can split

$$\mathcal{V}[\mu_{t_n}](z_n,\Omega_n) - \mathcal{V}[\mu_t](z,\Omega) = A_n + B_n,$$

where each term reads

$$A_n := \mathcal{V}[\mu_{t_n}](z_n, \Omega_n) - \mathcal{V}[\mu_{t_n}](z, \Omega),$$
$$B_n := \mathcal{V}[\mu_{t_n}](z, \Omega) - \mathcal{V}[\mu_t](z, \Omega).$$

Regarding the first term, the preceding part yields

$$A_n \leq |\Omega_n - \Omega| + CK\|\mu\|_{L^\infty_w(0,T;\mathcal{M}(\mathbb{T}\times\mathbb{R}))} d(z_n, z)^{1-2\alpha},$$

where the convergence $A_n \to 0$ is clear. On the other hand, the second term reads

$$B_n = -K \int_{\mathbb{T}\times\mathbb{R}} h(z\overline{z'})\, d_{(z',\Omega')}(\mu_{t_n} - \mu_t).$$

Now, the convergence $B_n \to 0$ follows from the definition of $\widetilde{\mathcal{C}}_{\mathcal{M}}$ in Definition 4.2.4 and the boundedness and continuity of the map

$$(z', \Omega') \in \mathbb{T} \times \mathbb{R} \longmapsto h(z\overline{z'}).$$

This amounts to the desired continuity property. Regarding the growth estimate notice that

$$\sup_{(z,\Omega)\in\mathbb{T}\times\mathbb{R}} \frac{|\mathcal{P}[\mu_t](z,\Omega)|}{1+|\Omega|} \leq \sup_{(z,\Omega)\in\mathbb{T}\times\mathbb{R}} \frac{|\Omega| + K\|\mu\|_{L^\infty_w(0,T;\mathcal{M}(\mathbb{T}\times\mathbb{R}))}\|h\|_{C(\mathbb{T})}}{1+|\Omega|} < \infty.$$

□

As a consequence of Lemmas 4.2.1 and 4.2.2 we can achieve a similar result for $\mathcal{V}[\mu]$. Given that it is a tangent vector field, let us recall the definition of Hölder-continuity of tangent vector fields along a complete Riemannian manifold.

Definition 4.2.9. *Consider a complete Riemannian manifold $(M, \langle\cdot,\cdot\rangle)$, fix $0 < \beta \leq 1$ and a tangent vector field $V : M \longrightarrow TM$. V is said to be β-Hölder continuous when there exists $C > 0$ such that*

$$\left|\tau[\gamma]_0^1(V_x) - V_y\right| \leq C d(x,y)^\beta,$$

for every $x, y \in M$ and any minimizing geodesic $\gamma : [0,1] \longrightarrow M$ joining x to y. The smallest such C is called the β-Hölder constant of the vector field V. Here $d(\cdot,\cdot)$ is the Riemannian distance (N.4) *and*

$$\tau[\gamma]_{s_1}^{s_2} : T_{\gamma(s_1)}M \longrightarrow T_{\gamma(s_2)}M,$$

stands for the parallel transport from $\gamma(s_1)$ to $\gamma(s_2)$ along the geodesic γ, see [111].

Corollary 4.2.10. *Consider $\alpha \in (0, \frac{1}{2})$, $K > 0$, and set $\mu \in \widetilde{\mathcal{C}}_{\mathcal{M}}$. Then,*

$$\frac{\mathcal{V}[\mu]}{1+|\Omega|} \in C([0,T], \mathfrak{X}C_b(\mathbb{T}\times\mathbb{R})).$$

In addition, there exists $C > 0$, which does not depend on μ, such that

$$\left|\tau_0^1[\widehat{\gamma}](\mathcal{V}[\mu_t](z_1,\Omega_1)) - \mathcal{V}[\mu_t](z_2,\Omega_2)\right| \leq |\Omega_1 - \Omega_2| + CK\|\mu\|_{L^\infty_w(0,T;\mathcal{M}(\mathbb{T}\times\mathbb{R}))} d(z_1,z_2)^{1-\alpha},$$

for every $t \in [0,T]$, each $(z_1,\Omega_1), (z_2,\Omega_2) \in \mathbb{T}\times\mathbb{R}$ and every minimizing geodesic $\widehat{\gamma} : [0,1] \longrightarrow \mathbb{T}\times\mathbb{R}$ joining (z_1,Ω_1) to (z_2,Ω_2).

Proof. The first part is clear by the Definition 4.2.3 along with Lemma 4.2.1. Let us focus on the last part where Lemma 4.2.2 will play a role. Set $t \in [0,T]$, $x = (z_1, \Omega_1)$, $y = (z_2, \Omega_2) \in \mathbb{T} \times \mathbb{R}$ and a minimizing geodesic $\widehat{\gamma} : [0,1] \longrightarrow \mathbb{T} \times \mathbb{R}$ joining x to y. Then,

$$\tau[\widehat{\gamma}]_0^1(\mathcal{V}[\mu_t](z_1, \Omega_1)) = (\mathcal{P}[\mu_t](z_1, \Omega_1)\, iz_2, 0).$$

Consequently,

$$\begin{aligned}|\tau[\gamma]_0^1(\mathcal{V}[\mu_t](z_1, \Omega_1)) - &\mathcal{V}[\mu_t](z_2, \Omega_2)| \\ &= |((\mathcal{P}[\mu_t](z_1, \Omega_1) - \mathcal{P}[\mu_t](z_2, \Omega_2))\, iz_2, 0)| = |\mathcal{P}[\mu_t](z_1, \Omega_1) - \mathcal{P}[\mu_t](z_2, \Omega_2)|,\end{aligned}$$

and the result clearly follows from Theorem 4.2.8. □

Remark 4.2.11. *If the tightness condition $\mu \in \widetilde{\mathcal{C}}_{\mathcal{M}}$ is deprived in Corollary 4.2.10 and it is replaced by the weaker condition $\mu \in \mathcal{C}_{\mathcal{M}}$, then time continuity might be lost. However, we still can obtain the following properties:*

1. *$\mathcal{P}[\mu_t](z, \Omega)$ is continuous in (z, Ω) for any $t \in [0,T]$.*
2. *$\mathcal{P}[\mu_t](z, \Omega)$ is measurable in t for any $(z, \Omega) \in \mathbb{T} \times \mathbb{T}$.*
3. *There exists a nonnegative $m \in L^1(0,T)$ so that*

$$\frac{\mathcal{P}[\mu_t](z, \omega)}{1 + |\Omega|} \le m(t),$$

for every $(t, z, \Omega) \in [0,T] \times \mathbb{T} \times \mathbb{R}$. Indeed

$$m(t) = \max\left\{1, K\|\mu\|_{L^\infty_w(0,T;\mathcal{M}(\mathbb{T}\times\mathbb{R}))}\|h\|_{C(\mathbb{T})}\right\}.$$

The above set of properties are called Caratheodory's conditions. Notice that time continuity is only useful when dealing with classical C^1 characteristics associated with the transport field $\mathcal{V}[\mu]$. However, in the case $\mu \in \mathcal{C}_{\mathcal{M}}$, where time continuity is missing but still Caratheodory's conditions hold true, the Caratheodory existence theorem guarantee the existence of Caratheodory solution, that is, absolutely continuous solutions that solve the characteristic system almost everywhere. For simplicity, let us skip it now although it will come into play in the critical regime, see Section 4.6.

Lemma 4.2.12. *Consider $\alpha \in (0, \frac{1}{2})$, $K > 0$ and fix $\mu \in \widetilde{\mathcal{C}}_{\mathcal{M}}$. For any $x_0 = (z_0, \Omega_0) \in \mathbb{T} \times \mathbb{R}$ let us consider the characteristic system issued at x_0, i.e.,*

$$\begin{cases} \dfrac{dX}{dt}(t; t_0, x_0) = \mathcal{V}[\mu_t](X(t; t_0, x_0)), \\ X(t_0; t_0, x_0) = x_0. \end{cases} \tag{4.2.7}$$

Then, (4.2.7) *enjoys at least one global-in-time C^1 solution $X(t; t_0, x_0) = (Z(t; t_0, z_0, \Omega_0), \Omega_0)$. Indeed, if we set $z_0 = e^{i\theta_0}$, for some $\theta_0 \in \mathbb{R}$, then*

$$Z(t; t_0, z_0, \Omega_0) = e^{i\Theta(t; t_0, \theta_0, \Omega_0)},$$

where $\Theta = \Theta(t; t_0, \theta_0, \Omega_0)$ is a global-in-time C^1 solution to

$$\begin{cases} \dfrac{d\Theta}{dt}(t; t_0, \theta_0, \Omega_0) = \mathcal{P}[\mu_t](\Theta(t; t_0, \theta_0, \Omega_0), \Omega_0), \\ \Theta(t_0; t_0, \theta_0, \Omega_0) = \theta_0. \end{cases} \tag{4.2.8}$$

Proof. The first part of the result is clear because $\frac{\mathcal{V}[\mu]}{1+|\Omega|}\in C([0,T],\mathfrak{X}C_b(\mathbb{T}\times\mathbb{R}))$ by virtue of Corollary 4.2.10. However, let us comment on the above representation in coordinates. By Theorem 4.2.8 one also has that $\mathcal{P}[\mu]$, regarded as a function in $\mathbb{R}\times\mathbb{R}$, is a continuous function with sub-linear growth. By the classical Peano theorem, there is at least one global-in-time solution $\Theta=\Theta(t;t_0,\theta_0,\Omega_0)$ to (4.2.8). Now, let us define $Z(t;t_0,z_0,\Omega_0)$ in terms of $\Theta(t;t_0,z_0,\Omega_0)$ like in the statement of this result. Then,

$$\frac{dZ}{dt}(t;t_0,z_0,\Omega_0)=ie^{i\Theta(t;t_0,\theta_0,\Omega_0)}\frac{d\Theta}{dt}(t;t_0,\theta_0,\Omega_0)=\mathcal{P}[\mu_t](Z(t;t_0,z_0,\Omega_0),\Omega_0)\,iZ(t;t_0,z_0,\Omega_0),$$

and, consequently, so defined $X(t;t_0,x_0)$ is a global-in-time solution to the characteristic system (4.2.7) thanks to the Definition 4.2.3 of the vector field $\mathcal{V}[\mu]$. □

Notice that in Theorem 4.2.8, a $(1-2\alpha)$-Hölder estimate for $\mathcal{P}[\mu]$ was obtained. Nevertheless, the infinite slope of h at each $\theta\in 2\pi\mathbb{Z}$ prevent us from a full Lipschitz-estimate for $\mathcal{P}[\mu]$. It is well known that Hölder continuity is not enough for the characteristic system (4.2.8) (or equivalently (4.2.7)) to enjoy a unique global-in-time C^1 solution. In the following, we will introduce the key concept that will amount to the forwards uniqueness result, namely, one-sided Lipschitz-continuity.

Definition 4.2.13. *Let $(M,\langle\cdot,\cdot\rangle)$ be a complete Riemannian manifold and consider a tangent vector field V along M. Then, we will say that V is one-sided Lipschitz when there exists a constant $L>0$ such that*

$$\left\langle V_y,\gamma'(1)\right\rangle-\left\langle V_x,\gamma'(0)\right\rangle\leq Ld(x,y)^2, \tag{4.2.9}$$

for every $x,y\in M$ and every minimizing geodesic $\gamma:[0,1]\longrightarrow M$ joining x to y.

Remark 4.2.14. *For such a minimizing geodesic γ as in Definition 4.2.13, we can associate the parallel transport from a point $\gamma(s_1)$ to $\gamma(s_2)$ along the geodesic γ is a linear isometry between the tangent spaces to M supported at such points*

$$\tau[\gamma]_{s_1}^{s_2}:T_{\gamma(s_1)}M\longrightarrow T_{\gamma(s_2)}M.$$

In addition, since γ is a geodesic, then the covariant derivative of γ vanishes, that is, $\frac{D\gamma'}{ds}=0$ and $\tau[\gamma]_{s_1}^{s_2}(\gamma'(s_1))=\gamma'(s_2)$. Then, the condition (4.2.9) *can be equivalently restated as follows*

$$-\left\langle\tau[\gamma]_0^1(V_x)-V_y,\gamma'(1)\right\rangle\leq Ld(x,y)^2. \tag{4.2.10}$$

To the best of our knowledge, such definition has not been clearly proposed previously in the literature as a generalization of the standard one-sided Lipschitz continuity in Euclidean spaces. For the sake of clarity, we list some of the main properties supporting its definition.

Proposition 4.2.15. *Let (M,g) be a complete Riemannian manifold and consider a tangent vector field $V:M\longrightarrow TM$ and a scalar differentiable function $\phi:M\longrightarrow\mathbb{R}$.*

1. *If V is Lipschitz-continuous, then V is one-sided Lipschitz.*
2. *If $M\equiv\mathbb{R}^d$, the d-dimensional Euclidean space, then V is one-sided Lipschitz in the sense* (4.2.9) *(equivalently* (4.2.10)*) if, and only if, it is one-sided Lipschitz in the standard sense, i.e.,*

   $$\langle V_x-V_y,x-y\rangle\leq L|x-y|^2,$$

   for every couple of vectors $x,y\in\mathbb{R}^d$.
3. *If ϕ is λ-convex, then $-\nabla\phi$ is one-sided Lipschitz with constant λ.*

Proof. Consider $x, y \in M$ and any minimizing geodesic $\gamma : [0,1] \longrightarrow M$ joining x to y. First, assume that V is Lipschitz-continuous with constant L. Then,

$$-\left\langle \tau[\gamma]_0^1(V_x) - V_y, \gamma'(1)\right\rangle \leq |\tau[\gamma]_0^1(V_x) - V_y|\,|\gamma'(1)| \leq L d(x,y)^2,$$

where we have used the Cauchy-Schwarz inequality, the Lipschitz continuity of V and that $|\gamma'(1)| = d(x,y)$ because γ is minimizing geodesic between x and y. Second, in the Euclidean case it is clear that there is a unique such minimizing geodesic, namely,

$$\gamma(s) = (1-s)x + sy = x + s(y-x),\ s \in [0,1].$$

Notice that $\gamma'(0) = \gamma'(1) = y - x$. Hence, our claim is clear. Finally, assume that ϕ is λ-convex (see, for instance, [9, 129, 268, 296]). Then, by definition we obtain

$$\phi(\gamma(s)) \leq (1-s)\phi(x) + s\phi(y) + \frac{\lambda}{2}(1-s)s d(x,y)^2,\ s \in [0,1].$$

Then, we find

$$\begin{aligned}
\frac{\phi(\gamma(s)) - \phi(x)}{s} &\leq \phi(y) - \phi(x) + \frac{\lambda}{2}(1-s)d(x,y)^2,\\
-\frac{\phi(\gamma(s)) - \phi(y)}{s-1} &\leq \phi(x) - \phi(y) + \frac{\lambda}{2} s d(x,y)^2.
\end{aligned}$$

Taking, limits as $s \to 0^+$ and $s \to 1^-$ respectively in the first and second expressions yields

$$\begin{aligned}
\left\langle \nabla\phi(x), \gamma'(0)\right\rangle &\leq \phi(y) - \phi(x) + \frac{\lambda}{2} d(x,y)^2,\\
-\left\langle \nabla\phi(y), \gamma'(1)\right\rangle &\leq \phi(x) - \phi(y) + \frac{\lambda}{2} d(x,y)^2.
\end{aligned}$$

Taking the sum of both terms, we get that $-\nabla\phi$ is one-sided Lipschitz with constant λ. □

Our next result will show that although $\mathcal{V}[\mu]$ is not fully Lipschitz-continuous with respect to $(z, \Omega) \in \mathbb{T}\times\mathbb{R}$, it is one-sided Lipschitz uniformly in $t \in [0,T]$. The cornerstone in such result is the following split of $-h$ into a decreasing and a Lipschitz-continuous part, see Fig. 4.1.

Lemma 4.2.16. *Consider $\alpha \in (0, \frac{1}{2})$ and set $\bar{h}$ and $\tilde{\theta} \in (0, \frac{\pi}{2})$ such that*

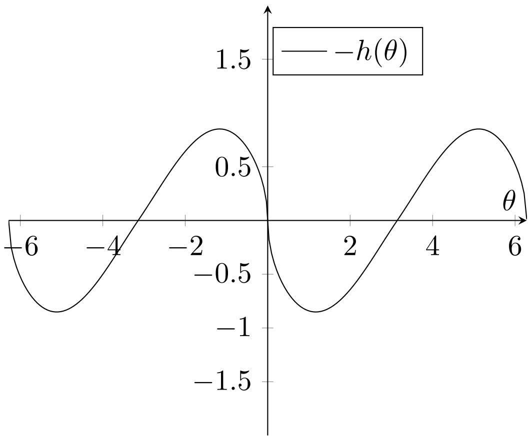
$$\bar{h} = \max_{0<\theta<\pi} h(\theta) \ \text{ and } \ 2\alpha \sin\tilde{\theta} = \tilde{\theta}\cos\tilde{\theta}.$$

Define the couple of functions $\delta, \lambda : [-2\pi, 2\pi] \longrightarrow \mathbb{R}$ as follows

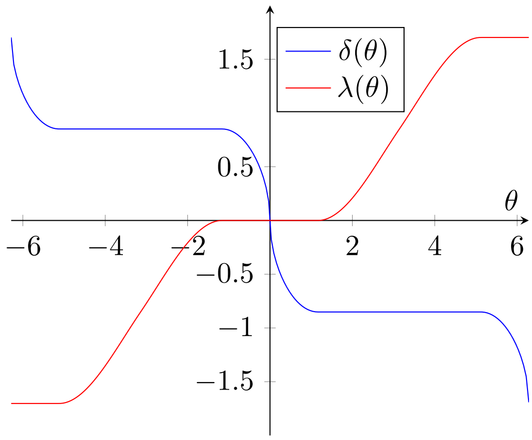
$$\delta(\theta) := \begin{cases}
2\bar{h} - h(\theta), & \theta \in [-2\pi, -2\pi + \tilde{\theta}),\\
\bar{h}, & \theta \in [-2\pi + \tilde{\theta}, -\tilde{\theta}),\\
-h(\theta), & \theta \in [-\tilde{\theta}, \tilde{\theta}],\\
-\bar{h}, & \theta \in (\tilde{\theta}, 2\pi - \tilde{\theta}],\\
-h(\theta) - 2\bar{h}, & \theta \in (2\pi - \tilde{\theta}, 2\pi],
\end{cases}$$

$$\lambda(\theta) := \begin{cases}
-2\bar{h}, & \theta \in [-2\pi, -2\pi + \tilde{\theta}),\\
-\bar{h} - h(\theta), & \theta \in [-2\pi + \tilde{\theta}, -\tilde{\theta}),\\
0, & \theta \in [-\tilde{\theta}, \tilde{\theta}],\\
\bar{h} - h(\theta), & \theta \in (\tilde{\theta}, 2\pi - \tilde{\theta}],\\
2\bar{h}, & \theta \in (2\pi - \tilde{\theta}, 2\pi].
\end{cases}$$

Then, the following properties hold true:



(a) $-h(\theta)$



(b) $\delta(\theta)$ and $\lambda(\theta)$

Figure 4.1: Plot of the function $-h(\theta)$ and the functions $\delta(\theta)$ and $\lambda(\theta)$ in the decomposition for the value $\alpha = 0.25$.

1. *δ is monotonically decreasing, λ is Lipschitz-continuous and*

$$-h(\theta) = \delta(\theta) + \lambda(\theta), \ \forall\, \theta \in [-2\pi, 2\pi].$$

2. *$-h$ is one-sided Lipschitz in $[-2\pi, 2\pi]$, i.e., there exists $L_0 > 0$ such that*

$$((-h)(\theta_1) - (-h)(\theta_2))\,(\theta_1 - \theta_2) \leq L_0(\theta_1 - \theta_2)^2.$$

See Figure 4.1, also see Figure 3.2 in the preceding Chapter 3 for a similar split of the kernel in the smaller fomain $[-\pi, \pi]$, that was applied to the agent-based system.

Lemma 4.2.17. *Consider $\alpha \in (0, \frac{1}{2})$, $K > 0$ and set $\mu \in \mathcal{C}_{\mathcal{M}}$. Then, we have*

$$(\mathcal{P}[\mu_t](\theta_1, \Omega_1) - \mathcal{P}[\mu_t](\theta_2, \Omega_2))\,(\theta_1 - \theta_2) \leq (\Omega_1 - \Omega_2)(\theta_1 - \theta_2) + KL_0\|\mu\|_{L^\infty_w(0,T;\mathcal{M}(\mathbb{T}\times\mathbb{R}))}(\theta_1 - \theta_2)^2,$$

for every $t \in [0, T]$, each $\theta_1, \theta_2 \in \mathbb{R}$ with $\theta_1 - \theta_2 \in [-\pi, \pi]$ and any $\Omega_1, \Omega_2 \in \mathbb{R}$. Here, the constant L_0 is the one-sided Lipschitz constant in Lemma 4.2.16.

Proof. By the Definition 4.2.3 we can state

$$\begin{aligned}&(\mathcal{P}[\mu_t](\theta_1, \Omega_1) - \mathcal{P}[\mu_t](\theta_2, \Omega_2))\,(\theta_1 - \theta_2)\\ &= (\Omega_1 - \Omega_2)(\theta_1 - \theta_2) + K \int_{(\theta_1-\pi, \theta_1+\pi]} \int_{\mathbb{R}} ((-h)(\theta_1 - \theta') - (-h)(\theta_2 - \theta'))(\theta_1 - \theta_2)\, d_{(\theta',\Omega')}\mu_t.\end{aligned}$$

For every $\theta' \in (\theta_1 - \pi, \theta_1 + \pi]$ we equivalently have $\theta_1 - \theta' \in (-\pi, \pi]$. Since $\theta_2 - \theta_1 \in [-\pi, \pi]$, we also obtain that $\theta_2 - \theta' \in (-2\pi, 2\pi]$. Hence, we are in the range of applicability of Lemma 4.2.16, that implies

$$\begin{aligned}&(\mathcal{P}[\mu_t](\theta_1, \Omega_1) - \mathcal{P}[\mu_t](\theta_2, \Omega_2))\,(\theta_1 - \theta_2)\\ &\qquad\leq (\Omega_1 - \Omega_2)(\theta_1 - \theta_2) + KL_0 \int_{(\theta_1-\pi, \theta_1+\pi]} \int_{\mathbb{R}} (\theta_1 - \theta_2)^2\, d_{(\theta',\Omega')}\mu_t. \qquad (4.2.11)\end{aligned}$$

□

Theorem 4.2.18. *Consider $\alpha \in (0, \frac{1}{2})$, $K > 0$ and set $\mu \in \mathcal{C}_{\mathcal{M}}$. Then, $\mathcal{V}[\mu]$ is one-sided Lipschitz in $\mathbb{T} \times \mathbb{R}$ uniformly in $t \in [0,T]$, i.e., there exists $L = L(\alpha, K, \mu) > 0$ such that*

$$\left\langle \mathcal{V}[\mu_t](z_2,\Omega_2), \widehat{\gamma}'(1) \right\rangle - \left\langle \mathcal{V}[\mu_t](z_1,\Omega_1), \widehat{\gamma}'(0) \right\rangle \leq L\, d((z_1,\Omega_1),(z_2,\Omega_2))^2,$$

for every $t \in [0,T]$, any $(z_1,\Omega_1),(z_2,\Omega_2) \in \mathbb{T}\times\mathbb{R}$ and each minimizing geodesic $\widehat{\gamma} : [0,1] \longrightarrow \mathbb{T}\times\mathbb{R}$ in the manifold $\mathbb{T}\times\mathbb{R}$ joining (z_1,Ω_1) to (z_2,Ω_2).

Proof. Fix any $t \in [0,T]$ and $(z_1,\Omega_1),(z_2,\Omega_2) \in \mathbb{T}\times\mathbb{R}$. Our first step will be to characterize the minimizing geodesics $\widehat{\gamma} : [0,1] \longrightarrow \mathbb{T}\times\mathbb{R}$ joining (z_1,Ω_1) to (z_2,Ω_2). Let us write $z_1 = e^{i\theta_1}$ and $z_2 = e^{i\theta_2}$ for some $\theta_1, \theta_2 \in \mathbb{R}$ and assume that $\theta_2 - \theta_1 \in (-\pi, \pi]$ without loss of generality.

• *Case 1*: $\theta_2 - \theta_1 \in (-\pi,\pi)$. In this case, the only minimizing geodesic reads

$$\widehat{\gamma}(s) = (e^{i(\theta_1 + s(\theta_2-\theta_1))}, \Omega_1 + s(\Omega_2 - \Omega_1)).$$

Notice that the directions of the geodesic at the endpoints are

$$\widehat{\gamma}'(0) = ((\theta_2-\theta_1)\, iz_1, \Omega_2-\Omega_1) \text{ and } \widehat{\gamma}'(1) = ((\theta_2-\theta_1)\, iz_2, \Omega_2-\Omega_1).$$

Therefore, we can write

$$\left\langle \mathcal{V}[\mu_t](z_2,\Omega_2), \widehat{\gamma}'(1) \right\rangle - \left\langle \mathcal{V}[\mu_t](z_1,\Omega_1), \widehat{\gamma}'(0) \right\rangle = (\mathcal{P}[\mu_t](z_2,\Omega_2) - \mathcal{P}[\mu_t](z_1,\Omega_1))\,(\theta_2-\theta_1).$$

• *Case 2*: $\theta_2 - \theta_1 = \pi$. Now z_1 and z_2 are antipodes and there are exactly two different minimizing geodesics, namely,

$$\begin{aligned}\widehat{\gamma}_+(s) &= (e^{i(\theta_1+\pi s)}, \Omega_1 + s(\Omega_2-\Omega_1)),\\ \widehat{\gamma}_-(s) &= (e^{i(\theta_1-\pi s)}, \Omega_1 + s(\Omega_2-\Omega_1)).\end{aligned}$$

Its directions at the endpoints read

$$\widehat{\gamma}_\pm'(0) = (\pm\pi\, iz_1, \Omega_2-\Omega_1) \text{ and } \widehat{\gamma}_\pm'(1) = (\pm\pi\, iz_2, \Omega_2-\Omega_1).$$

Then, it is clear that

$$\left\langle \mathcal{V}[\mu_t](z_2,\Omega_2), \widehat{\gamma}_\pm'(1) \right\rangle - \left\langle \mathcal{V}[\mu_t](z_1,\Omega_1), \widehat{\gamma}_\pm'(0) \right\rangle = \pm\,(\mathcal{P}[\mu_t](z_2,\Omega_2) - \mathcal{P}[\mu_t](z_1,\Omega_1))\,\pi.$$

No matter the case, we can always use Lemma 4.2.17 to arrive at

$$\left\langle \mathcal{V}[\mu_t](z_2,\Omega_2), \widehat{\gamma}'(1) \right\rangle - \left\langle \mathcal{V}[\mu_t](z_1,\Omega_1), \widehat{\gamma}'(0) \right\rangle \leq (\Omega_1-\Omega_2)(\theta_1-\theta_2) + KL_0\|\mu\|_{L^\infty(0,T;\mathcal{M}(\mathbb{T}\times\mathbb{R}))}(\theta_1-\theta_2)^2.$$

Notice that $\theta_1 - \theta_2 \in [-\pi,\pi]$ and, consequently, $|\theta_1-\theta_2| = |\theta_1-\theta_2|_o = d(z_1,z_2)$. Applying Young's inequality we arrive at the desired result for the value

$$L = \frac{1}{2} + KL_0\|\mu\|_{L^\infty_w(0,T;\mathcal{M}(\mathbb{T}\times\mathbb{R}))}.$$

□

We are now ready to complete the existence part in Lemma 4.2.12 for the characteristic system (4.2.7) with an appropriate notion of uniqueness, namely, one-sided uniqueness. In this way, we obtain the following full well-posedness result.

Theorem 4.2.19. *Consider $\alpha\in(0,\frac{1}{2})$, $K>0$ and fix $\mu\in\widetilde{\mathcal{C}}_{\mathcal{M}}$. The characteristic system* (4.2.7) *associated with the transport field $\mathcal{V}[\mu]$ enjoys a global-in-time C^1 solution that is unique forward-in-time for every given initial data $x_0=(z_0,\Omega_0)=(e^{i\theta_0},\Omega_0)\in\mathbb{T}\times\mathbb{R}$. Indeed, the same representation of the solution holds true, i.e.,*

$$X(t;t_0,x_0)=(Z(t;t_0,z_0,\Omega_0),\Omega_0)=(e^{i\Theta(t;t_0,\theta_0,\Omega_0)},\Omega_0),\ t\geq t_0,$$

where $\Theta(t;t_0,\theta_0,\Omega_0)$ is the unique forward-in-time C^1 solution to (4.2.8).

Although the proof is standard and relies on the one-sided Lipschitz condition in Theorem 4.2.18 and the weak differentiability properties of the squared distance in Appendix 4.B, we give a simple proof for the sake of completeness because it involves some delicate points.

Proof. Let us assume that there are two different solutions $x_1=x_1(t)$ and $x_2=x_2(t)$ to (4.2.7) with same initial data $x_1(t_0)=x_0=x_2(t_0)$. Define the following function

$$I(t):=\frac{1}{2}d(x_1(t),x_2(t))^2,\ t\geq t_0,$$

where $d(\cdot,\cdot)$ means the Riemannian distance in $\mathbb{T}\times\mathbb{R}$, see (N.4) and Appendix 4.B. Recall that the distance function $d(\cdot,\cdot)$ is Lipschitz-continuous, see Proposition 4.B.1. Since x_1 and x_2 are C^1 trajectories, then $I=I(t)$ is absolutely continuous. Hence, its derivative exists for almost every t and it agrees with the one-sided Dini upper derivative. Since Theorem 4.B.7 implies that the one-sided Dini upper directional derivatives ($\frac{d^+}{dt}$) of the squared distance are finite, then the chain rule yields

$$\frac{dI}{dt}\equiv\frac{d^+I}{dt}=d^+\left(\frac{1}{2}d^2_{x_2(t)}\right)_{x_1(t)}(\dot{x}_1(t))+d^+\left(\frac{1}{2}d^2_{x_1(t)}\right)_{x_2(t)}(\dot{x}_2(t)),$$

for almost every $t\geq t_0$. Recall that in Theorem 4.B.7 we also got an upper bound for such Dini directional derivatives that reads as follows

$$\begin{aligned}d^+\left(\frac{1}{2}d^2_{x_2(t)}\right)_{x_1(t)}\Big(\dot{x}_1(t)\Big)&\leq\inf_{\substack{w_1\in\exp^{-1}_{x_1(t)}(x_2(t))\\|w_1|=d(x_1(t),x_2(t))}}-\langle\dot{x}_1(t),w_1\rangle,\\ d^+\left(\frac{1}{2}d^2_{x_1(t)}\right)_{x_2(t)}\Big(\dot{x}_2(t)\Big)&\leq\inf_{\substack{w_2\in\exp^{-1}_{x_2(t)}(x_1(t))\\|w_2|=d(x_1(t),x_2(t))}}-\langle\dot{x}_2(t),w_2\rangle.\end{aligned}$$

Let us fix any minimizing geodesic $\widehat{\gamma}_t:[0,1]\longrightarrow\mathbb{T}\times\mathbb{R}$ joining $x_1(t)$ to $x_2(t)$, for every $t\geq t_0$. Then, we can choose $w_1=\widehat{\gamma}_t'(0)$ and $w_2=-\widehat{\gamma}_t'(1)$ in the above inequalities. Consequently,

$$\frac{dI}{dt}\leq\left\langle\dot{x}_2(t),\widehat{\gamma}_t'(1)\right\rangle-\left\langle\dot{x}_1(t),\widehat{\gamma}_t'(0)\right\rangle=\left\langle\mathcal{V}[\mu_t](x_2(t)),\widehat{\gamma}_t'(1)\right\rangle-\left\langle\mathcal{V}[\mu_t](x_1(t)),\widehat{\gamma}_t'(0)\right\rangle.$$

Using Theorem 4.2.18 we get the estimate

$$\frac{dI}{dt}\leq Ld(x_1(t),x_2(t)^2)=2L\,I(t),\ \text{ a.e. } t\geq t_0.$$

Since $I(t_0)=\frac{1}{2}d(x_1(0),x_2(0))^2=\frac{1}{2}d(x_0,x_0)^2=0$, Gronwall's Lemma amounts to the desired sided-uniqueness, namely,

$$x_1(t)=x_2(t),\ \forall\,t\geq t_0.$$

□

The same ideas as above can be used even if $x_1(0) \neq x_2(0)$ to derive some stability result of the flow. In fact, although $\mathcal{V}[\mu]$ is not fully Lipschitz continuous (but merely one-sided Lipschitz continuous), its characteristic flow is.

Corollary 4.2.20. *Consider $\alpha \in (0, \frac{1}{2})$, $K > 0$ and fix $\mu \in \widetilde{\mathcal{C}}_{\mathcal{M}}$. Let $X(t; 0, z, \Omega) = (Z(t; 0, z, \Omega), \Omega)$ be the flow of the characteristic system* (4.2.7) *associated with the transport field $\mathcal{V}[\mu]$. Then, X is Lipschitz-continuous in (z, Ω); namely, there exists $L = L(\alpha, K, \mu)$ with*

$$d(X(t; 0, (z_1, \Omega_1)) - X(t; 0, (z_2, \Omega_2))) \leq d((z_1, \Omega_1), (z_2, \Omega_2))e^{Lt},$$

for every $(z_1, \Omega_1), (z_2, \Omega_2) \in \mathbb{T} \times \mathbb{R}$ and each $t \in [0, T]$.

Again, like in Theorem 4.2.18, the constant L in Corollary 4.2.20 is

$$L = \frac{1}{2} + KL_0\|\mu\|_{L^\infty_w(0,T;\mathcal{M}(\mathbb{T}\times\mathbb{R}))}.$$

4.2.4 Types of measure-valued solutions

In the above Definition 4.2.6 and Proposition 4.2.7, the classical concept of weak measure-valued solutions $f \in \mathcal{AC}_{\mathcal{M}}$ to (4.2.5) was revisited. It clearly agrees with the standard concept of measure-valued solution for homogeneous conservative continuity equations (for instance, see [8] and references therein). Indeed, such concept requires a very weak regularity of the transport field $\mathcal{V}[f]$ in the continuity equation (4.2.5), specifically,

$$\int_0^T \int_C |\mathcal{V}[f_t]|\, df_t\, dt < \infty, \tag{4.2.12}$$

for each compact subset $C \subseteq \mathbb{T} \times \mathbb{R}$. Although such concepts are not usually considered when working on general manifolds like $\mathbb{T} \times \mathbb{R}$, straightforward ideas allow extending them from the Euclidean space $\mathbb{R}^d$ to Riemannian manifolds. Apart from that notion of solution, there are a couple of related concepts that we recall in the following.

Definition 4.2.21. *We will say that a time dependent measure $f \in \widetilde{C}_{\mathcal{M}}$ is a measure-valued solution to* (4.2.5) *in the sense of the characteristic flow when*

$$f_t = X_f(t; 0, \cdot)_{\#} f_0, \text{ for all } t \geq 0,$$

where X_f stands for the flow of the transport field $\mathcal{V}[f]$ as introduced in Theorem 4.2.19.

The following result is a straightforward consequence of the definition.

Proposition 4.2.22. *Let $f \in \widetilde{C}_{\mathcal{M}}$ be a solution to* (4.2.5) *in the sense of the characteristic flow. Then*

1. *$f \in \mathcal{AC}_{\mathcal{M}}$.*
2. *f is a weak measure-valued solution to* (4.2.5).

Definition 4.2.23. *Define the space $\mathcal{C}^1([0,T]) := C^1([0,T], \mathbb{T} \times \mathbb{R})$, consider $f \in \widetilde{C}_{\mathcal{M}}$ and the set*

$$\mathcal{S}_f([0,T]) := \left\{(x, \gamma) \in (\mathbb{T} \times \mathbb{R}) \times \mathcal{C}^1([0,T]) : \gamma \text{ is a characteristic of } \mathcal{V}[f] \text{ issued at } x\right\}.$$

Then, f is said to be a superposition solution to (4.2.5) *if $f(t = 0) = f_0$ and there exists some probability measure $\eta \in \mathcal{P}((\mathbb{T} \times \mathbb{R}) \times \mathcal{C}^1([0,T]))$ such that $\operatorname{supp} \eta \subseteq \mathcal{S}_f([0,T])$ and $f_t = f_t^\eta$ for all $t \in [0,T]$, where the measure f^η is defined as follows*

$$\langle f_t^\eta, \varphi \rangle = \int_{(\mathbb{T}\times\mathbb{R})\times\mathcal{C}^1([0,T])} \varphi(\gamma_t)\, d_{(x,\gamma)}\eta,$$

for any test function $\varphi \in C_b(\mathbb{T} \times \mathbb{R})$.

Mimicking the ideas in [8, Theorem 4.4] we obtain the following result.

Proposition 4.2.24. *Let $f \in \mathcal{AC}_{\mathcal{M}} \cap \mathcal{T}_{\mathcal{M}}$ be any weak measure-valued solution to* (4.2.5)*. Then, there exists some probability measure $\eta \in \mathcal{P}((\mathbb{T}\times\mathbb{R})\times\mathcal{C}^1([0,T]))$ supported on $\mathcal{S}_f([0,T])$ such that*

$$f_t \equiv f_t^\eta,\ t \geq 0.$$

In other words, f is a superposition solution.

Recall that in Remark 4.2.11, we mentioned that an analogue to Lemma 4.2.12 with $f \in \mathcal{C}_M$ instead of $f \in \widetilde{\mathcal{C}}_{\mathcal{M}}$ can be derived by means of Caratheodory characteristics. Then, Definitions 4.2.21 and 4.2.23 along with Propositions 4.2.22 and 4.2.24 have analogues with $f \in \mathcal{C}_{\mathcal{M}}$ only. Again, we will skip it here.

So far, we have not used any special property of the transport field other than those appearing in Remark 4.2.11. Recall that they guarantee both condition (4.2.12) (making sense of weak measure-valued solutions) and existence of Caratheodory characteristics (giving sense to solutions in the sense of the flow and superposition solutions). Let us see that under the forward-uniqueness property of the characteristic system, that follows in Theorem 4.2.19 from the one-sided uniqueness property of $\mathcal{V}[f]$, any superposition solution is also a solution in the sense of the characteristic flow.

Proposition 4.2.25. *Let $f \in \widetilde{C}_{\mathcal{M}}$ be a superposition solution to* (4.2.5)*. Then, f is a solution to* (4.2.5) *in the sense of the charactersitic flow.*

Proof. By definition, $f_t = f_t^\eta$ for $t \in [0,T]$, for some $\eta \in \mathcal{P}((\mathbb{T}\times\mathbb{R})\times\mathcal{C}^1([0,T]))$ with $\operatorname{supp}\eta \subseteq \mathcal{S}_f([0,T])$. Due to Theorem 4.2.19, the one-sided Lipschitz condition of $\mathcal{V}[f]$ implies one-sided uniqueness of the characteristic system (4.2.7). Then, $\mathcal{S}_f([0,T])$ agrees with the graph of the flow X_f in the following sense

$$\mathcal{S}_f([0,T]) = \{(x, X_f(\cdot\,;0,x)) : x \in \mathbb{T}\times\mathbb{R}\}.$$

Consequently, we can write

$$\langle f_t,\varphi\rangle = \langle f_t^\eta,\varphi\rangle = \int_{(\mathbb{T}\times\mathbb{R})\times\mathcal{C}^1([0,T])} \varphi(\gamma_t)\, d_{(x,\gamma)}\eta = \int_{(\mathbb{T}\times\mathbb{R})\times\mathcal{C}^1([0,T])} \varphi(X_f(t;0,x))\, d_{(x,\gamma)}\eta, \quad (4.2.13)$$

for every $t \in [0,T]$ and $\varphi \in C_b(\mathbb{T}\times\mathbb{R})$. Consider the projection

$$\begin{array}{rccc} \pi_x : & (\mathbb{T}\times\mathbb{R})\times\mathcal{C}^1([0,T]) & \longrightarrow & \mathbb{T}\times\mathbb{R} \\ & (x,\gamma) & \longmapsto & x, \end{array}$$

along with the marginal $\mu := (\pi_x)_{\#}\eta$. By the disintegration theorem (see Theorem F.4.1 below), let us consider the family $(\eta_x)_{x\in\mathbb{T}\times\mathbb{R}}$ of conditional probabilities or disintegrations of η. Then,

$$\int_{(\mathbb{T}\times\mathbb{R})\times\mathcal{C}^1([0,T])} \varphi\, d\eta = \int_{\mathbb{T}\times\mathbb{R}} \left(\int_{\mathcal{C}^1([0,T])} \varphi(x,\gamma)\, d_\gamma(\eta_x) \right) d\mu,$$

for any $\varphi \in C_b(\mathbb{T}\times\mathbb{R})$. In particular, when applied to (4.2.13) we arrive at

$$\langle f_t,\varphi\rangle = \int_{\mathbb{T}\times\mathbb{R}} \varphi(X_f(t;0,x))\, d_x\mu,$$

for any $\varphi \in C_b(\mathbb{T}\times\mathbb{R})$. Therefore,

$$f_t = X_f(t;0,\cdot)_{\#}\mu, \ \text{ for all } \ t \in [0,T].$$

Taking $t = 0$ we get $\mu \equiv f_0$. Then, f is a solution in the sense of the characteristic flow to (4.2.5). □

Remark 4.2.26. *The above Propositions 4.2.22, 4.2.24 and 4.2.25 guarantee that all the above three concepts of measure-valued solutions are equivalent in our problem because of the properties:*

1. *Caratheodory's conditions in Remark 4.2.11.*

2. $\mathcal{V}[f]$ *is one-sided Lipschitz-continuous uniformly in* $t \in [0, T]$.

4.3 Existence of weak measure-valued solutions

In this part, we shall derive existence of global-in-time measure valued solutions to (4.2.5) in the subcritical regime $\alpha \in (0, \frac{1}{2})$. The idea does nor rely on any regularization technique of the kernel. Instead, it will rely on a compactness argument as $N \to \infty$ for any sequences of empirical empirical measures associated with a sequence of solutions to the N-particle system (4.1.1)-(4.1.2) that initially approximates the given initial datum f_0 in Wasserstein distance. Such method will produce weak measure-valued solutions (equivalently, solutions in the sense of the characteristic flow) to (4.2.5) in the subcritical regime. Similar ideas will be analysed later in Sections 4.6 and 4.7 for the most singular regimes. Notice that the aforementioned compactness of the empirical measures becomes a first step towards the derivation of the full mean field limit, that will be proved later in Section 4.4 via a Dobrushin-type inequality.

The rest of this section consists of the following three parts. First, we will revisit the concept of empirical measures associated with a solution to the discrete model (4.1.1)-(4.1.2) and will show that they automatically are weak measure-valued solutions to (4.2.5). In the second part, we will obtain some a priori bounds implying the weak compactness of such sequence of empirical measures. The final step will be to identify the limit as a weak measure-valued solution to the continuous model (4.2.5).

4.3.1 Empirical measures

In this part we will recall the definition of empirical measures associated with a discrete solution to (4.1.1)-(4.1.2), see [59, 176, 198, 230, 226]. We will also inspect whether they provide measure-valued solutions to the macroscopic system (4.2.5).

Definition 4.3.1. *Fix* $N \in \mathbb{N}$ *and consider* N *oscillators with phases and natural frequencies given by the configurations*

$$\Theta_0^N = (\theta_{1,0}^N, \ldots, \theta_{N,0}^N) \text{ and } \{\Omega_i^N : i = 1, \ldots, N\}.$$

Let $\Theta^N(t) := (\theta_1^N(t), \ldots, \theta_N^N(t))$ *be the unique global-in-time classical solution to the discrete singular Kuramoto model* (4.1.1)-(4.1.2) *according to Theorem 3.3.5 in Chapter 3. Then, the associated empirical measures are given by* $\mu^N \in \mathcal{P}(\mathbb{T} \times \mathbb{R})$ *defined as follows*

$$\mu_t^N := \frac{1}{N} \sum_{i=1}^N \delta_{z_i^N(t)}(z) \otimes \delta_{\Omega_i^N}(\Omega),$$

where $z_i^N(t) := e^{i\theta_i^N(t)}$ *for any* $i = 1, \ldots, N$.

Theorem 4.3.2. *Consider* $\alpha \in (0, \frac{1}{2})$ *and* $K > 0$. *Fix* $N \in \mathbb{N}$ *and consider* N *oscillators with initial phases and natural frequencies given by the configurations*

$$\Theta_0^N = (\theta_{1,0}^N, \ldots, \theta_{N,0}^N) \text{ and } \{\Omega_i^N : i = 1, \ldots, N\}.$$

Let μ^N be the associated empirical measure according to Definition 4.3.1. Then, $\mu^N \in \mathcal{AC}_{\mathcal{M}} \cap \mathcal{T}_{\mathcal{M}}$ is a weak measure-valued solution to (4.2.5) *and, in addition,*

$$\left|\frac{d}{dt}\int_{\mathbb{T}\times\mathbb{R}} \varphi\, d\mu_t^N\right| \leq \left(\frac{1}{N}\sum_{i=1}^N |\Omega_i^N| + K\|h\|_{C(\mathbb{T})}\right)\|\nabla\varphi\|_{C_0(\mathbb{T}\times\mathbb{R})}, \tag{4.3.1}$$

for every $t \geq 0$ and every $\varphi \in C_0^1(\mathbb{T}\times\mathbb{R})$.

Proof. Let us first prove that $\mu^N \in \mathcal{AC}_{\mathcal{M}} \cap \mathcal{T}_{\mathcal{M}}$. Regarding the tightness condition notice that

$$\||\Omega|\mu_t^N\|_{\mathcal{M}(\mathbb{T}\times\mathbb{R})} = \frac{1}{N}\sum_{i=1}^N |\Omega_i^N|,$$

for every $t \geq 0$. Regarding the absolute continuity in time, set $\varphi \in C_c^\infty(\mathbb{T}\times\mathbb{R})$ and observe that

$$t \in [0,+\infty) \longmapsto \int_{\mathbb{T}}\int_{\mathbb{R}} \varphi\, d\mu_t^N = \frac{1}{N}\sum_{i=1}^N \varphi(\theta_i^N(t), \Omega_i^N) \tag{4.3.2}$$

is locally absolutely continuous (it is C^1 in fact). Indeed, taking derivatives in (4.3.2) yields

$$\frac{d}{dt}\int_{\mathbb{T}} \varphi\, d\mu_t^N = \frac{1}{N}\sum_{i=1}^N \frac{\partial\varphi}{\partial\theta}(\theta_i^N(t), \Omega_i^N)\dot{\theta}_i^N(t). \tag{4.3.3}$$

As explained in Remark 4.A.3, we assert that for $(z = e^{i\theta}, \Omega) \in \mathbb{T}\times\mathbb{R}$

$$\frac{\partial\varphi}{\partial\theta}(\theta,\Omega) = -ie^{-i\theta}\nabla_z\varphi(\theta,\Omega).$$

Then, the above equation (4.3.3) can be restated as follows

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{T}} \varphi\, d\mu_t^N &= \frac{1}{N}\sum_{i=1}^N \Re\left[\nabla_z\varphi(\theta_i^N(t), \Omega_i^N)(-ie^{-i\theta_i^N(t)})\dot{\theta}_i^N(t)\right]\\ &= \frac{1}{N}\sum_{i=1}^N \left\langle \nabla_z\varphi(\theta_i^N(t), \Omega_i^N), \frac{d}{dt}e^{i\theta_i^N(t)}\right\rangle,\end{aligned} \tag{4.3.4}$$

where $\Re$ means real part of the complex number. To describe the associated transport field, notice that

$$\mathcal{P}[\mu_t^N](\theta,\Omega) = \Omega - K\int_{(-\pi,\pi]}\int_{\mathbb{R}} h(\theta-\theta')\, d_{(\theta',\Omega')}\mu_t^N = \Omega - \frac{K}{N}\sum_{j=1}^N h(\theta - \theta_j^N(t)),$$

and, consequently,

$$\mathcal{P}[\mu_t^N](\theta_i^N(t), \Omega_i^N) = \Omega_i - \frac{K}{N}\sum_{j=1}^N h(\theta_i^N(t) - \theta_j^N(t)).$$

Since $\theta_i^N(t)$ are solutions to the discrete singular Kuramoto model, then we arrive at

$$\frac{d}{dt}(e^{i\theta_i^N(t)}, \Omega_i^N) = (ie^{i\theta_i^N(t)}\dot{\theta}_i^N(t), 0) = \mathcal{V}[\mu_t^N](\theta_i^N(t), \Omega_i). \tag{4.3.5}$$

Putting (4.3.5) into (4.3.4) implies

$$\frac{d}{dt}\int_{\mathbb{T}}\int_{\mathbb{R}}\varphi\,d\mu_t^N = \frac{1}{N}\sum_{i=1}^N \left\langle \nabla_{(z,\Omega)}\varphi, \mathcal{V}[\mu_t^N]\right\rangle\Big|_{(z,\Omega)=(e^{i\theta_i^N(t)},\Omega_i^N)}.$$

Then, it becomes apparent that μ_t^N is a weak solution, namely,

$$\frac{d}{dt}\int_{\mathbb{T}}\int_{\mathbb{R}}\varphi\,d\mu_t^N = \int_{\mathbb{T}}\int_{\mathbb{R}}\left\langle \nabla_{(z,\Omega)}\varphi, \mathcal{V}[\mu_t^N]\right\rangle\,d\mu_t^N. \tag{4.3.6}$$

Notice that all the above computations also makes sense for $\varphi \in C_0^1(\mathbb{T}\times\mathbb{R})$. By putting the sublinear growth of $\mathcal{V}[\mu^N]$ in Theorem 4.2.8 into (4.3.6), we obtain

$$\begin{aligned}\left|\frac{d}{dt}\int_{\mathbb{T}\times\mathbb{R}}\varphi\,d\mu_t^N\right| &\leq \|\nabla\varphi\|_{C_0(\mathbb{T}\times\mathbb{R})}\int_{\mathbb{T}\times\mathbb{R}}|\mathcal{V}[\mu_t^N]|\,d\mu_t^N\\ &= \|\nabla\varphi\|_{C_0(\mathbb{T}\times\mathbb{R})}\int_{\mathbb{T}\times\mathbb{R}}|\mathcal{P}[\mu_t^N]|\,d\mu_t^N\\ &\leq \|\nabla\varphi\|_{C_0(\mathbb{T}\times\mathbb{R})}\int_{\mathbb{T}\times\mathbb{R}}\left(|\Omega| + K\|h\|_{C(\mathbb{T})}\|\mu_t^N\|_{\mathcal{M}(\mathbb{T}\times\mathbb{R})}\right)\,d\mu_t^N.\end{aligned}$$

Then, the estimate (4.3.1) becomes true. □

4.3.2 A priori estimates and compactness

Our main goal now is to derive the required compactness allowing us to pass to the limit in the weak formulation in Definition 4.2.6. To such end, we will first derive some necessary estimates in the following result.

Corollary 4.3.3. *Consider $\alpha \in (0, \frac{1}{2})$ and $K > 0$. Set N oscillators with phases and natural frequencies given by the configurations*

$$\Theta_0^N = (\theta_{1,0}^N, \ldots, \theta_{N,0}^N) \text{ and } \{\Omega_i^N : i = 1, \ldots, N\},$$

for any $N \in \mathbb{N}$. Assume that there exists a constant $M_1 > 0$ that does not depend on N such that

$$\frac{1}{N}\sum_{i=1}^N |\Omega_i^N| \leq M_1, \tag{4.3.7}$$

for all $N \in \mathbb{N}$ and consider the associated empirical measures μ^N according to Definition 4.3.1. Then,

$$\begin{aligned}&\sup_{t\in[0,T]}\|\mu_t^N\|_{C_0^1(\mathbb{T}\times\mathbb{R})^*} \leq 1,\\ &\|\mu_{t_1}^N - \mu_{t_2}^N\|_{C_0^1(\mathbb{T}\times\mathbb{R})^*} \leq (M_1 + K\|h\|_{C(\mathbb{T})})|t_1 - t_2|,\end{aligned}$$

for every $N \in \mathbb{N}$ and every $t_1, t_2 \geq 0$.

We skip the proof that is a clear consequence of the assumption (4.3.7) along with the estimate (4.3.1) in Theorem 4.3.2. In the sequel, we will need a stronger version of (4.3.7). The following result introduces the required equi-sumability condition of the natural frequencies along with its relation with condition (4.3.7).

Proposition 4.3.4. *Let us consider a configuration of $N \in \mathbb{N}$ natural frequencies*

$$\{\Omega_i^N : i = 1, \ldots, N\} \subseteq \mathbb{R},$$

for every $N \in \mathbb{N}$.

1. *Assume that the following equi-sumability condition holds true*

$$\lim_{R\to+\infty} \sup_{N\in\mathbb{N}} \frac{1}{N} \sum_{\substack{1\leq i\leq N\\ |\Omega_i^N|\geq R}} |\Omega_i^N| = 0. \tag{4.3.8}$$

   Then the sumability condition (4.3.7) *is also fulfilled.*

2. *Fix $k > 1$ and assume that the following summability condition holds true*

$$\frac{1}{N}\sum_{i=1}^N |\Omega_i^N|^k \leq M_k,$$

   for some N-independent $M_k > 0$ and each $N \in \mathbb{N}$. Then, the following condition takes place

$$\lim_{R\to 0} \sup_{N\in\mathbb{N}} \frac{1}{N} \sum_{\substack{1\leq i\leq N\\ |\Omega_i^N|\geq R}} |\Omega_i^N|^m = 0, \tag{4.3.9}$$

   for every $1 \leq m < k$. In particular, the equi-sumability condition (4.3.8) *holds.*

Proof. Regarding the first assertion, fix any arbitrary $R > 0$ and notice that

$$\frac{1}{N}\sum_{i=1}^N |\Omega_i|^N = \frac{1}{N} \sum_{\substack{1\leq i\leq N\\ |\Omega_i^N|<R}} |\Omega_i^N| + \frac{1}{N} \sum_{\substack{1\leq i\leq N\\ |\Omega_i^N|\geq R}} |\Omega_i^N| \leq R + \frac{1}{N} \sup_{\substack{1\leq i\leq N\\ |\Omega_i^N|\geq R}} |\Omega_i^N|,$$

for every $N \in \mathbb{N}$. By virtue of (4.3.8), the right hand side is bounded with respect to N and

$$\sup_{N\in\mathbb{N}} \frac{1}{N}\sum_{i=1}^N |\Omega_i^N| \leq R + \sup_{N\in\mathbb{N}} \frac{1}{N} \sup_{\substack{1\leq i\leq N\\ |\Omega_i^N|\geq R}} |\Omega_i^N|. \tag{4.3.10}$$

The optimal M_1 can be achieved by minimizing the right hand side in (4.3.10) with respect to R. Regarding the second assertion, fix $1 \leq m < k$ and notice that

$$\frac{1}{N} \sum_{\substack{1\leq i\leq N\\ |\Omega_i^N|\geq R}} |\Omega_i^N|^m \leq \frac{1}{R^{k-m}} \frac{1}{N}\sum_{i=1}^N |\Omega_i^N|^k \leq \frac{M_k}{R^{k-m}},$$

for every $R > 0$ and $N \in \mathbb{N}$. Taking supremum with respect to N and limits as $R \to 0$ yields (4.3.9). □

Corollary 4.3.5. *Let us assume that the hypothesis in Corollary 4.3.3 and that the equi-summability condition* (4.3.8) *are fulfilled. Then, the associated empirical measures μ^N in Corollary 4.3.3 are also uniformly equicontinuous in $C([0, +\infty), \mathcal{P}_1(\mathbb{T} \times \mathbb{R}) - W_1)$.*

Proof. Consider any continuous test function φ so that

$$|\varphi(z,\Omega)|\leq C_\varphi(1+|\Omega|),\ \text{ for all } (z,\Omega)\in\mathbb{T}\times\mathbb{R},$$

for some $C_\varphi>0$. Recall the scaled cut-off functions $\xi_R=\xi_R(\Omega)$ in (N.2), for any $R>0$. Fix $\varepsilon>0$ and take $R>0$ large enough so that

$$C_\varphi\left(\frac{1}{R}+1\right)\frac{1}{N}\sum_{\substack{1\leq i\leq N\\ |\Omega_i^N|\geq R}}|\Omega_i^N|\leq\frac{\varepsilon}{6}. \tag{4.3.11}$$

Notice that $\varphi\,\xi_R\in C_0(\mathbb{T}\times\mathbb{R})$. Then, there exists $\widehat{\varphi}\in C_c^\infty(\mathbb{T}\times\mathbb{R})$ so that

$$\|\varphi\,\xi_R-\widehat{\varphi}\|_{C_0(\mathbb{T}\times\mathbb{R})}\leq\frac{\varepsilon}{6}. \tag{4.3.12}$$

By virtue of Corollary 4.3.3 there exists $\delta>0$ so that

$$\left|\int_{\mathbb{T}\times\mathbb{R}}\widehat{\varphi}\,d(\mu_{t_1}^N-\mu_{t_2}^N)\right|\leq\frac{\varepsilon}{3}, \tag{4.3.13}$$

for every $|t_1-t_2|\leq\delta$ and each $N\in\mathbb{N}$. Now, consider the following split

$$\int_{\mathbb{T}\times\mathbb{R}}\varphi\,d(\mu_{t_1}^N-\mu_{t_2}^N)=A_N(t_1,t_2)+B_N(t_1,t_2)+C_N(t_1,t_2),$$

where each term reads

$$\begin{aligned}
A_N(t_1,t_2)&:=\int_{\mathbb{T}\times\mathbb{R}}\varphi\,(1-\xi_R)\,d(\mu_{t_1}^N-\mu_{t_2}^N),\\
B_N(t_1,t_2)&:=\int_{\mathbb{T}\times\mathbb{R}}(\varphi\,\xi_R-\widehat{\varphi})\,d(\mu_{t_1}^N-\mu_{t_2}^N),\\
C_N(t_1,t_2)&:=\int_{\mathbb{T}\times\mathbb{R}}\widehat{\varphi}\,d(\mu_{t_1}^N-\mu_{t_2}^N).
\end{aligned}$$

First, (4.3.11) along with the equi-sumability condition (4.3.8) implies

$$|A_N(t_1,t_2)|\leq\frac{\varepsilon}{3},$$

for all $t_1,t_2\geq 0$ and $N\in\mathbb{N}$. Second, (4.3.12) amounts to

$$|B_N(t_1,t_2)|\leq\frac{\varepsilon}{3},$$

for all $t_1,t_2\geq 0$ and $N\in\mathbb{N}$. Finally, (4.3.13) yields

$$|C_N(t_1,t_2)|\leq\frac{\varepsilon}{3},$$

for all $|t_1-t_2|\leq\delta$ and $N\in\mathbb{N}$. Putting everything together ends the proof. □

Corollary 4.3.6. *Consider $\alpha\in(0,\frac{1}{2})$ and $K>0$. For any $N\in\mathbb{N}$, set N oscillators with phases and natural frequencies given by the configurations*

$$\Theta_0^N=(\theta_{1,0}^N,\ldots,\theta_{N,0}^N)\ \text{ and }\ \{\Omega_i^N:\,i=1,\ldots,N\}.$$

Consider the (forward-in-time) unique classical solution $\Theta^N(t) = (\theta_1(t), \ldots, \theta_N(t))$ to (4.1.1)-(4.1.2) *as given in Theorem 3.3.5 of Chapter 3 and set the corresponding empirical measures μ^N according to Definition 4.3.1. Assume that the equi-sumability condition* (4.3.8) *holds true, and take M_1 fulfilling* (4.3.7) *according to Proposition 4.3.4. Then, for every fixed $T > 0$, there exists a subsequence of μ^N, that we denote in the same way for simplicity, and a limiting measure $f \in \mathcal{AC}_{\mathcal{M}} \cap \mathcal{T}_{\mathcal{M}}$ such that*

$$\sup_{t\in[0,T]} \||\Omega|\, f_t\|_{\mathcal{M}(\mathbb{T}\times\mathbb{R})} \leq M_1, \tag{4.3.14}$$

$$\lim_{R\to+\infty} \sup_{t\in[0,T]} \||\Omega|\, \chi_{|\Omega|\geq R}\, f_t\|_{\mathcal{M}(\mathbb{T}\times\mathbb{R})} = 0, \tag{4.3.15}$$

and, in addition,

$$f \in W^{1,\infty}_w([0,T], C^1_0(\mathbb{T}\times\mathbb{R})^*) \cap C([0,T], \mathcal{P}_1(\mathbb{T}\times\mathbb{R}) - W_1),$$

for every fixed $T > 0$. Moreover,

$$\mu^N \to f \ \ in\ \ C([0,T], \mathcal{P}_1(\mathbb{T}\times\mathbb{R}) - W_1), \tag{4.3.16}$$

where W_1 means the Kantorovich–Rubinstein distance.

Proof. Recall that the Riesz representation theorem in Theorem A.0.11 in Appendix A allows substracting a subsequence that converges weakly-* in $L^\infty_w(0,T;\mathcal{M}(\mathbb{T}\times\mathbb{R}))$. What is more, we can use the uniform estimates in the above Corollary 4.3.3 along with the weak-star version of the Ascoli-Arzelà theorem in Appendix B to show that there exists some subsequence and $f \in L^\infty_w(0,T;\mathcal{M}(\mathbb{T}\times\mathbb{R})) \cap C([0,T], C^1_0(\mathbb{T}\times\mathbb{R})^*)$ so that

$$\mu^N \to f \ \text{ in } \ C([0,T], C^1_0(\mathbb{T}\times\mathbb{R})^* - \text{weak}\,*).$$

Recall that the embedding $C^1_0(\mathbb{T}\times\mathbb{R}) \hookrightarrow C_0(\mathbb{T}\times\mathbb{R})$ is continuous and dense. Then, we can improve the above convergence into

$$\mu^N \to f \ \text{ in } \ C([0,T], \mathcal{M}(\mathbb{T}\times\mathbb{R}) - \text{weak}\,*),$$

i.e.,

$$\lim_{N\to\infty} \sup_{t\in[0,T]} \left| \int_{\mathbb{T}\times\mathbb{R}} \varphi\, d(\mu^N_t - f^N_t) \right| = 0, \tag{4.3.17}$$

for all $\varphi \in C_0(\mathbb{T}\times\mathbb{R})$. In order to augment the weak-star convergence into the Wasserstein one, we will first show that the other two properties that can be inherited by the limit.

• *Step 1*: Corollary 4.3.3 yields

$$\left| \int_{\mathbb{T}\times\mathbb{R}} \varphi\, d(\mu^N_{t_1} - \mu^N_{t_2}) \right| \leq (M_1 + K\|h\|_{C(\mathbb{T})})\|\varphi\|_{C^1_0(\mathbb{T}\times\mathbb{R})}|t_1 - t_2|,$$

for every $t_1, t_2 \in [0,T]$, each $\varphi \in C^1_0(\mathbb{T}\times\mathbb{R})$, and any $N \in \mathbb{N}$. Taking limits as $N \to \infty$ and using (4.3.17), we can obtain

$$\left| \int_{\mathbb{T}\times\mathbb{R}} \varphi\, d(f_{t_1} - f_{t_2}) \right| \leq (M_1 + K\|h\|_{C(\mathbb{T})})\|\varphi\|_{C^1_0(\mathbb{T}\times\mathbb{R})}|t_1 - t_2|,$$

for every $\varphi \in C^1_0(\mathbb{T}\times\mathbb{R})$ and each $t_1, t_2 \in [0,T]$. Consequently, $f \in W^{1,\infty}_w([0,T], C^1_0(\mathbb{T}\times\mathbb{R})^*)$.

• *Step 2*: Again, recall the scaled cut-off functions $\xi_R = \xi_R(\Omega)$ in (N.2). Then, notice that the function

$$(z,\Omega)\in\mathbb{T}\times\mathbb{R}\longmapsto |\Omega|\,\xi_R(\Omega)$$

belongs to $C_0(\mathbb{T}\times\mathbb{R})$. Consequently, (4.3.17) implies

$$\begin{aligned}\||\Omega|\,f_t\|_{\mathcal{M}(\mathbb{T}\times[-R,R])}&\leq\int_{\mathbb{T}\times\mathbb{R}}|\Omega|\,\xi_R(\Omega)\,d_{(z,\Omega)}f_t=\lim_{N\to\infty}\int_{\mathbb{T}\times\mathbb{R}}|\Omega|\,\xi_R(\Omega)\,d_{(z,\Omega)}\mu_t^N\\&\leq\limsup_{N\to\infty}\||\Omega|\mu_t^N\|_{\mathcal{M}(\mathbb{T}\times\mathbb{R})}=\limsup_{N\to\infty}\frac{1}{N}\sum_{i=1}^N|\Omega_i^N|\leq M_1,\end{aligned}$$

for every $t\in[0,T]$, and any $R>0$. Taking limit $R\to+\infty$ entails (4.3.14). Similarly, take a couple $0<R<R'$, and consider the test function in $C_0(\mathbb{T}\times\mathbb{R})$ defined as follows

$$(z,\Omega)\in\mathbb{T}\times\mathbb{R}\longmapsto|\Omega|\,\xi_{R'}(\Omega)(1-\xi_{R/2}(\Omega)).$$

Then, an analogue argument yields

$$\begin{aligned}\||\Omega|\,\chi_{R\leq|\Omega|\leq R'}f_t\|_{\mathcal{M}(\mathbb{T}\times\mathbb{R})}&\leq\int_{\mathbb{T}\times\mathbb{R}}|\Omega|\,\xi_{R'}(\Omega)(1-\xi_{R/2}(\Omega))\,d_{(z,\Omega)}f_t\\&=\lim_{N\to\infty}\int_{\mathbb{T}\times\mathbb{R}}|\Omega|\,\xi_{R'}(\Omega)(1-\xi_{R/2}(\Omega))\,d_{(z,\Omega)}\mu_t^N\\&\leq\limsup_{N\to\infty}\||\Omega|\,\chi_{|\Omega|\geq R}\mu_t^N\|_{\mathcal{M}(\mathbb{T}\times\mathbb{R})}\leq\sup_{N\in\mathbb{N}}\frac{1}{N}\sum_{\substack{1\leq i\leq N\\|\Omega_i^N|\geq R}}|\Omega_i^N|.\end{aligned}$$

Taking limit when $R'\to+\infty$ yields

$$\sup_{t\in[0,T]}\||\Omega|\,\chi_{|\Omega|\geq R}\,f_t\|_{\mathcal{M}(\mathbb{T}\times\mathbb{R})}\leq\sup_{N\in\mathbb{N}}\frac{1}{N}\sum_{\substack{1\leq i\leq N\\|\Omega_i^N|\geq R}}|\Omega_i^N|,$$

for every $R>0$. Finally, we obtain (4.3.15), as $R\to+\infty$. Recall that μ^N are uniformly equicontinuous in $C([0,T],\mathcal{P}_1(\mathbb{T}\times\mathbb{R})-W_1)$ thanks to Corollary 4.3.5. Then, we similarly infer that $f\in C([0,T],\mathcal{P}_1(\mathbb{T}\times\mathbb{R})-W_1)$.

• *Step 3*: Now, take a continuous test function $\varphi\in C(\mathbb{T}\times\mathbb{R})$ with linear growth, that is,

$$|\varphi(z,\Omega)|\leq C_\varphi(1+|\Omega|),\ \forall\,(z,\Omega)\in\mathbb{T}\times\mathbb{R},$$

for some $C_\varphi>0$. The integral of interest can be split as follows

$$\left|\int_{\mathbb{T}\times\mathbb{R}}\varphi\,d(\mu_t^N-f_t)\right|\leq A_N^R(t)+B_N^R(t),$$

where each term reads

$$\begin{aligned}A_N^R(t)&:=\left|\int_{\mathbb{T}\times\mathbb{R}}\varphi\,\xi_R\,d(\mu_t^N-f_t)\right|,\\B_N^R(t)&:=\left|\int_{\mathbb{T}\times\mathbb{R}}\varphi(1-\xi_R)\,d(\mu_t^N-f_t)\right|.\end{aligned}$$

Fix $\varepsilon > 0$ and consider $R > 0$ large enough so that

$$C_\varphi \left(\frac{1}{R}+1\right)\left(\frac{1}{N}\sum_{\substack{1\leq i\leq N\\ |\Omega_i^N|\geq R}} |\Omega_i^N|\right) \leq \frac{\varepsilon}{4}.$$

This can be done by virtue of hypothesis (4.3.8). Then, it is clear that

$$\begin{aligned} B_N^R(t) &\leq C_\varphi \int_{|\Omega|\geq R} (1+|\Omega|)\, d_{(z,\Omega)}(|\mu_t^N|+|f_t|)\\ &\leq C_\varphi \left(\frac{1}{R}+1\right)\int_{|\Omega|\geq R} |\Omega|\, d_{(z,\Omega)}(|\mu_t^N|+|f_t|) \leq 2C_\varphi\left(\frac{1}{R}+1\right)\left(\frac{1}{N}\sum_{\substack{1\leq i\leq N\\ |\Omega_i^N|\geq R}} |\Omega_i^N|\right)\leq \frac{\varepsilon}{2}, \end{aligned}$$

for every $N \in \mathbb{N}$, and every $t \in [0, T]$. Also, notice that the following function

$$(z,\Omega) \in \mathbb{T}\times\mathbb{R} \longmapsto \varphi(z,\Omega)\, \xi_R(\Omega),$$

belongs to $C_0(\mathbb{T}\times\mathbb{R})$. Then, applying (4.3.17) to such function, there exists $N_0 \in \mathbb{N}$ so that

$$A_N^R(t) = \left|\int_{\mathbb{T}\times\mathbb{R}} \varphi\, \xi_R\, d(\mu_t^N - f_t)\right| \leq \frac{\varepsilon}{2},$$

for every $N \geq N_0$, and every $t \in [0, T]$. Putting everything together implies the uniform-in-time convergence against any continuous function with linear growth, or, equivalently (see [296, Definition 6.8, Theorem 6.9]), the desired uniform-in-time convergence in the Kantorovich-Rubinstein distance W_1. □

Remark 4.3.7. *The above convergence in the Kantorovich-Rubinstein distance W_1 can be improved to any other Wasserstein distance W_p with $p > 1$ when the equi-sumability condition* (4.3.8) *is replaced with the general p-equi-sumability condition* (4.3.9)*. Indeed, such assumption implies*

$$\sup_{t\in[0,T]} \||\Omega|^p f_t\|_{\mathcal{M}(\mathbb{T}\times\mathbb{R})} \leq M_p,$$
$$\lim_{R\to 0}\sup_{t\in[0,T]} \||\Omega|^p \chi_{|\Omega|\geq R}\, f_t\|_{\mathcal{M}(\mathbb{T}\times\mathbb{R})} = 0.$$

Moreover,

$$\mu^N \to f \ \text{ in } \ C([0,T], \mathcal{P}_p(\mathbb{T}\times\mathbb{R}) - W_p).$$

Lemma 4.3.8. *Consider $\alpha \in (0, \frac{1}{2})$ and $K > 0$. For every $N \in \mathbb{N}$, consider N oscillators with phases and natural frequencies given by the configurations*

$$\Theta_0^N = (\theta_{1,0}^N, \ldots, \theta_{N,0}^N) \ \text{ and } \ \{\Omega_i^N : i = 1, \ldots, N\}.$$

Assume the equi-sumability condition (4.3.8)*, set the associated empirical measures μ^N according to Definition 4.3.1 and any limit f according to Corollary 4.3.6. Then,*

$$\mathcal{V}[\mu^N] - \mathcal{V}[f] \longrightarrow 0 \ \text{ in } \ C([0,T], \mathfrak{X}C_b(\mathbb{T}\times\mathbb{R})).$$

Proof. Since both μ^N and f belong to $\widetilde{C}_{\mathcal{M}}$, then Corollary 4.2.10 guarantees that

$$\frac{\mathcal{V}[\mu^N]}{1+|\Omega|},\frac{\mathcal{V}[f]}{1+|\Omega|}\in C([0,T],\mathfrak{X}C_b(\mathbb{T}\times\mathbb{R})),$$

for every $N\in\mathbb{N}$. Consequently, the continuity of both vector fields is granted. Throughout the rest of the proof, we will show that the convergence is uniform. By the Stone–Weierstrass theorem (see [131, Theorem 4.45] or [263, Theorem 7.32]) we can approximate $h(\theta-\theta')$ by products of functions with separate variables, that is, there exists $m\in\mathbb{N}$ and $\phi_1,\ldots,\phi_m,\psi_1,\ldots,\psi_m\in C(\mathbb{T})$ depending on ε and h so that

$$\left|h(\theta-\theta')-\sum_{i=1}^m\phi_i(\theta)\psi_i(\theta')\right|\leq\frac{\varepsilon}{2K},\ \forall\,\theta,\theta'\in\mathbb{R},\tag{4.3.18}$$

for every fixed $\varepsilon>0$. For simplicity, let us define

$$\widehat{h}(\theta,\theta'):=\sum_{i=1}^m\phi_i(\theta)\psi_i(\theta'),\ \forall\,\theta,\theta'\in\mathbb{R}.$$

Then, we have

$$|\mathcal{V}[\mu_t^N](\theta,\Omega)-\mathcal{V}[f_t](\theta,\Omega)|=|\mathcal{P}[\mu_t^N](\theta,\Omega)-\mathcal{P}[f_t](\theta,\Omega)|\leq F_N(t,\theta,\Omega)+G_N(t,\theta,\Omega),$$

where each term reads

$$\begin{aligned}F_N(t,\theta,\Omega)&:=K\left|\int_{(-\pi,\pi]\times\mathbb{R}}\widehat{h}(\theta,\theta')\,d_{(\theta',\Omega')}(\mu_t^N-f_t)\right|,\\ G_N(t,\theta,\Omega)&:=K\left|\int_{(-\pi,\pi]\times\mathbb{R}}(h(\theta-\theta')-\widehat{h}(\theta,\theta'))\,d_{(\theta',\Omega')}(\mu_t^N-f_t)\right|.\end{aligned}$$

Regarding the second term, the bound (4.3.18) automatically implies that

$$G_N(t,\theta,\Omega)\leq\varepsilon,$$

for every $t\in[0,T]$ and $(\theta,\Omega)\in(-\pi,\pi]\times\mathbb{R}$. On the other hand, the first term can be bounded in the following way

$$F_N(t,\theta,\Omega)\leq K\sum_{i=1}^m\|\phi_i\|_{C(\mathbb{T})}\left|\int_{(-\pi,\pi]\times\mathbb{R}}\psi_i(\theta')\,d_{(\theta',\Omega')}(\mu_t^N-f_t)\right|.$$

Using (4.3.16) in Corollary 4.3.6, one obtains that

$$\limsup_{N\to\infty}\|\mathcal{V}[\mu^N]-\mathcal{V}[f]\|_{C([0,T],\mathfrak{X}C_b(\mathbb{T}\times\mathbb{R}))}\leq\varepsilon.$$

Since $\varepsilon>0$ is arbitrary, we conclude the proof of this result. □

4.3.3 Passing to the limit

Here we will show that any limit f obtained in Corollary 4.3.6 yields a weak measure-valued solution to (4.2.5), thus solving the initial value problem for (4.2.5) with any initial data in $\mathcal{P}_1(\mathbb{T} \times \mathbb{R})$. Our first step is to show that any such initial datum can be approximated by an empirical measure associated with a discrete configuration that fulfil the above equi-sumability condition (4.3.8). Although it follows from classical arguments, we will introduce the result in our particular setting for the sake of completeness.

Lemma 4.3.9. *Consider any $\mu \in \mathcal{P}_1(\mathbb{T} \times \mathbb{R})$. Then, there exist N oscillators with phases and natural frequencies given by the configurations*

$$\Theta^N = (\theta_1^N, \ldots, \theta_N^N) \text{ and } \{\Omega_i^N : i = 1, \ldots, N\},$$

for every $N \in \mathbb{N}$, verifying the equi-sumability condition (4.3.8) *such that the corresponding empirical measures $\mu^N \in \mathcal{P}(\mathbb{T} \times \mathbb{R})$ in Definition 4.3.1 verify*

$$\lim_{N \to \infty} W_1(\mu^N, \mu) = 0.$$

Proof. By a standard application of Kolmogorov's consistency theorem [292], there exists a probability space $(E, \mathcal{F}, P)$ and a sequence of random variables $\{X_k\}_{k \in \mathbb{N}}$ with values in $\mathbb{T} \times \mathbb{R}$, namely,

$$\begin{array}{rccc} X_k = (Z_k, \Omega_k): & E & \longrightarrow & \mathbb{T} \times \mathbb{R}, \\ & \xi & \longmapsto & X_k(\xi), \end{array}$$

so that X_k are all independent and identically distributed with law μ. Let us define the following random probability measure

$$\mu^N = \frac{1}{N} \sum_{k=1}^N \delta_{X_k} = \frac{1}{N} \sum_{k=1}^N \delta_{Z_k}(z) \otimes \delta_{\Omega_k}(\Omega).$$

A straightforward application of the strong Law of Large Numbers [291] shows that

$$\lim_{N \to \infty} \int_{\mathbb{T} \times \mathbb{R}} \varphi \, d\mu_\xi^N = \int_{\mathbb{T} \times \mathbb{R}} \varphi \, d\mu, \; P - \text{a.s.} \tag{4.3.19}$$

for every $\varphi \in L^1(\mathbb{T} \times \mathbb{R}, d\mu)$. Let us define for each $R > 0$ the functions

$$\varphi_R(z, \Omega) := |\Omega| \, \chi_{|\Omega| \geq R}.$$

By the assumptions, $\varphi_R \in L^1(\mathbb{T} \times \mathbb{R}, d\mu)$ and, consequently, there exits $\{R_n\}_{n \in \mathbb{T}} \subseteq \mathbb{R}^+$ such that

$$\int_{\mathbb{T} \times \mathbb{R}} \varphi_{R_n} \, d\mu \leq \frac{1}{2n}. \tag{4.3.20}$$

Let us also set a dense sequence $\{\psi_k\}_{k \in \mathbb{N}} \subseteq C_0(\mathbb{T} \times \mathbb{R})$ in $L^1(\mathbb{T} \times \mathbb{R}, d\mu)$. Note that $\psi_k \in L^1(\mathbb{T} \times \mathbb{R}, d\mu)$ for every $k \in \mathbb{N}$. Hence, we can apply the strong Law of Large Numbers (4.3.19) to the whole family of functions

$$\{\varphi_{R_n} : n \in \mathbb{N}\} \cup \{\psi_k : k \in \mathbb{N}\},$$

to obtain

$$\lim_{N \to \infty} \int_{\mathbb{T} \times \mathbb{R}} \varphi_{R_n} \, d\mu_\xi^N = \int_{\mathbb{T} \times \mathbb{R}} \varphi_{R_n} \, d\mu, \text{ for all } \xi \in E \setminus E_n, \tag{4.3.21}$$

$$\lim_{N\to\infty}\int_{\mathbb{T}\times\mathbb{R}}\psi_k\,d\mu_\xi^N=\int_{\mathbb{T}\times\mathbb{R}}\psi_k\,d\mu,\ \text{ for all }\xi\in E\setminus F_k, \tag{4.3.22}$$

for every $n,k\in\mathbb{N}$, where $E_n,F_k\subseteq E$ are P-negligible sets. Let us define the P-negligible set

$$E':=\bigcup_{n,k\in\mathbb{N}}E_n\cup F_k.$$

Then, both (4.3.21) and (4.3.22) simultaneously hold, for all $n,k\in\mathbb{N}$ and each $\xi\in E\setminus E'$. On the one hand, (4.3.21) implies that there exists $N_n\in\mathbb{N}$ such that

$$\left|\int_{\mathbb{T}\times\mathbb{R}}\varphi_{R_n}\,d(\mu_\xi^N-\mu)\right|\leq\frac{1}{2n}, \tag{4.3.23}$$

for every $n\in\mathbb{N}$, every $N\geq N_n$, and each $\xi\in E\setminus E'$. Putting (4.3.20) and (4.3.23) together yields

$$\int_{\mathbb{T}\times\mathbb{R}}\varphi_{R_n}\,d\mu_\xi^N\leq\frac{1}{n}, \tag{4.3.24}$$

for every $N\geq N_n$, each $\xi\in E\setminus E'$ and any $n\in\mathbb{N}$. Finally, let us pick a realization $\xi_0\in E\setminus E'$ and set

$$R_n':=\max\left\{R_n,\max_{1\leq k\leq N_n}|\Omega_k(\xi_0)|\right\},$$

for every $n\in\mathbb{N}$. Note that (4.3.24) amounts to

$$\sup_{N\in\mathbb{N}}\int_{\mathbb{T}\times\mathbb{R}}\varphi_R\,d\mu_{\xi_0}^N=\sup_{N\geq N_n}\int_{\mathbb{T}\times\mathbb{R}}\varphi_R\,d\mu_{\xi_0}^N\leq\frac{1}{n},$$

for each $R>R_n'$ and any $n\in\mathbb{N}$, thus yielding the equi-sumability condition (4.3.8) for

$$\{\Omega_k(\xi_0):\,1\leq k\leq N\},\ \ N\in\mathbb{N}.$$

In addition, (4.3.22) implies that

$$\mu_{\xi_0}^N\overset{*}{\rightharpoonup}\mu\ \text{ in }\ \mathcal{M}(\mathbb{T}\times\mathbb{R}), \tag{4.3.25}$$

as a consequence of the density of $\{\psi_k\}_{k\in\mathbb{N}}$ in $C_0(\mathbb{T}\times\mathbb{R})$. Let us improve such weak-star convergence into convergence in the Rubinstein-Kantorovich metric W_1. Consider any continuous test function φ with

$$|\varphi(z,\Omega)|\leq C(1+|\Omega|),\ \text{ for all }\ (z,\Omega)\in\mathbb{T}\times\mathbb{R},$$

for some constant $C>0$. Again, the main trick is to consider a sequence of truncations. To do so, recover the cut-off functions $\xi_R=\xi_R(\Omega)$ in (N.2) and consider the split

$$\int_{\mathbb{T}\times\mathbb{R}}\varphi\,d(\mu_{\xi_0}^N-\mu)=:A_N+B_N,$$

where each term reads

$$A_N:=\int_{\mathbb{T}\times\mathbb{R}}\varphi\,\xi_R\,d(\mu_{\xi_0}^N-\mu),$$
$$B_N:=\int_{\mathbb{T}\times\mathbb{R}}\varphi\,(1-\xi_R)\,d(\mu_{\xi_0}^N-\mu).$$

On the one hand, note that

$$B_N \leq C\int_{\mathbb{T}\times\mathbb{R}} (1+|\Omega|)\,d\mu_{\xi_0}^N + C\int_{\mathbb{T}\times\mathbb{R}} (1+|\Omega|)\,d\mu.$$

Fix $\varepsilon > 0$. Taking $R > 0$ large enough, the assumption $\mu \in \mathcal{P}_1(\mathbb{T}\times\mathbb{R})$ along with the equi-sumability condition (4.3.8) show that $B_N \leq \frac{\varepsilon}{2}$, for every $N \in \mathbb{N}$. For such R, note that $\varphi\,\xi_R \in C_0(\mathbb{T}\times\mathbb{R})$. Using the above weak-star convergence (4.3.25) of $\mu_{\xi_0}^N$, one obtains $N_0 \in \mathbb{N}$ so that $A_N \leq \frac{\varepsilon}{2}$, for every $N \geq N_0$. Putting everything together, we conclude the proof. □

We are now ready to obtain the mean field limit that, in particular, yields the following existence result.

Theorem 4.3.10. *Consider $\alpha \in (0, \frac{1}{2})$, $K > 0$ and set any initial datum $f_0 \in \mathcal{P}_1(\mathbb{T}\times\mathbb{R})$. Then, for every $T > 0$ there exists a weak measure-valued solution $f \in \mathcal{AC}_{\mathcal{M}} \cap \mathcal{T}_{\mathcal{M}}$ to the initial value problem* (4.2.5). *In addition,* (4.3.14)-(4.3.15) *holds and*

$$f \in W_w^{1,\infty}([0,T], C_0^1(\mathbb{T}\times\mathbb{R})^*) \cap C([0,T], \mathcal{P}_1(\mathbb{T}\times\mathbb{R}) - W_1).$$

Proof. Our first step is to take a discrete approximation like in Lemma 4.3.9. Namely, consider N oscillators with phases and natural frequencies given by the configurations

$$\Theta_0^N = (\theta_{1,0}^N, \ldots, \theta_{N,0}^N) \text{ and } \{\Omega_i^N : i = 1, \ldots, N\},$$

for every $N \in \mathbb{N}$ so that they verify the equi-sumability condition (4.3.8) and the associated empirical measures $\mu_t^N \in \mathcal{P}(\mathbb{T}\times\mathbb{R})$ in Definition 4.3.1 verify

$$\lim_{N\to\infty} W_1(\mu_0^N, f_0) = 0.$$

Using Theorem 4.3.2, we infer that μ^N are weak-measure valued solutions to (4.2.5) issued at μ_0^N. Then, they verify the following weak formulation (see Definition 4.2.6)

$$\int_0^T\int_{\mathbb{T}\times\mathbb{R}} \frac{\partial\varphi}{\partial t}\, d_{(z,\Omega)}\mu_t^N\,dt + \int_0^T\int_{\mathbb{T}\times\mathbb{R}} \left\langle \mathcal{V}[\mu_t^N], \nabla_{(z,\Omega)}\varphi\right\rangle\, d_{(z,\Omega)}\mu_t^N\,dt$$
$$= -\int_{\mathbb{T}\times\mathbb{R}} \varphi(0,z,\Omega) d_{(z,\Omega)}\mu_0^N, \quad (4.3.26)$$

for every $\varphi \in C_c^1([0,T)\times\mathbb{T}\times\mathbb{R})$. Using Corollary 4.3.6, consider any weak limit f of a subsequence of μ^N, that we still denote in the same way for simplicity. In particular, recall that

$$\mu^N \to f \text{ in } C([0,T], \mathcal{P}_1(\mathbb{T}\times\mathbb{R}) - W_1).$$

Now, we can pass to the limit in the weak formulation (4.3.26) as $N \to \infty$. Specifically, regarding the first and third term, the passage to the limit is clear by linearity. Regarding the nonlinear term, let us show that the following term vanishes in the limit $N \to \infty$

$$I_N := \int_0^T\int_{\mathbb{T}\times\mathbb{R}} \left\langle \mathcal{V}[\mu_t^N], \nabla_{(z,\Omega)}\varphi\right\rangle\, d\mu_t^N\,dt - \int_0^T\int_{\mathbb{T}\times\mathbb{R}} \left\langle \mathcal{V}[f_t], \nabla_{(z,\Omega)}\varphi\right\rangle\, df_t\,dt,$$

for any given $\varphi \in C_c^1([0,T)\times\mathbb{T}\times\mathbb{R})$. Indeed, consider the following split

$$I_N = I_N^1 + I_N^2,$$

where each term reads

$$I_N^1 := \int_0^T \int_{\mathbb{T}\times\mathbb{R}} \left\langle \mathcal{V}[\mu_t^N] - \mathcal{V}[f_t], \nabla_{(z,\Omega)}\varphi \right\rangle \, d\mu_t^N \, dt,$$
$$I_N^2 := \int_0^T \int_{\mathbb{T}\times\mathbb{R}} \left\langle \mathcal{V}[f_t], \nabla_{(z,\Omega)}\varphi \right\rangle \, d(\mu_t^N - f_t) \, dt.$$

On the one hand, note that

$$|I_N^1| \leq \|\varphi\|_{C_b([0,T)\times\mathbb{T}\times\mathbb{R})} \|\mathcal{V}[\mu^N] - \mathcal{V}[f]\|_{C([0,T],\mathfrak{X}C_b(\mathbb{T}\times\mathbb{R}))} \to 0,$$

by virtue of Corollary 4.3.6. On the other hand, notice that $\langle \mathcal{V}[f], \nabla\varphi\rangle \in C_c([0,T) \times \mathbb{T} \times \mathbb{R})$ thanks to Corollary 4.2.10. Then, we can also pass to the limit in I_N^2 to show that $I_n^2 \to 0$, thus ending the proof. □

4.4 Uniqueness and rigorous mean field limit

The purpose of this part is to derive an upper bound for the growth of some Wasserstein-type distance between any two weak measure-valued solutions of (4.2.5). This is a *Dobrushin-type inequality* that has long been studied to show stability in mean-field equations. It is originally devoted to R. Dobrushin and H. Neunzert when the ambient space is $\mathbb{R}^d$ and the kernel is Lipschitz, see [112, 230]. In that case, the bounded-Lipschitz distance fits with the Lipschitz-continuity property of the kernel. Mimicking the ideas in the above-mentioned paper, an analogue result has been explored for the classical Kuramoto-Sakaguchi equation in [58, 198], where the kernel is still Lipschitz-continuous (it agrees with the sine function). However, such approach fails in our case because the singularly-weighted kernel h is no longer Lipschitz-continuous but barely Hölder-continuous in the subcritical case. Indeed, it is even discontinuous in the critical and supercritical regimes, that will be studied later in forthcoming sections.

Very recently, the case of non-Lipschitz kernels has been explored for the aggregation equations in $\mathbb{R}^d$. This is a gradient-flow system governed by the negative gradient of a λ-convex potential, see [64, 67] and the introductory Chapter 1. In such case, the quadratic Wasserstein distance W_2 has been considered instead of the bounded-Lipschitz distance for Lipschitz interactions. In our case (4.2.5), we will not use any gradient structure. Also, the natural frequencies have introduce extra heterogeneities in the system that does not appears in the aggregation equation. That requires turning the ambient Euclidean space $\mathbb{R}^d$ for the aggregation equation into the Riemannian manifold $\mathbb{T} \times \mathbb{R}$, that involves both periodicity in θ and heterogeneity Ω. Despite the above differences with the classical Lipschitz mean-field models and the aggregation equation, we shall recover Dobrushin-type estimates.

On the one hand, we will prove such estimate for an adapted version of the quadratic Wasserstein distance that we call *fiberwise quadratic Wasserstein distance* and any couple of general weak measure-valued solutions issued at initial data with the same distribution of natural frequencies g. This will be the cornerstone in our uniqueness result. On the other hand, for initial data whose distribution of natural frequencies differ, we will obtain an analogue estimate for the *classical quadratic Wasserstein distance* as long as such distributions of natural frequencies have bounded second order moments. The latter estimate will be used to derive the rigorous mean field limit.

4.4.1 Stability of fiberwise quadratic Wasserstein distance and uniqueness result

We refer to Appendix F for the introduction of the above-mentioned fiberwise quadratic Wasserstein distance. Also, the necessary properties of optimal transport theory that we require to construct such distance have been provided in such appendix for the reader's convenience. To the best of our knowledge, this is the first time that this distance has been introduced in the literature and has been applied to derive stability estimates of kinetic systems with heterogeneities. For its relation with the classical quadratic distance and equivalent representations that will be used in forthcoming chapters, see also Appendix F.

In the following result, we show that the conditional probabilities $f(\cdot|\Omega)$ of a solution to (4.2.5) with respect to a given value of the natural frequency $\Omega \in \mathbb{R}$ behave in a fiberwise way in the sense that they propagate fiber by fiber along the characteristic flow of the system.

Lemma 4.4.1. *Consider $\alpha \in (0, \frac{1}{2})$, $K > 0$ and let $f \in \mathcal{AC}_{\mathcal{M}} \cap \mathcal{T}_{\mathcal{M}}$ be a weak mesure-valued solution to* (4.2.5) *with initial datum $f_0 \in \mathcal{P}(\mathbb{T} \times \mathbb{R})$, according to Theorem 4.3.10. Let $X(t; 0, z, \Omega) = (Z(t; 0, z, \Omega), \Omega)$ be the flow associated with the transport field $\mathcal{V}[f]$, according to Theorem 4.2.19. Then,*

1. *The solution remains normalized, i.e.,*
$$f_t \in \mathcal{P}(\mathbb{T} \times \mathbb{R}), \ \text{for all } t \geq 0.$$
2. *The Ω-marginal remains unchanged, i.e.,*
$$(\pi_\Omega)_{\#} f_t = (\pi_\Omega)_{\#} f_0 \equiv g, \ \text{for all } t \geq 0.$$
3. *The disintegrations or conditional probabilities $\{f_t(\cdot|\Omega)\}_{\Omega \in \mathbb{R}} \subseteq \mathcal{P}(\mathbb{T})$ with respect to $\Omega \in \mathbb{R}$ (see Theorem F.4.1) propagate through the flow, i.e.,*
$$f_t(\cdot|\Omega) = Z(t; 0, \cdot, \Omega)_{\#} f_0(\cdot|\Omega), \ \text{for all } t \geq 0, \ g\text{-a.e. } \Omega \in \mathbb{R}.$$

Here, π_Ω is the projection in Ω, see (N.1).

Proof. Recall that, as discussed in Remark 4.2.26, f is also a solution in the sense of the flow, that is $f_t = X(t; 0, \cdot)_{\#} f_0$, for every $t \geq 0$. Recall that it is characterized by the following identity
$$\int_{\mathbb{T} \times \mathbb{R}} \varphi(z, \Omega) \, d_{(z,\Omega)} f_t = \int_{\mathbb{T} \times \mathbb{R}} \varphi(Z(t; 0, z, \Omega), \Omega) \, d_{(z,\Omega)} f_0, \tag{4.4.1}$$
for any $\varphi \in C_b(\mathbb{T} \times \mathbb{R})$. In particular, taking $\varphi \equiv 1$ the first assertion becomes clear. Regarding the second assertion, consider any text function $\phi \in C_b(\mathbb{R})$ and notice the following chain of identities
$$\int_{\mathbb{R}} \phi(\Omega) \, d_\Omega[(\pi_\Omega)_{\#} f_t] = \int_{\mathbb{T} \times \mathbb{R}} \phi(\Omega) \, d_{(z,\Omega)} f_t = \int_{\mathbb{T} \times \mathbb{R}} \phi(\Omega) \, d_{(z,\Omega)} f_0 = \int_{\Omega} \phi(\Omega) d_\Omega[(\pi_\Omega)_{\#} f_0],$$
where we have used the bounded-continuous test function $\varphi(z, \Omega) = \phi(\Omega)$ in Equation (4.4.1). Hence, the second claim is apparent by definition. Finally, consider any $\phi \in C(\mathbb{T})$ and $\psi \in C_b(\mathbb{R})$ and define $\varphi(z, \Omega) = \phi(z)\psi(\Omega)$. Using the disintegration formula (F.4.1) in both members of (4.4.1) implies
$$\int_{\mathbb{R}} \psi(\Omega) \left(\int_{\mathbb{T}} \phi(z, \Omega) \, d_z f_t(\cdot|\Omega) \right) d_\Omega g = \int_{\mathbb{R}} \psi(\Omega) \left(\int_{\mathbb{T}} \phi(Z(t; 0, z, \Omega)) d_z f_0(\cdot|\Omega) \right) d_\Omega g.$$
Since ψ is arbitrary we get
$$\int_{\mathbb{T}} \phi(z, \Omega) \, d_z f_t(\cdot|\Omega) = \int_{\mathbb{T}} \phi(Z(t; 0, z, \Omega)) d_z f_0(\cdot|\Omega),$$
g-a.e. $\Omega \in \mathbb{R}$, for every $t \geq 0$, thus ending the proof of the result. $\square$

Theorem 4.4.2. *Consider $\alpha \in (0, \frac{1}{2})$, $K > 0$ and let $f^1, f^2 \in \mathcal{AC}_{\mathcal{M}} \cap \mathcal{T}_{\mathcal{M}}$ be weak measure-valued solutions to* (4.2.5) *with initial data $f_0^1, f_0^2 \in \mathcal{P}(\mathbb{T} \times \mathbb{R})$ according to Theorem 4.3.10. Let us set their distributions of natural frequencies $g^i = (\pi_\Omega)_\# f_0^i$ for $i = 1, 2$. If $g^1 \equiv g^2 =: g$, then*

$$W_{2,g}(f_t^1, f_t^2) \leq W_{2,g}(f_0^1, f_0^2)e^{2KL_0 t},$$

for every $t \geq 0$, where L_0 is the one-sided Lipschitz constant of $-h$ in Lemma 4.2.16.

Proof. Again, f^1, f^2 are solutions in the sense of the flow by Remark 4.2.26. For g-a.e. $\Omega \in \mathbb{R}$ fixed, let us consider the corresponding term of the family of disintegrations at the initial time, i.e., $f_0^1(\cdot|\Omega)$ and $f_0^2(\cdot|\Omega)$. Set an optimal transference plan from the former probability measure in $\mathbb{T}$ to the latter one, i.e.,

$$\mu_{0,\Omega} \in \Pi(f_0^1(\cdot|\Omega), f_0^2(\cdot|\Omega)) := \left\{\mu \in \mathcal{P}(\mathbb{T} \times \mathbb{T}) : (\pi_1)_\#\mu = f_0^1(\cdot|\Omega) \text{ and } (\pi_2)_\#\mu = f_0^2(\cdot|\Omega)\right\},$$

so that the 2-Wasserstein distance is attained

$$W_2(f_0^1(\cdot|\Omega), f_0^2(\cdot|\Omega))^2 = \int_{\mathbb{T}}\int_{\mathbb{T}} d(z_1, z_2)^2 d_{(z_1,z_2)}\mu_{0,\Omega}.$$

Here, we are denoting the projections $\pi_1(z, z') = z$ and $\pi_2(z, z') = z'$. The existence of an optimal transference plan is granted by Theorem F.1.3 in Appendix F (see also the textbooks [9, 268, 296]). Then, we can define a competitor transference plan at time t as push-forward of the initial one, that is,

$$\mu_{t,\Omega} := (Z_{f^1}(t; 0, \cdot, \Omega) \otimes Z_{f^2}(t; 0, \cdot, \Omega))_\#\mu_{0,\Omega} \in \mathcal{P}(\mathbb{T} \times \mathbb{T}),$$

where $X_{f^i}(t; 0, z, \Omega) = (Z_{f^i}(t; 0, z, \Omega), \Omega)$ is the characteristic flow associated with the transport field $\mathcal{V}[f^i]$ according to Theorem 4.2.19 for $i = 1, 2$. Notice that by definition

$$\begin{aligned}
(\pi_1)_\#\mu_{t,\Omega} &= Z_{f^1}(t; 0, \cdot, \Omega)_\#((\pi_1)_\#\mu_{0,\Omega}) = Z_{f^1}(t; 0, \cdot, \Omega)_\# f_0^1(\cdot|\Omega),\\
(\pi_2)_\#\mu_{t,\Omega} &= Z_{f^2}(t; 0, \cdot, \Omega)_\#((\pi_2)_\#\mu_{0,\Omega}) = Z_{f^2}(t; 0, \cdot, \Omega)_\# f_0^2(\cdot|\Omega).
\end{aligned}$$

Using the third statement in Lemma 4.4.1 we conclude that

$$(\pi_1)_\#\mu_{t,\Omega} = f_t^1(\cdot|\Omega) \text{ and } (\pi_2)_\#\mu_{t,\Omega} = f_t^2(\cdot|\Omega).$$

Thus, $\mu_{t,\Omega} \in \Pi(f_t^1(\cdot|\Omega), f_t^2(\cdot|\Omega))$ and, consequently,

$$\begin{aligned}
\frac{1}{2}W_2(f_t^1(\cdot|\Omega), f_t^2(\cdot|\Omega))^2 &\leq \int_{\mathbb{T}}\int_{\mathbb{T}} \frac{1}{2}d(z_1, z_2)^2\, d_{(z_1,z_2)}\mu_{t,\Omega}\\
&= \int_{\mathbb{T}}\int_{\mathbb{T}} \frac{1}{2}d(Z_{f^1}(t; 0, z_1, \Omega), Z_{f^2}(t; 0, z_2, \Omega))^2\, d_{(z_1,z_2)}\mu_{0,\Omega}.
\end{aligned}$$

Integrating the above inequality against g yields

$$\frac{1}{2}W_{2,g}(f_t^1, f_t^2)^2 \leq \int_{\mathbb{R}}\int_{\mathbb{T}}\int_{\mathbb{T}} \frac{1}{2}d(Z_{f^1}(t; 0, z_1, \Omega), Z_{f^2}(t; 0, z_2, \Omega))^2\, d_{(z_1,z_2)}\mu_{0,\Omega}\, d_\Omega g =: I(t).$$

We are interested in proving some Grönwall-type inequality for $I = I(t)$. For every $z_1 = e^{i\theta_1}, z_2 = e^{i\theta_2} \in \mathbb{T}$ and g-a.e. $\Omega \in \mathbb{R}$ fixed, let us define the trajectories

$$Z_1(t) = Z_{f^1}(t; 0, z_1, \Omega) = e^{i\Theta_1(t)} \text{ and } Z_2(t) = Z_{f^2}(t; 0, z_2, \Omega) = e^{i\Theta_2(t)},$$

where $\Theta_1(t) = \Theta_{f^1}(t;0,\theta_1,\Omega)$ and $\Theta_2(t) = \Theta_{f^2}(t;0,\theta_2,\Omega)$ are the unique solutions to (4.2.8) in Lemma 4.2.12 with initial data θ_1 and θ_2 respectively. Consider a minimizing geodesic $\gamma_t : [0,1] \longrightarrow \mathbb{T}$ joining $Z_1(t)$ to $Z_2(t)$, for every fixed $t \geq 0$. Since the map

$$t \longmapsto \frac{1}{2}d^2(Z_1(t), Z_2(t)),$$

is clearly absolutely continuous, we can use previous arguments like in the proof of Theorem 4.2.19 to achieve the following estimate

$$\frac{d}{dt}\frac{1}{2}d^2(Z_1(t),Z_2(t)) \leq -\left\langle \mathcal{P}[f_t^1](Z_1(t),\Omega)iZ_1(t), \gamma_t'(0)\right\rangle - \left\langle \mathcal{P}[f_t^2](Z_2(t),\Omega)iZ_2(t), -\gamma_t'(1)\right\rangle.$$

Recall that to obtain it we need to use the one-sided Dini upper directional differentiability of the squared distance in $\mathbb{T}$, see Appendix 4.B. Now, let us describe such geodesics in γ_t. The way to go is analogous to that in the proof of Lemma 4.2.17, namely, consider $\theta_{21}(t) := \overline{\Theta_2(t) - \Theta_1(t)}$, the representative of $\Theta_2(t) - \Theta_1(t)$ modulo 2π that lies in the interval $(-\pi, \pi]$. There are two different cases.

• *Case 1*: $\theta_{21}(t) \in (-\pi,\pi)$. In this case there exists only one such minimizing geodesic and it reads

$$\gamma_t(s) = e^{i(\Theta_1(t)+s\theta_{21}(t))},\ s \in [0,1].$$

Hence, the above inequality can be restated as

$$\frac{d}{dt}\frac{1}{2}d^2(Z_1(t),Z_2(t)) \leq \left(\mathcal{P}[f_t^2](\Theta_2(t),\Omega) - \mathcal{P}[f_t^1](\Theta_1(t),\Omega)\right)\theta_{21}(t).$$

• *Case 2*: $\theta_{21}(t) = \pi$. In this second case there are exactly two minimizing geodesics

$$\gamma_{t,\pm}(s) = e^{i(\Theta_1(t)\pm\pi s)},\ s \in [0,1].$$

In such case, the above inequality reads

$$\frac{d}{dt}\frac{1}{2}d^2(Z_1(t),Z_2(t)) \leq \left(\mathcal{P}[f_t^2](\Theta_2(t),\Omega) - \mathcal{P}[f_t^1](\Theta_1(t),\Omega)\right)(\pm\pi).$$

Putting everything together, we arrive at the following inequality

$$\begin{aligned}&\frac{d}{dt}\frac{1}{2}d(Z_{f^1}(t;0,z_1,\Omega), Z_{f^2}(t;0,z_2,\Omega))\\ &\quad\leq (\mathcal{P}[f_t^1](\Theta_{f^1}(t;0,\theta_1,\Omega),\Omega) - \mathcal{P}[f_t^2](\Theta_{f^2}(t;0,\theta_2,\Omega),\Omega))\overline{\Theta_{f^1}(t;0,\theta_1,\Omega) - \Theta_{f^2}(t;0,\theta_2,\Omega)},\end{aligned}$$

for every θ_1, θ_2 and almost every $t \geq 0$. By virtue of the dominated convergence theorem, we show that I is absolutely continuous and we take derivatives under integral sign to obtain

$$\begin{aligned}\frac{dI}{dt} \leq \int_{\mathbb{R}}\int_{(-\pi,\pi]}\int_{(-\pi,\pi]} &(\mathcal{P}[f_t^1](\Theta_{f^1}(t;0,\theta_1,\Omega),\Omega) - \mathcal{P}[f_t^2](\Theta_{f^2}(t;0,\theta_2,\Omega),\Omega))\\ &\times \overline{\Theta_{f^1}(t;0,\theta_1,\Omega) - \Theta_{f^2}(t;0,\theta_2,\Omega)}\, d_{(\theta_1,\theta_2)}\mu_{0,\Omega}\, d_\Omega g,\end{aligned} \tag{4.4.2}$$

for almost every $t \geq 0$. Now, we need to identify $\mathcal{V}[\mu]$ as push-forward of initial data. Indeed, notice that by (disintegration) Theorem F.4.1 and Lemma 4.4.1

$$\mathcal{P}[f_t^i](\theta,\Omega) = \Omega - K\int_{(-\pi,\pi]}\int_{\mathbb{R}} h(\theta-\theta')\, d_{(\theta',\Omega')}f_t^i$$

$$= \Omega - K \int_{\mathbb{R}} \left(\int_{(-\pi,\pi]} h(\theta - \theta')\, d_{\theta'} f_t^i(\cdot|\Omega') \right) d_{\Omega'} g$$
$$= \Omega - K \int_{\mathbb{R}} \left(\int_{(-\pi,\pi]} h(\theta - \Theta_{f^i}(t;0,\theta_i',\Omega'))\, d_{\theta_i'} f_0^i(\cdot|\Omega') \right) d_{\Omega'} g,$$

for $i = 1, 2$. Recall that $(\pi_1)_{\#}\mu_{0,\Omega'} = f_0^1(\cdot|\Omega')$ and $(\pi_2)_{\#}\mu_{0,\Omega'} = f_0^2(\cdot|\Omega')$. Then, we obtain

$$\mathcal{P}[f_t^1](\theta,\Omega) = \Omega - K \int_{\mathbb{R}} \left(\int_{(-\pi,\pi]} \int_{(-\pi,\pi]} h(\theta - \Theta_{f^1}(t;0,\theta_1',\Omega'))\, d_{(\theta_1',\theta_2')}\mu_{0,\Omega'} \right) d_{\Omega'} g, \tag{4.4.3}$$
$$\mathcal{P}[f_t^2](\theta,\Omega) = \Omega - K \int_{\mathbb{R}} \left(\int_{(-\pi,\pi]} \int_{(-\pi,\pi]} h(\theta - \Theta_{f^2}(t;0,\theta_2',\Omega'))\, d_{(\theta_1',\theta_2')}\mu_{0,\Omega'} \right) d_{\Omega'} g. \tag{4.4.4}$$

Putting (4.4.3)-(4.4.4) into (4.4.2), we obtain the following expression

$$\begin{aligned}\frac{dI}{dt} \leq -K \int_{((-\pi,\pi]^2\times\mathbb{R})^2} &(h(\Theta_{f^1}(t;0,\theta_1,\Omega) - \Theta_{f^1}(t;0,\theta_1',\Omega')) - h(\Theta_{f^2}(t;0,\theta_2,\Omega) - \Theta_{f^2}(t;0,\theta_2',\Omega')))\\ &\times \overline{\Theta_{f^1}(t;0,\theta_1,\Omega) - \Theta_{f^2}(t;0,\theta_2,\Omega)}\, d_{(\theta_1,\theta_2)}\mu_{0,\Omega}\, d_\Omega g\, d_{(\theta_1',\theta_2')}\mu_{0,\Omega'}\, d_{\Omega'} g\end{aligned} \tag{4.4.5}$$

for almost every $t \geq 0$. Now, let us change variables $(\theta_1, \theta_2, \Omega)$ with $(\theta_1', \theta_2', \Omega')$

$$\begin{aligned}\frac{dI}{dt} \leq -K \int_{((-\pi,\pi]^2\times\mathbb{R})^2} &-(h(\Theta_{f^1}(t;0,\theta_1,\Omega) - \Theta_{f^1}(t;0,\theta_1',\Omega')) - h(\Theta_{f^2}(t;0,\theta_2,\Omega) - \Theta_{f^2}(t;0,\theta_2',\Omega')))\\ &\times \overline{\Theta_{f^1}(t;0,\theta_1',\Omega') - \Theta_{f^2}(t;0,\theta_2',\Omega')}\, d_{(\theta_1,\theta_2)}\mu_{0,\Omega}\, d_\Omega g\, d_{(\theta_1',\theta_2')}\mu_{0,\Omega'}\, d_{\Omega'} g,\end{aligned} \tag{4.4.6}$$

for almost every $t \geq 0$, where the antisymmetry of the kernel h around the origin has been used. Taking the mean value of both expressions (4.4.5) and (4.4.6) yields

$$\begin{aligned}\frac{dI}{dt} \leq \frac{K}{2} \int_{((-\pi,\pi]^2\times\mathbb{R})^2} &-(h(\Theta_{f^1}(t;0,\theta_1,\Omega) - \Theta_{f^1}(t;0,\theta_1',\Omega')) - h(\Theta_{f^2}(t;0,\theta_2,\Omega) - \Theta_{f^2}(t;0,\theta_2',\Omega')))\\ &\times \left(\overline{\Theta_{f^1}(t;0,\theta_1,\Omega) - \Theta_{f^2}(t;0,\theta_2,\Omega)} - \overline{\Theta_{f^1}(t;0,\theta_1',\Omega') - \Theta_{f^2}(t;0,\theta_2',\Omega')}\right)\\ &\times d_{(\theta_1,\theta_2)}\mu_{0,\Omega}\, d_\Omega g\, d_{(\theta_1',\theta_2')}\mu_{0,\Omega'}\, d_{\Omega'} g,\end{aligned} \tag{4.4.7}$$

for almost every $t \geq 0$. Denote

$$\Theta_1 := \Theta_{f^1}(t;0,\theta_1,\Omega), \qquad \Theta_1' := \Theta_{f^1}(t;0,\theta_1',\Omega'),$$
$$\Theta_2 := \Theta_{f^2}(t;0,\theta_2,\Omega), \qquad \Theta_2' := \Theta_{f^2}(t;0,\theta_2',\Omega'),$$

for almost every $t \geq 0$, each $\theta_1, \theta_2, \theta_1', \theta_2' \in (-\pi, \pi]$ and any Ω, Ω' fixed, then the integrand reads

$$((-h)(\Theta_1 - \Theta_1') - (-h)(\Theta_2 - \Theta_2'))\left(\overline{\Theta_1 - \Theta_2} - \overline{\Theta_1' - \Theta_2'}\right).$$

Let us now make a choice of representatives modulo 2π for such phases $\widehat{\Theta}_1, \widehat{\Theta}_2, \widehat{\Theta}_1', \widehat{\Theta}_2' \in \mathbb{R}$ with

$$\widehat{\Theta}_1 - \widehat{\Theta}_2 \in (-\pi, \pi], \qquad \widehat{\Theta}_1' - \widehat{\Theta}_2' \in (-\pi, \pi],$$
$$\widehat{\Theta}_1 - \widehat{\Theta}_1' \in [-2\pi, 2\pi], \qquad \widehat{\Theta}_2 - \widehat{\Theta}_2' \in [-2\pi, 2\pi].$$

Then, the integrand can be rewritten as follows

$$((-h)(\widehat{\Theta}_1-\widehat{\Theta}_1') - (-h)(\widehat{\Theta}_2 - \widehat{\Theta}_2'))\left((\widehat{\Theta}_1 - \widehat{\Theta}_2) - (\widehat{\Theta}_1' - \widehat{\Theta}_2')\right)$$
$$= ((-h)(\widehat{\Theta}_1 - \widehat{\Theta}_1') - (-h)(\widehat{\Theta}_2 - \widehat{\Theta}_2'))\left((\widehat{\Theta}_1 - \widehat{\Theta}_1') - (\widehat{\Theta}_2 - \widehat{\Theta}_2')\right).$$

Since all the terms lie in $[-2\pi, 2\pi]$, then Lemma 4.2.16 yields

$$\left((-h)(\widehat{\Theta}_1 - \widehat{\Theta}_1') - (-h)(\widehat{\Theta}_2 - \widehat{\Theta}_2')\right)\left((\widehat{\Theta}_1 - \widehat{\Theta}_2) - (\widehat{\Theta}_1' - \widehat{\Theta}_2')\right)$$
$$\leq L_0\,|(\widehat{\Theta}_1 - \widehat{\Theta}_1') - (\widehat{\Theta}_2 - \widehat{\Theta}_2')|^2 \leq 2L_0\,(|\widehat{\Theta}_1 - \widehat{\Theta}_2|^2 + |\widehat{\Theta}_1' - \widehat{\Theta}_2'|^2)$$
$$= 2L_0\,\left(|\Theta_1 - \Theta_2|_o^2 + |\Theta_1' - \Theta_2'|_o^2\right).$$

Thus, it becomes apparent that

$$\frac{dI}{dt} \leq 4KL_0\,I, \quad \text{for a.e. } t \geq 0.$$

By virtue of Grönwall's lemma, we end up with

$$I(t) \leq I(0)e^{4KL_0\,t},\ t \geq 0.$$

Notice that

$$I(0) = \int_{\mathbb{R}}\left(\int_{\mathbb{T}}\int_{\mathbb{T}}\frac{1}{2}d(z_1,z_2)^2\,d_{(z_1,z_2)}\mu_{0,\Omega}\right)\,d_\Omega g = \frac{1}{2}W_{2,g}(f_0^1, f_0^2)^2,$$

which ends the proof. □

Remark 4.4.3. *In Theorem 4.4.2, we have not shown any differential inequality for the Wasserstein distance $W_{2,g}(f_t^1, f_t^2)$ between two measure valued solutions to* (4.1.3). *However, it is clear that we can come back to some sort of differential inequality. Specifically, notice that Theorem 4.4.2 amounts to the inequality*

$$\frac{W_{2,g}(f_t^1, f_t^2) - W_{2,g}(f_{t_0}^1, f_{t_0}^2)}{(t-t_0)} \leq W_{2,g}(f_{t_0}^1, f_{t_0}^2)\frac{e^{2KL_0(t-t_0)-1}}{t-t_0},$$

for any $t \geq t_0 \geq 0$. Then, taking $\limsup$ *as $t \searrow t_0$ in the left hand side and using L'Hôpital rule in the right hand side, we obtain*

$$\frac{d^+}{dt}W_{2,g}(f_t^1, f_t^2) \leq 2KL_0 W_{2,g}(f_t^1, f_t^2),$$

for any $t \geq 0$, where $\frac{d^+}{dt}$ is the one sided upper Dini derivative (recall Definition 4.B.4 in Appendix 4.B).

As a clear consequence, we obtain the following uniqueness result for general initial data.

Corollary 4.4.4. *Consider $\alpha \in (0, \frac{1}{2})$, $K > 0$ and let $f^1, f^2 \in \mathcal{AC}_{\mathcal{M}} \cap \mathcal{T}_{\mathcal{M}}$ be weak measure-valued solutions to* (4.2.5) *with initial data $f_0^1, f_0^2 \in \mathcal{P}(\mathbb{T}\times\mathbb{R})$. If $f_0^1 = f_0^2$, then*

$$f_t^1 = f_t^2, \ \text{for every } t \geq 0.$$

4.4.2 Stability of quadratic Wasserstein distance and mean field limit

When the distributions of natural frequencies of both solutions do not agree, then the metric space $(\mathcal{P}_g(\mathbb{T}\times\mathbb{R}), W_{2,g})$ cannot be used. In such general case, we will simply resort on the standard quadratic Wasserstein distance W_2 in both variables (z,Ω). Nevertheless, such approach requires the solutions to lie in $\mathcal{P}_2(\mathbb{T}\times\mathbb{R})$. The next result shows that we only need to require that on the initial datum.

Lemma 4.4.5. *Consider $\alpha\in(0,\frac{1}{2})$, $K>0$ and let $f\in\mathcal{AC}_{\mathcal{M}}\cap\mathcal{T}_{\mathcal{M}}$ be a weak measure-valued solution to* (4.2.5)*. Assume that the distribution $g=(\pi_\Omega)_\# f$ of natural frequencies has bounded second order moment, i.e., $\Omega^2 g\in\mathcal{M}(\mathbb{R})$, then*

$$\sup_{t\geq 0}\|\Omega^2 f_t\|_{\mathcal{M}(\mathbb{T}\times\mathbb{R})}\leq\|\Omega^2 g\|_{\mathcal{M}(\mathbb{R})}<\infty.$$

Proof. Consider the scaled cut-off functions $\{\xi_R\}_{R>0}$ in (N.2) and the compactly supported test functions $\varphi_R(z,\Omega)=\xi_R(\Omega)\Omega^2$. Since f is a solution in the sense of the flow by Remark 4.2.26, then we claim

$$\begin{aligned}\int_{\mathbb{T}\times\mathbb{R}}\varphi_R(z,\Omega)\,d_{(z,\Omega)}f_t&=\int_{\mathbb{T}\times\mathbb{R}}\varphi_R(Z_f(t;0,z,\Omega),\Omega)\,d_{(z,\Omega)}f_0\\&=\int_{\mathbb{T}\times\mathbb{R}}\xi_R(\Omega)\Omega^2\,d_{(R,\Omega)}f_0\\&=\int_{\mathbb{R}}\xi_R(\Omega)\Omega^2\,d_\Omega g,\end{aligned}$$

for every $R>0$. On the one hand, the right hand side has a limit by virtue of the dominated convergence theorem. Then, Fatou's lemma implies

$$\int_{\mathbb{T}\times\mathbb{R}}\Omega^2\,d_{(z,\Omega)}f_t\leq\liminf_{R\to+\infty}\int_{\mathbb{T}\times\mathbb{R}}\varphi_R(z,\Omega)\,d_{(z,\Omega)}f_t=\lim_{R\to+\infty}\int_{\mathbb{R}}\xi_R(\Omega)\Omega^2\,d_\Omega g=\int_{\mathbb{R}}\Omega^2\,d_\Omega g,$$

thus, proving our assertion. ☐

Theorem 4.4.6. *Consider $\alpha\in(0,\frac{1}{2})$, $K>0$ and let $f^1,f^2\in\mathcal{AC}_{\mathcal{M}}\cap\mathcal{T}_{\mathcal{M}}$ be weak measured-valued solutions to* (4.2.5) *with initial data $f_0^1,f_0^2\in\mathcal{P}_2(\mathbb{T}\times\mathbb{R})$. Then,*

$$W_2(f_t^1,f_t^2)\leq e^{\left(\frac{1}{2}+2KL_0\right)t}W_2(f_0^1,f_0^2),$$

for every $t\geq 0$, where L_0 is the one-sided Lipschitz constant of $-h$ in Lemma 4.2.16.

The proof resembles that in Theorem 4.4.2. However, instead we use a full quadratic Wasserstein distance in both variables (z,Ω), i.e., take $M=\mathbb{T}\times\mathbb{R}$ in Appendix F to construct the Wasserstein space $(\mathcal{P}_2(\mathbb{T}\times\mathbb{R}),W_2)$. That makes the proof comparable to that in [67] for the aggregation equation except for the fact that our system is no longer a gradient flow due to the heterogeneity imposed by the natural frequencies Ω. Again, the one-sided Lipschitz property in Lemma 4.2.16 will be the cornerstone in the proof.

Proof. Since $f_0^1,f_0^2\in\mathcal{P}_2(\mathbb{T}\times\mathbb{R})$, then Theorem F.1.3 in Appendix F ensures the existence of an optimal transference plan μ_0 joining them both, i.e.,

$$\mu_0\in\mathcal{P}(f_0^1,f_0^2):=\left\{\mu\in\mathcal{P}((\mathbb{T}\times\mathbb{R})\times(\mathbb{T}\times\mathbb{R})):\,(\pi_1)_\#\mu=f_0^1\ \text{ and }\ (\pi_2)_\#\mu=f_0^2\right\},$$

such that

$$W_2(f_0^1, f_0^2)^2 = \int_{\mathbb{T}\times\mathbb{R}} \int_{\mathbb{T}\times\mathbb{R}} d((z_1, \Omega_1), (z_2, \Omega_2))^2 \, d_{((z_1,\Omega_1),(z_2,\Omega_2))}\mu_0.$$

Now the projection are $\pi_1((z, \Omega), (z', \Omega')) = (z, \Omega)$ and $\pi_2((z, \Omega), (z', \Omega')) = (z', \Omega')$. Again, we can construct a competitor at time t via push-forward, namely,

$$\mu_t := (X_{f^1}(t; 0, \cdot) \otimes X_{f^2}(t; 0, \cdot))_{\#}\mu_0 \in \mathcal{P}((\mathbb{T} \times \mathbb{R}) \times (\mathbb{T} \times \mathbb{R})),$$

where $X_{f^i}(t; 0, z, \Omega) = (Z_{f^i}(t; 0, z, \Omega), \Omega)$ for $i = 1, 2$ is the characteristic flow associated with the transport field $\mathcal{V}[f^i]$ according to Theorem 4.2.19. Since $\mu_t \in \Pi(f_t^1, f_t^2)$, then

$$\begin{aligned}\frac{1}{2}W_2(f_t^1, f_t^2)^2 &\leq \int_{\mathbb{T}\times\mathbb{R}} \int_{\mathbb{T}\times\mathbb{R}} \frac{1}{2}d((z_1, \Omega_1), (z_2, \Omega_2))^2 \, d_{((z_1,\Omega_1),(z_2,\Omega_2))}\mu_t \\ &= \int_{\mathbb{T}\times\mathbb{R}} \int_{\mathbb{T}\times\mathbb{R}} \frac{1}{2}d(X_{f^1}(t; 0, z_1, \Omega_1), X_{f^2}(t; 0, z_2, \Omega_2))^2 \, d_{((z_1,\Omega_1),(z_2,\Omega_2))}\mu_0 =: I(t).\end{aligned}$$

Again, we seek a Gönwal-type inequality for I, that in turns would yield the claimed estimate on the Wasserstein distance. Fix $(z_1 = e^{i\theta_1}, \Omega_1), (z_2 = e^{i\theta_2}, \Omega_2) \in \mathbb{T} \times \mathbb{R}$ and define

$$\Theta_1(t) := \Theta_{f^1}(t; 0, \theta_1, \Omega_1) \text{ and } \Theta_2(t) := \Theta_{f^2}(t; 0, \theta_2, \Omega_2),$$

the unique forward-in-time solutions to (4.2.8) in Lemma 4.2.12. Also, consider the following curves in $\mathbb{T}$

$$\begin{aligned}Z_1(t) &:= Z_{f^1}(t; 0, z_1, \Omega_1) = e^{i\Theta_1(t)}, \\ Z_2(t) &:= Z_{f^2}(t; 0, z_2, \Omega_2) = e^{i\Theta_2(t)},\end{aligned}$$

and the associated curves in $\mathbb{T} \times \mathbb{R}$,

$$\begin{aligned}X_1(t) &:= X_{f^1}(t; 0, z_1, \Omega_1) = (Z_1(t), \Omega_1), \\ X_2(t) &:= X_{f^2}(t; 0, z_2, \Omega_2) = (Z_2(t), \Omega_2).\end{aligned}$$

Set a minimizing geodesic $\widehat{\gamma}_t : [0, 1] \longrightarrow \mathbb{T} \times \mathbb{R}$ joining $X_1(t)$ to $X_2(t)$, for every fixed $t > 0$. Again, the following map

$$t \longmapsto \frac{1}{2}d^2(X_1(t), X_2(t)),$$

is absolutely continuous at least. Taking one-sided upper Dini directional derivatives given in Definition 4.B.4 of Appendix 4.B entails

$$\frac{d}{dt}\frac{1}{2}d^2(X_1(t), X_2(t)) \leq -\left\langle \mathcal{V}[f_t^1](X_1(t)), \widehat{\gamma}_t'(0)\right\rangle - \left\langle \mathcal{V}[f_t^2](X_2(t)), -\widehat{\gamma}_t'(1)\right\rangle,$$

for almost every $t \geq 0$. Let us now consider $\theta_{21}(t) := \overline{\Theta_2(t) - \Theta_1(t)}$, the representative of $\Theta_2(t) - \Theta_1(t)$ modulo 2π that lies in $(-\pi, \pi]$. Again, we distinguish two cases:

- *Case 1:* $\theta_{21}(t) \in (-\pi, \pi)$. In this case, the only minimizing geodesic reads

$$\widehat{\gamma}_t(s) = (\gamma_t(s), \Omega_1 + s(\Omega_2 - \Omega_1)) = (e^{i(\Theta_1(t)+s\theta_{21}(t))}, \Omega_1 + s(\Omega_2 - \Omega_1)), \; s \in [0, 1].$$

Then, the inequality reads

$$\frac{d}{dt}\frac{1}{2}d^2(X_1(t), X_2(t)) \leq (\mathcal{P}[f_t^2](\Theta_2(t), \Omega_2) - \mathcal{P}[f_t^1](\Theta_1(t), \Omega_1))\theta_{21}(t).$$

- *Case 2:* $\theta_{21}(t) = \pi$. In that second case there are exactly two minimizing geodesics

$$\widehat{\gamma}_{t,\pm}(s) = (\gamma_{t,\pm}(s), \Omega_1 + s(\Omega_2 - \Omega_1)) = (e^{i(\Theta_1(t)\pm\pi s)}, \Omega_1 + s(\Omega_2 - \Omega_1)),\ s \in [0,1].$$

Then, we restate the above inequality as follows

$$\frac{d}{dt}\frac{1}{2}d^2(X_1(t), X_2(t)) \leq (\mathcal{P}[f_t^2](\Theta_2(t), \Omega_2) - \mathcal{P}[f_t^1](\Theta_1(t), \Omega_1))(\pm\pi),$$

for almost every $t \geq 0$. To sum up, we achieve the following estimate

$$\begin{aligned}&\frac{d}{dt}\frac{1}{2}d^2(X_{f^2}(t;0,z_1,\Omega_1), X_{f^2}(t;0,z_2,\Omega_2))\\ &\leq (\mathcal{P}[f_t^1](\Theta_{f^1}(t;0,\theta_1,\Omega_1),\Omega_1) - \mathcal{P}[f_t^2](\Theta_{f^2}(t;0,\theta_2,\Omega_2),\Omega_2))\overline{\Theta_{f^1}(t;0,\theta_1,\Omega_1) - \Theta_{f^2}(t;0,\theta_1,\Omega_2)},\end{aligned}$$

for every $\theta_1, \theta_2, \Omega_1, \Omega_2 \in \mathbb{R}$ and almost every $t \geq 0$. Using the dominated convergence theorem, we show that I is absolutely continuous and taking derivatives under the integral sign implies

$$\begin{aligned}\frac{dI}{dt} \leq \int_{(-\pi,\pi]\times\mathbb{R}} \int_{(-\pi,\pi]\times\mathbb{R}} &(\mathcal{P}[f_t^1](\Theta_{f^1}(t;0,\theta_1,\Omega_1),\Omega_1) - \mathcal{P}[f_t^2](\Theta_{f^2}(t;0,\theta_2,\Omega_2),\Omega_2))\\ &\times \overline{\Theta_{f^1}(t;0,\theta_1,\Omega_1) - \Theta_{f^2}(t;0,\theta_1,\Omega_2)}\, d_{((\theta_1,\Omega_1),(\theta_2,\Omega_2))}\mu_0,\end{aligned} \tag{4.4.8}$$

for almost every $t \geq 0$. Also, note that

$$\begin{aligned}\mathcal{P}[f_t^i](\theta,\Omega) &= \Omega - \int_{(-\pi,\pi]}\int_{\mathbb{R}} h(\theta - \theta')\, d_{(\theta',\Omega')}f_t^i\\ &= \Omega - K\int_{(-\pi,\pi]}\int_{\mathbb{R}} h(\theta - \Theta_{f^i}(t;0,\theta_i',\Omega_i'))\, d_{(\theta_i',\Omega_i')}f_0^i,\end{aligned}$$

for $i = 1, 2$. Since $(\pi_1)_{\#}\mu_0 = f_0^1$ and $(\pi_2)_{\#}\mu_0 = f_0^2$, then

$$\mathcal{P}[f_t^1](\Theta,\Omega) = \Omega - K\int_{(-\pi,\pi]\times\mathbb{R}}\int_{(-\pi,\pi]\times\mathbb{R}} h(\theta - \Theta_{f^1}(t;0,\theta_1',\Omega_1'))\, d_{((\theta_1',\Omega_1'),(\theta_2',\Omega_2'))}\mu_0, \tag{4.4.9}$$

$$\mathcal{P}[f_t^2](\Theta,\Omega) = \Omega - K\int_{(-\pi,\pi]\times\mathbb{R}}\int_{(-\pi,\pi]\times\mathbb{R}} h(\theta - \Theta_{f^2}(t;0,\theta_2',\Omega_2'))\, d_{((\theta_1',\Omega_1'),(\theta_2',\Omega_2'))}\mu_0. \tag{4.4.10}$$

Putting (4.4.9)-(4.4.10) into (4.4.8) amounts to

$$\begin{aligned}\frac{dI}{dt} &\leq \int_{((-\pi,\pi]\times\mathbb{R})^4} (\Omega_1 - \Omega_2)\,\overline{\Theta_{f^1}(t;0,\theta_1,\Omega_1) - \Theta_{f^2}(t;0,\theta_2,\Omega_2)}\, d_{((\theta_1,\Omega_1),(\theta_2,\Omega_2))}\mu_0\, d_{((\theta_1',\Omega_1'),(\theta_2',\Omega_2'))}\mu_0\\ &- K\int_{((-\pi,\pi]\times\mathbb{R})^4} (h(\Theta_{f^1}(t;0,\theta_1,\Omega_1) - \Theta_{f^1}(t;0,\theta_1',\Omega_1')) - h(\Theta_{f^2}(t;0,\theta_2,\Omega_2) - \Theta_{f^2}(t;0,\theta_2',\Omega_2')))\\ &\qquad\times \overline{\Theta_{f^1}(t;0,\theta_1,\Omega_1) - \Theta_{f^2}(t;0,\theta_2,\Omega_2)}\, d_{((\theta_1,\Omega_1),(\theta_2,\Omega_2))}\mu_0\, d_{((\theta_1',\Omega_1'),(\theta_2',\Omega_2'))}\mu_0,\end{aligned} \tag{4.4.11}$$

for almost every $t \geq 0$. By virtue of the Young inequality in the first term and an analogue symmetrization trick to that in (4.4.7) in the second term, we obtain

$$\begin{aligned}&\frac{dI}{dt} \leq I(t)\\ &- \frac{K}{2}\int_{((-\pi,\pi]\times\mathbb{R})^4} (h(\Theta_{f^1}(t;0,\theta_1,\Omega_1) - \Theta_{f^1}(t;0,\theta_1',\Omega_1')) - h(\Theta_{f^2}(t;0,\theta_2,\Omega_2) - \Theta_{f^2}(t;0,\theta_2',\Omega_2')))\end{aligned}$$

$$\times \left(\overline{\Theta_{f^1}(t;0,\theta_1,\Omega_1)-\Theta_{f^2}(t;0,\theta_2,\Omega_2)}-\overline{\Theta_{f^1}(t;0,\theta_1,\Omega_1)-\Theta_{f^2}(t;0,\theta_2,\Omega_2)}\right)$$
$$\times\, d_{((\theta_1,\Omega_1),(\theta_2,\Omega_2))}\mu_0\, d_{((\theta_1',\Omega_1'),(\theta_2',\Omega_2'))}\mu_0,$$

for almost every $t \geq 0$. Mimicking the idea in Theorem 4.4.2 that uses the one-sided Lipschitz property of $-h$ in Lemma 4.2.16 implies

$$\frac{dI}{dt} \leq (1+4KL_0)I, \quad \text{for a.e. } t \geq 0.$$

Using Grönwall's lemma

$$I(t) \leq I(0)e^{(1+4KL_0)t},\ t \geq 0.$$

Finally, notice that

$$I(0) = \int_{\mathbb{T}\times\mathbb{R}}\int_{\mathbb{T}\times\mathbb{R}} \frac{1}{2} d((z_1,\Omega_1),(z_2,\Omega_2))^2\, d_{((z_1,\Omega_1),(z_2,\Omega_2))}\mu_0 = \frac{1}{2}W_2(f_0^1,f_0^2)^2,$$

and that ends the proof. □

Remark 4.4.7. *Again, we recover the differential inequality*

$$\frac{d^+}{dt}W_2(f_t^1,f_t^2) \leq \left(\frac{1}{2}+2KL_0\right)W_2(f_t^1,f_t^2),$$

for every $t \geq 0$.

The above Theorem 4.4.6 along with Theorem 4.3.2 implies the rigorous local-in-time mean field limit as depicted in the following result.

Corollary 4.4.8. *Consider* $\alpha \in (0,\frac{1}{2})$, $K > 0$ *and let* $f \in \mathcal{AC}_{\mathcal{M}} \cap \mathcal{T}_{\mathcal{M}}$ *be the unique weak measure-valued solution in the sense of the flow to* (4.2.5) *with initial datum* $f_0 \in \mathcal{P}_2(\mathbb{T}\times\mathbb{R})$. *Consider* N *oscillators with phases and natural frequencies given by the configurations*

$$\Theta_0^N = (\theta_{1,0}^N,\ldots,\theta_{N,0}^N) \ \textit{and}\ \{\Omega_i^N : i=1,\ldots,N\},$$

for every $N \in \mathbb{N}$. *Let* $\Theta^N(t) := (\theta_1^N(t),\ldots,\theta_N^N(t))$ *be the unique global-in-time classical solution to the discrete singular Kuramoto model according to [241, Theorem 3.1] and define the associated empirical measures in* $\mathbb{T}\times\mathbb{R}$

$$\mu_t^N := \frac{1}{N}\sum_{i=1}^N \delta_{z_i^N(t)}(z)\otimes\delta_{\Omega_i^N}(\Omega),$$

where $z_i^N(t) := e^{i\theta_i^N(t)}$. *If* $\lim_{N\to\infty} W_2(\mu_0^N,f_0)=0$, *then,*

$$\lim_{N\to\infty}\sup_{t\in[0,T]} W_2(\mu_t^N,f_t)=0, \ \textit{for all}\ T>0.$$

4.5 Global phase-synchronization of identical oscillators in finite time

In this section we shall analyze the dynamics and emergent behavior of solutions to the macroscopic model (4.2.5) in the subcritical regime of the parameter, i.e., $\alpha \in (0,\frac{1}{2})$. Our goal is to extend the result in Theorem 3.5.4 of the above Chapter 3 towards the kinetic equation.

Remark 4.5.1. *Consider $\Theta(t) = (\theta_1, \ldots, \theta_N(t))$ a solution to (4.1.1)-(4.1.2) in the subcritical regime $\alpha \in (0, \frac{1}{2})$ with $\Omega_i = 0$ for all $i = 1, \ldots, N$ and consider its average phase*

$$\theta_{av}(t) := \frac{1}{N} \sum_{i=1}^{N} \theta_i(t), \ t \geq 0.$$

We shall sometimes regard such quantity, originally living in the real line, as its projection over the unit circle, that is, $z_{av}(t) = e^{i\theta_{av}(t)}$. Notice that thanks to the dynamics of (3.3.1)(3.3.2), we know that it a conserved quantity of the system. Then, an equivalent statement of Theorem 3.5.4 is that if the identical oscillators are initially confined to a half circle, then

$$\theta_i(t) = \theta_{av}(0), \ \text{for all } \ t \geq T_c,$$

where $T_c = \frac{D(\Theta_0)^{1-2\alpha}}{2\alpha K h(D(\Theta_0))}$.

Our intuition is that the above should remain true for measure-valued solutions to the macroscopic system (4.2.5) due to the fact that T_c does not depend on the number N of oscillators. Before introducing the main result of this section, we need to define an analogue concept of average phase at the continuum level. Notice that centers of mass can be built on any Riemannian manifold more general that the Euclidean space.

Definition 4.5.2. *Let $(M, \langle\cdot,\cdot\rangle)$ be a Riemannian manifold and consider a probability measure $\mu \in \mathcal{P}_2(M)$. A point $x_0 \in M$ is called a center of mass for μ if it solves the minimization problem*

$$\min_{x \in M} \int_M \frac{1}{2} d(x, x_0)^2 \, d_x\mu, \tag{4.5.1}$$

where $d(\cdot,\cdot)$ stands for the Riemannian distance on M.

If $M = \mathbb{R}^d$ it is clear that (4.5.1) has a unique solution and it agrees with the standard definition of center of mass in $\mathbb{R}^d$, i.e.,

$$x_0 = \int_{\mathbb{R}^d} x \, d_x\mu,$$

Indeed, one simple way to show uniqueness is by strict convexity of the function under consideration, that is strictly convex because the squared distance so is in the Euclidean space. However, for general manifold this is not true and one might find that the center of mass may be non-unique depending on the probability measure μ.

Lemma 4.5.3. *Let $(M, \langle\cdot,\cdot\rangle)$ be a complete Riemannian manifold and set*

$$\begin{aligned} \iota_M &: \text{ injectivity radius of } M, \\ \Delta_M &: \text{ upper bound of the sectional curvatures of } M. \end{aligned}$$

Define the following parameter

$$\delta_M := \min\left\{\iota_M, \frac{\pi}{\sqrt{\Delta_M}}\right\}, \tag{4.5.2}$$

and consider $\mu \in \mathcal{P}_2(M)$ with $\operatorname{supp}\mu \subseteq \mathbb{B}_r(x_0)$ (where $\mathbb{B}_r(x_0) = \exp_{x_0}(B_r(0))$ is the geodesic ball of radius r centered at some x_0). If $r < \frac{\delta_M}{2}$, then there is a unique center of mass of μ and it lies in $\mathbb{B}_r(x_0)$.

See [2] for a comprehensive presentation of this topic and also references therein. In particular, notice that if $M=\mathbb{T}$, then $\iota_{\mathbb{T}}=\pi$, $\Delta_{\mathbb{T}}=1$ and consequently, $\delta_{\mathbb{T}}=\pi$ (see (4.5.2)). This shows the uniqueness of center of mass along with its location on the unit torus if $\mu\in\mathcal{P}(\mathbb{T})$ has support with diameter smaller that π. This is the content of the following result, where we also obtain a explicit formula for the center of mass.

Proposition 4.5.4. *Let us consider $\mu\in\mathcal{P}(\mathbb{T})$ and assume that $D_0=\operatorname{diam}(\operatorname{supp}\mu)<\pi$. Then, there exists a unique center of mass of μ and it reads $Z_{av}=e^{i\Theta_{av}}$, with*

$$\Theta_{av}:=\int_{[\theta_*,\theta^*]}\theta d_\theta\mu,$$

where $C=\{e^{i\theta}\,:\,\theta\in[\theta_*,\theta^*]\}$ is the geodesic convex hull of $\operatorname{supp}\mu$, that is the smallest geodesically convex subset of $\mathbb{T}$ containing $\operatorname{supp}\mu$.

Notice that the above C exists and is unique by virtue of the assumption $D_0<\pi$. When $D_0\geq\pi$ there might be two (or any) such smallest geodesically convex sets. Moreover, Θ_{av} is not uniquely defined, but it depends on the choice of representatives that we make for θ_* and θ^*. However, Z_{av} is uniquely defined since all those representatives agree modulo 2π.

Proof of Proposition 4.5.4. • *Step 1*: Reducing to the set C.

Consider the function

$$\mathcal{F}(z):=\int_{\mathbb{T}}\frac{1}{2}d(z',z)^2\,d_{z'}\mu,$$

for any $z\in\mathbb{T}$. By virtue of Lemma 4.5.3 we obtain that

$$\min_{z\in\mathbb{T}}\mathcal{F}(z)=\min_{z\in C}\mathcal{F}(z). \tag{4.5.3}$$

• *Step 2*: Solving the problem on C.

Let us call $\Theta_{av}\in[\theta_*,\theta^*]$ and $Z_{av}=e^{i\theta_{av}}$ the unique center of mass of μ and consider any $\theta\in[\theta_*,\theta^*]$ and the associated $z=e^{i\theta}\in C$. Then,

$$\mathcal{F}(z)=\int_{(\theta-\pi,\theta+\pi]}\frac{(\theta-\theta')^2}{2}\,d_{\theta'}\mu=\int_{[\theta_*,\theta^*]}\frac{(\theta-\theta')^2}{2}\,d_{\theta'}\mu.$$

Since $\theta=\Theta_{av}$ is a minimizer of the above function, by differentiation we obtain that

$$\int_{[\theta_*,\theta^*]}(\Theta_{av}-\theta')\,d_{\theta'}\mu=0,$$

and that ends the proof. □

Definition 4.5.5. *Consider $f_0\in\mathcal{P}(\mathbb{T}\times\mathbb{R})$. Assume that $D_0=\operatorname{diam}(\operatorname{supp}\rho_0)<\pi$ and let us set*

$$C_0=\{e^{i\theta}\,:\,\theta\in[\theta_*(0),\theta^*(0)]\},$$

with $D_0=\theta^*(0)-\theta_*(0)$ to be the geodesic convex hull of $\operatorname{supp}\rho_0$ in $\mathbb{T}$. We will define the average phase of the initial configuration by the center of mass in $\mathbb{T}$ of the density ρ_0, that is, $Z_{av}(0):=e^{i\Theta_{av}(0)}$ where

$$\Theta_{av}(0):=\int_{[\theta_*(0),\theta^*(0)]}\int_{\mathbb{R}}\theta\,d_{(\theta,\Omega)}f_0=\int_{[\theta_*(0),\theta^*(0)]}\theta\,d_\theta\rho_0.$$

For the reader convenience, let us introduce an alternative representation of $Z_{av}(0)$.

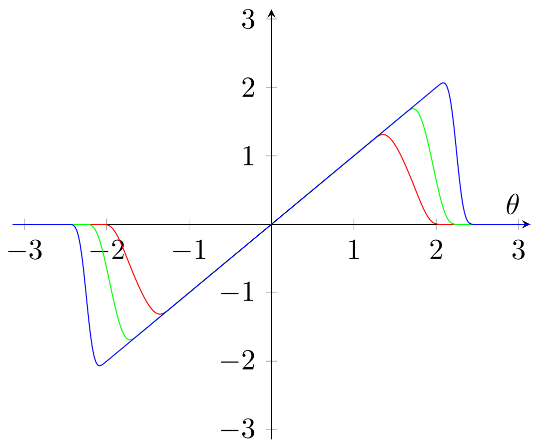


Figure 4.2: Plot of $\vartheta_{\theta_0,\varepsilon}$ with $\theta_0 = -\pi$ and values $\varepsilon = 1$ (blue), $\varepsilon = 0.8$ (green) and $\varepsilon = 0.6$ (red) respectively.

Lemma 4.5.6 (Periodified phase). *Let us consider the following cut-off function*

$$\xi_{\delta_1,\delta_2}(r) := \begin{cases} 1, & r \in [0,\delta_1), \\ \frac{1}{1+\exp\left(\frac{2r-(\delta_1+\delta_2)}{(\delta_2-r)(r-\delta_1)}\right)}, & r \in [\delta_1,\delta_2), \\ 0, & r \in [\delta_2,+\infty), \end{cases}$$

for any $0 < \delta_1 < \delta_2$*. Fix any* $\varepsilon > 0$ *and define (see Figure 4.2)*

$$\vartheta_{\theta_0,\varepsilon}(\theta) := \theta\, \xi_{\pi-2\varepsilon,\pi-\varepsilon}(|\theta-\theta_0-\pi|),\ \theta \in [\theta_0,\theta_0+2\pi].$$

Then, $\vartheta_{\theta_0,\varepsilon}(\theta)$ *is a periodic function of class* C^∞ *such that* $|\vartheta_\varepsilon(\theta)| \leq |\theta|$ *and*

$$\vartheta_{\theta_0,\varepsilon}(\theta) = \begin{cases} \theta, & |\theta-\theta_0-\pi| \leq \pi-2\varepsilon, \\ 0, & \pi-\varepsilon \leq |\theta-\theta_0-\pi| \leq \pi. \end{cases}$$

Lemma 4.5.7. *Consider* $f_0 \in \mathcal{P}(\mathbb{T}\times\mathbb{R})$*. Assume that* $D_0 = \operatorname{diam}(\operatorname{supp}\rho_0) < \pi$ *and fix any point away from the support* $z_0 = e^{i\theta_0} \in \mathbb{T}\setminus\operatorname{supp}\rho_0$*. Then, there exists* $\varepsilon_0 > 0$ *such that*

$$\Theta_{av}(0) = \int_{(-\pi,\pi]} \vartheta_{\theta_0,\varepsilon}(\theta)\, d_\theta\rho_0 = \int_{\mathbb{T}} \vartheta_{\theta_0,\varepsilon}(z)\, d_z\rho_0,\ \forall\, \varepsilon \in (0,\varepsilon_0).$$

Theorem 4.5.8. *Consider any initial datum* $f_0 \in \mathcal{P}(\mathbb{T}\times\mathbb{R})$ *with identical distribution of natural frequencies, namely,* $g = (\pi_\Omega)_\# f_0 = \delta_0(\Omega)$*, where* π_Ω *is the projection* (N.1)*. Let* $f = f_t$ *be the unique global-in-time weak measure-valued solution to* (4.2.5) *issued at* f_0 *with* $\alpha \in (0,\frac{1}{2})$ *and assume* $D_0 := \operatorname{diam}(\operatorname{supp}\rho_0) < \pi$*. Then,*

$$f_t = f_\infty \text{ for all } t \geq T_c,$$

where $T_c = \frac{D_0^{1-2\alpha}}{2\alpha K h(D_0)}$ *and the equilibrium* f_∞ *is given by the monopole* $f_\infty := \delta_{Z_{av}(0)}(z)\otimes\delta_0(\Omega)$*.*

Proof. Consider the smallest geodesically convex subset C_0 of $\mathbb{T}$ that contains $\operatorname{supp}\rho_0$ as given in Definition 4.5.5. Let us recall that in the existence Theorem 4.3.10, the weak measure-valued solution f arose as the limit

$$\mu^N \to f \text{ in } \mathcal{P}_1(\mathbb{T}\times\mathbb{R}) - W_1, \tag{4.5.4}$$

of some sequence of empirical measures

$$\mu_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{z_i^N(t)}(z)\delta_{\Omega_i^N}(\Omega),$$

with $z_i^N(t) = e^{i\theta_i^N(t)}$. Recall that $\Theta^N(t) = (\theta_1^N(t),\ldots,\theta_N^N(t))$ are global classical solutions to (4.1.1)-(4.1.2). Without loss of generality, Lemma 4.3.9 allows assuming that $\Omega_i^N = 0$ for all $i = 1,\ldots,N$ and $N \in \mathbb{N}$. Also, the same result allows ensuring that μ_0^N can be taken so that $\operatorname{supp}\mu_0^N \subseteq C_0^\delta$, where

$$C_0^\delta := \left\{e^{i\theta} : \theta \in \left[\theta_*(0) - \frac{\delta}{2}, \theta^*(0) + \frac{\delta}{2}\right]\right\},$$

and δ is any arbitrary value $\delta \in (0, \pi - D_0)$. By virtue of Theorem 3.5.4 and Remark 4.5.1, we obtain that

$$\Theta_i^N(t) = z_{av}^N(0), \text{ for all } t \geq T_c^\delta,$$

for every $i = 1,\ldots,N$, every $N \in \mathbb{N}$ and T_c^δ is given by

$$T_c^\delta = \frac{(D_0+\delta)^{1-2\alpha}}{2\alpha K h(D_0+\delta)}.$$

Hence, we obtain that

$$\mu_t^N = \delta_{z_{av}^N(0)}(z)\delta_0(\Omega), \text{ for all } t \geq T_c^\delta. \tag{4.5.5}$$

Consider $z_0 \in \mathbb{T}\setminus\operatorname{supp}\rho_0$ and $\varepsilon_0 > 0$ as given in Lemma 4.5.7. Then, it is clear that

$$\Theta_{av}(0) = \int_{\mathbb{T}\times\mathbb{R}} \vartheta_{\theta_0,\varepsilon}(z)\, d_{(z,\Omega)} f_0 = \lim_{N\to\infty}\int_{\mathbb{T}\times\mathbb{R}} \vartheta_{\theta_0,\varepsilon}(z)\, d_{(z,\Omega)}\mu_0^N = \lim_{N\to\infty}\theta_{av}^N(0). \tag{4.5.6}$$

Taking limits in (4.5.5) and using (4.5.4) and (4.5.6) we obtain that

$$f_t = \delta_{Z_{av}(0)}(z)\delta_0(\Omega), \text{ for all } t \geq T_c^\delta.$$

Since $\delta \in (0, \pi - D_0)$, the result holds true. □

4.6 The critical regime

This section is devoted to the critical case, $\alpha = \frac{1}{2}$. Notice that most of the ideas in Sections 4.2, 4.3, 4.4 and 4.5 break down due to the jump discontinuity of the kernel h. However, an appropriate concept of measure-valued solution can be considered, thus yielding well-posedness of (4.2.5) like in Theorem 4.3.10, analogue Dobrushin-type estimates in Wasserstein distance like in Theorems 4.4.2 and 4.4.6, rigorous mean field limit and a similar analysis of the dynamics like in Theorem 4.5.8.

4.6.1 Solutions in the sense of the Filippov flow

In this section, we will extend the above existence results in Section 4.3 to the critical regime. Note that the vector field $\mathcal{V}[\mu]$ in Definition 4.2.3 does not make sense when $\alpha = \frac{1}{2}$ for general finite Radon measures $\mu \in \mathcal{M}(\mathbb{T}\times\mathbb{R})$. The reason relies on the fact that in such case the kernel h is no longer continuous, but it exhibits a jump discontinuity at $\theta \in 2\pi\mathbb{Z}$. Consequently, Definition 4.2.3 is wrong due to the possible presence of atoms in the Radon measure μ. Hence, the above regularity theory for $\mathcal{V}[\mu]$ in Section 4.2 fails when $\alpha = \frac{1}{2}$ and it requires to be adapted.

This part will be split in several steps. First, we will adapt Definition 4.2.3 so that the transport field also makes sense for $\alpha = \frac{1}{2}$. Second, although the classical theory cannot solve the characteristic system due to the discontinuity of the transport field, we can obtain a generalized concept of solutions to the characteristic system. Namely, we will consider Filippov solutions of the characteristic system and will show that they exist and are global and unique forward-in-time. Third, we will show that the unique forward-in-time solutions to the discrete model (4.1.1)-(4.1.2) that were obtained in Theorem 3.3.12 in Chapter 3 give rise to *solutions in the sense of the Filippov flow* to the macroscopic system (4.2.5) by considering the corresponding empirical measures like in Definition 4.3.1. To end this part, we derive appropriate a priori bounds on the sequence of empirical measures so that we can pass to the limit and obtain solutions in the sense of the Filippov flow to the macroscopic system (4.2.5) for any general initial datum.

Definition 4.6.1. *Consider $\alpha = \frac{1}{2}$ and $K > 0$. We will formally define the function $\mathcal{P}[\mu]$ and the tangent vector field $\mathcal{V}[\mu]$ along the manifold $\mathbb{T} \times \mathbb{R}$ given by*

$$\mathcal{P}[\mu](\theta, \Omega) := \Omega - K \int_{\mathbb{T}\setminus\{e^{i\theta}\}} \int_{\mathbb{R}} h(\theta - \theta')\, d_{(\theta',\Omega')}\mu,$$
$$\mathcal{V}[\mu](z, \Omega) := (\mathcal{P}[\mu](z, \Omega)\, iz, 0),$$

where $\mu \in \mathcal{M}(\mathbb{T} \times \mathbb{R})$ is any finite Radon measure.

Notice that in this definition for the critical case, we have removed $z' = e^{i\theta}$. It is consistent with that of the subcritical case in Definition 4.2.3 because, in such case, $h(0) = 0$. Moreover, since the function

$$z' \in \mathbb{T} \setminus \{z\} \longmapsto h(z\bar{z}'),$$

is bounded and continuous, then the above integral also makes sense in the critical case.

The Filippov set-valued tangent vector field

In this part we will introduce the concept of *Filippov set-valued tangent vector field* associated with a given single-valued tangent vector field. It will be required in order to define the appropriate concept of generalized characteristic trajectories of (4.2.5). The definition is a standard translation of the standard concept for the Euclidean case [14, 130] to the Riemannian case via local charts. We will first give a general definition that is valid for any complete Riemannian manifold and extends the classical *Filippov convexification* or *Filippov set-valued map* of a given measurable vector field in the Euclidean space [14, 130]. Our second step will be to identify a clear characterization in our particular Riemannian manifold $\mathbb{T} \times \mathbb{R}$.

Definition 4.6.2. *Let $(M, \langle\cdot,\cdot\rangle)$ be a d-dimensional Riemannian manifold. Fix $x_0 \in M$, and set a local chart $(\mathcal{U}, \varphi)$ around x_0 and any local frame $U_1, \ldots, U_d \in \mathfrak{X}C^\infty(\mathcal{U})$ on $\mathcal{U}$. For any locally essentially bounded tangent vector field $V \in \mathfrak{X}L^\infty_{loc}(M)$, let us consider its local expression*

$$V = f_1 U_1 + \cdots f_d U_d \text{ in } \mathcal{U},$$

where $f_1, \ldots, f_d \in L^\infty_{loc}(\mathcal{U})$. Define $\mathcal{D} := \varphi(\mathcal{U}) \subseteq \mathbb{R}^d$, $F_i := f_i \circ \varphi^{-1}$ and the vector field

$$F = (F_1, \ldots, F_d) : \mathcal{D} \subseteq \mathbb{R}^d \longrightarrow \mathbb{R}^d.$$

Then, we define the Filippov set-valued tangent vector field at x_0 by the formula

$$\mathcal{K}[V]_{x_0} = \left\{ \sum_{i=1}^d \beta_i U_i(x_0) : \beta = (\beta_1, \ldots, \beta_d) \in \mathcal{K}[F](\varphi(x_0)) \right\},$$

where $\mathcal{K}[F](\varphi(x_0))$ is the Filippov set-valued map associated with F, that is,

$$\mathcal{K}[F](\varphi(x_0)) := \bigcap_{\delta>0} \bigcap_{|\mathcal{N}|=0} \overline{\text{co}}(F(B_\delta(x_0) \setminus \mathcal{N})).$$

Recall its main properties in Appendix D and also see [14, 130].

Remark 4.6.3. *Notice that given any local chart $(\mathcal{U}, \varphi)$ around x_0 with coordinates $\varphi = (x_1, \ldots, x_d)$, we can construct the associated local frame whose basis of tangent vector fields read as follows*

$$U_i := \frac{\partial}{\partial x_i}, \text{ for all } i = 1, \ldots, d.$$

In such particular local frame we can set the local coordinates of the vector field to be

$$V = f_1 \frac{\partial}{\partial x_1} + \cdots f_d \frac{\partial}{\partial x_d} \text{ in } \mathcal{U}.$$

The associated representation of the Filippov set-valued tangent field then reads

$$\mathcal{K}[V]_{x_0} = \left\{ \sum_{i=1}^d \beta_i \left.\frac{\partial}{\partial x_i}\right|_{x_0} : \beta = (\beta_1, \ldots, \beta_d) \in \mathcal{K}[F](\varphi(x_0)) \right\}.$$

Our interest in considering general local frames will be clarified later in Lemma 4.6.17.

Let us show that the hypothesis $V \in \mathfrak{X}L^\infty_{loc}(M)$ guarantee the independence of the above definition on the given local charts and frames. We will require the following technical result.

Lemma 4.6.4. *Consider a couple of open set $\mathcal{D}_1, \mathcal{D}_2 \subseteq \mathbb{R}^d$, a C^1 diffeomorphism $\Phi : \mathcal{D}_1 \longrightarrow \mathcal{D}_2$ and a measurable vector field $F : \mathcal{D}_2 \longrightarrow \mathbb{R}^d$. Then, the following formula takes place*

$$\mathcal{K}[F \circ \Phi](x) = \mathcal{K}[F](\Phi(x)),$$

for every $x \in \mathcal{D}_1$.

Proof. Since Φ is a diffeomorphism, then one inclusion follows from the other one when applied to Φ^{-1}. Then, we will just focus on one inclusion; specifically: $\mathcal{K}[F \circ \Phi] \subseteq \mathcal{K}[F] \circ \Phi$. Consider $x \in \mathcal{D}_1$, set any arbitrary $\delta > 0$ and any negligible set $\mathcal{N} \subseteq \mathcal{D}_2$. Since Φ is a diffeomorphism there exists $\delta' > 0$ and another negligible set $\mathcal{N}' \subseteq \mathcal{D}_1$ so that

$$\Phi(B_{\delta'}(x) \setminus \mathcal{N}') \subseteq B_\delta(\Phi(x)) \setminus \mathcal{N}.$$

Consequently,

$$\mathcal{K}[F \circ \Phi](x) \subseteq \overline{\text{co}}((F \circ \Phi)(B_{\delta'}(x) \setminus \mathcal{N}')) \subseteq \overline{\text{co}}(F(B_\delta(\Phi(x)) \setminus \mathcal{N})).$$

Since δ and $\mathcal{N}$ are arbitrary, then the desired inclusion holds true. □

Lemma 4.6.5. *Consider an open set $\mathcal{D} \subseteq \mathbb{R}^d$, a continuous map $A : \mathcal{D} \longrightarrow \mathcal{M}_d(\mathbb{R})$ and a locally essentially bounded vector field $F : \mathcal{D} \longrightarrow \mathbb{R}^d$. Then, the following formula takes place*

$$\mathcal{K}[AF](x) = A(x)\mathcal{K}[F][x],$$

for every $x \in \mathcal{D}$.

Proof. Fix $x_0 \in \mathcal{D}$. The proof is split into two parts that correspond to each of the inclusions

$$A(x_0)\mathcal{K}[F](x_0) \subseteq \mathcal{K}[AF](x_0) \text{ and } \mathcal{K}[AF](x_0) \subseteq A(x_0)\mathcal{K}[F](x_0).$$

• *Step 1*: Consider $z_0 \in A(x_0)\mathcal{K}[F](x_0)$ and set $y_0 \in \mathcal{K}[F](x_0)$ so that $z_0 = A(x_0)y_0$. Fix any arbitrary $\delta > 0$ and any negligible set $\mathcal{N} \subseteq \mathcal{D}$. Our goal here is to show that $z_0 \in \overline{\text{co}}(AF)(B_\delta(x_0) \setminus \mathcal{N})$. Consider any small enough $\delta' > 0$ so that $\delta' < \delta$ and $F \in L^\infty(B_{\delta'}(x_0), \mathbb{R}^d)$. Take any arbitrary $n \in \mathbb{N}$ and note that $y_0 \in \overline{\text{co}}(F(B_{\delta'/n}(x_0)\setminus\mathcal{N}))$. Then, there exists $y_n \in \text{co}(F(B_{\delta'/n}(x_0)\setminus\mathcal{N}))$ so that $|y_n - y_0| \leq \frac{1}{n}$. In addition, Caratheodory's theorem of convex analysis provides

$$q_n^1, \ldots, q_n^{d+1} \in B_{\delta'/n}(x_0) \setminus \mathcal{N} \text{ and } \lambda_n^1, \ldots, \lambda_n^{d+1} \in [0,1],$$

so that $\sum_{k=1}^{d+1} \lambda_n^k = 1$ and y_n can be written as follows

$$y_n = \sum_{k=1}^{d+1} \lambda_n^k F(q_n^k),$$

for every $n \in \mathbb{N}$. Let us define the following vectors

$$z_n := \sum_{k=1}^{d+1} \lambda_n^k A(q_n^k) F(q_n^k),$$

for every $n \in \mathbb{N}$. By definition, $z_n \in \text{co}((AF)(B_\delta(x_0) \setminus \mathcal{N}))$. Let us see that $z_n \to z_0$ and that ends the proof. To that end, let us split as follows

$$\begin{aligned} z_n &= \sum_{k=1}^{d+1} \lambda_n^k (A(q_n^k) - A(x_0)) F(q_n^k) + \sum_{k=1}^{d+1} \lambda_n^k A(x_0) F(q_n^k) \\ &= \sum_{k=1}^{d+1} \lambda_n^k (A(q_n^k) - A(x_0)) F(q_n^k) + A(x_0) y_n. \end{aligned}$$

On the one hand, the second term converges towards $A(x_0)y_0 = z_0$. On the other hand, let us note that the first one vanishes as $n \to \infty$. Indeed, taking norms, it can be bounded above by

$$\sum_{k=1}^{d+1} \lambda_n^k |A(q_n^k) - A(x_0)| \|F\|_{L^\infty(B_{\delta'}(x_0), \mathbb{R}^d)}.$$

Now the convergence to zero becomes apparent since the coefficients λ_n^k sum up to 1 and $A(q_n^k) \to A(x_0)$ for every $k = 1, \ldots, d+1$ by virtue of the continuity of A along with the convergence $q_n^k \to x_0$ as $n \to \infty$.

• *Step 2*: Now, consider $z_0 \in \mathcal{K}[AF](x_0)$, fix any arbitrary $\delta > 0$ and any negligible set $\mathcal{N} \subseteq \mathcal{D}$. Our goal is to show that there exists $y_0 \in \overline{\text{co}}(F(B_\delta(x_0) \setminus \mathcal{N}))$ so that $z_0 = A(x_0)y_0$. Consider a small enough $\delta' > 0$ so that $\delta' < \delta$ and $F \in L^\infty(B_{\delta'}(x_0), \mathbb{R}^d)$. Take any arbitrary $n \in \mathbb{N}$ and note that $z_0 \in \overline{\text{co}}((AF)(B_{\delta'/n}(x_0) \setminus \mathcal{N}))$. Then, there exists $z_n \in \text{co}((AF)(B_{\delta'/n}(x_0) \setminus \mathcal{N}))$ so that $|z_n - z_0| \leq \frac{1}{n}$. Again, Caratheodory's theorem implies the existence of

$$q_n^1, \ldots, q_n^{d+1} \in B_{\delta'/n}(x_0) \setminus \mathcal{N} \text{ and } \lambda_n^1, \ldots, \lambda_n^{d+1} \in [0,1],$$

so that $\sum_{k=1}^{d+1} \lambda_n^k = 1$ and y_n can be written as follows

$$z_n = \sum_{k=1}^{d+1} \lambda_n^k A(a_n^k) F(q_n^k),$$

for every $n \in \mathbb{N}$. Now, consider the following vectors

$$z_n := \sum_{k=1}^{d+1} \lambda_n^k F(q_n^k),$$

for every $n \in \mathbb{N}$. Notice that $y_n \in \operatorname{co}(F(B_\delta(x_0) \setminus \mathcal{N}))$. By the boundedness of F in $B_{\delta'}(x_0)$ it is clear that $\{y_n\}_{n\in\mathbb{N}}$ is a bounded sequence. Hence, Weierstrass theorem, provides a subsequence $y_{\sigma(n)}$ that converges towards some $y_0 \in \overline{\operatorname{co}}(F(B_\delta(x_0) \setminus \mathcal{N}))$. In addition, let us note that the following split holds true

$$\begin{aligned} A(x_0)y_{\sigma(n)} &= \sum_{k=1}^{d+1} \lambda_{\sigma(n)}^k A(x_0) F(q_{\sigma(n)}^k) \\ &= \sum_{k=1}^{d+1} \lambda_{\sigma(n)}^k (A(x_0) - A(q_{\sigma(n)}^k)) F(q_{\sigma(n)}^k) + \sum_{k=1}^{d+1} \lambda_{\sigma(n)}^k A(q_{\sigma(n)}^k) F(q_{\sigma(n)}^k) \\ &= \sum_{k=1}^{d+1} \lambda_{\sigma(n)}^k (A(x_0) - A(q_{\sigma(n)}^k)) F(q_{\sigma(n)}^k) + z_{\sigma(n)}, \end{aligned}$$

for every $n \in \mathbb{N}$. Taking limits as $n \to \infty$ and using the boundedness of F in $B_{\delta'}(x_0)$ and the continuity of A show that $A(x_0)y_0 = z_0$ and that ends the proof. □

Remark 4.6.6. *As a simple consequence of the above Lemmas 4.6.4 and 4.6.5 we will show that the above Definition 4.6.2 does not depend on the chosen local frames. To that end, consider a couple of local frames $(\mathcal{U}, \varphi, U_1, \ldots, U_d)$ and $(\mathcal{W}, \psi, W_1, \ldots, W_d)$. Write V in both basis*

$$V = \sum_{i=1}^{d} f_i U_i = \sum_{j=1}^{d} g_j W_j.$$

Let us set the matrix P of change from basis $\{W_1, \ldots, W_d\}$ to $\{U_1, \ldots, U_d\}$, i.e., the continuous map $P : \mathcal{U} \cap \mathcal{W} \longrightarrow \mathcal{M}_d(\mathbb{R})$ whose coordinates $P = (p_{ij})_{1\leq i,j\leq d}$ fulfil

$$W_j = \sum_{i=1}^{d} p_{ij} U_i \ \text{ in } \mathcal{U} \cap \mathcal{W},$$

for all $j = 1, \ldots, d$. Then, the change of basis formula provide the relations

$$f_i = \sum_{j=1}^{d} p_{ij} g_j \ \text{ in } \mathcal{U} \cap \mathcal{W}, \tag{4.6.1}$$

for all $i = 1, \ldots, d$. Consider $F_i = f_i \circ \varphi^{-1}$ and $G_i = g_j \circ \psi^{-1}$ and the associated Filippov maps

$$\begin{aligned} \mathcal{K}[F](\varphi(x_0)) &= \bigcap_{\delta>0} \bigcap_{|\mathcal{N}|=0} \overline{\operatorname{co}}(F(B_\delta(\varphi(x_0)) \setminus \mathcal{N})), \\ \mathcal{K}[G](\psi(x_0)) &= \bigcap_{\delta>0} \bigcap_{|\mathcal{N}|=0} \overline{\operatorname{co}}(G(B_\delta(\psi(x_0)) \setminus \mathcal{N})). \end{aligned}$$

Consider the associated Filippov set-valued tangent fields, where we use upper scripts for distinction

$$\mathcal{K}[V]_{x_0}^{\varphi} = \left\{ \sum_{i=1}^{d} \beta_i U_i(x_0) : \beta = (\beta_1, \ldots, \beta_d) \in \mathcal{K}[F](\varphi(x_0)) \right\},$$

$$\mathcal{K}[V]_{x_0}^{\psi} = \left\{ \sum_{j=1}^{d} \gamma_j V_j(x_0) : \gamma = (\gamma_1, \ldots, \gamma_d) \in \mathcal{K}[G](\psi(x_0)) \right\}.$$

Our goal is to show that $\mathcal{K}[V]_{x_0}^{\varphi} = \mathcal{K}[V]_{x_0}^{\psi}$. *On the one hand, recall that* (4.6.1) *amounts to*

$$F \circ (\varphi \circ \psi^{-1}) = (P \circ \psi^{-1})\, G \;\; in \;\; \psi(\mathcal{U} \cap \mathcal{W}), \tag{4.6.2}$$

Taking $\mathcal{D}_1 = \psi(\mathcal{U} \cap \mathcal{V}), \mathcal{D}_2 = \varphi(\mathcal{U} \cap \mathcal{V})$ *along with* $\Phi = \varphi \circ \psi^{-1}$ *and* $A = P \circ \psi^{-1}$ *we can apply Lemmas 4.6.4 and 4.6.5 to* (4.6.2) *and we infer that*

$$\mathcal{K}[F](\varphi(x_0)) = P(x_0)\, \mathcal{K}[G](\psi(x_0)).$$

Then, the claimed independence on the chosen local frame follows.

We are interested in introducing the Filippov set-valued tangent vector in $\mathbb{T} \times \mathbb{R}$ associated with the transport fields in Definition 4.6.1 for the critical regime $\alpha = \frac{1}{2}$. Notice that it can be done since $\mathcal{V}[\mu]$ is locally essentially bounded in $\mathbb{T} \times \mathbb{R}$ for any given $\mu \in \mathcal{M}(\mathbb{T} \times \mathbb{R})$.

Proposition 4.6.7. *Consider* $\alpha = \frac{1}{2}$, $K > 0$ *and any* $\mu \in \mathcal{M}(\mathbb{T} \times \mathbb{R})$. *Then, the Filippov set-valued tangent field associated with* $\mathcal{V}[\mu]$ *reads as follows*

$$\mathcal{K}[\mathcal{V}[\mu]](z_0, \Omega_0) = \{(p\, i z_0, 0) : \, p \in \mathcal{K}[\mathcal{P}[\mu](\cdot, \Omega_0)](\theta_0)\},$$

for every $(z_0 = e^{i\theta_0}, \Omega_0) \in \mathbb{T} \times \mathbb{R}$. *Here, the set* $\mathcal{K}[\mathcal{P}[\mu](\cdot, \Omega_0)](\theta_0)$ *is nothing but the standard Filippov set-valued map at* $\theta = \theta_0$ *associated with the map* $\theta \in \mathbb{R} \longmapsto \mathcal{P}[\mu](e^{i\theta}, \Omega_0)$ *with fixed* Ω_0.

Proof. Consider any $(z_0 = e^{i\theta_0}, \Omega_0)$ and take some local chart around it, e.g.,

$$(\theta, \Omega) \in (\theta_0 - \pi, \theta_0 + \pi) \times \mathbb{R} \longmapsto (e^{i\theta}, \Omega).$$

Consider the local frame $\frac{\partial}{\partial \theta}$ and $\frac{\partial}{\partial \Omega}$, see Remark 4.6.3. Then, we can write the transport field $\mathcal{V}[\mu]$ in local coordinates as follows

$$\mathcal{V}[\mu](z, \Omega) = \mathcal{P}[\mu](z, \Omega)(iz, 0) + 0\,(0, 1),$$

for every $z \in \mathbb{T} \setminus \{\bar{z}_0\}$ and $\Omega \in \mathbb{R}$. Set the vector field

$$\begin{array}{rccc} F: & (\theta_0 - \pi, \theta_0 + \pi) \times \mathbb{R} & \longrightarrow & \mathbb{R}^2, \\ & (\theta, \Omega) & \longmapsto & (\mathcal{P}[\mu](e^{i\theta}, \Omega), 0). \end{array}$$

Then, Definition 4.6.2 shows that

$$\mathcal{K}[\mathcal{V}[\mu]](z_0, \Omega_0) = \{(p\, i z_0, q) : \, (p, q) \in \mathcal{K}[F](\theta_0, \Omega_0)\}.$$

Notice that the q-component in any vector in $(p, q) \in \mathcal{K}[F](\theta_0, \Omega_0)$ vanishes by virtue of the definition of F and that F is continuous in the second variable. A straightforward computation then shows that

$$\mathcal{K}[F_1](\theta_0, \Omega_0) = \mathcal{K}[F_1(\cdot, \Omega_0)](\theta_0) \equiv \mathcal{K}[\mathcal{P}[\mu](\cdot, \Omega_0)](\theta_0),$$

and that ends the proof. □

The Filippov flow of the transport field

In this part we will recover some regularity properties of the Filippov set-valued vector field that extend those in Subsection 4.2.3 to the critical case and will be useful throughout the next parts. Our final goal will be to show the well-posedness of a Filippov flow associated with the transport field $\mathcal{V}[\mu]$ for $\alpha = \frac{1}{2}$.

Lemma 4.6.8. *Consider $\alpha = \frac{1}{2}$, $K > 0$ and $\mu \in L^\infty_w(0,T;\mathcal{M}(\mathbb{T}\times\mathbb{R}))$. Then,*

$$\frac{\mathcal{P}[\mu]}{1+|\Omega|} \in L^\infty([0,T]\times\mathbb{T}\times\mathbb{R}).$$

Proof. By analogue arguments to those in Corollary 4.2.10 we show that

$$\operatorname{ess\,sup}_{(t,z,\Omega)\in[0,T]\times\mathbb{T}\times\mathbb{R}} \frac{|\mathcal{P}[\mu_t](z,\Omega)|}{1+|\Omega|} \leq \sup_{(z,\Omega)\in\mathbb{T}\times\mathbb{R}} \frac{|\Omega| + K\|\mu\|_{L^\infty(0,T;\mathcal{M}(\mathbb{T}\times\mathbb{R}))}\|h\|_{L^\infty(\mathbb{T})}}{1+|\Omega|} < \infty,$$

what proves the claimed results. □

Recall that in Subsection 4.2.3 the existence of classical solutions in the subcritical case when $\mu \in \widetilde{\mathcal{C}}_\mathcal{M}$ relied on the following continuity property (see Corollary 4.2.10)

$$\frac{\mathcal{V}[\mu]}{1+|\Omega|} \in C([0,T],\mathfrak{X}C_b(\mathbb{T}\times\mathbb{R})).$$

There, time continuity came from the continuity of μ in $C([0,T],\mathcal{M}(\mathbb{T}\times\mathbb{R})-\text{narrow})$ and was lost when $\mu \in \mathcal{C}_\mathcal{M}$ only. Nontheless, as noticed in Remark 4.2.11, Caratheodory's conditions (in particular continuity in z) were found under such weaker assumptions where tightness is deprived. Such conditions prove sufficient to obtain solutions in the sense of Caratheodory. Unfortunately, in our setting for $\alpha = \frac{1}{2}$, we expect to lose continuity both in t and z due to the jump discontinuity of h. Indeed, note that since the microscopic system shows global phase synchronization in finite time under certain regimes of the natural frequencies (see Chapter 3), it is possible that Dirac masses emerge and gain mass as times goes on. It justifies that time continuity might also be lost at certain times and the velocity field becomes discontinuous at such phase value. Our next result clarifies to what extend time continuity can be preserved, see [67, Lemma A.1] for similar results.

Theorem 4.6.9. *Consider $\alpha = \frac{1}{2}$, $K > 0$ and let $\{\mu_n\}_{n\in\mathbb{N}}$ and μ be in $\mathcal{P}(\mathbb{T}\times\mathbb{R})$ so that*

$$\mu_n \to \mu \ \text{ in } \ \mathcal{P}(\mathbb{T}\times\mathbb{R})-\textit{narrow}.$$

Then, the following convergence takes place

$$\lim_{n\to\infty}\sup_{\Omega\in\mathbb{R}} |\mathcal{V}[\mu_n](z,\Omega) - \mathcal{V}[\mu](z,\Omega)| = 0,$$

for each continuity point $z \in \mathbb{T}$ of the marginal measure $(\pi_z)_\#\mu$, where π_z is the projection (N.1). *In particular, it happens a.e. in $\mathbb{T}$.*

Proof. Let us define $\rho_n := (\pi_z)_\#\mu_n$ and $\rho := (\pi_z)_\#\mu$. Then, notice that

$$|\mathcal{V}[\mu_n](z,\Omega) - \mathcal{V}[\mu](z,\Omega)| = |\mathcal{P}[\mu_n](z,\Omega) - \mathcal{P}[\mu](z,\Omega)| = \left| K\int_{(-\pi,\pi]\setminus\{\theta\}} h(\theta-\theta')\,d_{\theta'}(\rho_n-\rho)\right|,$$

for $z = e^{i\theta} \in \mathbb{T}$ such that $\theta \in (-\pi, \pi]$ and $\Omega \in \mathbb{R}$. Define

$$F_n(\theta) := K \int_{(-\pi,\pi]\setminus\{\theta\}} h(\theta - \theta')\, d_{\theta'}(\rho_n - \rho) = F_n^1(\theta) + F_n^2(\theta) + F_n^3(\theta),$$

for every $\theta \in (-\pi, \pi]$ and $n \in \mathbb{N}$, where each term reads

$$\begin{aligned}
F_n^1(\theta) &:= K \int_{(-\pi,\pi]\setminus\{\theta\}} h_\varepsilon(\theta - \theta')\, d_{\theta'}(\rho_n - \rho),\\
F_n^2(\theta) &:= K \int_{(-\pi,\pi]\setminus\{\theta\}} \chi_{|\theta'-\theta|_o \geq \varepsilon^{1/2}} (h(\theta - \theta') - h_\varepsilon(\theta - \theta'))\, d_{\theta'}(\rho_n - \rho),\\
F_n^3(\theta) &:= K \int_{(-\pi,\pi]\setminus\{\theta\}} \chi_{|\theta'-\theta|_o < \varepsilon^{1/2}} (h(\theta - \theta') - h_\varepsilon(\theta - \theta'))\, d_{\theta'}(\rho_n - \rho).
\end{aligned}$$

Here $\varepsilon > 0$ is any fixed but arbitrary parameter. First, notice that $\rho_n \to \rho$ narrow in $\mathcal{P}(\mathbb{T})$. Since h_ε is continuous, then,

$$\lim_{n\to\infty} F_n^1(\theta) = 0,$$

for every $\theta \in (-\pi, \pi]$. Second, by a clear application of the mean value theorem, we obtain

$$|h(\theta) - h_\varepsilon(\theta)| \leq \frac{1}{2} \frac{\varepsilon}{|\theta|_o}, \tag{4.6.3}$$

for all $\theta \in \mathbb{R} \setminus 2\pi\mathbb{Z}$. As an application of (4.6.3) we obtain the upper bound

$$\limsup_{n\to\infty} |F_n^2(\theta)| \leq \varepsilon^{1/2}, \tag{4.6.4}$$

for all $\theta \in (-\pi, \pi]$. Finally, consider any cut-off function $\xi \in C_c^\infty([0, +\infty))$ like in (N.2) and notice that

$$\begin{aligned}
|F_n^3(\theta)| &\leq 2 \int_{(-\pi,\pi]\setminus\{\theta\}} \xi\left(\frac{|\theta'-\theta|_o}{\varepsilon^{1/2}}\right) d_{\theta'}(\rho_n + \rho)\\
&\leq 2 \int_{(-\pi,\pi]} \xi\left(\frac{|\theta'-\theta|_o}{\varepsilon^{1/2}}\right) d_{\theta'}(\rho_n - \rho) + 4 \int_{(-\pi,\pi]} \chi_{|\theta'-\theta|_o \leq \varepsilon^{1/2}}\, d_{\theta'}\rho.
\end{aligned}$$

Consequently,

$$\limsup_{n\to\infty} |F_n^3(\theta)| \leq 4 \int_{(-\pi,\pi]} \chi_{|\theta'-\theta|_o \leq \varepsilon^{1/2}}\, d_{\theta'}\rho. \tag{4.6.5}$$

Putting (4.6.3), (4.6.4) and (4.6.5) together yields

$$\limsup_{n\to\infty} |F_n(\theta)| \leq \varepsilon^{1/2} + 4 \int_{(-\pi,\pi]} \chi_{|\theta'-\theta|_o \leq \varepsilon^{1/2}}\, d_{\theta'}\rho,$$

for every $\varepsilon > 0$. The fist term vanishes as $\varepsilon \to 0$. Since θ is a continuity point of ρ, i.e., $\rho(\{\theta\}) = 0$, so does the second term. Thus,

$$\limsup_{n\to\infty} |F_n(\theta)| = 0,$$

and that ends the first part. The second part is clear and follows, for instance, from the Lebesgue decomposition theorem. $\square$

Corollary 4.6.10. *Consider $\alpha = \frac{1}{2}$, $K > 0$ and let μ be in $C([0,T], \mathcal{P}(\mathbb{T}\times\mathbb{R}) - narrow)$. Then,*

$$\lim_{\tau\to t}\sup_{\Omega\in\mathbb{R}} |\mathcal{V}[\mu_\tau](z,\Omega) - \mathcal{V}[\mu_t](z,\Omega)| = 0, \tag{4.6.6}$$

for almost every $(t,z)\in[0,T]\times\mathbb{T}$.

Proof. Theorem 4.6.9 provides a negligible set $\mathcal{N}_t \subseteq \mathbb{T}$ so that (4.6.6) takes place, for every $z\in\mathbb{T}\setminus\mathcal{N}_t$ and each $t\in[0,T]$. Consider the set

$$\mathcal{N} := \{(t,z):\, t\in[0,T],\, z\in\mathcal{N}_t\}.$$

It is a negligible set of $\mathbb{T}\times\mathbb{R}$ and the claimed convergence takes place in $([0,T]\times\mathbb{T})\setminus\mathcal{N}$. □

Similarly we obtain.

Corollary 4.6.11. *Consider $\alpha = \frac{1}{2}$, $K > 0$ and set $\{\mu^n\}_{n\in\mathbb{N}}$ and μ in $C([0,T], \mathcal{P}(\mathbb{T}\times\mathbb{R}) - narrow)$ so that the following narrow convergence takes place*

$$\mu^n \to \mu \ \text{ in } \ C([0,T], \mathcal{P}(\mathbb{T}\times\mathbb{R}) - narrow).$$

Then, we obtain that

$$\lim_{n\to\infty}\sup_{\Omega\in\mathbb{R}} |\mathcal{V}[\mu_t^n](z,\Omega) - \mathcal{V}[\mu_t](z,\Omega)| = 0,$$

for almost every $(t,z)\in[0,T]\times\mathbb{T}$.

In the sequel, we will resort on the concept of solutions in the sense of Filippov, to be understood in the sense of absolutely continuous solutions that solve the differential inclusion into the Filippov set-valued tangent field almost everywhere. See Appendix D for the necessary tools to construct such solutions and also [14, 130, 205, 249]. Indeed, using Lemma D.2.3 in Appendix D, we derive the following existence result.

Lemma 4.6.12. *Consider $\alpha = \frac{1}{2}$, $K > 0$ and fix $\mu \in L^\infty(0,T;\mathcal{M}(\mathbb{T}\times\mathbb{R}))$. For any $x_0 = (z_0,\Omega_0)\in\mathbb{T}\times\mathbb{R}$ let us consider the characteristic system issued at x_0, i.e.,*

$$\begin{cases} \dfrac{dX}{dt}(t;t_0,x_0) \in \mathcal{K}[\mathcal{V}[\mu_t]](X(t;t_0,x_0)), \\ X(t_0;t_0,x_0) = x_0. \end{cases} \tag{4.6.7}$$

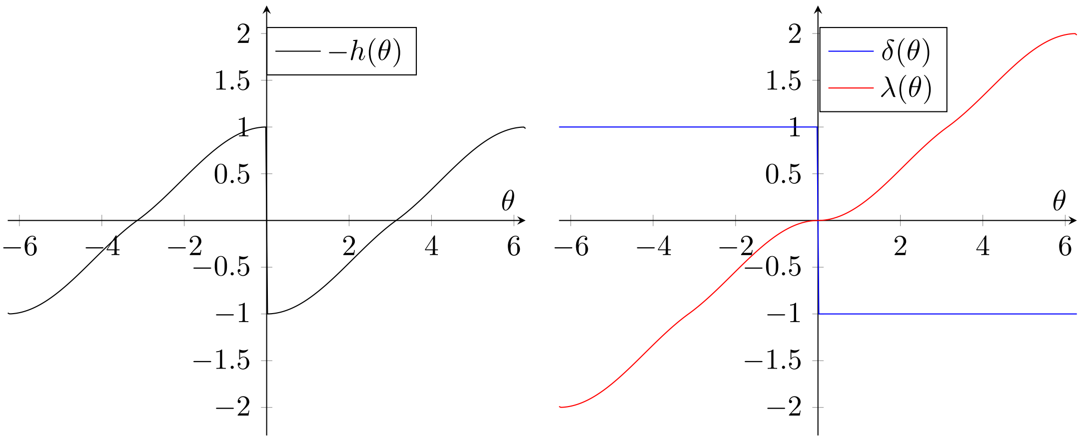
Then, (4.6.7) *has at least one global-in-time Filippov solution $X(t;t_0,x_0) = (Z(t;t_0,z_0,\Omega_0),\Omega_0)$. Indeed, if we set $z_0 = e^{i\theta_0}$ for some $\theta_0\in\mathbb{R}$, then*

$$Z(t;t_0,z_0,\Omega_0) = e^{i\Theta(t;t_0,\theta_0,\Omega_0)},$$

where $\Theta = \Theta(t;t_0,\theta_0,\Omega_0)$ is a global-in-time Filippov solution to

$$\begin{cases} \dfrac{d\Theta}{dt}(t;t_0,\theta_0,\Omega_0) \in \mathcal{K}[\mathcal{P}[\mu_t](\cdot,\Omega_0)](\Theta(t;t_0,\theta_0,\Omega_0)), \\ \Theta(t_0;t_0,\theta_0,\Omega_0) = \theta_0. \end{cases} \tag{4.6.8}$$

Our next step is to show the uniqueness of the above Filippov trajectories. Our approach mimics that in Subsection 4.2.3, specifically, we will show an analogue decomposition to that in Lemma 4.2.16 (see also Figure 4.3) that implies one-sided Lipschitz condition. Also see Figure 3.4 in the preceding Chapter 3 for a similar split of the kernel in the smaller fomain $[-\pi,\pi]$, that was applied to the agent-based system.



(a) $-h(\theta)$ (b) $\delta(\theta)$ and $\lambda(\theta)$

Figure 4.3: Plot of the function $-h(\theta)$ and the functions $\delta(\theta)$ and $\lambda(\theta)$ in the decomposition for the value $\alpha = 0.5$.

Lemma 4.6.13. *Consider $\alpha = \frac{1}{2}$ and define the couple of functions $\delta, \lambda : [-2\pi, 2\pi] \longrightarrow \mathbb{R}$ as follows*

$$\delta(\theta) := \begin{cases} -1 - h(\theta), & \theta \in [-2\pi, 0), \\ 1 - h(\theta), & \theta \in (0, 2\pi]. \end{cases}$$

$$\lambda(\theta) := \begin{cases} 1, & \theta \in [-2\pi, 0), \\ -1, & \theta \in (0, 2\pi]. \end{cases}$$

Then, the following properties hold true

1. *δ is monotonically decreasing, λ is Lipschitz-continuous and*

$$-h(\theta) = \delta(\theta) + \lambda(\theta), \ \forall \theta \in [-2\pi, 2\pi].$$

2. *$-h$ is one-sided Lipschitz in $[-2\pi, 2\pi]$, i.e., there exists $L_0 > 0$ such that*

$$((-h)(\theta_1) - (-h)(\theta_2))\,(\theta_1 - \theta_2) \leq L_0(\theta_1 - \theta_2)^2.$$

Hence, we are ready to introduce the following results.

Lemma 4.6.14. *Consider $\alpha \in \frac{1}{2}$, $K > 0$ and set $\mu \in L^\infty(0, T; \mathcal{M}(\mathbb{T} \times \mathbb{R}))$. Then, for L_0 given in Lemma 4.6.13 we obtain that*

$$(\mathcal{P}[\mu_t](\theta_1, \Omega_1) - \mathcal{P}[\mu_t](\theta_2, \Omega_2))\,(\theta_1 - \theta_2) \leq (\Omega_1 - \Omega_2)(\theta_1 - \theta_2) + K L_0 \|\mu\|_{L^\infty(0,T;\mathcal{M}(\mathbb{T}\times\mathbb{R}))}(\theta_1 - \theta_2)^2,$$

for every $t \in [0, T]$, each $\theta_1, \theta_2 \in \mathbb{R}$ with $\theta_1 - \theta_2 \in [-\pi, \pi]$ and any $\Omega_1, \Omega_2 \in \mathbb{R}$.

Theorem 4.6.15. *Consider $\alpha = \frac{1}{2}$, $K > 0$ and set $\mu \in L^\infty(0, T; \mathcal{M}(\mathbb{T} \times \mathbb{R}))$. Then, $\mathcal{V}[\mu]$ is one-sided Lipschitz in $\mathbb{T} \times \mathbb{R}$ uniformly in $t \in [0, T]$, i.e., there exists $L = L(\alpha, K, \mu) > 0$ such that*

$$\left\langle \mathcal{V}[\mu_t](z_2, \Omega_2), \widehat{\gamma}'(1) \right\rangle - \left\langle \mathcal{V}[\mu_t](z_1, \Omega_1), \widehat{\gamma}'(0) \right\rangle \leq L\, d((z_1, \Omega_1), (z_2, \Omega_2))^2,$$

for every $t \in [0, T]$, any $(z_1, \Omega_1), (z_2, \Omega_2) \in \mathbb{T} \times \mathbb{R}$ and each minimizing geodesic $\widehat{\gamma} : [0, 1] \longrightarrow \mathbb{T} \times \mathbb{R}$ in the manifold $\mathbb{T} \times \mathbb{R}$ joining (z_1, Ω_1) to (z_2, Ω_2).

The proofs follow similar arguments to those in Lemma 4.2.17 and Theorem 4.2.18, then we omit them. Let us finally transform the point-wise one-sided Lipschitz condition into an analogue one-sided condition in the multivalued sense for the associated Filippov set-valued map.

Definition 4.6.16. *Let $(M, \langle\cdot,\cdot\rangle)$ be a complete Riemannian manifold and $V : M \longrightarrow 2^{TM}$ a set-valued tangent vector field along M. Then, we will say that V is one-sided Lipschitz (in multivalued sense) when there exists a constant $L > 0$ such that*

$$\left\langle v_y, \gamma'(1)\right\rangle - \left\langle v_x, \gamma'(0)\right\rangle \leq L d(x,y)^2,$$

for each $x, y \in M$, each $v_x \in X_x$, $v_y \in X_y$ and any minimizing geodesic $\gamma : [0,1] \longrightarrow M$ joining x to y.

Lemma 4.6.17. *Let $(M, \langle\cdot,\cdot\rangle)$ be a complete Riemannian manifold and consider an essentially locally bounded tangent field $V \in \mathfrak{X}L^\infty_{loc}(M)$. If V is one-sided Lipschitz a.e. with constant $L > 0$, then so is its Filippov set-valued tangent field $\mathcal{K}[V]$ given by Definition 4.6.2 with same constant L.*

The starting point of the proof is that the analogue result with the general $(M, \langle\cdot,\rangle)$ replaced by the Euclidean space with the flat metric is clearly true and, in particular, was shown in Lemma D.1.6. Then, we will use the local description in coordinates appearing in Definition 4.6.2 to augment our local result into a global one.

Proof. Consider $x, y \in M$, set any minimizing geodesic $\gamma : [0,1] \longrightarrow M$ joining x to y and fix any $v_x \in \mathcal{K}[V]_x$ and $v_y \in \mathcal{K}[V]_y$. Our goal is to show that

$$\left\langle v_y, \gamma'(1)\right\rangle - \left\langle v_y, \gamma'(0)\right\rangle \leq L d(x,y)^2. \tag{4.6.9}$$

• *Step 1*: Local normal orthonormal frame around x.

We can construct a local frame $(\mathcal{U}, \varphi, E_1, \ldots, E_d)$ centered at x where $\{E_1, \ldots, E_d\}$ is an orthonormal basis at each point and $(\mathcal{U}, \varphi)$ is a normal neighborhood. The existence of normal neighborhoods around any points is classical in Riemannian Geometry (see [111, 191, 246]) and follows from the inverse function theorem. Indeed, let us consider the injectivity radius of M at x, that is defined by

$$\iota(x) := \operatorname{dist}(x, \operatorname{cut}(x)) = \sup\{\delta > 0 :\ \exp_x : \mathbb{B}_\delta(0) \longrightarrow \exp_x(\mathbb{B}_\delta(0)) \text{ is a diffeomorphism}\},$$

that is a positive number. Then, any $\delta < \iota(x)$ provides a geodesic ball $\exp_x(\mathbb{B}_\delta(0))$. Take $\mathcal{U} = \exp_x(\mathbb{B}_\delta(0))$ as normal neighborhood of x.

Regarding the local chart, let us consider $\varphi := (\exp_x \circ \mathcal{L})^{-1}$ for any linear isometry $\mathcal{L} : \mathbb{R}^d \longrightarrow T_xM$, that maps the standard basis $\{e_1, \ldots, e_d\}$ of $\mathbb{R}^d$ into some orthonormal basis $\{u_1, \ldots, u_d\}$ of T_xM, i.e., $\mathcal{L}(e_i) = u_i$ for every $i = 1, \ldots, d$. Obviously, the associated local frame $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d}$ is not necessarily orthonormal in the full $\mathcal{U}$. In fact, only locally flat Riemannian manifolds (like $M = \mathbb{T}\times\mathbb{R}$) can enjoy such property. Nevertheless, we can augment $\{u_1, \ldots, u_d\}$ into a complete local frame on $\mathcal{U}$ through the next procedure. Consider the unique minimizing geodesic $\gamma_{x,z} : [0,1] \longrightarrow M$ joining x to z, i.e.,

$$\gamma_{x,z}(s) := \exp_x(s \exp_x^{-1}(z)),\ s \in [0,1],$$

for any $z \in \mathcal{U}$. Then, we can define the local frame at z via paralleling transporting $\{u_1, \ldots, u_d\}$ along $\gamma_{x,z}$, that is,

$$U_i(z) := \tau[\gamma_{x,z}]_0^1(u_i),$$

for all $i = 1, \ldots, d$. Since the parallel transport is a linear isometry between tangent spaces we recover the local orthonormal character of the frame $U_1, \ldots, U_d$. In addition, the following property holds true by definition

$$\nabla_{\gamma'_{x,z}(s)} U_i(\gamma_{x,z}(s)) = 0,$$

for all $s \in [0,1]$ and every $z \in \mathcal{U}$. In other words, the tangent fields U_i are all parallel along any radial geodesic issued at x within the normal neighborhood $\mathcal{U}$.

• *Step 2*: One-sided Lipschitz property of F.

Using such local frame $(\mathcal{U}, \varphi, E_1, \ldots, E_d)$, we can locally write the tangent field V as follows

$$V = \sum_{i=1}^{d} f_i E_i \text{ in } \mathcal{U}.$$

Again, define $F_i = f_i \circ \varphi^{-1}$ and the vector field $F = (F_1, \ldots, F_d)$. Our goal is to see that F is one-sided-Lipschitz at $\varphi(x) = 0$ with constant L. To that end, set any $z \in \mathcal{U}$ and consider the local coordinates

$$\begin{aligned}
V_x &= \sum_{i=1}^{d} F_i(\varphi(x)) E_i(x) = \sum_{i=1}^{d} F_i(\varphi(x)) e_i, \\
V_z &= \sum_{i=1}^{d} F_i(\varphi(z)) E_i(z) = \sum_{i=1}^{d} F_i(\varphi(z)) \tau[\gamma_{x,z}]_0^1(e_i) = \tau[\gamma_{x,z}] \left(\sum_{i=1}^{d} F_i(\varphi(z)) e_i \right).
\end{aligned}$$

Since V is one-sided Lipschitz at x with constant L, then

$$\langle V_z, \gamma'_{x,z}(1) \rangle - \langle V_x, \gamma'_{x,z}(0) \rangle \leq L\, d(x,z)^2.$$

Notice that $\gamma'_{x,z}(0) = \exp_x^{-1}(z) = \mathcal{L}(\varphi(z))$ and, consequently,

$$\begin{aligned}
(F(\varphi(z)) - F(\varphi(x))) \cdot (\varphi(z) - \varphi(0)) &= \langle \tau[\gamma_{x,z}]_1^0(V_z) - V_x, \gamma'_{x,z}(0) \rangle \\
&= \langle V_z, \gamma'_{x,z}(1) \rangle - \langle V_x, \gamma'_{x,z}(0) \rangle \\
&\leq L\, d(x,z)^2 = L\, |\varphi(z) - \varphi(x)|^2,
\end{aligned}$$

for every $z \in \mathcal{U}$. As advanced in the above comment, Lemma D.1.6 shows that $\mathcal{K}[F]$ is one-sided Lipschitz continuous at $\varphi(x)$ with constant L, i.e.,

$$(\beta^z - \beta^x) \cdot (\varphi(z) - \varphi(x)) \leq L |\varphi(z) - \varphi(x)|^2, \tag{4.6.10}$$

for every $z \in \mathcal{U}$ and every $\beta^x \in \mathcal{K}[F](\varphi(x))$ and $\beta^z \in \mathcal{K}[F](\varphi(z))$.

• *Step 3*: Local result.

In this step we will assume that the whole minimizing geodesic γ joining x to y lies in $\mathcal{U}$, i.e., $\gamma(s) \in \mathcal{U}$ for all $s \in [0,1]$. By the uniqueness, we obtain $\gamma = \gamma_{x,y}$. In that case, we write

$$\begin{aligned}
v_x &= \sum_{i=1}^{d} \beta_i U_i(x) = \sum_{i=1}^{d} \beta_i u_i, \\
v_y &= \sum_{i=1}^{d} \gamma_i U_i(y) = \sum_{i=1}^{d} \gamma_i \tau[\gamma_{x,y}]_0^1(u_i) = \tau[\gamma_{x,y}]_0^1 \left(\sum_{i=1}^{d} \gamma_i u_i \right),
\end{aligned}$$

for some coefficients $\beta^x = (\beta_1^x, \ldots, \beta_d^x) \in \mathcal{K}[F](\varphi(x))$ and $\beta^y = (\beta_1^y, \ldots, \beta_d^y) \in \mathcal{K}[F](\varphi(y))$. Note that $\gamma'(0) = \exp_x^{-1}(y) = \mathcal{L}(\varphi(y))$ and γ is a geodesic. Then, we infer

$$\langle v_y, \gamma'(1)\rangle - \langle v_x, \gamma'(0)\rangle = \left\langle \sum_{i=1}^{d} (\beta_i^y - \beta_i^x) u_i, \gamma'(0) \right\rangle = (\beta^y - \beta^x) \cdot (\varphi(y) - \varphi(0)).$$

Hence, (4.6.9) follows from (4.6.10).

• *Step 4*: Global result.

Now, let us assume that the minimizing geodesic γ does not necessarily lies in the normal neighborhood $\mathcal{U}$. Notice that for every $s \in [0,1]$ we can repeat *Step 1* to find a normal orthonormal frame $(\mathcal{U}^s, \varphi^s, E_1^s, \ldots, E_d^s)$ around $\gamma(s)$. Indeed, recall that $\mathcal{U}^s = \exp_{\gamma(s)}(\mathbb{B}_{\delta_s}(0))$, for any $0 < \delta < \iota(\gamma(s))$ and each $s \in [0,1]$. Notice that we can choose all the δ_s to be the same δ independently on s. To such end, recall that the injectivity radius $\iota : M \longrightarrow \mathbb{R}^+$ is a continuous function, see [191, Proposition 2.1.10]. Since $\gamma([0,1])$ is compact, then

$$\delta_0 := \min_{s \in [0,1]} \iota(\gamma(s)) > 0,$$

and we choose any $0 < \delta < \delta_0$ as the radii of each geodesic ball about $\gamma(s)$. Fix $n \in \mathbb{N}$ the smallest integer with $d(x,y) \leq k\delta$ and define the numbers

$$s_k = \begin{cases} k\delta, & k = 0, \ldots, n-1, \\ 1, & k = n. \end{cases}$$

Since γ is a minimizing geodesic we infer

$$\gamma([s_k, s_{k+1}]) \subseteq \mathcal{U}^{s_k},$$

for all $k = 0, \ldots, n-1$. Hence, for all $k = 0, \ldots, n-1$ the piece of geodesic $\gamma([s_k, s_{k+1}])$ satisfies the same properties as in *Step 1* for the normal local orthonormal frame $(\mathcal{U}^{s_k}, \varphi^{s_k}, E_1^{s_k}, \ldots, E_d^{s_k})$. Let us consider any $v_{\gamma(s_k)} \in \mathcal{K}[V]_{\gamma(s_k)}$, for every $k = 1, \ldots, n-1$. Thus, we have

$$\langle v_{\gamma(s_{k+1})}, \gamma'(s_{k+1})\rangle - \langle v_{\gamma(s_k)}, \gamma'(s_k)\rangle \leq L\, d(\gamma(s_k), \gamma(s_{k+1}))^2,$$

for every $k = 0, \ldots, n-1$. Then,

$$\begin{aligned} \langle v_y, \gamma'(1)\rangle - \langle v_x, \gamma'(0)\rangle &= \sum_{k=0}^{n-1} \langle v_{\gamma(s_{k+1})}, \gamma'(s_{k+1})\rangle - \langle v_{\gamma(s_k)}, \gamma'(s_k)\rangle \\ &\leq L \sum_{k=0}^{n-1} d(\gamma(s_k), \gamma(s_{k+1}))^2 \leq L \left(\sum_{k=0}^{n-1} d(\gamma(s_k), \gamma(s_{k+1})) \right)^2 = L\, d(x,y)^2, \end{aligned}$$

where we have used that γ is minimizing in the last step. □

As a direct consequence of Theorem 4.6.15 and Lemma 4.6.17 we obtain the following result.

Corollary 4.6.18. *Consider $\alpha = \frac{1}{2}$, $K > 0$ and set $\mu \in L_w^\infty(0,T; \mathcal{M}(\mathbb{T} \times \mathbb{R}))$. Then, the Filippov set-valued vector field $\mathcal{K}[\mathcal{V}[\mu]]$ associated with the transport field $\mathcal{V}[\mu]$ is one-sided Lipschitz in $\mathbb{T} \times \mathbb{R}$ in multivalued sense uniformly in $t \in [0,T]$, i.e., there exists $L = L(\alpha, K, \mu) > 0$ such that*

$$\langle v_{(z_2,\Omega_2)}, \widehat{\gamma}'(1)\rangle - \langle v_{(z_1,\Omega_1)}, \widehat{\gamma}'(0)\rangle \leq L d((z_1,\Omega_1),(z_2,\Omega_2))^2,$$

for every $t \in [0,T]$, each (z_1,Ω_1), $(z_2,\Omega_2) \in \mathbb{T} \times \mathbb{R}$, any couple $v_{(z_1,\Omega_1)} \in \mathcal{K}[\mathcal{V}[\mu_t]](z_1,\Omega_1)$ and $v_{(z_2,\Omega_2)} \in \mathcal{K}[\mathcal{V}[\mu_t]](z_2,\Omega_2)$, and each minimizing geodesic $\widehat{\gamma} : [0,1] \longrightarrow \mathbb{T} \times \mathbb{R}$ joining the point (z_1,Ω_1) to the point (z_2,Ω_2).

We are now ready to recover a one-sided uniqueness property for Filippov characteristics solving (4.6.7).

Theorem 4.6.19. *Consider $\alpha = \frac{1}{2}$, $K > 0$ and fix $\mu \in L^\infty_w(0,T;\mathcal{M}(\mathbb{T}\times\mathbb{R}))$. The characteristic system* (4.6.7) *associated with the transport field $\mathcal{V}[\mu]$ enjoys a global-in-time absolutely continuous Filippov solution that is unique forward-in-time for every given initial data $x_0 = (z_0,\Omega_0) = (e^{i\theta_0},\Omega_0) \in \mathbb{T}\times\mathbb{R}$. Indeed, the same representation of the solution holds true, i.e.,*

$$X(t;t_0,x_0) = (Z(t;t_0,z_0,\Omega_0),\Omega_0) = (e^{i\Theta(t;t_0,\theta_0,\Omega_0)},\Omega_0),\ t \geq t_0,$$

where $\Theta(t;t_0,\theta_0,\Omega_0)$ is the unique forward-in-time absolutely continuous Filippov solution to (4.6.8).

We will say that $X(\cdot\,;t_0,x_0)$ is the *Filippov characteristic* issued at x_0 at time t_0 and $X(t;t_0,\cdot)$ is the *Filippov flow* from t_0 to t. The proof is a simple adaptation to that of Theorem 4.2.19, when classical solutions are replaced with Filippov solutions. Again, the proof relies on the one-sided Lipschitz condition in multivalued sense for the Fillippov set-valued tangent field associated with the transport field (recall Theorem 4.6.15). Also, the weak differentiability properties of the squared distance in Appendix 4.B will be used. For clarity, we just provide a sketch.

Proof. Consider two different solutions in the sense of Filippov $x_1 = x_1(t)$ and $x_2 = x_2(t)$ to the characteristic system (4.6.7) issued at $x_1(t_0) = x_0 = x_2(t_0)$ and define

$$I(t) := \frac{1}{2} d(x_1(t),x_2(t))^2,\ t \geq t_0.$$

Since the Filippov trajectories are at least locally absolutely continuous, then so is $I = I(t)$. Again, we compute it in terms of the one-sided Dini upper derivative

$$\frac{dI}{dt} \equiv \frac{d^+I}{dt} = d^+\left(\frac{1}{2}d^2_{x_2(t)}\right)_{x_1(t)}(\dot{x}_1(t)) + d^+\left(\frac{1}{2}d^2_{x_1(t)}\right)_{x_2(t)}(\dot{x}_2(t)),$$

for almost every $t \geq t_0$. By virtue of Theorem 4.B.7

$$d^+\left(\frac{1}{2}d^2_{x_2(t)}\right)_{x_1(t)}(\dot{x}_1(t)) \leq \inf_{\substack{w_1\in \exp^{-1}_{x_1(t)}(x_2(t))\\ |w_1| = d(x_1(t),x_2(t))}} -\langle \dot{x}_1(t),w_1\rangle,$$

$$d^+\left(\frac{1}{2}d^2_{x_1(t)}\right)_{x_2(t)}(\dot{x}_2(t)) \leq \inf_{\substack{w_2\in \exp^{-1}_{x_2(t)}(x_1(t))\\ |w_2| = d(x_1(t),x_2(t))}} -\langle \dot{x}_2(t),w_2\rangle.$$

Fix a minimizing geodesic $\widehat{\gamma}_t : [0,1] \longrightarrow \mathbb{T}\times\mathbb{R}$ joining $x_1(t)$ to $x_2(t)$, for almost every $t \geq t_0$. Then, we can choose $w_1 = \widehat{\gamma}_t'(0)$ and $w_2 = -\widehat{\gamma}_t'(1)$ in the above inequalities. Consequently,

$$\frac{dI}{dt} \leq \left\langle \dot{x}_2(t),\widehat{\gamma}_t'(1)\right\rangle - \left\langle \dot{x}_1(t),\widehat{\gamma}_t'(0)\right\rangle,\ \text{ a.e. } t \geq 0,$$

where, recall that $\dot{x}_1(t) \in \mathcal{K}[\mathcal{V}[\mu_t]](x_1(t))$ and $\dot{x}_2(t) \in \mathcal{K}[\mathcal{V}[\mu_t]](x_2(t))$. Then, Theorem 4.2.18 implies

$$\frac{dI}{dt} \leq L d(x_1(t),x_2(t)^2) = 2L\, I(t),\ \text{ a.e. } t \geq t_0,$$

and Grönwall's lemma concludes the proof. □

Empirical measures as solutions in the sense of the Filippov flow

Our next goal is to recover an analogue of Theorem 4.3.2 for the critical regime $\alpha = \frac{1}{2}$, where the classical solutions to (4.1.1)-(4.1.2) are considered in Filippov's sense as it was introduced in Theorem 3.3.12 in the above Chapter 3. More specifically, we will no longer obtain weak-measure valued solutions like in the preceding theorem for the subcritical regime, but *measure valued solutions in the sense of the Filippov flow*, to be introduced in the sequel. Throughout this part, we shall use the notation in Chapter 3 to describe collision classes, formation of clusters and sticking. We refer to Subsection 3.2.3 for an easier readability.

Theorem 4.6.20. *Consider $\alpha = \frac{1}{2}$ and $K > 0$. Fix $N \in \mathbb{N}$ and consider N oscillators with initial phases and natural frequencies given by the configurations*

$$\Theta_0^N = (\theta_{1,0}^N, \ldots, \theta_{N,0}^N) \text{ and } \{\Omega_i^N : i = 1, \ldots, N\}.$$

Consider the (forward-in-time) unique solution $\Theta^N(t) = (\theta_1^N(t), \ldots, \theta_t^N)$ to (4.1.1)-(4.1.2) in the sense of Filippov associated with such initial configuration as given in Theorem 3.3.12 and set the corresponding empirical measures μ^N according to Definition 4.3.1. Then, $\mu^N \in \mathcal{AC}_{\mathcal{M}} \cap \mathcal{T}_{\mathcal{M}}$ and

$$X^N(t; 0, \cdot)_{\#}\mu_0^N = \mu_t^N, \tag{4.6.11}$$

for every $t \geq 0$, where $X^N(t; 0, \cdot)$ is the Filippov flow associated with the transport field $\mathcal{V}[\mu^N]$. In addition, an analogue of (4.3.1) holds true, namely,

$$\left| \frac{d}{dt} \int_{\mathbb{T}\times\mathbb{R}} \varphi \, d\mu_t^N \right| \leq \left(\frac{1}{N} \sum_{i=1}^N |\Omega_i^N| + K \right) \|\nabla\varphi\|_{C_0(\mathbb{T}\times\mathbb{R})}, \tag{4.6.12}$$

for almost every $t \geq 0$ and every $\varphi \in C_0^1(\mathbb{T}\times\mathbb{R})$.

Proof. The proof that $\mu^N \in \mathcal{AC}_{\mathcal{M}} \cap \mathcal{T}_{\mathcal{M}}$ is parallel to that of Theorem 4.3.2. However, the time regularity is much tighter now. Specifically, fix any $\varphi \in C_c^\infty(\mathbb{T}\times\mathbb{R})$ and consider the map

$$t \in [0, +\infty) \longmapsto \int_{\mathbb{T}\times\mathbb{R}} \varphi \, d\mu_t^N = \frac{1}{N} \sum_{i=1}^N \varphi(\theta_i^N(t), \Omega_i^N). \tag{4.6.13}$$

Since φ is Lipschitz-continuous and $\Theta^N = \Theta^N(t)$ is a Filippov solution (then, it is merely absolutely continuous), such map is not necessarily C^1 like in the subcritical case, but it is locally absolutely continuous at least. The tightness condition is again clear since μ^N has a bounded first order Ω-moment uniformly-in-time like in the subcritical regime. Our next goal is to prove that (4.6.11) holds true. Take any $\varphi \in C_b(\mathbb{T}\times\mathbb{R})$ and note that, by definition,

$$\int_{\mathbb{T}\times\mathbb{R}} \varphi \, d(X^N(t; 0, \cdot)_{\#}\mu_0^N) = \frac{1}{N} \sum_{i=1}^N \varphi(X^N(t; 0, z_i^N(0), \Omega_i^N)).$$

Consequently, we just must check that

$$X^N(t; 0, z_i^N(0), \Omega_i^N) = (z_i^N(t), \Omega_i^N),$$

for every $t \geq 0$. By the one-sided uniqueness of Filippov trajectories (recall Theorem 4.6.19), we only need to show that

$$\frac{d}{dt}(z_i^N, \Omega_i^N) \in \mathcal{K}[\mathcal{V}[\mu_t^N]](z_i^N, \Omega_i^N).$$

Using the representation of the Filippov set-valued tangent field in Proposition 4.6.7, notice that we equivalently need to show that

$$\dot{\theta}_i^N \in \mathcal{K}[\mathcal{P}[\mu_t^N](\cdot,\Omega_i^N)](\theta_i^N), \tag{4.6.14}$$

for almost every $t \geq 0$. Using the notation in Subsection 3.2.3, we obtain

$$\mathcal{P}[\mu_t^N](\theta,\Omega_i^N) = \Omega_i^N - \frac{K}{N}\sum_{\substack{j=1\\ \bar{\theta}_j^N(t)\neq\bar{\theta}}}^{N} h(\theta-\theta_j^N(t)) = \Omega_i^N - \frac{K}{N}\sum_{\substack{k=1\\ \bar{\theta}_{\iota_k}^N(t)\neq\bar{\theta}}}^{\kappa^N(t)} n_k(t)h(\theta-\theta_{\iota_k}^N(t)),$$

where $\bar{\theta}$ is the representative of θ in $(-\pi,\pi]$ modulo 2π. Then, one clearly obtains that

$$\mathcal{K}[\mathcal{P}[\mu_t^N](\cdot,\Omega_i^N)](\theta) = \left\{\Omega_i^N - \frac{K}{N}\sum_{\substack{j=1\\ \bar{\theta}_j^N(t)\neq\bar{\theta}}}^{N} h(\theta-\theta_j^N(t)) - \frac{K}{N}\sum_{\substack{j=1\\ \bar{\theta}_j^N(t)=\bar{\theta}}}^{N} \widehat{y}_i \,:\, \widehat{y}_i\in[-1,1]\right\}. \tag{4.6.15}$$

In particular, if we evaluate (4.6.15) at $\theta = \theta_i^N(t)$ we obtain the set

$$\begin{aligned}&\mathcal{K}[\mathcal{P}[\mu_t^N](\cdot,\Omega_i^N)](\theta_i^N(t))\\ &\quad=\left\{\Omega_i^N - \frac{K}{N}\sum_{j\notin\mathcal{C}_i(\Theta^N(t))} h(\theta_i^N(t)-\theta_j^N(t)) - \frac{K}{N}\#\mathcal{C}_i(\Theta^N(t))\widehat{y}_i \,:\, \widehat{y}_i\in[-1,1]\right\}.\end{aligned} \tag{4.6.16}$$

On the other hand, since $\Theta^N = \Theta^N(t)$ is a Filippov solution to (4.1.1)-(4.1.2), then the characterization in Proposition 3.3.7 for the Filippov set-valued map associated with the discrete system shows that there exists $Y^N = (y_{ij}^N)_{1\leq i,j\leq N}$ with $y_{ij}^N \in L^\infty(0,+\infty)$ and $Y^N(t) \in \mathrm{Skew}_N([-1,1])$ for a.e. $t \geq 0$ so that

$$\dot{\theta}_i^N(t) = \Omega_i^N - \frac{K}{N}\sum_{j\notin\mathcal{C}_i(\Theta^N(t))} h(\theta_i^N(t)-\theta_j^N(t)) - \frac{K}{N}\sum_{j\in\mathcal{C}_i(\Theta^N(t))} y_{ij}^N(t), \tag{4.6.17}$$

for almost every $t \geq 0$ and every $i = 1,\ldots,N$. Let us define the following mean value

$$\widehat{y}_i^N(t) := \frac{1}{\#\mathcal{C}_i(\Theta^N(t))}\sum_{j\in\mathcal{C}_i(\Theta^N(t))} y_{ij}^N(t) \in [-1,1].$$

Then, such choice along with the representation (4.6.16) and (4.6.17) imply that (4.6.14) holds true. Finally, let us conclude (4.6.12). Fix any $\varphi \in C_0^1(\mathbb{T}\times\mathbb{R})$ and take derivatives almost everywhere in (4.6.13)

$$\frac{d}{dt}\int_{\mathbb{T}\times\mathbb{R}} \varphi\, d\mu_t^N = \frac{d}{dt}\left(\frac{1}{N}\sum_{i=1}^N \varphi(\theta_i^N(t),\Omega_i^N)\right) = \frac{1}{N}\sum_{i=1} \frac{\partial\varphi}{\partial\theta}(\theta_i^N(t),\Omega_i^N)\dot{\theta}_i^N(t).$$

Since $\theta_i^N(t)$ are solutions in the sense of Filippov to the discrete singular system (4.1.1)-(4.1.2), then

$$|\dot{\theta}_i^N(t)| \leq |\Omega_i^N| + K, \text{ for a.e. } t \geq 0,$$

and that ends the proof. □

Notice that we can repeat all the ideas of the compactness result in Corollary 4.3.6 for $\alpha = \frac{1}{2}$.

Corollary 4.6.21. *Consider $\alpha = \frac{1}{2}$ and $K > 0$. For every $N \in \mathbb{N}$, set N oscillators with initial phases and natural frequencies given by the configurations*

$$\Theta_0^N = (\theta_{1,0}^N, \ldots, \theta_{N,0}^N) \text{ and } \{\Omega_i^N : i = 1, \ldots, N\}.$$

Consider the (forward-in-time) unique Filippov solution $\Theta^N(t) = (\theta_1(t), \ldots, \theta_N(t))$ to (4.1.1)-(4.1.2) *as given in Theorem 3.3.12 of Chapter 3 and set the corresponding empirical measures μ^N according to Definition 4.3.1. Assume that the equi-sumability condition* (4.3.8) *holds true and take M_1 fulfilling* (4.3.7) *according to Proposition 4.3.4. Then, for every fixed $T > 0$, there exists a subsequence of μ^N, that we denote in the same way for simplicity, and a limiting measure $f \in \mathcal{AC}_{\mathcal{M}} \cap \mathcal{T}_{\mathcal{M}}$ such that* (4.3.14) *and* (4.3.15) *hold true. In addition,*

$$f \in W_w^{1,\infty}([0,T], C_0^1(\mathbb{T} \times \mathbb{R})^*) \cap C([0,T], \mathcal{P}_1(\mathbb{T} \times \mathbb{R}) - W_1),$$

for every fixed $T > 0$ and

$$\mu^N \to f \text{ in } C([0,T], \mathcal{P}_1(\mathbb{T} \times \mathbb{R}) - W_1),$$

where W_1 means the Kantorovich–Rubinstein distance.

Notice that the strong uniform convergence in Lemma 4.3.8 cannot hold in the critical regime. Nevertheless, the sequence $|\mathcal{V}[\mu^N] - \mathcal{V}[f]|$ is essentially uniformly bounded. Consequently it enjoys a subsequence, that we denote in the same way, so that it converges weakly-star in L^∞. Using a standard application of Banach–Saks' theorem, we claim that the weak-star limit agrees with the a.e. limit in Corollary 4.6.11, thus obtaining the following result.

Corollary 4.6.22. *Under the assumptions in Corollary 4.3.6 the following convergence takes place*

$$|\mathcal{V}[\mu^N] - \mathcal{V}[f]| \overset{*}{\rightharpoonup} 0 \text{ in } L^\infty((0,+\infty) \times \mathbb{T} \times \mathbb{R}).$$

Before showing our final result, that allows concluding the mean field limit and the existence of measure-valued solutions in the sense of the Filippov flow, we need to guarantee that the above weak-star convergence implies uniform convergence of the flows. In the sequel, we adapt the result in [32, Theorem 4.3] to vector fields in Riemannian manifolds.

Lemma 4.6.23. *Let $(M, \langle \cdot, \cdot \rangle)$ be a complete Riemannian manifold and consider $\{V^n\}_{n \in \mathbb{N}}$ and V in $L^\infty((0,+\infty), \mathfrak{X}L^\infty_{loc}(M))$ that are one-sided Lipschitz with same constant $L > 0$. Assume that*

$$|V^n - V| \rightharpoonup 0 \text{ in } L^1_{loc}((0,+\infty) \times \mathbb{T} \times \mathbb{R}).$$

Then, the associated Filippov flows $X^n = X^n(t;0,x)$ and $X = X(t;0,x)$ verify

$$X^n \to X \text{ in } C_{loc}((0,+\infty) \times \mathbb{T} \times \mathbb{R}).$$

Since the proof is clear, we omit it. It simply relies on Definition 4.6.2, where the Filippov set-valued tangent field is introduced using local coordinates, and the analogue result in [32, Theorem 4.3] for quasi-monotone operators in the Euclidean spaces.

Theorem 4.6.24. *Consider $\alpha = \frac{1}{2}$, $K > 0$ and set any initial datum $f_0 \in \mathcal{P}_1(\mathbb{T} \times \mathbb{R})$. Then, for every $T > 0$ there exists a measure-valued solution $f \in \mathcal{AC}_{\mathcal{M}} \cap \mathcal{T}_{\mathcal{M}}$ in the sense of the Filippov flow to the initial value problem* (4.2.5), *i.e.,*

$$X(t;0,\cdot)_{\#} f_t = f_0, \text{ for all } t \in [0,T],$$

where $X = X(t;0,z,\Omega)$ is the Filippov flow of $\mathcal{V}[f]$. In addition, (4.3.14)-(4.3.15) *holds true and*

$$f \in W_w^{1,\infty}([0,T], C_0^1(\mathbb{T} \times \mathbb{R})^*) \cap C([0,T], \mathcal{P}_1(\mathbb{T} \times \mathbb{R}) - W_1).$$

Proof. Consider a discrete approximation like in Lemma 4.3.9. Namely, consider N oscillators with phases and natural frequencies given by the configurations

$$\Theta_0^N = (\theta_{1,0}^N, \ldots, \theta_{N,0}^N) \text{ and } \{\Omega_i^N : i = 1, \ldots, N\},$$

for every $N \in \mathbb{N}$ so that they verify the equi-sumability condition (4.3.8) and the associated empirical measures $\mu_t^N \in \mathcal{P}(\mathbb{T} \times \mathbb{R})$ in Definition 4.3.1 verify

$$\lim_{N\to\infty} W_1(\mu_0^N, f_0). \tag{4.6.18}$$

By virtue of Theorem 4.6.20, μ^N are measure-valued solutions to (4.2.5) in the sense of the Filippov flow issued at μ_0^N, i.e.,

$$X^N(t; 0, \cdot)_{\#}\mu_0^N = \mu_t^N, \text{ for all } t \geq 0. \tag{4.6.19}$$

By Corollary 4.6.21, there exists a limiting measure f so that

$$\mu^N \to f \text{ in } C([0,T], \mathcal{P}_1(\mathbb{T} \times \mathbb{R}) - W_1).$$

Using Corollary 4.6.22 we claim that

$$|\mathcal{V}[\mu^N] - \mathcal{V}[\mu]| \overset{*}{\rightharpoonup} 0 \text{ in } L^\infty((0, +\infty) \times \mathbb{T} \times \mathbb{R}).$$

In particular, the convergence takes place weakly in $L^1_{loc}((0, +\infty) \times \mathbb{T} \times \mathbb{R})$. Using Lemma 4.6.23 we obtain

$$X^N \to X \text{ in } C_{loc}((0, +\infty) \times \mathbb{T} \times \mathbb{R}). \tag{4.6.20}$$

Let us finally show that we can pass to the limit in (4.6.19). Writing (4.6.19) in weak form against any test function $\varphi \in C_c(\mathbb{T} \times \mathbb{R})$ we can write

$$\int_{\mathbb{T}\times\mathbb{R}} \varphi(X^N(t; 0, z, \Omega))\, d_{(z,\Omega)}\mu_0^N = \int_{\mathbb{T}\times\mathbb{R}} \varphi(z, \Omega)\, d_{(z,\Omega)}\mu_t^N,$$

for every $t \geq 0$. First, it is clear that by the above convergence of the empirical measures we can pass to the limit in the right hand side. Regarding the left hand side, we need to prove that the following sequence

$$I^N(t) := \int_{\mathbb{T}\times\mathbb{R}} \varphi(X^N(t; 0, z, \Omega))\, d_{(z,\Omega)}\mu_0^N - \int_{\mathbb{T}\times\mathbb{R}} \varphi(X(t; 0, z, \Omega))\, d_{(z,\Omega)} f_0,$$

vanishes as $N \to 0$ for every $t \geq 0$. Consider the following split

$$I^N(t) = A_R^N(t) + B_R^N(t) + C^N(t),$$

for any $R > 0$, where each of the terms reads

$$A_R^N(t) := \int_{\mathbb{T}\times\mathbb{R}} \xi\left(\frac{|\Omega|}{R}\right) (\varphi(X^N(t; 0, x, \Omega)) - \varphi(X(t; 0, z, \Omega)))\, d_{(z,\Omega)}\mu_0^N,$$

$$B_R^N(t) := \int_{\mathbb{T}\times\mathbb{R}} \left(1 - \xi\left(\frac{|\Omega|}{R}\right)\right) (\varphi(X^N(t; 0, x, \Omega)) - \varphi(X(t; 0, z, \Omega)))\, d_{(z,\Omega)}\mu_0^N,$$

$$C^N(t) := \int_{\mathbb{T}\times\mathbb{R}} \varphi(X(t; 0, z, \Omega))\, d_{(z,\Omega)}(\mu_0^N - f_0).$$

Here $\xi \in C_c([0,+\infty))$ is again any cut-off function like in (N.2). First, (4.6.20) guarantees that

$$\lim_{N\to\infty} A_R^N(t) = 0,$$

for each $R > 0$ and every $t \geq 0$. Second, notice that

$$B_R^N(t) \leq 2\|\varphi\|_{C_0(\mathbb{T}\times\mathbb{R})}\||\Omega|\chi_{|\Omega|\geq R}\mu_0^N\|_{\mathcal{M}(\mathbb{T}\times\mathbb{R})} \leq 2\|\varphi\|_{C_0(\mathbb{T}\times\mathbb{R})}\frac{1}{N}\sum_{\substack{1\leq i\leq N\\ |\Omega_i^N|\geq R}}|\Omega_i^N|,$$

for every $N \in \mathbb{N}$ and each $t \geq 0$. Third (4.6.18) shows that

$$\lim_{N\to\infty} C^N(t) = 0,$$

for all $t \geq 0$. Putting everything together yields,

$$\limsup_{t\to\infty} I^N(t) \leq 2\|\varphi\|_{C_0(\mathbb{T}\times\mathbb{R})}\frac{1}{N}\sum_{\substack{1\leq i\leq N\\ |\Omega_i^N|\geq R}}|\Omega_i^N|,$$

for every $R > 0$. Therefore, the equi-sumability condition (4.3.8) ends the proof. □

4.6.2 Uniqueness of solutions in the sense of the Filippov flow

In this part, we shall show that the ideas in Theorem 4.4.2 of Subsection 4.4.1 for the bound on the fiberwise quadratic Wasserstein distance can be extended to the critical regime. As a byproduct, we will recover an uniqueness result for solutions in the sense of the Filippov flow to the non-linear transport equation (4.2.5). Our proof relies on an approximation of the discontinuous kernel h through the regularized kernels h_ε. In the sequel, we provide some technical lemma that will be used along the proof.

Definition 4.6.25. *Consider $\alpha = \frac{1}{2}$, $K > 0$ and $\varepsilon > 0$. We will formally define the function $\mathcal{P}_\varepsilon[\mu]$ and the tangent vector field $\mathcal{V}_\varepsilon[\mu]$ along the manifold $\mathbb{T}\times\mathbb{R}$ given by*

$$\mathcal{P}_\varepsilon[\mu](\theta,\Omega) := \Omega - K\int_{\mathbb{T}}\int_{\mathbb{R}} h_\varepsilon(\theta-\theta')\,d_{(\theta',\Omega')}\mu,$$
$$\mathcal{V}_\varepsilon[\mu](z,\Omega) := (\mathcal{P}_\varepsilon[\mu](z,\Omega)\,iz, 0),$$

where $\mu \in \mathcal{M}(\mathbb{T}\times\mathbb{R})$ is any finite Radon measure.

In a similar way to the decompositions in Lemmas 4.2.16 and 4.6.26 for the kernels of the subcritical and critical regimes, we need an appropriate split of the regularized kernels of the critical case in a consistent way so that we obtain a common one-sided-Lipschitz constant both for h and h_ε. Notice that the standard Lipschitz constant of h_ε, obtained via a uniform bound of the first derivative h_ε', should be avoided as it blows up when $\varepsilon \to 0$.

Lemma 4.6.26. *Consider $\alpha = \frac{1}{2}$, $\varepsilon > 0$ and set $\bar{h}_\varepsilon$ and $\tilde{\theta}_\varepsilon \in (0, \frac{\pi}{2})$ such that*

$$\bar{h}_\varepsilon := \max_{0\leq\theta\leq\pi} h_\varepsilon(\theta) \;\text{ and }\; \tilde{\theta}_\varepsilon \tan(\tilde{\theta}_\varepsilon) - \tilde{\theta}_\varepsilon^2 = \varepsilon^2.$$

Define the couple of functions $\delta_\varepsilon, \lambda_\varepsilon : [-2\pi, 2\pi] \longrightarrow \mathbb{R}$ as follows

$$\delta_\varepsilon(\theta) := \begin{cases} 2\bar{h}_\varepsilon - h_\varepsilon(\theta), & \theta \in [-2\pi, -2\pi + \tilde{\theta}_\varepsilon), \\ \bar{h}_\varepsilon, & \theta \in [-2\pi + \tilde{\theta}_\varepsilon, -\tilde{\theta}_\varepsilon), \\ -h_\varepsilon(\theta), & \theta \in [-\tilde{\theta}_\varepsilon, \tilde{\theta}_\varepsilon], \\ -\bar{h}_\varepsilon, & \theta \in (\tilde{\theta}_\varepsilon, 2\pi - \tilde{\theta}_\varepsilon], \\ -h_\varepsilon(\theta) - 2\bar{h}_\varepsilon, & \theta \in (2\pi - \tilde{\theta}_\varepsilon, 2\pi], \end{cases}$$

$$\lambda_\varepsilon(\theta) := \begin{cases} -2\bar{h}_\varepsilon, & \theta \in [-2\pi, -2\pi + \tilde{\theta}_\varepsilon), \\ -\bar{h}_\varepsilon - h_\varepsilon(\theta), & \theta \in [-2\pi + \tilde{\theta}_\varepsilon, -\tilde{\theta}_\varepsilon), \\ 0, & \theta \in [-\tilde{\theta}_\varepsilon, \tilde{\theta}_\varepsilon], \\ \bar{h}_\varepsilon - h_\varepsilon(\theta), & \theta \in (\tilde{\theta}_\varepsilon, 2\pi - \tilde{\theta}_\varepsilon], \\ 2\bar{h}_\varepsilon, & \theta \in (2\pi - \tilde{\theta}_\varepsilon, 2\pi]. \end{cases}$$

Then, there exists $0 < L_\varepsilon \leq -\inf_{\theta\in(0,\pi)} h'(\theta)$ such that following properties hold true

1. *δ_ε is monotonically decreasing, Λ_ε is Lipschitz-continuous with constant L_ε and*

$$-h_\varepsilon(\theta) = \delta_\varepsilon(\theta) + \lambda_\varepsilon(\theta), \ \forall \theta \in [-2\pi, 2\pi].$$

2. *$-h_\varepsilon$ is one-sided Lipschitz in $[-2\pi, 2\pi]$ with same constant L_ε, namely,*

$$((-h_\varepsilon)(\theta_1) - (-h_\varepsilon)(\theta_2))\,(\theta_1 - \theta_2) \leq L_\varepsilon(\theta_1 - \theta_2)^2.$$

Proof. Everything is clear except, at most, the estimate of the Lipschitz constant of λ_ε. Since such function is piecewise smooth, it is enough to compute the Lipschitz constant on any of the pieces of its domain. We will only focus on the interval $(\tilde{\theta}_\varepsilon, 2\pi - \tilde{\theta}_\varepsilon)$ where the functions is not constant. In the other non-constant piece the estimate follows from similar arguments. Let us compute

$$\lambda'_\varepsilon(\theta) = \frac{d}{d\theta}\left(\bar{h}_\varepsilon - h_\varepsilon(\theta)\right) = -h'_\varepsilon(\theta) = \frac{1}{(\varepsilon^2 + |\theta|_o^2)^{1/2}}\left[\frac{|\theta|_o \sin|\theta|_o}{\varepsilon^2 + |\theta|_o^2} - \cos\theta\right],$$

for every $\theta \in (\tilde{\theta}_\varepsilon, 2\pi - \tilde{\theta}_\varepsilon)$. Since λ_ε is increasing in the whole interval $(\tilde{\theta}_\varepsilon, 2\pi - \tilde{\theta}_\varepsilon)$ as a consequence of the definition of $\tilde{\theta}_\varepsilon$, then λ'_ε is non-negative along it and we conclude that

$$\begin{aligned} \sup_{\theta\in(\tilde{\theta}_\varepsilon, 2\pi-\tilde{\theta}_\varepsilon)} |\lambda'_\varepsilon(\theta)| &= \sup_{\theta\in(\tilde{\theta}_\varepsilon, 2\pi-\tilde{\theta}_\varepsilon)} \lambda'_\varepsilon(\theta) = \sup_{\theta\in(\tilde{\theta}_\varepsilon, 2\pi-\tilde{\theta}_\varepsilon)} \frac{1}{(\varepsilon^2 + |\theta|_o^2)^{1/2}}\left[\frac{|\theta|_o \sin|\theta|_o}{\varepsilon^2 + |\theta|_o^2} - \cos\theta\right] \\ &\leq \sup_{\theta\in(\tilde{\theta}_\varepsilon, 2\pi-\tilde{\theta}_\varepsilon)} \frac{1}{|\theta|_o}\left[\frac{|\theta|_o \sin|\theta|_o}{|\theta|_o^2} - \cos\theta\right] = \sup_{\theta\in(\tilde{\theta}_\varepsilon, 2\pi-\tilde{\theta}_\varepsilon)} (-h'(\theta)) = -\inf_{\theta\in(\tilde{\theta}_\varepsilon, 2\pi-\tilde{\theta}_\varepsilon)} h'(\theta). \end{aligned}$$

The proof then follows from the mean value theorem. ☐

Theorem 4.6.27. *Consider $\alpha = \frac{1}{2}$, $K > 0$ and let $\{\mu^\varepsilon\}_{\varepsilon>0}$ and μ be in $\mathcal{P}(\mathbb{T}\times\mathbb{R})$ so that*

$$\mu^\varepsilon \to \mu \ \text{ in } \ \mathcal{P}(\mathbb{T}\times\mathbb{R}) - \text{narrow}.$$

Then, the following convergence takes place

$$\lim_{\varepsilon\to 0} \sup_{\Omega\in\mathbb{R}} |\mathcal{V}_\varepsilon[\mu^\varepsilon](z,\Omega) - \mathcal{V}[\mu](z,\Omega)| = 0,$$

for each continuity point $z \in \mathbb{T}$ of the marginal measure $(\pi_z)_{\#}\mu$. In particular, it happens a.e. in $\mathbb{T}$.

Proof. Let us consider

$$F_\varepsilon(\theta) := |\mathcal{V}_\varepsilon[\mu^\varepsilon](\theta,\Omega) - \mathcal{V}[\mu](\theta,\Omega)| = \left| K \int_{(-\pi,\pi]\setminus\{\theta\}} h_\varepsilon(\theta-\theta')\,d_{\theta'}\rho_\varepsilon - K \int_{(-\pi,\pi]\setminus\{\theta\}} h(\theta-\theta')\,d_{\theta'}\rho \right|,$$

for every $\theta \in (-\pi,\pi]$ and $\varepsilon > 0$, where $\rho_\varepsilon = (\pi_z)_\#\mu_\varepsilon$ and $\rho = (\pi_z)_\#\mu$. Let us consider the following spit

$$F_\varepsilon(\theta) \leq F^1_\varepsilon(\theta) + F^2_{\varepsilon,\delta}(\theta) + F^3_{\varepsilon,\delta}(\theta),$$

for every $\delta > \varepsilon^{1/2}$, where each term looks like

$$\begin{aligned}
F^1_\varepsilon(\theta) &:= \left| K \int_{(-\pi,\pi]\setminus\{\theta\}} h(\theta-\theta')\,d_{\theta'}(\rho_\varepsilon - \rho) \right|,\\
F^2_{\varepsilon,\delta}(\theta) &:= \left| K \int_{(-\pi,\pi]\setminus\{\theta\}} \chi_{|\theta-\theta'|_o \geq \delta}(h_\varepsilon(\theta-\theta') - h(\theta-\theta'))\,d\rho_\varepsilon \right|,\\
F^3_{\varepsilon,\delta}(\theta) &:= \left| K \int_{(-\pi,\pi]\setminus\{\theta\}} \chi_{|\theta-\theta'| < \delta}(h_\varepsilon(\theta-\theta') - h(\theta-\theta'))\,d\rho_\varepsilon \right|.
\end{aligned}$$

Let us fix $\theta \in (-\pi,\pi]$ any continuity point of ρ. On the one hand, Theorem 4.6.9 implies that

$$\lim_{\varepsilon\to 0} F^1_\varepsilon(\theta) = 0.$$

Second, the estimate (4.6.3) in the proof of such result entails

$$F^2_{\varepsilon,\delta}(\theta) \leq \frac{K}{2}\frac{\varepsilon}{\delta} \leq \frac{K\varepsilon^{1/2}}{2},$$

for every $\delta > \varepsilon^{1/2}$. Then, taking limits $\varepsilon \to 0$ we obtain

$$\lim_{\varepsilon\to 0} F^2_{\varepsilon,\delta}(\theta) = 0.$$

To deal with the third term, let us consider any cut-off function $\xi \in C_c([0,+\infty))$ like in (N.2). Then,

$$\begin{aligned}
F^3_{\varepsilon,\delta}(\theta) &\leq K \int_{(-\pi,\pi]\setminus\{\theta\}} \xi\left(\frac{|\theta-\theta'|_o}{\delta}\right) d\rho_\varepsilon\\
&= K \int_{(-\pi,\pi]\setminus\{\theta\}} \xi\left(\frac{|\theta-\theta'|_o}{\delta}\right) d(\rho_\varepsilon - \rho) + K \int_{(-\pi,\pi]\setminus\{\theta\}} \xi\left(\frac{|\theta-\theta'|_o}{\delta}\right) d\rho\\
&\leq K \int_{(-\pi,\pi]\setminus\{\theta\}} \xi\left(\frac{|\theta-\theta'|_o}{\delta}\right) d(\rho_\varepsilon - \rho) + K \int_{(-\pi,\pi]\setminus\{\theta\}} \chi_{|\theta-\theta'|_o \geq 2\delta}\,d\rho.
\end{aligned}$$

Taking limits yields

$$\limsup_{\varepsilon\to 0} F^3_{\varepsilon,\delta}(\theta) \leq K \int_{(-\pi,\pi]\setminus\{\theta\}} \chi_{|\theta-\theta'|_o \geq 2\delta}\,d\rho.$$

Putting everything together, we obtain

$$\limsup_{\varepsilon\to 0} F_\varepsilon(\theta) \leq K \int_{(-\pi,\pi]\setminus\{\theta\}} \chi_{|\theta-\theta'|_o \geq 2\delta}\,d\rho,$$

for any arbitrary $\delta > 0$. Since θ is a continuity point of ρ we conclude the desired result by taking limits $\delta \to 0$. ☐

Theorem 4.6.28. *Consider $\alpha = \frac{1}{2}$, $K > 0$ and let $f^1, f^2 \in \mathcal{AC}_{\mathcal{M}} \cap \mathcal{T}_{\mathcal{M}}$ be solutions in the sense of the Filippov flow to* (4.2.5) *with initial data $f_0^1, f_0^2 \in \mathcal{P}(\mathbb{T}\times\mathbb{R})$ according to Theorem 4.6.24. Let us set their distributions of natural frequencies $g^i = (\pi_\Omega)_\# f_0^i$ for $i = 1, 2$. If $g := g^1 \equiv g^2$, then*

$$W_{2,g}(f_t^1, f_t^2) \le W_{2,g}(f_0^1, f_0^2)e^{2KL_0 t},$$

for every $t \ge 0$, where L_0 is the one-sided Lipschitz constant of $-h$ in Lemma 4.6.13 and $W_{2,g}$ is the fiberwise quadratic Wasserstein distance in Proposition F.4.2.

Proof. Using similar arguments to those in Section 4.3 for the Lipschitz-continuous regularized kernel h_ε, we can construct global classical solutions $f^{1,\varepsilon}, f^{2,\varepsilon} \in \mathcal{AC}_{\mathcal{M}} \cap \mathcal{T}_{\mathcal{M}}$ to the regularized systems

$$\left\{\begin{array}{l}\dfrac{\partial f^{1,\varepsilon}}{\partial t} + \operatorname{div}(\mathcal{V}_\varepsilon[f^{1,\varepsilon}]f^{1,\varepsilon}) = 0,\\ f_0^{1,\varepsilon} = f_0^1,\end{array}\right. \quad \text{and} \quad \left\{\begin{array}{l}\dfrac{\partial f^{2,\varepsilon}}{\partial t} + \operatorname{div}(\mathcal{V}_\varepsilon[f^{2,\varepsilon}]f^{2,\varepsilon}) = 0,\\ f_0^{2,\varepsilon} = f_0^2.\end{array}\right.$$

For g-a.e. $\Omega \in \mathbb{R}$ fixed, let us consider the corresponding term of the family of disintegrations at the initial time, i.e., $f_0^1(\cdot|\Omega)$ and $f_0^2(\cdot|\Omega)$. Set an optimal transference plan from the former probability measure in $\mathbb{T}$ to the latter one, i.e.,

$$\mu_{0,\Omega} \in \Pi(f_0^1(\cdot|\Omega), f_0^2(\cdot|\Omega)) := \left\{\mu \in \mathcal{P}(\mathbb{T}\times\mathbb{T}) : (\pi_1)_\#\mu = f_0^1(\cdot|\Omega) \text{ and } (\pi_2)_\#\mu = f_0^2(\cdot|\Omega)\right\},$$

so that the 2-Wasserstein distance is attained

$$W_2(f_0^1(\cdot|\Omega), f_0^2(\cdot|\Omega))^2 = \int_{\mathbb{T}}\int_{\mathbb{T}} d(z_1, z_2)^2 d_{(z_1,z_2)}\mu_{0,\Omega}.$$

Again, we are denoting the projections $\pi_1(z, z') = z$ and $\pi_2(z, z') = z'$. We then can define the following competitor transference plans at time t

$$\begin{aligned}\mu_{t,\Omega} &:= (Z_{f^1}(t; 0, \cdot, \Omega) \otimes Z_{f^2}(t; 0, \cdot, \Omega))_\#\mu_{0,\Omega} \in \mathcal{P}(\mathbb{T}\times\mathbb{T}),\\ \mu_{t,\Omega}^\varepsilon &:= (Z_{f^{1,\varepsilon}}(t; 0, \cdot, \Omega) \otimes Z_{f^{2,\varepsilon}}(t; 0, \cdot, \Omega))_\#\mu_{0,\Omega} \in \mathcal{P}(\mathbb{T}\times\mathbb{T}),\end{aligned}$$

where $X_{f^i}(t; 0, z, \Omega) = (Z_{f^i}(t; 0, z, \Omega), \Omega)$ is the Filippov flow associated with the transport field $\mathcal{V}[f^i]$ for $i = 1, 2$ according to Theorem 4.6.15 and $X_{f^{i,\varepsilon}}(t; 0, z, \Omega) = (Z_{f^{i,\varepsilon}}(t; 0, z, \Omega), \Omega)$ is the classical flow of $\mathcal{V}_\varepsilon[f^{i,\varepsilon}]$ according to Theorem 4.2.18. Notice that $\mu_{t,\Omega}, \mu_{t,\Omega}^\varepsilon \in \Pi(f_t^1(\cdot|\Omega), f_t^2(\cdot|\Omega))$. Hence,

$$\begin{aligned}\frac{1}{2}W_2(f_t^1(\cdot|\Omega), f_t^2(\cdot|\Omega))^2 &\le \int_{\mathbb{T}}\int_{\mathbb{T}} \frac{1}{2}d(z_1, z_2)^2\, d_{(z_1,z_2)}\mu_{t,\Omega}\\ &= \int_{\mathbb{T}}\int_{\mathbb{T}} \frac{1}{2}d(Z_{f^1}(t; 0, z_1, \Omega), Z_{f^2}(t; 0, z_2, \Omega))^2\, d_{(z_1,z_2)}\mu_{0,\Omega}.\end{aligned}$$

Integrating the above inequality against g yields

$$\frac{1}{2}W_{2,g}(f_t^1, f_t^2)^2 \le \int_{\mathbb{R}}\int_{\mathbb{T}}\int_{\mathbb{T}} \frac{1}{2}d(Z_{f^1}(t; 0, z_1, \Omega), Z_{f^2}(t; 0, z_2, \Omega))^2\, d_{(z_1,z_2)}\mu_{0,\Omega}\, d_\Omega g =: I(t).$$

Let us define

$$I_\varepsilon(t) := \int_{\mathbb{R}}\int_{\mathbb{T}}\int_{\mathbb{T}} \frac{1}{2}d(Z_{f^{1,\varepsilon}}(t; 0, z_1, \Omega), Z_{f^{2,\varepsilon}}(t; 0, z_2, \Omega))^2\, d_{(z_1,z_2)}\mu_{0,\Omega}\, d_\Omega g.$$

Since the kernel h_ε is globally Lipschitz-continuous, we can mimic the proof of Theorem 4.4.2 and obtain

$$I_\varepsilon(t) \leq I_\varepsilon(0) e^{4KL_\varepsilon t},$$

for every $t \geq 0$, where L_ε is the one sided-Lipschitz constant of $-h_\varepsilon$ in Lemma 4.6.26. On the one hand, $I_\varepsilon(0) = I(0) = \frac{1}{2} W_{2,g}(f_0^1, f_0^2)$. On the other hand, recall that

$$L_\varepsilon \leq - \inf_{\theta \in (0,\pi)} h'(\theta) = L_0,$$

for every $\varepsilon > 0$. Consequently,

$$I_\varepsilon(t) \leq \frac{1}{2} W_{2,g}(f_0^1, f_0^2)^2 e^{4KL_0 t},$$

for every $t \geq 0$ and $\varepsilon > 0$. Our last goal is to show that $\lim_{\varepsilon \to 0} I_\varepsilon(t) = I(t)$. Consider the scaled cut-off functions $\xi_R = \xi_R(\Omega)$ in (N.2) and define the decomposition

$$I_\varepsilon(t) - I(t) = A_\varepsilon^R(t) + B_\varepsilon^R(t),$$

where both terms are given by the formulas

$$\begin{aligned} A_\varepsilon^R(t) := \int_\mathbb{R} \int_\mathbb{T} \int_\mathbb{T} \xi_R(\Omega) \frac{1}{2} \big(& d(Z_{f^{1,\varepsilon}}(t;0,z_1,\Omega), Z_{f^{2,\varepsilon}}(t;0,z_2,\Omega))^2 \\ & - d(Z_{f^1}(t;0,z_1,\Omega), Z_{f^2}(t;0,z_2,\Omega))^2 \big) \, d_{(z_1,z_2)}\mu_{0,\Omega} \, d_\Omega g, \\ B_\varepsilon^R(t) := \int_\mathbb{R} \int_\mathbb{T} \int_\mathbb{T} (1 - \xi_R(\Omega)) \frac{1}{2} \big(& d(Z_{f^{1,\varepsilon}}(t;0,z_1,\Omega), Z_{f^{2,\varepsilon}}(t;0,z_2,\Omega))^2 \\ & - d(Z_{f^1}(t;0,z_1,\Omega), Z_{f^2}(t;0,z_2,\Omega))^2 \big) \, d_{(z_1,z_2)}\mu_{0,\Omega} \, d_\Omega g. \end{aligned}$$

Since the vector fields $\mathcal{V}_\varepsilon[f^{i,\varepsilon}]$ and $\mathcal{V}[f^i]$ are all essentially uniformly bounded, then Theorem 4.6.27, the Alaoglu–Bourbaki theorem and a standard application of Banach–Saks' theorem show that

$$|\mathcal{V}_\varepsilon[f^{i,\varepsilon}] - \mathcal{V}[f^i]| \overset{*}{\rightharpoonup} 0 \text{ in } L^\infty((0,+\infty) \times \mathbb{T} \times \mathbb{R}),$$

for every $i = 1, 2$. Then, Lemma 4.6.23 implies that

$$X_{f^{i,\varepsilon}} \to X_{f^i} \text{ in } C_{\text{loc}}([0,+\infty) \times \mathbb{T} \times \mathbb{R}),$$

for every $i = 1, 2$. Since the squared distance is uniformly continuous, then we claim that

$$\lim_{\varepsilon \to 0} A_\varepsilon^R(t) = 0,$$

for every $R > 0$ and each $t \geq 0$. On the other hand,

$$|B_\varepsilon^R(t)| \leq \pi^2 \|\chi_{|\Omega| \geq 2R} g\|_{\mathcal{M}(\mathbb{R})},$$

for every $R > 0$ and each $t \geq 0$. Putting everything together yields

$$\limsup_{\varepsilon \to 0} |I_\varepsilon(t) - I(t)| \leq \pi^2 \|\chi_{|\Omega| \geq 2R} g\|_{\mathcal{M}(\mathbb{R})}.$$

Taking limits $R \to +\infty$ concludes the proof by tightness of g. □

Remark 4.6.29. *Again, the above implies the following differential inequality*

$$\frac{d^+}{dt} W_{2,g}(f_t^1, f_t^2) \leq 2KL_0 W_{2,g}(f_1^t, f_2^t), \text{ for all } t \geq 0.$$

Corollary 4.6.30. *Consider $\alpha = \frac{1}{2}$, $K > 0$ and let $f^1, f^2 \in \mathcal{AC}_\mathcal{M} \cap \mathcal{T}_\mathcal{M}$ be solutions in the sense of the Filippov flow to the non-linear transport equation* (4.2.5) *with initial data $f_0^1, f_0^2 \in \mathcal{P}(\mathbb{T} \times \mathbb{R})$. If $f_0^1 = f_0^2$, then*

$$f_t^1 = f_t^2, \text{ for every } t \geq 0.$$

4.6.3 Mean field limit towards solutions in the sense of the Filippov flow

This part is devoted to adapt the bound of the quadratic Wasserstein distance in 4.4.6 of Subsection 4.4.2 to the critical case. As a byproduct, we will recover a quantitative version of the local-in-time mean field limit towards solutions in the sense of the Filippov flow.

Theorem 4.6.31. *Consider $\alpha = \frac{1}{2}$, $K > 0$ and let $f^1, f^2 \in \mathcal{AC}_{\mathcal{M}} \cap \mathcal{T}_{\mathcal{M}}$ be solutions in the sense of the Filippov flow to the non-linear transport equation* (4.2.5) *with initial data $f_0^1, f_0^2 \in \mathcal{P}_2(\mathbb{T} \times \mathbb{R})$. Then, there exists $C = C(\alpha, K, f_0^1, f_0^2) > 0$ such that*

$$W_2(f_t^1, f_t^2) \leq e^{\left(\frac{1}{2}+2KL_0\right)t} W_2(f_0^1, f_0^2),$$

for every $t \geq 0$, where L_0 is the one-sided Lipschitz constant of $-h$ in Lemma 4.6.13.

The proof follows a similar approximation argument like in Theorem 4.6.28 and we omit it.

Corollary 4.6.32. *Consider $\alpha = \frac{1}{2}$, $K > 0$ and let $f \in \mathcal{AC}_{\mathcal{M}} \cap \mathcal{T}_{\mathcal{M}}$ be the unique solution in the sense of the Filippov flow to* (4.2.5) *with initial datum $f_0 \in \mathcal{P}_2(\mathbb{T} \times \mathbb{R})$. Consider N oscillators with initial phases and natural frequencies given by the configurations*

$$\Theta_0^N = (\theta_{1,0}^N, \ldots, \theta_{N,0}^N) \text{ and } \{\Omega_i^N : i = 1, \ldots, N\},$$

for every $N \in \mathbb{N}$. Let $\Theta^N(t) := (\theta_1^N(t), \ldots, \theta_N^N(t))$ be the unique global-in-time Filippov solution to the discrete singular Kuramoto model according to Theorem 3.3.12 in Chapter 3 and define the associated empirical measures in $\mathbb{T} \times \mathbb{R}$

$$\mu_t^N := \frac{1}{N} \sum_{i=1}^{N} \delta_{z_i^N(t)}(z) \otimes \delta_{\Omega_i^N}(\Omega),$$

where $z_i^N(t) := e^{i\theta_i^N(t)}$. If $\lim_{N\to\infty} W_2(\mu_0^N, f_0) = 0$, then

$$\lim_{N\to\infty} \sup_{t\in[0,T]} W_2(\mu_t^N, f_t) = 0, \text{ for all } T > 0.$$

4.6.4 Global phase-synchronization of identical oscillators in finite time

Recall that in the agent-based system (4.1.1)-(4.1.2) enjoys finite-time global phase synchronization of identical oscillators, see Theorem 3.5.1 in Chapter 3. Mimicking the ideas in Theorem 4.5.8, we obtain the following analogue.

Theorem 4.6.33. *Consider any initial datum $f_0 \in \mathcal{P}(\mathbb{T} \times \mathbb{R})$ with identical distribution of natural frequencies, namely, $g = (\pi_\Omega)_\# f_0 = \delta_0(\Omega)$, where π_Ω is the projection* (N.1)*. Let $f = f_t$ be the unique global-in-time measure-valued solution in the sense of the Filippov flow to* (4.2.5) *issued at f_0 with $\alpha = \frac{1}{2}$ and assume $D_0 := \operatorname{diam}(\operatorname{supp} \rho_0) < \pi$. Then,*

$$f_t = f_\infty \text{ for all } t \geq T_c,$$

where $T_c = \frac{D_0}{Kh(D_0)}$ and the equilibrium f_∞ is given by the monopole $f_\infty := \delta_{Z_{av}(0)}(z) \otimes \delta_0(\Omega)$.

4.7 The supercritical regime

This part is devoted to the derivation of weak measure-valued solutions via a different technique. Namely, we will explore a singular hyperbolic limit of vanishing inertia type on a kinetic second order regularized system. Once the regularized model with inertia is presented, we will introduce an appropriate scaling where the inertia term is neglected and singularization of the weights emerges as the scaling parameter ε tends to zero in a sort of *overdamped* or *Smoluchowski limit*. The proposed scaling is reminiscent of the ideas in Chapter 2, see also [22, 57, 126, 175, 232, 252]. As a consequence of the rigorous hydrodynamic limit $\varepsilon \to 0$, we obtain weak measure-valued solutions of the macroscopic singular system in the supercritical regime $\alpha \in (\frac{1}{2}, 1)$ for identical oscillators, i.e., $g = \delta_0$. Of course, the idea also works in the most regular regime $\alpha \in (0, \frac{1}{2})$ even for non-identical oscillators, that is $g \neq \delta_0$, thus recovering the above weak measure-valued solutions in Theorems 4.3.10.

4.7.1 Second order regularized system

This part is structured as follows. First, we will introduce the agent-based second order Kuramoto model with inertia endowed with regular weighted coupling, frequency damping and noise. Second, we will recall the derivation of the Vlasov–McKean kinetic equation associated with the second order stochastic regular system.

The second order agent-based system

Let us consider the following scaled second order stochastic system for the dynamics of the N oscillators under the effect of inertia, frequency damping and noise:

$$\begin{cases} d\theta_i = \omega_i dt, \\ \varepsilon d\omega_i = \Omega_i dt + \dfrac{\nu}{N} \displaystyle\sum_{j=1}^{N} h_\varepsilon(\theta_j - \theta_i) dt - \omega_i dt + \sqrt{2\varepsilon}\, dW_t^i, \\ \theta_i(0) = \theta_{i,0},\ \omega_i(0) = \omega_{i,0}. \end{cases} \tag{4.7.1}$$

for $i = 1, \ldots, N$. Here, $W^i = W_t^i$ are independent Brownian motions. Again, $\theta_i = \theta_i(t)$ are phase values of the signals whilst $\omega_i = \omega_i(t)$ yields the evolution of their frequencies, to be distinguished from the static natural frequencies Ω_i. We have introduced an inertia term modulated by the "inertia parameter" $\varepsilon > 0$ that makes the transient to the original first-order dynamics faster as $\varepsilon \to 0$. In turns, the noise disappears and singularity in the coupling functions h_ε emerges as $\varepsilon \to 0$. Hence, we formally recover the singular Kuramoto model as the reduced first order dynamics when $\varepsilon = 0$. Introducing inertia is not an artificial method and one can indeed find a large literature concerning the original second order Kuramoto model with inertia as a suitable model of synchronization of coupled oscillators, see [76, 77, 78, 79] and references therein.

The second order Vlasov-McKean equation

As mentioned in Subsection 4.2.1 and the introductory Chapter 1, classical mean field and propagation of chaos methods [163, 164, 176, 177, 178, 181, 216, 217, 281] allow deriving the kinetic equation as $N \to \infty$ associated with the stochastic agent based model (4.7.1):

$$\frac{\partial P^\varepsilon}{\partial t} + \omega \frac{\partial P^\varepsilon}{\partial \theta}$$

$$= \frac{1}{\varepsilon}\frac{\partial}{\partial\omega}\left((\omega-\Omega)P^\varepsilon + \frac{\partial P^\varepsilon}{\partial\omega} + K\int_{(-\pi,\pi]\times\mathbb{R}\times\mathbb{R}} h_\varepsilon(\theta-\theta')P^\varepsilon(\theta',\omega',\Omega')P^\varepsilon\,d\theta'\,d\omega'\,d\Omega'\right), \quad (4.7.2)$$

for every $t\geq 0$, $\theta\in(-\pi,\pi]$, $\omega\in\mathbb{R}$ and $\Omega\in\mathbb{R}$. Here, $P^\varepsilon(t,\theta,\omega,\Omega)$ describes the probability distribution of finding an oscillator at time t with phase θ, frequency ω and natural frequency Ω, respectively. Again, we endow (4.7.2) with periodic boundary conditions

$$P^\varepsilon(t,-\pi,\omega,\Omega) = P^\varepsilon(t,\pi,\omega,\Omega). \quad (4.7.3)$$

Using Appendix 4.A to identify $\theta\in(-\pi,\pi]$ with $z\in\mathbb{T}$ via the formula $z=e^{i\theta}$. An important fact to be remarked is that the interaction term in (4.7.2) can be simplified. Specifically, consider the associated macroscopic quantities

$$\begin{aligned} f^\varepsilon(t,\theta,\Omega) &:= \int_{\mathbb{R}} P^\varepsilon\,d\omega,\\ \rho^\varepsilon(t,\theta) &:= \int_{\mathbb{R}^2} P^\varepsilon\,d\omega\,d\Omega = \int_{\mathbb{R}} f^\varepsilon\,d\Omega,\\ g(\Omega) &:= \int_{\mathbb{T}\times\mathbb{R}} P^\varepsilon\,d\theta\,d\omega = \int_{\mathbb{T}} f^\varepsilon\,d\theta, \end{aligned} \quad (4.7.4)$$

and note that such term is the following convolution

$$\int_{(-\pi,\pi]\times\mathbb{R}\times\mathbb{R}} h_\varepsilon(\theta-\theta')P^\varepsilon(t,\theta',\omega',\Omega')\,d\theta'\,d\omega'\,d\Omega' = (h_\varepsilon * \rho^\varepsilon)(t,\theta,\omega,\Omega),$$

where the convolution is considered as periodic functions, that is, as functions in $\mathbb{T}$. Then, (4.7.2)-(4.7.3) can be restated as follows

$$\frac{\partial P^\varepsilon}{\partial t} + \omega\frac{\partial P^\varepsilon}{\partial\theta} = \frac{1}{\varepsilon}\frac{\partial}{\partial\omega}\left((\omega-\Omega)P^\varepsilon + \frac{\partial P^\varepsilon}{\partial\omega} + K(h_\varepsilon*\rho^\varepsilon)P^\varepsilon\right). \quad (4.7.5)$$

Let us now set the desired hypothesis on the sequence of initial data f_ε^0 and the distribution g_ε of natural frequencies. Depending on the degree of integrability that we wish to recover on the limiting distribution, i.e., $g\in L^p(\mathbb{R})$ or just for $g\in\mathcal{M}(\mathbb{R})$, we respectively assume that either

$$\left\{\begin{array}{l} f_\varepsilon^0 = f_\varepsilon^0(\theta,\omega,\Omega)\geq 0 \text{ and } f_\varepsilon^0\in C_c^\infty(\mathbb{T}\times\mathbb{R}\times\mathbb{R}),\\ \|f_\varepsilon^0\|_{L^1(\mathbb{T}\times\mathbb{R}\times\mathbb{R})} = 1 \text{ and } f_\varepsilon^0 \overset{*}{\rightharpoonup} f^0 \text{ in } \mathcal{M}(\mathbb{T}\times\mathbb{R}),\\ \|f_\varepsilon^0\|_{L^p_\Omega(\mathbb{R};L^1_{(\theta,\omega)}(\mathbb{T}\times\mathbb{R}))}\leq C_0 \text{ and } f_\varepsilon^0 \overset{*}{\rightharpoonup} f^0 \text{ in } L^p_w(\mathbb{R},\mathcal{M}(\mathbb{T})),\\ \|\Omega^2 g_\varepsilon\|_{L^1(\mathbb{R})}\leq V_0 \text{ and } \frac{1}{2}\|\omega^2 f_\varepsilon^0\|_{L^1(\mathbb{T}\times\mathbb{R})}\leq E_0, \end{array}\right. \quad (4.7.6)$$

fulfil for some $1<p<\infty$, or the limiting assumptions with $p\to 1$ hold true, i.e.,

$$\left\{\begin{array}{l} f_\varepsilon^0 = f_\varepsilon^0(\theta,\omega,\Omega)\geq 0 \text{ and } f_\varepsilon^0\in C_c^\infty(\mathbb{T}\times\mathbb{R}\times\mathbb{R}),\\ \|f_\varepsilon^0\|_{L^1(\mathbb{T}\times\mathbb{R}\times\mathbb{R})} = 1 \text{ and } f_\varepsilon^0 \overset{*}{\rightharpoonup} f^0 \text{ in } \mathcal{M}(\mathbb{T}\times\mathbb{R}),\\ \|\Omega^2 g_\varepsilon\|_{L^1(\mathbb{R})}\leq V_0 \text{ and } \frac{1}{2}\|\omega^2 f_\varepsilon^0\|_{L^1(\mathbb{T}\times\mathbb{R})}\leq E_0. \end{array}\right. \quad (4.7.7)$$

Again, we are using the notation L^p_w for the weak-* Lebesgue–Bochner spaces in Appendix A.

Remark 4.7.1. *Note that both* (4.7.7) *and* (4.7.6) *indirectly impose some assumptions on* $g_\varepsilon = (\pi_\Omega)_\# f_\varepsilon$ *and* $g = (\pi_\Omega)_\# g$*. Specifically, the following properties take place:*

1. *If* (4.7.7) *holds, then the following properties of* g_ε *fulfil*

$$\begin{cases} g_\varepsilon = g_\varepsilon(\Omega) \geq 0, \\ g_\varepsilon \in C_c^\infty(\Omega), \\ \|g_\varepsilon\|_{L^1(\mathbb{R})} = 1, \\ g_\varepsilon \overset{*}{\rightharpoonup} g \ \ in \ \ \mathcal{M}(\mathbb{R}). \end{cases} \tag{4.7.8}$$

2. *Similarly, if in addition* (4.7.6) *holds then not only do we recover* (4.7.8) *but also*

$$\begin{cases} \|g_\varepsilon\|_{L^p(\mathbb{R})} \leq C_0, \\ g_\varepsilon \rightharpoonup g \ \ in \ \ L^p(\mathbb{R}), \end{cases} \tag{4.7.9}$$

due to the fact that $L^p_w(0,T;\mathcal{M}(\mathbb{T})) \equiv L^{p'}(0,T;C(\mathbb{T}))^*$*, see Theorem A.0.11 in Appendix A.*

Notice that under the assumptions (4.7.7) or (4.7.6), for the above compactly supported initial data f_ε^0, classical techniques assure the existence of a global-in-time classical solution $f_\varepsilon = f_\varepsilon(t,\theta,\omega,\Omega)$ to such system (4.7.5). The remaining parts are structured as follows. First, we introduce the hierarchy of frequency moments that for positive ε is not a closed system, as usual. Second, we introduce some a priori bounds. Finally, we will show that the above a priori bounds allow passing to the limit in a weak sense, closing such hierarchy of frequency moments and obtaining weak measure-valued solutions to the singular Kuramoto model in the subcritical regime $\alpha \in (0, \frac{1}{2})$.

4.7.2 A priori estimates

Apart from (4.7.4), we will be concerned with the following set of frequency moments of the distribution function $P^\varepsilon = P^\varepsilon(t,\theta,\omega,\Omega)$

$$\begin{aligned} j^\varepsilon(t,\theta,\Omega) &:= \int_{\mathbb{R}} \omega P^\varepsilon(t,\theta,\omega,\Omega)\,d\omega, \\ \mathcal{S}^\varepsilon(t,\theta,\Omega) &:= \int_{\mathbb{R}} \omega^2 P^\varepsilon(t,\theta,\omega,\Omega)\,d\omega, \\ \mathcal{T}^\varepsilon(t,\theta,\Omega) &:= \int_{\mathbb{R}} \omega^3 P^\varepsilon(t,\theta,\omega,\Omega)\,d\omega. \end{aligned}$$

The corresponding hierarchy of frequency moments can be easily derived from (4.7.5) if one multiplies it by 1, ω and ω^2 and integrates with respect to ω. The regularity of the global-in-time solution P^ε along with the periodicity conditions with respect to θ yields the equations

$$\frac{\partial f^\varepsilon}{\partial t} + \frac{\partial j^\varepsilon}{\partial \theta} = 0, \tag{4.7.10}$$

$$\varepsilon\frac{\partial j^\varepsilon}{\partial t} + \varepsilon\frac{\partial \mathcal{S}^\varepsilon}{\partial \theta} + j^\varepsilon - \Omega f^\varepsilon + (h_\varepsilon * \rho^\varepsilon) f^\varepsilon = 0, \tag{4.7.11}$$

$$\varepsilon\frac{\partial \mathcal{S}^\varepsilon}{\partial t} + \varepsilon\frac{\partial \mathcal{T}_\varepsilon}{\partial \theta} + 2(\mathcal{S}^\varepsilon - \Omega j^\varepsilon) - 2f^\varepsilon + 2(h_\varepsilon * \rho^\varepsilon) j^\varepsilon = 0. \tag{4.7.12}$$

Let us now focus on deriving some a priori estimates of the system (4.7.5). To such end, we first introduce the primitive function of the kernel, that will give some insight about the inter-particle interactions of our system.

Definition 4.7.2. *Let us define*

$$W_\varepsilon(\theta) := \int_0^\theta h_\varepsilon(\theta')\,d\theta' = \int_0^{\bar\theta} h_\varepsilon(\theta')\,d\theta',$$

for every $\alpha \in (0,1)$ and $\varepsilon > 0$. As usual, $\bar\theta$ denotes the representative modulo 2π of θ in $(-\pi,\pi]$.

By definition, W_ε enjoy nice regularity properties in the whole range of the parameter $\alpha \in (0,1)$ by virtue of the mild singularity of h_ε, although we will focus on $\alpha \in (\frac{1}{2},1)$ in this sections.

Proposition 4.7.3. *The following properties hold true*

1. *W_ε is 2π-periodic.*
2. *If $\alpha \in \left(0,\frac{1}{2}\right)$ and $1 \le p < \frac{1}{2\alpha}$, there exists a positive constant $M_{\alpha,p}$ such that*
$$\|W_\varepsilon\|_{W^{2,p}(\mathbb{T})} \le M_{\alpha,p},\ \forall \varepsilon > 0.$$
3. *If $\alpha = \frac{1}{2}$, there exists a positive constant M such that*
$$\|W_\varepsilon\|_{W^{1,\infty}(\mathbb{T})} \le M,\ \forall \varepsilon > 0.$$
4. *If $\alpha \in \left(\frac{1}{2},1\right)$ and $1 \le p < \frac{1}{2\alpha-1}$, there exists a positive constant $M_{\alpha,p}$ such that*
$$\|W_\varepsilon\|_{W^{1,p}(\mathbb{T})} \le M_{\alpha,p},\ \forall \varepsilon > 0.$$
5. *W_ε is a primitive function of h_ε.*
6. *$W_\varepsilon \ge 0$ and the identity only holds at $\theta \in 2\pi\mathbb{Z}$.*

Remark 4.7.4. *Recall that the Sobolev embedding theorem entails the compact inclusion*

$$W^{1,p}(\mathbb{T}) \subset\subset C(\mathbb{T}),\ \forall\, p > 1.$$

In particular, there is some constant $M_\alpha > 0$ that does not depend on $\varepsilon > 0$ such that

$$\|W_\varepsilon\|_{C(\mathbb{T})} \le M_\alpha,\ \textit{for all}\ \ \varepsilon > 0.$$

Lemma 4.7.5. *Consider the strong solution P^ε to* (4.7.5) *whose initial data P_0^ε fulfil the assumptions* (4.7.7)*. Then, the next formula holds true for every $\varepsilon > 0$*

$$\frac{d}{dt}\left(\varepsilon \int_{\mathbb{T}}\int_{\mathbb{R}} \mathcal{S}^\varepsilon\,d\theta\,d\Omega + K\int_{\mathbb{T}} (W_\varepsilon * \rho^\varepsilon)\rho^\varepsilon\,d\theta\right) + 2\int_{\mathbb{T}}\int_{\mathbb{R}} \mathcal{S}^\varepsilon\,d\theta\,d\Omega = 2\int_{\mathbb{T}}\int_{\mathbb{R}} \Omega j^\varepsilon\,d\theta\,d\Omega + 2.$$

Proof. Let us integrate (4.7.12) with respect to θ and Ω to obtain

$$\frac{d}{dt}\left(\varepsilon \int_{\mathbb{T}}\int_{\mathbb{R}} \mathcal{S}^\varepsilon\,d\theta\,d\Omega\right) + 2\int_{\mathbb{T}}\int_{\mathbb{R}} (h_\varepsilon * \rho^\varepsilon) j^\varepsilon\,d\theta\,d\Omega + 2\int_{\mathbb{T}}\int_{\mathbb{R}} \mathcal{S}^\varepsilon\,d\theta\,d\Omega = 2\int_{\mathbb{T}}\int_{\mathbb{R}} \Omega j^\varepsilon\,d\theta\,d\Omega + 2.$$

The only effort to be done is to identify the second term. To such end, we notice that $\frac{\partial W_\varepsilon}{\partial\theta} = h_\varepsilon$ and consequently,

$$\int_{\mathbb{T}}\int_{\mathbb{R}} (h_\varepsilon * \rho^\varepsilon) j^\varepsilon\,d\theta\,d\Omega = \int_{\mathbb{T}}\int_{\mathbb{R}} \left(\frac{\partial W_\varepsilon}{\partial\theta} * \rho^\varepsilon\right) j^\varepsilon\,d\theta\,d\Omega = \int_{\mathbb{T}}\int_{\mathbb{R}} \frac{\partial}{\partial\theta}(W_\varepsilon * \rho^\varepsilon) j^\varepsilon\,d\theta\,d\Omega$$

$$= -\int_{\mathbb{T}}\int_{\mathbb{R}} (W_\varepsilon * \rho^\varepsilon)\frac{\partial j^\varepsilon}{\partial \theta}\, d\theta\, d\Omega = \int_{\mathbb{T}}\int_{\mathbb{R}} (W_\varepsilon * \rho^\varepsilon)\, \frac{\partial f^\varepsilon}{\partial t}\, d\theta\, d\Omega,$$

where an integration by parts and the continuity equation (4.7.10) have been used. Now, let us restate the last term

$$\int_{\mathbb{T}}\int_{\mathbb{R}} (W_\varepsilon * \rho^\varepsilon)\, \frac{\partial f^\varepsilon}{\partial t}\, d\theta\, d\Omega = \int_{\mathbb{T}^2}\int_{\mathbb{R}^2} W_\varepsilon(\theta-\theta')\frac{\partial f^\varepsilon}{\partial t}(t,\theta,\Omega) f^\varepsilon(t,\theta',\Omega')\, d\theta\, d\theta'\, d\Omega\, d\Omega'$$
$$= \frac{d}{dt}\int_{\mathbb{T}} (W_\varepsilon * \rho^\varepsilon)\rho^\varepsilon\, d\theta - \int_{\mathbb{T}^2}\int_{\mathbb{R}^2} W_\varepsilon(\theta-\theta') f^\varepsilon(t,\theta,\Omega)\frac{\partial f^\varepsilon}{\partial t}(t,\theta',\Omega')\, d\theta\, d\theta'\, d\Omega\, d\Omega'.$$

Since W_ε is an even function, then the left hand side agrees with the second term of the right hand side and consequently,

$$2\int_{\mathbb{T}} (h_\varepsilon * \rho^\varepsilon) j^\varepsilon\, d\theta\, d\Omega = \frac{d}{dt}\int_{\mathbb{T}} (W_\varepsilon * \rho^\varepsilon)\rho^\varepsilon\, d\theta.$$

This ends the proof of this lemma. □

Theorem 4.7.6. *Consider the strong solution* P^ε *to* (4.7.5) *whose initial data* P_0^ε *fulfill the assumptions* (4.7.6)*, for some* $p \in [1,+\infty)$*. Then, the following estimates*

$$\begin{aligned} \|f^\varepsilon\|_{L^\infty(0,T;L^p(\mathbb{R},L^1(\mathbb{T})))} &\le C_0,\\ \|j^\varepsilon\|_{L^2(0,T;L^q(\mathbb{R},L^1(\mathbb{T})))} &\le C_0^{1/2}\left(2\varepsilon E_0 + K M_\alpha + T(V_0+2)\right)^{1/2},\\ \|\mathcal{S}^\varepsilon\|_{L^1(0,T;L^1(\mathbb{T}\times\mathbb{R}))} &\le 2\varepsilon E_0 + K M_\alpha + T(V_0+2), \end{aligned}$$

hold for every $\varepsilon > 0$*, where* $q := \frac{2p}{1+p}$.

Proof. • *Step 1:* By integration with respect to θ in the continuity equation (4.7.5) we achieve

$$\frac{d}{dt}\int_{\mathbb{T}} f^\varepsilon\, d\theta = 0 \implies \int_{\mathbb{T}} f^\varepsilon\, d\theta = \int_{\mathbb{T}} f_0^\varepsilon\, d\theta = g^\varepsilon.$$

Then, the first a priori estimate for f^ε follows from Remark 4.7.1 (with $C_0 = 1$ if $p=1$).

• *Step 2:* Using the Cauchy–Schwarz and Young inequalities on the first term in the right hand side of the formula obtained in Lemma 4.7.5 we arrive at

$$\frac{d}{dt}\left(\varepsilon\int_{\mathbb{T}}\int_{\mathbb{R}} \mathcal{S}^\varepsilon\, d\theta\, d\Omega + K\int_{\mathbb{T}} (W_\varepsilon * \rho^\varepsilon)\rho^\varepsilon\, d\theta\right) + 2\int_{\mathbb{T}}\int_{\mathbb{R}} \mathcal{S}^\varepsilon\, d\theta\, d\Omega \le \|\Omega^2 g^\varepsilon\|_{L^1(\mathbb{R})} + \int_{\mathbb{T}}\int_{\mathbb{R}} \mathcal{S}^\varepsilon\, d\theta\, d\Omega + 2.$$

Now, let us integrate with respect to time in $[0,T]$. Using the fundamental theorem of calculus and neglecting the terms corresponding to time $t=T$ (notice that $W_\varepsilon \ge 0$ by Remark 4.7.4), we obtain

$$\int_0^T\int_{\mathbb{T}}\int_{\mathbb{R}} \mathcal{S}^\varepsilon\, dt\, d\theta\, d\Omega \le \varepsilon_0\int_{\mathbb{T}}\int_{\mathbb{R}} \mathcal{S}_0^\varepsilon\, d\theta\, d\Omega + K\int_{\mathbb{T}} (W_\varepsilon * \rho_0^\varepsilon) f_0^\varepsilon\, d\theta + T\left(\|\Omega^2 g^\varepsilon\|_{L^p(\mathbb{R})} + 2\right).$$

Using the assumptions (4.7.6) and the uniform-in-ε bound of W_ε in Remark 4.7.4, we obtain the third a priori estimate.

• *Step 3:* The second a priori estimate is a consequence of the first one that we can obtain by interpolation in L^p spaces. Indeed,

$$\int_{\mathbb{T}} |j^\varepsilon|\,d\theta \leq \int_{\mathbb{T}}\int_{\mathbb{R}} |\omega| f^\varepsilon\,d\omega \leq \left(\int_{\mathbb{T}} f^\varepsilon\,d\theta\right)^{1/2}\left(\int_{\mathbb{T}} \mathcal{S}^\varepsilon\,d\theta\right)^{1/2} = (g^\varepsilon)^{1/2}\left(\int_{\mathbb{T}} \mathcal{S}^\varepsilon\,d\theta\right)^{1/2},$$

where the Cauchy-Schwarz inequality has been used. Define $q = \frac{2p}{1+p}$ and notice that

$$\frac{1}{2p} + \frac{1}{2} = \frac{1}{q}.$$

Then, we can take L^q norms in the above expression and use the generalized Hölder inequality with exponents $2p$ and 2 to obtain

$$\|j^\varepsilon(t,\cdot)\|_{L^q(\mathbb{R},L^1(\mathbb{T}))} \leq \|g^\varepsilon\|_{L^p(\mathbb{R})}^{1/2} \|\mathcal{S}^\varepsilon(t,\cdot)\|_{L^1(\mathbb{T}\times\mathbb{R})}^{1/2},$$

for every $t \in [0,T]$. Finally, let us take L^2 norms with respect to time to achieve the second a priori estimate for j^ε in $L^2(0,T;L^q(\mathbb{R},L^1(\mathbb{T})))$. □

4.7.3 Compactness of the regularized system

In this part, we will derive the corresponding weak-star compactness as a consequence of Theorem 4.7.6. We will do it for both type of assumptions (4.7.7) and (4.7.6) that we can assume on the initial data. Then, we obtain the following two Corollaries for each of the two cases:

Corollary 4.7.7. *Consider the strong solution P^ε to* (4.7.5) *whose initial data P_0^ε fulfill the assumptions* (4.7.7). *Then, there exists $f \in L_w^\infty(0,T;\mathcal{M}(\mathbb{T}\times\mathbb{R}))$ and $j \in L_w^2(0,T;\mathcal{M}(\mathbb{T}\times\mathbb{R}))$ such that*

$$\begin{aligned} f^\varepsilon &\overset{*}{\rightharpoonup} f \ \text{ in } \ L_w^\infty(0,T;\mathcal{M}(\mathbb{T}\times\mathbb{R})),\\ j^\varepsilon &\overset{*}{\rightharpoonup} j \ \text{ in } \ L_w^2(0,T;\mathcal{M}(\mathbb{T}\times\mathbb{R})),\end{aligned}$$

up to a subsequence that we denote the same for simplicity.

Corollary 4.7.8. *Consider the strong solution P^ε to* (4.7.5) *whose initial data P_0^ε fulfill the assumptions* (4.7.6), *for some $1 < p < \infty$. Then, the above weak-star limits $f \in L_w^\infty(0,T;\mathcal{M}(\mathbb{T}\times\mathbb{R}))$ and $j \in L_w^2(0,T;\mathcal{M}(\mathbb{T}\times\mathbb{R}))$ in Corollary 4.7.7 also satisfy*

$$\begin{aligned} f^\varepsilon &\overset{*}{\rightharpoonup} f \ \text{ in } \ L_w^\infty(0,T;L_w^p(\mathbb{R},\mathcal{M}(\mathbb{T}))),\\ j^\varepsilon &\overset{*}{\rightharpoonup} j \ \text{ in } \ L_w^2(0,T;L_w^{\frac{2p}{1+p}}(\mathbb{R},\mathcal{M}(\mathbb{T}))).\end{aligned}$$

The proofs are straightforward consequences of the a priori estimates in the above Theorem 4.7.6 along with the Alaoglu–Bourbaki theorem and the Riesz representation theorem for Lebesgue–Bochner spaces in A.0.11, then we skip them. Indeed, apart from the above weak convergence in time, we can recover a stronger convergence result of the density f^ε. This is the content of the next result.

Theorem 4.7.9. *Consider the strong solution P^ε to* (4.7.5) *with initial data P_0^ε.*

1. *If P_0^ε fulfils the assumptions* (4.7.7), *then*

$$f^\varepsilon \to f \ \text{ in } \ C([0,T],\mathcal{M}(\mathbb{T}\times\mathbb{R}) - \textit{narrow}).$$

2. *If P_0^ε fulfils the assumptions* (4.7.6), *for some* $1 < p < \infty$, *then*

$$f^\varepsilon \to f \ \ in \ \ C([0,T], L^p_w(\mathbb{R}, \mathcal{M}(\mathbb{T})) - weak\,*).$$

Proof. Since both proofs are similar, we just focus on the most involved one, that is, the first one. The second result follows a parallel train of thoughts. Let use the continuity equation (4.7.10) that we write in weak form against a test function $\varphi(t,\theta,\Omega) = \eta(t)\phi(\theta,\Omega)$, where η and ϕ are smooth and compactly supported

$$\begin{aligned}
&\int_0^T \frac{\partial \eta}{\partial t}\int_{\mathbb{T}}\int_{\mathbb{R}} f^\varepsilon(t,\theta,\Omega)\phi(\theta,\Omega)\,dt\,d\theta\,d\Omega\\
&\qquad = -\int_0^T \int_{\mathbb{T}}\int_{\mathbb{R}} j^\varepsilon(t,\theta,\Omega)\eta(t)\frac{\partial\phi}{\partial\theta}(\theta,\Omega)\,dt\,d\theta\,d\Omega\\
&\qquad\qquad \leq \|j^\varepsilon\|_{L^2(0,T;L^1(\mathbb{T}\times\mathbb{R}))}\|\phi\|_{W_0^{1,\infty}(\mathbb{T}\times\mathbb{R})}.
\end{aligned}$$

Then, the standard characterization of Sobolev spaces yields

$$\begin{aligned}
&\left\|\int_{\mathbb{T}}\int_{\mathbb{R}} f^\varepsilon(\cdot,\theta,\Omega)\phi(\theta,\Omega)\,d\theta\,d\Omega\right\|_{H^1(0,T)}\\
&\qquad \leq (T^{1/2}\|f^\varepsilon\|_{L^\infty(0,T;L^1(\mathbb{T}\times\mathbb{R}))} + \|j^\varepsilon\|_{L^2(0,T;L^1(\mathbb{T}\times\mathbb{R}))})\|\phi\|_{W_0^{1,\infty}(\mathbb{T}\times\mathbb{R})}.
\end{aligned}$$

By the Sobolev embedding theorem and the first and second estimates in Theorem 4.7.6 we can also claim that there exists $C > 0$ that does not depend neither on ε nor in the chosen test functions so that

$$\left|\int_{\mathbb{T}}\int_{\mathbb{R}} (f^\varepsilon(t_1,\theta,\Omega) - f^\varepsilon(t_2,\theta,\Omega))\phi(\theta,\Omega)\,d\theta\,d\Omega\right| \leq C|t_1-t_2|^{1/2}\|\phi\|_{W_0^{1,\infty}(\mathbb{T}\times\mathbb{R})},$$

for every $t_1, t_2 \in [0,T]$ and each $\varepsilon > 0$. Since the chosen test functions are arbitrary, then we obtain

$$\|f^\varepsilon(t_1,\cdot,\cdot) - f^\varepsilon(t_2,\cdot,\cdot)\|_{W^{-1,1}(\mathbb{T}\times\mathbb{R})} \leq C|t_1-t_2|^{1/2},$$

for every $t_1, t_2 \in [0,T]$ and each $\varepsilon > 0$. Also notice that

$$\|f^\varepsilon\|_{C([0,T],W^{-1,1}(\mathbb{T}\times\mathbb{R}))} \leq C,$$

for every $\varepsilon > 0$ for another constant C that, without loss of generality, we denote in the same way. Such assertion is nothing but a consequence of the first a priori estimate in Theorem 4.7.6 and the chain of continuous embeddings

$$L^1(\mathbb{T}\times\mathbb{R}) \hookrightarrow \mathcal{M}(\mathbb{T}\times\mathbb{R}) \hookrightarrow W^{-1,1}(\mathbb{T}\times\mathbb{R}),$$

where the last embedding is a consequence of the dense embedding of $W_0^{1,\infty}(\mathbb{T}\times\mathbb{R})$ into $C_0(\mathbb{T}\times\mathbb{R})$. Then, we can use the weak-* verison of the Ascoli–Arzelà theorem to the space

$$C([0,T], (W_0^{1,\infty}(\mathbb{T}\times\mathbb{R}))^*) \cong C([0,T], W^{-1,1}(\mathbb{T}\times\mathbb{R})),$$

(see Appendix B) to and obtain a subsequence of f_ε that we denote in the same way so that

$$f^\varepsilon \to f \ \text{ in } \ C([0,T], W^{-1,1}(\mathbb{T}\times\mathbb{R}) - \text{weak}*).$$

By the first estimate in Theorem 4.7.6 along with the above-mentioned density of $W_0^{1,\infty}(\mathbb{T}\times\mathbb{R})$ into $C_0(\mathbb{T}\times\mathbb{R})$ we can actually improve the above convergence into

$$f^\varepsilon \to f \ \text{ in } \ C([0,T],\mathcal{M}(\mathbb{T}\times\mathbb{R})-\text{weak}*). \tag{4.7.13}$$

Let us finally show that the above weak-star convergence can be improved into narrow convergence in the spaces of finite Radon measures $\mathcal{M}(\mathbb{T}\times\mathbb{R})$. To such end, fix any test function $\varphi\in C_b(\mathbb{T}\times\mathbb{R})$ and recall the cut-off functions ξ_R in (N.2). Then, we obtain

$$\begin{aligned}
&\sup_{t\in[0,T]}\left|\int_{\mathbb{T}\times\mathbb{R}}\varphi(\theta,\Omega)\,d_{(\theta,\Omega)}(f_\varepsilon-f)\right|\\
&\leq \sup_{t\in[0,T]}\left|\int_{\mathbb{T}\times\mathbb{R}}\varphi(\theta,\Omega)(1-\xi_R(\Omega))\,d_{(\theta,\Omega)}(f_\varepsilon-f)\right|+\sup_{t\in[0,T]}\left|\int_{\mathbb{T}\times\mathbb{R}}\varphi(\theta,\Omega)\xi_R(\Omega)\,d_{(\theta,\Omega)}(f_\varepsilon-f)\right|\\
&\leq \frac{1}{R}\|\varphi\|_{C_b(\mathbb{T}\times\mathbb{R})}\sup_{t\in[0,T]}\int_{\mathbb{T}\times\mathbb{R}}|\Omega|\,d_{(\theta,\Omega)}(f_t^\varepsilon+f_t)+\sup_{t\in[0,T]}\left|\int_{\mathbb{T}\times\mathbb{R}}\varphi(\theta,\Omega)\xi_R(\Omega)\,d_{(\theta,\Omega)}(f_\varepsilon-f)\right|.
\end{aligned}$$

By virtue of the above convergence (4.7.3) and the following uniform tightness condition

$$\int_{\mathbb{T}}\int_{\mathbb{R}}|\Omega|f^\varepsilon\,d\theta\,d\Omega=\int_{\mathbb{T}}\int_{\mathbb{R}}|\Omega|f_0^\varepsilon\,d\theta\,d\Omega\leq V_0^{\frac{1}{2}},$$

for every $\varepsilon>0$ and $t\in[0,T]$, we can take $\limsup$ in the above inequality as $\varepsilon\to 0$ to obtain

$$\limsup_{\varepsilon\to 0}\sup_{t\in[0,T]}\left|\int_{\mathbb{T}\times\mathbb{R}}\varphi(\theta,\Omega)\,d_{(\theta,\Omega)}(f_\varepsilon-f)\right|\leq\frac{2V_0^{1/2}}{R}\|\varphi\|_{C_b(\mathbb{T}\times\mathbb{R})}.$$

Since $R>0$ is arbitrary we conclude that

$$f^\varepsilon\to f \ \text{ in } C([0,T],\mathcal{M}(\mathbb{T}\times\mathbb{R})-\text{narrow}).$$

□

As a simple consequence, we can prove that the tensor product $f^\varepsilon\otimes f^\varepsilon$ also enjoys good compactness properties.

Corollary 4.7.10. *Consider the strong solution P^ε to* (4.7.5) *with initial data P_0^ε.*

1. *If P_0^ε fulfils the assumptions* (4.7.7), *then*

$$f^\varepsilon\otimes f^\varepsilon \overset{*}{\rightharpoonup} f\otimes f \ \text{ in } \ L^\infty_w(0,T;\mathcal{M}(\mathbb{T}^2\times\mathbb{R}^2)-\textit{narrow}).$$

2. *If f_ε^0 fulfils the assumptions* (4.7.6), *for some $1<p<\infty$, then*

$$f^\varepsilon\otimes f^\varepsilon \overset{*}{\rightharpoonup} f\otimes f \ \text{ in } \ L^\infty_w(0,T;L^p_w(\mathbb{R}^2,\mathcal{M}(\mathbb{T}^2))-\textit{weak}\,*).$$

Our next step is to show that we can pass to the limit in the balance laws for the phase density and phase current (4.7.10)-(4.7.11). To such end, let us write them in weak form by multiplication against any test function $\varphi\in C_0^1([0,T)\times\mathbb{T}\times\mathbb{R})$ and integrate by parts:

$$\int_0^T\int_{\mathbb{T}}\int_{\mathbb{R}}f^\varepsilon\frac{\partial\varphi}{\partial t}\,dt\,d\theta\,d\Omega+\int_0^T\int_{\mathbb{T}}\int_{\mathbb{R}}j^\varepsilon\frac{\partial\varphi}{\partial\theta}\,dt\,d\theta\,d\Omega=-\int_{\mathbb{T}}\int_{\mathbb{R}}f_0^\varepsilon\,\varphi(0,\cdot,\cdot)\,d\theta\,d\Omega, \tag{4.7.14}$$

$$\varepsilon \int_0^T \int_{\mathbb{T}} \int_{\mathbb{R}} j^\varepsilon \frac{\partial \varphi}{\partial t}\, dt\, d\theta\, d\Omega + \varepsilon \int_0^T \int_{\mathbb{T}} \int_{\mathbb{R}} \mathcal{S}^\varepsilon \frac{\partial \varphi}{\partial \theta}\, dt\, d\theta\, d\Omega = -\varepsilon \int_{\mathbb{T}} \int_{\mathbb{R}} j_0^\varepsilon\, \varphi(0,\cdot,\cdot)\, d\theta\, d\Omega$$
$$+ \int_0^T \int_{\mathbb{T}} \int_{\mathbb{R}} (j^\varepsilon - \Omega f^\varepsilon)\, \varphi\, dt\, d\theta\, d\Omega + \int_0^T \int_{\mathbb{T}} \int_{\mathbb{R}} (h_\varepsilon * \rho^\varepsilon) f^\varepsilon \varphi\, dt\, d\theta\, d\Omega. \tag{4.7.15}$$

Note that under the weak assumptions (4.7.7), Corollary 4.7.7 and the a priori estimate of $\mathcal{S}_\varepsilon$ in Theorem 4.7.6 allow passing to the limit all the terms except at most two of them; namely, the nonlinear term and the term involving Ωf^ε. Let us first address the latter one and we shall discuss about the most difficult convergence result for the nonlinear term later.

Proposition 4.7.11. *Consider the strong solution P^ε to* (4.7.5) *with initial data P_0^ε.*

1. *If P_0^ε fulfil the assumptions* (4.7.7)*, then*

$$\|\Omega f^\varepsilon\|_{L^\infty(0,T;L^1(\mathbb{T}\times\mathbb{R}))} \leq V_0^{1/2},$$
$$\|\Omega f\|_{L^\infty_w(0,T;\mathcal{M}(\mathbb{T}\times\mathbb{R}))} \leq V_0^{1/2}.$$

Moreover, the following weak-star convergence takes place

$$\Omega f^\varepsilon \overset{*}{\rightharpoonup} \Omega f \ \ in \ \ L^\infty_w(0,T;\mathcal{M}(\mathbb{T}\times\mathbb{R})).$$

2. *If P_0^ε fulfil the assumptions* (4.7.6)*, for some $1<p<\infty$, then*

$$\|\Omega f^\varepsilon\|_{L^\infty(0,T;L^{\frac{2p}{1+p}}(\mathbb{R},L^1(\mathbb{T})))} \leq V_0^{1/2} C_0^{1/2},$$
$$\|\Omega f\|_{L^\infty_w(0,T;L_w^{\frac{2p}{1+p}}(\mathbb{R},\mathcal{M}(\mathbb{T})))} \leq V_0^{1/2} C_0^{1/2}.$$

Moreover, the following weak-star convergence takes place

$$\Omega f^\varepsilon \overset{*}{\rightharpoonup} \Omega f \ \ in \ \ L^\infty_w(0,T;L_w^{\frac{2p}{1+p}}(\mathbb{R},\mathcal{M}(\mathbb{T}))).$$

Proof. Since both proofs are similar, we just focus on the second one. Notice that by the continuity equation we obtain that

$$\frac{d}{dt}\int_{\mathbb{T}} |\Omega| f^\varepsilon\, d\theta = 0 \implies \int_{\mathbb{T}} |\Omega| f^\varepsilon\, d\theta = \int_{\mathbb{T}} |\Omega| f_0^\varepsilon\, d\theta = |\Omega| g^\varepsilon.$$

Again, we can take $L^{\frac{2p}{1+p}}$-norms and use the generalized Hölder inequality to arrive at

$$\|f^\varepsilon(t,\cdot)\|_{L^{\frac{2p}{1+p}}(\mathbb{R},L^1(\mathbb{T}))} \leq \|\Omega^2 g^\varepsilon\|_{L^1(\mathbb{R})}^{1/2} \|g^\varepsilon\|_{L^p(\mathbb{R})}^{1/2},$$

for every $t\in[0,T]$. Taking supreme yields the desired estimate of Ωf^ε by virtue of the assumptions (4.7.6).

Regarding the limiting estimate, let us first note that

$$(t,\Omega)\in(0,T)\times\mathbb{R} \longmapsto \|\Omega f_t(\cdot,\Omega)\|_{\mathcal{M}(\mathbb{T})} = |\Omega| \|f_t(\cdot,\Omega)\|_{\mathcal{M}(\mathbb{T})}$$

belongs to $L^1_{\text{loc}}((0,T)\times\mathbb{R})$. In order to get the $L^\infty_w(0,T;L^q_w(\mathbb{R},\mathcal{M}(\mathbb{R})))$-estimate with $q=\frac{2p}{1+p}$, we consider a test function $\varphi\in L^1(0,T;L^{q'}(\mathbb{R},C(\mathbb{T})))$ and, without loss of generality, assume that it takes the form $\varphi(t,\theta,\Omega)=\eta(t)\phi(\theta)\psi(\Omega)$ with appropriate test functions of separate variables $\eta\in L^1(0,T)$, $\psi\in C_c(\mathbb{R})$ and $\phi\in C(\mathbb{T})$. Then,

$$\begin{aligned}\int_0^T\eta(t)\int_{\mathbb{R}}\psi(\Omega)\left(\int_{\mathbb{T}}\phi(z)|\Omega|\,d_zf_t(\cdot,\Omega)\right)\,d\Omega\,dt&=\int_0^T\int_{\mathbb{R}}|\Omega|\psi(\Omega)\left(\int_{\mathbb{T}}\phi(z)\,d_zf_t(\cdot,\Omega)\right)\,d\Omega\,dt\\&=\lim_{\varepsilon\to 0}\int_0^T\int_{\mathbb{R}}|\Omega|\psi(\Omega)\left(\int_{\mathbb{T}}\phi(z)f_t^\varepsilon\,d_z(\cdot,\Omega)\right)\,d\Omega\,dt\\&\leq\limsup_{\varepsilon\to 0}\|\Omega f^\varepsilon\|_{L^\infty(0,T;L^q(\mathbb{R},L^1(\mathbb{T})))}\|\varphi\|_{L^1(0,T;L^{q'}(\mathbb{R},C(\mathbb{T})))},\end{aligned}$$

where we have used the above estimate for Ωf^ε and the weak-star convergence in Corollary 4.7.8. Since φ is arbitrary we can conclude the desired estimate on Ωf by the Riesz representation theorem for weak Lebesgue–Bochner spaces, see Theorem A.0.11.

The proof of the convergence result follows similar arguments. Fix a test function $\varphi\in L^1(0,T;L^{q'}(\mathbb{R},C(\mathbb{T})))$. By density, we can assume that $\varphi\in C_c((0,T)\times\mathbb{T}\times\mathbb{R})$. Then,

$$\begin{aligned}&\int_0^T\int_{\mathbb{T}}\int_{\mathbb{R}}\varphi(t,\theta,\Omega)(\Omega f^\varepsilon(t,\theta,\Omega)-\Omega f_t(\theta,\Omega))\,dt\,d\theta\,d\Omega\\&\quad=\int_0^T\int_{\mathbb{T}}\int_{\mathbb{R}}\Omega\varphi(t,\theta,\Omega)(f^\varepsilon(t,\theta,\Omega)-f_t(\theta,\Omega))\,dt\,d\theta\,d\Omega.\end{aligned}$$

Since $\Omega\varphi(t,\theta,\Omega)$ belongs to $L^1(0,T;L^{q'}(\mathbb{R},C(\mathbb{T}))$, then we can apply Corollary (4.7.8) and obtain that the last integral converges towards zero when $\varepsilon\to 0$, thus concluding the proof. □

Notice that the above result allows passing to the limit in the term Ωf^ε in (4.7.15). Then, the only term that remains to be studied is the nonlinear one.

4.7.4 Convergence of the nonlinear term and hydrodynamic limit

In this part, we will discuss about the nonlinear term (4.7.15). On the one hand, we will show that we cannot pass to the limit for general g. The main reason is that the proposed cancellation property would fail for $\alpha\in(\frac{1}{2},1)$. On the other hand, we will show that it proves useful in the identical case $g=\delta_0$. Although we will not comment on the more regular regime $\alpha\in(0,\frac{1}{2})$ here, it is clear that $h_\varepsilon*\rho^\varepsilon$ converges strongly for general g although f^ε is just narrowly convergent. In such way, we can recover the weak measure-valued solutions to (4.2.5) like in Section 4.3, but we will skip it here for simplicity.

The main idea is to write the nonlinear term in (4.7.15) appropriately using a well known symmetrization idea, like in Chapter 2. Specifically, note that

$$\begin{aligned}&\int_0^T\int_{\mathbb{T}}\int_{\mathbb{R}}(h_\varepsilon*\rho^\varepsilon)f^\varepsilon\varphi\,dt\,d\theta\,d\Omega\\&\quad=\int_0^T\int_{\mathbb{T}^2\times\mathbb{R}^2}h_\varepsilon(\theta-\theta')\varphi(t,\theta,\Omega)f^\varepsilon(t,\theta,\Omega)f^\varepsilon(t,\theta',\Omega')\,dt\,d\theta\,d\theta'\,d\Omega\,d\Omega'\\&\qquad=-\int_0^T\int_{\mathbb{T}^2\times\mathbb{R}^2}h_\varepsilon(\theta-\theta')\varphi(t,\theta',\Omega')f^\varepsilon(t,\theta,\Omega)f^\varepsilon(t,\theta',\Omega')\,dt\,d\theta\,d\theta'\,d\Omega\,d\Omega',\end{aligned}$$

where we have changed variables (θ,Ω) with (θ',Ω') and we have used the antisymmetry of the kernel h_ε in the last line. Taking the mean value of both expressions we obtain

$$\int_0^T\int_{\mathbb{T}}\int_{\mathbb{R}}(h_\varepsilon*\rho^\varepsilon)f^\varepsilon\varphi\,dt\,d\theta\,d\Omega$$
$$=\int_0^T\int_{\mathbb{T}^2\times\mathbb{R}^2}H_\varepsilon^\varphi(t,\theta,\theta',\Omega,\Omega')f^\varepsilon(t,\theta,\Omega)f^\varepsilon(t,\theta',\Omega')\,dt\,d\theta\,d\theta'\,d\Omega\,d\Omega',\quad (4.7.16)$$

where the function H_ε^φ reads

$$H_\varepsilon^\varphi(t,\theta,\theta',\Omega,\Omega'):=\frac{1}{2}h_\varepsilon(\theta-\theta')(\varphi(t,\theta,\Omega)-\varphi(t,\theta',\Omega')).$$

Notice that when $\varepsilon\to 0$, the above function H_0^φ is not continuous at $\theta=\theta'$ unless $\Omega=\Omega'$. Here, the Lipschitz continuity of the test function φ in the variable θ plays a role to cancel the full singularity of h in the whole range $\alpha\in(\frac{1}{2},1)$. Then, it is not clear at all that (4.7.16) makes sense in the limit $\varepsilon\to 0$, as the limiting measure f is expected to have atoms. However, we can solve the problem at least for identical oscillators, i.e., $g=\delta_0$. This is the content of the main result in this part.

Theorem 4.7.12. *Fix $\alpha\in(\frac{1}{2},1)$ and consider the strong solution P^ε to* (4.7.5) *whose initial data P_0^ε fulfil the assumptions* (4.7.7)*. Assume that the limiting oscillators are all identical, that is, $g=\delta_0$. Then, there is a subsequence of f^ε, denoted in the same way, and $f\in\mathcal{AC}_{\mathcal{M}}\cap\mathcal{T}_{\mathcal{M}}$ so that*

$$f^\varepsilon\to f\ \text{ in }\ C([0,T],\mathcal{M}(\mathbb{T}\times\mathbb{R})-\textit{narrow}),$$

and its associated macroscopic density ρ is a weak measure-valued solution (in the symmetrized sense) to (4.2.5) *in the identical case*

$$\begin{cases}\dfrac{\partial\rho}{\partial t}-\dfrac{\partial}{\partial\theta}((h*\rho)\rho)=0,\\ \rho(0,\cdot,\cdot)=\rho_0.\end{cases}$$

Specifically, in weak form

$$\int_0^T\int_{\mathbb{T}}\frac{\partial\phi}{\partial t}\,d_\theta\rho-\int_0^T\int_{\mathbb{T}^2}\frac{1}{2}h(\theta-\theta')\left(\frac{\partial\phi}{\partial\theta}(t,\theta)-\frac{\partial\phi}{\partial\theta}(t,\theta')\right)\,d_{(\theta,\theta')}(\rho\otimes\rho)=-\int_{\mathbb{T}}\phi(0,\theta)d_\theta\rho_0,$$

for any test function $\phi\in C_c^1([0,T)\times\mathbb{T})$. If in addition (4.7.6) *holds true, for some $1<p<\infty$, then*

$$f^\varepsilon\to f\ \text{ in }\ C([0,T],L_w^p(\mathbb{R},\mathcal{M}(\mathbb{T}))-\textit{weak}\,*).$$

Proof. Let us recover the weak formulation (4.7.15) and use the symmetrized version (4.7.16) of the nonlinear term. Then, we obtain

$$\varepsilon\int_0^T\int_{\mathbb{T}}\int_{\mathbb{R}}j^\varepsilon\frac{\partial\varphi}{\partial t}\,dt\,d\theta\,d\Omega+\varepsilon\int_0^T\int_{\mathbb{T}}\int_{\mathbb{R}}\mathcal{S}^\varepsilon\frac{\partial\varphi}{\partial\theta}\,dt\,d\theta\,d\Omega$$
$$=-\varepsilon\int_{\mathbb{T}}\int_{\mathbb{R}}j_0^\varepsilon\,\varphi(0,\cdot,\cdot)\,d\theta\,d\Omega+\int_0^T\int_{\mathbb{T}}\int_{\mathbb{R}}(j^\varepsilon-\Omega f^\varepsilon)\,\varphi\,dt\,d\theta\,d\Omega$$
$$+\int_0^T\int_{\mathbb{T}^2}\int_{\mathbb{R}^2}H_\varepsilon^\varphi(t,\theta,\theta',\Omega,\Omega')f^\varepsilon(t,\theta,\Omega)f^\varepsilon(t,\theta',\Omega')\,dt\,d\theta\,d\theta'\,d\Omega\,d\Omega',\quad (4.7.17)$$

for any test function $\varphi\in C_c^1([0,T)\times\mathbb{T}\times\mathbb{R})$. The main idea is to get rid of the Ω variable in (4.7.17), as it becomes irrelevant in the limit $\varepsilon\to 0$ due to the assumed hypothesis $g=\delta_0$. To simplify notation, denote

$$\widehat{j}^\varepsilon(t,\theta):=\int_{\mathbb{R}}j^\varepsilon(t,\theta,\Omega)\,d\Omega=(\pi_\theta)_\# j^\varepsilon,\qquad \widehat{\mathcal{S}}^\varepsilon(t,\theta):=\int_{\mathbb{R}}\mathcal{S}^\varepsilon(t,\theta,\omega)\,d\Omega=(\pi_\theta)_\#\mathcal{S}^\varepsilon,$$

$$\widehat{j}_t(\theta) := (\pi_\theta)_{\#} j_t.$$

Recall that Theorem 4.7.9, Corollaries 4.7.10, 4.7.7 along with Theorem 4.7.6 respectively imply

$$\begin{array}{l}\rho^\varepsilon \to \rho \ \text{ in } \ C([0,T],\mathcal{M}(\mathbb{T})-\text{narrow}),\\ \widehat{j}^\varepsilon \overset{*}{\rightharpoonup} \widehat{j} \ \text{ in } \ L^2_w(0,T;\mathcal{M}(\mathbb{T})-\text{narrow}),\\ \|\widehat{\mathcal{S}}^\varepsilon\|_{L^1(0,T;L^1(\mathbb{T}))} \leq 2\varepsilon E_0 + KM_\alpha + T(V_0+2),\\ \rho^\varepsilon\otimes\rho^\varepsilon \overset{*}{\rightharpoonup} \rho\otimes\rho \ \text{ in } \ L^\infty_w(0,T;\mathcal{M}(\mathbb{T}^2)).\end{array} \tag{4.7.18}$$

Now, we can take $\varphi(t,\theta,\Omega)=\phi(t,\theta)$ in (4.7.17) with $\phi\in C^1_c([0,T)\times\mathbb{T})$. Notice that it can be done without loss of generality by virtue of the tightness a priori estimates in Proposition 4.7.11, thus yielding

$$\begin{aligned}\varepsilon\int_0^T\int_{\mathbb{T}} \widehat{j}^\varepsilon\frac{\partial\phi}{\partial t}\,dt\,d\theta + \varepsilon\int_0^T\int_{\mathbb{T}} \widehat{\mathcal{S}}^\varepsilon\frac{\partial\phi}{\partial\theta}\,dt\,d\theta\\ = -\varepsilon\int_{\mathbb{T}}\widehat{j}^\varepsilon_0\,\phi(0,\cdot)\,d\theta + \int_0^T\int_{\mathbb{T}}\widehat{j}^\varepsilon\,\phi\,dt\,d\theta - \int_0^T\int_{\mathbb{T}}\int_{\mathbb{R}}\Omega f^\varepsilon\,\phi\,dt\,d\theta\,d\Omega\\ + \int_0^T\int_{\mathbb{T}^2}\widehat{H}^\phi_\varepsilon(t,\theta,\theta')\rho^\varepsilon(t,\theta)\rho^\varepsilon(t,\theta')\,dt\,d\theta\,d\theta',\end{aligned} \tag{4.7.19}$$

for the bounded and continuous function

$$\widehat{H}^\phi_\varepsilon(t,\theta,\theta') = \frac{1}{2}h_\varepsilon(\theta-\theta')(\phi(t,\theta)-\phi(t,\theta')).$$

Then, (4.7.18) clearly allows passing to the limit in all the terms of (4.7.19) (including the non-linear term)

$$\int_0^T\int_{\mathbb{T}}\phi\,d_\theta\widehat{j}\,dt = \int_0^T\int_{\mathbb{T}}\int_{\mathbb{R}}\Omega\,d_{(\theta,\Omega)}f\,dt + \int_0^T\int_{\mathbb{T}^2}H^\phi_0(t,\theta,\theta')d_{(\theta,\theta')}\rho\otimes\rho. \tag{4.7.20}$$

To finish, let us just identify the first term in the right hand side of (4.7.20), which has not been closed in terms of the macroscopic quantities ρ and $\widehat{j}$ yet. To such end, notice that $\rho^\varepsilon\to\rho$ in $C([0,T],\mathcal{M}(\mathbb{T})-\text{narrow})$ and also $g^\varepsilon \overset{*}{\rightharpoonup} g\equiv\delta_0$ narrowly. Similarly, notice that $f^\varepsilon\to f$ in $C([0,T],\mathcal{M}(\mathbb{T}\times\mathbb{R})-\text{narrow})$. Hence, uniqueness implies

$$f_t(\theta,\Omega) = \rho_t(\theta)\otimes\delta_0(\Omega), \ \text{ for all } \ t\in[0,T].$$

Consequently,

$$\int_0^T\int_{\mathbb{T}}\int_{\mathbb{R}}\Omega\,d_{(\theta,\Omega)}f\,dt = \int_0^T\int_{\mathbb{T}}\int_{\mathbb{R}}\Omega\,d_{(\theta,\Omega)}(\rho_t\otimes\delta_0) = 0,$$

thus ending the proof. □

Appendices

4.A Periodic finite Radon measures

In this appendix we recall the relations between some spaces of finite Radon measures, that are used throughout the whole thesis, in general, and this chapter in particular. Our aim is to appropriately set the space of measures with periodic properties to be used. First, note that the spaces of densities $L^1(\mathbb{T}),\ L^1(-\pi,\pi),\ L^1((-\pi,\pi])$ and $L^1([-\pi,\pi])$ can be mixed up since such functions are defined on the whole torus $\mathbb{T}$ except at most at $-1+0\,i \equiv (-1,0)$. However, it is no longer valid for the spaces of finite Radon measures:

$$\mathcal{M}(\mathbb{T}),\ \mathcal{M}(-\pi,\pi),\ \mathcal{M}((-\pi,\pi]) \text{ and } \mathcal{M}([-\pi,\pi]).$$

The main reason is that all such spaces contain the Dirac mass δ_π except the second one. Also, the last measure space might duplicate the Dirac masses at $\delta_{-\pi}$ and δ_π, that can be identified in $\mathcal{M}(\mathbb{T})$ though. Naturally, one has to rule out such doubling of point masses by appropriately setting the good spaces. The main idea is to note that if one unfolds the torus $\mathbb{T}$ into the interval $(-\pi,\pi]$, then each measure in $\mathcal{M}(\mathbb{T})$ can be identified with a measure in $\mathcal{M}((-\pi,\pi])$ and conversely. This is the content of the next straightforward result:

Theorem 4.A.1. *The next Banach spaces of finite Radon measures are topologically isomorphic when endowed with the total variation norm:*

$$\mathcal{M}(\mathbb{T}) \cong \mathcal{M}_p([-\pi,\pi]) \cong \mathbb{R} \oplus_1 \mathcal{M}(-\pi,\pi) \cong \mathcal{M}((-\pi,\pi]).$$

Although it is straightforward, we will sketch the proof of such result for the readers convenience.

4.A.1 Periodic functions

Before we sketch the proof, we will first introduce some natural identification between Banach spaces of regular functions with analogue periodicity properties.

Definition 4.A.2. *For any derivable function $f:\mathbb{T}\longrightarrow\mathbb{R}$ along $\mathbb{T}$, we define the associated derivatives*

$$\begin{aligned}\frac{\partial f}{\partial z}(e^{i\theta}) &:= -ie^{-i\theta}\frac{d}{d\theta}f(e^{i\theta}),\\ \frac{\partial f}{\partial \bar{z}}(e^{i\theta}) &:= ie^{i\theta}\frac{d}{d\theta}f(e^{i\theta}),\end{aligned}$$

Remark 4.A.3. *The above derivatives have long been used in complex analysis. Consider $z=e^{i\theta}$, $\bar{z}=e^{-i\theta}$ and define $g(z)=f(\bar{z})$. Then, the motivation underlying the above definition is simply a formal chain rule, namely,*

$$\begin{aligned}\frac{d}{d\theta}f(e^{i\theta}) &= \frac{\partial f}{\partial z}\frac{dz}{d\theta} = ie^{i\theta}\frac{\partial f}{\partial z},\\ \frac{d}{d\theta}f(e^{i\theta}) &= \frac{d}{d\theta}g(e^{-i\theta}) = \frac{\partial g}{\partial \bar{z}}\frac{d\bar{z}}{d\theta} = \frac{\partial f}{\partial \bar{z}}\frac{\partial \bar{z}}{\partial \bar{z}}\frac{d\bar{z}}{d\theta} = -ie^{-i\theta}\frac{\partial f}{\partial \bar{z}}.\end{aligned}$$

Notice that one can go from one to the other by taking complex conjugation, i.e., $\frac{\partial f}{\partial \bar{z}} = \overline{\frac{\partial f}{\partial z}}$. Using only one of the above derivatives we can recover the other one (along with the full differential map), thus avoiding redundancy of information. Indeed, $\mathbb{T}$ is a Riemannian manifold with the standard metric and one can easily check that

$$\langle \nabla f(z), iz\rangle = df_z(iz) = \frac{\partial f}{\partial z}iz \implies \nabla f(z) = \langle \nabla f(z), iz\rangle\, iz = -z^2\frac{\partial f}{\partial z} = \frac{\partial f}{\partial \bar{z}}.$$

Definition 4.A.4. *We will set the next Banach spaces of test functions:*

1. *$C^1(\mathbb{T})$ will denote the Banach space of continuously differentiable functions $f : \mathbb{T} \longrightarrow \mathbb{R}$. It is endowed with the complete norm*

$$\|f\|_{C^1(\mathbb{T})} := \|f\|_{C(\mathbb{T})} + \left\|\frac{\partial f}{\partial z}\right\|_{C(\mathbb{T})}.$$

2. *$C^1_p([-\pi, \pi])$ will denote the Banach space of continuously differentiable functions $g : [-\pi, \pi] \longrightarrow \mathbb{R}$ such that g and its derivatives have same values at the endpoints of the interval. It is endowed with the complete norm*

$$\|g\|_{C^1_p([-\pi,\pi])} := \|g\|_{C([-\pi,\pi])} + \left\|\frac{dg}{d\theta}\right\|_{C([-\pi,\pi])}.$$

3. *As it is usual, $C^1_0((-\pi, \pi])$ (respectively, $C^1_0(-\pi, \pi)$) denote the Banach space of continuously differentiable functions such that g and its derivatives vanish at $-\pi$ (respectively at $-\pi$ and π). It is endowed with the preceding complete norm.*

The spaces $C(\mathbb{T})$, $C_p([-\pi, \pi])$, $C_0((-\pi, \pi])$ and $C_0(-\pi, \pi)$ are also Banach spaces when endowed with the uniform norm. Similarly, spaces with higher order derivatives can also be considered.

By definition, the next results are clear:

Proposition 4.A.5. *For every $f \in C^1(\mathbb{T})$, let us define $\Phi[f](\theta) := f(e^{i\theta})$. Then, the following map is an isometric isomorphism*

$$\begin{array}{rccc} \Phi: & C(\mathbb{T}) & \longrightarrow & C_p([-\pi,\pi]), \\ & f & \longmapsto & \Phi[f]. \end{array}$$

Proposition 4.A.6. *For every $g \in C_p([-\pi, \pi])$, let us define*

$$\begin{aligned} \Psi_1[g] &:= g(-\pi) = g(\pi) \in \mathbb{R}, \\ \Psi_2[g] &:= g - \Psi_1[g] \in C_0(-\pi,\pi). \end{aligned}$$

Then, the next map is a topological isomorphism

$$\begin{array}{rccc} \Psi: & C_p([-\pi,\pi]) & \longrightarrow & \mathbb{R} \oplus_\infty C_0(-\pi,\pi), \\ & g & \longmapsto & \Psi[g] := (\Psi_1[g], \Psi_2[g]). \end{array}$$

Proposition 4.A.7. *Let us set any cut-off function $\eta \in C([-\pi, \pi])$ such that $\eta(-\pi) = 0$, $\eta(\pi) = 1$ and $0 \leq \eta \leq 1$. For every $g \in C_0((-\pi, \pi])$, let us define*

$$\begin{aligned} \Lambda_1^\eta[g] &:= g(\pi) \in \mathbb{R}, \\ \Lambda_2^\eta[g] &:= g - \Lambda_1^\eta[g]\eta \in C_0(-\pi,\pi). \end{aligned}$$

Then, the next map is a topological isomorphism

$$\begin{array}{rccc} \Lambda^\eta: & C_0((-\pi,\pi]) & \longrightarrow & \mathbb{R} \oplus_\infty C_0(-\pi,\pi), \\ & g & \longmapsto & (\Lambda_1^\eta[g], \Lambda_2^\eta). \end{array}$$

4.A.2 Periodic measures

Definition 4.A.8. *As it is usual, taking duals of the Banach spaces in Definition 4.A.4 we arrive at the next Banach spaces of finite Radon measures endowed with the (dual) total variation norm:*

$$\begin{aligned}\mathcal{M}(\mathbb{T}) &:= C(\mathbb{T})^*,\\ \mathcal{M}_p([-\pi,\pi]) &:= C_p([-\pi,\pi])^*,\\ \mathcal{M}(-\pi,\pi) &:= C_c(-\pi,\pi)^*,\\ \mathcal{M}((-\pi,\pi]) &:= C_c((-\pi,\pi])^*.\end{aligned}$$

Proof of Theorem 4.A.1. The proof is just a simple consequence of Propositions 4.A.5, 4.A.6 and 4.A.7 that follows from taking dual operators to Φ, Ψ and Λ^η

$$\begin{array}{rccc}\Phi^*: & \mathcal{M}_p([-\pi,\pi]) & \longrightarrow & \mathcal{M}(\mathbb{T}),\\ & \mu & \longmapsto & \Phi^*(\mu),\\ \Psi^*: & \mathbb{R}\oplus_1\mathcal{M}(-\pi,\pi) & \longrightarrow & \mathcal{M}_p([-\pi,\pi]),\\ & (b,\nu) & \longrightarrow & \Psi^*(b,\nu),\\ (\Lambda^\eta)^*: & \mathbb{R}\oplus_1\mathcal{M}(-\pi,\pi) & \longmapsto & \mathcal{M}((-\pi,\pi]),\\ & (b,\nu) & \longmapsto & (\Lambda^\eta)^*(b,\nu).\end{array}$$

Indeed, given $\mu\in\mathcal{M}_p([-\pi,\pi])$, $b\in\mathbb{R}$ and $\nu\in\mathcal{M}(-\pi,\pi)$ and setting $f\in C(\mathbb{T})$ and $g\in C_p([-\pi,\pi])$, the duality read

$$\begin{aligned}\langle\Phi^*(\mu),f\rangle &= \langle\mu,\Phi[f]\rangle\,,\\ \langle\Psi^*(b,\nu),g\rangle &= \Psi_1[g]b+\langle\nu,\Psi_2[g]\rangle\,,\\ \langle(\Lambda^\eta)^*(b,\nu),g\rangle &= \Lambda_1^\eta[g]b+\langle\nu,\Lambda_2^\eta[g]\rangle\,.\end{aligned}$$

□

Remark 4.A.9. *In particular, the next identification takes place under the above topological isomorphisms:*

$$\begin{array}{ccccc}\mathcal{M}(\mathbb{T}) & \cong & \mathbb{R}\oplus\mathcal{M}(-\pi,\pi) & \cong & \mathcal{M}((-\pi,\pi]),\\ \delta_{(-1,0)} & \equiv & (1,0\,d\theta) & \equiv & \delta_\pi.\end{array}$$

Notice that we have only provided topological isomorphisms between the above spaces of measures in Theorem 4.A.1. In particular, we can identify the spaces $\mathcal{M}(\mathbb{T})$ with $\mathcal{M}((-\pi,\pi])$ via the composition

$$\mathfrak{I}^\eta := (\Lambda^\eta)^*\circ(\Psi^*)^{-1}\circ(\Phi^*)^{-1} = ((\Psi\circ\Phi)^{-1}\circ\Lambda^\eta)^*.$$

Specifically, it means that, given $\mu\in\mathcal{M}(\mathbb{T})$, the measure $\mathfrak{I}^\eta[\mu]\in\mathcal{M}((-\pi,\pi])$ acts as follows

$$\langle\mathfrak{I}^\eta[\mu],\phi\rangle = \left\langle\mu,f_\phi^\eta\right\rangle,$$

for any $\phi\in C_0((-\pi,\pi])$, where the continuous function $f_\phi^\eta := ((\Psi\circ\Phi)^{-1}\circ\Lambda^\eta)[\phi]\in C(\mathbb{T})$ reads

$$f_\phi^\eta(e^{i\theta}) = \phi(\theta)+(1-\eta(\theta))\phi(\pi),\ \theta\in(-\pi,\pi].$$

Although $\mathfrak{I}^\eta$ is not an isometry, we still can do better and introduce a simpler isometry between the particular spaces $\mathcal{M}(\mathbb{T})$ and $\mathcal{M}((-\pi,\pi])$. This will be the content of our last result.

Theorem 4.A.10. *Let us consider the bijective and continuous mapping*

$$\begin{array}{rccc}\iota: & (-\pi,\pi] & \longrightarrow & \mathbb{T} \\ & \theta & \longmapsto & e^{i\theta}.\end{array}$$

Then, the associated push-forward mapping is a surjective isometry

$$\begin{array}{rccc}\iota_\#: & \mathcal{M}((-\pi,\pi]) & \longrightarrow & \mathcal{M}(\mathbb{T}) \\ & \mu & \longmapsto & \iota_\#\mu.\end{array}$$

Proof. First, it is clear that $\iota_\#$ is bijective because so is ι. Let us now prove that it is a linear isometry. On the one hand, consider any $\mu\in\mathcal{M}((-\pi,\pi])$. Then,

$$\begin{aligned}\|\iota_\#\mu\|_{\mathcal{M}(\mathbb{T})} &= \sup_{\|\varphi\|_{C(\mathbb{T})}\leq 1}\int_{\mathbb{T}}\varphi\,d(\iota_\#\mu) = \sup_{\|\varphi\|_{C(\mathbb{T})}\leq 1}\int_{(-\pi,\pi]}(\varphi\circ\iota)\,d\mu\\ &\leq \sup_{\|\phi\|_{C_0((-\pi,\pi])}\leq 1}\int_{(-\pi,\pi]}\phi\,d\mu = \|\mu\|_{\mathcal{M}((-\pi,\pi])}.\end{aligned}$$

On the other hand, let us show the reverse inequality. Take any $\phi\in C_0((-\pi,\pi])$ with the property $\|\phi\|_{C_0((-\pi,\pi])}\leq 1$. By density, we can assume that $\phi\in C_c((-\pi,\pi])$. Then, there exists some $M_0\in(-\pi,\pi)$ such that

$$\phi(\theta)=0,\quad \text{for all } \theta\in(-\pi,M_0].$$

For every $\varepsilon\in(0,\pi+M_0)$ let us consider a cut-off function $\eta_\varepsilon\in C([-\pi,\pi])$ with $0\leq\eta_\varepsilon\leq 1$ such that

$$\eta_\varepsilon(-\pi)=0 \quad\text{and}\quad \eta_\varepsilon|_{[-\pi+\varepsilon,\pi]}\equiv 1.$$

For any $\varepsilon>0$, let us define

$$f_\varepsilon(e^{i\theta}) := f_\phi^{\eta_\varepsilon}(e^{i\theta}) = \phi(\theta)+(1-\eta_\varepsilon)\phi(\pi),\ \theta\in(-\pi,\pi],$$

as in the proof of Theorem 4.A.1. Then, it is clear that

$$\int_{(-\pi,\pi]}\phi\,d\mu = \int_{(-\pi,\pi]}(f_\varepsilon\circ\iota)\,d\mu - \phi(\pi)\int_{(-\pi,\pi]}(1-\eta_\varepsilon)\,d\mu = \int_{\mathbb{T}}f_\varepsilon\,d(\iota_\#\mu)-\phi(\pi)\int_{(-\pi,\pi]}(1-\eta_\varepsilon)\,d\mu.$$

Notice that due to the boundedness of $1-\eta_\varepsilon$ we have that it belongs to $L^1(\mu)$ and the above terms make sense. In fact

$$\int_{(-\pi,\pi]}\phi\,d\mu \leq \int_{\mathbb{T}}|f_\varepsilon|\,d|\iota_\#\mu| + \|\phi\|_{C_0((-\pi,\pi])}\left|\int_{(-\pi,\pi]}(1-\eta_\varepsilon)\,d\mu\right|,$$

for every $\varepsilon>0$. Regarding the first term, it is clear that $\|f_\varepsilon\|_{C(\mathbb{T})}=\|\varphi\|_{C(\mathbb{T})}\leq 1$ for every $\varepsilon\in(0,\pi+M_0)$. On the other hand, notice that the second term vanishes as $\varepsilon\to 0$ due to the dominated convergence theorem. Then, putting both facts together we can conclude

$$\int_{(-\pi,\pi]}\phi\,d\mu\leq\|\iota_\#\mu\|_{\mathcal{M}(\mathbb{T})},$$

and it ends the proof. □

4.B Differentiability properties of the squared distance

In this appendix we will revisit some well known results about (non)-differentiability of the squared distance in a Riemannian manifold. Most of them are folklore in Riemannian Geometry and require no comment. Nevertheless, we will comment on the appropriate concept of differentiability that we are interested in, namely, the one-sided Dini upper derivative.

Proposition 4.B.1. *Let $(M, \langle\cdot,\cdot\rangle)$ be a d-dimensional complete Riemannian manifold and consider the Riemannian distance as recalled in* (N.4). *Fix any $y \in M$ and define the distance function $d_y : M \to \mathbb{R}$ towards some point $y \in M$ by the rule*

$$d_y(x) = d(x, y),\ \forall\, x \in M.$$

Then, following properties hold true:

1. *d_y is Lipschitz in M.*
2. *d_y is derivable at almost every $y \in M$.*
3. *d_y is derivable in $M \setminus \{\mathrm{cut}(y) \cup \{y\}\}$ and*
$$(\nabla d_y)(x) = -\frac{\exp_x^{-1}(y)}{|\exp_x^{-1}(y)|}.$$
4. *$\frac{1}{2}d_y^2$ fails to be everywhere directionally derivable unless M is diffeomorphic to the flat space $\mathbb{R}^d$. However, it is derivable at any $x \in M \setminus (\mathrm{cut}(x) \cup \{x\})$ and*
$$\nabla\left(\frac{1}{2}d_y^2\right)(x) = -\exp_x^{-1}(y).$$

Here, $\mathrm{cut}(x)$ denotes the cut locus of the point x in M and $\exp_x : T_xM \to M$ is nothing but the Riemannian exponential map at such x.

Since the proofs are standard, we do not provide proofs here. They can be found in any textbook of Riemannian Geometry. Instead, we just focus on the later assertion that is the less apparent one. According to such result not only $\frac{1}{2}d_y^2$ fails to be derivable at some points $x \in \mathrm{cut}(y)$, but also the lateral directional derivatives might not agree at points $x \in M$ that can be joined with y through several minimizing geodesics. The proof is apparently hidden in the litterature; however, one can find a short proof following simple arguments in [304].

Remark 4.B.2. *Just to illustrate a meaningful example, consider $M = \mathbb{T}$ with the (standard) induced metric. Recall that for $z_1 = e^{i\theta_1}$ and $z_2 = w^{i\theta_2}$ where $\theta_1, \theta_2 \in \mathbb{R}$ one has the clear identity*

$$d(z_1, z_2) = |\theta_1 - \theta_2|_o.$$

Also, consider some $z = e^{i\theta}$, where $\theta \in \mathbb{R}$, and its antipode $\bar{z} = -z = e^{i(\theta+\pi)}$. For every $\omega > 0$, one can define a geodesic with speed ω as follows

$$\gamma_{z,\omega}(s) := e^{i(\theta+\omega s)},\ s \in \mathbb{R},$$

Indeed, $\gamma_{z,\omega}(0) = z$ and it is minimizing in any interval whose length is not larger than $\frac{\pi}{\omega}$. Consequently,

$$\frac{1}{2}d_{\bar{z}}^2(\gamma_\omega(s)) = \begin{cases} \frac{1}{2}(\pi + \omega s)^2, & s \in (-\frac{\pi}{\omega}, 0], \\ \frac{1}{2}(\pi - \omega s)^2, & s \in [0, \frac{\pi}{\omega}). \end{cases}$$

Hence, both one-sided derivatives exist but they differ from each other, namely,

$$\left.\frac{d}{ds}\right|_{s=0^{\mp}} \frac{1}{2} d_{\bar{z}}^2(\gamma_{z,\omega}(s)) = \pm\omega\pi.$$

Apart from the above differentiability properties, others have been explored in the literature with applications to optimal mass transportation and Wasserstein distances. We address some of them in the following result, see [129, Proposition 2.9], [211, Proposition 6], [296, Third Appendix of Chapter 10] for more details.

Proposition 4.B.3. *Let $(M, \langle\cdot,\cdot\rangle)$ be a complete Riemannian manifold, fix $y \in M$ and the squared distance function $\frac{1}{2}d_y^2$ towards the point y. Then,*

1. *$\frac{1}{2}d_y^2$ is superdifferentiable at every $x \in M$ and for every $w \in \exp_x^{-1}(y)$ with $|w| = d(x,y)$ one has that $-w$ is an upper gradient, i.e.,*

$$\frac{1}{2}d_y^2(\exp_x(v)) \leq \frac{1}{2}d_y^2(x) - \langle w, v\rangle + o(|v|), \text{ as } |v| \to 0.$$

2. *If in addition M has non-negative sectional curvatures then $\frac{1}{2}d_y^2$ is 1-semiconcave, i.e.,*

$$\frac{1}{2}d_y^2(\gamma(s)) \geq (1-s)\frac{1}{2}d_y^2(x_1) + s\frac{1}{2}d_y^2(x_2) + s(1-s)\frac{1}{2}d^2(x_1,x_2),\ s \in [0,1],$$

for any couple $x_1, x_2 \in M$ and any geodesic $\gamma : [0,1] \longrightarrow M$ joining x_1 to x_2.

Both superdifferentiability and semiconcavity are locally equivalent, see [296] for further information. On the one hand, the non-negativity condition on the sectional curvatures is required in order to obtain uniform estimates in term of the quadratic modulus of semiconcavity and it is certainly a very imposing hypothesis. On the other hand, the superdifferentiability is not enough for our purpose since the $o(|v|)$ term in the right hand-side is not necessarily uniform. For the purposes in this thesis we will resort on an slightly different tool coming form non-smooth analysis, namely, the *one-sided upper Dini directional derivative* of a function.

Definition 4.B.4. *Let $(M, \langle\cdot,\cdot\rangle)$ be a finite-dimensional complete Riemannian manifold and consider some function $f : M \to \mathbb{R}$, any $x \in M$ and any direction $v \in T_xM$. Then, the one-sided upper and lower Dini derivatives of f at x in the direction v stand for*

$$(d^+f)_x(v) := \left.\frac{d^+}{ds}\right|_{s=0} f(\exp_x(sv)) = \limsup_{s\to 0^+} \frac{f(\exp_x(sv)) - f(x)}{s},$$
$$(d_+f)_x(v) := \left.\frac{d_+}{ds}\right|_{s=0} f(\exp_x(sv)) = \liminf_{s\to 0^+} \frac{f(\exp_x(sv)) - f(x)}{s}.$$

By definition both derivatives are ordered

$$-\infty \leq (d_+f)_x(v) \leq (d^+f)_x(v) \leq +\infty,$$

and we will say that f is one-sided upper (respectively lower) Dini derivable at x in the direction v if the corresponding one-sided upper (respectively lower) Dini derivative is finite. If in addition both derivatives agree, then f is also one-sided directionally derivable at x in the direction v in the standard sense and all the derivatives agree.

An important fact to be remarked is that the geodesic $s \mapsto \exp_x(sv)$ has been chosen as a representative of a curve with direction v at x. However, one might have taken any other C^1 curve and apparently it would have provided a "different" definition of directional derivative. Since it will be of interest for our purposes, let us show that any such curve representing the direction v at x could have been chosen, yielding the same definition.

Lemma 4.B.5. *Let $(M, \langle\cdot,\cdot\rangle)$ be a Riemannian manifold and fix any $x \in M$ and any couple $v, w \in T_xM$. Consider a couple of C^1 curves $\gamma_1 : (-s_0, s_0) \to M$ and $\gamma_2 : (-s_0, s_0) \to M$ such that $\gamma_1(0) = x = \gamma_2(0)$, $\gamma_1'(0) = v$ and $\gamma_2'(0) = w$. Then,*

$$\limsup_{s\to 0} \frac{d(\gamma_1(s), \gamma_2(s))}{s} \leq |v - w|.$$

Proof. Although the proof is standard, we provide a simple proof for the sake of completeness. Consider some $R > 0$ smaller enough than the radius of injectivity at x (e.g. half of it) and consider the ball $B_R(0) \subseteq T_xM$ along with the associated geodesic ball $\mathbb{B}_R(x) := \exp_x(B_R(0))$. By definition one has that $\exp_x : B_R(0) \longrightarrow \mathbb{B}_R(x)$ is a diffeomorphism. Without loss of generality, we will assume that s_0 is small enough so that $\gamma_i(s) \in \mathbb{B}_R(x)$ for all $s \in (-s_0, s_0)$ and $i = 1, 2$. Hence, we can define the curves in $B_R(0)$

$$\bar{\gamma}_i(s) := \exp_x^{-1}(\gamma_i(s)),\ s \in (-s_0, s_0),\ i = 1, 2.$$

Equivalently, $\gamma_i(s) = \exp_x(\bar{\gamma}_i(s))$ and taking derivatives one has

$$\gamma_i'(0) = (d\exp_x)_0(\bar{\gamma}_i'(0)) \Longrightarrow \bar{\gamma}_1'(0) = v \text{ and } \bar{\gamma}_2'(0) = w,$$

where we have used that $(d\exp_x)_0$ is nothing but the identity map in T_xM and $\gamma_1'(0) = v$, $\gamma_2'(0) = w$. Also, consider the interpolating curves between $\bar{\gamma}_1(s)$ and $\bar{\gamma}_2(s)$

$$\bar{\gamma}_s(\varepsilon) = (1-\varepsilon)\bar{\gamma}_1(s) + \varepsilon\gamma_2(s),\ \varepsilon \in [0, 1],$$

for every $s \in (-s_0, s_0)$. They have associated interpolating curves between $\gamma_1(s)$ and $\gamma_2(s)$

$$\gamma_s(\varepsilon) = \exp_x(\bar{\gamma}_s(\varepsilon)) = \exp_x((1-\varepsilon)\gamma_1(s) + \varepsilon\gamma_2(\varepsilon)),\ \varepsilon \in [0, 1].$$

Then, we can estimate

$$\frac{d(\gamma_1(s), \gamma_2(s))}{s} = \frac{d(\exp_x(\bar{\gamma}_1(s)), \exp_x(\bar{\gamma}_2(s)))}{s} \leq [\exp_x]_{C^{0,1}(\mathbb{B}_R(x))} \frac{|\bar{\gamma}_1(s) - \bar{\gamma}_2(s)|}{s},$$

where $[\,\cdot\,]_{C^{0,1}(B_R(x))}$ stands for the Lipschitz constant in $B_R(x)$. Taking $\limsup$ and using that $\bar{\gamma}_1$ and $\bar{\gamma}_2$ are both derivable at $s = 0$ with derivatives v and w respectively we obtain

$$\limsup_{s\to 0} \frac{d(\gamma_1(s), \gamma_2(s))}{s} \leq [\exp_x]_{C^{0,1}(\mathbb{B}_R(x))}|v - w|.$$

Now, notice that $R > 0$ can be chosen arbitrarily small. Taking $\liminf$ with respect to such radius we obtain

$$\limsup_{s\to 0} \frac{d(\gamma_1(s), \gamma_2(s))}{s} \leq \liminf_{R\to 0}\, [\exp_x]_{C^{0,1}(\mathbb{B}_R(x))}|v - w|.$$

Also, the mean value theorem implies

$$\liminf_{R\to 0}\, [\exp_x]_{C^{0,1}(\mathbb{B}_R(x))} \leq \liminf_{R\to 0} \sup_{\xi \in B_R(0)} |(d\exp_x)_\xi|_{T_x^*M}.$$

Since $(d\exp_x)_0$ agrees with the identity map in T_xM, that has operator norm equals to 1, we deduce the following estimate

$$\liminf_{R\to 0}\,[\exp_x]_{C^{0,1}(\mathbb{B}_R(x))}\leq 1,$$

and that ends the proof of our result. $\square$

As a consequence we obtain the following characterizations of the one-sided Dini derivatives.

Theorem 4.B.6. *Let $(M,\langle\cdot,\cdot\rangle)$ be a complete Riemannian manifold, consider some $x\in M$ and $v\in T_xM$ and choose any C^1 curve $\gamma:(-s_0,s_0)\longrightarrow M$ such that $\gamma(0)=x$ and $\gamma'(0)=v$. If $f:M\longrightarrow\mathbb{R}$ is locally Lipschitz map around x, then*

$$\begin{aligned}(d^+f)_x(v)&=\limsup_{s\to 0^*}\frac{f(\gamma(s))-f(x)}{s},\\ (d_+f)_x(v)&=\liminf_{s\to 0^*}\frac{f(\gamma(s))-f(x)}{s}.\end{aligned}$$

Proof. Let us define the auxiliary C^1 curve $\widetilde{\gamma}(s)=\exp_x(sv)$ for $s\in(-s_0,s_0)$. According to the above Definition 4.B.4 we only need to prove that

$$\lim_{s\to 0^+}\frac{f(\gamma(s))-f(\widetilde{\gamma}(s))}{s}=0.$$

Consider R smaller enough that the radius of injectivity at x and set $B_R(0)\subseteq T_xM$ along with the geodesic ball $\mathbb{B}_R(x):=\exp_x(B_R(0))$ that is a relatively compact set. Consider L_R the Lipschitz constant of f in $\overline{\mathbb{B}}_R(x)$ and assume that s_0 is small enough so that $\gamma(s),\widetilde{\gamma}(s)\in\overline{\mathbb{B}}_R(x)$ for every $s\in(-s_0,s_0)$. Thus,

$$\left|\frac{f(\gamma(s))-f(\widetilde{\gamma}(s))}{s}\right|\leq L_R\frac{d(\gamma(s),\widetilde{\gamma}(s))}{s},\ s\in(-s_0,s_0).$$

Since $\gamma'(0)=v=\widetilde{\gamma}'(0)$ we conclude the proof of this result by virtue of Lemma 4.B.5. $\square$

We are now ready to give a simple proof of the one-sided uper Dini directional differentiability of the distance.

Theorem 4.B.7. *Let $(M,\langle\cdot,\cdot\rangle)$ be a complete Riemannian manifold, fix $y\in M$ and the squared distance $\frac{1}{2}d_y^2$ towards the point y. Then, $\frac{1}{2}d_y^2$ is one-sided upper Dini directionally derivable in all M and*

$$d^+\left(\frac{1}{2}d_y^2\right)_x(v)\leq\inf_{\substack{w\in\exp_x^{-1}(y)\\|w|=d(x,y)}}-\langle v,w\rangle\,,$$

for any $x\in M$ and any direction $v\in T_xM$.

Proof. Consider $x\in M$ and $v\in T_xM$ and set $w\in\exp_x^{-1}(y)$ with $|w|=d(x,y)$. Also, consider a minimizing geodesic $\gamma_0:[0,1]\longrightarrow M$ such that $\gamma_0(0)=x$, $\gamma_0(1)=y$ and $\gamma_0'(0)=w$. Then, there exists some smooth variation $\{\gamma_s\}_{s\in(-s_0,s_0)}$ of γ_0 such that

$$\gamma_s(0)=\exp_x(sv)\ \text{ and }\ \gamma_s(1)=y,$$

for every $s \in (-s_0, s_0)$. Such a variation enjoys an associated variational field

$$V(\xi) := \frac{\partial \gamma}{\partial s}(s=0,\xi) = \left.\frac{d}{ds}\right|_{s=0} \gamma_s(\xi),\ \xi \in [0,1].$$

Also, we can define the associated energy functional

$$\mathcal{E}[\gamma_s] := \frac{1}{2}\int_0^1 |\gamma_s'(\xi)|^2\, d\xi.$$

Then, the first variation formula of energy (see [111, Proposition 2.4]) around the geodesic γ_0 amounts to

$$\left.\frac{d}{ds}\right|_{s=0} \mathcal{E}[\gamma_s] = \left\langle \gamma_0'(1), V(1)\right\rangle - \left\langle \gamma_0'(0), V(0)\right\rangle = \left\langle \gamma_0'(1), 0\right\rangle - \langle w, v\rangle = -\langle w, v\rangle\,.$$

Since γ_0 is minimizing then we can equivalently restate

$$\lim_{s\to 0} \frac{\mathcal{E}[\gamma_s] - \frac{1}{2}d_y^2(x)}{s} = -\langle w, v\rangle\,, \tag{4.B.1}$$

and it is clear that

$$\frac{\frac{1}{2}d_y^2(\exp_x(sv)) - \frac{1}{2}d_y^2(x)}{s} \le \frac{\mathcal{E}[\gamma_s] - \frac{1}{2}d_y^2(x)}{s},\ s \in (-s_0, s_0),$$

Taking $\limsup$ as $s \to 0^+$ and using the above formula (4.B.1) implies

$$d^+\left(\frac{1}{2}d_y^2\right)_x (v) \le -\langle v, w\rangle\,,$$

where $w \in \exp_x^{-1}(y)$ such that $|w| = d(x,y)$ is arbitrary. This ends the proof. □

CHAPTER 5

On the trend to global equilibrium for Kuramoto oscillators

Contents

5.1 Introduction

In the present chapter, we quantify the rate of convergence to the global equilibrium for C^1 solutions to the Kuramoto–Sakaguchi equation from generic initial data, providing a first quantitative result in this context. As a by-product, we derive a quantitative statistical estimate, on the rate of concentration for the original agent-based Kuramoto model. Such a model was introduced by Y. Kuramoto several decades [195, 196] ago and is one of the paradigms to study collective synchronization phenomena in biological and mechanical systems in nature. It has gained extensive attention from the physics and mathematics community, see [1, 11, 24, 45, 58, 75, 109, 145, 157, 155, 209, 248, 280, 297].

The main motivation to perform our study on the Kuramoto–Sakaguchi equation is threefold. First, such a model has become a starting point for a broad family of models in collective dynamics. Historically, many of the central analytical techniques developed to study such models were first applied to the Kuramoto model and later generalized to the rest of the field. Second, the Kuramoto model provides a concrete example of a gradient flow structure in which the energy functional is not convex. Such lack of convexity generates challenges to use theory of gradient flows to derive rates of convergence. Third, we are interested in quantifying the relaxation time of a nondeterministic event. Indeed, in a large coupling strength regime, one expects relaxation to the global equilibrium of the particle system with almost sure probability. However, such relaxation fails for some well prepared initial data.

In the case of identical oscillators, the Kuramoto–Sakaguchi equation exhibits a gradient flow structure in the space of probability measure under the Wasserstein distance. Nowadays, it is well-known that transportation distances between measures can be successfully used to study evolutionary equations. More precisely, one of the most surprising achievements of [180, 237, 236] has been that many evolutionary equations of the form

$$\frac{\partial \rho}{\partial t} = \operatorname{div}\left(\nabla\rho + \rho\nabla V + \rho(\nabla W * \rho)\right),$$

can be seen as gradient flows of some entropy functional in the spaces of probability measures with respect to the Wasserstein distance. When such entropy functionals are convex with respect to the Wasserstein distance, such an interpretation allows proving entropy estimates and functional inequalities (see [296] for more details on this area). Such tools, in turns, can be used to obtain convergence rates and stability estimates of the corresponding equations.

There are two main difficulties when one tries to use such a theory in the Kuramoto–Sakaguchi equation. First, even in the identical case, as for the Kuramoto model, the entropy functional associated with the equation does not satisfy the necessary convexity hypothesis. Second, in the nonidentical case, the Wasserstein gradient flow structure of the equation is not available. On the other hand, the Kuramoto–Sakaguchi equation has the virtue that the broad family of unstable equilibria is characterized easily. Thus, it provides an ideal setting in which to develop techniques to attack the lack of convexity.

In this chapter, we adapt the techniques developed by L. Desvilletes and C. Villani in [104] to derive quantitative convergence rates for a nonconvex gradient flow in the particular context of the Kuramoto–Sakaguchi equation. We hope that this provides insight on how to attack this difficulty in more general situations.

The rest of the section is devoted to introduce the model. Then, we will recall the current state of the art regarding the asymptotics of the system in a strong coupling strength regime. Finally, we will state our main result, the proof of which will be the object of the rest of the chapter.

5.1.1 The Kuramoto model

The Kuramoto model governs the synchronization dynamics of N oscillators - each identified by its phase and natural frequency pair $(\theta_i(t), \Omega_i)$ in $\mathbb{T} \times \mathbb{R}$. Such dynamics is given by the system

$$\begin{cases} \dot{\theta}_i = \Omega_i + \dfrac{K}{N} \displaystyle\sum_{j=1}^{N} \sin(\theta_j - \theta_i), \\ \theta_i(0) = \theta_{i,0}, \end{cases} \tag{5.1.1}$$

for $i = 1, \cdots, N$. The large crowd dynamics, $N \to \infty$, is captured by the kinetic description, given by the Kuramoto–Sakaguchi equation, which governs the probability distribution of oscillators $f(t, \theta, \Omega)$ at $(t, \theta, \Omega) \in \mathbb{R}^+ \times \mathbb{T} \times \mathbb{R}$

$$\begin{cases} \dfrac{\partial f}{\partial t} + \dfrac{\partial}{\partial \theta}(v[f]f) = 0, \quad (\theta, \Omega) \in \mathbb{T} \times \mathbb{R}, \, t \geq 0, \\ f(0, \theta, \Omega) = f_0(\theta, \Omega), \quad (\theta, \Omega) \in \mathbb{T} \times \mathbb{R}. \end{cases} \tag{5.1.2}$$

We denote the velocity field by $v[f]$, that is,

$$v[f](t, \theta, \Omega) := \Omega + K \int_{\mathbb{T}} \sin(\theta' - \theta) \rho(t, \theta') \, d\theta', \tag{5.1.3}$$

and we define

$$\rho(t, \theta) := \int_{\mathbb{R}} f(t, \theta, \Omega) \, d\Omega, \qquad g(\Omega) := \int_{\mathbb{T}} f(t, \theta, \Omega) \, d\theta = \int_{\mathbb{T}} f_0(\theta, \Omega) \, d\theta.$$

Here, K is the positive coupling strength and measures the degree of the interaction between oscillators, and ρ and g respectively describe the macroscopic phase density and natural frequency distribution. The rigorous derivation from (5.1.1) to (5.1.2) was done by Lancellotti [198] using Neunzert's method [230].

5.1.2 The gradient flow structure and stationary solutions

The Kuramoto model in $\mathbb{T}^N$ can be lifted to a dynamical system in $\mathbb{R}^N$. J. L. van Hemmen and W. F. Wreszinki [290] observed that by doing this the Kuramoto model can be formulated as a gradient flow of the energy

$$V(\Theta) = -\frac{1}{N} \sum_{j=1}^{N} \Omega_j \theta_j + \frac{K}{2N^2} \sum_{k,j=1}^{N} \Big(1 - \cos(\theta_j - \theta_k) \Big), \tag{5.1.4}$$

under the metric of $\mathbb{R}^N$ induced by the scaled inner product

$$\langle v, w \rangle_N = \frac{v \cdot w}{N}. \tag{5.1.5}$$

Here, $\Theta = (\theta_1, ..., \theta_N)$, v, and w belong to $\mathbb{R}^N$. Specifically, (5.1.1) solves the gradient flow problem

$$\begin{cases} \dot{\Theta}(t) = -\nabla_N V(\Theta(t)), \\ \Theta(0) = \Theta_0, \end{cases} \tag{5.1.6}$$

where ∇_N denotes the gradient with respect to the scaled inner product. Let us also recall that if we define the order parameters $\Theta \longmapsto r(\Theta), \phi(\Theta)$ by the relation

$$r(\Theta)e^{i\phi(\Theta)} = \frac{1}{N}\sum_{k=1}^{N} e^{i\theta_k},$$

then, we have that the potential reads

$$V(\Theta) = -\frac{1}{N}\sum_{j=1}^{N}\Omega_j\theta_j + \frac{K}{2}\big(1 - r^2(\Theta)\big), \tag{5.1.7}$$

and the gradient slope take the form

$$|\nabla_N V(\Theta)|_N^2 = \frac{1}{N}\sum_{j=1}^{N}\Big|\Omega_j - Kr\sin(\theta_j - \phi)\Big|^2. \tag{5.1.8}$$

The main interest of the order parameter is that $r(\Theta)$ represents a measure of coherence for the ensemble of oscillators. Specifically, when $r(\Theta)$ is close to 1, then all the phases θ_i within Θ tend to be synchronized around the same phase value. Moreover, using them we can rewrite system (5.1.1) as follows

$$\dot{\theta}_i = \Omega_i - Kr\sin(\theta_i - \phi),$$

for every $i = 1, \cdots, N$. Without lost of generality we may assume that, the natural frequencies are centered, i.e.,

$$\frac{1}{N}\sum_{i=1}^{N}\Omega_i = 0. \tag{5.1.9}$$

We observe that such a condition is not restrictive because we can always perform a linear change of the reference frame to guarantee it. However, such condition is necessary to show the existence of stationary states and we shall assume it throughout the chapter. For any such a stationary state Θ_∞ so that $r_\infty > 0$, we must have that $\nabla V(\Theta_\infty) = 0$. Using (5.1.8), we readily obtain that at equilibria the following condition holds

$$\max_{1\leq j\leq N}|\Omega_j| \leq Kr_\infty,$$

and phases θ_j must take some of the following two forms

$$\begin{aligned} \theta_{j,\infty} &= \phi_\infty + \arcsin\left(\frac{\Omega_j}{Kr_\infty}\right), \\ \theta_{j,\infty} &= \phi_\infty + \pi - \arcsin\left(\frac{\Omega_j}{Kr_\infty}\right), \end{aligned} \tag{5.1.10}$$

for every $j = 1, \ldots, N$.

In the same spirit, the Hessian operator of the potential V is given by

$$\langle D_N^2 V(\Theta)v, v\rangle_N = \frac{K}{N}\sum_{j=1}^{N} r\cos(\theta_j - \phi)|v_j|^2 - K\left|\frac{1}{N}\sum_{j=1}^{N} v_j e^{i\theta_j}\right|^2, \tag{5.1.11}$$

$D_N^2 V$ denotesthe Hessian operator with respect to the scaled inner product (5.1.5) and $v = (v_1, ...v_N)$ is contained in $\mathbb{R}^N$. From this, after accounting for the rotational invariance of the model, we deduce that the stable equilibrium must satisfy that

$$\theta_{j,\infty} = \phi + \arcsin\left(\frac{\Omega_j}{Kr_\infty}\right),$$

for every $j = 1, \ldots, N$.

Remark 5.1.1. *When $r = 0$ there are plenty more equilibria. In the identical case it can be shown that they are non-isolated even after accounting for rotation invariance.*

For the Kuramoto–Sakaguchi equation, in the case of identical oscillators, the equation enjoys a Wasserstein gradient flow structure (we refer the reader to Appendix A from [154]). In the nonidentical case, this structure is not strictly available. Nonetheless, in our analysis, we use several techniques and objects inspired by theory of gradient flows in the space of probability measures. Similarly, if we consider the continuous version of the order parameters

$$Re^{i\phi} = \int_{\mathbb{T}\times\mathbb{R}} e^{i\theta} f(t,\theta,\Omega)\, d\theta\, d\Omega, \tag{5.1.12}$$

equation (5.1.2) can be restated as follows

$$\begin{cases} \dfrac{\partial f}{\partial t} + \dfrac{\partial}{\partial \theta}(\Omega f - KR\sin(\theta-\phi) f) = 0, & (\theta,\Omega) \in \mathbb{T}\times\mathbb{R},\ t \geq 0, \\ f(0,\theta,\Omega) = f_0(\theta,\Omega), & (\theta,\Omega) \in \mathbb{T}\times\mathbb{R}. \end{cases}$$

Again, without loss of generality, we can assume that g is centered as well, i.e.,

$$\int_{\mathbb{R}} \Omega g(\Omega)\, d\Omega = 0. \tag{5.1.13}$$

Again, this is a necessary condition for equilibria to exist and we shall assume it throughout the chapter. For any such equilibria f_∞ with corresponding $R_\infty > 0$, we obtain that

$$\operatorname{supp} g \subseteq [-KR_\infty, KR_\infty],$$

and f_∞ takes the form

$$f_\infty(\theta,\Omega) = g^+(\Omega) \otimes \delta_{\vartheta^+(\Omega)}(\theta) + g^-(\Omega) \otimes \delta_{\vartheta^-(\Omega)}(\theta), \tag{5.1.14}$$

where,

$$g = g^- + g^+,$$

for some non-negative g^-, and g^+ and

$$\vartheta^+(\Omega) = \phi_\infty + \arcsin\left(\frac{\Omega}{KR_\infty}\right),$$
$$\vartheta^-(\Omega) = \phi_\infty + \pi - \arcsin\left(\frac{\Omega}{KR_\infty}\right),$$

for each $\Omega \in \operatorname{supp} g$. As it will become apparent later along the chapter, the stable equilibria with $R_\infty > 0$ correspond to the case $g^- = 0$ where there is no antipodal mass.

5.1.3 Statement of the problem and main results

By direct inspection of the Hessian of the energy (5.1.11), one can see that, in a large coupling strength regime, out of all of the possible equilibria up to rotations; there is only one that is stable. That is the equilibrium in which the Hessian operator is strictly positive on the subspace orthogonal to rotations. One expects that with probability one, the system (5.1.1) should converge to such equilibria if the coupling strength is sufficiently large. Such phenomenon has been widely observed in numerical simulations. However, to the date, this result is absent from the literature. It has only been verified for restricted initial configurations where all of the oscillators are constrained in an arc of the circle [75].

There have been many approaches in the literature to show the convergence of the system to the critical points of (5.1.11) in the large coupling strength regime. Since stable equilibria have oscillators contained within an interval of size less than π, convergence results have been mainly addressed in the particular case where initial data is originally confined to such a basin of attraction, namely a half-circle. Specifically, in [75, 145] a system of differential inequalities was found for the phase and frequency diameter, that yields the convergence of the system to a phase-locked state. Recall that (5.1.1) is a gradient flow (5.1.6) governed by a potential energy (5.1.4). In [157, 202] the authors derived the convergence to phase-locked states using Łojasiewicz gradient's inequality for analytic potentials [204] and it was used to obtain convergence rates (after some unquantified initial time) in some particular cases where the Łojasiewicz exponent can be explicitly computed. For general initial data along the whole circle, the literature is rare and the main contribution is [147], but rates are not available. One of the main difficulties when trying to use standard theory from dynamical systems to show this is the fact that critical points of (5.1.11) are not isolated (see Remark 5.1.1).

In the continuum case, accumulation of oscillators in the hemisphere opposite of the order parameter was excluded in [154]. However, convergence towards a stationary solution was not established yet for generic initial data. See [58] for a particular proof when the phase diameter is smaller than π. Additionally, see [24] for a description of the equilibrium in the kinetic case, where a conditional convergence result is presented, without rates. To date, regarding generic initial data, there are only arguments based on compactness that do not give any bound on the rate of convergence.

Our goal here is precisely to investigate the long-time relaxations of solutions to the global equilibrium. We are interested in the study of rates of convergence for the Kuramoto–Sakaguchi equation towards the stable equilibria from generic initial data. Additionally, we wish to derive constructive bounds for this convergence and use them to obtain quantitative information about the convergence of the particle system to the global equilibria as well. There are several reasons why one may be interested in explicit bounds on the rate of convergence. In particular, one may look for the qualitative properties of solutions. More importantly, only after getting convergence rates, we can use the dynamics of the kinetic equations to deduce quantitative statistical information about the particle system.

The first thing that one might be tempted to do is to apply linearization techniques around the equilibria. This analysis has been done in [106, 107, 108, 109], and is connected with the methods in Landau Damping. However, there is a fundamental reason not to be content with that analysis, which has to do with the nature of linearization. Quoting L. Desvillettes and C. Villani :

This technique is likely to provide excellent estimates of convergence only after the solution has entered a narrow neighborhood of the equilibrium state, narrow enough that only linear terms are prevailing in the equation. But by nature, it cannot say anything about the time needed to enter such a *neighborhood; the later has to be estimated by techniques which would be well-adapted to the nonlinear equation.*

Here is where our contribution takes places, and this is why we shall not rely on linearization techniques. Instead, we shall stick as close as possible to the physical mechanism of entropy production. Our main result is here:

Theorem 5.1.2. *Let f_0 be contained in $C^1(\mathbb{T}\times\mathbb{R})$ and let g be compactly supported in $[-W, W]$. Consider the unique global-in-time classical solution $f = f(t,\theta,\Omega)$ to* (5.1.2)*. Then, there exists a universal constant C such that if*

$$\frac{W}{K} \leq CR_0^3, \tag{5.1.15}$$

then we can find a time T_0 with the property that

$$T_0 \lesssim \frac{1}{KR_0^2}\log\left(1 + W^{1/2}\|f_0\|_{L^2} + \frac{1}{R_0}\right), \tag{5.1.16}$$

and

$$W_{2,g}(f_t, f_\infty) \lesssim e^{-\frac{1}{40}K(t-T_0)}, \text{ for all } t \text{ in } [T_0,\infty).$$

Here, $W_{2,g}$ represents the fiberwise Wasserstein distance that we introduced in Chapter 4 (see also Appendix F) and f_∞ is the unique global equilibrium of the Kuramoto–Sakaguchi equation up to rotations (see Proposition 5.3.8).

In the above theorem and throughout the rest of the chapter, given two function h_1 and h_2 involving the different parameters in our system, we say that $h_1 \lesssim h_2$ if there exists a universal constant C such that $h_1 \leq Ch_2$. Since our argument is constructive, every time we use such a notation, we could compute C explicitly. Additionally, because we often deal with absolutely continuous measures, by abuse of notation, we will sometimes use f to denote the measure $f\,dx$.

As a direct consequence of our main theorem, we obtain the following quantitative concentration estimate for the particle system.

Corollary 5.1.3. *Let μ_t^N be a sequence of empirical measures associated to solutions of the particle system* (5.1.1) *starting at independent and identically distributed random initial data with law f_0 (see Section 5.6 for further details). Assume that f_0, R_0, K, and W satisfy the hypotheses of Theorem 5.1.2 and let L be an interval with diameter $2/5$ centered around the phase ϕ_∞ of the global equilibrium f_∞. Then, there exists a positive time T_0 satisfying* (5.1.16) *and an integer N^* with the property that*

$$\log N^* \lesssim \frac{1}{R_0^2}\log\left(1 + W^{1/2}\|f_0\|_{L^2} + \frac{1}{R_0}\right),$$

and for any $N \geq N^*$ and any s contained in the interval

$$\left[T_0, T_0 + \frac{1}{25K}\log\left(\frac{N}{N^*}\right)\right],$$

we can quantify the probability of mass concentration and diameter contraction of the particle system with N oscillators. Indeed, we have that

$$\mathbb{P}\Big(\forall\, t \geq s, \exists\, L_s^N(t) \subseteq \mathbb{T}:\ L_s^N(s) = L \text{ and (M)} - \text{(D) holds}\Big) \geq 1 - C_1 e^{-C_2 N^{\frac{1}{2}}}.$$

Here, conditions (M) *and* (D) *yield mass concentration and diameter contraction. More precisely, such properties are given by*

$$\mu_t^N(L_s^N(t)\times\mathbb{R}) \geq 1-\frac{1}{5}e^{-\frac{1}{20}K(s-T_0)}, \qquad \textit{for every } t \textit{ in } [s,\infty), \tag{M}$$

$$\operatorname{diam}(L_s^N(t)) \leq \max\left\{\frac{4}{5}e^{-\frac{K}{20}(t-s)}, 12\frac{W}{K}\right\}, \qquad \textit{for every } t \textit{ in } [s,\infty). \tag{D}$$

Additionally, C_1 and C_2 are universal positive constants which could be explicitly computed.

5.1.4 Ingredients

The proof of Theorem 5.1.2 is the first quantitative proof for the relaxation problem for Kuramoto oscillators with generic initial data. It is intricate but rests on a few well-identified principles. Such principles apply with a lot of generalities to many variants of the Kuramoto model. The proof builds upon the following ingredients.

- A quantitative entropy production estimate inspired by the formal Riemannian calculus of the probability measures under the Wasserstein distance, first introduced by F. Otto in [237], which we address in Subsection 5.2.3 and Section 5.5. See also [154, Appendix A] for an overview in the context of Kuramoto–Sakaguchi with identical oscillators.
- The fiberwise Wasserstein distance $W_{2,g}$ introduced in Appendix F of the thesis and also used in the above Chapter 4. Such a distance is well adapted to the nonlinear problem. By using this distance, in Section 5.3 we will derive new logarithmic Sobolev and Talagrand type inequalities associated with it (see [238]).
- A quantitative instability estimate excluding the equilibria with mass in the opposite pole of the order parameter, that we derive in Section 5.4. A form of such an estimate was originally presented in [154], but we use a more refined version in this work.
- A new estimate on the norms of the solution on sets evolving by the flow of the continuity equation that allows us to propagate information along the different parts of the system. We discuss these estimates in Subsection 5.4.1.

For pedagogical reasons, before entering into the details of the proof, we shall provide first a summary of the strategy. Such a summary will be the objective of the next section.

5.2 Strategy

In this section, we shall describe the plan of the proof of Theorem 5.1.2, and the system of differential inequalities upon which our estimates of convergence are based.

Two of the most attractive features of our proof are the fact that it follows the intuition derived from the mechanism of entropy production, and it is systematic. Additionally, it capitalizes on the behavior observed in numerical simulations under a large coupling strength regime.

We shall overcome three crucial difficulties. First, the order parameter R defined in (5.1.12) is not monotonic and when it vanishes so does the mean-field force between particles. Additionally, our description of the equilibria is only valid when it is positive (this difficulty plays an essential role in the particle system as well). The second difficulty is the fact that Kuramoto–Sakaguchi equation tends to concentrate the density, which produces exponential growth of the global L^p norms for $p > 1$. The third difficulty, related to the second one, is that a large family of equilibria with mass in the opposite hemisphere of the order parameter appears in which the entropy production vanishes.

In the particle system (5.1.1), the potential function V plays the role of the entropy. Consequently, since the particle system is a gradient flow (see (5.1.6)), we have that

$$\frac{d}{dt}V(\Theta(t)) = -|\nabla_N V(\Theta(t))|_N^2.$$

Thus, we can see from this expression that when the particle system slope $|\nabla_N V(\Theta(t))|_N^2$, is large, then the potential function $V(\Theta(t))$ should decrease locally. To quantify the rate of increase of the slope, the starting point is the Hessian operator (5.1.11) of the energy functional for the particle system. Such an expression implies that $D_N^2 V(\Theta(t))$ is bounded from above (as a quadratic form) by $Kr(\Theta(t))$, that is,

$$\langle D_N^2 V(\Theta(t))v, v\rangle_N \leq Kr(\Theta(t))|v|_N^2,$$

for any $(v_1, \ldots, v_N)$ in $\mathbb{R}^N$, which implies the differential inequality

$$-2Kr(\Theta(t))|\nabla_N V(\Theta(t))|_N^2 \leq \frac{d}{dt}|\nabla_N V(\Theta(t))|_N^2 \leq 2K|\nabla_N V(\Theta(t))|^2,$$

along solutions of the Kuramoto model (5.1.1). Notice that by (5.1.7)

$$\frac{V(\Theta(t))}{K} \quad \text{and} \quad 1 - r^2(t),$$

are related up to lower-order terms that can be neglected thanks to condition (5.1.15). Similarly, considering the time derivative of the above quantities, we have that the following two expressions

$$\frac{|\nabla_N V(\Theta(t))|_N^2}{K} \quad \text{and} \quad \frac{dr^2}{dt}(t),$$

should also differ by a lower-order term that, again, can be controlled using (5.1.15). This justifies that, in the large coupling strength regime, we indistinctly call $\frac{dR^2}{dt}$ and $|\nabla_N V(\Theta(t))|_N^2$ the dissipation.

In the continuous case, those objects were extended to the setting of the Kuramoto–Sakaguchi equation (5.1.2) with identical oscillators using the Riemannian structure introduced by F. Otto for the space of probability measures (see [154, Appendix A]). However, in the non-identical case the Kuramoto–Sakaguchi equation (5.1.2) is not a Wasserstein gradient flow and this presents an obstacle to try to use the above objects. By analogy, let us define the continuum analog of the particles' slope (5.1.8) given, by,

$$\mathcal{I}[f] := \int_{\mathbb{T}\times\mathbb{R}} (\Omega - KR\sin(\theta - \phi))^2 \, f \, d\theta \, d\Omega. \tag{5.2.1}$$

We shall again call this quantity the dissipation. Indeed, notice that taking derivatives in (5.1.12), one clearly obtains the following dynamics of the order parameters

$$\begin{aligned} \dot{R} &= -\int_{\mathbb{T}\times\mathbb{R}} \sin(\theta - \phi)(\Omega - KR\sin(\theta - \phi)) f \, d\theta \, d\Omega, \\ \dot{\phi} &= \frac{1}{R}\int_{\mathbb{T}\times\mathbb{R}} \cos(\theta - \phi)(\Omega - KR\sin(\theta - \phi)) f \, d\theta \, d\Omega. \end{aligned} \tag{5.2.2}$$

Using it, we will show, in Lemma 5.3.12, that dissipation and time derivative of the order parameter are again related up to lower-order terms that can be controlled by condition (5.1.15), i.e.,

$$\mathcal{I}[f_t] - W^2 \leq K\frac{d}{dt}(R^2) \leq 3\,\mathcal{I}[f_t] + W^2.$$

Indeed, in Corollary 5.3.2 we show that we can again control the growth of the dissipation in the continuous description in a similar way, namely,

$$-2KR\mathcal{I}[f] \leq \frac{d}{dt}\mathcal{I}[f] \leq 2K\mathcal{I}[f].$$

In Subsection 5.2.3, we will describe how this relationship along with the principle of entropy production, can be used to provide a universal lower bound of $R(t)$ of the form λR_0, for some λ in $(0,1)$. In fact, we will show that by making K sufficiently large we can make λ as close to one as needed.

5.2.1 Displacement concavity and entropy production

Before entering into the details of the entropy production principle, we set some necessary notation. We define a dynamic neighborhood of the order parameter ϕ and its antipode as follows.

Definition 5.2.1. *Given an angle α in $(0, \frac{\pi}{2})$, we denote by $L^+_\alpha(t)$ the interval (arc) in $\mathbb{T}$ that is centered around $\phi(t)$, and has a diameter $\pi - 2\alpha$, that is,*

$$L^+_\alpha(t) = \left(\phi(t) - \frac{\pi}{2} + \alpha, \phi(t) + \frac{\pi}{2} - \alpha\right).$$

Similarly, we denote by $L^-_\alpha(t)$ the interval (arc) in $\mathbb{T}$ of the same diameter that is centered around the antipode $\phi(t) + \pi$, that is,

$$L^-_\alpha(t) = \left(\phi(t) + \frac{\pi}{2} + \alpha, \phi(t) + \frac{3\pi}{2} - \alpha\right).$$

In this way, $L^+_\alpha(t) \cup L^-_\alpha(t)$ is a neighborhood of the average phase and its antipode.

Also, here and throughout the rest of the chapter, given a measurable set $B \subseteq \mathbb{T}$ we define

$$\rho_t(B) = \int_B \rho(t,\theta)\, d\theta,$$

and more generally, we will, let

$$\rho(A(t)) = \int_{A(t)} \rho(t,\theta)\, d\theta,$$

for any time-dependent family of measurable sets $t \to A(t)$.

Now we describe the entropy production principle in our context. Roughly speaking, it will quantify the following fact:

If at some time t the system is far from the family of equilibria with positive order parameter, then the order parameter will increase a lot in the next few instants of time.

To make it rigorous, let us come back to the dissipation functional (5.2.1). As for the particle system (5.1.8), notice that $\mathcal{I}[f]$ vanishes if, and only if, f is an equilibrium. Hence, $\mathcal{I}[f]$ can be thought of a natural measure of how close a given f is to the family of equilibria (5.1.14). Notice that such expression of equilibria (5.1.14) guarantees that, by our assumption (5.1.15) on $\frac{W}{K}$, all the possible equilibria in our analysis have phase support confined to small arcs centered around ϕ and its antipode $\phi+\pi$. Since the diameter of the neighborhood can be made arbitrarily small due to hypothesis (5.1.15), then we can fix any small enough value of α for the size of the neighborhood $L_\alpha^+(t)\cup L_\alpha^-(t)$. For simplicity, we will set $\alpha=\pi/6$ all along the chapter.

The entropy production principle then shows that, in the large coupling strength regime, if entropy production is small (i.e., the time derivative of the order parameter is small), then most of the mass of the system lies in the neighborhood $L_\alpha^+(t)\cup L_\alpha^-(t)$ of $\phi(t)$ and its antipode. Specifically, in the proof of Proposition 5.3.14, we will quantify such assertion as follows

$$\rho(\mathbb{T}\backslash L_\alpha^+(t)\cup L_\alpha^-(t))\leq \frac{1}{KR^2\cos^2\alpha}\frac{d}{dt}R^2+\frac{W^2}{K^2R^2\cos^2\alpha}. \tag{5.2.3}$$

In other words, (5.2.3) suggests that when f is sufficiently far from the family of equilibria (5.1.14) (i.e. it has enough mass outside the time-dependent neighborhood $L_\alpha^+(t)\cup L_\alpha^-(t)$), then the dissipation $\mathcal{I}[f]$ is large. Consequently, the time derivative of the order parameter is large in this case as well, and this produces an entropy production of the system.

In the Lemma below, we quantify the corresponding gain in the order parameter.

Lemma 5.2.2. ***(Semiconcavity and entropy production)*** *Assume that f_0 is contained in $C^1(\mathbb{T}\times\mathbb{R})$ and that g is compactly supported in $[-W,W]$. Consider the unique global-in-time classical solution $f=f(t,\theta,\Omega)$ to* (5.1.2). *Let $\alpha=\pi/6$, t_0 be a positive time, and λ be contained in $(0,1)$. Additionally, suppose that*

$$\sqrt{2}R_0\geq R(t_0)>\lambda R_0\quad \textit{and}\quad \dot{R}(t_0)\geq\frac{K}{4}\cos^2\alpha\lambda^3R_0^3.$$

Then, there exists a universal constant C such that if

$$\frac{W}{K}\leq C\lambda^2R_0^2, \tag{5.2.4}$$

then,

$$R^2(t_0+d)-R^2(t_0)\geq\frac{1}{40}\lambda^4R_0^3. \tag{5.2.5}$$

Moreover, we can select d in such a way that

$$d\leq\frac{1}{3KR_0}\log 10,$$

and

$$R\leq\frac{3}{2}R_0\quad in\quad [t_0,t_0+d].$$

5.2.2 Small dissipation regime and lower bound of R

When the dissipation is large, the above entropy production principle quantifies the gain of the order parameter in the next few instants of time. Regarding the reverse regime with small dissipation, Proposition 5.3.14 in Section 5.3 will show that when $\dot{R}$ is below a critical threshold, we achieve the following differential inequality

$$\frac{d}{dt}R^2>\frac{K}{2}\Big(-R^3+[\lambda R_0+\frac{3}{5}(1-\lambda)R_0]R^2-\frac{3}{5}(1-\lambda)\lambda^2R_0^3\Big), \tag{5.2.6}$$

which hold in any time interval $[t_1, t_2]$ such that

$$\dot{R}(t) \leq \frac{K}{4} \cos^2 \alpha \lambda^3 R_0^3 \quad \text{in} \quad [t_1, t_2].$$

The estimate (5.2.3) will be crucial to derive such a proposition. Additionally, note that the right-hand side of (5.2.6) vanishes when $R = \lambda R_0$. In Corollary 5.3.15, we will combine this inequality with the above entropy production in Lemma 5.2.2 to quantify a universal lower bound $R(t) \geq \lambda R_0$ of the order parameter.

5.2.3 Instability of the antipodal equilibria

The main obstacle to use the above entropy production estimate to show the convergence to the global equilibrium is the fact that it does not exclude the possibility that $\dot{R}$ may vanish or alternate signs over long periods. To overcome such difficulty we need to quantify the instability of the antipodal equilibrium, that roughly speaking states the following:

If the system is eventually close enough to a critical point and such a critical point has mass in the opposite hemisphere of the order parameter, then the system would depart from such equilibria and mass will leave the opposite hemisphere exponentially fast.

To quantify this instability, let us first introduce some necessary notation. We consider a smooth regularization of the characteristic function of $L_\alpha^-(t)$ as follows

$$\chi_{\alpha,\delta_0}^-(\theta) = \xi_{\alpha,\delta_0}(\theta - \phi - \pi),$$

where $\delta_0 > 0$ is a small fixed parameter and ξ_{α,δ_0} is a smooth regularization of the characteristic function of $[-(\frac{\pi}{2} - \alpha), (\frac{\pi}{2} - \alpha)]$, namely,

$$\xi_{\alpha,\delta_0}(r) := \begin{cases} 1, & \text{if } |r| \leq \frac{\pi}{2} - \alpha, \\ \dfrac{1}{1 + \exp\left(\frac{2|r| - (\pi - 2\alpha + \delta_0)}{(\frac{\pi}{2} - \alpha + \delta_0 - |r|)(|r| - \frac{\pi}{2} + \alpha)}\right)}, & \text{if } \frac{\pi}{2} - \alpha \leq |r| \leq \frac{\pi}{2} - \alpha + \delta_0, \\ 0, & \text{if } |r| \geq \frac{\pi}{2} - \alpha + \delta_0. \end{cases} \tag{5.2.7}$$

As for α, we can take δ_0 as small as desired. For notational simplicity we will set

$$\xi_\alpha := \xi_{\alpha,1/2} \quad \text{and} \quad \chi_\alpha^- := \chi_{\alpha,1/2}^-.$$

Additionally, we will use the notation

$$f_t^2(B) = \int_A f^2(t, \theta, \Omega) \, d\theta \, d\Omega,$$

for any measurable set $B \subseteq \mathbb{T}$ and, more generally,

$$f^2(\varphi) = \int \xi(t, \theta, \Omega) f^2 \, d\theta \, d\Omega,$$

for any function $\varphi : \mathbb{R}^+ \times \mathbb{T} \times \mathbb{R} \to \mathbb{R}$. Bearing all the above notation in mind, the main inequality quantifying the instability of equilibria with antipodal mass reads as follows

$$\frac{d}{dt} f^2(\chi_\alpha^-(t)) \leq -KR \sin \alpha f^2(\chi_\alpha^-(t))) + 4K f_t^2(\mathbb{T}) \left[\frac{W}{K} + \sqrt{\frac{2\dot{R}}{KR} + \frac{1}{R^2} \frac{W^2}{K^2}} - R \cos \alpha \right]^+.$$

Although this inequality is a variant of an estimate previously introduced in [154], we prove it in Proposition 5.4.1 because it fits better the approach in this chapter.

Notice that when the system is close enough to an equilibrium so that the dissipation is below a critical threshold, the second term of this inequality vanishes and, indeed, it establishes the instability of equilibria with antipodal mass. However, when one tries to use such inequality to quantify the convergence rates, but the dissipation is not sufficiently small, one sees that the term $f_t^2(\mathbb{T})$ represents an obstacle. Specifically, it stands to reason that one can produce examples in which $f_t^2(\mathbb{T})$ grows exponentially fast because the Kuramoto–Sakaguchi equation concentrates mass. We solve this difficulty by adopting a Lagrangian viewpoint in which we analyze norms of the solution along sets evolving according to the continuity equation. That is the content of the next subsection.

5.2.4 Sliding norms

The key ingredient that allows us to relate the different functionals appearing in our estimates is the notion of sliding norms along the flow of the continuity equation. For this purpose, we let $X(t;t_0,\theta,\Omega) = (\Theta(t;t_0,\theta,\Omega),\Omega)$ be the forward flow associated to the continuity equation (5.1.2) like in Chapter 4. That is

$$\begin{cases} \dfrac{d}{dt}\Theta(t;t_0,\theta,\Omega) = KR(t)\sin(\Theta(t;t_0,\theta,\omega)-\phi), \\ \Theta(t_0;t_0,\theta,\Omega) = \theta, \end{cases}$$

for any $t, t_0 \geq 0$. Throughout this chapter we will use the following shorter notation

$$X_{t_0,t}(\theta,\Omega) \equiv X(t;t_0,\theta,\Omega) \text{ and } \Theta_{t_0,t}(\theta,\Omega) \equiv \Theta(t;t_0,\theta,\Omega),$$

for every $(\theta,\Omega) \in \mathbb{T}\times\mathbb{R}$ and every $t, t_0 \in \mathbb{R}$. Using such notation, for any measurable subset $A \subseteq \mathbb{T}\times\mathbb{R}$ we will denote

$$A_{t_0,t} := X_{t_0,t}(A),$$

for each $t \geq t_0 \geq 0$. Similarly, given a measurable set $B \subseteq \mathbb{T}$, we will denote

$$B_{t_0,t} := \Theta_{t_0,t}(B\times[-W,W]),$$

for each $t \geq t_0 \geq 0$. In particular, notice that $B_{t_0,t}$ is nothing but the projection on $\mathbb{T}$ of the set $(B\times[-W,W])_{t_0,t}$. Finally, if the above subsets depend on time, i.e. $A = A(t)$ and $B = B(t)$, then we will sometimes simplify our notation as follows

$$A(t_0)_t \equiv A(t_0)_{t_0,t} \text{ and } B(t_0)_t \equiv B(t_0)_{t_0,t},$$

for every $t \geq t_0 \geq 0$. Now, we are a position to state our sliding norm estimate which is given by

$$\frac{d}{dt}f^2(A_{t_0,t}) \leq KR\Big(\sup_{(\theta,\Omega)\in A_{t_0,t}} \cos(\theta-\phi(t))\Big)f^2(A_{t_0,t}),$$

and holds for any measurable set $A \subseteq \mathbb{T}\times\mathbb{R}$. We prove such inequality in Lemma 5.4.2. To use this inequality effectively, one must obtain a control on the dynamics of sets evolving according to the characteristic flow, both in the large and small dissipation regime. We perform this analysis in Subsection 5.4.1.

5.2.5 The system of differential inequalities

All the above-mentioned bounds lead to a system of coupled differential inequalities and functional inequalities. For convenience, let us recast it explicitly here:

$$\frac{d}{dt}f^2(A_{t_0,t}) \leq KR\Bigg(\sup_{(\theta,\Omega)\in A_{t_0,t}} \cos(\theta-\phi(t))\Bigg)f^2(A_{t_0,t}), \tag{5.2.8}$$

$$-2KR\mathcal{I}[f] \leq \frac{d}{dt}\mathcal{I}[f] \leq 2K\mathcal{I}[f], \tag{5.2.9}$$

$$\mathcal{I}[f_t]-W^2 \leq K\frac{d}{dt}R^2 \leq 3\mathcal{I}[f_t]+W^2, \tag{5.2.10}$$

$$\frac{d}{dt}f^2(\chi_\alpha^-(t)) \leq -KR\sin\alpha f^2(\chi_\alpha^-(t))) + 4Kf_t^2(\mathbb{T})\left[\frac{W}{K}+\sqrt{\frac{2\dot{R}}{KR}+\frac{1}{R^2}\frac{W^2}{K^2}}-R\cos\alpha\right]^+, \tag{5.2.11}$$

$$\frac{d}{dt}R^2 > K\Bigg(-R^3+[\lambda R_0+\frac{3}{5}(1-\lambda)R_0]R^2-\frac{3}{5}(1-\lambda)\lambda^2R_0^3\Bigg), \tag{5.2.12}$$

where the first inequality holds for any measurable set $A\subseteq\mathbb{T}\times\mathbb{R}$, the last inequality holds in any interval $[t_1,t_2]$ satisfying the hypotheses of Proposition 5.3.14, and all of the other inequalities above holds for every t in $[0,\infty)$.

The goal of such a system is to derive an explicit bound on the time T_0 in Theorem 5.1.2. To achieve this, we use two main components. On the one hand, we study the dynamics of sets along the characteristic flow in Subsection 5.4.1. On the other hand, we recover the approach developed by L. Desvillettes and C. Villani in [104] in our setting. Such an argument is described in detail in Section 5.5 and it consists of performing a subdivision into time intervals subordinated to different scales of values of the order parameter. Such intervals are classified into intervals where the dissipation is above and below a certain threshold. If the dissipation is large on an interval, we use the lower bound (5.2.9) in the form of our entropy production estimate to quantify the increase of the order parameter. Conversely, if the dissipation is small, we use (5.2.11) to quantify the departure of the system from the family of equilibria with antipodal mass. To do this effectively, we communicate information between the different regimes using inequality (5.2.8) and our analysis on the dynamics of sets from Subsection 5.4.1.

As a result of the above analysis, we obtain the following corollary:

Corollary 5.2.3. *Let f_0 be contained in $C^1(\mathbb{T}\times\mathbb{R})$ and let g be compactly supported in $[-W,W]$. Consider the unique global-in-time classical solution $f=f(t,\theta,\Omega)$ to* (5.1.2) *and let $\beta=\pi/3$. Then, there exists a universal constant C such that if*

$$\frac{W}{K}\leq CR_0^3,$$

then we can find a time T_0 with the property that

$$T_0 \lesssim \frac{1}{KR_0^2}\log\left(1+W^{1/2}\|f_0\|_{L^2}+\frac{1}{R_0}\right),$$

and

$$R(t)\geq\frac{3}{5} \quad \textit{and} \quad \rho\big(\mathbb{T}\backslash L_\beta^+(t)\big)\leq e^{-\frac{1}{20}K(t-T_0)}, \tag{5.2.13}$$

for every t in $[T_0,\infty)$.

Such a Corollary is the starting point of the last part of our strategy.

5.2.6 Local displacement convexity and Talagrand type inequalities

At the particle level, we see that the Hessian operator (5.1.11) is positive definite in the subspace orthogonal to rotation whenever the oscillators are strictly contained one a suitable interval. As mentioned in Section 5.1, the classical theory of gradient flows allows deriving convergence rates towards equilibrium when the energy is strictly convex. Thus, once the mass enters exponentially fast to the region of convexity after T_0, one may hope to recover such a convergence result for our system. Indeed, inspired by the arguments in [238] on their proof of the logarithmic Sobolev and Talagrand inequalities, we derive analogous inequalities that yield the exponential convergence result and uniqueness of the global equilibrium. Since our system is not a Wasserstein gradient flow, we derive such inequalities for the fiberwise transportation distance $W_{2,g}$, that has been introduced in Chapter 4 (also see Appendix F) and is well adapted to the nonlinear problem. The proof of such inequalities is the content of the next section.

5.3 Functional inequalities

As discussed before, the proof of Theorem 5.1.2 will be split into two distinguished parts that capture two qualitatively different features of the dynamics of Kuramoto–Sakaguchi equation (5.1.2). Firstly, recall that from many preceding works (see e.g., [24, 58, 154]) it is apparent that the entropy functional of the equation does not satisfy the necessary convexity properties for the classical theory of gradient flows to work and show convergence towards the global equilibrium. Thus, we need to prove, using different tools, that the dynamics of the equation itself drives the system towards an appropriate "convexity area" exponentially fast after some quantified time $T_0 > 0$. This is the content of Corollary 5.2.3 where such a convexity area is described by a dynamic neighborhood of the order parameter ϕ.

The proof of such result is postponed to forthcoming sections and becomes the cornerstone of this chapter. We devote this part to study the other main feature of the dynamics. Specifically, we show that although the system is not a Wasserstein gradient flow, the generalized dissipation functional that has been introduced in (5.2.1) satisfies an appropriate Hessian-type inequality after the solution has entered into the concentration regime quantified in Corollary 5.2.3. The final step is inspired in [238] about the derivation of the logarithmic Sobolev and Talagrand inequalities for gradient flows in then Wasserstein space. Indeed, we shall show that despite the fact that our system is not a Wasserstein gradient flow due to the presence of heterogeneities introduced by Ω, some dissipation-transportation inequality still can be achieved for an adequate distance on the space of probability measures. Such inequality along with the exponential decay of the dissipation guarantee the exponential convergence to the global equilibrium in Theorem 5.1.2.

To start, we first study the dynamics of the dissipation functional (5.2.1) along the flow of the Kuramoto–Sakaguchi equation.

Theorem 5.3.1. *Assume that f_0 is contained in $C^1(\mathbb{T}\times\mathbb{R})$ and g is compactly supported in $[-W, W]$. Consider the unique global-in-time classical solution $f = f(t,\theta,\Omega)$ to* (5.1.2). *Then,*

$$\frac{d}{dt}\mathcal{I}[f] = -K\int_{\mathbb{T}^2\times\mathbb{R}^2} \big((\Omega - KR\sin(\theta-\phi)) - (\Omega' - KR\sin(\theta'-\phi))\big)^2 \times \cos(\theta-\theta')f(t,\theta,\Omega)f(t,\theta',\Omega')\,d\theta\,d\theta'\,d\Omega\,d\Omega'.$$

Proof. Taking derivatives yields the Wasserstein two terms

$$\frac{d}{dt}\mathcal{I}[f] = I_1 + I_2,$$

where each of them takes the form

$$I_1 := 2\int_{\mathbb{T}\times\mathbb{R}} (\Omega - KR\sin(\theta-\phi))(-K\dot{R}\sin(\theta-\phi) + KR\cos(\theta-\phi)\dot{\phi})f\,d\theta\,d\Omega,$$
$$I_2 := \int_{\mathbb{T}\times\mathbb{R}} (\Omega - KR\sin(\theta-\phi))^2\partial_t f\,d\theta\,d\Omega.$$

Let us use (5.2.2) and substitute the formulas for $\dot{R}$ and $\dot{\phi}$ in each term. By doing this, we get that

$$\begin{aligned} I_1 &= 2K\int_{\mathbb{T}^2\times\mathbb{R}^2} (\Omega - KR\sin(\theta-\phi))(\Omega' - KR\sin(\theta'-\phi)) \\ &\quad\times(\sin(\theta-\phi)\sin(\theta'-\phi) - \cos(\theta-\phi)\cos(\theta'-\phi))f(t,\theta,\Omega)f(t,\theta',\Omega')\,d\theta\,d\theta'\,d\Omega\,d\Omega' \\ &= 2K\int_{\mathbb{T}^2\times\mathbb{R}^2} (\Omega - KR\sin(\theta-\phi))(\Omega' - KR\sin(\theta'-\phi)) \\ &\qquad\times\cos(\theta-\theta')f(t,\theta,\Omega)f(t,\theta',\Omega')\,d\theta\,d\theta'\,d\Omega\,d\Omega', \end{aligned} \tag{5.3.1}$$

and

$$\begin{aligned} I_2 &= \int_{\mathbb{T}\times\mathbb{R}} \partial_\theta\left[(\Omega - KR\sin(\theta-\phi))^2\right](\Omega - KR\sin(\theta-\phi))f\,d\theta\,d\Omega \\ &= -2K\int_{\mathbb{T}\times\mathbb{R}} (\Omega - KR\sin(\theta-\phi))^2 R\cos(\theta-\phi)f\,d\theta\,d\Omega, \end{aligned}$$

where we have used the Kuramoto–Sakaguchi equation (5.1.2) and integration by parts. Notice that by definition of the order parameter (5.1.12), we obtain

$$R\cos(\theta-\phi) = \int_{\mathbb{T}\times\mathbb{R}} \cos(\theta-\theta')f(t,\theta',\Omega')\,d\theta'\,d\Omega'. \tag{5.3.2}$$

Using such identity in the above formula for I_2 implies

$$I_2 = -2K\int_{\mathbb{T}^2\times\mathbb{R}^2} (\Omega - KR\sin(\theta-\phi))^2\cos(\theta-\theta')f(t,\theta,\Omega)f(t,\theta',\Omega')\,d\theta\,d\theta'\,d\Omega\,d\Omega'. \tag{5.3.3}$$

Let us now change variables (θ,Ω) with (θ',Ω') in (5.3.3) and take the mean value of both expressions for I_2. Since the cosine is an even function, we equivalently write

$$\begin{aligned} I_2 = -K\int_{\mathbb{T}^2\times\mathbb{R}^2} &((\Omega - KR\sin(\theta-\phi))^2 + (\Omega' - KR\sin(\theta'-\phi))^2) \\ &\times\cos(\theta-\theta')f(t,\theta,\Omega)f(t,\theta',\Omega')\,d\theta\,d\theta'\,d\Omega\,d\Omega'. \end{aligned} \tag{5.3.4}$$

Finally, putting (5.3.1) and (5.3.4) together and completing the square yield the desired result. □

As a consequence of the previous theorem, we obtain the following quantitative behavior of the dissipation.

Corollary 5.3.2. *Assume that f_0 is contained in $C^1(\mathbb{T}\times\mathbb{R})$ and g is compactly supported in $[-W,W]$. Consider the unique global-in-time classical solution $f = f(t,\theta,\Omega)$ to* (5.1.2). *Then,*

$$-2KR\mathcal{I}[f] \leq \frac{d}{dt}\mathcal{I}[f] \leq 2K\mathcal{I}[f], \tag{5.3.5}$$

for all $t \geq 0$. *In particular,*

$$\mathcal{I}[f](t_0)e^{-2K\int_{t_0}^t R(s)\,ds} \leq \mathcal{I}[f](t) \leq \mathcal{I}[f](t_0)e^{2K(t-t_0)},$$

for all $t \geq t_0 \geq 0$.

Proof. Note that the second chain of inequalities follows from by integration on (5.3.5) with respect to time. Then, we focus on the proof of (5.3.5), that we divide in two steps associated with the upper and lower bound respectively.

• *Step 1*: Upper bound.
Using Theorem 5.3.1 and bounding $\cos(\theta-\theta')$ by 1, we achieve the following upper bound for the derivative of the dissipation functional along f:

$$\begin{aligned}\frac{d}{dt}\mathcal{I}[f] \leq &\int_{\mathbb{T}^2\times\mathbb{R}^2}\big((\Omega - KR\sin(\theta-\phi)) - (\Omega' - KR\sin(\theta'-\phi))\big)\big)^2 f(t,\theta,\Omega)f(t,\theta',\Omega')\,d\theta\,d\theta'\,d\Omega\,d\Omega' \\ &= 2K\int_{\mathbb{T}\times\mathbb{R}}(\Omega - KR\sin(\theta-\phi))^2 f\,d\theta\,d\Omega - 2K\left(\int_{\mathbb{T}\times\mathbb{R}}(\Omega - KR\sin(\theta-\phi))f\,d\theta\,d\Omega\right)^2.\end{aligned}$$

Using the definition (5.1.12) of R and ϕ along with the assumption (5.1.13), we clearly obtain that the second term vanishes and we conclude the upper bound.

• *Step 2*: Lower bound.
Again, we shall use Theorem 5.3.1 and expand the square to obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{I}[f] = &-2K\int_{\mathbb{T}^2\times\mathbb{R}^2}(\Omega - KR\sin(\theta-\phi))^2\cos(\theta-\theta')f(t,\theta,\Omega)f(t,\theta',\Omega')\,d\theta\,d\theta'\,d\Omega\,d\Omega' \\ &+ 2K\int_{\mathbb{T}^2\times\mathbb{R}^2}(\Omega - KR\sin(\theta-\phi))(\Omega' - KR\sin(\theta'-\phi))\cos(\theta-\theta') \\ &\qquad\qquad \times f(t,\theta,\Omega)f(t,\theta',\Omega')\,d\theta\,d\theta'\,d\Omega\,d\Omega' \\ &= -2KR\int_{\mathbb{T}\times\mathbb{R}}(\Omega - KR\sin(\theta-\phi))^2\cos(\theta-\phi)f\,d\theta\,d\Omega \\ &\quad + 2K\left|\int_{\mathbb{T}\times\mathbb{R}}(\Omega - KR\sin(\theta-\phi))e^{i(\theta-\phi)}f\,d\theta\,d\Omega\right|^2 \\ &\geq -2KR\int_{\mathbb{T}\times\mathbb{R}}(\Omega - KR\sin(\theta-\phi))^2 f\,d\theta\,d\Omega,\end{aligned}$$

where in the second identity we have used (5.3.2) while in the last inequality we have bounded $\cos(\theta-\theta')$ by 1 and we have neglected the non-negative term. Hence, the desired result follows. □

5.3.1 The fiberwise Wasserstein distance and relation to dissipation

The fiberwise Wasserstein distance $W_{2,g}$ was presented in Chapter 4 (see also Appendix F) through a gluing procedure of the standard quadratic Wasserstein distance in $\mathbb{T}$ between conditional probabilities at any fiber $\Omega \in \mathbb{R}$. Since it will be used throughout this section, we refer the reader to that Appendix. In particular, we call the attention of its definition in terms of conditional probabilities in Proposition F.4.2. Also, we recall its Benamou–Brenier representation in Proposition F.4.3 and its relation with the standard quadratic Wasserstein distance on the product $\mathbb{T} \times \mathbb{R}$ in Proposition F.4.4. Indeed, according to such last result, the fiberwise Wasserstein distance dominates the quadratic Wassetstein distance. Before we move to the heart of the matter, let us remark the following essential fact.

Remark 5.3.3. *The classical quadratic Wasserstein distance W_2 in $\mathcal{P}_2(\mathbb{T} \times \mathbb{R})$ is defined as the transportation cost associated with the standard Riemannian distance in the product space $\mathbb{T} \times \mathbb{R}$. That is,*

$$W_2(\mu^1, \mu^2) = \left(\inf_{\gamma \in \Pi(\mu^1, \mu^2)} \int_{\mathbb{T}^2 \times \mathbb{R}^2} (d(\theta, \theta')^2 + (\Omega - \Omega')^2) \, d\gamma \right)^{1/2},$$

for any $\mu^1, \mu^2 \in \mathcal{P}_2(\mathbb{T} \times \mathbb{R})$, where $d(\theta, \theta')$ denotes the canonical Riemannian distance between any θ and θ' in $\mathbb{T}$. For our purposes, such a distance is not appropriate as it is not dimensionally correct. Indeed, θ and Ω have different physical units and considering W_2 causes problems to derive asymptotic behavior of solutions in a large coupling strength regime.

Notice that W_2 was used though in Theorems of Chapter 4. However, this is not a problem because we only used them to derive the mean field limit of the particle system towards the kinetic equation. By doing so, we only move N and K is regarded as a fixed coefficient that plays no real role.

The above remark suggests considering the following correction of the classical quadratic Wasserstein distance in $\mathcal{P}_2(\mathbb{T} \times \mathbb{R})$.

Definition 5.3.4 (Scaled quadratic Wasserstein distance). *Let us consider the scaled Riemannian distance on the product space $\mathbb{T} \times \mathbb{R}$, i.e.,*

$$d_K((\theta, \Omega), (\theta', \Omega')) = \left(d(\theta, \theta')^2 + \frac{(\Omega - \Omega')^2}{K^2} \right)^{\frac{1}{2}}.$$

We define the scaled quadratic Wasserstein distance on $\mathcal{P}_2(\mathbb{T} \times \mathbb{R})$ by the transportation costs associated with the above scaled Riemannian distance, that is,

$$SW_2(\mu_0^1, \mu^2) = \left(\inf_{\gamma \in \Pi(\mu^1, \mu^2)} \int_{\mathbb{T}^2 \times \mathbb{R}^2} \left(d(\theta, \theta')^2 + \frac{(\Omega - \Omega')^2}{K^2} \right) d\gamma \right)^{1/2},$$

for any $\mu, \nu \in \mathcal{P}_2(\mathbb{T} \times \mathbb{R})$.

Indeed, a completely analogue argument to Proposition F.4.4 provides the following relation.

Proposition 5.3.5. *Consider $g \in \mathcal{P}_2(\mathbb{T})$. Then we obtain*

$$SW_2(\mu^1, \mu^2) \leq W_{2,g}(\mu^1, \mu^2),$$

for any $\mu^1, \mu^2 \in \mathcal{P}_g(\mathbb{T} \times \mathbb{R})$.

We are now ready to state the main relation between this fiberwise transportation distance (F.4.2) and the dissipation functional (5.2.1).

Lemma 5.3.6. *Assume that f_0 is contained in $C^1(\mathbb{T} \times \mathbb{R})$ and g is compactly supported in $[-W, W]$. Consider the unique global-in-time classical solution $f = f(t, \theta, \Omega)$ to (5.1.2). Then,*

$$\frac{d}{ds} \frac{1}{2} W_{2,g}(f_t, f_s)^2 \leq \mathcal{I}[f]^{\frac{1}{2}} W_{2,g}(f_t, f_s),$$

for every $t \geq 0$ and almost every $s \geq 0$.

Proof of Lemma 5.3.6. Since f satisfies the Kuramoto–Sakaguchi equation (5.1.2), then each conditional probability with respect to $\Omega \in \mathbb{T}$ verifies the following continuity equation

$$\frac{\partial}{\partial t} f(\theta|\Omega) + \operatorname{div}_\theta((\Omega - KR\sin(\theta-\phi))e^{i\theta} f(\theta|\Omega)) = 0,$$

for all $t \geq 0$ and $\theta \in \mathbb{T}$. That is, the disintegrations themselves are driven by the following tangent transport field

$$\theta \in \mathbb{T} \longmapsto v_t^\Omega(\theta) := (\Omega - KR\sin(\theta-\phi))e^{i\theta}.$$

Since f is smooth, it is clear that the family $s \in [0,+\infty) \longmapsto f_s(\cdot|\Omega)$ is locally absolutely continuous with respect to the quadratic Wasserstein distance on $\mathbb{T}$. This clearly guarantees that the following function is also locally absolutely continuous

$$s \in [0,+\infty) \longrightarrow W_2(f_t(\cdot|\Omega), f_s(\cdot|\Omega))^2,$$

for every $\Omega \in \operatorname{supp} g$, see [9, Theorem 8.4.6] or [296, Theorem 23.9]. In particular, we can take derivatives almost everywhere and obtain the formula

$$\frac{d}{ds}\frac{1}{2} W_2(f_t(\cdot|\Omega), f_s(\cdot|\Omega))^2 = -\int_{\mathbb{T}} \left\langle v_s^\Omega(\theta), \nabla\psi^{s,t}_{\tau=0}(\theta,\Omega)\right\rangle f_s(\theta|\Omega)\, d\theta, \tag{5.3.6}$$

for almost every $t \geq 0$, where the family $\tau \in [0,1] \longmapsto (h^{s,t}_\tau, \psi^{s,t}_\tau)$ has been chose according to (F.4.3) in Proposition F.4.3 so that it represents a Wasserstein geodesic joining the conditional probabilities of f_s to those of f_t. By the dominated convergence theorem, we can then show that the following function is also absolutely continuous

$$s \in [0,+\infty) \longrightarrow W_{2,g}(f_t, f_s)^2.$$

Integrating by parts and using (5.3.6) we obtain that

$$\begin{aligned}\frac{d}{ds}\frac{1}{2} W_{2,g}(f_t,f_s)^2 &= -\int_{\mathbb{T}\times\mathbb{R}} \left\langle v_s^\Omega(\theta), \nabla\psi^{s,t}_{\tau=0}(\theta,\Omega)\right\rangle f_s(\theta|\Omega) g(\Omega)\, d\theta\, d\Omega \\ &= -\int_{\mathbb{T}\times\mathbb{R}} \left\langle v_s^\Omega(\theta), \nabla\psi^{s,t}_{\tau=0}(\theta,\Omega)\right\rangle f_s(\theta,\Omega)\, d\theta\, d\Omega.\end{aligned} \tag{5.3.7}$$

Using the Cauchy–Schwarz inequality in (5.3.7) along with the definition of the dissipation function (5.2.1) and the representation of the fiberwise quadratic Wasserstein distance in Proposition F.4.3 of Appendix F we obtain that

$$\frac{d}{ds}\frac{1}{2} W_{2,g}(f_t,f_s)^2 \leq \mathcal{I}[f]^{\frac{1}{2}} W_{2,g}(f_t,f_s),$$

for almost every $s \geq 0$. Hence, the desired result follows. □

As a direct consequence of the above Lemma, we obtain the following dissipation-transportation inequality.

Corollary 5.3.7. *Assume that f_0 is contained in $C^1(\mathbb{T}\times\mathbb{R})$ and g is compactly supported in $[-W, W]$. Consider the unique global-in-time classical solution $f = f(t,\theta,\Omega)$ to* (5.1.2)*. Then,*

$$W_{2,g}(f_t,f_s) \leq \int_t^s \mathcal{I}[f_\tau]^{1/2}\, d\tau, \quad \textit{for all} \ \ s \geq t.$$

5.3.2 Convergence and uniqueness of the global equilibria

In this section, we shall show the claimed result about convergence to the global equilibria. Before we proceed with the proof, let us first show that such equilibrium is unique up to phase rotations. That result is not new and was first proved in [58] via a strict contractivity estimate in such region of convexity for an appropriate Wasserstein distance $\widetilde{W}_p$ in $\mathcal{P}_2([0,2\pi)\times\mathbb{R})$. Notice that the geometry of $\mathbb{T}$ has been disregarded in $\widetilde{W}_p$. Indeed, the distance $\widetilde{W}_2$ is strictly larger $W_{2,g}$ because the geometry of the $\mathbb{T}$ reduces the transportation cost of mass between phases separated by distances larger that π (when viewed in the real line). We show that the uniqueness result is also true using this new fiberwise distance and we leave the full study of similar strict contractivity of $W_{2,g}$ to future works.

Proposition 5.3.8. *Let f_∞ and f'_∞ be stationary measure-valued solutions to* (5.1.2) *and assume that they have the same distribution g of natural frequencies and,* $\operatorname{diam}(\operatorname{supp}_\theta f_\infty)$ *and* $\operatorname{diam}(\operatorname{supp}_\theta f'_\infty)$ *are less than $\pi/2$. Then, they agree up to phase rotations, that is, there exists a constant $c\in\mathbb{R}$ such that*

$$f'_\infty(\theta,\Omega)=f_\infty(\theta-c,\Omega).$$

Proof. For any $c\in\mathbb{R}$ we consider the rotation operator in the variable θ

$$\mathcal{T}_c[f'_\infty](\theta,\Omega):=f'_\infty(\theta-c,\Omega),$$

and define the following optimization problem

$$\min_{c\in\mathbb{R}} W_{2,g}(f_\infty,\mathcal{T}_c[f'_\infty])^2. \tag{5.3.8}$$

Such minimum of (5.3.8) exists from straightforward arguments and will be achieved at some $c=c_0\in\mathbb{R}$. Without loss of generality, let us assume that $c_0=0$. Indeed, otherwise we can replace f'_∞ with $\mathcal{T}_{c_0}[f'_\infty]$ and it does not change thesis of this result. On the one hand, let us consider the following continuity equation

$$\begin{cases} \frac{\partial}{\partial s}f'_s+\operatorname{div}_\theta(e^{i\theta}f'_s)=0,\\ f'_{s=0}=f'_\infty,\end{cases} \tag{5.3.9}$$

whose solution clearly describes the above family of phase shifts, namely, $f'_s=\mathcal{T}_s[f'_\infty]$. Since $W_{2,g}(f_\infty,f'_\infty)$ minimizes the problem (5.3.8), then we obtain a critical value at $c=0$, i.e.,

$$\left.\frac{d}{ds}\right|_{s=0} W_{2,g}(f_\infty,f'_s)^2=0. \tag{5.3.10}$$

Let us write down condition (5.3.10) more explicitly. Indeed, consider a Wasserstein geodesic that joins the conditional probability $f'_\infty(\cdot|\Omega)$ to $f'_s(\cdot|\Omega)$ and represent it through a family

$$\tau\in[0,T]\longrightarrow(h^s_\tau,\psi^s_\tau)\ \text{ with }\ \begin{array}{c} h^s_{\tau=0}(\cdot|\Omega)=f'_\infty(\cdot|\Omega),\\ h^s_{\tau=1}(\cdot|\Omega)=f'_s(\cdot|\Omega),\end{array} \tag{5.3.11}$$

as in (F.4.3) in Proposition F.4.3 of Appendix F. Here, notice that (F.4.3) only holds in the viscosity/distributional sense due to the fact that equilibria have atoms. In particular, we cannot guarantee that ψ_τ has second order derivatives (that we require in the sequel). Such fact can be handled by nowadays standard regularization arguments. In particular, notice that the dissipation functional $\mathcal{I}[f]$ is continuous with respect to $W_{2,g}$, which makes it well behaved with respect to regularizations.

Now observe that, by construction $f_s'(\cdot|\Omega)$, verifies the continuity equation (5.3.9) that is driven by the trivial tangent transport field $\theta \in \mathbb{T} \longrightarrow e^{i\theta}$. Then, the same ideas in the proof of Lemma 5.3.6 (see [9, Theorem 8.4.6] or [296, Theorem 23.9]), we obtain

$$\frac{d}{ds}\bigg|_{s=0} \frac{1}{2} W_2(f_\infty(\cdot|\Omega), f_s'(\cdot|\Omega))^2 = \int_{\mathbb{T}} \left\langle e^{i\theta}, \nabla_\theta \psi_{\tau=1}^{s=0}(\theta,\Omega)\right\rangle \, d_\theta f_\infty'(\cdot|\Omega).$$

for almost every $s \geq 0$. Taking integrals in Ω against g and using (5.3.10) we obtain

$$\int_{\mathbb{T}\times\mathbb{R}} \left\langle e^{i\theta}, \nabla_\theta \psi_{\tau=1}^{s=0}\right\rangle \, d_{(\theta,\Omega)} f_\infty' = 0.$$

Indeed, using the equations for $h_\tau^{s=0}$ and $\varphi_\tau^{s=0}$ in (F.4.3), it is clear that the above implies

$$\int_{\mathbb{T}\times\mathbb{R}} \left\langle e^{i\theta}, \nabla_\theta \psi_{\tau}^{s=0}\right\rangle \, d_{(\theta,\Omega)} h_\tau^{s=0} = 0, \tag{5.3.12}$$

for every $\tau \in [0,1]$. On the other hand, by hypothesis f_∞ and f_∞' verify the (stationary) Kuramoto–Sakaguchi equation (5.1.2), that is,

$$\begin{aligned}
\frac{\partial}{\partial t} f_\infty + \operatorname{div}_\theta((\Omega - KR_\infty \sin(\theta - \phi_\infty))e^{i\theta} f_\infty) = 0,\\
\frac{\partial}{\partial t} f_\infty' + \operatorname{div}_\theta((\Omega - KR_\infty' \sin(\theta - \phi_\infty'))e^{i\theta} f_\infty') = 0.
\end{aligned}$$

Since the solutions are stationary, then we can again use the same ideas as before to arrive at the identity

$$\begin{aligned}
0 = \frac{d}{dt}\frac{1}{2} W_2(f_\infty(\cdot|\Omega), f_\infty'(\cdot|\Omega))^2 = &\int_{\mathbb{T}} \left\langle (\Omega - KR_\infty' \sin(\theta - \phi_\infty'))e^{i\theta}, \nabla_\theta \psi_{\tau=1}^{s=0}(\cdot,\Omega)\right\rangle \, d_\theta f_\infty'(\cdot|\Omega)\\
&- \int_{\mathbb{T}} \left\langle (\Omega - KR_\infty \sin(\theta - \phi_\infty))e^{i\theta}, \nabla_\theta \psi_{\tau=0}^{s=0}(\cdot,\Omega)\right\rangle \, d_\theta f_\infty(\cdot|\Omega),
\end{aligned}$$

Here on we shall omit the superscripts $s = 0$ of $h_\tau^{s=0}$ and $\psi_\tau^{s=0}$ for simplicity, as it is clear from the context. Then, integrating against g and using the fundamental theorem of calculus in τ yields

$$\int_0^1 \frac{d}{d\tau} \int_{\mathbb{T}\times\mathbb{R}} \left\langle (\Omega - KR_\tau \sin(\theta - \phi_\tau))e^{i\theta}, \nabla_\theta \varphi_\tau \right\rangle \, d_{(\theta,\Omega)} h_\tau \, d\tau = 0, \tag{5.3.13}$$

where R_τ and ϕ_τ are order parameters associated with the displacement interpolation h_τ. Let us now expand the derivative in (5.3.13) and use the Hamilton–Jacobi equation for ψ_τ and the continuity equation for h_τ in (F.4.3). Then we obtain that

$$A + B + C = 0,$$

where each term reads

$$\begin{aligned}
A &:= \int_0^1 \int_{\mathbb{T}\times\mathbb{R}} \left\langle \nabla_\theta \left(-\frac{1}{2}|\nabla_\theta \psi_\tau|^2\right), (\Omega - KR_\tau \sin(\theta - \phi_\tau)e^{i\theta} \right\rangle \, d_{(\theta,\Omega)} h_\tau \, d\tau,\\
B &:= \int_0^1 \int_{\mathbb{T}\times\mathbb{R}} \left\langle \frac{d}{d\tau}\left[\Omega - KR_\tau \sin(\theta - \phi_\tau)\right] e^{i\theta}, \nabla_\theta \psi_\tau \right\rangle \, d_{(\theta,\Omega)} h_\tau \, d\tau,\\
C &:= \int_0^1 \int_{\mathbb{T}\times\mathbb{R}} \left\langle \nabla_\theta \left\langle \nabla_\theta \psi_\tau, (\Omega - KR_\tau \sin(\theta - \phi_\tau))e^{i\theta}\right\rangle, \nabla_\theta \psi_\tau \right\rangle \, d_{(\theta,\Omega)} h_\tau \, d\tau.
\end{aligned}$$

On the one hand, taking the sum of A and C we can simplify into

$$\begin{aligned}A+C &= -K\int_0^1\int_{\mathbb{T}\times\mathbb{R}} R_\tau\cos(\theta-\phi_\tau)\,|\nabla_\theta\psi_\tau|^2\,d_{(\theta,\Omega)}h_\tau\,d\tau\\ &= -K\int_0^1\int_{\mathbb{T}\times\mathbb{R}}\int_{\mathbb{T}\times\mathbb{R}}\cos(\theta-\theta')\,|\nabla_\theta\psi_\tau|^2\,d_{(\theta,\Omega)}h_\tau\,d_{(\theta',\Omega')}h_\tau\,d\tau\\ &= -\frac{K}{2}\int_0^1\int_{\mathbb{T}\times\mathbb{R}}\int_{\mathbb{T}\times\mathbb{R}}\cos(\theta-\theta')\left(|\nabla_\theta\psi_\tau(\theta,\Omega)|^2+\left|\nabla_\theta\psi_\tau(\theta',\Omega')\right|^2\right)d_{(\theta,\Omega)}h_\tau\,d_{(\theta',\Omega')}h_\tau\,d\tau,\end{aligned} \tag{5.3.14}$$

where in the second line we have used the properties of the order parameters R_τ and ϕ_τ of the interpolation h_τ, namely

$$\begin{aligned}R_\tau &= \int_{\mathbb{T}\times\mathbb{R}}\cos(\theta'-\phi_\tau)\,d_{(\theta',\Omega')}h_\tau,\\ 0 &= \int_{\mathbb{T}\times\mathbb{R}}\sin(\theta'-\phi_\tau)\,d_{(\theta',\Omega')}h_\tau.\end{aligned}$$

and in the third line we have used a clear symmetrization argument. Let us now differentiate with respect to τ and use the continuity equation for h_τ to obtain the formulas

$$\begin{aligned}\frac{dR_\tau}{d\tau} &= -\int_{\mathbb{T}\times\mathbb{R}}\sin(\theta'-\phi_\tau)\left\langle e^{i\theta'},\nabla_\theta\psi_\tau(\theta',\Omega')\right\rangle d_{(\theta',\Omega')}h_\tau,\\ R_\tau\frac{d\phi_\tau}{d\tau} &= \int_{\mathbb{T}\times\mathbb{R}}\cos(\theta'-\phi_\tau)\left\langle e^{i\theta'},\nabla_\theta\psi_\tau(\theta',\Omega')\right\rangle d_{(\theta',\Omega')}h_\tau.\end{aligned}$$

Then, the term B can be written as follows

$$\begin{aligned}B &= \int_0^1\int_{\mathbb{T}\times\mathbb{R}}\left\langle e^{i\theta},\nabla_\theta\psi_\tau\right\rangle\frac{d}{d\tau}\left(-K\frac{dR_\tau}{d\tau}\sin(\theta-\phi_\tau)+KR_\tau\frac{d\phi_\tau}{d\tau}\cos(\theta-\phi_\tau)\right)d_{(\theta,\Omega)}h_\tau\,d\tau\\ &= K\int_0^1\int_{\mathbb{T}\times\mathbb{R}}\int_{\mathbb{T}\times\mathbb{R}}\cos(\theta-\theta')\left\langle e^{i\theta},\nabla_\theta\psi_\tau(\theta,\Omega)\right\rangle\left\langle e^{i\theta'},\nabla_\theta\psi_\tau(\theta',\Omega')\right\rangle d_{(\theta,\Omega)}h_\tau\,d_{(\theta',\Omega')}h_\tau\,d\tau\end{aligned} \tag{5.3.15}$$

Putting the formulas(5.3.14) and (5.3.15) into (5.3.13) entails

$$\begin{aligned}0 = -\frac{K}{2}\int_0^1\int_{\mathbb{T}\times\mathbb{R}}\int_{\mathbb{T}\times\mathbb{R}}\cos(\theta-\theta')\left(\left\langle e^{i\theta},\nabla_\theta\psi_\tau(\theta,\Omega)\right\rangle-\left\langle e^{i\theta'},\nabla_\theta\psi_\tau(\theta',\Omega')\right\rangle\right)^2&\\ \times\, d_{(\theta,\Omega)}h_\tau\,d_{(\theta',\Omega')}h_\tau\,d\tau.&\end{aligned} \tag{5.3.16}$$

Since there exists $0<\delta<\pi/2$ such that

$$\operatorname{diam}(\operatorname{supp}_\theta f_\infty)<\delta\ \text{ and }\ \operatorname{diam}(\operatorname{supp}_\theta f'_\infty)<\delta.$$

The same is true for the interpolations h_τ and, consequently. Indeed, this is a consequence of the monotone rearrangement property of the 1-dimensional transport on each fiber. Hence, we can take upper bounds in (5.3.16) and obtain that

$$0\leq -\frac{K}{2}\cos\delta\int_0^1\int_{\mathbb{T}\times\mathbb{R}}\int_{\mathbb{T}\times\mathbb{R}}\left(\left\langle e^{i\theta},\nabla_\theta\psi_\tau(\theta,\Omega)\right\rangle-\left\langle e^{i\theta'},\nabla_\theta\psi_\tau(\theta',\Omega')\right\rangle\right)^2 d_{(\theta,\Omega)}h_\tau\,d_{(\theta',\Omega')}h_\tau\,d\tau$$

$$= -K\cos\delta \int_0^1 \int_{\mathbb{T}\times\mathbb{R}} |\nabla_\theta \psi_\tau|^2 \, d_{(\theta,\Omega)} h_\tau \, d\tau + K\cos\delta \int_0^1 \left(\int_{\mathbb{T}\times\mathbb{R}} \left\langle e^{i\theta}, \nabla_\theta \psi_\tau \right\rangle \, d_{(\theta,\Omega)} h_\tau \right)^2 d\tau.$$

Notice that the condition (5.3.12) allows neglecting the second term. Also, notice that the cosine has positive sign and hence,

$$\nabla_\theta \psi_\tau^{s=0} = 0, \text{ for } d\tau \otimes h_\tau^{s=0}\text{-a.e. } (\tau,\theta,\Omega) \in [0,1]\times\mathbb{T}\times\mathbb{R}.$$

In particular, the continuity equation for $h_\tau^{s=0}$ implies that

$$f_\infty = h_\tau^{s=0} = f'_\infty, \text{ for all } \tau \in [0,1],$$

thus ending the proof. □

We now come back to the proof of Theorem 5.1.2. First, we show that once the concentration regime in Corollary 5.2.3 takes place, Theorem 5.3.1 guaranteed that the dissipation decays exponentially fast.

Corollary 5.3.9. *Assume that f_0 is contained in $C^1(\mathbb{T}\times\mathbb{R})$ and g is compactly supported in $[-W,W]$ and centered (i.e.,* (5.1.13)*). Consider the unique global-in-time classical solution $f = f(t,\theta,\Omega)$ to* (5.1.2)*. Then, the following holds true*

$$\frac{d\mathcal{I}[f]}{dt} \leq -2K\cos(\beta)\,\mathcal{I}[f] + 24K(W+K)^2 \rho_t(\mathbb{T}\setminus L_\beta^+(t)),$$

for every $t \geq 0$.

Proof. Set $\beta = \frac{\pi}{3}$ and use Theorem 5.3.1 to split the derivative of the dissipation functional into two parts as follows

$$\frac{d\mathcal{I}[f]}{dt} = I_1 + I_2,$$

where each factor reads

$$\begin{aligned} I_1 = -K &\int_{L_\beta^+(t)\times L_\beta^+(t)\times\mathbb{R}\times\mathbb{R}} \big((\Omega - KR\sin(\theta-\phi)) - (\Omega' - KR\sin(\theta'-\phi))\big)^2 \\ &\times \cos(\theta-\theta') f(t,\theta,\Omega) f(t,\theta',\Omega')\, d\theta\, d\theta'\, d\Omega\, d\Omega', \\ I_2 = -K &\int_{((\mathbb{T}\times\mathbb{T})\setminus(L_\beta^+(t)\times L_\beta^+(t)))\times\mathbb{R}\times\mathbb{R}} \big((\Omega - KR\sin(\theta-\phi)) - (\Omega' - KR\sin(\theta'-\phi))\big)^2 \\ &\times \cos(\theta-\theta') f(t,\theta,\Omega) f(t,\theta',\Omega')\, d\theta\, d\theta'\, d\Omega\, d\Omega'. \end{aligned}$$

On the one hand, it is clear that

$$\begin{aligned} I_1 &\leq -K\cos(\beta) \int_{L_\beta^+(t)\times L_\beta^+(t)\times\mathbb{R}\times\mathbb{R}} \big((\Omega - KR\sin(\theta-\phi)) - (\Omega' - KR\sin(\theta'-\phi))\big)^2 \\ &\qquad \times f(t,\theta,\Omega) f(t,\theta',\Omega')\, d\theta\, d\theta'\, d\Omega\, d\Omega' \\ &= -K\cos(\beta) \int_{\mathbb{T}^2\times\mathbb{R}^2} \big((\Omega - KR\sin(\theta-\phi)) - (\Omega' - KR\sin(\theta'-\phi))\big)^2 \\ &\qquad \times f(t,\theta,\Omega) f(t,\theta',\Omega')\, d\theta\, d\theta'\, d\Omega\, d\Omega' \\ &\quad + K\cos(\beta) \int_{((\mathbb{T}\times\mathbb{T})\setminus(L_\beta^+(t)\times L_\beta^+(t)))\times\mathbb{R}\times\mathbb{R}} \big((\Omega - KR\sin(\theta-\phi)) - (\Omega' - KR\sin(\theta'-\phi))\big)^2 \\ &\qquad \times f(t,\theta,\Omega) f(t,\theta',\Omega')\, d\theta\, d\theta'\, d\Omega\, d\Omega' \\ &=: I_{11} + I_{12}, \end{aligned} \tag{5.3.17}$$

where in the second identity we have added and subtracted the second term in order to complete an integral in $\mathbb{T}^2 \times \mathbb{R}^2$. Indeed, notice that doing so and using (5.1.13) we get

$$\begin{aligned} I_{11} &= -K\cos(\beta)\int_{\mathbb{T}^2\times\mathbb{R}^2} \big((\Omega - KR\sin(\theta-\phi) - (\Omega' - KR\sin(\theta'-\phi)))\big)^2 \\ &\qquad\qquad \times f(t,\theta,\Omega)f(t,\theta',\Omega')\,d\theta\,d\theta'\,d\Omega\,d\Omega' \\ &= -2K\cos(\beta)\int_{\mathbb{T}\times\mathbb{R}} (\Omega - KR\sin(\theta-\phi))^2 f\,d\theta\,d\Omega = -2K\cos(\beta)\mathcal{I}[f]. \end{aligned}$$

Here, we have used the cancellation of the crossed term after we expand the square appearing in the first factor. Let us call $I_3 = I_{12} + I_2$ and notice that

$$\begin{aligned} I_3 \leq 2K\int_{((\mathbb{T}\times\mathbb{T})\setminus(L_\beta^+(t)\times L_\beta^+(t)))\times\mathbb{R}\times\mathbb{R}} &\big((\Omega - KR\sin(\theta-\phi) - (\Omega' - KR\sin(\theta'-\phi)))\big)^2 \\ &\times f(t,\theta,\Omega)f(t,\theta',\Omega')\,d\theta\,d\theta'\,d\Omega\,d\Omega'. \end{aligned}$$

In other words, we achieved the estimate

$$\frac{d\mathcal{I}[f]}{dt} \leq -2K\cos(\beta)\,\mathcal{I}[f] + I_3. \tag{5.3.18}$$

Our last goal is to estimate the remainder I_3. Define the following time-dependent sets

$$\begin{aligned} A_1 &:= L_\beta^+(t) \times (\mathbb{T}\setminus L_\beta^+(t)) \times \mathbb{R}\times\mathbb{R}, \\ A_2 &:= (\mathbb{T}\setminus L_\beta^+(t)) \times L_\beta^+(t) \times \mathbb{R}\times\mathbb{R}, \\ A_3 &:= (\mathbb{T}\setminus L_\beta^+(t)) \times (\mathbb{T}\setminus L_\beta^+(t)) \times \mathbb{T}\times\mathbb{R}. \end{aligned}$$

Since we have that $((\mathbb{T}\times\mathbb{T})\setminus(L_\beta^+(t)\times L_\beta^+(t)))\times\mathbb{R}\times\mathbb{R} = A_1\cup A_2\cup A_3$, then we can split I_3 as follows

$$I_3 \leq I_{31} + I_{32} + I_{33},$$

where each integral takes the following form

$$\begin{aligned} I_{3i} := 2K\int_{A_i} &\big((\Omega - KR\sin(\theta-\phi) \\ &- (\Omega' - KR\sin(\theta'-\phi))\big) f(t,\theta,\Omega)f(t,\theta',\Omega')\,d\theta\,d\theta'\,d\Omega\,d\Omega', \end{aligned}$$

for every $i = 1, 2$. Changing variables we observe that $I_{31} = I_{32}$. Then we can focus on estimating I_{31} and I_{33} only. Notice that the integrand can be bounded as follows

$$\big((\Omega - KR\sin(\theta-\phi)) - (\Omega' - KR\sin(\theta'-\phi))\big)^2 \leq 4(W+K)^2.$$

Then, we obtain

$$I_{31}(t) \leq 8K(W+K)^2\rho_t(\mathbb{T}\setminus L_\beta^+(t)),$$

for every $t \geq 0$. Exactly the same argument allows estimating I_{33} and obtaining an identical bound. Putting everything together into (5.3.18) finishes the proof. □

Now, we can apply Gronwall's lemma in order to derive the desired quantitative estimate on the decay rate of the dissipation.

Corollary 5.3.10. *Assume that f_0 is contained in $C^1(\mathbb{T}\times\mathbb{R})$ and g is compactly supported in $[-W,W]$ and centered (i.e., (5.1.13)). Consider the unique global-in-time classical solution $f=f(t,\theta,\Omega)$ to (5.1.2). Then, there is a universal constant C such that if*

$$\frac{W}{K}\leq CR_0^3,$$

then there exists a time T_0 with the property that

$$T_0\lesssim\frac{1}{KR_0^2}\log\left(1+\frac{1}{R_0}+W^{1/2}\|f_0\|_{L^2}\right),$$

and

$$\mathcal{I}[f_t]\lesssim K^2e^{-\frac{1}{20}K(t-T_0)},$$

for all t in $[T_0,\infty)$.

Proof. Let us adjust C small enough so that we meet the hypotheses of Corollary 5.2.3. Then, there exists such a time T_0 so that

$$\rho_t(\mathbb{T}\setminus L_\alpha^+(t))\leq Me^{-\frac{1}{20}K(t-T_0)},$$

for every $t\geq T_0$ and some universal constant M. This along with Corollary 5.3.9 implies

$$\frac{d}{dt}\mathcal{I}[f]\leq -2K\cos(\beta)\mathcal{I}[f]+24K(W+K)^2Me^{-\frac{1}{20}K(t-T_0)},$$

for any $t\geq T_0$. Integrating the inequality, we obtain that

$$\begin{aligned}\mathcal{I}[f_t]&\leq\mathcal{I}[f_{T_0}]e^{-2K\cos(\beta)(t-T_0)}+\frac{24K(W+K)^2M}{2K\cos(\beta)-\frac{1}{20}K}\left(e^{-\frac{K}{20}(t-T_0)}-e^{-2K\cos(\beta)(t-T_0)}\right),\\&\lesssim (W+K)^2e^{-\frac{K}{20}(t-T_0)}\lesssim K^2e^{-\frac{K}{20}(t-T_0)},\end{aligned}$$

where in the second inequality we have used that

$$\mathcal{I}[f_{T_0}]\leq (W+K)^2,$$

by the definition (5.2.1) and in the second inequality we have used the hypothesis on $\frac{W}{K}$. ☐

Using the transportation-dissipation inequality in Corollary 5.3.7 and the above exponential decay of the dissipation in Corollary 5.3.10 we obtain the following result.

Corollary 5.3.11. *Assume that the hypotheses in Corollary 5.3.10 hold true. Then,*

$$W_{2,g}(f_t,f_s)\lesssim e^{-\frac{1}{40}K(t-T_0)}-e^{-\frac{1}{40}K(s-T_0)},$$

for every $s\geq t\geq T_0$.

We are now ready to conclude the proof of the main theorem of this chapter.

Proof of Theorem 5.1.2.

• *Step 1* Convergence.
By the above Corollary 5.3.11, the net $(f_t)_{t\geq 0}$ verifies the Cauchy condition in the metric space $(\mathcal{P}_g(\mathbb{T}\times\mathbb{R}),W_{2,g})$. Notice that it is a complete metric space. Consequently, there exists some

probability measure $f_\infty \in \mathcal{P}_g(\mathbb{T}\times\mathbb{R})$ such that $W_{2,g}(f_t, f_\infty) \to 0$ as $t \to \infty$. Taking limits in the inequality in Corollary 5.3.11 as $s \to \infty$ yields

$$W_{2,g}(f_t, f_\infty) \lesssim e^{-\frac{1}{40}K(t-T_0)}, \tag{5.3.19}$$

for every $t \geq T_0$ and using the order relation in Proposition 5.3.5 between the standard quadratic Wasserstein distance and the fiberwise quadratic Wasserstein distance concludes the exponential convergence in Theorem 5.1.2.

• *Step 2* Uniqueness of the equilibrium.
Notice that, in particular, f_∞ is an equilibrium of the Kuramoto–Sakaguchi equation (5.1.2) and the asymptotic concentration estimate in Corollary 5.2.3 guarantees that

$$\operatorname{diam}(\operatorname{supp}_\theta f_\infty) \leq \beta = \frac{\pi}{3} < \frac{\pi}{2}.$$

Hence, by Proposition 5.3.8 it is unique up to phase shifts. □

5.3.3 Semiconcavity, entropy production estimate and lower bound of R

The main objective of this part is the proof of the entropy production estimate Lemma 5.2.2. As a byproduct in Corollary 5.3.15 we will obtain a universal lower bound on the order parameter. Before we begin the proof of the entropy production estimate, we will need a relationship between the time derivative of the order parameter and the dissipation (5.2.10). That is the content of the following lemma.

Lemma 5.3.12. *Assume that f_0 is contained in $C^1(\mathbb{T}\times\mathbb{R})$ and that g is compactly supported in $[-W, W]$. Then, the inequality*

$$\mathcal{I}[f_t] - W^2 \leq K\frac{d}{dt}(R^2) \leq 3\,\mathcal{I}[f_t] + W^2, \tag{5.3.20}$$

holds.

Proof. By (5.2.2) we have that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}KR^2 &= -\int KR\sin(\theta-\phi)(\Omega - KR\sin(\theta-\phi))f\,d\theta d\Omega\\ &= \mathcal{I}[f] - \int \Omega(\Omega - KR\sin(\theta-\phi))f\,d\theta d\Omega.\end{aligned}$$

Consequently, by young's inequality, we obtain that

$$\frac{1}{2}\frac{d}{dt}KR^2 \leq \mathcal{I}[f] + \frac{1}{2}\int(\Omega - KR\sin(\theta-\phi))^2 f\,d\theta d\Omega + \frac{1}{2}\int \Omega^2 f\,d\theta d\Omega,$$

and

$$\frac{1}{2}\frac{d}{dt}KR^2 \geq \mathcal{I}[f] - \frac{1}{2}\int(\Omega - KR\sin(\theta-\phi))^2 f\,d\theta d\Omega - \frac{1}{2}\int \Omega^2 f\,d\theta d\Omega.$$

Hence, the desired result follows. □

Now we are ready to prove our entropy production estimate in Lemma 5.2.2.

Proof of Lemma 5.2.2. Without loss of generality, we can assume that

$$R < \frac{3}{2}R_0 \quad \text{in} \quad \left[t_0, t_0 + \frac{1}{3KR_0}\log 10\right]. \tag{5.3.21}$$

Otherwise, if this condition fails for some s in the above interval, then we set $d = s - t_0$ and (5.2.5) would follow. Thanks to the inequalities (5.3.5) and (5.3.20), we arrive at the following estimate

$$\begin{aligned}\frac{dR^2}{dt} &\geq \frac{\mathcal{I}[f_t]}{K} - \frac{W^2}{K} \geq \frac{\mathcal{I}[f_{t_0}]e^{-3KR_0(t-t_0)}}{K} - \frac{W^2}{K}\\ &\geq \frac{1}{K}\left(\frac{K}{3}\frac{dR^2}{dt}\bigg|_{t=t_0} - \frac{W^2}{3}\right)e^{-3KR_0(t-t_0)} - \frac{W^2}{K}\\ &= \frac{2}{3}R(t_0)\dot{R}(t_0)e^{-3KR_0(t-t_0)} - \frac{4W^2}{3K}\\ &\geq \frac{K}{6}\cos^2\alpha\lambda^3R_0^3R(t_0)e^{-3KR_0(t-t_0)} - \frac{4W^2}{3K}.\end{aligned}$$

Let us integrate the above inequality on the interval $[t_0, t_0 + d]$ for some d in $[0, \frac{1}{3KR_0}\log 10)$, which we will choose appropriately after the calculations below. By doing this and using (5.3.21), we deduce that

$$R^2(t_0 + d) - R^2(t_0) \geq \frac{1}{18}\cos^2\alpha\lambda^4R_0^3\left[1 - e^{-3KR_0d}\right] - \frac{4}{3}\frac{W^2}{K}d.$$

Thus, by choosing $d = \frac{1}{3KR_0}\log 10$, we obtain that

$$R^2(t_0 + d) - R^2(t_0) \geq \frac{1}{20}\cos^2\alpha\lambda^4R_0^3 - \frac{4}{9}\frac{W^2}{K^2R_0}\log 10.$$

Consequently, by selecting C appropriately in (5.2.4) we conclude that

$$R^2(t_0 + d) - R^2(t_0) \geq \frac{1}{21}\cos^2\alpha\lambda^4R_0^3.$$

Hence, since $\alpha = \pi/6$ the desired result follows. □

Before showing the lower bound in the order parameter, we will need control in its angular velocity in the small dissipation regime. We achieve this in the following lemma

Lemma 5.3.13. *Assume that f_0 is contained in $C^1(\mathbb{T}\times\mathbb{R})$ and that g is compactly supported in $[-W, W]$. Consider the unique global-in-time classical solution $f = f(t, \theta, \Omega)$ to* (5.1.2)*. Then, we have that*

$$|\dot{\phi}| \leq \frac{1}{R}\sqrt{K\frac{d}{dt}R^2 + W^2}.$$

Proof. By (5.2.2), and Jensen inequality, we have that

$$\begin{aligned} R|\dot{\phi}| &\leq \int |\cos(\theta-\phi)(\Omega - KR\sin(\theta-\phi))| f\,d\theta\,d\Omega \\ &\leq \int |(\Omega - KR\sin(\theta-\phi))| f\,d\theta\,d\Omega \\ &\leq \left(\int |(\Omega - KR\sin(\theta-\phi))|^2 f d\theta d\Omega\right)^{\frac{1}{2}} \\ &= I^{\frac{1}{2}} \\ &\leq \sqrt{K\frac{d}{dt}R^2 + W^2}, \end{aligned}$$

where in the last inequality, we have used (5.3.20). Thus, the desired result follows. □

We will derive a global lower bound on the order parameter as an application of the entropy production estimate (5.2.2). To achieve this, we consider the following lemma, which controls the rate at which the order parameter can decrease.

Lemma 5.3.14. ***(Rate of decrease and mass monotonicity)*** *Let λ be contained in $(2/3, 1)$, assume that f_0 is contained $C^1(\mathbb{T}\times\mathbb{R})$ and that g is compactly supported in $[-W, W]$. Consider the unique global-in-time classical solution $f = f(t, \theta, \Omega)$ to* (5.1.2)*. Additionally, let γ be a positive number in $(\pi/6, \pi/2)$, and let α be as specified in Section 5.2. Then, we have that*

$$\frac{d}{dt}R^2 \geq \frac{KR^2\cos^2\gamma}{2}\left(1 - \frac{2W^2}{K^2R^2\cos^2\gamma} - \frac{R}{\sin\gamma} - \frac{1+\sin\gamma}{\sin\gamma}f(\chi_\alpha^-)\right), \tag{5.3.22}$$

and

$$\frac{d}{dt}f(\chi_\alpha^-) \leq 4K\left[\frac{W}{K} + \sqrt{\frac{2\dot{R}}{KR} + \frac{1}{R^2}\frac{W^2}{K^2}} - R\cos\alpha\right]^+, \tag{5.3.23}$$

for all $t \geq 0$.

Moreover, suppose that $\dot{R}(t_0) \leq 0$, $R(t_0) \geq R_0$,

$$\dot{R} \leq \frac{K\cos^2\alpha\lambda^3R_0^3}{4} \quad \textit{in} \quad [t_0, t_0+d] \textit{ and } \quad \cos^2\gamma = \frac{1-\lambda}{5}R_0,$$

for some non-negative numbers d and t_0. Then, there exist a universal constant C such that if we take

$$\frac{W}{K} \leq C(1-\lambda)\lambda^2R_0^2, \tag{5.3.24}$$

then,

$$\frac{d}{dt}R^2 > \frac{K\cos^2\gamma}{2\sin\gamma}\left(-R^3 + [\lambda R_0 + \frac{3}{5}(1-\lambda)R_0]R^2 - \frac{3}{5}(1-\lambda)\lambda^2R_0^3\right), \tag{5.3.25}$$

in $[t_0, t_0+d]$. Consequently,

$$R \geq \lambda R_0 \quad \textit{in} \quad [t_0, t_0+d).$$

Proof. We divide the proof into the following steps:

• *Step 1*: Derivation of estimate (5.3.23).

Recall that $\chi_\alpha^-(\theta) = \xi_\alpha(\theta - \phi - \pi)$, with ξ_α as defined in (5.2.7). Then, by direct computation, we have that

$$\begin{aligned}
\frac{d}{dt} f(\chi_\alpha^-) &= \frac{d}{dt}\int_{\mathbb{T}\times\mathbb{R}} \xi_\alpha(\theta-\phi-\pi) f \, d\theta\, d\Omega \\
&= \int_{\mathbb{T}\times\mathbb{R}} \xi_\alpha'(\theta-\phi-\pi)[\Omega - KR\sin(\theta-\phi) - \dot{\phi}] f\, d\theta\, d\Omega \\
&\leq f(|\xi_\alpha'|)[W + |\dot{\phi}|] + KR\int_{\mathbb{T}\times\mathbb{R}} \xi_\alpha'(\theta-\phi-\pi)\sin(\theta-\phi-\pi) f \; d\theta\, d\Omega \\
&\leq f(|\xi_\alpha'|)[W + |\dot{\phi}| - KR\cos\alpha] \\
&\leq f(|\xi_\alpha'|)\left[W + \frac{1}{R}\sqrt{2KR\frac{d}{dt}R + W^2} - KR\cos\alpha\right].
\end{aligned}$$

Notice that in the last inequality we have used Lemma 5.3.13 in order to estimate $|\dot{\phi}|$ and the only thing that remains to show is the bound of the second term in the third line. Firstly, the support of $\xi_\alpha'(\theta - \phi - \pi)$ consists of $S^+ \cup S^-$ where each set stands for

$$S^+ := \left[\phi + \frac{3\pi}{2} - \alpha, \phi + \frac{3\pi}{2} - \alpha + \frac{1}{2}\right] \quad \text{and} \quad S^- := \left[\phi + \frac{\pi}{2} + \alpha - \frac{1}{2}, \phi + \frac{\pi}{2} + \alpha\right].$$

Since $\xi_\alpha'(\theta - \phi - \pi)$ is non-increasing in S^+ and non-decreasing in S^-, we then obtain

$$\begin{aligned}
\theta \in S^+ &\implies \xi_\alpha'(\theta-\phi-\pi) \leq 0 \;\text{ and }\; \sin(\theta-\phi-\pi) \geq \cos\alpha, \\
\theta \in S^- &\implies \xi_\alpha'(\theta-\phi-\pi) \geq 0 \;\text{ and }\; \sin(\theta-\phi-\pi) \leq -\cos\alpha.
\end{aligned}$$

Consequently,

$$\xi_\alpha'(\theta-\phi-\pi)\sin(\theta-\phi-\pi) \leq -|\xi_\alpha'(\theta-\phi-\pi)|\cos\alpha,$$

for all $\theta \in S^+ \cup S^-$, thus yielding the aforementioned bound Hence, (5.3.23) follows.

• *Step 2:* Derivation of estimate (5.3.22).
By the first equation in (5.2.2), we obtain the following lower bound on $\dot{R}$

$$\begin{aligned}
\frac{K}{2}\frac{d}{dt}R^2 &= -\int_{\mathbb{T}\times\mathbb{R}} KR\sin(\theta-\phi)(\Omega - KR\sin(\theta-\phi)) f \; d\theta\, d\Omega \\
&\geq \int_{\mathbb{T}\times\mathbb{R}} (KR\sin(\theta-\phi))^2 f \; d\theta\, d\Omega - \int \Omega(KR\sin(\theta-\phi)) f \; d\theta\, d\Omega \\
&\geq \frac{1}{2}\int_{\mathbb{T}\times\mathbb{R}} (KR\sin(\theta-\phi))^2 f \; d\theta\, d\Omega - \frac{W^2}{2} \\
&\geq \frac{1}{2}K^2R^2\cos^2\gamma f(\mathbb{T}\setminus(L_\gamma^+(t)\cup L_\gamma^-(t)) - \frac{W^2}{2}.
\end{aligned}$$

Then, we obtain

$$f(\mathbb{T}\setminus(L_\gamma^+(t)\cup L_\gamma^-(t)) \leq \frac{1}{KR^2\cos^2\gamma}\frac{d}{dt}R^2 + \frac{W^2}{K^2R^2\cos^2\gamma}. \tag{5.3.26}$$

Additionally, using a similar argument on (5.1.12), where we split the integral into the sectors

L_γ^+, L_γ^- and $\mathbb{T}\setminus(L_\gamma^+\cup L_\gamma^-)$, allows getting the lower bound

$$\begin{aligned}
R &\geq \sin\gamma\, f(L_\gamma^+) - \sin\gamma\, f(\mathbb{T}\setminus(L_\gamma^+\cup L_\gamma^-)) - f(L_\gamma^-)\\
&= \sin\gamma\left(1 - f(L_\gamma^-) - f(\mathbb{T}\setminus(L_\gamma^+\cup L_\gamma^-))\right) - \sin\gamma\, f(\mathbb{T}\setminus(L_\gamma^+\cup L_\gamma^-)) - f(L_\gamma^-)\\
&= \sin\gamma - 2\sin\gamma\, f(\mathbb{T}\setminus(L_\gamma^+\cup L_\gamma^-)) - (1+\sin\gamma)f(L_\gamma^-)\\
&\geq \sin\gamma - 2\sin\gamma\left(\frac{1}{KR^2\cos^2\gamma}\frac{d}{dt}R^2 + \frac{W^2}{K^2R^2\cos^2\gamma}\right) - (1+\sin\gamma)f(L_\gamma^-).
\end{aligned}$$

Here, we have used the estimate (5.3.26) in the last inequality. Then, (5.3.22) follows.

• *Step 3*: Upper bound on $f(L_\gamma^-)$.
Let us first achieve a lower bound of $f(L_\gamma^+)$. To such end, we use a similar procedure and reverse the inequalities that we have considered in the preceding step. Specifically, notice that a similar split in (5.1.12) allows obtaining

$$\begin{aligned}
R &\leq f(L_\gamma^+) + \sin\gamma f(\mathbb{T}\setminus(L_\gamma^+\cup L_\gamma^-)) - \sin\gamma f(L_\gamma^-)\\
&= f(L_\gamma^+) + \sin\gamma f(\mathbb{T}\setminus(L_\gamma^+\cup L_\gamma^-)) - \sin\gamma(1 - f(L_\gamma^+) - f(\mathbb{T}\setminus(L_\gamma^+\cup L_\gamma^-)))\\
&= (1+\sin\gamma)f(L_\gamma^+) + 2\sin\gamma f(\mathbb{T}\setminus(L_\gamma^+\cup L_\gamma^-)) - \sin\gamma.
\end{aligned}$$

In particular, we obtain the lower bound

$$f(L_\gamma^+) \geq \frac{R}{1+\sin\gamma} - \frac{2\sin\gamma}{1+\sin\gamma}f(\mathbb{T}\setminus(L_\gamma^+\cup L_\gamma^-)) + \frac{\sin\gamma}{1+\sin\gamma}.$$

Hence, we obtain the upper bound

$$\begin{aligned}
f(L_\gamma^-) &= 1 - f(L_\gamma^+) - f(\mathbb{T}\setminus(L_\gamma^+\cup L_\gamma^-))\\
&\leq 1 - \frac{\sin\gamma}{1+\sin\gamma} - \frac{R}{1+\sin\gamma} - \frac{1-\sin\gamma}{1+\sin\gamma}f(\mathbb{T}\setminus(L_\gamma^+\cup L_\gamma^-))\\
&\leq \frac{1}{1+\sin\gamma} - \frac{R}{1+\sin\gamma}.
\end{aligned} \tag{5.3.27}$$

Notice that since $\dot{R}(t_0)\leq 0$ we can select C appropriately in 5.3.24 to guarantee that

$$\frac{W}{K} + \sqrt{\frac{2\dot{R}(t_0)}{KR(t_0)} + \frac{1}{R(t_0)^2}\frac{W^2}{K^2}} - R(t_0)\cos\alpha \leq \frac{W}{K} + \sqrt{\frac{1}{R_0^2}\frac{W^2}{K^2}} - \lambda R_0\cos\alpha < 0. \tag{5.3.28}$$

Then, estimate (5.3.23) implies

$$\left.\frac{d}{dt}\right|_{t=t_0} f(\chi_\alpha^-)(t) \leq 0.$$

By continuity, and, inequalities (5.3.23) and (5.3.28), $f(\chi_\alpha^-)(t)$ remains non increasing along $[t_0, t_0+\delta]$ for small enough $\delta > 0$. Hence, we obtain that

$$\begin{aligned}
f(L_\gamma^-)(t) &\leq f(\chi_\alpha^-)(t)\\
&\leq f(\chi_\alpha^-)(t_0)\\
&\leq f(L_\gamma^-)(t_0) + f(\mathbb{T}\setminus(L_\gamma^+\cup L_\gamma^-))(t_0)\\
&\leq \frac{1}{1+\sin\gamma} - \frac{R(t_0)}{1+\sin\gamma} + \frac{W^2}{K^2\cos^2\gamma R^2(t_0)},
\end{aligned} \tag{5.3.29}$$

for all t in $[t_0, t_0+\delta]$. Here, we have used the estimates (5.3.26) and (5.3.27) along with the hypothesis $\dot{R}(t_0) \leq 0$.

• *Step 4:* Derivation of (5.3.25) and lower bound of R in $[t_0, t_0+\delta]$.
Putting the last estimate (5.3.29) and (5.3.22) together, we obtain the differential inequality

$$\begin{aligned} \frac{dR^2}{dt} &\geq \frac{KR^2\cos^2\gamma}{2\sin\gamma}\left[R(t_0) - R - (1-\sin\gamma) - \frac{2\sin\gamma W^2}{K^2\cos^2\gamma R^2} - \frac{1+\sin\gamma W^2}{K^2\cos^2\gamma R^2(t_0)}\right] \\ &> \frac{K\cos^2\gamma}{2\sin\gamma}\left[-R^3 + b(t_0)R^2 - c(t_0)\right], \end{aligned} \tag{5.3.30}$$

for all t in $[t_0, t_0+\delta]$. Here, the coefficients read

$$\begin{aligned} b(t_0) &:= R(t_0) - \cos^2\gamma - \frac{2W^2}{K^2\cos^2\gamma R^2(t_0)}, \\ c(t_0) &:= \frac{2W^2}{K^2\cos^2\gamma}. \end{aligned}$$

Notice that in the last inequality in (5.3.30) we have used

$$1-\sin\gamma < \cos^2\gamma, \quad \sin\gamma < 1, \quad \text{and} \quad 1+\sin\gamma < 2.$$

By making C smaller if necessary in (5.3.24) we can guarantee that

$$\begin{aligned} b(t_0) &= R(t_0) - \cos^2\gamma - \frac{2W^2}{K^2\cos^2\gamma R^2(t_0)} \\ &\geq R_0 - \frac{(1-\lambda)}{5}R_0 - 10\frac{W^2}{K^2}\frac{1}{R_0^3(1-\lambda)} \\ &\geq R(t_0) - \frac{2(1-\lambda)}{5}R_0 \\ &= \lambda R_0 + (1-\lambda)R_0 - \frac{2(1-\lambda)}{5}R_0 \\ &= \lambda R_0 + \frac{3}{5}(1-\lambda)R_0. \end{aligned}$$

Arguing in a similar way and making C smaller if necessary in (5.3.24), we can guarantee that

$$\begin{aligned} c(t_0) &:= \frac{2W^2}{K^2\cos^2\gamma} \\ &= \left(\frac{W}{K}\right)^2 \frac{10}{(1-\lambda)R_0} \\ &\leq \frac{3}{5}(1-\lambda)\lambda^2 R_0^3. \end{aligned}$$

Consequently, we have that

$$\frac{d}{dt}R^2 > \frac{K\cos^2\gamma}{2\sin\gamma}\Big[-R^3 + \big[\lambda R_0 + \frac{3}{5}(1-\lambda)R_0\big]R^2 - \frac{3}{5}(1-\lambda)\lambda^2 R_0^3\Big].$$

in $[t_0, t_0+\delta]$. Since λR_0, is the biggest root of the polynomial

$$p(r) = -r^3 + [\lambda R_0 + \frac{3}{5}(1-\lambda)R_0]r^2 - \frac{3}{5}(1-\lambda)\lambda^2 R_0^3,$$

we obtain desire lower bound $R \geq \lambda R_0$ in $[t_0, t_0 + \delta]$ by an elementary continuity method argument (we can see that λR_0 is the biggest root of p from the inequality $p(0) < 0$ and the fact that λ being contained in $(2/3, 1)$ implies that $p'(\lambda R_0) < 0)$).

• *Step 5*: Propagation of (5.3.25) and the lower bound of R in $[t_0, t_0 + d]$.
The main idea is supported by a continuity method. We proceed by contradiction. Specifically, define the time

$$t_* := \inf \left\{ t \in (t_0 + \delta, t_0 + d] : \frac{d}{dt} R^2 < \frac{K \cos^2 \gamma}{2 \sin \gamma} p(R) \right\},$$

and assume that $t^* < t_0 + d$. Notice that, by definition, it implies

$$\frac{d}{dt} R^2 \geq \frac{K \cos^2 \gamma}{2 \sin \gamma} p(R), \quad \text{for all } \ t \in [t_0, t_*].$$

In particular, by the same ideas in Step 4, we have that

$$R(t) \geq \lambda R_0, \quad \text{for all } \ t \in [t_0, t_*].$$

By (5.3.23) and the fact that

$$\dot{R} \leq \frac{K \cos^2 \alpha \lambda^3 R_0^3}{4} \quad \text{in} \quad [t_0, t_0 + d],$$

making C smaller in (5.3.24) if necessary, we can guarantee that,

$$\begin{aligned} &\frac{W}{K} + \sqrt{\frac{2\dot{R}(t)}{KR(t)} + \frac{1}{R(t)^2} \frac{W^2}{K^2}} - R(t) \cos \alpha \\ &\qquad \leq \frac{W}{K} + \sqrt{\frac{\lambda^2 R_0^2 \cos^2 \alpha}{2} + \frac{1}{\lambda^2 R_0^2} \frac{W^2}{K^2}} - \lambda R_0 \cos \alpha < 0, \end{aligned} \tag{5.3.31}$$

for all t in $[t_0, t_*]$. In particular the, by (5.3.23) and continuity we have that $f(\chi_\alpha^-)$ is non increasing in $[t_0, t_* + \delta_*]$ and some small enough $\delta_* > 0$. Hence, we can repeat the train of thoughts in *Step 4* to extend the upper bound of $f(\chi_\gamma^-)(t)$ in (5.3.29) to the larger interval $[t_0, t_* + \delta_*]$. Again, the same ideas as in *Step 4* imply that

$$\frac{d}{dt} R^2 > \frac{K \cos^2 \gamma}{2 \sin \gamma} p(R), \quad \text{for all } \ t \in [t_0, t_* + \delta_*],$$

and it contradicts the definition of t_*. □

We close this section by showing that we can obtain a universal lower bound on the order parameter. That is the objective of the following corollary.

Corollary 5.3.15. *Suppose that* $1 - \lambda$ *is contained in* $(0, R_0/120)$. *Then, there exists a universal constant* C *such that if*

$$\frac{W}{K} < C \lambda^2 (1 - \lambda) R_0^2, \tag{5.3.32}$$

then, we have that

$$R \geq \lambda R_0,$$

for every t *in* $[0, \infty)$.

Proof. We begin by choosing C small enough so that it can be taken simultaneously as the corresponding universal constants in Lemma 5.2.2 and 5.3.14. We claim that either one of the following two conditions holds:

(i) We have that $\dot{R} < K/4\lambda^3 R_0^3 \cos^2\alpha$ in $[0,\infty)$.

(ii) There exist a time t^* and an increasing and strictly positive universal function h, satisfying that $R \geq \lambda R_0$ in $[0, t^*]$ and $R(t^*)^2 \geq R_0^2 + h(R_0)$.

We divide the proof of the corollary into two steps. The second of which is the proof of the claim.

• *Step 1*: We show how the claim implies the Corollary.
To see this, we use the following iterative argument based on the fact that R is bounded and the system is autonomous. If condition (ii) of the claim holds, we use the fact that the system is autonomous in time to translate the initial condition of the system to be the configuration at t^*. Since by assumption the value of the order parameter at t^* is bigger than R_0 we are free to apply the claim again with the same value of C to the corresponding shifted initial condition. We can do this iteratively as many time as needed provided that condition (ii) still holds after the time translation.

To conclude this step, note that since R is bounded and the function h is positive, increasing, and universal condition (ii) can hold consecutively after each time translation only a finite number of times. Hence, after finitely many time shifts, condition (i) will hold. Finally, once condition (i) holds, the global lower bound follows by applying Lemma 5.3.14.

• *Step 2*: We show the claim.
For this purpose suppose that (i) does not hold, that is the set

$$\left\{ t \geq 0 \ : \ \dot{R}(t) \geq K \frac{\lambda^3 R_0^3 \cos^2\alpha}{4} \right\},$$

is not empty. To show that (ii) holds in this case, let us consider the smallest time t_1 such that $\dot{R}(t_1) \geq K/4\lambda^3 R_0^3 \cos^2\alpha$. Now, let t_2 denote the biggest time t_2, bigger or equal to t_1, such that $\dot{R} \geq K/4\lambda^3 R_0^3 \cos^2\alpha$ in $[t_1, t_2]$. Notice that the existence of t_2 follows by the boundedness of R.

Now, observe that, by definition of t_1 Lemma 5.3.14 implies that $R \geq \lambda R_0$ in $[0, t_1]$. Moreover, by construction

$$\dot{R} \geq \frac{K}{4} \lambda^3 R_0^3 \cos^2\alpha \quad \text{in} \quad [t_1, t_2].$$

Consequently, $R \geq R(t_1) \geq \lambda R_0$ in $[t_1, t_2]$. Now, we consider two cases:

∘ *Case 2.1.* $R(t_2) \leq \sqrt{2} R_0$.
In this case, observe that Lemma 5.2.2 implies that we can find a constant d such that

$$R^2(t_2 + d) - R^2(t_2) = \frac{\lambda^4}{40} R_0^3.$$

Consequently, by our assumptions on λ, we have that

$$\begin{aligned}
R(t_2+d)^2 &= R(t_2)^2 + \frac{\lambda^4}{40}R_0^3 \\
&\geq \lambda^2 R_0^2 + \frac{\lambda^4}{40}R_0^3 \\
&\geq R_0^2 + \frac{\lambda^4}{40}R_0^3 - (1-\lambda^2)R_0^2 \\
&> R_0^2 + \left(\frac{5}{240}R_0 - 2(1-\lambda)\right)R_0^2 \\
&> R_0^2 + \frac{1}{240}R_0^3.
\end{aligned}$$

Here, on the third line, we have used the fact that $\lambda^4 > 9/10$. Thus, the desired result follows by setting $t^* = t_2 + d$ and

$$h(r) := \frac{r^3}{240}.$$

∘ *Case 2.2.* $R(t_2) > \sqrt{2}R_0$.
In this case, we obtain that $R(t_2)^2 - R_0^2 > R_0^2 > R_0^3/240$. Hence, the desired result holds for $t^* = t_2$. □

5.4 Instability of antipodal equilibria and sliding norms

We now start implementing the program outlined in Sections 5.2.3 and 5.2.4. To do this, we first derive inequalities (5.2.8) and (5.2.11).

Proposition 5.4.1. ***(Instability of antipodal equilibria)*** *Assume that f_0 is contained in $C^1(\mathbb{T}\times\mathbb{R})$ and g is compactly supported in $[-W,W]$. Consider the unique global-in-time classical solution $f = f(t,\theta,\Omega)$ to* (5.1.2) *and let α be as specified in Section 5.2. Then, we have that*

$$\frac{d}{dt}f^2(\chi_\alpha^-(t)) \leq -KR\sin\alpha f^2(\chi_\alpha^-(t))) + 4Kf^2(\mathbb{T})\left[\frac{W}{K} + \sqrt{\frac{2\dot{R}}{KR} + \frac{1}{R^2}\frac{W^2}{K^2}} - R\cos\alpha\right]^+,$$

and

$$\frac{d}{dt}f^2(\mathbb{T}) \leq KRf^2(\mathbb{T}). \tag{5.4.1}$$

Moreover, with the hypothesis (5.3.24) *and notation from Proposition 5.3.14, if $[t_1,t_2]$ is a time interval such that*

$$\dot{R} \leq K\frac{\lambda^3 R_0^3\cos^2\alpha}{4} \qquad in \quad [t_1,t_2], \tag{5.4.2}$$

then, we have that

$$\frac{d}{dt}f^2(L_\alpha^-(t)) \leq -K\lambda R_0\sin\alpha f^2(L_\alpha^-(t))) \qquad in \quad [t_1,t_2]. \tag{5.4.3}$$

Proof. We begin with the first inequality in the Proposition. Arguing as in Step 1 of the proof of

Lemma 5.3.14 we obtain that

$$
\begin{aligned}
\frac{d}{dt}\int_{\mathbb{T}\times\mathbb{R}}\chi_\alpha^-(\theta-\phi+\pi)f^2\,d\theta\,d\Omega &= \int_{\mathbb{T}\times\mathbb{R}}\dot\phi\chi_\alpha^{-\prime}(\theta-\phi+\pi)f^2\,d\theta\,d\Omega\\
&\quad+2\int_{\mathbb{T}\times\mathbb{R}}\chi_\alpha^-(\theta-\phi+\pi)f\partial_t f\,d\theta\,d\Omega\\
&=\int_{\mathbb{T}\times\mathbb{R}}[\dot\phi+2\Omega-2KR\sin(\theta-\phi)]\chi_\alpha^{-\prime}(\theta-\phi+\pi)f^2\,d\theta\,d\Omega\\
&\quad+2\int_{\mathbb{T}\times\mathbb{R}}\chi_\alpha^-(\theta-\phi+\pi)[\Omega-KR\sin(\theta-\phi)]f\partial_\theta f\,d\theta\,d\Omega\\
&\le\int_{\mathbb{T}\times\mathbb{R}}[\dot\phi+\Omega-KR\sin(\theta-\phi)]\chi_\alpha^{-\prime}(\theta-\phi+\pi)f^2\,d\theta\,d\Omega\\
&\quad-\int_{\mathbb{T}\times\mathbb{R}}\chi_\alpha^-(\theta-\phi+\pi)KR\sin\alpha f^2\,d\theta\,d\Omega.
\end{aligned}
\tag{5.4.4}
$$

The first inequality in the proposition follows from Lemma 5.3.13 and the same arguments as in *Step 1* from Proposition 5.3.14. Inequality (5.4.1) follows from similar arguments to those of (5.4.4) by replacing χ_α with the constant function that is equal to one in $\mathbb{T}$. Finally, to derive inequality (5.4.3), recalling the notation introduced in Section 5.2.3, replacing χ_α^- with $\chi_{\alpha,\varepsilon}^-$ in (5.4.4) and arguing as in *Step 1* from Proposition 5.3.14 we get that

$$
\frac{d}{dt}f^2(\chi_{\alpha,\varepsilon}^-(t))\le -KR\sin\alpha f^2(\chi_{\alpha,\varepsilon}^-(t)))+KC_{\varepsilon,\alpha}f^2(\mathbb{T})\left[\frac{W}{K}+\sqrt{\frac{2\dot R}{KR}+\frac{1}{R^2}\frac{W^2}{K^2}}-R\cos\alpha\right]^+.
$$

Now, we observe that as in (5.3.31), we can see that the second term of the above inequality vanishes on the interval $[t_1,t_2]$. Consequently, such a term is independent of ε and thus (5.4.3) follows by letting $\varepsilon\to 0$. □

A form of the above Lemma was one of the main tools used to derive the main result in [154]. However, to obtain our convergence rates, we work with a sliding version of the L^2 norm. Such sliding norms allow us to propagate the above estimate analog the flow of the continuity equation. This technique turns out to be one of the crucial components in our arguments in Section 5.5.

Lemma 5.4.2. ***(Sliding norms)*** *Assume that f_0 is contained in $C^1(\mathbb{T}\times\mathbb{R})$ and g is compactly supported. Consider the unique global-in-time classical solution $f=f(t,\theta,\Omega)$ to (5.1.2). Then, for any measurable set A we have that*

$$
\frac{d}{dt}f^2(A_{t_0,t})\le KR\Big(\sup_{(\theta,\Omega)\in A_{t_0,t}}\cos(\theta-\phi(t))\Big)f^2(A_{t_0,t}).
$$

Proof. By the change of variable theorem, we have that

$$
\begin{aligned}
\frac{d}{dt}\frac{1}{2}\int_{A_{t_0,t}} f^2\,d\theta\,d\Omega &= \frac{d}{dt}\Big|_{t=t_0}\frac{1}{2}\int_A f_t^2(\Theta_{t_0,t}(\theta,\Omega),\Omega)\partial_\theta\Theta_{t_0,t}\,d\theta\,d\Omega \\
&= \int_A f_t(\Theta_{t_0,t}(\theta,\Omega),\Omega)\big[\partial_t f(\Theta_{t_0,t}(\theta,\Omega),\Omega) \\
&\quad + \dot{\Theta}_{t_0,t}(\theta,\Omega)\partial_\theta f(\Theta_{t_0,t_0}(\theta,\Omega),\Omega)\big]\partial_\theta\Theta_{t_0,t}\,d\theta\,d\Omega \\
&\quad - \frac{1}{2}KR\int_A \cos(\Theta_t(\theta,\Omega)-\phi)\partial_\theta\Theta_{t_0,t}f^2\,d\theta\,d\Omega \\
&= \int_A f_t(\Theta_{t_0,t}(\theta,\Omega),\Omega)\big[-\partial_\theta(\Omega f - KR\sin(\Theta_{t_0,t}(\theta,\Omega)-\phi)f) \\
&\quad + (\Omega - KR\sin(\Theta_{t_0,t}(\theta,\Omega)-\phi)\partial_\theta f(\Theta_{t_0,t}(\theta,\Omega),\Omega)\big]\partial_\theta\Theta_{t_0,t}\,d\theta\,d\Omega \\
&\quad + \frac{1}{2}KR\int_A \cos(\Theta_{t_0,t}(\theta,\Omega)-\phi)f^2\partial_\theta\Theta_{t_0,t}\,d\theta\,d\Omega \\
&= \frac{1}{2}KR\int_A \cos(\Theta_{t_0,t}(\theta,\Omega)-\phi)f_t^2(\theta,\Omega)\partial_\theta\Theta_{t_0,t}\,d\theta\,d\Omega.
\end{aligned}
$$

where for t and each Ω, $\partial_\theta\Theta_{t_0,t}(\cdot,\Omega)$ denotes the Jacobian of the map $\theta \to \Theta_{t_0,t}(\theta,\Omega)$. Hence, the desired result follows. □

To make full use of the above control, we need to understand the dynamics of the Lagrangian flow associated with the continuity equation. That is the objective of the next part.

5.4.1 Emergence of attractor sets

In this section, we will show the emergence of time-dependent sets that will act as attractors along the characteristic flow. Such sets, in combination with our analysis on sliding norms in the previous section, will allow us to propagate information between the different parts of the system.

Before showing the emergence of attractor sets, we state the following Lemma, which we will repeatedly use throughout the rest of the chapter. Additionally, in this part, we will use the notation introduced in Subsection 5.2.3.

Lemma 5.4.3 (Emergence of invariant sets). *Assume that f_0 is contained in $C^1(\mathbb{T}\times\mathbb{R})$, g has compact support in $[-W,W]$, and consider the unique global-in-time classical solution to* (5.1.2) $f = f(t,\theta,\Omega)$. *Let $t_0 \geq 0$ be an initial time in $[0,\infty)$ and $L \subseteq \mathbb{T}$ be an interval. Now, assume that, initially we have that*

$$\rho_{t_0}(L) \geq m, \quad \text{and} \quad p = \inf_{\theta,\theta'\in L}\cos(\theta-\theta'),$$

for some positive numbers m and p in $(0,1)$. Additionally, suppose that

$$mp-(1-m) \geq \sigma \quad \text{and} \quad \frac{W^2}{K^2} \leq \frac{(1-p)\sigma^2}{4}, \tag{5.4.5}$$

for some $\sigma > 0$. Then, if we set

$$\underline{P}(t) = \inf_{\theta,\theta'\in L_{t_0,t}}\cos(\theta-\theta'),$$

the following bounds hold true

$$\rho(L_{t_0,t}) \geq m, \tag{5.4.6}$$

$$\inf_{\theta\in L_{t_0,t}} R\cos(\theta-\phi) \geq m\underline{P} - (1-m), \tag{5.4.7}$$

$$1-\underline{P}(t) \leq \max\left((1-p)e^{-\frac{K\sigma}{4}(t-t_0)}, \frac{4}{\sigma^2}\frac{W^2}{K^2}\right), \tag{5.4.8}$$

for every t in $[t_0,\infty)$

Proof. The proof of (5.4.8) is based on a continuity method argument that holds under the condition (5.4.5). Such an argument is based on inequalities (5.4.6), (5.4.7), and

$$\frac{dP}{dt} \geq 2K\sqrt{1-P^2}\left[R\left(\inf_{\theta\in L_{t_0,t}}\cos(\theta-\phi)\right)\sqrt{\frac{1-P}{2}} - \frac{W}{K}\right], \quad \forall t\geq t_0, \tag{5.4.9}$$

which hold when

$$P = \cos(\Theta_{s,t}(\theta,\Omega) - \Theta_{s,t}(\theta',\Omega')), \tag{5.4.10}$$

for any $s\geq t_0$ such that $t\geq s$, and any couple of points (θ,Ω) and (θ',Ω') contained in $L_{t_0,s}\times[-W,W]$.

We will first proof inequality (5.4.8) first and then prove the remaining inequalities afterward. Indeed, let us define t' as the supremum of the set of times $t^*\geq t_0$ such that inequality (5.4.8) holds, for every t in $[t_0,t^*]$. We begin by noting that, by continuity

$$1-\underline{P}(t') = \max\left((1-p)e^{-\frac{K\sigma}{4}(t'-t_0)}, \frac{4}{\sigma^2}\frac{W^2}{K^2}\right).$$

Now, we must prove that there exists $\delta>0$ such that (5.4.8) holds in $[t_0,t'+\delta]$. More precisely, our goal is to show that there exists a uniform time $\delta>0$ such that for any pair of characteristics starting at $L_{t_0,t'}\times[-W,W]$ we have that the corresponding P (given by 5.4.10) satisfies that $1-P$ is bounded by the right-hand side of (5.4.8) in $[t',t'+\delta]$.

To do this, let $s=t'$ in the definition of P. Now observe that by (5.4.7) and (5.4.9), when $t=t'$, we have that

$$\begin{aligned}\left.\frac{dP}{dt}\right|_{t=t'} &\geq 2K\sqrt{1-P^2}\left[R\left(\inf_{\theta\in L_{t_0,t'}}\cos(\theta-\phi)\right)\sqrt{\frac{1-P}{2}} - \frac{W}{K}\right]\\ &\geq 2K\sqrt{1-P^2}\left[[m\underline{P}-(1-m)]\sqrt{\frac{1-P}{2}} - \frac{W}{K}\right]\\ &\geq 2K\sqrt{1+P}\left[\frac{\sqrt{2}}{2}\sigma(1-P) - \frac{W}{K}\sqrt{1-P}\right].\end{aligned} \tag{5.4.11}$$

Here, all the time-dependent expressions are evaluated at $t=t'$. Additionally, in the last inequality, we have used our assumption that (5.4.8) holds on the interval $[t_0,t']$, which together with (5.4.5) implies the uniform lower bound $p\leq\underline{P}$. Now, let (θ,Ω) and (θ',Ω') be any couple of points contained in $L_{t_0,t'}\times[-W,W]$ such that the corresponding P satisfies that

$$1-P(t') = 1-\underline{P}(t') = \max\left((1-p)e^{-\frac{K\sigma}{4}(t'-t_0)}, \frac{4}{\sigma^2}\frac{W^2}{K^2}\right). \tag{5.4.12}$$

Note that since L is compact, then $L_{t_0,t'}\times[-W,W]$ is compact as well. Thus, the set of such pairs (θ,Ω) and (θ',Ω') in $L_{t_0,t'}$ whose corresponding P (obtained via (5.4.10)) satisfies (5.4.12) is a compact set as well. We shall denote such a set by $\mathcal{P} \subseteq L_{t_0,t'} \times [-W,W] \times L_{t_0,t'} \times [-W,W]$. To continue our proof observe that by using the assumption (5.4.12), we get that

$$\frac{\sigma\sqrt{1-P(t')}}{2} \geq \frac{W}{K},$$

for any couple of characteristics in $\mathcal{P}$ and, consequently, by (5.4.11) we obtain that

$$\begin{aligned}\frac{d}{dt}\bigg|_{t=t'}(1-P) &\leq -2K\sqrt{1+P}\left[\frac{\sqrt{2}}{2}\sigma(1-P) - \frac{\sigma(1-P(t'))}{2}\right] \\ &\leq -\frac{2}{5}K\sigma((1-\underline{P}(t')). \\ &< \begin{cases} -\frac{2}{5}K\sigma(1-p)e^{-\frac{K\sigma}{4}(t'-t_0)} & \text{if } \ \frac{4}{\sigma^2}\frac{W^2}{K^2} < 1-\underline{P}(t'), \\ 0 & \text{if } \ \frac{4}{\sigma^2}\frac{W^2}{K^2} = 1-\underline{P}(t'), \end{cases}\end{aligned}$$

Since the right-hand side of the above inequality is uniform in the set of pairs in $\mathcal{P}$ and the set $\mathcal{P}$ is compact, we can find $\varepsilon > 0$ such that if $\mathcal{P}_\varepsilon$ is an ε-neighborhood of $\mathcal{P}$, then we have that

$$\begin{aligned}\frac{d}{dt}\bigg|_{t=t'}(1-\cos(\Theta_{t',t}(\theta,\Omega) - \Theta_{t',t}(\theta',\Omega'))) &\leq -\frac{1}{3}K\sigma((1-\underline{P}(t')) \\ &< \begin{cases} -\frac{K}{4}\sigma(1-p)e^{-\frac{K\sigma}{4}(t'-t_0)}, & \text{if } \ \frac{4}{\sigma^2}\frac{W^2}{K^2} < 1-\underline{P}(t'), \\ 0 & \text{if } \ \frac{4}{\sigma^2}\frac{W^2}{K^2} = 1-\underline{P}(t'), \end{cases}\end{aligned} \tag{5.4.13}$$

for any (θ,Ω),(θ',Ω') in $\mathcal{P}_\varepsilon$. This implies the existence of δ and thus concludes the continuity method argument. Indeed, for characteristics with initial data in $\mathcal{P}_\varepsilon$ the existence of the time interval $[t', t'+\delta)$, follows by the fact that the inequality in (5.4.13) is strict and uniform in $\mathcal{P}_\varepsilon$. Similarly, for characteristics in $(L_{t_0,t'} \times [-W,W] \times L_{t_0,t'} \times [-W,W])\backslash\mathcal{P}_\varepsilon$, the existence of the uniform time δ follows by the fact that the characteristics have uniformly bounded speed and ε provides a uniform separation distance. Indeed, by continuity and compactness, we can find a uniform time neighborhood of t', in which the infimum for $\underline{P}$ is attained in $\mathcal{P}_{\varepsilon/2}$, and we have already shown the existence of δ in such a case.

Hence, to complete the proof of the lemma it suffices to derive inequalities (5.4.6), (5.4.7), and (5.4.9). We achieve this in the following steps:

• *Step 1*: Proof of inequalities (5.4.6) and (5.4.7).
Inequality (5.4.6) follows from the fact that the continuity equation preserves the mass of sets along the characteristic flow. To derive inequality (5.4.7), we observe that

$$\begin{aligned}\inf_{\theta\in L_{t_0,t}} R\cos(\theta-\phi) &= \inf_{\theta\in L_{t_0,t}} \left\langle e^{i\theta}, \int_{\mathbb{T}\times\mathbb{R}} e^{i\theta'} f'\,d\theta'\,d\Omega' \right\rangle \\ &\geq \inf_{\theta\in L_{t_0,t}} \int_{\mathbb{T}\times\mathbb{R}} \cos(\theta-\theta')f'\,d\theta'\,d\Omega' \\ &\geq \inf_{\theta\in L_{t_0,t}} \left[\int_{(L\times[-W,W])_{t_0,t}} \cos(\theta-\theta')f'\,d\theta'\,d\Omega' \right. \\ &\quad \left. + \int_{\mathbb{T}\times\mathbb{R}\backslash(L\times[-W,W])_{t_0,t}} \cos(\theta-\theta')f'\,d\theta'\,d\Omega'\right] \\ &\geq m\underline{P} - (1-m).\end{aligned} \tag{5.4.14}$$

This completes Step 1.

• *Step 2*: Proof of inequality (5.4.9).
To obtain (5.4.9) let us fix t in $[t_0,\infty)$, and let (θ,Ω) and (θ',Ω') be contained in $L\times[-W,W]$. Additionally, let us set

$$\Theta(s):=\Theta_{t_0,s}(\theta,\Omega),\ \text{and}\ \Theta'(s):=\Theta_{t_0,s}(\theta',\Omega').$$

Then,

$$\begin{aligned}\frac{d}{ds}\Big|_{s=t}\cos(\Theta-\Theta')&=-\sin(\Theta-\Theta')(\dot\Theta-\dot\Theta')\\ &=-\sin(\Theta-\Theta')\left((\Omega-\Omega')-KR(\sin(\Theta-\phi)-\sin(\Theta'-\phi))\right)\\ &=-\sin(\Theta-\Theta')\left[(\Omega-\Omega')-2KR\cos\left(\frac{\Theta+\Theta'}{2}-\phi\right)\sin\left(\frac{\Theta-\Theta'}{2}\right)\right]\\ &=-2\cos\left(\frac{\Theta-\Theta'}{2}\right)\left[(\Omega-\Omega')\sin\left(\frac{\Theta-\Theta'}{2}\right)\right.\\ &\qquad\left.-2KR\cos\left(\frac{\Theta+\Theta'}{2}-\phi\right)\sin^2\left(\frac{\Theta-\Theta'}{2}\right)\right]\\ &\geq 4KR\cos\left(\frac{\Theta-\Theta'}{2}\right)\left[\cos\left(\frac{\Theta+\Theta'}{2}-\phi\right)\frac{1-\cos(\Theta-\Theta')}{2}\right.\\ &\qquad\left.-\frac{W}{KR}\sqrt{\frac{1-\cos(\Theta-\Theta')}{2}}\right],\end{aligned}\tag{5.4.15}$$

where we have used several standard trigonometric formulas. Now, notice that

$$\frac{\Theta_{t_0,t}(\theta,\Omega)+\Theta_{t_0,t}(\theta',\Omega')}{2}\quad\text{belongs to}\quad L_{t_0,t},$$

since it is a convex combination of two points in $L_{t_0,t}$. Thus, when $s=t_0$ (5.4.9) follows by standard trigonometric identities. In the case when s is contained in $[t_0,t]$ we can easily derive (5.4.9) by the same argument and the semigroup property of the characteristic flow. □

As a first application of the above lemma, we quantify below the first time in which the system forms an attractor.

Lemma 5.4.4 (First invariant set). *Assume that f_0 is contained in $C^1(\mathbb{T}\times\mathbb{R})$ and g has compact support in $[-W,W]$. Consider the unique global-in-time classical solution to* (5.1.2) *$f=f(t,\theta,\Omega)$ and let us set an angle $0<\gamma<\frac{\pi}{2}$ so that*

$$\cos^2\gamma=\frac{1}{30}R_0.\tag{5.4.16}$$

Then, we can find a universal constant C such that if

$$\frac{W}{K}\leq CR_0^2,\tag{5.4.17}$$

then there exists a positive time T_{-1} satisfying that

$$T_{-1}\lesssim\frac{1}{KR_0^3},\tag{5.4.18}$$

and the bounds

$$\rho(L_\gamma^+(T_{-1})_t) \geq \frac{1+\frac{4}{5}R_0}{2}, \tag{5.4.19}$$

$$\inf_{\theta\in L_\gamma^+(T_{-1})_t} R\cos(\theta-\phi) \geq \frac{3}{5}R_0, \tag{5.4.20}$$

$$\inf_{\theta,\theta'\in L_\gamma^+(T_{-1})_t} \cos(\theta-\theta') \geq 1-\frac{1}{15}R_0, \tag{5.4.21}$$

hold true for every t in $[T_{-1},\infty)$.

Proof. Define the time

$$T_{-1} = \inf\left\{t\geq 0:\ \dot R \leq \frac{KR_0^3}{4\cdot 30^2}\right\}, \tag{5.4.22}$$

and note that by construction, (5.4.18) follows by the fact that R is bounded by 1 and the fundamental theorem of calculus. The proof of the remaining parts of the Lemma will follow directly from an application of Lemma 5.4.3 by setting $L = L_\gamma^+(T_{-1})$ and $t_0 = T_{-1}$. To verify the corresponding hypotheses, first, we begin by controlling the mass in $L_\gamma^+(T_{-1})$. Indeed, by the decomposition of the integral (5.1.12) that defines R into three parts L_γ^+, L_γ^-, and $\mathbb{T}\setminus(L_\gamma^+\cup L_\gamma^-)$, we obtain the inequality:

$$R \leq (1+\sin\gamma)\rho(L_\gamma^+) - \sin\gamma + 2\sin\gamma\rho(\mathbb{T}\setminus(L_\gamma^+\cup L_\gamma^-)). \tag{5.4.23}$$

Consequently, using (5.3.26) to control $\rho(\mathbb{T}\setminus(L_\gamma^+\cup L_\gamma^-))$, we deduce that

$$\begin{aligned}
\rho(L_\gamma^+) &\geq \frac{R+\sin\gamma}{1+\sin\gamma} - \frac{2\sin\gamma}{1+\sin\gamma}\rho(\mathbb{T}\setminus(L_\gamma^+\cup L_\gamma^-))\\
&\geq \frac{1}{1+\sin\gamma}\left[R+\sin\gamma - 2\left(\frac{1}{KR^2\cos^2\gamma}\frac{d}{dt}R^2 + \frac{W^2}{K^2R^2\cos^2\gamma}\right)\right]\\
&= \frac{1}{1+\sin\gamma}\left[R+1+(\sin\gamma-1) - 2\left(\frac{2\dot R}{KR\cos^2\gamma} + \frac{W^2}{K^2R^2\cos^2\gamma}\right)\right].
\end{aligned}$$

for any $t\geq 0$. Then, evaluating the above expression at $t = T_{-1}$, using the fact that by construction $R(T_{-1})\geq R_0$, and selecting $C < 1/30)$ in (5.4.17), we deduce that

$$\begin{aligned}
\rho(L_\gamma^+(T_{-1})) &\geq \frac{1}{2}\left[R_0+1+(\sin\gamma-1) - 2\left(\frac{2\dot R(T_{-1})}{KR\cos^2\gamma} + \frac{W^2}{K^2R^2\cos^2\gamma}\right)\right]\\
&\geq \frac{1}{2}\left[R_0+1-\frac{R_0}{30} - 2\left(30\frac{2\dot R(T_{-1})}{KR_0^2} + \frac{30}{R_0}\frac{W^2}{K^2R_0^2}\right)\right]\\
&\geq \frac{1}{2}\left(1+\frac{4}{5}R_0\right),
\end{aligned} \tag{5.4.24}$$

where we have used the fact that $1-\sin\gamma \leq 1-\sin\gamma^2 = \cos^2\gamma = \frac{R_0}{30}$.
Second, we estimate the infimum of the cosine of the difference of angles in $L_\gamma^+(T_{-1})$, that is

$$\begin{aligned}
\inf_{\theta,\theta'\in L_\gamma^+(T_{-1})} \cos(\theta-\theta') &= \cos(\pi-2\gamma)\\
&= \cos\left(2\left(\frac{\pi}{2}-\gamma\right)\right) = 2\cos^2\left(\frac{\pi}{2}-\gamma\right)-1 = 2\sin^2\gamma - 1 = 1-\frac{1}{15}R_0.
\end{aligned} \tag{5.4.25}$$

Finally, considering Lemma 5.4.3 with $m = \rho(L_\gamma^+(T_{-1}))$ and $p = \cos(\pi - 2\gamma)$, and using the bounds in (5.4.24) and (5.4.25), we obtain

$$\begin{aligned} mp - (1-m) &\geq \frac{1 + \frac{4R_0}{5}}{2}\left(1 - \frac{1}{15}R_0\right) + \left(\frac{\frac{4R_0}{5} - 1}{2}\right) \\ &\geq \frac{1}{2}\left(\frac{8}{5}R_0 - \frac{1}{15}R_0 - \frac{4}{75}R_0^2\right) > \frac{3}{5}R_0. \end{aligned}$$

Thus, the desired result follows by applying Lemma 5.4.3 with $\sigma = \frac{3}{5}R_0$ and noticing that the hypothesis in (5.4.5) follows by the assumption (5.4.17) taking C small enough. □

In the next corollary, we shall explain in which sense the sets whose formation we showed above have an attractive property. Before stating it we will need the following notation:

Definition 5.4.5. *Given positive times $t_0 \leq t_1$, we will define the new time-dependent interval in $[t_1, \infty)$, which will be a dynamic neighborhood of $L_\gamma^+(t_0)_{t_1}$, as follows. First, we define*

$$\left(L_\gamma^+(t_0)_{t_1}\right)_\epsilon = \left\{\theta \in \mathbb{T} \;:\; \inf_{\theta^* \in L_\gamma^+(t_0)_{t_1}} \cos(\theta - \theta^*) \geq 1 - \epsilon\right\},$$

for any ϵ in $[R_0/15, 1)$. Second, using the notation in Subsection 5.2.3, we will denote the θ-projection of the image of $\left(L_\gamma^+(t_0)_{t_1}\right)_\epsilon \times [-W, W]$ under the characteristic flow as follows

$$\left(L_\gamma^+(t_0)_{t_1}\right)_{\epsilon,t} := \Theta_{t_1,t}\big(\left(L_\gamma^+(t_0)_{t_1}\right)_\epsilon \times [-W, W]\big),$$

for any $t > t_1$. When t_0 is clear from the context, we will avoid referring to it in the above notation.

Now, we are ready to state the corollary.

Corollary 5.4.6 (Emergence of attractor sets). *Assume the hypothesis in Lemma 5.4.4 and consider non-negative times $t \geq t_1 \geq T_{-1}$ and $\epsilon = R_0/15$. Then, there exists a universal constant C such that if*

$$\frac{W}{K} \leq CR_0^2, \tag{5.4.26}$$

then we obtain the bounds

$$\rho\big((L_\gamma^+(T_{-1})_{t_1})_{\epsilon,t}\big) \geq \frac{1 + \frac{4}{5}R_0}{2}, \tag{5.4.27}$$

$$\inf_{\theta \in (L_\gamma^+(T_{-1})_{t_1})_{\epsilon,t}} R\cos(\theta - \phi) \geq \frac{1}{2}R_0, \tag{5.4.28}$$

$$\inf_{\theta,\theta' \in (L_\gamma^+(T_{-1})_{t_1})_{\epsilon,t}} \cos(\theta - \theta') \geq 1 - \frac{1}{3}R_0, \tag{5.4.29}$$

hold true for every t in $[t_1, \infty)$.

Proof. We will show how to select C appropriately at the end of the proof, for the moment, let us make it small enough so that we can use Lemma 5.4.4. The proof will follow directly from Lemma 5.4.3 by setting $L := L^+_\gamma(T_{-1})_{t_1,\epsilon}$. To verify the corresponding hypotheses; first, we begin by controlling the mass in L. Indeed, by Lemma 5.4.4 we have that

$$\rho(L^+_\gamma(T_{-1})_{t_1,\epsilon}) \geq \rho\big(L^+_\gamma(T_{-1})_{t_1}\big) \geq \frac{1+\frac{4}{5}R_0}{2}. \tag{5.4.30}$$

Second, we estimate the infimum over the cosine of the difference of angles in $L^+_\gamma(T_{-1})_{t_1,\epsilon}$ for this purpose let $\bar{\theta}$ be contained in $L^+_\gamma(T_{-1})_{t_1}$. Then, for any θ and θ' in $\big(L^+_\gamma(T_{-1})_{t_1}\big)_\epsilon$, we have that

$$\begin{aligned}
\cos\big(\theta-\theta'\big) &= \cos\big(\theta-\bar{\theta}+\bar{\theta}-\theta'\big)\\
&= \cos\big(\theta-\bar{\theta}\big)\cos\big(\theta'-\bar{\theta}\big) - \sin\big(\theta-\bar{\theta}\big)\sin(\theta'-\bar{\theta})\\
&\geq \left[1-\frac{1}{15}R_0\right]^2 - \left[1-\left[1-\frac{1}{15}R_0\right]^2\right]\\
&\geq 2\left[1-\frac{1}{15}R_0\right]^2 - 1 \geq 1-\frac{1}{3}R_0.
\end{aligned}$$

Thus, since θ and θ' were arbitrary, we deduce that

$$\inf_{\theta,\theta'\in L^+_\gamma(T_{-1})_{t_1,\epsilon}} \cos(\theta-\theta') \geq 1-\frac{1}{3}R_0. \tag{5.4.31}$$

Finally, considering $m = \frac{1+\frac{4}{5}R_0}{2}$ and $p = 1-\frac{1}{3}R_0$ and using the bounds in (5.4.30) and (5.4.31), we obtain that

$$\begin{aligned}
mp-(1-m) &\geq \frac{1+\frac{4R_0}{5}}{2}\left(1-\frac{1}{3}R_0\right) + \left(\frac{\frac{4R_0}{5}-1}{2}\right)\\
&\geq \frac{1}{2}\left(\frac{8}{5}R_0-\frac{1}{3}R_0-\frac{4}{15}R_0^2\right)\\
&> \frac{1}{2}\left(\frac{24-5-4}{15}\right)R_0 = \frac{1}{2}R_0.
\end{aligned}$$

Therefore, the desired result follows by choosing C appropriately in (5.4.26) so that (5.4.5) holds and applying Lemma 5.4.3 with $\sigma = \frac{R_0}{2}$. □

In the next Lemma, we derive an estimate that we will use in Section 5.5. Such an estimate shows that if the entropy production vanishes over sufficiently long intervals of time, then L^2 norm of the solution in $\mathbb{T}\backslash\big(L^+_\gamma(T_{-1})_t\big)_\epsilon$, will begin to decrease exponentially.

Lemma 5.4.7. *Let $[t_1,t_2]$ be a time interval in $[T_{-1},\infty)$, such that*

$$\dot{R} \leq K\frac{\lambda^3R_0^3\cos^2\alpha}{4} \quad \textit{and} \quad R < 2R_0, \quad \textit{in} \quad [t_1,t_2].$$

with α as specified in Section 5.2. Assume that $\epsilon = R_0/15$ and λ is contained in $(2/3,1)$. Then there exists a universal constant C and some $\delta > 0$ such that if

$$\frac{W}{K} \leq C\lambda^2R_0^2(1-\lambda), \quad \textit{and} \quad t_2-t_1 \geq \delta \tag{5.4.32}$$

then, we have that

$$f^2(\mathbb{T}\backslash\big(L^+_\gamma(T_{-1})_t\big)_\epsilon) \leq f^2(L^-_\alpha(t_1))e^{K\left(2\delta R_0-\frac{(t-t_1-\delta)R_0\sin\alpha}{2}\right)}, \tag{5.4.33}$$

for all t in $[t_1+\delta, t_2]$*. Moreover, we can choose* δ *so that*

$$\delta \lesssim \frac{1}{K\lambda R_0 \cos^2\alpha} + \frac{\sin\alpha}{K\lambda R_0}\log\frac{1}{R_0}. \tag{5.4.34}$$

Proof. We will show how to select C appropriately at the end of the proof, for the moment, let us make it small enough so that we can use Lemma 5.3.14 and Lemma 5.4.4. The proof is based on Lemma 5.3.14, Proposition 5.4.1, Lemma 5.4.4, and the following differential inequalities:

$$\begin{aligned} \frac{d}{dt}\underline{P} &\geq K\lambda R_0\sqrt{1-\underline{P}^2}\left(\sqrt{1-\underline{P}^2}-\frac{4\cos\alpha}{5}\right), \qquad && t\in[t_1,t_2]\cap\{|\underline{P}|\leq \sin\alpha\}, \\ \frac{d}{dt}(1-P) &\leq -\frac{1}{4}\sin\alpha K\lambda R_0(1-P), && t\in[s,t_2]\cap\{P\leq 1-R_0/15\}. \end{aligned} \tag{5.4.35}$$

Such inequalities hold when $\underline{P}=\cos(\underline{\Theta}_{r,t}(\theta,\Omega)-\phi)$ for any r and for any θ satisfying that $\cos(\theta-\phi(r))=-\sin\alpha$ in $[t_1,t_2]$, and when $P=\cos(\Theta_{r',t}(\theta,\Omega)-\Theta_{T_{-1},t}(\theta',\Omega'))$ for any r' in $[t_1,t_2]$ and any θ and θ' such that $\cos(\theta-\phi(r'))\geq\sin\alpha$ and θ' is contained in $L^+_\gamma(T_{-1})$. Here, Ω and Ω' are contained in $[-W,W]$.

We claim such inequalities imply that there exists $\delta>0$ satisfying (5.4.34) such that

$$\mathbb{T}\backslash\big(L^+_\gamma(T_{-1})_s\big)_\epsilon \subseteq L^-_\alpha(s-\delta)_s,$$

for any s in $[t_1+\delta,t_2]$. Here, we are using the notation introduced in Subsection 5.2.3 and in Definition 5.4.5. We divide the proof into three steps, the second of which will be the proof of the claim:

• *Step 1*: We show that the claim implies (5.4.33).
To achieve this let s be contained in $[t_1+\delta,t_2]$. Then, using Lemma 5.3.14 and Proposition 5.4.1, on the interval $[t_1,s-\delta]$ we obtain that

$$f^2(L^-_\alpha(s-\delta))\leq f^2(L^-_\alpha(t_1))e^{-K(\frac{(s-\delta-t_1)KR_0\sin\alpha}{2})}.$$

Consequently, once the claim is proved, the lemma would follow by the above inequality and Lemma 5.4.2.

• *Step 2*: We show how the inequalities in (5.4.35) imply the claim.
Consider a time r contained in $[t_1,t_2-\delta]$. Since we are assuming that $\underline{P}(r)=-\sin\alpha$, the first inequality in (5.4.35) implies that there exists $\underline{\delta}>0$ such that

$$\frac{d}{dt}\underline{P}\geq\frac{K\lambda R_0\cos^2\alpha}{5}\quad\text{in}\quad[r,r+\underline{\delta}].$$

Consequently, in particular we can find $\underline{\delta}$ such that the above property holds, $\underline{P}(r+\underline{\delta})=\sin\alpha$ and

$$\underline{\delta}\leq\frac{10\alpha}{K\lambda R_0\cos^2\alpha}.$$

By the definition of $\underline{P}$ this implies that

$$\mathbb{T}\backslash L^+_\alpha(s)\subseteq L^-_\alpha(s-\underline{\delta})_s,$$

for any s in $[t_1+\delta,t_2]$. To derive such implication, we have set $s=r+\underline{\delta}$. Consequently, if we let θ be any element $\mathbb{T}\backslash L^-_\alpha(s-\underline{\delta})_s$ and we set $r'=r+\underline{\delta}$ in the definition of P then, by Lemma 5.4.4

and construction, we have that $P(r') > -1$. Moreover, by integrating the second inequality in (5.4.35) we have that we can find $\overline{\delta} > 0$ such that $P(s + \underline{\delta} + \overline{\delta}) \geq 1 - \frac{R_0}{15}$ and

$$\overline{\delta} \lesssim \frac{\sin\alpha}{K\lambda R_0} \log \frac{1}{R_0}.$$

Thus, by the construction of P we obtain that

$$\mathbb{T}\backslash \big(L^+_\gamma(T_{-1})_{r+\underline{\delta}+\overline{\delta}}\big)_\epsilon \subseteq \mathbb{T}\backslash L^+_\alpha(r+\underline{\delta})_{r+\underline{\delta}+\overline{\delta}}.$$

Consequently, the claim follows by selecting $s = r + \underline{\delta} + \overline{\delta}$ and $\delta = \underline{\delta} + \overline{\delta}$.

• *Step 3*: We derive (5.4.35).
Let us denote:

$$\underline{\Theta} = \Theta_{r,t}(\theta,\Omega), \quad \Theta = \Theta_{r',t}(\theta,\Omega), \quad \text{and,} \quad \Theta' = \Theta_{T_{-1},t}(\theta',\Omega'),$$

for $\theta, \theta', \Omega, \Omega'$ as described in the above definition of $\underline{P}$ and P. To derive the first inequality, observe that thanks to Lemma 5.3.13 and our assumption on $\dot{R}$, we can select the constant in (5.4.32) appropriately so that we can guarantee that

$$\begin{aligned}
\frac{d}{dt}\cos\left(\underline{\Theta} - \phi\right) &= -\sin(\underline{\Theta} - \phi(t))(\dot{\underline{\Theta}} - \dot{\phi}) \\
&= -\sin(\underline{\Theta} - \phi)(\Omega - KR\sin(\underline{\Theta} - \phi)) - \dot{\phi}) \\
&\geq -|\sin(\underline{\Theta} - \phi)| \left(\frac{1}{R}\sqrt{K\frac{d}{dt}R^2 + W^2} + W - KR|\sin(\underline{\Theta} - \phi)| \right) \\
&\geq |\sin(\underline{\Theta} - \phi)| \left(KR|\sin(\Theta - \phi)| - \frac{4K\lambda R_0 \cos\alpha}{5} \right).
\end{aligned}$$

Here, in the third inequality, we have used Lemma 5.3.13. Consequently, $\underline{P}$ satisfies the inequality

$$\frac{d}{dt}\underline{P} \geq K\lambda R_0 \sqrt{1 - \underline{P}^2}\left(\sqrt{1 - \underline{P}^2} - \frac{4\cos\alpha}{5}\right).$$

Thus, the first inequality in (5.4.35) follows. Finally, to derive the second inequality we use the same argument in the derivation of (5.4.9) to obtain that

$$\frac{dP}{dt} \geq 2K\sqrt{1-P^2}\left[R\cos\left(\frac{\Theta+\Theta'}{2} - \phi\right)\sqrt{\frac{1-P}{2}} - \frac{W}{K}\right] \quad \text{in} \quad [t_1, t_2],$$

Now, using the same arguments as in the proof of inequality in (5.4.35) and equation (5.4.20), we obtain that

$$\begin{aligned}
\cos\left(\frac{\Theta+\Theta'}{2} - \phi\right) &= \cos\left(\frac{(\Theta-\phi)+(\Theta'-\phi)}{2}\right) \\
&\geq \frac{\cos(\Theta-\phi)+\cos(\Theta'-\phi)}{2} \geq \frac{\sin(\alpha)+\frac{3}{5}R_0}{2} \geq \frac{\sin\alpha}{2}.
\end{aligned}$$

Here, we have used the fact that the first inequality in (5.4.35) implies that $\cos(\Theta - \phi) \geq \sin\alpha$ in $[r', t_2]$. Thus, we deduce that, whenever, $1 - P \geq R_0/15$, we have that

$$\begin{aligned}
\frac{dP}{dt} &\geq 2K\sqrt{1-P^2}\left[\frac{R_0\lambda\sin\alpha}{2}\sqrt{\frac{1-P}{2}} - \frac{W}{K}\right] \\
&\geq 2K\sqrt{1+P}\left[\frac{\sqrt{2}}{4}R_0\lambda\sin\alpha(1-P) - \frac{W}{K}\sqrt{1-P}\right].
\end{aligned}$$

Consequently, by choosing C appropriately in (5.4.32) so that

$$\frac{W}{K}\sqrt{1-P} < CR_0^2 < \frac{R_0\lambda\sin\alpha}{20}(1-P) \quad \text{whenever} \quad 1-P \geq R_0/15,$$

we can guarantee that

$$\frac{d}{dt}P \geq \frac{K\lambda R_0}{4}(1-P), \quad \text{whenever} \quad P \leq 1-\frac{R_0}{15}.$$

Hence, the desired result follows. □

We close this section with a lemma that will allow us to control the L^2 norm of the solution in $\mathbb{T}\backslash\big(L_\gamma^+(T_{-1})_t\big)_\epsilon$ in the intervals of high entropy production.

Lemma 5.4.8. *Let $[t_1,t_2]$ be a time interval contained in $[T_{-1},\infty)$ with the property that*

$$R < 2R_0, \quad in \quad [t_1,t_2].$$

Then, we have that

$$f^2(L_\alpha^-(t)) \leq f^2(\mathbb{T}\backslash\big(L_\gamma^+(T_{-1})_{t_1}\big)_\epsilon)e^{2KR_0(t-t_1)},$$

for all $t\in[t_1,t_2]$.

Proof. This Lemma follows directly from Lemma 5.4.2 and Corollary 5.4.6. □

5.5 Average entropy production via differential inequalities

In this section, we analyze the system of inequalities presented in Subsection 5.2.5 and derived in Sections 5.3 and 5.4. We shall demonstrate that this system implies the control on the time T_0 presented in Theorem 5.1.2. We begin by describing a subdivision of the interval $[0,T_0]$ inspired by the treatment of L. Desvillettes and C. Villani in [104].

We first subordinate the subdivision to different scales of values of the order parameter. Then, we classify the intervals (of such subdivision) into intervals where dissipation is above and below a certain threshold. Such threshold depends on the scale of the order parameter.

5.5.1 The subdivision

Now, we give the precise construction of our subdivision. Before we enter into details, we shall introduce further notation that we will use along this part.

• *The dyadic hierarchy*.
Let us consider an auxiliary time partition into subintervals $[r_k,r_{k+1})$ whose endpoints are enumerated in the sequence $\{r_k\}_{k\in\mathbb{N}}$. Such a partition will be used in this part and is set according to a dyadic behavior of the square of the order parameter R^2. Namely, such sequence provides the first times at which R^2 doubles its value. To such an end, let us set $R_0=R(0)$ and $r_0=0$. Additionally, assume that R_k and r_k are given for certain $k\in\mathbb{N}$ and let us define

$$R_{k+1}^2 = 2R_k^2 \text{ and } r_{k+1} := \inf\{t\geq r_k:\ R^2(t)\geq 2R_k^2 = R_{k+1}^2\}. \tag{5.5.1}$$

Since R is bounded by 1, then the sequence consists of finitely many terms

$$0=r_0<r_1<\cdots<r_{k_*}<r_{k_*+1}=\infty.$$

Here and throughout this section, we will assume that

$$\frac{W}{K} \leq CR_0^3 \text{ and } 1-\lambda \leq \frac{\cos^2\alpha}{180} R_0, \tag{5.5.2}$$

with C small enough so that all the results in Subsections 5.3.3 and Section 5.4 hold (note that our assumption in λ implies the lower bound $\lambda > 179/180$ and thus we can suppress λ from the previous constraints on the universal constant C). Now, let us set

$$\mu_k := \frac{\cos^2\alpha}{4}\lambda^3 R_k^3,\ d_k := \frac{1}{3KR_k}\log 10, \text{ and } \delta_k := \frac{1}{KR_k}\log\left(\frac{1}{R_k}\right). \tag{5.5.3}$$

Observe that (5.5.2) implies that $\frac{W}{K} \leq C\lambda^2(1-\lambda)R_k^2$, for any $k = 0,\ldots,k_*$ with the same universal constant C. In particular, we can use Lemma 5.3.14 and obtain that

$$R(t) \geq \lambda R_k, \text{ for all } t \text{ in } [r_k, r_{k+1}). \tag{5.5.4}$$

• *Initial time of the subdivision.*
Let us use Lemma 5.4.4 to define the corresponding times of formation of attractors that is, we set

$$T_{-1}^k := \inf\left\{t \geq r_k : \frac{dR}{dt} \leq KQR_k^3\right\}, \tag{5.5.5}$$

where $k = 0,\ldots,k_*$ and Q is chosen so that we meet condition (5.4.22) when one applies Lemma 5.4.4 after translating the system in time. Here, for each k, we select the time translation so that the configuration of the system at time r_k is the new initial condition (recall that, by the definition of r_k, we can use Lemma 5.4.4 with the same universal constant C. Then, we define

$$t_0 := \min\{T_{-1}^k : k = 0,\ldots,k_*\}, \tag{5.5.6}$$

and also

$$k_0 := \max\{k \in \mathbb{Z}_0^+ : r_k \leq t_0\}.$$

Notice that since t_0 is the first time in the subdivision, Lemma 5.4.4 and Corollary 5.4.6 will apply at any later step. Thus, we will obtain a controlled behavior of the characteristic flow close to the attractor set $(L_\gamma^+(t_0)_t)_\epsilon$. Here, and throughout the rest of this section we will choose γ by the condition

$$\cos^2\gamma = \frac{1}{30}R_{k_0}. \tag{5.5.7}$$

We have done so according to condition (5.4.16).

• *The subdivision.*
Subordinated to the "dyadic" sequence $\{r_k\}_{k=0}^{k_*}$, we will construct the sequence of times $\{t_l\}_{l\in\mathbb{N}}$ describing the subdivision in the following way. We start at the time t_0 specified in Lemma 5.5.1. Assume that for some l in $\mathbb{N}$ the time t_l is given and let us proceed with the construction of t_{l+1}. First, consider the only $k(l)$ in $\{0,\ldots,k_*\}$ such that t_l is contained in $[r_{k(l)}, r_{k(l)+1})$. Then, we will distinguish between two different situations:

1. If $\dot{R}(t_l) < K\mu_{k(l)}$, then we set

$$t_{l+1} := \sup\{t \in [t_l, r_{k(l)+1}) : \dot{R}(s) < K\mu_{k(l)},\ \forall\, s \in [t_l, s)\}. \tag{5.5.8}$$

2. If $\dot{R}(t_l) \geq K\mu_{k(l)}$, then we first compute

$$\widetilde{t}_{l+1} := \sup\{t \in [t_l, r_{k(l)+1}) : \dot{R}(s) \geq K\mu_{k(l)}, \forall s \in [t_l, s)\}, \tag{5.5.9}$$

and set t_{l+1} via the following correction:

$$t_{l+1} = \begin{cases} \widetilde{t}_{l+1} + d_{k(l)} & \text{if } \widetilde{t}_{l+1} + d_{k(l)} \leq r_{k(l)+1}, \\ r_{k(l)+1} & \text{otherwise.} \end{cases} \tag{5.5.10}$$

• *The good and the bad sets.*
We can think of the intervals $[t_l, t_{l+1})$ obeying the above first item as *bad sets* as they are subject to "small" slope of the order parameter. On the contrary, those sets obeying the second item can be thought of *good sets*, as they involve "large" slope of the order parameter in comparison with the critical value $K\mu_{k(l)}$. The critical value itself depends on the size of $R^2_{k(l)}$ in the above dyadic hierarchy as depicted in (5.5.3). For this reason, we shall collect all the indices l of good and bad sets associated to the index k of the dyadic hierarchy as follows.

$$\begin{aligned} G_k &:= \{l \in \mathbb{Z}_0^+ : t_l \in [r_k, r_{k+1}) \text{ and } \dot{R}(t_l) \geq K\mu_k\}, \\ B_k &:= \{l \in \mathbb{Z}_0^+ : t_l \in [r_k, r_{k+1}) \text{ and } \dot{R}(t_l) < K\mu_k\}, \end{aligned} \tag{5.5.11}$$

for every $k = 0, \ldots, k_*$. Equivalently, we will say that $[t_l, t_{l+1})$ is *of type* G_k if $l \in G_k$ and it is *of type* B_k if $l \in B_k$. For notational purposes, we will denote their sizes

$$\begin{aligned} g_k &:= \#G_k, \\ b_k &:= \#B_k, \end{aligned} \tag{5.5.12}$$

for every $k = 0, \ldots, k_*$. Notice that as a consequence of the definition (5.5.11), after any interval of type B_k whose closure is properly contained in $[r_k, r_{k+1})$ there is an interval of type G_k. The reverse statement is not necessarily true. Namely, notice that for any l in G_k, we need first to compute the interval $[t_l, \widetilde{t}_{l+1})$ according to (5.5.9) and later we extend it into the interval of type G_k $[t_l, t_{l+1})$. Unfortunately, the slope $\dot{R}$ can both grow or decrease in $[\widetilde{t}_{t+1}, t_{l+1})$ and we then lose the control of what is next: either G_k or B_k set. Nevertheless, this is enough to show that

$$b_k \leq g_k + 1, \quad \text{for all } k = 0, \ldots, k_*. \tag{5.5.13}$$

Of course, by definition $g_0 = \cdots = g_{k_0 - 1} = 0$. The size of g_k for $k = k_0, \ldots, k_*$ will be estimated in Lemma 5.5.3. Finally, for notational simplicity, we shall sometimes enumerate the indices in G_k in an increasing manner, namely,

$$G_k = \{l_m^k : m = 1, \ldots, g_k\},$$

where $\{l_m^k\}_{1 \leq m \leq g_k}$ is an increasing sequence for each $k = 0, \ldots, k_*$.

Bound of the size of t_0.

By Lemma 5.4.4 we have that that each T_{-1}^k can be estimated via (5.4.18). However, we will show that our dyadic choice allows us to get a sharper estimate of t_0. More specifically, the cubic exponent for R_0 in (5.4.18) can be relaxed to a quadratic one. This is the content of the following Lemma.

Lemma 5.5.1 (Bound of t_0). *Let t_0 be defined as above and suppose condition* (5.5.2) *holds. Then, we have that*

$$t_0 \lesssim \frac{1}{KR_0^2}.$$

Proof. By construction, it is clear that $k_0 \le k_*$. By the fundamental theorem of calculus and the definition of t_0, we obtain that

$$R(r_{k+1}) - R(r_k) = \int_{r_k}^{r_{k+1}} \dot{R}(t)\,dt \ge KQR_k^3(r_{k+1} - r_k),$$

and

$$R(t_0) - R(r_{k_0}) = \int_{k_0}^{t_0} \dot{R}(t)\,dt \ge KQR_{k_0}^3(t_0 - r_{k_0}),$$

for any $k = 0, \dots, k_0 - 1$. Here, we have used the fact that $r_k \le t_0 \le T_{-1}^k$ for every $k = 0, \dots, k_0$. By estimate (5.5.6) and the definition of T_{-1}^k in (5.5.5) we can control the time derivative of the order parameter in the above integrals. Using the dyadic definition of r_k we arrive at the bounds

$$r_{k+1} - r_k \le Q\frac{(R(r_{k+1}) - R(r_k))}{KR_k^3} \le \frac{1}{2}\frac{Q}{KR_k^2}, \tag{5.5.14}$$

and

$$t_0 - r_{k_0} \le \frac{Q(R(t_0) - R(r_{k_0}))}{KR_{k_0}^3} \le \frac{1}{2}\frac{Q}{KR_{k_0}^2}, \tag{5.5.15}$$

for any $k = 0, \dots, k_0 - 1$. To conclude the proof of the lemma, we represent t_0 via a telescopic sum

$$t_0 = t_0 - r_{k_0} + \sum_{k=0}^{k_0-1}(r_{k+1} - r_k) \le \frac{1}{2}\frac{Q}{KR_{k_0}^2}\sum_{k=0}^{k_0}\left(\frac{1}{2}\right)^k \le \frac{Q}{KR_0^2}.$$

□

Gain vs loss

In the forthcoming parts, we compare the growth of the order parameter R along intervals of type G_k with its loss on intervals of type B_k. To do this precisely, for each k in $\{k_0, ..., k_*\}$, we have to give special consideration to the last interval of the subdivision in each $[r_k, r_{k+1})$. We will denote such terminal intervals by $[t_{l(k)}, t_{l(k)+1})$ in such a way that $t_{l(k)}$ is in $[r_k, r_{k+1})$ and $t_{l(k)+1} = r_{k+1}$. We will use the ideas in Collorary 5.3.15. In the following Lemma, we will see that assumption (5.5.2) implies that the loss in R^2 in smaller than $4/5$ of the gain (except on possibly the last interval of $[t_{l(k)}, t_{l(k)+1})$.)

Lemma 5.5.2 (Gain vs loss). *Assume that condition* (5.5.2) *holds. Then we have that*

$$R^2(t_l) - R^2(t_{l+1}) \le \frac{4}{5}\left(R^2(t_{l_m^k+1}) - R^2(\tilde{t}_{l_m^k+1})\right) \le \frac{4}{5}\left(R^2(t_{l_m^k+1}) - R^2(t_{l_m^k})\right),$$

for any l in B_k and any l_m^k in $G_k \backslash l(k)$.

Proof. Thanks to Corollary 5.3.15 and Lemma 5.2.2 we have that

$$R^2(t_l) - R^2(t_{l+1}) \le (1 - \lambda^2)R^2(t_l) \le 4(1 - \lambda)R_k^2 \text{ and } R^2\big(t_{l_m^k+1}\big) - R^2\big(\tilde{t}_{l_m^k+1}\big) \ge \frac{1}{40}\lambda^4 R_k^3.$$

In particular, our thesis holds true as long as one checks the inequality

$$4(1-\lambda) \leq \frac{1}{50}\lambda^4 R_k.$$

Such inequality is true due to our choice of λ. Here, we have used the fact that $\alpha = \pi/6$ and condition (5.5.2) implies that $\lambda > 179/180$. □

Number of intervals of type G_k

Our objective here is to obtain an estimate on the numbers g_k for $k = k_0, \ldots, k_*$. Recall that due to (5.5.13), this will yield a control in the number of sets of type B_k.

Lemma 5.5.3 (Bound on g_k). *Assume that condition* (5.5.2) *holds. Then, we have that*

$$\max(b_k, g_k) \lesssim \frac{1}{R_k}.$$

Proof. To prove this, recall that by Lemma 5.2.2, we have that

$$\sum_{l=G_k\setminus l(k)} \big(R^2(t_{l+1}) - R^2(t_l)\big) \geq (g_k - \chi_{\{l(k)\in G_k\}})\frac{\lambda^4 R_k^3}{40}. \tag{5.5.16}$$

Thus, Lemma 5.5.2 implies

$$\sum_{l\in B_k} (R^2(t_{l+1}) - R^2(t_l)) \geq -g_k \frac{\lambda^4 R_k^3}{50}. \tag{5.5.17}$$

Taking the sum of both the oscillations at good and bad sets, we recover a telescopic sum involving the evaluation of R^2 at the largest and smallest of the times t_l in $[r_k, r_{k+1})$. Recall that by construction, the oscillation of R^2 in $[t_{l(k)}, t_{l(k)+1})$ is positive, independently on whether $l(k)$ is in B_k or G_k. By doing this, we obtain that

$$R_{k+1}^2 - \lambda^2 R_k^2 \geq \frac{g_k}{200}\lambda^4 R_k^3 - \frac{1}{40}\chi_{\{l(k)\in G_k\}}\lambda^4 R_k^3.$$

Hence, we deduce the bound

$$g_k \leq \frac{200(2-\lambda^2)R_k^2}{R_k^3} + 5. \tag{5.5.18}$$

Here, we have used the fact that assumption (5.5.2) implies that $\lambda > 179/180$. Hence, the desired result follows. □

Sum of lengths of intervals of type G_k.

In this section, we control the total diameter of the intervals in G_k. To do this we will consider the sets $\mathring{G}_k$ and $\mathring{B}_k$. The set $\mathring{G}_k$ is obtained by deleting the biggest element from G_k if the last interval in $[r_k, r_{k+1})$ is of type G_k. Otherwise, we let $\mathring{G}_k = G_k$. On the other hand, the set $\mathring{B}_k$ is obtaining by deleting the last element in B_k in the case where the intervals in $[r_k, r_{k+1})$ do not end with two or more intervals of type G_k. Otherwise, we let $\mathring{B}_k = B_k$. Now, we are ready to state our control.

Lemma 5.5.4. *The sum of the lengths of the interval* $[t_{l_m^k}, t_{l_m^k+1}]$ *satisfies*

$$\sum_{m=1}^{g_k} \left(t_{l_m^k+1} - t_{l_m^k}\right) \lesssim \frac{1}{KR_k^2}.$$

Proof. Let us first bound the length of each time interval $[t_{l_m^k}, t_{l_m^k+1})$ of type G_k for $m = 1, \ldots, g_k$. Notice that as defined in (5.5.10), we have the identity

$$t_{l_m^k+1} - t_{l_m^k} = (\widetilde{t}_{l_m^k+1} - t_{l_m^k}) + d_k. \tag{5.5.19}$$

Our next goal is to estimate the first term. To such end, we shall use the idea in Lemma 5.5.3 and the fundamental theorem of calculus to write

$$R(\widetilde{t}_{l_m^k+1}) - R(t_{l_m^k}) = \int_{t_{l_m^k}}^{\widetilde{t}_{l_m^k+1}} \dot{R}(t)\,dt \geq \frac{\cos^2\alpha}{4} K\lambda^3 R_k^3 \left(\widetilde{t}_{l_m^k+1} - t_{l_m^k}\right),$$

for all $m = 1, \ldots, g_k$. Here, we have used (5.5.9) to bound the time derivativeof R. Hence, we obtain

$$\widetilde{t}_{l_m^k+1} - t_{l_m^k} \leq \frac{4}{\cos^2\alpha\, K\lambda^3 R_k^3}\left(R(\widetilde{t}_{l_m^k+1}) - R(t_{l_m^k})\right), \tag{5.5.20}$$

for all $m = 1, \ldots, g_k$. By summing over all the intervals of type $\mathring{G}_k$ we obtain that

$$\begin{aligned}\sum_{l\in\mathring{G}_k} \left(\widetilde{t}_{l+1} - t_l\right) &\leq \frac{4}{\cos^2\alpha\, K\lambda^3 R_k^3}\sum_{l\in\mathring{G}_k}\left(R(\widetilde{t}_{l+1}) - R(t_l)\right)\\ &= \frac{4}{\cos^2\alpha\, K\lambda^3 R_k^3}\sum_{l\in\mathring{G}_k}\left[\left(R(t_{l+1}) - R(t_{l_m^k})\right) - \left(R(t_{l+1}) - R(\widetilde{t}_{l+1})\right)\right],\end{aligned} \tag{5.5.21}$$

Let us add and subtract to the first term in (5.5.21) the oscillations of R over all the sets of type $\mathring{B}_k$. Notice that after doing so the first term becomes a telescopic sum of evaluations of R at points t_l in $[r_k, r_{k+1})$ and it can be easily bounded by the oscillation of R between the largest and smallest t_l that lie in $[r_k, r_{k+1})$. In turns, it can be easily bounded by $R_{k+1} - \lambda R_k$ due to the definition of r_{k+1} in (5.5.1) and the lower bound of the order parameter given by (5.5.4). Then, we obtain

$$\begin{aligned}\sum_{l\in\mathring{G}_k} \left(\widetilde{t}_{l+1} - t_l\right) &\leq \frac{4}{\cos^2\alpha\, K\lambda^3 R_k^3}(R_{k+1} - \lambda R_k)\\ &\quad - \frac{4}{\cos^2\alpha\, K\lambda^3 R_k^3}\left[\sum_{l\in\mathring{B}_k}\left(R(t_l) - R(t_{l+1})\right) + \sum_{l\in\mathring{G}_k}\left(R(t_{l+1}) - R(\widetilde{t}_{l+1})\right)\right].\end{aligned} \tag{5.5.22}$$

Our goal is to show that the term in the second line is non-positive. Indeed, let us use lemmas 5.2.2 and 5.5.2 in the second term of (5.5.22) to obtain that

$$\begin{aligned}\sum_{l\in\mathring{G}_k} \left(\widetilde{t}_{l+1} - t_l\right) &\leq \frac{4(2-\lambda)}{K\cos^2\alpha\,\lambda^3 R_k^2} - \frac{4}{5\cos^2\alpha\, K\lambda^3 R_k^3}\sum_{l\in\mathring{G}_k}\left(R(t_{l+1}) - R(\widetilde{t}_{l+1})\right)\\ &\leq \frac{4(2-\lambda)}{\cos^2\alpha\, K\lambda^3 R_k^2}.\end{aligned}$$

Hence, by lemmas 5.2.2 and 5.5.3 and (5.5.19) we deduce that

$$\sum_{m=1}^{g_k}\left(t_{l_m^k+1}-t_{l_m^k}\right)\leq d_k g_k+\tilde{t}_{l_{g_k+1}^k}-t_{l_{g_k}^k}+\sum_{l\in\mathring{G}_k}\left(\widetilde{t}_{l+1}-t_l\right)\lesssim\frac{1}{KR_k^2},$$

where we have used (5.5.20) and our usual bound on the oscillation to control the difference

$$\tilde{t}_{l_{g_k+1}^k}-t_{l_{g_k}^k}.$$

Thus, the desired result follows. □

Growth of $f^2(\mathbb{T}\setminus(L_\gamma^+(t_0)_t)_\epsilon)$

Our goal here is to control the growth of $f^2(\mathbb{T}\setminus(L_\gamma^+(t_0)_t)_\epsilon)$ in each interval $[r_k,r_{k+1})$, where the parameter ϵ of the neighborhood is set once for all as follows

$$\epsilon:=\frac{R_0}{15}.$$

Notice that ϵ has been set so that the attractive property in Corollary 5.4.6 holds true. To initialize the iterative method, we need to control $f^2(\mathbb{T}\setminus(L_\gamma^+(t_0)_t)_\epsilon)$ at $t=t_0$. Hence, we begin by providing a control of the growth of $f_t^2(\mathbb{T})$ during the transient $[0,t_0]$.

Lemma 5.5.5. *Assume condition* (5.5.2) *holds. Then, we have that*

$$\|f_{t_0}\|_2^2\leq\|f_0\|_2^2e^{\frac{4Q}{R_0}}.$$

Proof. Thanks to Proposition 5.4.1 we obtain that

$$\|f_{t_0}\|_2^2\leq\|f_0\|_2^2\exp\left(K\int_0^{t_0}R(s)\,ds\right).$$

Then, the main objective is to estimate the time integral of the order parameter. To that end, observe that

$$\begin{aligned}\int_0^{t_0}R(s)\,ds&=\sum_{k=0}^{k_0-1}\int_{r_k}^{r_{k+1}}R(s)\,ds+\int_{r_{k_0}}^{t_0}R(s)\,ds\\&\leq\sum_{k=0}^{k_0-1}R_{k+1}(r_{k+1}-r_k)+R_{k_0+1}(t_0-r_{k_0})\\&\leq Q\sum_{k=0}^{k_0}\frac{R_k}{KR_k^2}=Q\sum_{k=0}^{k_0}\frac{1}{KR_0}\left(\frac{\sqrt{2}}{2}\right)^k\leq\frac{4Q}{KR_0}.\end{aligned}$$

Notice that we have used (5.5.14) and (5.5.15) to estimate the lengths of the intervals $[r_k,r_{k+1})$. Hence, the desired result follows. □

Let us now begin our study on the primary goal of this section. To do this, let us introduce the following notation that we will use in this part. Define the parameters

$$D_k:=\max(b_k,g_k)(\delta_k+d_k)+\sum_{l=1}^{g_k}(\tilde{t}_{l_m^k+1}-t_{l_m^k}),\tag{5.5.23}$$

for any $k = k_0, \dots, k_*$. Notice that its size can be controlled in the following way due to lemmas 5.5.3 and 5.5.4 and the values in (5.5.3):

$$\begin{aligned} D_k &\lesssim \frac{1}{KR_k^2} + \frac{1}{R_k}\left[\frac{1}{KR_k}\log\left(\frac{1}{R_k}\right) + \frac{1}{KR_k}\right] \\ &\lesssim \frac{1}{KR_k^2}\log\left(1+\frac{1}{R_k}\right). \end{aligned} \tag{5.5.24}$$

Let us also introduce the following sequence of functions $\{F_k\}_{k=k_0}^{k_*}$. We proceed by induction. For $k = k_0$, we define

$$F_{k_0}(t) := \begin{cases} \|f_0\|_{L^2}^2 e^{\frac{4Q}{R_0}} e^{2KR_{k_0}(t-t_0)}, & \text{for } t \in [t_0, t_0 + D_{k_0}], \\ \|f_0\|_{L^2}^2 e^{\frac{4Q}{R_0}} e^{2KR_{k_0}D_{k_0}} e^{-K\frac{R_{k_0}\sin\alpha}{2}(t-t_0-D_{k_0})}, & \text{for } t \in [t_0 + D_{k_0}, r_{k_0+1}). \end{cases}$$

Assume that F_{k-1} is given in the interval $[r_{k-1}, r_k)$ and let us define F_k in the interval $[r_k, r_{k+1})$ through the formula

$$F_k(t) := \begin{cases} F_{k-1}(r_k) e^{2KR_k(t-r_k)}, & \text{for } t \in [r_k, r_k + D_k], \\ F_{k-1}(r_k) e^{2KR_kD_k} e^{-K\frac{R_k\sin\alpha}{2}(t-r_k-D_k)}, & \text{for } t \in [r_k + D_k, r_{k+1}). \end{cases}$$

Lemma 5.5.6. *Assume condition* (5.5.2) *holds, then we have that*

$$F_k(t) \le ||f_0||_2^2 e^{\frac{B}{KR_0}\log\left(1+\frac{1}{R_0}\right)}, \ t \in [r_k, r_{k+1}),$$

for some universal constant B and for each $k = k_0, \dots, k_*$.

Proof. By definition, we note that

$$F_k(t) \le F_{k-1}(r_k) e^{2KR_kD_k}, \ \text{ for all } \ t \in [r_k, r_{k+1}),$$

and for every $k = k_0 + 1 \dots, k_*$. Also, notice that by contruction, we have that

$$F_{k_0}(r_{k_0+1}) \le ||f_0||_2^2 \, e^{\frac{4Q}{R_0}} e^{2KR_{k_0}D_{k_0}}.$$

Then, a simple induction shows that

$$F_k(t) \le ||f_0||_2^2 \, e^{\frac{4Q}{R_0}} \prod_{q=k_0}^{k} e^{2KR_qD_q} = ||f_0||_2^2 \exp\left(\frac{4Q}{R_0} + \sum_{q=k_0}^{k} 2KR_qD_q\right). \tag{5.5.25}$$

Finally, let us use the bound (5.5.24) on the above sum to achieve

$$\sum_{q=k_0}^{k} 2KD_qR_q \lesssim \sum_{q=k_0}^{k} \frac{R_q}{KR_q^2}\log\left(1+\frac{1}{R_q}\right) \lesssim \frac{1}{KR_0}\log\left(1+\frac{1}{R_0}\right)\sum_{q=k_0}^{k}\left(\frac{\sqrt{2}}{2}\right)^q.$$

Hence, the desired result follows. □

The sequence $\{F_k\}_{k=k_0}^{k_*}$ has been constructed as a barrier in order to control the map $t \to f^2(\mathbb{T} \setminus (L_\gamma^+(t_0))_t)_\epsilon)$ at each interval $[r_k, r_{k+1})$. We achieve this in the following theorem. Such a theorem is the main result in this section. As a byproduct, we derive corollaries 5.5.8 and 5.2.3, which provide the basis for our discussion in Subsection 5.3.1 and Section 5.3.2.

Theorem 5.5.7. *Assume that condition* (5.5.2) *holds, then we have that*

$$f^2(\mathbb{T}\setminus(L_\gamma^+(t_0)_t)_\epsilon)\leq F_k(t),\ t\in[r_k,r_{k+1}),$$

for each $k=k_0,\dots,k_*$.

Proof. We proceed by induction:

• *Step 1*: Base case ($k=k_0$).
Notice that the inequality is true at $t=t_0$ thanks to Lemma 5.5.5. Let us now look at each of the intervals of type G_{k_0} and B_{k_0} and quantify the growth or decay rate of $f^2(\mathbb{T}\setminus(L_\gamma^+(t_0)_t)_\epsilon)$ via lemmas 5.4.2, 5.4.7 and 5.4.8. Specifically, we shall distinguish between three different scenarios for each interval $[t_l,t_{l+1})$ with t_l in $[r_{k_0},r_{k_0+1})$:

1. If the interval is of type G_{k_0}, then $\dot{R}(t_l)\geq K\mu_{k_0}$ and Lemma 5.4.7 cannot be used to quantify a decrease estimate of the L^2 norm. Fortunately, we can at least use Lemma 5.4.8 on the sliding L^2 norm in combination with Corollary 5.4.6 to obtain that

$$\begin{aligned}f^2\big(L_\alpha^-(t)\big)&\leq f^2(\mathbb{T}\setminus(L_\gamma^+(t_0)_{t_l})_{\epsilon,t})\\&\leq f^2(\mathbb{T}\setminus(L_\gamma^+(t_0)_{t_l})_\epsilon)e^{2KR_{k_0}(t-t_l)}\\&\leq f^2(\mathbb{T}\setminus(L_\gamma^+(t_0)_{t_l})_\epsilon)e^{2KR_{k_0}(t_{l+1}-t_l)},\end{aligned}$$

   for every t in $[t_l,t_{l+1})$.

2. If the interval is of type B_{k_0}, then two different possibilities can take place: either $[t_l,t_{l+1})$ is small or it is large.

   (a) If $[t_l,t_{l+1})$ is small (i.e., $t_{l+1}-t_l\leq\delta_{k_0}$), then Lemma 5.4.7 cannot be used either. Then, we have to rely on a similar argument to that of type G_k, and it implies

$$\begin{aligned}f^2\big(L_\alpha^-(t)\big)&\leq f^2(\mathbb{T}\setminus(L_\gamma^+(t_0)_{t_l})_\epsilon)e^{2KR_{k_0}(t-t_l)}\\&\leq f^2(\mathbb{T}\setminus(L_\gamma^+(t_0)_{t_l})_\epsilon)e^{2KR_{k_0}\delta_{k_0}},\end{aligned}$$

   for every t in $[t_l,t_{l+1})$.

   (b) Finally, if $[t_l,t_{l+1})$ is large (i.e., $t_{l+1}-t_l>\delta_{k_0}$) then, we can apply Lemma 5.4.7. However, notice that it can only be applied for t in $[t_l+\delta_{k_0},t_{l+1})$ and, in the remaining part of the interval $[t_l,t_l+\delta_{k_0})$ we can only apply the same argument as before supported by Lemma 5.4.2 about sliding L^2 norm. Specifically, for any t in $[t_l,t_l+\delta_k)$ Lemma 5.4.2 implies

$$f^2\big(L_\alpha^-(t)\big)\leq f^2(\mathbb{T}\setminus(L_\gamma^+(t_0)_{t_l})_\epsilon)e^{2K\delta_{k_0}R_{k_0}},$$

   Now, for any t in $[t_l+\delta_k,t_{l+1})$ lemmas 5.4.7 and 5.4.8 yield

$$\begin{aligned}&f^2(\mathbb{T}\backslash(L_\gamma^+(t_0)_t)_\epsilon)\\&\quad\leq f^2(L_\alpha^-(t_l))e^{K\left(2R_{k_0}\delta_{k_0}-\frac{(t-t_l-\delta_{k_0})R_{k_0}\sin\alpha}{2}\right)}\\&\quad\leq f_{t_l}^2(\mathbb{T}\setminus(L_\gamma^+(t_0)_{t_l})_\epsilon)e^{K\left(2R_{k_0}\delta_{k_0}-\frac{(t-t_l-\delta_{k_0})R_0\sin\alpha}{2}\right)}.\end{aligned}$$

Bearing all those possibilities in mind, let us now show the inequality for F_{k_0} in (t_0, r_{k_0+1}). Fix any time t in (t_0, r_{k_0+1}) and consider the index

$$p := \max\{l \in \mathbb{N} :\ t_l \le t\}.$$

Then, we shall repeat the above classification at each $[t_l, t_{l+1})$ with l in $\{0, \ldots, p-1\}$ ending with $[t_p, t)$. Also, let us split the indices of intervals of type B_{k_0} into two parts corresponding to small or large intervals as in the above discussion, namely,

$$\begin{aligned} B_{k_0}^S &:= \{l \in B_{k_0} :\ t_{l+1} - t_l \le \delta_{k_0}\}, \\ B_{k_0}^L &:= \{l \in B_{k_0} :\ t_{l+1} - t_l > \delta_{k_0}\}. \end{aligned}$$

Notice that we then have the disjoint union

$$\{0, \ldots, p-1\} = G_{k_0,p} \cup B_{k_0,p}^S \cup B_{k_0,p}^L,$$

where $G_{k_0,p} = G_{k_0} \cap \{0, \ldots, p-1\}$, $B_{k_0,p}^S = B_{k_0}^S \cap \{0, \ldots, p-1\}$, and

$$B_{k_0,p}^L = B_{k_0}^L \cap \{0, \ldots, p-1\}.$$

By applying the above discussion in a recursive way, we obtain that

$$\begin{aligned} &f_{t_p}^2(\mathbb{T} \setminus (L_\gamma^+(t_0)_{t_p})_\epsilon) \\ &\quad \le f_{t_0}^2(\mathbb{T}) \exp\Bigg\{ 2R_{k_0}K\Bigg[\sum_{l \in G_{k_0,p}} (t_{l+1} - t_l) + \sum_{l \in B_{k_0,p}^S} \delta_{k_0}\Bigg] \\ &\qquad\qquad + \sum_{l \in B_{k_0,p}^L} \left(2R_{k_0}\delta_{k_0} - \frac{(t_{l+1} - t_l - \delta_{k_0})R_{k_0} \sin\alpha}{2}\right)\Bigg\}. \end{aligned} \tag{5.5.26}$$

Similarly, for any t in $(t_p, t_p + \delta_{k_0})$ we have that

$$\begin{aligned} &f^2(\mathbb{T} \setminus (L_\gamma^+(t_0)_t)_\epsilon) \\ &\le f_{t_p}^2(\mathbb{T} \setminus (L_\gamma^+(t_0)_{t_p})_\epsilon) \exp\{2KR_{k_0}[(t_{p+1} - t_p)\chi_{\{p \in G_{k_0}\}} \\ &\qquad\qquad + \delta_{k_0}\chi_{\{p \in B_{k_0}^S\}} + \delta_{k_0}\chi_{\{p \in B_{k_0}^L\}}]\} \\ &\le f_{t_p}^2(\mathbb{T} \setminus (L_\gamma^+(t_0)_{t_p})_\epsilon) \exp\left\{2KR_{k_0}\left[(t_{p+1} - t_p)\chi_{\{p \in G_{k_0}\}} + \delta_{k_0}\chi_{\{p \in B_{k_0}\}}\right]\right\}. \end{aligned} \tag{5.5.27}$$

Thus, for any t in $[t_p + \delta_{k_0}, t_{p+1})$ we obtain that

$$\begin{aligned} &f^2(\mathbb{T} \setminus (L_\gamma^+(t_0)_t)_\epsilon) \\ &\le f_{t_p}^2(\mathbb{T} \setminus (L_\gamma^+(t_0)_{t_p})_\epsilon) \exp\left\{2KR_{k_0}\left[(t_{p+1} - t_p)\chi_{\{p \in G_{k_0}\}} + \delta_{k_0}\chi_{\{p \in B_{k_0}^S\}}\right.\right. \\ &\qquad\qquad \left.\left. + \left(\delta_{k_0} - \frac{(t - t_p - \delta_{k_0})\sin\alpha}{4}\right)\chi_{\{p \in B_{k_0}^L\}}\right]\right\} \\ &\le f_{t_p}^2(\mathbb{T} \setminus (L_\gamma^+(t_0)_{t_p})_\epsilon) \exp\left\{2KR_{k_0}\left[(t_{p+1} - t_p)\chi_{\{p \in G_{k_0}\}} + \delta_{k_0}\chi_{\{p \in B_{k_0}\}}\right.\right. \\ &\qquad\qquad \left.\left. - \frac{(t - t_p - \delta_{k_0})R_0 \sin\alpha}{4}\chi_{\{p \in B_{k_0}^L\}}\right]\right\}. \end{aligned} \tag{5.5.28}$$

Putting (5.5.26), (5.5.27) and (5.5.28) together and recalling D_k in (5.5.23) implies

$$
\begin{aligned}
&f_t^2(\mathbb{T} \setminus (L_\gamma^+(t_0)_t)_\epsilon) \\
&\quad \leq f_{t_0}^2(\mathbb{T}) \exp\left\{ 2KD_{k_0}R_{k_0} - \sum_{l\in B_{k_0,p}} K\frac{(t_{l+1}-t_l)R_{k_0}\sin\alpha}{2} \right. \\
&\qquad\qquad\qquad \left. - K\frac{(t-t_p)R_{k_0}\sin\alpha}{2}\chi_{\{p\in B_{k_0}\}} \right\},
\end{aligned} \tag{5.5.29}
$$

where we have absorbed the δ_{k_0} in the las term into D_{k_0}. On the other hand, notice that we can recover t from the following telescopic sum

$$
\begin{aligned}
t &= t - t_p + \sum_{l=0}^{p-1}(t_{l+1}-t_l) + t_0 \\
&= t_0 + (t-t_p)\chi_{\{p\in G_{k_0}\}} + (t-t_p)\chi_{\{p\in B_{k_0}\}} + \sum_{l\in G_{k_0,p}}(t_{l+1}-t_l) + \sum_{l\in B_{k_0,p}}(t_{l+1}-t_l) \\
&\leq t_0 + D_{k_0} + (t-t_p)\chi_{\{p\in B_{k_0}\}} + \sum_{l\in B_{k_0,p}}(t_{l+1}-t_l).
\end{aligned}
$$

Consequently,

$$
-(t-t_p)\chi_{\{p\in B_{k_0}\}} - \sum_{l\in B_{k_0,p}}(t_{l+1}-t_l) \leq -(t-t_0-D_{k_0}),
$$

which can be used to bound the last two terms in the above exponential of (5.5.29). Then, we obtain,

$$
f_t^2(\mathbb{T} \setminus (L_\gamma^+(t_0)_t)_\epsilon) \leq \begin{cases} f_{t_0}^2(\mathbb{T})e^{2KD_{k_0}}, & \text{for } t\in(t_p, t_p+\delta_{k_0}), \\ f_{t_0}^2(\mathbb{T})e^{2KD_{k_0}-\frac{KR_0\sin\alpha}{2}(t-t_0-D_{k_0})}, & \text{for } t\in[t_p+\delta_{k_0}, t_{p+1}). \end{cases} \tag{5.5.30}
$$

Notice that the worst situation is the one where there is no intermediate fall-off, that is, $B_{k_0,p}^L = \emptyset$. Since such scenario dominates all the other possibilities, we shall restrict to it without loss of generality. This amounts to the chain of inequalities

$$
\begin{aligned}
t_p + \delta_{k_0} &= t_0 + \delta_{k_0} + \sum_{l\in G_{k_0,p}}(t_{l+1}-t_l) + \sum_{l\in B_{k_0,p}^S}(t_{l+1}-t_l) + \sum_{l\in B_{k_0,p}^L}(t_{l+1}-t_l) \\
&\leq t_0 + \sum_{l\in G_{k_0}}(t_{l+1}-t_l) + \max(g_k, b_k)\delta_k \leq t_0 + D_{k_0},
\end{aligned}
$$

that is, $t_p + \delta_{k_0} \leq t_0 + D_{k_0}$, that leads to restating (5.5.30) as follows

$$
f_t^2(\mathbb{T} \setminus (L_\gamma^+(t_0)_t)_\epsilon) \leq \begin{cases} f_{t_0}^2(\mathbb{T})e^{2KD_{k_0}R_{k_0}}, & \text{for } t\in(t_0, t_0+D_{k_0}), \\ f_{t_0}^2(\mathbb{T})e^{2KD_{k_0}R_{k_0}-\frac{KR_{k_0}\sin\alpha}{2}(t-t_0-D_{k_0})}, & \text{for } t\in[t_0+D_{k_0}, r_{k_0+1}). \end{cases}
$$

Finally, use Lemma 5.5.5 to relate the L^2 norm at $t = t_0$ and at $t = 0$. Thus, we have showed the claimed bound.

• *Step 2*: Inductive hypothesis.
Let us assume that for certain $k_0 < k < k_*$ we have

$$
f^2(\mathbb{T} \setminus (L_\gamma^+(t_0)_t)_\epsilon) \leq F_q(t),\ t\in[r_q, r_{q+1}),
$$

for any $q < k$.

• *Step 3*: Induction step.
The proof for the index k becomes a simple consequence of the inductive hypothesis where we need to apply again Lemmas 5.4.2, 5.4.7 and 5.4.8 repeatedly in the spirit as in *Step 1* for the base step. □

As a consequence of Theorem 5.5.7 we obtain the following two Corollaries.

Corollary 5.5.8. *Suppose assumption* (5.5.2) *holds. Then, we have that*

$$r_{k+1} - r_k \lesssim \frac{1}{KR_k}\frac{1}{R_0}\log\left(1 + \frac{1}{R_0} + W^{1/2}||f_0||_2\right),$$

for any $k \leq k_*$.

Proof. Thanks to (5.5.24), we may assume, without lost of generality that $r_{k+1} - r_k \geq D_k$. Now, observe that, by Theorem 5.5.7 and (5.5.25) we have that

$$\begin{aligned} f^2(\mathbb{T}\backslash\big(L_\gamma^+(t_0)\big)_\epsilon) &\leq F_k(t) \\ &\leq \|f_0\|_{L^2}^2\, e^{\frac{4Q}{R_0}} \left(\prod_{q=k_0}^{k} e^{2KR_qD_q}\right) e^{-K\frac{R_k\sin\alpha}{2}(t-r_k-D_k)}. \\ &\leq \|f_0\|_{L^2}^2\, e^{\frac{Q'}{R_0}\log\left(1+\frac{1}{R_0}\right)}\, e^{-K\frac{R_k\sin\alpha}{2}(t-r_k-D_k)}, \end{aligned}$$

for every t in $[r_k + D_k, r_{k+1})$ and some universal constant Q'. On the other hand, by Jensen inequality, we have that

$$\rho(\mathbb{T}\backslash(L_\gamma^+(t_0)_t)_\epsilon) \leq \sqrt{4\pi W f^2(\mathbb{T}\backslash(L_\gamma^+(t_0)_t)_\epsilon)}.$$

Consequently, if we let $m(s) = 1 - \rho(\mathbb{T}\backslash(L_\gamma^+(t_0)_s)_\epsilon)$, using Theorem 5.5.7, we deduce that

$$1 - m(s) \leq 2\sqrt{\pi}||f_0||_2 e^{\frac{Q'}{2KR_0}\log\left(1+\frac{1}{R_0}\right)} e^{-K\frac{R_k\sin\alpha}{4}(s-r_k-D_k)}. \tag{5.5.31}$$

For any s in $[D_k + r_k, r_{k+1}]$. On the other hand, by Lemmas 5.4.3, 5.4.4, and Corollary 5.4.6, if we let

$$P(t) = \inf_{\theta,\theta'\in\left(L_\gamma^+(t_0)_s\right)_{\epsilon,t}} \cos(\theta - \theta'), \tag{5.5.32}$$

we then obtain that

$$1 - P(t) \leq \max\left[\frac{1}{3}R_{k_0}e^{-\frac{K}{8}R_{k_0}(t-s)}, \frac{16}{R_{k_0}^2}\frac{W^2}{K^2}\right],$$

for every t in $[s, r_{k+1}]$. Additionally, using Lemmas 5.4.3 and 5.4.4, and Corollary 5.4.6, if we let $L = \left(L_\gamma^+(t_0)_s\right)_\epsilon$ then we have that

$$\begin{aligned} R(t) &\geq \inf_{\theta,\theta'\in L_{s,t}} R\cos(\theta - \theta') \\ &\geq m(s)P(t) - (1 - m(s)) \\ &= \big(1 - (1 - m(s))\big)P(t) - (1 - m(s)) \\ &\geq P(t) - 2(1 - m(s)) \\ &\geq 1 - (1 - P(t)) - 4\sqrt{\pi}W^{\frac{1}{2}}\|f_0\|_{L^2}e^{\frac{Q'}{2R_0}\log\left(1+\frac{1}{R_0}\right)}e^{-K\frac{R_k\sin\alpha}{4}(s-r_k-D_k)}. \end{aligned} \tag{5.5.33}$$

Now, observe that, by construction

$$\frac{\sqrt{2}}{2} \geq R \quad \text{in} \quad [r_k, r_{k+1}).$$

Consequently, by (5.5.31) and (5.5.32), if we set

$$t = r_{k+1} \text{ and } s = r_{k+1} - \frac{8}{KR_{k_0}} \log \frac{1}{10R_{k_0}},$$

in (5.5.33), and make C smaller within the constrains of (5.5.2) if necessary, we obtain that

$$\frac{1}{3} R_{k_0} e^{-\log \frac{1}{10R_{k_0}}} + 4\sqrt{\pi} W^{\frac{1}{2}} \|f_0\|_{L^2} e^{\frac{Q'}{2R_0} \log\left(1+\frac{1}{R_0}\right)} e^{-K \frac{R_k \sin\alpha}{4} \left(r_{k+1} - r_k - D_k - \frac{8}{KR_{k_0}} \log \frac{1}{10R_{k_0}}\right)} \geq 1 - \frac{\sqrt{2}}{2}. \quad (5.5.34)$$

Thus,

$$4\sqrt{\pi} ||f_0||_2 W^{1/2} e^{\frac{C_1}{R_0} \log\left(1+\frac{1}{R_0}\right)} e^{-K \frac{R_k \sin\alpha}{4} (r_{k+1} - r_k - D_k)} \geq 1 - \frac{\sqrt{2}}{2} - \frac{1}{30} \geq \frac{1}{10}.$$

for some universal constant C_1. Hence,

$$\frac{4}{KR_k \sin\alpha} \log\left(40\sqrt{\pi} W^{\frac{1}{2}} \|f_0\|_{L^2}\right) + \frac{4C_1}{KR_0} \frac{1}{R_k \sin\alpha} \log\left(1 + \frac{1}{R_0}\right) + D_k \geq r_{k+1} - r_k.$$

Consequently, using (5.5.24) the desired result follows. □

5.5.2 Emergence of mass concentration

We now prove Corollary 5.2.3. Apart from what is stated there, we will indeed show that

$$\rho\left(\mathbb{T} \backslash \left(L_\gamma^+(t_0)_s\right)_{\epsilon,t}\right) \leq e^{-\frac{1}{10} K \sin\alpha (t - T_0)}, \quad (5.5.35)$$
$$\left(L_\gamma^+(t_0)_s\right)_{\epsilon,t} \subseteq L_\beta^+(t),$$

for every t in $[T_0, \infty)$. Here,

$$s = t - \frac{8}{KR_{k_*}} \log \frac{1}{40R_{k_*}}.$$

Additionally, recall that γ was chosen in (5.5.7).

Proof of Corollary 5.2.3. We begin by showing the first equation in (5.2.13). To do this, we control r_{k_*} via the following telescopic sum and Corollary 5.5.8

$$\begin{aligned}
r_{k_*} &= t_0 + \sum_{k=k_0}^{k_*} r_{k+1} - r_k \\
&\lesssim \frac{1}{KR_0^2} + \sum_{k=k_0}^{k_*} \frac{1}{KR_k} \frac{1}{R_0} \log\left(1 + \frac{1}{R_0} + W^{1/2} ||f_0||_2\right) \\
&\lesssim \frac{1}{KR_0^2} + \sum_{k=k_0}^{k_*} \left(\frac{\sqrt{2}}{2}\right)^k \frac{1}{KR_0^2} \log\left(1 + \frac{1}{R_0} + W^{1/2} ||f_0||_2\right). \\
&\lesssim \frac{1}{KR_0^2} \log\left(1 + \frac{1}{R_0} + W^{1/2} ||f_0||_2\right).
\end{aligned}$$

Consequently, by construction, to guarantee the first equation in (5.2.13) it suffices to take,

$$r_{k_*} \leq T_0 \lesssim \frac{1}{KR_0^2} \log \left(1 + \frac{1}{R_0} + W^{1/2}||f_0||_2\right).$$

Indeed, recall that by definition $R(r_{k_*}) \geq \sqrt{2}/2$ and consequently, by (5.5.4) we have that

$$R(t) \geq \frac{\sqrt{2}}{2}\lambda \geq \frac{3}{5},$$

for every t in $[r_{k_*}, \infty)$. Now, we proceed to show that we can guarantee the second equation in 5.2.13 by selecting T_0 within the desired constraints. To achieve this, we argue as in equation (5.5.33) and (5.5.34) from the proof of Corollary 5.5.8, with

$$s = t - \frac{8}{KR_{k_*}} \log \frac{1}{40R_{k_*}},$$

to obtain that,

$$\begin{aligned} &\rho\big(\mathbb{T}\backslash\big(L_\gamma^+(t_0)_s\big)_{\epsilon,t}\big) \\ &\qquad \leq 4\sqrt{\pi}W^{1/2}\|f_0\|_{L^2} e^{\frac{Q'}{2R_0}\log\left(1+\frac{1}{R_0}\right)} e^{-K\frac{R_{k_*}\sin\alpha}{4}\left(t-r_{k_*}-D_{k_*}-\frac{8}{KR_{k_*}}\log\frac{1}{40R_{k_*}}\right)}, \end{aligned} \tag{5.5.36}$$

and, additionally,

$$\begin{aligned} \inf_{\theta\in\left(L_\gamma^+(t_0)_s\right)_{\epsilon,t}} \cos(\theta-\phi) &\geq 1 - \frac{1}{3}R_{k_*} e^{-\log\frac{1}{40R_{k_*}}} \\ &\quad - 4\sqrt{\pi}W^{1/2}\|f_0\|_{L^2} e^{\frac{Q'}{2R_0}\log\left(1+\frac{1}{R_0}\right)} e^{-K\frac{R_{k_*}\sin\alpha}{4}\left(t-r_{k_*}-D_{k_*}-\frac{8}{KR_{k_*}}\log\frac{1}{40R_{k_*}}\right)}, \end{aligned} \tag{5.5.37}$$

for any t in $\left[r_{k_*} + D_{k_*} + \frac{8}{KR_{k_*}}\log\frac{1}{40R_{k_*}}, \infty\right)$. Thus, since $R_{k_*} \geq \sqrt{2}/2$ we see that, if we choose T_0 in such a way that

$$\begin{aligned} &\frac{1}{KR_0^2}\log\left(1+\frac{1}{R_0}+W^{1/2}||f_0||_2\right) \gtrsim T_0 \\ &\qquad \geq \frac{4}{KR_{k_*}\sin\alpha}\left[\log\frac{4\sqrt{\pi}W^{1/2}\|f_0\|_{L^2}}{R_{k_*}/120}\right] + \frac{Q'+16}{2KR_0}\log\left(1+\frac{1}{40R_0}\right) + r_{k_*} + D_{k_*}. \end{aligned} \tag{5.5.38}$$

then, we can guarantee that condition (5.5.35) holds for every t in $[T_0, \infty)$. Indeed, by (5.5.36), such a choice of T_0 together with Lemma 5.4.3 and Corollary 5.4.6 implies that

$$\begin{aligned} \inf_{\theta\in\left(L_\gamma^+(t_0)_s\right)_{\epsilon,t}} \cos(\theta-\phi) &\geq \frac{59}{60}, \\ \rho\big(\mathbb{T}\backslash\big(L_\gamma^+(t_0)_s\big)_{\epsilon,t}\big) &\leq \frac{1}{120}e^{-K\frac{R_{k_*}\sin\alpha}{4}(t-T_0)}, \end{aligned} \tag{5.5.39}$$

for every t in $[T_0, \infty)$. Consequently, the desired result follows from the fact that (5.5.39) in particular implies that $\big(L_\gamma^+(t_0)_s\big)_{\epsilon,t} \subseteq L_\beta^+(t)$ for $\beta = \frac{\pi}{3}$. □

5.6 Applications to the particle system

The main objective of this section is to prove Corollary 5.1.3. Before we proceed with the proof, let us introduce some necessary tools and notation. Along this section, we will set a probability density f_0 that belongs to C^1 and will assume that g has compact support in $[-W, W]$. Indeed, we will assume that f_0, K and W satisfies the hypotheses of Theorem 5.1.2. Also, we will consider the unique global-in-time classical solution $f = f(t, \theta, \Omega)$ to (5.1.2).

Definition 5.6.1 (The random empirical measures). *By the consistency theorem of Kolmogorov (see [292, Theorem 3.5]), let us consider a probability space $(E, \mathcal{F}, \mathbb{P})$ and set some sequence of random variables for $k \in \mathbb{N}$*

$$(\theta_k(0), \Omega_k(0)) : E \longrightarrow \mathbb{T} \times \mathbb{R},$$

that are i.i.d. with law f_0. For every $N \in \mathbb{N}$, let us consider the random variables

$$t \longmapsto (\theta_1^N(t), \Omega_1(0)), \ldots, (\theta_N^N(t), \Omega_N(0))$$

solving the agent-based system (5.1.1) *issued at the above random initial data. Then, we define the associated random empirical measures as follows*

$$\mu_t^N := \frac{1}{N} \sum_{i=1}^{N} \delta_{(\theta_i^N(t), \Omega_i(0))}(\theta, \Omega), \tag{5.6.1}$$

for every $t \geq 0$.

The proof of Corollary 5.1.3 gathers three different tools:

- First, we shall use our main Theorem 5.1.2, that quantifies the rate of convergence of the solution $f = f(t, \theta, \Omega)$ towards the global equilibrium f_∞ as $t \to \infty$.
- Second, we require a *concentration inequality* to quantify the law of large numbers. More specifically, we need to quantify the rate of convergence in probability $\mathbb{P}$ of μ_0^N towards f_0 as the number of oscillators N tends to infinity.
- Finally, in order to propagate the above quantification for larger times, we require some *stability estimate for the transportation distance* between μ_t^N and f_t.

Those tools will allow us to quantify a time in which a sufficient number of oscillators of the particle system is concentrated around a neighborhood of the support of the global equilibrium f_∞. This, along with Lemma 5.4.3 (which also holds for the particle system), will guarantee that the concentration property of oscillators propagates for larger times. Additionally, we will derive the contraction of the diameter if the configuration of oscillators. Before beginning the rigorous proof, let us elaborate on the concentration and stability inequalities.

5.6.1 Wasserstein concentration inequality

It is apparent from the literature that the above random empirical measures μ_0^N in Definition 5.6.1 approximate the initial datum f_0 as $N \to \infty$. Specifically, by the strong Law of Large Numbers (see [291]) we obtain that

$$\mu_0^N \overset{*}{\rightharpoonup} f_0, \quad \mathbb{P}\text{-a.s},$$

in the narrow topology of $\mathcal{P}(\mathbb{T}\times\mathbb{R})$ as $N\to\infty$. Unfortunately, this is not enough for our purposes as we seek quantitative estimates for the rate of convergence. Such a quantitative control is called *concentration inequality* and there have been many approaches to it in the literature. Most of them require some special structure on the initial data f_0 and the sequence of random empirical measures μ_0^N, see [38, 39, 41]. Specifically, some transportation-entropy inequality is required. To the best of our knowledge, the first result where those assumptions on can be removed was recently introduced in [132]. In our particular setting, it reads as follows.

Lemma 5.6.2. *Let f_0 be contained in $\mathcal{P}(\mathbb{T}\times\mathbb{R})$ be any probability measure with a distribution of natural frequencies $g=(\pi_\Omega)_\# f_0$ and assume that*

$$\mathcal{E}(g):=\int_{\mathbb{R}} e^{\Omega^4}\,dg<\infty. \tag{5.6.2}$$

Take any sequence $\{(\theta_k(0),\Omega_k(0))\}_{k\in\mathbb{N}}$ of i.i.d. random variables with law f_0 and set the random empirical measures μ_0^N according to Definition 5.6.1. Then,

$$\mathbb{P}\left(W_2(\mu_0^N,f_0)\geq\varepsilon\right)\leq C_1e^{-C_2N\varepsilon^4},$$

for every $\varepsilon>0$ and N in $\mathbb{N}$. Here, C_1 and C_2 are two positive constants that depend neither on ε nor on N, but only depend on $\mathcal{E}(g)$.

Proof. Take $d=2$, $p=2$, $\gamma=1$ and $\beta=4$ in [132, Theorem 2]. □

In the above result, we used the classical quadratic Wasserstein distance W_2, namely,

$$W_2(\mu_0^N,f_0)=\left(\inf_{\gamma\in\Pi(\mu_0^N,f_0)}\int_{\mathbb{T}^2\times\mathbb{R}^2}(d(\theta,\theta')^2+(\Omega-\Omega')^2)\,d\gamma\right)^{1/2}.$$

However, as discussed in Remark 5.3.3 in Section 5.3, such distance is not appropriate for this problem due to the fact that the standard quadratic distance on the product Riemannian manifold $\mathbb{T}\times\mathbb{R}$ provides a cost functional which is not dimensionally correct. Indeed, we corrected such situation by scaling Ω. Let us recall the scaled quadratic Wasserstein distance (see Definition 5.3.4),

$$SW_2(\mu_0^N,f_0)=\left(\inf_{\gamma\in\Pi(\mu,\nu)}\int_{\mathbb{T}^2\times\mathbb{R}^2}\left(d(\theta,\theta')^2+\frac{(\Omega-\Omega')^2}{K^2}\right)\,d\gamma\right)^{1/2}.$$

Let us note that by scaling, we can straightforwardly adapt the above Lemma 5.6.2 to the right transportation distance SW_2.

Remark 5.6.3. *Consider the dilation map of the variable Ω with factor K*

$$\mathcal{D}_K(\Omega):=\frac{\Omega}{K},\ \text{for } \Omega\in\mathbb{R}.$$

Then, we can define the following scaled objects:

$$f_{0,K}:=(Id\otimes\mathcal{D}_K)_\# f_0\ \text{ and }\ \mu_{0,K}^N:=(Id\otimes\mathcal{D}_K)_\#\mu_0^N.$$

Notice that $f_{0,K}$ is contained in $\mathcal{P}(\mathbb{T}\times\mathbb{R})$ and the empirical measures $\mu_{0,K}^N$ are i.i.d. variables with law $f_{0,K}$. Interestingly, we obtain the relation

$$SW_2(\mu_0^N,f_0)=W_2(\mu_{0,K}^N,f_{0,K}).$$

Then, by applying Lemma 5.6.2 and the above Remark to the scaled objects, we obtain the following result.

Lemma 5.6.4. *Let f_0 be a probability density in $C^1(\mathbb{T}\times\mathbb{R})$, assume that the distribution of natural frequencies $g=(\pi_\Omega)_\# f_0$ has compact support in $[-W,W]$ and that condition* (5.1.15) *in Theorem 5.1.2 holds true. Take any sequence $\{(\theta_k(0),\Omega_k(0))\}_{k\in\mathbb{N}}$ of i.i.d. random variables with law f_0 and set the random empirical measures μ_0^N according to Definition 5.6.1. Then,*

$$\mathbb{P}\left(SW_2(\mu_0^N,f_0)\geq\varepsilon\right)\leq C_1\exp\left(-C_2N\varepsilon^4\right), \tag{5.6.3}$$

for every $\varepsilon>0$ and N in $\mathbb{N}$. Here, C_1 and C_2 are two positive universal constants.

Remark 5.6.5. *Notice that, according to Lemma 5.6.2, the above C_1 and C_2 only depend upon $\mathcal{E}(g_K)$ where $g_K:=\mathcal{D}_{K\#}g$. Since g has compact support in $[-W,W]$ we obtain that*

$$1\leq\mathcal{E}(g_K)\leq e^{\frac{W^4}{K^4}},$$

so that C_1 and C_2 will ultimately depend only on $\frac{W}{K}$. However, under the assumptions (5.1.15) *in Theorem 5.1.2 $\frac{W}{K}$ is smaller than a universal constant. Consequently, $\mathcal{E}(g_K)$ can be made smaller than a universal constant arbitrarily close to* 1*. This justifies that C_1 and C_2 can be considered universal constants.*

5.6.2 Wasserstein stability estimate

In Theorems 4.4.6 and 4.6.31 of Chapter 4, we proved the following Dobrushin-type stability estimate for the classical quadratic Wasserstein distance in $\mathbb{T}\times\mathbb{R}$

$$W_2(f_t,\bar{f}_t)\leq e^{\left(2K+\frac{1}{2}\right)t}W_2(f_0,\bar{f}_0), \tag{5.6.4}$$

for any two measured valued solution of the Kuramoto model with singular weights (4.2.5). In particular, notice that the same idea holds true for any two measured valued solution to (5.1.2). However, notice that units are not correct in the above inequality due to the fact that W_2 is not dimensionally correct in this problem (recall Remark 5.3.3). Since this time, a correct dimension of the Wasserstein distance is necessary (as we tackle the regime where the physical parameter K is large compared to W), we shall swap the role of W_2 in the above results with its correct scaled version SW_2 (see Definition 5.3.4) to recover the following result.

Lemma 5.6.6. *Consider $K>0$ and let f and $\widetilde{f}$ be weak measured-valued solutions to* (5.1.2) *with initial data f_0 and $\widetilde{f}_0\in\mathcal{P}_2(\mathbb{T}\times\mathbb{R})$. Then, we have that*

$$SW_2(f_t,\widetilde{f}_t)\leq e^{\frac{5}{2}Kt}SW_2(f_0,\widetilde{f}_0),$$

for every $t\geq 0$.

The proof follows the same lines as Theorems 4.4.6 and 4.6.31, then we omit it.

5.6.3 Probability of mass concentration and diameter contraction

This part is devoted to derive the proof of Corollary 5.1.3 supported by all the above results.

Let L and $L_{1/2}$ be intervals of diameter $2/5$ and $1/5$ centered around the order parameter ϕ_∞ of f_∞. Recall that by Corollary 5.2.13 we obtain

$$R_\infty=\lim_{t\to\infty}R(t)\geq 3/5.$$

Looking at the structure of the stable equilibria f_∞ in (5.1.14) (that corresponds to $g^- = 0$, that is, no antipodal mass), we observe that for any (θ, Ω) in $\operatorname{supp} f_\infty$ we have the relation

$$\theta = \phi_\infty + \arcsin\left(\frac{\Omega}{KR_\infty}\right).$$

In particular,

$$|\theta - \phi_\infty| \leq \arcsin\left(\frac{W}{KR_\infty}\right) \leq \arcsin\left(\frac{5}{3}\frac{W}{K}\right).$$

Then, we can select C in (5.1.15), so that we have that

$$\operatorname{supp} f_\infty \subseteq L_{\frac{1}{2}} \times [-W, W]. \tag{5.6.5}$$

Notice that the choice of the diameter of L is somehow arbitrary and is subordinated to the size of the universal constant C in Theorem 5.1.2 (the smaller C, the smaller the diameter of L). For simplicity, we have set it to $2/5$ but it can be generalized to sharper values. We divide the proof into the following steps:

• *Step a:* We control the mass of μ_t^N and f_t in $\mathbb{T}\backslash L$, namely,

$$\mu_t^N((\mathbb{T}\backslash L) \times \mathbb{R}) \leq 25\, SW_2(\mu_t^N, f_\infty)^2, \tag{5.6.6}$$

$$\rho_t(\mathbb{T}\backslash L) \leq 25\, SW_2(f_t, f_\infty)^2, \tag{5.6.7}$$

for any $t > 0$.

Fix $t > 0$ and let $\gamma_t \in \mathcal{P}\big((\mathbb{T} \times \mathbb{R}) \times (\mathbb{T} \times \mathbb{R})\big)$ be an optimal transport plan between μ_t^N and f_∞ for the scaled Wasserstein distance SW_2. Then, we have that

$$\begin{aligned}
SW_2(\mu_t^N, f_\infty)^2 &= \int_{(\mathbb{T}\times\mathbb{R})^2} d_K((\theta,\Omega),(\theta',\Omega'))^2 d\gamma_t \\
&\geq \int_{((\mathbb{T}\backslash L)\times\mathbb{R})\times\left(L_{1/2}\times\mathbb{R}\right)} d(\theta,\theta')^2 d\gamma_t \\
&\geq \frac{1}{25}\gamma_t\big((\mathbb{T}\backslash L)\times\mathbb{R}\big)\times\big(L_{1/2}\times\mathbb{R}\big) \\
&= \frac{1}{25}\Big[\gamma_t\big(((\mathbb{T}\backslash L)\times\mathbb{R})\times(\mathbb{T}\times\mathbb{R})\big) - \gamma_t\big(((\mathbb{T}\backslash L)\times\mathbb{R})\times\ ((\mathbb{T}\backslash L_{1/2})\times\mathbb{R})\big)\Big] \\
&\geq \frac{1}{25}\Big[(\gamma_t\big(((\mathbb{T}\backslash L)\times\mathbb{R})\times(\mathbb{T}\times\mathbb{R})\big) - \gamma_t\big((\mathbb{T}\times\mathbb{R})\times\big((\mathbb{T}\backslash L_{1/2})\times\mathbb{R}\big)\big)\Big] \\
&= \frac{1}{25}\Big[\mu_t^N((\mathbb{T}\backslash L)\times\mathbb{R}) - f_\infty((\mathbb{T}\backslash L_{1/2})\times\mathbb{R})\Big].
\end{aligned}$$

Thus, using the inclusion (5.6.5), we observe that the second term in the last line of the above inequality vanishes and we obtain (5.6.6). Similarly, using the above argument with μ_t^N replaced with f_t, we deduce that (5.6.7).

• *Step b:* We claim that we can select T_0 satisfying that

$$T_0 \lesssim \frac{1}{KR_0^2}\log\left(1 + W^{1/2}\|f_0\|_{L^2} + \frac{1}{R_0}\right), \tag{5.6.8}$$

and with the additional property that

$$SW_2(f_t, f_\infty) \leq \frac{1}{\sqrt{500}} e^{-\frac{1}{40}K(t-T_0)}, \tag{5.6.9}$$

for every t in $[T_0, \infty)$.

To show this, take Q_1 large enough and T_0 verifying

$$T_0 \leq \frac{Q_1}{KR_0^2} \log\left(1 + W^{1/2}\|f_0\|_{L^2} + \frac{1}{R_0}\right),$$

so that we meet the constraints in Theorem 5.1.2. Then, using (5.3.19) and Proposition 5.3.5 we obtain that

$$SW_2(f_t, f_\infty) \leq Q_2 e^{-\frac{1}{40}K(t-T_0)}, \tag{5.6.10}$$

for all t in $[T_0, \infty)$ and some universal constant Q_2. Notice that by taking Q_1 large enough, we can make Q_2 arbitrarily small (e.g. $Q_2 = \frac{1}{\sqrt{500}}$). This concludes the proof of the claim.

• *Step c:* We compute N in $\mathbb{N}$ and $d_N > 0$ for each $N \geq N^*$ so that

$$\mathbb{P}\left(SW_2(\mu_t^N, f_t) \leq \frac{1}{\sqrt{500}} e^{-\frac{1}{40}K(t-T_0)}\right) \geq 1 - C_1 e^{-C_2 N^{\frac{1}{2}}}, \tag{5.6.11}$$

for any t in $[T_0, T_0 + d_N]$ and any $N \geq N^*$.

First, for each N in $\mathbb{N}$ let us set the scale

$$\varepsilon_N := N^{-\frac{1}{8}}. \tag{5.6.12}$$

Now, we define N^* as follows

$$N^* := \min\left\{N \in \mathbb{N} : \varepsilon_N e^{\frac{5K}{2}T_0} \leq \frac{1}{\sqrt{500}}\right\}, \tag{5.6.13}$$

so that, by definition, we get the bound

$$N^* \geq 500^4 e^{20KT_0}.$$

Fix any $N \geq N^*$. Notice that N^* has been defined in (5.6.13) so that there exists $d_N > 0$ with the property

$$\varepsilon_N e^{\frac{5K}{2}(T_0+d_N)} = \frac{1}{\sqrt{500}} e^{-\frac{1}{40}Kd_N}, \tag{5.6.14}$$

Indeed, by dividing (5.6.14) over (5.6.13), we can quantify d_N in terms of N^* as follows

$$\frac{\varepsilon_N}{\varepsilon_{N^*}} e^{\frac{5K}{2}d_N} \geq e^{-\frac{1}{40}Kd_N}.$$

Consequently, we have that

$$d_N \geq \frac{5}{101K} \log \frac{N}{N^*}.$$

By construction, letting $\varepsilon = \varepsilon_N$ in the concentration inequality (5.6.3) of Lemma 5.6.4, we obtain the following quantification

$$\mathbb{P}\left(SW_2(\mu_0^N, f_0) \geq \varepsilon_N\right) \leq C_1 e^{-C_2 N^{\frac{1}{2}}}, \tag{5.6.15}$$

for every $N \in \mathbb{N}$. Thus, by monotonicity of the exponential function, we conclude that for any $t \in [T_0, T_0 + d_N]$ we have that

$$\begin{aligned}
C_1 e^{-C_2 N^{\frac{1}{2}}} &\geq \mathbb{P}\left(SW_2(\mu_0^N, f_0) \geq \varepsilon_N\right) \\
&\geq \mathbb{P}\left(SW_2(\mu_t^N, f_t) \geq \varepsilon_N e^{\frac{5K}{2}t}\right) \\
&\geq \mathbb{P}\left(SW_2(\mu_t^N, f_t) \geq \varepsilon_N e^{\frac{5K}{2}(T_0+d_N)}\right) \\
&= \mathbb{P}\left(SW_2(\mu_t^N, f_t) \geq \frac{1}{\sqrt{500}} e^{-\frac{1}{40}K d_N}\right) \\
&\geq \mathbb{P}\left(SW_2(\mu_t^N, f_t) \geq \frac{1}{\sqrt{500}} e^{-\frac{1}{40}K(t-T_0)}\right),
\end{aligned}$$

where in the first inequality we have used the concentration inequality (5.6.15), in the second one we have used the stability estimate in Lemma 5.6.6 and the remaining ones follow from our choice of d_N in (5.6.14) and t in $[T_0, T_0 + d_N]$. That ends the proof of (5.6.11).

• *Step d:* We quantify the probability of mass concentration of μ_t^N in the interval L, namely,

$$\mathbb{P}\left(\mu_t^N(L \times \mathbb{R}) \geq 1 - \frac{1}{5} e^{-\frac{1}{20}K(t-T_0)}\right) \geq 1 - C_1 e^{-C_2 N^{\frac{1}{2}}}, \tag{5.6.16}$$

for every t in $[T_0, T_0 + d_N)$ and any $N \geq N^*$.

Now, by (5.6.6), (5.6.9) and triangular inequality we have that

$$\begin{aligned}
\mu_t^N((\mathbb{T}\backslash L) \times \mathbb{R}) &\leq 25\, SW_2(\mu_t^N, f_\infty)^2 \\
&\leq 50 \left[SW_2(\mu_t^N, f_t)^2 + SW_2(f_t, f_\infty)^2\right] \\
&\leq 50 \left[SW_2(\mu_t^N, f_t)^2 + \frac{1}{500} e^{-\frac{1}{20}K(t-T_0)}\right],
\end{aligned}$$

for every t in $[T_0, T_0 + d_N)$. Hence, we obtain

$$\mu_t^N(L \times \mathbb{R}) \geq 1 - \frac{1}{10} e^{-\frac{3}{10}K(t-T_0)} - 50\, SW_2(\mu_t^N, f_t)^2,$$

for each t in $[T_0, T_0 + d_N]$. This, along with (5.6.11) concludes the proof of (5.6.16)

• *Step e:* We quantify the probability of mass concentration and diameter contraction along the time interval $[s, \infty)$ for any s in $[T_0, T_0 + d_N]$.

We are now ready to finish the proof of Corollary 5.1.3. Let us consider $N \geq N^*$, s in $[T_0, T_0 + d_N]$, and any realization of the random empirical measure μ^N (recall Definition 5.6.1) so that the condition within (5.6.16) holds. Hence, by construction, we obtain that at such realization

$$p := \inf_{\theta, \theta' \in L} \cos(\theta - \theta') \geq \frac{4}{5} \quad \text{and} \quad m := \mu_s^N(L \times \mathbb{R}) \geq 1 - \frac{1}{5} e^{-\frac{1}{20}K(s-T_0)} \geq \frac{4}{5}.$$

Then, we obtain the relation

$$mp - (1-m) = \frac{4}{5} \cdot \frac{4}{5} - \left(1 - \frac{4}{5}\right) = \frac{11}{25}.$$

In particular, take $\sigma := 2/5$ and notice that the above relations along with the assumption (5.1.15) in Theorem 5.1.2 guarantee the condition (5.4.5) within the hypotheses of Lemma 5.4.3. Notice that such result also holds true for the particle system. Consequently, it asserts that for such realization of μ^N we can consider a time-dependent interval $L_s^N(t)$ with $t \geq s$ so that $L_s^N(s) = L$ and

$$\begin{aligned} \mu_t^N(L_s^N(t) \times \mathbb{R}) &\geq 1 - \frac{1}{5}e^{-\frac{1}{20}K(s-T_0)}, \\ 1 - \inf_{\theta,\theta' \in L_s^N(t)} \cos(\theta - \theta') &\leq \max\left\{\frac{1}{5}e^{-\frac{K}{10}(t-s)}, 25\frac{W^2}{K^2}\right\}, \end{aligned} \tag{5.6.17}$$

for any $t \geq s$. Indeed, we have that $L_s^N(t) = \pi_\theta(X_{s,t}^N(L \times [-W, W]))$, where $X_{s,t}^N$ represents the flow of the particle system, that is, the flow of $v[\mu^N]$. Our final goal is to simplify the last condition in (5.6.17). To such an end, let us consider $D_s^N(t) := \operatorname{diam}(L_s^N(t))$ and notice that such inequality implies that

$$2\frac{(D_s^N(t))^2}{5} \leq 1 - \cos(D_s^N(t)) \leq \max\left\{\frac{1}{5}e^{-\frac{K}{10}(t-s)}, 25\frac{W^2}{K^2}\right\}, \tag{5.6.18}$$

for any $t \geq s$. In particular, we obtain (D). Thus, Corollary 5.1.3 follows.

Part III

Knotted vortex structures in Fluid Mechanics and stability results

CHAPTER 6

Stability results of generalized Beltrami fields and knotted vortex structures in the Euler equations

Contents

6.1 Introduction to the problem

Beltrami fields, that is, three dimensional vector fields whose curl is proportional to the field itself, are a particularly important class of smooth stationary solutions of the three-dimensional incompressible Euler equations:

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla p, \\ \operatorname{div} u = 0. \end{cases}$$

In a way, what makes them so special is the celebrated structure theorem of Arnold [13], which asserts that, under suitable technical hypotheses, the velocity field of a smooth stationary solution to the Euler equations is either a Beltrami field or "laminar", in the sense that it admits a regular first integral whose smooth level sets provide "layers" to which the fluid flow is tangent. In fluid mechanics, a Beltrami field is interpreted as a fluid whose velocity field is parallel to its vorticity.

Understanding the knot and link type of stream lines and tubes in stationary fluids has also attracted the attention of many researchers, both from the theoretical and the experimental points of view [115, 116, 190, 301], because knotted stationary vortex structures turned out to play a key role in the so called Lagrangian theory of turbulence. From a numerical point of view, the description of the flows in the literature that allow for arbitrary vortex structures is mainly based on an active vector formulation of Euler's equations (see [86] and the references therein). The existence of knotted and linked vortex lines and tubes in stationary solutions to the Euler equations was established in [115, 116] using *strong Beltrami fields*, that is, Beltrami fields with a constant proportionaly factor $\lambda \in \mathbb{R}\setminus\{0\}$:

$$\operatorname{curl} u = \lambda u. \tag{6.1.1}$$

Notice that the Beltrami fields in [115, 116] can be assumed to fall off as $1/|x|$ at infinity, and that this decay rate is optimal (see the global obstructions in the form of a Liouville-type theorem in [72, 227]). Concrete examples of Beltrami fields with constant proportionality factor are the ABC flows, whose analysis has yielded considerable insight into the aforementioned phenomenon of Lagrangian turbulence [113].

The main objective of this chapter is to study the existence, regularity and stability results of *generalized Beltrami fields* (i.e., Beltrami fields with nonconstant proportionality factor). This vector fields play a fundamental role in the understanding of turbulence. The idea that turbulent flows can be understood as a superposition of Beltrami flows has already been proposed in [87, 242]. They are also relevant in magnetohydrodynamics in the context of vanishing Lorentz force (*force-free fields*) and they can be used to model magnetic relaxation, which is relevant in some astrophysical applications [182, 193, 219, 218]. Indeed, to the best of our knowledge there are just a handful of explicit examples, all of which have Euclidean symmetries, and the analysis of Beltrami fields with nonconstant factor has proved extremely hard. The heart of the matter is that, as it was recently proved in [117], the equation for a generalized Beltrami field,

$$\left\{\begin{array}{l} \operatorname{curl} u = fu, \\ \operatorname{div} u = 0, \end{array}\right. \tag{6.1.2}$$

does not admit any nontrivial solution, even locally, for a "generic" nonconstant function f. In a very precise sense, it shows that Beltrami fields with a nonconstant factor are rare and such obstruction is of a purely local nature. These results have been carefully stated in the introductory Chapter 1.

One of the aims of this chapter is to show that, although generalized Beltrami fields are indeed rare, one can still prove some kind of partial stability result. Specifically, we will show that for each nontrivial Beltrami field, there are "many" close enough nonconstant proportionality factor with associated close nontrivial generalized Beltrami fields. The stabilility result is "partial" in the sense that a "full" stability result cannot be expected since the space of factors that enjoy nontrivial generalized Beltrami fields does not contain any ball in the $C^{k,\alpha}$ norm by the above-mentioned obstructions. The analysis of stability can be crucial to shed some light on the interactions between the different scales in the study of relevant configurations in a fully turbulent state. More concretely, we shall prove two stability results for generalized Beltrami fields:

The first one (Theorem 6.4.7) is an "almost global" perturbation result for strong Beltrami fields defined on $\mathbb{R}^3$. Roughly speaking, it asserts that given any nontrivial solution of (6.1.1) on $\mathbb{R}^3$ with optimal fall-off at infinity (i.e., $1/|x|$) and any arbitrarily small ball G, there are infinitely many nonconstant factors f, as close to the constant λ as one wishes in $C^{k,\alpha}(\mathbb{R}^3)$, such that the corresponding equation (6.1.2) admits nontrivial solutions on the complement $\mathbb{R}^3 \setminus \overline{G}$. This can be combined with the results in [116, 115] about robustness of the strong Beltrami fields therein to construct almost global Beltrami fields with a nonconstant factor that feature vortex lines and vortex tubes of arbitrarily complicated topology (Theorem 6.5.1). The second stability result (Theorem 6.6.3) states an analogue for perturbations of nontrivial Beltrami fields with both constant or nonconstant factor defined in a small enough open set where the field does not to vanish.

The point of these stability results is that the perturbation of the initial proportionality factor is defined by recursively propagating a two-variable function along the integral curves of a velocity vector field. Indeed, the flexibility in choosing the proportionality factor is granted by the method of our proof. Notice that the idea of constructing the proportionality factor by dragging along the integral curves of a field is somehow inherent to the problem, as the incompressibility condition $\operatorname{div} u = 0$ implies that, if it is nonconstant, the factor f must be a first integral of the generalized Beltrami field, i.e.,

$$u(x) \cdot \nabla f(x) = 0, \text{ for all } x \in \mathbb{R}^3.$$

6.2 The modified Grad–Rubin method

Let us outline the key aspects of the proofs. For concreteness, since all the ideas involved in the proof of the local partial stability result are essentially present in that of the almost global theorem, we shall restrict to discussing only discuss the latter result here. As we have already mentioned, the point of the partial stability result is to develop a perturbation technique allowing us to deform the initial factor f, which, for the purpose of this discussion, can be taken to be a nonzero constant λ. This requires analyzing a related boundary value problem, namely, the *Neumann boundary value problem for the inhomogeneous Beltrami equation* with constant proportionality factor λ in an exterior domains. To our best knowledge, this problem has not been directly studied in the literature. Our analysis is based on a boundary integral equation method for complex-valued solutions which requires some potential theory estimates for generalized volume and single layer potentials and an analysis of the decay properties and radiation conditions of the solutions. They will be determined through the natural connections between the complex-valued solutions of the Beltrami, Helmholtz and Maxwell systems.

One of the fist results where a perturbation method for Beltrami fields was obtained is [182]. In such paper, the authors showed that one can perturb Beltrami fields with very specific factor $\lambda = 0$ (that is, a harmonic field) defined in an exterior domain and construct a generalized Beltrami field with a small nonconstant factor. Apart from the smallness assumption on f, one of the inconveniences of their proof is that the perturbed fields and factors are of low regularity (of class $C^{1,\alpha}$ and $C^{0,\alpha}$, respectively). In view of the relevance and important applications of Beltrami fields with nonzero λ, we have striven to extend the result for harmonic fields to general Beltrami fields, and also to show the existence of perturbations of arbitrarily high regularity (the field will be in $C^{k+1,\alpha}$ and the factor in $C^{k,\alpha}$ for any fixed integer k). It should be stressed that the passing from $\lambda = 0$ to $\lambda \neq 0$ is not a trivial matter, since the behavior of the equations at infinity is completely different. Indeed, oversimplifying a little, for $\lambda = 0$ the behavior of the fields at infinity is that of a harmonic function, so one gets uniqueness simply from a decay condition, while for nonzero λ, Beltrami fields solve Helmhotz's equation, so radiation conditions must be specified to obtain uniqueness. Since we consider that they are of independent interest, we will present a detailed treatment of these topics in Section 6.3 and Appendix H of this thesis.

The gist of the proof of the almost global partial stability result for strong Beltrami fields is to study the convergence in $C^{k,\alpha}$ of an iterative scheme that, roughly speaking, takes the form

$$\begin{cases} \nabla\varphi_n \cdot u_n = 0, & x \in \Omega, \\ \varphi_n = \varphi^0, & x \in \Sigma, \end{cases} \qquad \begin{cases} \operatorname{curl} u_{n+1} - \lambda u_{n+1} = \varphi_n u_n, & x \in \Omega, \\ u_{n+1} \cdot \eta = u_0 \cdot \eta, & x \in S. \end{cases}$$

Here, Ω stands for an exterior domain with smooth boundary S, η is its outward unit normal vector field and Σ is some open subset of the boundary. We will call it *modified Grad–Rubin method*, see [6, 33] for the original Grad–Rubin method in the setting of force-free fields perturbations of harmonic fields. Such iterative method will be started up with an initial strong Beltrami field u_0 of constant proportionality factor λ (which indeed can be assumed to exhibit knotted and linked vortex structures according to [115, 116]) and we will prescribe the value φ^0 of the perturbation of the proportionality factor λ over subset Σ. Notice that $\{\varphi_n\}_{n\in\mathbb{N}}$ and $\{u_n\}_{n\in\mathbb{N}}$ are taken in a consistent way so that whenever we get some compactness and take limits φ and u, then φ is formally a global first integral of u and such vector field verifies the Beltrami equation (6.1.2) with $f = \lambda + \varphi$.

Consequently, our approach will be based on the study of two distinguished problems: 1) *stationary transport equations* along stream tubes, and 2) inhomogeneous problems of div-curl type that we will call *inhomogeneous Beltrami equations* and which are intimately linked to the Helmhotz equation. In fact, regarding the second problem, we will start with the complex-valued fundamental solution of the Helmholtz equation in $\mathbb{R}^3$

$$\Gamma_\lambda(x) = \frac{e^{i\lambda|x|}}{4\pi|x|},\ x \in \mathbb{R}^3\setminus\{0\},$$

and will arrive at a representation formula of Helmholtz–Hodge type for their complex-valued solutions. In doing so, it will be necessary to specify the optimal decay and radiation conditions that allow handling the associated generalized volume and single layer potentials. Indeed, such conditions will read as follows

$$\int_{\partial B_R(0)} |u(x)|\, d_xS = o(R^2), \quad R \to +\infty, \tag{6.2.1}$$

$$\int_{\partial B_R(0)} \left| i\frac{x}{R} \times u(x) - u(x)\right| d_xS = o(R), \quad R \to +\infty. \tag{6.2.2}$$

Here, (6.2.1) is nothing but a weak decay condition of the velocity field u in L^1 and (6.2.2) will be called the L^1 *Silver–Müller–Beltrami radiation condition* (L^1 SMB) and will be deduced from both the classical Sommerfeld and Silver–Müller radiation conditions, whose connections with the Helmholtz equation and the Maxwell system are classical.

Summing up, we will be interested in analyzing the existence and uniqueness of complex-valued smooth solutions with high order Hölder-type regularity of the general *Neumann boundary value problem for the inhomogeneous Beltrami equation* (NIB)

$$\begin{cases} \operatorname{curl} u - \lambda u = w, & x \in \Omega, \\ u \cdot \eta = g, & x \in \Omega, \\ + L^1 \text{ decay property (6.2.1)}, \\ + L^1 \text{ SMB radiation condition (6.2.2)}. \end{cases} \tag{6.2.3}$$

Notice that although we were originally interested in real-valued Beltrami fields, we will be concerned with complex-valued solutions to (6.2.3) and we will then take real parts to obtain the real-valued ones. The reason to do it is twofold. Firstly, this will allow us to employ a representation formula for complex-valued *radiating* fields. Secondly, this presents no problems related to the application to knotted structures as one can realize the fields in [115, 116] as real parts of complex-valued radiating Beltrami fields. Problem (6.2.3) was previously studied in [193], where the author proved C^1 regularity results in bounded domains. We introduce some potential theory estimates of high order for generalized potentials associated with inhomogeneous kernels in exterior domains and adapt the boundary integral method to the unbounded setting. We will also improve regularity from C^1 to $C^{k+1,\alpha}$.

Consequently, our final modified Grad–Rubin method is described as follows:

$$\left\{\begin{array}{ll}\nabla\varphi_n\cdot u_n=0, & x\in\Omega,\\ \varphi_n=\varphi^0, & x\in\Sigma,\end{array}\right. \qquad \left\{\begin{array}{ll}\operatorname{curl} v_{n+1}-\lambda v_{n+1}=\varphi_n u_n, & x\in\Omega,\\ v_{n+1}\cdot\eta=u_0\cdot\eta, & x\in S,\\ +\,L^1 \text{ Decay property } (6.2.1), & \\ +\,L^1 \text{ SBM radiation condition } (6.2.2), & \end{array}\right. \tag{6.2.4}$$

where $u_n=\Re v_n$ are the real parts of the complex-valued solutions v_n. The compactness of $\{u_n\}_{n\in\mathbb{N}}$ in $C^{k+1,\alpha}(\overline{\Omega},\mathbb{R}^3)$ follows from some Schauder estimates of Equation (6.2.3) in Hölder spaces. Similarly, $\{\varphi_n\}_{n\in\mathbb{N}}$ will be shown to be compact in $C^{k,\alpha}(\overline{\Omega})$ too. Concerning the application to solutions u_0 with knotted vortex structures of the type constructed in [115, 116], we will see that the solution u inherits the knotted vortex structures from u_0 (up to a small deformation) by virtue of structural stability. This is a straightforward consequence of the fact that u can be chosen close to u_0 as long as the prescribed value φ^0 is small enough.

The rest of this chapter is organized as follows. On the one hand, Problem (6.2.3) will be studied in Section 6.3 by extending the results in [193, 229, 298]. By comparison with the *vector-valued divergence-free Helmhotz equation*, the *reduced Maxwell system* and the Beltrami equation, we will deduce the apropriate radiation and decay conditions. The SMB radiation condition (6.2.2) will then be connected with the classical Silver–Müller and Sommerfeld radiation conditions and we will then present a representation formula of Helmholtz–Hodge type which involves these radiation conditions and that will be extremely useful to obtain our existence, uniqueness and regularity results. On the other hand, the analysis of the linear transport equations in the left hand side of (6.2.4) is obtained in Section 6.4 and we shall then conclude the convergence of the iterative scheme by putting both components together. In Section 6.5 we combine the above results to construct small perturbations of the constant proportionality factor λ leading to nontrivial generalized Beltrami fields that exhibit the same kind of knots and links and so to construct stationary solutions to the Euler equations. The local partial stability result for generalized Beltrami fields will be discussed in Section 6.6. Finally, Appendix 6.A contains the technicalities of the proof of regularity for parametrizations of stream tubes of velocity fields with Hölder-type regularity, that is introduced in Proposition 6.4.3.

We recall that, in order to support the involved regularity results, Appendix H of this thesis provides the necessary Hölder estimates of high order for volume and single layer potentials associated with the inhomogeneous kernel $\Gamma_\lambda(x)$. The underlying ideas can be adapted to many other general inhomogeneous kernels with an appropriate decay at infinity. Also, we refer to Appendix G for some useful list of identities regarding the differential operators ∇_S, div_S and curl_S on hypersurfaces of the Euclidean space that will be systematically used in this chapter.

Specific notation for this chapter

We shall consider domains G, Ω and surfaces S fulfilling the properties below:

- G is a C^{k+5} bounded domain homeomorphic to an Euclidean ball and containing the origin, i.e., $0 \in G$.
- $\Omega := \mathbb{R}^3 \setminus \overline{G}$ is its exterior domain and $S := \partial\Omega = \partial G$ is the boundary surface. (6.2.5)
- η denotes the outward unit normal vector field of S.

We remark that most of our results hold under weaker assumption on the boundary regularity (specifically $C^{k+1,\alpha}$ boundaries). However, there are certain very specific results which require S to be at least C^{k+5}, because higher order derivatives of the normal vector field η are involved, see for instance Theorem H.2.1) in Appendix .

Along this chapter, we will mainly use the inhomogeneous Hölder spaces of higher order $C^{k,\alpha}(\overline{\Omega})$. They have been set in the introductory section *Conventions and notation* of this thesis. To avoid confussion, we recall them here. Let us agree to say that $C^k(\Omega)$ is the space of functions of class C^k on Ω with finite C^k norm (meaning that all their derivatives up to order k are bounded). We will replace Ω by $\overline{\Omega}$ when the function and all its derivatives up to order k can be continuously extended to the closure of Ω. The space $C^{k,\alpha}(\Omega)$ is the Hölder space with exponent $\alpha \in (0,1)$ and k-th order regularity endowed with the inhomogeneous norm (N.5). Vector-valued analogs of these spaces are denoted in the usual fashion, e.g. $C^{k,\alpha}(\Omega, \mathbb{R}^3)$. We will use similar notation $C^k(S)$, $C^{k,\alpha}(S)$ for functions defined on S and $\mathfrak{X}^{k,\alpha}(S) \equiv \mathfrak{X}^{k,\alpha}(S, \mathbb{R}^3)$ for tangent vector fields along S of regularity $C^{k,\alpha}$.

6.3 Neumann problem for the inhomogeneous Beltrami equation

In this section we analyze the existence and uniqueness of solutions in $C^{k+1,\alpha}$ of the NIB problems (6.2.3) arising in the modified Grad–Rubin iterative method (6.2.4). The key tool is a representation formula of Helmholtz–Hodge type for its solutions, which we will combine with the well-posedness of the underlying boundary integral equation for the tangential components in the space of $C^{k+1,\alpha}$ tangent vector fields to the boundary. For this we will need to deal with some regularity results for high order derivatives of generalized volume and single layer potentials arising in the classical potential theory, which will require some potential-theoretic estimates for inhomogeneous singular integral kernels that are relegated to Appendix H for simplicity of exposition. Regarding the representation formula, we will introduce and discuss in detail the weakest decay and radiation conditions under which this formula holds (namely, (6.2.1) and (6.2.2)), as this topic is of independent interest. Notice that many other radiation conditions have been used in the literature for related models: the natural one for the scalar complex-valued Helmholtz equation is the *Sommerfeld radiation condition* and those of the reduced Maxwell system are called the *Silver–Müller radiation conditions* (SM) (see e.g. [84, 85, 228, 302]).

Let us first recall some previous results in the literature about the exterior NIB boundary value problem (6.2.3). Although the same problem is studied in [193] for bounded domains and C^1 vector fields, the technique that we present in this section has not been studied in the case of exterior domains and $C^{k,\alpha}$-regularity. We recall that in [193] it was essential to assume that λ is "regular" with respect to the interior problem. This is the case when λ is not a Dirichlet eigenvalue of the Laplacian in the interior domain, or if it is a simple eigenvalue whose eigenfunction has non-zero mean. Then, such a condition is not restrictive and it holds generically, as it can be seen e.g. by considering arbitrarily small rescalings of the domain. Related results for exterior domains are proved in [229]. Indeed, the technique used in bounded domain by [298] and [193] (for $\lambda = 0$ and $\lambda \neq 0$, respectively) goes through to the case of $\lambda = 0$ and exterior domains via sharp estimates of harmonic volume and single layer potentials in $C^{1,\alpha}$. In our case λ is a nonzero constant, which leads to inhomogeneous kernels where the estimates in unbounded domains are much harder to obtain.

There is some literature regarding Laplace's equation in less regular settings (e.g. L^p data and Lipschitz domains). For C^1 domains, [93, 94] solved it via the analysis of harmonic measures and [123] introduced a method of layer potentials. The latter looks like the method that we propose and is supported by Fredholm's theory: some boundary singular integral operator is shown to be compact and one to one in the C^1 setting, leading to biyectivity and an useful lower estimate that entails the well posedness. For purely Lipschitz domains, compactness does no longer hold [122] whilst biyectivity is preserved [95]. Regarding non-symmetric elliptic operators $L = -\operatorname{div}(A(x)\nabla\cdot)$ in the half-space $(x,t) \in \mathbb{R}^n \times \mathbb{R}^+$, the well posedness of the Dirichlet problem with L^p data [171] follows from the method of "ε-approximability" and the absolute continuity of the L-harmonic measure with respect to the surface measure.

Let us now analyze the representation formula, the radiation conditions and some existence and uniqueness results for the scalar complex-valued Helmholtz equation. We will introduce some classical notation and powerful tools like the *far field pattern* of a *radiating* solution not only in the homogeneous setting but also in the inhomogeneous one. All these results will be later used and extended to the NIB problem (6.2.3) in the subsequent parts of this section.

6.3.1 Inhomogeneous Helmholtz equation in the exterior domain

The Helmholtz equation with wave number $\lambda \in \mathbb{R}$ in the exterior domain Ω stands for the elliptic PDE

$$\Delta a + \lambda^2 a = 0, \ \ x \in \Omega,$$

where the unknown is a possibly complex-valued scalar function $a \in C^2(\Omega, \mathbb{C})$. This equation arises in acoustic and electromagnetic mathematics [85, 228] and in the study of Beltrami fields arising either from the incompressible Euler equation or from the force-free field system of magnetohydrodynamics. Indeed, it can be derived from (6.1.1) by taking curl and noting that Beltrami fields are divergence-free when $\lambda \neq 0$.

This relation with the Beltrami equation suggests studying the representation formulas, radiation conditions and uniqueness result for the Helmholtz equation.

Definition 6.3.1. *We will say that a complex-valued scalar function $a \in C^1(\Omega, \mathbb{C})$ verifies*

- *the L^1 Sommerfeld radiation condition if*

$$\int_{\partial B_R(0)} \left| \nabla a(y) \cdot \frac{y}{R} - i\lambda a(y) \right| \, d_y S = o\left(R\right), \ \ R \to +\infty. \tag{6.3.1}$$

- *the L^1 decay property at infinity if*

$$\int_{\partial B_R(0)} |a(y)| \, d_y S = o(R^2), \ \text{when } R \to +\infty. \tag{6.3.2}$$

Other stronger radiation conditions may be assumed to obtain representation formulas and certain uniqueness results [85, 228]. For instance, the L^2 Sommerfeld radiation condition

$$\int_{\partial B_R(0)} \left|\nabla a(y)\cdot\frac{y}{R} - i\lambda a(y)\right|^2 d_yS = o(1),\quad R\to+\infty, \tag{6.3.3}$$

implies (6.3.1) and, in turns, the classical (L^∞) Sommerfeld radiation condition

$$\sup_{y\in\partial B_R(0)} \left|\nabla a(y)\cdot\frac{y}{R} - i\lambda a(y)\right| = o\left(\frac{1}{R}\right),\quad R\to+\infty, \tag{6.3.4}$$

implies (6.3.3). There is another stronger link between the L^2 and L^1 conditions that will be exhibited in the next results. The proof follows from a simple expansion of the square in the L^2 condition (6.3.3) and an integration by parts argument in the Helmholtz equation multiplied by the solution itself.

Remark 6.3.2. *Let $a\in C^2(\Omega,\mathbb{C})\cap C^1(\overline{\Omega},\mathbb{C})$ be any complex-valued solution to the Helmholtz equation such that (6.3.3) holds. Then*

$$\lim_{R\to+\infty}\int_{\partial B_R(0)}\left(\left|\frac{\partial a}{\partial\eta}\right|^2+\lambda^2|a|^2\right)d_xS = -2\lambda\Im\left(\int_S a\frac{\partial\overline{a}}{\partial\eta}\,d_xS\right).$$

In particular, (6.3.3) $\Rightarrow$ (6.3.1) + (6.3.2) *for each complex-valued solution of the Helmholtz equation.*

Before showing that this radiation condition leads to the aforementioned formula, let us analyze it in the case of the fundamental solution to the 3-D Helmholtz equation,

$$\Gamma_\lambda(x) = \frac{e^{i\lambda|x|}}{4\pi|x|} = \frac{\cos(\lambda|x|)}{4\pi|x|} + i\frac{\sin(\lambda|x|)}{4\pi|x|}. \tag{6.3.5}$$

Taking derivatives we obtain

$$\nabla\Gamma_\lambda(x) = \left(i\lambda - \frac{1}{|x|}\right)\Gamma_\lambda(x)\frac{x}{|x|}. \tag{6.3.6}$$

Then, a straightforward inductive argument shows that all the partial derivatives of $\Gamma_\lambda(x)$ up to second order verify an even stronger version of the Sommerfeld radiation condition (6.3.4). Hence we easily infer:

Proposition 6.3.3. *The fundamental solution of the Helmholtz equation, together with its partial derivatives up to order 2 satisfy the identities*

$$\begin{aligned}
\nabla\Gamma_\lambda\cdot\frac{x}{|x|} - i\lambda\Gamma_\lambda &= -\frac{\Gamma_\lambda(x)}{|x|},\\
\nabla\left(\frac{\partial\Gamma_\lambda}{\partial x_i}\right)\cdot\frac{x}{|x|} - i\lambda\frac{\partial\Gamma_\lambda}{\partial x_i} &= \left(\frac{2}{|x|}-i\lambda\right)\Gamma_\lambda\frac{x_i}{|x|^2},\\
\nabla\left(\frac{\partial^2\Gamma_\lambda}{\partial x_i\partial x_j}\right)\cdot\frac{x}{|x|} - i\lambda\frac{\partial^2\Gamma_\lambda}{\partial x_i\partial x_j} &= -\nabla\left(\frac{\partial\Gamma_\lambda}{\partial x_i}\right)\cdot\frac{\partial}{\partial x_j}\left(\frac{x}{|x|}\right) + \frac{\partial}{\partial x_j}\left(\left(\frac{2}{|x|}-i\lambda\right)\Gamma_\lambda\frac{x_i}{|x|^2}\right),
\end{aligned}$$

for every $i,j\in\{1,2,3\}$. Consequently,

$$\sup_{x\in\partial B_R(0)}\left|\nabla(D^\gamma\Gamma_\lambda)(x)\cdot\frac{x}{R} - i\lambda D^\gamma\Gamma_\lambda(x)\right| = O\left(\frac{1}{R^2}\right),\ \textit{for } R\to+\infty,$$

for every multi-index with $|\gamma|\le 2$.

In particular, $\Gamma_\lambda(x)$ together with its partial derivatives up to order two verify the (L^∞) Sommerfeld radiation condition (6.3.4). It is then an easy task to obtain new complex-valued solutions to the homogeneous Helmholtz equation enjoying such radiation condition through the definition of the generalized single layer potentials associated with the kernel $\Gamma_\lambda(x)$.

Proposition 6.3.4. *Let a be the generalized single layer potential with density $\zeta \in C(S)$ associated with the Helmholtz equation, i.e.,*

$$a(x) := (\mathcal{S}_\lambda \zeta)(x) = \int_S \Gamma_\lambda(x-y)\zeta(y)\, d_yS,$$

for every $x \in \Omega$. Then, a solves the homogeneous Helmholtz equation $\Delta a + \lambda^2 a = 0$ in the exterior domain Ω. Moreover, a and all its partial derivatives up to second order verify the Sommerfeld radiation condition (6.3.4).

The same result remains true for generalized volume potential with compactly supported densities. In this case, radiating solutions for the inhomogeneous complex-valued Helmholtz equation can be obtained.

Proposition 6.3.5. *Let a be the generalized volume potential with density $\zeta \in C_c(\overline{\Omega})$ associated with the Helmholtz equation, i.e.,*

$$a(x) := (\mathcal{N}_\lambda \zeta)(x) = \int_\Omega \Gamma_\lambda(x-y)\zeta(y)\, d_yS,$$

for every $x \in \Omega$. Then, a solves the inhomogeneous Helmholtz equation $-(\Delta a + \lambda^2 a) = \zeta$ in the exterior domain Ω. Moreover, a and all its partial derivatives up to second order verify the Sommerfeld radiation condition (6.3.4).

To establish the representation formula for the inhomogeneous Helmholtz equation, we study the radiation conditions for the volume and single layer potentials, as well as its decay properties at infinity (see Theorem C.0.4 in Appendix C for the fall-off at infinity of the fractional integral operator $I_\beta f$). Using it, the above result permit obtaining a Stokes-type formula to represent the solutions to the inhomogeneous Helmholtz equation. We will skip the proof, since it is completely analogous to the more important result for complex-valued solutions of the inhomogeneous Beltrami equation that we present in the next subsection (Theorem 6.3.11). We also refer to [85, Theorem 2.4] and [228, Theorem 3.1.1] for a proof with more restrictive radiation conditions that can be recovered from the next stronger version via Remark 6.3.2.

Theorem 6.3.6. *Let $a \in C^2(\Omega, \mathbb{C}) \cap C^1(\overline{\Omega}, \mathbb{C})$ be any function which verifies the L^1 Sommerferld radiation condition (6.3.1) and the L^1 decay property at infinity (6.3.2). Assume that $\Delta a + \lambda^2 a = O(|x|^{-\rho})$ when $|x| \to +\infty$, for some exponent $2 < \rho < 3$. Then,*

$$\begin{aligned} a(x) = &- \int_\Omega \Gamma_\lambda(x-y)(\Delta a(y) + \lambda^2 a(y))\, dy \\ &+ \int_S \frac{\partial \Gamma_\lambda(x-y)}{\partial \eta(y)} a(y)\, d_yS \\ &- \int_S \Gamma_\lambda(x-y) \frac{\partial a}{\partial \eta}(y)\, d_yS, \end{aligned} \tag{6.3.7}$$

for every $x \in \Omega$ and, as a consequence,

$$a = O(|x|^{-(\rho-2)}), \text{ when } |x| \to +\infty.$$

Indeed, when $\Delta a + \lambda^2 a$ has compact support, we recover the optimal decay at infinity, namely,

$$a = O(|x|^{-1}), \text{ when } |x| \to +\infty.$$

The decay property follow from Theorem C.0.4 and they may also be found in [85, 228]. Notice that the decay rates $|x|^{-(\rho-2)}$ (for the inhomogeneous equation) and $|x|^{-1}$ (for the homogeneous one) are straighforward consequences of the representation formula.

An immediate consequence of the representation formulas in Theorem 6.3.6 is that a *far field pattern* at infinity exists for each solution to the Helmholtz equation (see [85] for details). It is a very powerful tool since it provides a description of the asymptotic behavior at infinity and easy uniqueness criteria for radiating solutions. Although most of the literature is only devoted to far field patterns of complex-valued radiating solutions to the homogeneous Helmholtz equation, our problem concerns the inhomogeneous setting. For this, consider any solution $a \in C^2(\Omega, \mathbb{C}) \cap C^1(\overline{\Omega}, \mathbb{C})$ to the inhomogeneous Helmholtz equation

$$-(\Delta a + \lambda^2 a) = f, \qquad x \in \Omega,$$

where f is compactly supported in $\overline{\Omega}$ and a verifies both the decay condition (6.3.2) and the L^1 Sommerfeld radiation condition (6.3.1). Then, Theorem 6.3.6 leads to

$$a(x) = \int_\Omega \Gamma_\lambda(x-y) f(y)\,dy + \int_S \frac{\partial \Gamma_\lambda(x-y)}{\partial \eta(y)} a(y)\,d_yS - \int_S \Gamma_\lambda(x-y) \frac{\partial a}{\partial \eta}(y)\,d_yS.$$

Consider the compact subset $K := \operatorname{supp} f$ and notice the asymptotic behavior

$$\begin{aligned} \Gamma_\lambda(x-y) &= \Gamma_\lambda(x) \left\{ e^{-i\lambda \frac{x}{|x|}\cdot y} + O\left(\frac{1}{|x|}\right) \right\}, \qquad \text{when } |x| \to +\infty, \\ \frac{\partial \Gamma_\lambda(x-y)}{\partial \eta(y)} &= \Gamma_\lambda(x) \left\{ \frac{\partial e^{-i\lambda \frac{x}{|x|}\cdot y}}{\partial \eta(y)} + O\left(\frac{1}{|x|}\right) \right\}, \quad \text{when } |x| \to +\infty, \end{aligned}$$

where $O\left(|x|^{-1}\right)$ is uniform in $y \in K \cup S$ in the first formula and uniform in $y \in S$ in the second one. From here we deduce the asymptotic behavior

$$a(x) = \Gamma_\lambda(x) \left\{ a_\infty\left(\frac{x}{|x|}\right) + O\left(\frac{1}{|x|}\right) \right\}, \text{ when } |x| \to +\infty, \tag{6.3.8}$$

where a_∞ is called the *far field pattern* of a, and reads as

$$a_\infty(\sigma) = \int_\Omega e^{-i\lambda\sigma\cdot y} f(y)\,dy + \int_S \frac{\partial e^{-i\lambda\sigma\cdot y}}{\partial \eta(y)} a(y)\,d_yS - \int_S e^{-i\lambda\sigma\cdot y} \frac{\partial a}{\partial \eta}(y)\,d_yS,$$

for each point $\sigma \in \partial B_1(0)$. It is apparent that a_∞ is uniquely determined from formula (6.3.8). Hence, we can define the following well-defined linear and one to one map

$$\begin{array}{ccc} \mathcal{D}_\infty & \longrightarrow & C^\infty(\partial B_1(0)) \\ a & \longmapsto & a_\infty, \end{array} \tag{6.3.9}$$

where the domain of the *far field pattern mapping* is

$$\mathcal{D}_\infty := \{a \in C^2(\Omega, \mathbb{C}) \cap C^1(\overline{\Omega}, \mathbb{C}) \,:\, \Delta a + \lambda^2 a \text{ has compact support and } (6.3.1) - (6.3.2) \text{ hold}\}.$$

A similar reasoning leads to an explicit formula for the far field pattern of the derivatives of a, namely,

$$(\nabla a)_\infty(\sigma) = i\lambda a_\infty(\sigma)\sigma, \ \forall \sigma \in \partial B_1(0). \tag{6.3.10}$$

The split in (6.3.8) ensures that

$$\lim_{R\to+\infty}\int_{\partial B_R(0)}|a(x)|^2\,dx=\frac{1}{4\pi}\int_{\partial B_1(0)}|a_\infty(\sigma)|^2\,d_\sigma S. \tag{6.3.11}$$

By *Rellich' Lemma* [85, Lemma 2.11], the only complex-valued solution $a\in C^2(\Omega,\mathbb{C})$ to the exterior homogeneous Helmholtz equation such that the limit in the left hand side of (6.3.11) becomes zero is the zero function identically. Therefore, whenever a solution a to the homogeneous Helmholtz equation verifies (6.3.1)-(6.3.2) and its far field pattern a_∞ vanishes, then a vanishes everywhere. This can be used, in combination with Remark 6.3.2, to achieve the following uniqueness result, that, in particular, can be used for Dirichlet and Neumann boundary value problems in the exterior domain (see [85, Theorem 2.12]).

Lemma 6.3.7. *Consider any solution $a\in C^2(\Omega,\mathbb{C})\cap C^1(\overline{\Omega},\mathbb{C})$ to the complex-valued homogeneous Helmholtz equation in the exterior domain Ω fulfilling the L^2 Sommerfeld radiation condition (6.3.3). Then, a verifies*

$$\lambda\Im\left(\int_S a(x)\frac{\partial\overline{a}}{\partial\eta}(x)\,d_xS\right)\leq 0.$$

If the equality holds, then a vanishes everywhere in Ω.

In the case of vector-valued solutions, the decay property and radiation conditions can be considered componentwise. For instance, given any vector-valued solution $u\in C^2(\Omega,\mathbb{C}^3)\cap C^1(\overline{\Omega},\mathbb{C}^3)$ to

$$-(\Delta u+\lambda^2 u)=F,\qquad x\in\Omega,$$

where F is compactly supported, then the decay property and radiation condition read

$$\int_{\partial B_R(0)}|u(x)|\,dx=o(R^2),\quad \text{when } R\to+\infty,$$
$$\int_{\partial B_R(0)}\left|\operatorname{Jac}u(x)\frac{x}{R}-i\lambda u(x)\right|\,dx=o(R),\quad \text{when } R\to+\infty.$$

One can wonder whether there are more natural radiation conditions for vector-valued solutions to Helmholtz equation, see [84, Theorem 4.13] and [302, Section 5, Theorem 2]. Straightforward computations using $\operatorname{curl}(\operatorname{curl}u)-\nabla(\operatorname{div}u)-\lambda^2u=F$ in Ω, and $\operatorname{curl}(\operatorname{curl}(\operatorname{curl}u))-\lambda^2\operatorname{curl}u=\operatorname{curl}F$ in Ω show that the terms associated with the far field patterns vanish and we obtain the radiation conditions

$$\sup_{x\in\partial B_R(0)}\left|\frac{x}{R}\times\operatorname{curl}u(x)-\operatorname{div}u(x)\frac{x}{R}+i\lambda u(x)\right|=o\left(\frac{1}{R}\right),\ \text{ when } R\to+\infty,$$
$$\sup_{x\in\partial B_R(0)}\left|\lambda\frac{x}{R}\times u(x)+i\operatorname{curl}u(x)\right|=o\left(\frac{1}{R}\right),\ \text{ when } R\to+\infty.$$

When u is a divergence-free solution to the Helmholtz equation (as in our case), the radiation condition are simpler and read

$$\sup_{x\in\partial B_R(0)}\left|\frac{x}{R}\times\operatorname{curl}u(x)+i\lambda u(x)\right|=o\left(\frac{1}{R}\right),\ \text{ when } R\to+\infty, \tag{6.3.12}$$

$$\sup_{x\in\partial B_R(0)}\left|\lambda\frac{x}{R}\times u(x)+i\operatorname{curl}u(x)\right|=o\left(\frac{1}{R}\right),\ \text{ when } R\to+\infty. \tag{6.3.13}$$

6.3.2 Inhomogeneous Beltrami equation in the exterior domain

Now, we move to the complex-valued inhomogeneous Beltrami equation. In order to understand where the natural radiation condition (6.2.2) comes from, we will connect three different systems that will provide an appropriate terminology. The heuristic idea is summarized in Figure 6.1. Through the relations between the vector fields u and B in the left hand side of such

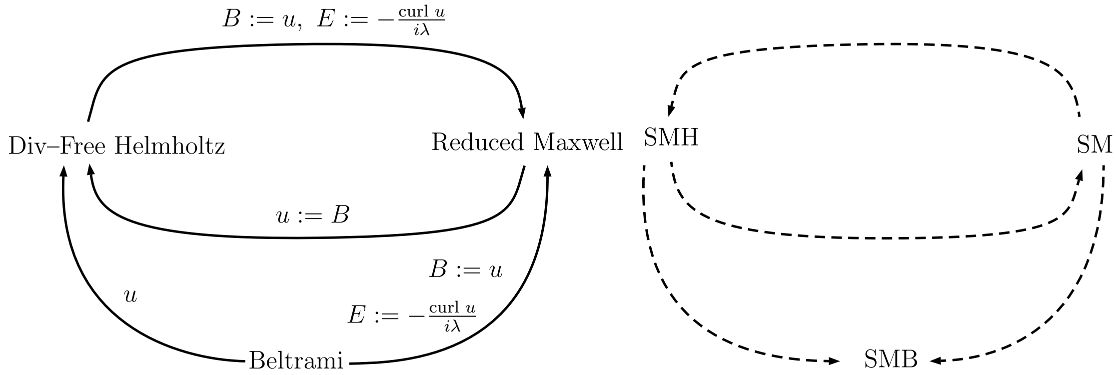


Figure 6.1: Sketch of the connections between the three related models: *divergence-free Helmholtz equation*, *reduced Maxwell system* and *Beltrami equation*. The picture in the left shows the bonds between such models whilst the picture in the right exhibits the associated relations between its natural radiation conditions.

pictures, we find (see [85, Theorem 6.4] and [302]) that the divergence-free Helmholtz equation and the reduced Maxwell system [85, Definition 6.5] are completely equivalent, i.e.,

$$\left\{\begin{array}{ll} \Delta u + \lambda^2 u = 0, & x \in \Omega, \\ \operatorname{div} u = 0, & x \in \Omega, \end{array}\right. \Longleftrightarrow \left\{\begin{array}{ll} \operatorname{curl} E - i\lambda B = 0, & x \in \Omega, \\ \operatorname{curl} B + i\lambda E = 0, & x \in \Omega. \end{array}\right.$$

In order that the solutions to this system could be represented through the classical *Stratton–Chu formulas* [85, Theorem 6.6], the *Silver–Müller radiation conditions* (SM) have to be considered:

$$\sup_{x\in\partial B_R(0)} \left|B(x) \times \frac{x}{R} - E(x)\right| = o\left(\frac{1}{R}\right), \quad \text{when } R \to +\infty,$$
$$sup_{x\in\partial B_R(0)} \left|E(x) \times \frac{x}{R} + B(x)\right| = o\left(\frac{1}{R}\right), \quad \text{when } R \to +\infty.$$

Due to our choice of B and E, the SM radiation conditions leads to (6.3.12)–(6.3.13) again. Thus, the natural radiation conditions for the divergence-free vector-valued Helmholtz equation are actually a consequence of the SM radiation conditions for the reduced Maxwell system. Therefore, we will call them the *Silver–Müller–Helmholtz radiation conditions* (SMH).

Let us now consider the case of the Beltrami equation

$$\operatorname{curl} u - \lambda u = 0, \quad x \in \Omega.$$

When $\lambda \neq 0$, then u is a solution to the divergence-free Helmholtz equation, and consequently it also solves the reduced Maxwell system. Therefore, one may want to transfer the SMH or the original SM radiation condition to the Beltrami framework. An easy substitution in (6.3.12) and (6.3.13) leads to the *Silver–Müller–Beltrami radiation condition* (SMB):

$$\sup_{x\in\partial B_R(0)} \left|i\frac{x}{R} \times u(x) - u(x)\right| = o\left(\frac{1}{R}\right), \text{ when } R \to +\infty.$$

It might seem that the only connection between the Beltrami equation and the divergence-free vector-valued Helmholtz equation is the first implication sketched in Figure 6.1, but the connection is actually much stronger. The reason is the following. Given any solution u to the Beltrami equation, it is obviously a solution to the divergence-free Helmholtz equation. The point is that, conversely, given any solution $\widehat{u}$ to the divergence-free Helmholtz equation,

$$u := \frac{\operatorname{curl}\widehat{u} + \lambda\widehat{u}}{2\lambda}. \tag{6.3.14}$$

is a solution to the Beltrami equation, and all the solutions can be constructed this way.

In view of this converse relation, it is natural to wonder about the radiation conditions that one should assume on $\widehat{u}$ in order for u to verify the SMB radiation condition. For this, notice that

$$i\frac{x}{R} \times u(x) - u(x) = \frac{i}{2\lambda}\left(\frac{x}{R} \times \operatorname{curl}\widehat{u}(x) + i\lambda\widehat{u}(x)\right) + \frac{i}{2\lambda}\left(\lambda\frac{x}{R} \times \widehat{u}(x) + i\operatorname{curl}\widehat{u}(x)\right),$$

for every $x \in \partial B_R(0)$. Therefore, the SMB radiation condition on u is recovered form the SMH radiation conditions on $\widehat{u}$, so all the possible links between the three models and its corresponding radiation conditions in Figure 6.1 follow.

Remark 6.3.8. *The complex-valued Beltrami fields u satisfying the SMB radiation condition take the form (6.3.14) for some solution $\widehat{u}$ of the divergence-free Helmholtz equation satisfying the SMH radiation conditions.*

Definition 6.3.9. *We will say that u verify*

1. *the L^1 Silver–Müller–Beltrami condition if*

$$\int_{\partial B_R(0)} \left|i\frac{x}{R} \times u(x) - u(x)\right| d_xS = o(R), \text{ when } R \to +\infty; \tag{6.3.15}$$

2. *the L^1 decay property at infinity if*

$$\int_{\partial B_R(0)} |u(x)|\, d_xS = o(R^2), \text{ when } R \to +\infty. \tag{6.3.16}$$

Analogously to the case of the Helmholtz equation, one might consider the L^2 SMB radiation condition

$$\int_{\partial B_R(0)} \left|i\frac{x}{R} \times u(x) - u(x)\right|^2 d_xS = o(1),\ R \to +\infty, \tag{6.3.17}$$

or the (L^∞) SMB radiation condition

$$\sup_{x\in\partial B_R(0)} \left|i\frac{x}{R} \times u(x) - u(x)\right| = o\left(\frac{1}{R}\right),\ R \to +\infty. \tag{6.3.18}$$

As in the Helmholtz equation, similar reasonings yield the next remark that links (6.3.17) to (6.3.15) and (6.3.16).

Remark 6.3.10. *Let $u \in C^1(\overline{\Omega}, \mathbb{C}^3)$ be any complex-valued solution to the Beltrami equation such that (6.3.17) holds. Then*

$$\lim_{R\to+\infty} \int_{\partial B_R(0)} \left(\left|\frac{x}{R} \times u(x)\right|^2 + |u(x)|^2\right) d_xS = 2\Im\left(\int_S \overline{u}(x) \cdot (\eta(x) \times u(x))\, d_xS\right).$$

In particular, (6.3.17) $\Rightarrow$ (6.3.15) + (6.3.16) *for each complex-valued solution of the Beltrami equation.*

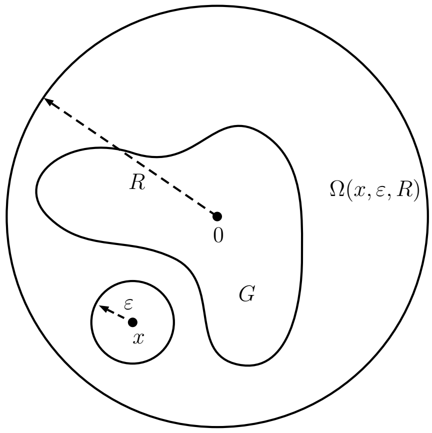


Figure 6.2: Domain $\Omega(x,\varepsilon,R)$.

In the next result we show the desired decomposition theorem of Helmholtz–Hodge type is proved under the above L^1 decay and radiation hypotheses:

Theorem 6.3.11. *Let $u \in C^1(\overline{\Omega},\mathbb{C}^3)$ be any vector field which verifies the L^1 SMB condition (6.3.15) and the decay condition* (6.3.16)*. Assume that* $\operatorname{div} u$, $\operatorname{curl} u - \lambda u = O(|x|^{-\rho})$ *when* $|x| \to +\infty$ *for* $2 < \rho < 3$*. Then, u can be decomposed as* $u(x) = -\nabla\phi(x) + \operatorname{curl} A(x) + \lambda A(x)$, *for every* $x \in \Omega$, *where ϕ and A are the scalar and vector fields*

$$\phi(x) = \int_\Omega \Gamma_\lambda(x-y)\operatorname{div} u(y)\,dy + \int_S \Gamma_\lambda(x-y)\eta(y)\cdot u(y)\,d_yS,$$
$$A(x) = \int_\Omega \Gamma_\lambda(x-y)(\operatorname{curl} u(y) - \lambda u(y))\,dy + \int_S \Gamma_\lambda(x-y)\eta(y)\times u(y)\,d_yS.$$

As a consequence,

$$u = O(|x|^{-(\rho-2)}),\ \textit{when}\ |x| \to +\infty.$$

Indeed, when both $\operatorname{div} u$ *and* $\operatorname{curl} u - \lambda u$ *are compactly supported, one obtains the optimal decay at infinity, namely,*

$$u = O(|x|^{-1}),\ \textit{when}\ |x| \to +\infty,$$

and u satisfies the Sommerfeld radiation condition (6.3.4) componentwise.

Proof. Consider any $x \in \Omega$ and fix any couple of radii $\varepsilon_0, R_0 > 0$ such that

$$\overline{B}_{\varepsilon_0}(x) \subseteq \Omega\ \text{ and }\ \overline{B}_{\varepsilon_0}(x) \cup \overline{G} \subseteq B_{R_0}(0).$$

Define the subdomain $\Omega(x,\varepsilon,R) := \Omega \cap (B_R(0)\setminus \overline{B}_\varepsilon(x))$ for $R > R_0$ and $\varepsilon > \varepsilon_0$, as in Figure 6.2.

Let $e \in \mathbb{C}^3$ be fixed. Since Γ_λ solves the scalar homogeneous Helmholtz equation outside the origin, then $\Gamma_\lambda e$ is a solution to the vector-valued Helmholtz equation too. Therefore, the following identity

$$0 = -\int_{\Omega(x,\varepsilon,R)} (\Delta(\Gamma_\lambda(x-y)e) + \lambda^2(\Gamma_\lambda(x-y)e))\cdot u(y)\,dy$$

holds. As in the classical Helmholtz–Hodge theorem, having in mind $\operatorname{curl}(\operatorname{curl}) = \nabla(\operatorname{div}) - \Delta$, removing the dot product by e, and subtracting and adding appropriate terms, we obtain the following formula

$$0 = -\int_{\partial\Omega(x,\varepsilon,R)} \nabla_x\Gamma_\lambda(x-y)(\nu\cdot u)(y)\,d_yS + \int_{\Omega(x,\varepsilon,R)} \nabla_x\Gamma_\lambda(x-y)\operatorname{div} u(y)\,dy$$

$$
\begin{aligned}
&+\int_{\partial\Omega(x,\varepsilon,R)} \nabla_x\Gamma_\lambda(x-y)\times(\nu\times u)(y)\,d_yS-\int_{\Omega(x,\varepsilon,R)}\nabla_x\Gamma_\lambda(x-y)\times(\operatorname{curl}u-\lambda u)(y)\,dy\\
&+\lambda\left(-\int_{\Omega(x,\varepsilon,R)}\Gamma_\lambda(x-y)(\operatorname{curl}u-\lambda u)(y)\,dy+\int_{\partial\Omega(x,\varepsilon,R)}\Gamma_\lambda(x-y)(\nu\times u)(y)\,d_yS\right).
\end{aligned}
\tag{6.3.19}
$$

Taking limits when $\varepsilon\to 0$ and $R\to+\infty$ shows that the volume integrals converges to the integral over the whole exterior domain due to the dominated convergence theorem, the fall-off of the Riesz potential in Theorem C.0.4 and the hypotheses on $\operatorname{div}u$ and $\operatorname{curl}u-\lambda u$:

$$
\begin{aligned}
\int_{\Omega(x,\varepsilon,R)}\nabla_x\Gamma_\lambda(x-y)\operatorname{div}u(y)\,dy&\longrightarrow\int_\Omega\nabla_x\Gamma_\lambda(x-y)\operatorname{div}u(y)\,dy,\\
\int_{\Omega(x,\varepsilon,R)}\nabla_x\Gamma_\lambda(x-y)\times(\operatorname{curl}u-\lambda u)(y)\,dy&\longrightarrow\int_\Omega\nabla_x\Gamma_\lambda(x-y)\times(\operatorname{curl}u-\lambda u)(y)\,dy,\\
\int_{\Omega(x,\varepsilon,R)}\Gamma_\lambda(x-y)(\operatorname{curl}u-\lambda u)(y)\,dy&\longrightarrow\int_\Omega\Gamma_\lambda(x-y)(\operatorname{curl}u-\lambda u)(y)\,dy,
\end{aligned}
$$

when $\varepsilon\to 0$ and $R\to+\infty$. Regarding the boundary integrals, it is worth splitting them into the three connected components of the boundary surface of $\Omega(x,\varepsilon,R)$, that is, $\partial\Omega(x,\varepsilon,R)=S\cup\partial B_\varepsilon(x)\cup\partial B_R(0)$. Since the integrals over S are not relevant in the limit $\varepsilon\to 0$ and $R\to+\infty$, we focus on the two remaining terms. On the one hand, using (6.3.6) and *Lagrange's formula* $v=(e\cdot v)\,e-e\times(e\times v)$, for any unitary vector e and any general vector v, the boundary terms over the sphere $\partial B_\varepsilon(x)$ can be written as

$$
\begin{aligned}
I_\varepsilon:=&-\left(i\lambda-\frac{1}{\varepsilon}\right)\frac{e^{i\lambda\varepsilon}}{4\pi\varepsilon}\int_{\partial B_\varepsilon(x)}u(y)\,d_yS-\lambda\frac{e^{i\lambda\varepsilon}}{4\pi\varepsilon}\int_{\partial B_\varepsilon(x)}\frac{y-x}{\varepsilon}\times u(y)\,d_yS\\
=&i\lambda\frac{e^{i\lambda\varepsilon}}{4\pi\varepsilon}\int_{\partial B_\varepsilon(x)}\left(i\frac{y-x}{\varepsilon}\times u(y)-u(y)\right)\,d_yS+\frac{e^{i\lambda\varepsilon}}{4\pi\varepsilon^2}\int_{\partial B_\varepsilon(x)}u(y)\,d_yS.
\end{aligned}
$$

Consequently, the first term converges to zero as $\varepsilon\to 0$ while the second term converges to $u(x)$ due to the properties of the mean value over spheres of continuous functions.

In addition, the boundary terms over $\partial B_R(0)$ may also be written in a similar way

$$
\begin{aligned}
I_R:=&\int_{\partial B_R(0)}\left\{-\nabla_x\Gamma_\lambda(x-y)\frac{y}{R}\cdot u(y)+\nabla_x\Gamma_\lambda(x-y)\times\left(\frac{y}{R}\times u(y)\right)+\lambda\Gamma_\lambda(x-y)\frac{y}{R}\times u(y)\right\}d_yS\\
=&\int_{\partial B_R(0)}\left(i\lambda-\frac{1}{|x-y|}\right)\frac{e^{i\lambda|x-y|}}{4\pi|x-y|}\frac{y-x}{|y-x|}\frac{y}{R}\cdot u(y)\,d_yS\\
&-\int_{\partial B_R(0)}\left\{\left(i\lambda-\frac{1}{|x-y|}\right)\frac{e^{i\lambda|x-y|}}{4\pi|x-y|}\frac{y-x}{|y-x|}\times\left(\frac{y}{R}\times u(y)\right)+\lambda\frac{e^{i\lambda|x-y|}}{4\pi|x-y|}\frac{y}{R}\times u(y)\right\}d_yS.
\end{aligned}
$$

Lagrange's formula for the triple vector product cannot be directly applied since $B_R(0)$ is not centered at x. See Remark 6.3.12 below for the behavior of this boundary integrals if we had defined $\Omega(x,\varepsilon,R)=\Omega\cap B_R(x)\cap(\mathbb{R}^3\setminus\overline{B}_\varepsilon(x))$ instead of $\Omega(x,\varepsilon,R)=\Omega\cap B_R(0)\cap(\mathbb{R}^3\setminus\overline{B}_\varepsilon(x))$. Adding and subtracting appropriate terms in order to apply Lagrange's formula for the triple vector product

$$
I_R:=-i\lambda\int_{\partial B_R(0)}\frac{e^{i\lambda|x-y|}}{4\pi|x-y|}\left(i\frac{y}{R}\times u(y)-u(y)\right)\,d_yS-\int_{\partial B_R(0)}\frac{e^{i\lambda|x-y|}}{4\pi|x-y|^2}u(y)\,d_yS
$$

$$+\int_{\partial B_R(0)} \left(i\lambda - \frac{1}{|x-y|}\right) \frac{e^{i\lambda|x-y|}}{4\pi|x-y|} \left(\frac{y-x}{|y-x|} - \frac{y}{R}\right) \frac{y}{R} \cdot u(y)\, d_yS$$
$$-\int_{\partial B_R(0)} \left(i\lambda - \frac{1}{|x-y|}\right) \frac{e^{i\lambda|x-y|}}{4\pi|x-y|} \left(\frac{y-x}{|y-x|} - \frac{y}{R}\right) \times \left(\frac{y}{R} \times u(y)\right) d_yS.$$

Then, a mean value argument leads to the following bound of the norm of I_R for $R > |x|$

$$\begin{aligned}|I_R| \leq & \frac{|\lambda|}{4\pi(R-|x|)} \int_{\partial B_R(0)} \left| i\frac{y}{R} \times u(y) - u(y)\right| d_yS \\ & + \frac{1}{4\pi(R-|x|)^2} \int_{\partial B_R(0)} |u(y)|\, d_yS + \frac{2C|x|}{4\pi(R-|x|)^2} \int_{\partial B_R(0)} |u(y)|\, d_yS. \end{aligned} \tag{6.3.20}$$

Thereby, $I_R \to 0$ when $R \to +\infty$, thanks to the L^1 SMB radiation condition (6.3.15) and the decay property (6.3.16).

Now that we have the representation formula in the statement of the theorem, the asymptotic behavior at infinity follows from the decay properties of the Riesz potential in Theorem C.0.4 and the componentwise Sommerfeld radiation condition in the compactly supported case is a direct consequence of Propositions 6.3.4 and 6.3.5. □

Remark 6.3.12. *Consider $\Omega(x,\varepsilon,R) = \Omega \cap B_R(x) \cap (\mathbb{R}^3 \setminus \overline{B}_\varepsilon(x))$ instead of $\Omega(x,\varepsilon,R) = \Omega \cap B_R(0) \cap (\mathbb{R}^3 \setminus \overline{B}_\varepsilon(x))$ in Eq. (6.3.19). We can argue in the same way both for the boundary terms over $\partial B_\varepsilon(x)$ and for those over $\partial B_R(x)$. Then, the former has already been studied in the above proof and the later reads*

$$\begin{aligned} I_R := & \left(i\lambda - \frac{1}{R}\right) \frac{e^{i\lambda R}}{4\pi R} \int_{\partial B_R(x)} u(y)\, d_yS + \lambda \frac{e^{i\lambda R}}{4\pi R} \int_{\partial B_R(x)} \frac{y-x}{R} \times u(y)\, d_yS \\ = & -i\lambda \frac{e^{i\lambda R}}{4\pi R} \int_{\partial B_R(x)} \left(i\frac{y-x}{\varepsilon} \times u(y) - u(y)\right) d_yS - \frac{e^{i\lambda R}}{4\pi R^2} \int_{\partial B_R(x)} u(y)\, d_yS. \end{aligned} \tag{6.3.21}$$

Therefore, the same representation theorem might have been obtained from the following radiation and decay conditions

$$\int_{\partial B_R(x)} \left(i\frac{y-x}{\varepsilon} \times u(y) - u(y)\right) d_yS = o(R), \qquad \text{when } R \to +\infty,$$
$$\int_{\partial B_R(x)} u(y)\, d_yS = o(R^2), \qquad \text{when } R \to +\infty,$$

for every $x \in \Omega$. The hypotheses are stronger than (6.3.15) and (6.3.16) in the sense that they have to be assumed on every $x \in \Omega$. However, they are weaker in the sense that norms can be removed here. Therefore, one might take advantage of certain geometric cancellations of our vector fields to ensure these conditions.

An obvious but interesting feature of the above boundary terms is that in both cases, when $\Omega(x,\varepsilon,R) = \Omega \cap B_R(0) \cap (\mathbb{R}^3 \setminus \overline{B}_\varepsilon(x))$ (6.3.20) and $\Omega(x,\varepsilon,R) = \Omega \cap B_R(x) \cap (\mathbb{R}^3 \setminus \overline{B}_\varepsilon(x))$ (6.3.21), the harmonic case $\lambda = 0$ does not need to prescribe any radiation condition at infinity, as it is the case in the classical Helmholtz–Hodge theorem and in [229, 298].

Again, Remark 6.3.10 and the Rellich' lemma [85, Lemma 2.11] yields an uniqueness result, which is similar to that for the reduced Maxwell system in [85, Theorem 6.10]:

Lemma 6.3.13. *Consider any solution $u \in C^1(\overline{\Omega}, \mathbb{C}^3)$ to the complex-valued homogeneous Beltrami equation in the exterior domain satisfying the L^2 SMB radiation condition (6.3.17). Then, u verifies the inequality*

$$\Im\left(\int_S \overline{u}(x) \cdot (\eta(x) \times u(x))\, d_x S\right) \geq 0.$$

If the equality holds, then u vanishes everywhere in Ω.

To conclude, let us state the existence result for the complex-valued inhomogeneous Beltrami equation that will be needed in the modified Grad–Rubin iterative scheme in Section 6.4. Since this iterative method only involves compactly supported inhomogeneities, we will focus on this case although it is easy to extend it to general inhomogeneous terms with an appropriate fall off at infinity.

Theorem 6.3.14. *Let $0 \neq \lambda \in \mathbb{R}$ be any constant that is not a Dirichlet eigenvalue of the Laplace operator in the interior domain, $w \in C_c^{k,\alpha}(\overline{\Omega}, \mathbb{C}^3)$ and $g \in C^{k+1,\alpha}(S, \mathbb{C})$ such that $\operatorname{div} w \in C^{k,\alpha}(\overline{\Omega}, \mathbb{C})$ and the following compatibility condition*

$$\int_S (\lambda g + w \cdot \eta)\; dS = 0 \tag{6.3.22}$$

is satisfied. Consider any solution $\xi \in \mathfrak{X}^{k+1,\alpha}(S, \mathbb{C}^3)$ to the boundary integral equation

$$\left(\frac{1}{2}I - T_\lambda\right)\xi = \mu,\; x \in S, \tag{6.3.23}$$

where $T_\lambda \xi$ and μ are defined by

$$(T_\lambda \xi)(x) = \int_S \eta(x) \times (\nabla_x \Gamma_\lambda(x-y) \times \xi(y))\; d_y S + \lambda \int_S \Gamma_\lambda(x-y)\eta(x) \times \xi(y)\, d_y S, \tag{6.3.24}$$

$$\begin{aligned}\mu(x) = \frac{1}{\lambda}\int_\Omega \eta(x) \times \nabla_x \Gamma_\lambda(x-y) \operatorname{div} w(y)\, dy - \int_S \eta(x) \times \nabla_x \Gamma_\lambda(x-y) g(y)\, d_y S \\ + \int_\Omega \eta(x) \times (\nabla_x \Gamma_\lambda(x-y) \times w(y))\, dy + \lambda \int_S \Gamma_\lambda(x-y)\eta(x) \times w(y)\, d_y S.\end{aligned} \tag{6.3.25}$$

Define the complex-valued vector field

$$u(x) := -\nabla \phi(x) + \operatorname{curl} A(x) + \lambda A(x),\; x \in \Omega, \tag{6.3.26}$$

where ϕ and A stand for the scalar and vector fields

$$\phi(x) = -\frac{1}{\lambda}\int_\Omega \Gamma_\lambda(x-y) \operatorname{div} w(y)\, dy + \int_S \Gamma_\lambda(x-y) g(y)\, d_y S, \tag{6.3.27}$$

$$A(x) = \int_\Omega \Gamma_\lambda(x-y) w(y)\, dy + \int_S \Gamma_\lambda(x-y)\xi(y)\, d_y. \tag{6.3.28}$$

Then, u is a complex-valued solution to the exterior NIB problem

$$\left\{\begin{array}{ll} \operatorname{curl} u - \lambda u = w, & x \in \Omega, \\ u \cdot \eta = g, & x \in \Omega, \\ + L^1 \textit{ SMB radiation condition } (6.3.15), & \\ + L^1 \textit{ decay property } (6.3.16). & \end{array}\right. \tag{6.3.29}$$

Furthermore, the decay and radiation conditions are stronger since u behaves as $O\left(|x|^{-1}\right)$ at infinity and the Sommerfeld radiation condition (6.3.4) holds componentwise.

Proof. Since the divergence of any solution u can be recovered from the equation through the identity $\operatorname{div} u = -\frac{1}{\lambda}\operatorname{div} w$, then one arrives at the next expression for the candidate to be a solution to (6.3.29)

$$u(x) = -\nabla\phi(x) + \operatorname{curl} A(x) + \lambda A(x),$$

where ϕ and A are defined as follows

$$\begin{aligned}\phi(x) &= -\frac{1}{\lambda}\int_\Omega \Gamma_\lambda(x-y)\operatorname{div} w(y)\,dy + \int_S \Gamma_\lambda(x-y)g(y)\,d_yS,\\ A(x) &= \int_\Omega \Gamma_\lambda(x-y)w(y)\,dy + \int_S \Gamma_\lambda(x-y)\eta(y)\times u_+(y)\,d_yS.\end{aligned}$$

Consider $\xi := \eta \times u_+$, where $u_\pm$ denotes the limits of u at S from Ω and G respectively. In order to obtain a more manageable formula for ξ, one can use the well known *jump relations* for the derivatives of a single layer potential associated with the fundamental solution to the Helmholtz equation, $\Gamma_\lambda(x)$ (see e.g. [84]). This formulas lead to the following identity

$$\begin{aligned}u_\pm(x) =& \frac{1}{\lambda}\int_\Omega \nabla_x\Gamma_\lambda(x-y)\operatorname{div} w(y)\,dy - \mathrm{PV}\int_S \nabla_x\Gamma_\lambda(x-y)g(y)\,d_yS\\ &+ \int_\Omega \nabla_x\Gamma_\lambda(x-y)\times w(y)\,dy + \mathrm{PV}\int_S \nabla_x\Gamma_\lambda(x-y)\times\xi(y)\,d_yS\\ &+ \lambda\int_\Omega \Gamma_\lambda(x-y)w(y)\,dy + \lambda\int_S \Gamma_\lambda(x-y)\xi(y)\,d_yS \pm \frac{1}{2}\eta(x)g(x) \mp \frac{1}{2}\eta(x)\times\xi(x),\end{aligned} \tag{6.3.30}$$

where PV stands for the Cauchy principal value integral. It is clear that the sum of the last two term in last line is actually $\pm\frac{1}{2}u_\pm(x)$. Consequently, one can take cross products by $\eta(x)$ and arrive at the boundary integral equation in (6.3.23) for the tangential component ξ. There, we have intentionally avoided the PV signs because the $\eta(x)$ factor in such integrals provides certain geometrical cancellations (see Appendix H) leading to absolutely convergent integrals.

Now, let us show that the field u so defined is a solution to (6.3.29) as long as ξ solves the boundary integral equation (6.3.23). We will prove later that ξ is unique and, consequently, (6.3.29) is uniquely solvable. First, let us obtain some PDEs for the potentials ϕ and A both in the interior and the exterior domain. Since volume and single layer potentials are indeed complex-valued solutions to such PDEs, we have

$$\Delta\phi + \lambda^2\phi = \begin{cases} \frac{1}{\lambda}\operatorname{div} w, & x\in\Omega\\ 0, & x\in G\end{cases} \qquad \Delta A + \lambda^2 A = \begin{cases} -w, & x\in\Omega\\ 0, & x\in G\end{cases} \tag{6.3.31}$$

Therefore,

$$\begin{aligned}\operatorname{curl} u - \lambda u &= \nabla(\operatorname{div} A) - \Delta A + \lambda\operatorname{curl} A + \lambda\nabla\phi - \lambda\operatorname{curl} A - \lambda^2 A\\ &= -(\Delta A + \lambda^2 A) + \nabla\underbrace{(\operatorname{div} A + \lambda\phi)}_{a}.\end{aligned}$$

A direct substitution of (6.3.31) leads to the following PDE for u at any side of the boundary surface S:

$$\operatorname{curl} u - \lambda u = \begin{cases} w + \nabla a, & x\in\Omega,\\ \nabla a, & x\in G.\end{cases} \tag{6.3.32}$$

In order to show that u solves (6.3.29), it remains to check that ∇a is zero in the exterior domain and u satisfies the boundary condition $u_+\cdot\eta = g$ (the decay and radiation conditions will be studied later). To this end, it might be useful to find first a PDE for a. The same reasoning as above shows that a solves the homogeneous Helmholtz equation both in Ω and in G, specifically

$$\Delta a + \lambda^2 a = \operatorname{div}(\Delta A) + \lambda\Delta\phi + \lambda^2 \operatorname{div} A + \lambda^3\phi = \operatorname{div}(\Delta A + \lambda^2 A) + \lambda(\Delta\phi + \lambda^2\phi) = 0. \quad (6.3.33)$$

Let us show first the jump relations for the scalar potential a. Straightforward computations on the explicit formulas for ϕ and A involving the divergence theorem lead to

$$\begin{aligned}
a(x) &= \operatorname{div} A(x) + \lambda\phi(x)\\
&= \int_\Omega \{\nabla_x\Gamma_\lambda(x-y)\cdot w(y) - \Gamma_\lambda(x-y)\operatorname{div} w(y)\}\,dy\\
&+ \int_S \{\nabla_x\Gamma_\lambda(x-y)\cdot \xi(y) + \lambda\Gamma_\lambda(x-y)g(y)\}\,d_yS\\
&= -\int_\Omega \operatorname{div}_y(\Gamma_\lambda(x-y)w(y))\,dy + \int_S \nabla_x\Gamma_\lambda(x-y)\cdot\xi(y)\,d_yS + \lambda\int_S \Gamma_\lambda(x-y)g(y)\,d_yS\\
&= \int_S \Gamma_\lambda(x-y)(\lambda g(y) + w(y)\cdot\eta(y))\,d_yS + \int_S \nabla_x\Gamma_\lambda(x-y)\cdot\xi(y)\,d_yS.
\end{aligned}$$

Finally, notice that $\nabla_x\Gamma_\lambda(x-y)\cdot\xi(y) = -(\nabla_S)_y[\Gamma_\lambda(x-y)]\cdot\xi(y)$ for every $y\in S$ because of ξ being a tangent vector field along S. Hence, the integration by parts formula over S yields the next simpler expression for a:

$$a(x) = \int_S \Gamma_\lambda(x-y)\left(\lambda g(y) + w(y)\cdot\eta(y) + \operatorname{div}_S\xi(y)\right)\,d_yS,$$

For the sake of clarity, we refer to Appendix G for some of the main properties of the operators ∇_S, div_S and curl_S that shall be used throughout this proof. Then, a is nothing but a new single layer potential. As such, the first and second jumps relations read

$$a_+ - a_- \equiv 0, \qquad \left(\frac{\partial a}{\partial\eta}\right)_+ - \left(\frac{\partial a}{\partial\eta}\right)_- \equiv -\left(\lambda g + w\cdot\eta + \operatorname{div}_S\xi\right), \quad (6.3.34)$$

on the surface S. In particular, a is continuous across S but its normal derivative exhibits a jump discontinuity with height $\lambda g + w\cdot\eta + \operatorname{div}_S\xi$. The same kind of reasoning yields the jump relation for u

$$u_+ - u_- = g\,\eta - \eta\times\xi,\ x\in S. \quad (6.3.35)$$

Consequently, the boundary integral equation (6.3.23) along with the jump relation (6.3.35) ensure that

$$\eta\times u_+ = \xi, \quad \eta\times u_- = 0, \quad (6.3.36)$$

on S. Regarding a, let us show that it is indeed constant on S and to this end, define the next vector field in the interior domain G:

$$v := \lambda u + \nabla a, \quad x\in G.$$

Notice that v is a strong Beltrami field with factor λ by virtue of (6.3.32). Then, one can repeat the same kind of uniqueness criterion as in Lemma 6.3.13 in the simpler bounded setting, specifically

$$\lambda\int_G |v|^2\,dx = \int_G \overline{v}\cdot\operatorname{curl} v\,dx = \int_G \operatorname{div}(v\times\overline{v})\,dx = \int_S (\eta\times v)\cdot\overline{v}\,dS.$$

Now, notice that we can substitute both v and $\overline{v}$ in the above formula with its tangential parts thanks to the presence of a cross product by the unit normal vector field η and

$$-\eta \times (\eta \times v) = -\lambda \eta \times (\eta \times u_-) + \nabla_S a = \nabla_S a,$$

by virtue of (6.3.36). Thereby, an integration by parts leads again to

$$\lambda \int_G |v|^2 \, dx = \int_S (\eta \times \nabla_S a) \cdot \nabla_S \overline{a} \, dS = -\int_S a \, \overline{\operatorname{curl}_S (\nabla_S a)} \, dS = 0,$$

where the well know formula $\operatorname{curl}_S \nabla_S = 0$ has been used in the last step. Consequently, v vanishes everywhere in G and, in particular, $\nabla_S a \equiv 0$, i.e., $a_\pm \equiv a_0 = \text{const}$ on S.

We will next prove that a vanishes everywhere in the exterior domain Ω using the uniqueness result in Lemma 6.3.7. Notice that since a can be written as a sum of volume and single layer potentials with compactly supported densities together with its first order partial derivatives, then a satisfies a stronger Sommerfeld radiation condition due to Propositions 6.3.4 and 6.3.5. Consequently, this lemma can be applied. We therefore want to show that

$$\Im \left(\int_S a_+ \left(\frac{\partial \overline{a}}{\partial \eta} \right)_+ dS \right) = 0 \,. \tag{6.3.37}$$

To derive (6.3.37), we first pass from the exterior to the interior trace values using the jump relations (6.3.34)

$$\int_S a_+ \left(\frac{\partial \overline{a}}{\partial \eta} \right)_+ dS = a_0 \int_S (\lambda g + w \cdot \eta + \operatorname{div}_S \xi) \, dS + \int_S a_- \left(\frac{\partial \overline{a}}{\partial \eta} \right)_- dS =_I + II.$$

On the one hand, I becomes zero because of the divergence theorem over surfaces and the compatibility condition (6.3.22) in the hypothesis. On the other hand, integrate by parts in II to arrive at

$$II := \int_S \operatorname{div} (a \nabla \overline{a}) \, dS = \int_G |\nabla a|^2 \, dx + \int_G a \Delta \overline{a} \, dx = \int_G |\nabla a|^2 \, dx - \lambda^2 \int_G |a|^2 \, dx,$$

where the Helmholtz equation (6.3.33) has being used. Therefore, one arrives at

$$\Im \left(\int_S a_+ \left(\frac{\partial \overline{a}}{\partial \eta} \right)_+ dS \right) = \Im \left(\int_G |\nabla a|^2 \, dx - \lambda^2 \int_G |a|^2 \, dx \right) = 0,$$

and consequently $a = 0$ in Ω and u solves the inhomogeneous Beltrami equation.

Before proving the boundary condition and the decay and radiation properties, let us show that a also vanishes in the interior domain. On the one hand, a solves the homogeneous Helmholtz equation in such domain and it also satisfies the interior homogeneous Dirichlet conditions in S since $a_- = a_+$ on S and $a = 0$ in Ω. Moreover, λ is prevented from being a Dirichlet eigenvalue of the Laplacian in the interior domain, so a also vanishes in G. In particular, the jumps relations (6.3.34) yields

$$\lambda g + w \cdot \eta + \operatorname{div}_S \xi \equiv 0. \tag{6.3.38}$$

Furthermore, since u is now a solution to the next inhomogeneous Beltrami equation, $\operatorname{curl} u - \lambda u = w, \quad x \in \Omega$, taking trace values at S one gets $\eta \cdot (\operatorname{curl} u)_+ - \lambda \eta \cdot u_+ = w \cdot \eta$. Now, one can write the first term in an intrinsic way through $\eta \cdot (\operatorname{curl} u)_+ = -\operatorname{div}_S(\eta \times u_+) = -\operatorname{div}_S \xi$, and, consequently, we have

$$\eta \cdot u_+ + w \cdot \eta + \operatorname{div}_S \xi \equiv 0. \tag{6.3.39}$$

Then, comparing (6.3.38) and (6.3.39) entails the boundary condition $\eta \cdot u_+ = g$.

Finally, let us show the decay and radiation conditions on u. First, since

$$\Gamma_\lambda(x), \nabla\Gamma_\lambda(x) = O\left(|x|^{-1}\right), \quad \text{when } |x| \to +\infty,$$

and w has compact support, then u enjoys the optimal decay $u = O\left(|x|^{-1}\right)$ when $|x| \to +\infty$ according to Theorem C.0.4. Second, as u is again a sum of single and volume layer potential associated with the Helmholtz equation along with some partial derivatives, then u satisfies Sommerfeld radiation condition componentwise thanks to Propositions 6.3.4 and 6.3.5. Therefore, one can show that u verifies SMH conditions (6.3.12) and (6.3.13). Since $\operatorname{curl} u - \lambda u = w$ and w is compactly supported, then u actually satisfies the strong SMB radiation condition and this finishes the proof. □

6.3.3 Well-posedness of the boundary integral equation

One should also notice that, in addition to the uniqueness result proved in Theorem 6.3.14, we will also need a study of the regularity of the solution, which is obviously in $C^1(\overline{\Omega}, \mathbb{C}^3)$ by the decomposition (6.3.26). We will prove in this next subsection that the regularity assumptions on the data w and g actually leads to $C^{k+1,\alpha}(\Omega, \mathbb{C}^3)$ regularity on u. Some necessary potential theoretic estimates have been relegated to Appendix H of the thesis to streamline the exposition. Let us start by studying the well-posedness of (6.3.23) using the Riesz–Fredholm theory for compact operators, which follows easily from our previous estimates:

Proposition 6.3.15. *The linear operator $T_\lambda : \mathfrak{X}^{k+1,\alpha}(S) \longrightarrow \mathfrak{X}^{k+1,\alpha}(S)$ is compact.*

Proof. The gain of regularity proved in Theorem H.2.1 implies that T_λ defines a continuous linear operator

$$T_\lambda : \mathfrak{X}^{k,\alpha}(S) \longrightarrow \mathfrak{X}^{k+1,\alpha}(S).$$

Since $\mathfrak{X}^{k+1,\alpha}(S) \hookrightarrow \mathfrak{X}^{k,\alpha}(S)$ is compact by the Ascoli–Arzelà theorem, the proposition follows. □

The proposition ensures that it is possible to apply Riesz–Fredholm theory to the operator $\frac{1}{2}I - T_\lambda$. In particular, $\frac{1}{2}I - T_\lambda$ is one to one if, and only if, it is onto, i.e.,

$$\operatorname{Ker}\left(\frac{1}{2}I - T_\lambda\right) = 0 \Longleftrightarrow \operatorname{Im}\left(\frac{1}{2}I - T_\lambda\right) = \mathfrak{X}^{k+1,\alpha}(S).$$

As it is hard to show explicitly that such operator is onto, let us equivalently show that it is one to one. This is a consequence of the uniqueness Lemma 6.3.13 and the existence Theorem 6.3.14.

Proposition 6.3.16. *The bounded linear operator $\frac{1}{2}I - T_\lambda$ on $\mathfrak{X}^{k+1,\alpha}(S)$ is one to one and onto. Consequently, the boundary integral equation (6.3.23) has a unique solution $\xi \in \mathfrak{X}^{k+1,\alpha}(S)$ for any $\mu \in \mathfrak{X}^{k+1,\alpha}(S)$.*

Proof. According to the preceding argument, we only have to show that $\operatorname{Ker}\left(\frac{1}{2}I - T_\lambda\right) = 0$. To this end, let us consider an arbitrary $\xi \in \operatorname{Ker}\left(\frac{1}{2}I - T_\lambda\right)$ and show that $\xi \equiv 0$. By definition, $\xi \in \mathfrak{X}^{k+1,\alpha}(S)$ solves the boundary integral equation $\frac{1}{2}\xi - T_\lambda\xi = 0$ on S. Define $u(x) := \operatorname{curl} A(x) + \lambda A(x)$, where A is the vector potential $A(x) := \int_S \Gamma_\lambda(x-y)\xi(y)\, d_yS$. Thus, Theorem 6.3.14 for $w \equiv 0$ and $g \equiv 0$ leads to a solution $u \in C^1(\overline{\Omega}, \mathbb{C}^3)$ to the homogeneous Beltrami equation in Ω

$$\begin{cases} \operatorname{curl} u = \lambda u, & x \in \Omega, \\ \eta \cdot u_+ = 0, & x \in S, \end{cases}$$

that satisfies the Dirichlet boundary condition $\eta\times u_+ = \xi$ on S and the SMB radiation condition.

We would like to show that this boundary value problem has a unique solution, but this does not follow directly from Lemma 6.3.13. However, since $\eta\cdot u_+ = 0$ on S, then $u_+ = -\eta\times(\eta\times u_+)$ on S and we have the following relation between the curl operator on S, curl_S, and the curl operator on $\mathbb{R}^3$:

$$\operatorname{curl}_S u_+ = \operatorname{curl}_S\left(-\eta\times(\eta\times u_+)\right) = \eta\cdot\operatorname{curl} u_+ = \lambda\,\eta\cdot u_+ = 0,$$

see Appendix G. Since S is homeomorphic to a sphere, Poincaré's lemma shows that u_+ has a potential $\psi\in C^2(S)$ on the surface, $u_+ = \nabla_S\psi$ on S, where ∇_S stands for the gradient vector on the surface S. Consequently,

$$\Im\left(\int_S \overline{u}_+\cdot(\eta\times u_+)\,dS\right) = \Im\left(\int_S \overline{\nabla_S\psi}\cdot(\eta\times\nabla_S\psi)\,dS\right) = -\Im\left(\int_S \overline{\operatorname{curl}_S(\nabla_S\psi)}\psi\,dS\right) = 0.$$

The identity follows from an integration by parts on S and the classical property $\operatorname{curl}_S(\nabla_S\psi) = 0$. Therefore, Lemma 6.3.13 yields the desired result. □

Remark 6.3.17. *The importance of the above result lies on the following facts.*

1. *First, the existence part of the above result ensures that it is possible to choose some ξ solving (6.3.23). Obviously, it is essential to rigurously establish the existence Theorem 6.3.14.*

2. *Second, the uniqueness result shows that since ξ can be uniquely chosen, then (6.3.29) has a unique solution too.*

3. *Finally, it provides a very useful estimate for the subsequent result. Since $\frac{1}{2}I - T_\lambda$ is linear, continuous and bijective, then $\left(\frac{1}{2}I - T_\lambda\right)^{-1}$ is continuous by virtue of the Banach isomorphism theorem. Consequently, there exists a positive constant c (which depends on G and λ) such that*

$$c\|\xi\|_{C^{k+1,\alpha}(S)} \leq \left\|\left(\frac{1}{2}I - T_\lambda\right)\xi\right\|_{C^{k+1,\alpha}(S)}, \tag{6.3.40}$$

for any $\xi\in\mathfrak{X}^{k+1,\alpha}(S)$.

We conclude by proving the following regularity result for the solution u of (6.3.29) according to Theorem 6.3.14. It is an immediate consequence of the decomposition (6.3.26), the estimates for the volume and single layer potentials in Appendix H (Lemmas H.1.10 and H.1.1) and the estimate (6.3.40).

Corollary 6.3.18. *Assume that the hypothesis in Theorem 6.3.14 are satisfied, fix any $R>0$ such that $\overline{G}\subseteq B_R(0)$ and assume that the closure of $\Omega_R := B_R(0)\setminus\overline{G}$ contains the support of w. Then, there exists some nonnegative constant $C_0 = C_0(k,\alpha,G,R,\lambda)$ such that the next estimate*

$$\|u\|_{C^{k+1,\alpha}(\Omega)} \leq C_0\left\{\|w\|_{C^{k,\alpha}(\Omega)} + \|\operatorname{div} w\|_{C^{k,\alpha}(\Omega)} + \|g\|_{C^{k+1,\alpha}(S)}\right\}. \tag{6.3.41}$$

holds. In particular, not only does u belong to $C^1(\overline{\Omega},\mathbb{C}^3)$, but also to $C^{k+1,\alpha}(\overline{\Omega},\mathbb{C}^3)$.

6.3.4 Optimal fall-off in exterior domains

It is worth discussing the differences between the optimal fall-off $|x|^{-1}$ of the solutions to inhomogeneous Beltrami equation and that of the solutions of the div-curl problem. First, it is well know that the exterior Neumann boundary value problem associated with the div-curl system

$$\begin{cases} \operatorname{curl} u = w, & x \in \Omega, \\ \operatorname{div} u = f, & x \in \Omega, \\ u \cdot \eta = g, & x \in S, \\ u = O(|x|^{1-\rho}), & x \in \Omega, \end{cases} \tag{6.3.42}$$

where $w, f = O(|x|^{-\rho})$ and $\rho \in (1, 3)$, is uniquely solvable when appropriate regularity spaces are considered (see [182, 229]) and w has zero flux in the exterior domain. Moreover, the solution inherits the optimal fall-off $|x|^{-2}$ when w and f are assumed to have compact support. In particular, any harmonic field ($w = 0, f = 0$) so obtained decays at infinity as $|x|^{-2}$. Such result is an easy consequence of the Helmholtz–Hodge representation formula in [229, Theorem 4.1] and the natural fall-off of the fundamental solution of the Laplace equation, $\Gamma_0(x)$.

In our case, the exterior Neumann boundary value problem associated with the inhomogeneous Beltrami equation (6.3.29) has an associated representation formula of Helmholtz–Hodge type (6.3.26) that transfers the "optimal fall-off" $|x|^{-1}$ to the solution in Theorem 6.3.14 when w is assumed to have compact support. Let us show that it is indeed the optimal decay rate. To this end, assume that u solves the equation

$$\operatorname{curl} u - \lambda u = w, \quad x \in \Omega,$$

(not necessarily fulfilling neither (6.3.16) nor (6.3.15)) for some divergence-free vector field w. Then, the solution u is divergence-free too. Hence, taking curl in the inhomogeneous Beltrami equation, we are led to the vector-valued Helmholtz equation

$$-(\Delta u + \lambda^2 u) = \lambda w + \operatorname{curl} w, \quad x \in \Omega.$$

Consider $K := \operatorname{supp} w \subseteq \overline{\Omega}$ and notice that $\lambda w + \operatorname{curl} w$ is also compactly supported in K. Imagine that u decayed as $|x|^{-(1+\varepsilon)}$ for some small $\varepsilon > 0$. Hence, a straightforward computation leads to

$$\lim_{R \to +\infty} \int_{\partial B_R(0)} |u(x)|^2 = 0.$$

Consequently, Rellich's Lemma would show that u vanishes outside some sufficiently large ball centered at the origin and containing K. Then, the unique continuation principle of the Helmholtz equation allow proving that u is also compactly supported in K (see [203] for the study of such property in many other linear PDEs with constants coefficients). In particular, g would vanish outside $K \cap S$. In an equivalent way, the next result holds.

Corollary 6.3.19. *Let $u \in C^{k+1,\alpha}(\overline{\Omega}, \mathbb{R}^3)$ be a solution to*

$$\operatorname{curl} u - \lambda u = w, \quad x \in \Omega,$$

for a divergence-free compactly supported w and some $\lambda \in \mathbb{R} \setminus \{0\}$. If u is transverse to S at some point outside the support of w, then u cannot decay faster than $|x|^{-1}$ at infinity.

The above Corollary can be interpreted in two different ways. First, it establishes the optimal fall-off of a "transverse" strong Beltrami field ($w = 0$). Second, it also deals with some kind of "transverse" generalized Beltrami fields in exterior domains ($w = \varphi u$) that will be of a great interest in our work. We restrict to the second result since it contains the first one as a particular case.

Corollary 6.3.20. *Let $u \in C^{k+1,\alpha}(\overline{\Omega}, \mathbb{R}^3)$ be a generalized Beltrami field, i.e.,*

$$\begin{cases} \operatorname{curl} u - f u = 0, & x \in \Omega, \\ \operatorname{div} u = 0, & x \in \Omega, \end{cases}$$

whose proportionality factor is a compactly supported perturbation of a constant proportionality factor $\lambda \in \mathbb{R}\setminus\{0\}$, i.e., $f = \lambda + \varphi$ for some $\varphi \in C_c^{k,\alpha}(\overline{\Omega})$. If u is transverse to S at some point outside the support of the perturbation φ, then u cannot decay faster than $|x|^{-1}$ at infinity.

Remark 6.3.21. *In particular, the above result leads to the natural counterpart for exterior domain of the Liouville theorem in [72, 227] about the fall-off of entire generalized Beltrami fields. Such theorem states that there is no globally defined generalized Beltrami field decaying faster than $|x|^{-1}$ at infinity. As many others Liouville type results, it strongly depends on the solution being defined in the whole $\mathbb{R}^3$. In our case we remove this hypothesis but, in return, we need to argue with generalized Beltrami fields with constant proportionality factor outside a compact set enjoying some trasversality condition on the boundary surface of the exterior domain.*

6.4 An iterative scheme for strong Beltrami fields

Our objective in this section is to set the iterative scheme that we will use to establish the partial stability of strong Beltrami fields that will yield the existence of almost global Beltrami fields with a non-constant factor and complex vortex structures.

6.4.1 Further notation and preliminaries

On the differentiable surface S, we will consider local charts of the same regularity as S (that is, maps μ covering open subsets $\Sigma \subseteq S$ of the form

$$\mu: \quad D \quad \longrightarrow \quad \mathbb{R}^3,$$

where $\mu(D) = \Sigma$ and D is a disk in the plane). We will assume μ to be a local parametrization up to the boundary so that μ can be homeomorphically extended to the closure $\overline{D}$, $\overline{\Sigma} = \mu(\overline{D})$.

We will also consider the corresponding C^k and $C^{k,\alpha}$ spaces of functions defined on a coordinate neighborhood Σ of S provided with a local chart μ. Up to the degree of smoothness of the surface, by compactness they are known to be independent of the choice of the chart, so one can write

$$C^k(\Sigma) := \{f : \Sigma \longrightarrow \mathbb{R} : \ f \circ \mu \in C^k(D)\}, \text{ and } C^{k,\alpha}(\Sigma) := \{f : \Sigma \longrightarrow \mathbb{R} : \ f \circ \mu \in C^{k,\alpha}(D)\}$$

and similarly for spaces on $\overline{\Sigma}$. These spaces can be respectively endowed with the complete norms

$$\|f\|_{C^k(\Sigma,\mu)} := \|f \circ \mu\|_{C^k(D)}, \qquad \|f\|_{C^{k,\alpha}(\Sigma,\mu)} := \|f \circ \mu\|_{C^{k,\alpha}(D)},$$

where the dependence on μ will be removed if it is apparent from the context.

An useful result is *Calderón's extension theorem* for $C^{k,\alpha}$ functions, see e.g. [138, Lemma 6.37]:

Proposition 6.4.1. *Let $O \subseteq \mathbb{R}^3$ be a $C^{k,\alpha}$ domain with bounded boundary ∂O, and let O' be any open subset such that $\overline{O} \subseteq O'$. Then, there exists a linear operator*

$$\mathcal{P}: \quad C^{k,\alpha}(\overline{O}) \quad \longrightarrow \quad C^{k,\alpha}(\overline{O'}),$$

$P(f) \equiv \overline{f}$, such that

1. *$\mathcal{P}$ is an extension operator, i.e., $P(f)|_O = f$, $\forall f \in C^{k,\alpha}(\overline{O})$.*

2. *The support of $\mathcal{P}(f)$ is contained in the open subset O' for evey $f \in C^{k,\alpha}(\overline{\Omega})$.*

3. *$\mathcal{P}$ is continuous in the $C^{k,\alpha}$ topology, i.e.,*
$$\|\mathcal{P}(f)\|_{C^{k,\alpha}(O')} \leq C_{\mathcal{P}} \|f\|_{C^{k,\alpha}(O)}, \ \forall f \in C^{k,\lambda}(\overline{O}).$$

4. *$\mathcal{P}$ is also continuous in the C^m topology for any $0 \leq m \leq k$, i.e.,*
$$\|\mathcal{P}(f)\|_{C^{m}(O')} \leq C_{\mathcal{P}} \|f\|_{C^{m}(O)}, \ \forall f \in C^{k,\alpha}(\overline{O}).$$

In the above inequalities, $C_{\mathcal{P}}$ stands for a constant which depends on k, O and O'.

To describe the stream lines and tubes associated with a velocity field $u \in C^{k+1,\alpha}(\overline{\Omega}, \mathbb{R}^3)$ in presence of a boundary surface which u is not tangent to, it is convenient to consider an extension of the field to obtain the following characterization from the Picard–Lindelöf theorem on Hölder spaces:

Proposition 6.4.2. *Let $O \subseteq \mathbb{R}^3$ be a $C^{k+1,\alpha}$ bounded domain, where $k \geq 0$ and $0 < \alpha \leq 1$. Consider any vector field $u \in C^{k+1,\alpha}(\overline{O}, \mathbb{R}^3)$, its associated extension $\overline{u} = \mathcal{P}(u) \in C^{k+1,\alpha}(\mathbb{R}^3, \mathbb{R}^3)$ according to Proposition 6.4.1, any point $x_0 \in \mathbb{R}^3$ and an initial time $t_0 \in \mathbb{R}$. Consider the associated characteristic system*
$$\begin{cases} \dfrac{dX}{dt}(t; t_0, x_0) = \overline{u}(X(t; t_0, x_0)), \quad t \in \mathbb{R}, \\ X(t_0; t_0, x_0) = x_0. \end{cases} \tag{6.4.1}$$
Then, such problem has a unique global-in-time solution $X(t; t_0, x_0)$. In addition, $X(t; t_0, \cdot)$ is a C^{k+1} global diffeomorphism of the Euclidean space for every $t, t_0 \in \mathbb{R}$ and its inverse is $X(t_0; t, \cdot)$. The solutions to these problems represent the stream lines of the extended velocity field $\overline{u}$.

Since a sharper result will be discussed later, we omit the proof.

Consider any vector field $u \in C^{k+1,\alpha}(\overline{\Omega}, \mathbb{R}^3)$, its extension $\overline{u} = \mathcal{P}(u) \in C^{k+1,\alpha}(\mathbb{R}^3, \mathbb{R}^3)$ according to Calderón's extension theorem and its associated flow map $X(t; t_0, x_0)$ through Proposition 6.4.2. Then, for any $x_0 \in \overline{\Omega}$ we shall define $T(x_0) \geq 0$ to be the largest time for which the stream line $X(t; 0, x_0),\ t > 0$ remains inside the open subset Ω, i.e.,
$$T(x_0) := \sup\{T > 0 : \ X(t; 0, x_0) \in \Omega, \ \text{ for all } \ t \in (0, T)\}.$$
Notice that, by definition, $X(t; 0, x_0),\ 0 < t < T(x_0)$ represents a stream line of u, i.e., it solves
$$\begin{cases} \dfrac{dX}{dt}(t; t_0, x_0) = u(X(t; t_0, x_0)), \quad 0 < t < T(x_0), \\ X(0; 0, x_0) = x_0. \end{cases}$$
We will also consider stream tubes which emanate from the surface S. Specifically, consider an open subset $\Sigma \subseteq S$ together with a local chart $\mu : D \longrightarrow S$. The stream tube of u which emanates from Σ is the collection of all stream lines of u radiating from the points in the open subset Σ, i.e.,
$$\mathcal{T}(\Sigma, u) := \{X(t; 0, \mu(s)) : \ s \in D, \ 0 < t < T(\mu(s))\}.$$
For our purpose, it will also be useful to consider truncations at "height" $T > 0$ of the above tube, i.e.,
$$\mathcal{T}(\Sigma, u, T) := \{X(t; 0, \mu(s)) : \ s \in D, \ 0 < t < \min\{T, T(\mu(s))\}\}.$$
Notice that in order for a stream line of u to be well defined, it is necessary that the velocity field points towards the exterior domain. The same condition leads to well defined stream tubes emanating from Σ.

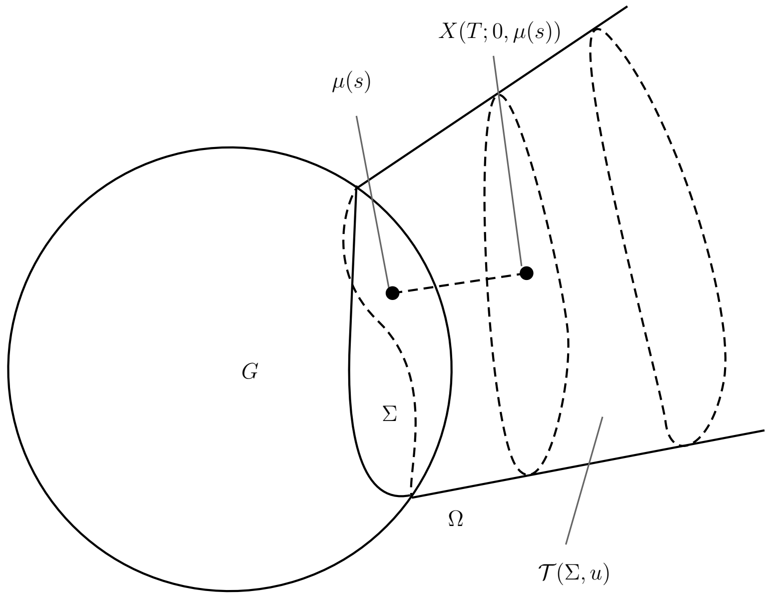


Figure 6.3: Stream lines and tubes of the velocity field u.

Proposition 6.4.3. *Consider G, Σ, and μ verying the hypothesis (6.2.5), $u \in C^{k+1,\alpha}(\overline{\Omega}, \mathbb{R}^3)$, and assume that the vector field u points towards the exterior domain at any point of Σ, i.e., there exits a positive $\rho_0 > 0$ such that $u \cdot \eta \geq \rho_0$ on Σ. Then, $\mathcal{T}(\Sigma, u)$ emanates from Σ and it is smoothly foliated by streamlines of u. Specifically, let us define the following map*

$$\begin{array}{rccc} \phi: & \mathcal{D}(\Sigma,u) & \longrightarrow & \mathcal{T}(\Sigma,u) \\ & (t,s) & \longmapsto & \phi(t,s) := X(t;0,\mu(s)), \end{array}$$

where the domain is the straight tube

$$\mathcal{D}(\Sigma,u) := \{(t,s) : \, s \in D, \; 0 < t < T(\mu(s))\}.$$

Then, the following properties hold true:

1. $T(\mu(s)) > 0$*, for each* $s \in D$.
2. ϕ *is bijective.*
3. ϕ *is a* C^{k+1} *diffeomorphism.*
4. $\operatorname{Jac}(\phi)$ *and* $\operatorname{Jac}(\phi)^{-1}$ *belongs to* $C^{k,\alpha}$ *locally in t, i.e.,*

$$\|\operatorname{Jac}(\phi)\|_{C^{k,\alpha}(\overline{\mathcal{D}(\Sigma,\mu,T)})}, \|\operatorname{Jac}(\phi)^{-1}\|_{C^{k,\alpha}(\overline{\mathcal{T}(\Sigma,\mu,T)})} \leq \kappa\left(\|u\|_{C^{k+1,\alpha}(\overline{\Omega})}, T\right),$$

for every positive number T, where $\kappa : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+$ is an increasing function with respect to each variable, and we denote, the truncated straight tube at "height" T as follows

$$\mathcal{D}(\Sigma,u,T) := \{(t,s) : \, s \in D, \; 0 < t < \min\{T, T(\mu(s))\}\}.$$

For an easier readability, the proof is postponed to Appendix 6.A.

The analysis in the next sections requires stream tubes of u that are bounded and have both ends on S. These structures were considered (although its existence was not proved) in [182]. In our setting, we will say that the stream tube of u arising from Σ is a (ρ_0, T, δ)-stream tube of u when

- $u \cdot \eta \geq \rho_0$ on Σ.
- For every $s \in D$ there exist two associated positive numbers $0 < T_0(s), T_\delta(s) < \frac{T}{2}$ such that $X(T_0(s); 0, \mu(s)) \in S$ and $X(T_\delta(s); 0, \mu(s)) \in S_\delta$.

Here ρ_0, T, δ are positive constants which measure the initial angle of the streams lines over Σ, the time at which the whole tube has returned to the surface and the depth that the stream lines achieve into the interior domain G, while S_δ stands for the boundary of the subdomain of G made of the points in G at distance at least δ from S, i.e., $G_\delta := \{x \in G : \text{dist}(x, S) > \delta\}$ (see Figure 6.4).

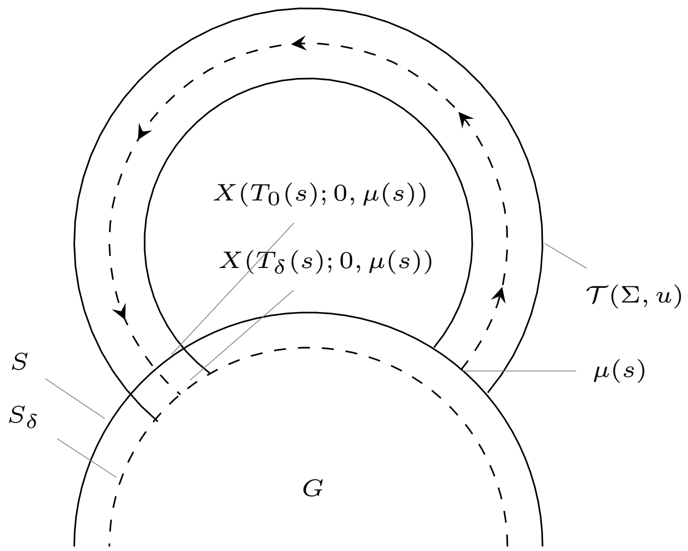


Figure 6.4: (ρ_0, T, δ)-stream tube of u.

Since a stream tube consists of integral curves, the diameter of a (ρ_0, T, δ)-stream tube is bounded in terms of the sup norm of the vector field, the flow time T and the diameter at time 0 as

$$\text{diam}(\mathcal{T}(\Sigma, u)) \leq T\|u\|_{C^0(\Omega)} + \text{diam}(\Sigma). \tag{6.4.2}$$

A detailed proof of such assertion can be found in [182, Lemma 4.6]. In a similar way, [182, Lemma 4.7] provides a criterion to obtain "almost" (ρ_0, T, δ)-stream tubes for velocity fields which are "close enough" to any other given velocity field enjoying this kind of stream tubes. This merely asserts that, as it is well known, a C^0-small perturbation of the initial vector field will not prevent the integral curves of the perturbed field from intersecting a surface to which the initial flow was transverse. This can be written as follows:

Lemma 6.4.4. *Let G, Σ, μ verify (6.2.5) and consider $u_1, u_2 \in C^{k+1,\alpha}(\overline{\Omega}, \mathbb{R}^3)$. Define the associated stream tubes $\mathcal{T}_i := \mathcal{T}(\Sigma, u_i)$ emanating from Σ and suppose that $\mathcal{T}_1$ is a (ρ_0, T, δ)-stream tube of u_1 and the following two assumption are fulfilled:*

1. *$u_1 \cdot \eta = u_2 \cdot \eta$ on Σ.*

2. *There exits* $\theta \in (0,1)$ *such that*

$$\|u_1 - u_2\|_{C^0(\Omega)} < 2\frac{(1-\theta)\delta}{C_{\mathcal{P}}T} e^{-\frac{1}{2}C_{\mathcal{P}}T\|u_1\|_{C^1(\Omega)}},$$

where $C_{\mathcal{P}}$ *is the constant in Calderón's extension theorem (see Proposition 6.4.1),*

Then, $\mathcal{T}_2$ *is also a* $(\rho_0, T, \theta\delta)$*-stream tube of* u_2.

6.4.2 Iterative scheme

In this section we discuss the *Grad–Rubin iterative method* (see, the review [301]) used to obtain nonlinear force-free fields in the magnetohydrodynamical setting. An implementation of the Grad–Rubin method was obtained through the decomposition of the Beltrami equation with small proportionality factor f into a hyperbolic part, which transports the proportionality factor f along the magnetic field lines, and an elliptic one, to correct the magnetic field step by step using Ampere's law [6]. This method was used in [33] to obtain small perturbations of harmonic fields in bounded domains, leading to a strategy to generate generalized Beltrami fields with small non-constant proportionality factors. It was also analyzed in [182] to obtain small perturbations of harmonic fields in exterior domains. The $C^{0,\alpha}$ regularity of the small proportionality factors and the $C^{1,\alpha}$ regularity of the magnetic fields were also addressed in such paper. A natural question is to ascertain whether these results can be adapted to get perturbations of strong Beltrami fields with any constant proportionality factor $\lambda \neq 0$.

Assume that u_0 is a strong Beltrami field with constant proportionality factor in the exterior domain Ω. We will restrict ourselves to strong Beltrami fields u_0 with optimal decay at infinity, say $|x|^{-1}$ (in contrast with the shap fall-off for harmonic fields $|x|^{-2}$). Now, we would like to solve

$$\begin{cases} \operatorname{curl} u = (\lambda + \varphi)u, & x \in \Omega, \\ \operatorname{div} u = 0, & x \in \Omega, \\ u \cdot \eta = u_0 \cdot \eta, & x \in S, \\ |u(x)| \leq \frac{C}{|x|}, & x \in \Omega, \end{cases} \tag{6.4.3}$$

where φ is a "small" perturbation of the constant proportionality factor λ. To solve this problem, we move the term λu in the equation for $\operatorname{curl} u$ from the inhomogeneous side, to the homogeneous one and we propose the following modification of the classical Grad–Rubin iterative method

$$\begin{cases} \operatorname{curl} u_{n+1} - \lambda u_{n+1} = \varphi_n u_n, & x \in \Omega, \\ u_{n+1} \cdot \eta = u_0 \cdot \eta, & x \in S, \\ |u_{n+1}(x)| \leq \frac{C}{|x|}, & x \in \Omega, \end{cases} \qquad \begin{cases} \nabla \varphi_n \cdot u_n = 0, & x \in \Omega, \\ \varphi_n = \varphi^0, & x \in \Sigma. \end{cases} \tag{6.4.4}$$

We have intentionally removed the divergence-free conditions $\operatorname{div} u_{n+1} = 0$ in the left hand side. The reason is twofold. First, note that if one computes the divergence in the first equation and assumes $\lambda \neq 0$, one recovers $\operatorname{div} u_{n+1} = -\frac{1}{\lambda}\operatorname{div}(\varphi_n u_n)$ from the first equation. Therefore, it is an easy task to check that as soon as u_0 is divergence-free and φ_n is a fist integral of u_n, then u_{n+1} is also divergence-free in each step of the iteration. Second, as it has been shown in the preceding section, the exterior inhomogeneous Beltrami equation is generally an overdetermined system if one also prescribes the value of the divergence of the vector field. In particular, for the inhomogeneous Beltrami equation to have nontrivial divergence-free solutions it is necessary that the inhomogeneity is also divergence-free.

The inhomogeneous Beltrami equations in the left hand side was studied in the preceding section through the analysis of the complex-valued solutions satisfying both the L^1 decay condition (6.3.16) and the L^1 SMB radiation condition (6.3.15). The stationary problem along a (ρ_0, T, δ)-stream tube of u_n in the right hand side of (6.4.4) will be studied in the $C^{k+1,\alpha}$ setting in the next subsection. Finally, we will glue both steps to show the convergence of the *modified Grad–Rubin iterative method* in Equation (6.2.4) at the end of this section.

6.4.3 Linear transport problem

We begin with the steady transport equations along (ρ_0, T, δ)-stream tubes in the right hand side of (6.2.4). The main idea to find a solution is to transport φ^0 along the foliated stream tube in Proposition 6.4.3 and to check that this definition leads to regular enough factors f_n of u_n due to the regularity of the tube.

Theorem 6.4.5. *Let G, Σ, μ satisfy the hypotheses (6.2.5), consider any $u \in C^{k+1,\alpha}(\overline{\Omega}, \mathbb{R}^3)$ such that $\mathcal{T}(\Sigma, u)$ is a (ρ_0, T, δ)-stream tube of such a velocity field and assume that $\varphi^0 \in C_c^{k+1,\alpha}(\Sigma)$. Consider the first integral equation associated with u*

$$\begin{cases} u \cdot \nabla\varphi = 0 & \text{in } \Omega \\ \varphi = \varphi^0 & \text{on } \Sigma. \end{cases} \tag{6.4.5}$$

Then, there exists an unique solution φ along $\mathcal{T}(\Sigma, u)$, its support lies in the closure of $\mathcal{T}(\Sigma, u)$ and it can be extended to a global solution in Ω with zero value outside $\mathcal{T}(\Sigma, u)$. Moreover, it belongs to $C^{k+1,\alpha}(\overline{\Omega})$ and the estimate

$$\|\varphi\|_{C^{k+1,\alpha}(\Omega)} \leq \|\varphi^0\|_{C^{k+1,\alpha}(\Sigma)}\, \kappa\left(\|u\|_{C^{k+1,\alpha}(\Omega)}, T\right)$$

holds, for some continuous and separately increasing function $\kappa : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+$.

Proof. The proof of this result can be found in [182, Lemmas 4.8, 4.9 and 5.2] for the restricted case $k = 0$. Let us then sketch the proof of the general case $k \neq 0$. Define the Calderón extension of u, $\overline{u} := \mathcal{P}(u)$, according to Proposition 6.4.1 and denote its flow map by $X(t; t_0, x_0)$.

- *Step 1.* Uniqueness.

  Notice that as long as φ is a smooth first integral of u, then

$$\frac{d}{dt}\varphi(X(t; 0, \mu(s))) = (\overline{u} \cdot \nabla\varphi)(X(t; 0, \mu(s))) = (u \cdot \nabla\varphi)(X(t; 0, \mu(s))) = 0,$$

for every $(t, s) \in \mathcal{D}(\Sigma, u)$. Therefore, $\varphi(x) = \varphi^0(\mu(s(x)))$ for every $x \in \mathcal{T}(\Sigma, u)$, where here on we will denote

$$(t(x), s(x)) = \phi^{-1}(x),\ x \in \mathcal{T}(\Sigma, u).$$

- *Step 2.* Existence.

  Noitce that the previous formula for φ defines a smooth function in $\mathcal{T}(\Sigma, u)$ (by virtue of the bijectivity and regularity of the parametrization ϕ in Proposition 6.4.2) which obviously solves (6.4.5) along the stream tube. Furthermore, with the exception of the endpoints, it is compactly supported in the interior of the tube. The extension of φ by zero outside the tube yields a global smooth solution of (6.4.5) in Ω.

- *Step 3.* Bound for $\|\varphi\|_{C^{k+1,\alpha}(\Omega)}$.

Since φ is extended by zero outside the tube, let us equivalently obtain an estimate for $\|\varphi\|_{C^{k+1,\alpha}(\mathcal{T}(\Sigma,u))}$. To this end, let us fix any multi-index $\gamma = (\gamma_1,\gamma_2,\gamma_3)$ such that $|\gamma| \leq k+1$ and note that

$$D^\gamma \varphi(x) = \gamma! \sum_{(l,\beta,\delta)\in\mathcal{D}(\gamma)} (D^\delta(\varphi^0\circ\mu))(s(x)) \prod_{r=1}^{l} \frac{1}{\delta_r!}\left(\frac{1}{\beta_r!}D^{\beta_r}s(x)\right)^{\delta_r}.$$

for every $x \in \mathcal{T}(\Sigma,u)$. The above formula is nothing but a chain rule for high order partial derivatives in high dimension. It can be found in [206], also see (N.3) in the introductory part of *Conventions and notation* of this thesis. Here, $\mathcal{D}(\gamma)$ stands for the set of all possible decompositions

$$\gamma = \sum_{r=1}^{l} |\delta_r|\beta_r,$$

where δ_r, β_r are multi-indices, $\delta := \sum_{r=1}^{l}\delta_r$ and for every $r = 1,\ldots,l-1$ there exists some $i_r \in \{1,2,3\}$ such that $(\beta_r)_i = (\beta_{r+1})_i$ for every $i \neq i_r$ and $(\beta_r)_{i_r} < (\beta_{r+1})_{i_r}$. First of all, it is necessary to know how to handle $D^{\beta_r}s(x)$. To this end, note that $\mathrm{Jac}(\phi^{-1})(x) = \mathrm{Jac}(\phi)^{-1}(\phi^{-1}(x))$ and, consequently,

$$D^\rho(\mathrm{Jac}(\phi^{-1})_{i,j})(x) = \sum_{n=1}^{n_\rho} \prod_{\substack{\beta\in\Gamma_n\\1\leq p,q\leq 3}} A^{i,j}_{n,p,q}(\rho,\beta)(D^\beta(\mathrm{Jac}(\phi)^{-1}_{p,q}))(\phi^{-1}(x)),$$

for every multi-index ρ such that $|\rho| \leq k$. Here, $A^{i,j}_{n,p,q}(\rho,\beta)$ stand for constant coefficients and Γ_n is a set of 3-multi-indices of order at most $|\rho| \leq k$. Expanding the products of sums by distributivity, each term in $D^\gamma\varphi$ takes the form

$$(D^\delta(\varphi^0\circ\mu))(s(x)) \prod_{\substack{\beta\in\Gamma\\1\leq p,q\leq 3}} B^{i,j}_{p,q}(\gamma,\beta)(D^\beta(\mathrm{Jac}(\phi)^{-1}_{p,q}))(\phi^{-1}(x)),$$

where Γ is a set of multi-indices with degree at most k. The first factor can be bounded by $\|\varphi^0\|_{C^{k+1,\alpha}(\Sigma)}$ whilst the terms in the second factor are bounded by $\kappa(\|u\|_{C^{k+1,\alpha}(\Omega)},T)$ as stated in Proposition 6.4.2. Hence, it is clear that

$$\|\varphi\|_{C^{k+1}(\Omega)} \leq \|\varphi^0\|_{C^{k+1,\alpha}(\Sigma)}\kappa(\|u\|_{C^{k+1,\alpha}(\Omega)},T).$$

Finally, for any multi-index with maximum order $k+1$, the α-Hölder seminorm of $D^\gamma\varphi$ can be estimated as follows. Take $x_1, x_2 \in \mathcal{T}(\Sigma,u)$ and appropriately add and subtract the crossed terms. Since $D^\delta(\varphi^0\circ\mu)$ is bounded by $\|\varphi^0\|_{C^{k+1,\alpha}(\Sigma)}$ and $D^\beta(\mathrm{Jac}(\phi)^{-1}_{pq})$ is bounded by $\kappa(\|u\|_{C^{k+1,\alpha}(\Omega)},T)$, then it only remains to obtain estimates for

$$\begin{aligned} I :&= \left.(D^\delta(\varphi^0\circ\mu))(s(x))\right|_{x_1}^{x_2},\\ II :&= \left.(D^\beta(\mathrm{Jac}(\phi)^{-1}_{p,q}))(\phi^{-1}(x))\right|_{x_1}^{x_2}. \end{aligned}$$

First, we distinguish the cases $|\delta| < k+1$ and $|\delta| = k+1$. In the former case, the mean value theorem, the estimates in Proposition 6.4.2 for $\mathrm{Jac}(\phi)^{-1}$ and the estimate (6.4.2) of the diameter of the stream tube $\mathcal{T}(\Sigma,u)$ yield the upper bound

$$I \leq \|\varphi^0\|_{C^{k+1,\alpha}(\Sigma)}\kappa(\|u\|_{C^{k+1,\alpha}(\Omega)},T)|x_1 - x_2|$$

$$\leq \|\varphi^0\|_{C^{k+1,\alpha}(\Sigma)}\kappa(\|u\|_{C^{k+1,\alpha}})(T\|u\|_{C^0(\Omega)} + \text{diam}(\Sigma))^{1-\alpha}|x_1 - x_2|^\alpha.$$

In the later case, the α-Hölder continuity of $D^\delta(\varphi^0 \circ \mu)$ gives rise to an analogous estimate

$$I \leq \|\varphi^0\|_{C^{k+1,\alpha}(\Sigma)}\kappa\left(\|u\|_{C^{k+1,\alpha}(\Omega)}, T\right)^\alpha |x_1 - x_2|^\alpha.$$

Second, note that $D^\beta(\text{Jac}(\phi)^{-1}_{p,q})$ is α-Hölder continuous with Hölder's constant that can be bounded above by $\kappa(\|u\|_{C^{k+1,\alpha}(\Omega)}, T)$ by virtue of Proposition 6.4.2. Thus,

$$II \leq \kappa(\|u\|_{C^{k+1,\alpha}(\Omega)}, T)|\phi^{-1}(x_1) - \phi^{-1}(x_2)|^\alpha.$$

The mean value theorem then leads to the desired upper estimate

$$|D^\gamma\varphi(x_1) - D^\gamma\varphi(x_2)| \leq \kappa(\|u\|_{C^{k+1,\alpha}(\Omega)}, T)|x_1 - x_2|^\alpha,$$

by accounting for an appropriately modification of the function κ if needed. □

Apart from existence and uniqueness results of (6.4.5), we also need some stability property for the problem (6.4.5) with respect to the generating vector field u. In such a way, after we show some compactness on $\{u_n\}_{n\in\mathbb{N}}$ in $C^{k+1,\alpha}(\overline{\Omega}, \mathbb{R}^3)$ we readily obtain further compactness of $\{\varphi_n\}_{n\in\mathbb{N}}$ in $C^{k,\alpha}(\overline{\Omega})$.

Corollary 6.4.6. *Let G, Σ, μ satisfy the properties (6.2.5). Consider any couple of vector fields $u_1, u_2 \in C^{k+1,\alpha}(\overline{\Omega}, \mathbb{R}^3)$, and denote by $\mathcal{T}_1 := \mathcal{T}(\Sigma, u_1)$ and $\mathcal{T}_2 := \mathcal{T}(\Sigma, u_2)$ the associated stream tubes which emanate from Σ. Assume that $\mathcal{T}_i$ is a (ρ_0, T, δ_i)-stream tube of u_i. Consider any boundary data $\varphi^0 \in C_c^{k+1,\alpha}(\Sigma)$ and the solutions φ_1 and φ_2 (according to Theorem 6.4.5) to each transport problem associated with u_1 and u_2 respectively:*

$$\begin{cases} \nabla\varphi_1 \cdot u_1 = 0, & x \in \Omega, \\ \varphi_1 = \varphi^0, & x \in \Sigma, \end{cases} \qquad \begin{cases} \nabla\varphi_2 \cdot u_2 = 0, & x \in \Omega, \\ \varphi_2 = \varphi^0, & x \in \Sigma. \end{cases}$$

Then,

$$\|\varphi_1 - \varphi_2\|_{C^{k,\alpha}(\Omega)} \leq \|\varphi^0\|_{C^{k+1,\alpha}(\Sigma)} \cdot \kappa\left(\|u_1\|_{C^{k+1,\alpha}(\Omega)}, T\right) \cdot \kappa\left(\|u_2\|_{C^{k+1,\alpha}(\Omega)}, T\right) \|u_1 - u_2\|_{C^{k+1,\alpha}(\Omega)},$$

where $\kappa : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+$ is continuous, separately increasing and does not depend on u_i, φ^0 or T.

The proof follows the same train of thoughts as Theorem 6.4.5 and we omit it here. We refer the reader to [182, Lemma 5.3] where it was proved for $C^{1,\alpha}$ regularity and to the master thesis [253, Corolario 2.4.4], where it has been extended to $C^{k+1,\alpha}$ regularity.

6.4.4 Limit of the approximate solutions

The existence and uniqueness results in Theorems 6.4.5 and 6.3.14 together with the stability result for the transport problem in Corollary 6.4.6 now allow us to take the limit as $n \to +\infty$ in the modified Grad–Rubin iterative scheme (6.2.4). Therefore, we obtain a generalized Beltrami field which is close to the initial strong Beltrami field and whose proportionality factor is a non-constant small enough perturbation of the initial constant proportionality factor λ:

Theorem 6.4.7. *Let G, Σ, μ satisfy the hypotheses (6.2.5) and assume that $0 \neq \lambda \in \mathbb{R}$ is not a Dirichlet eigenvalue of Laplace operator in the interior domain G. Consider any complex-valued strong Beltrami field $v_0 \in C^{k+1,\alpha}(\overline{\Omega}, \mathbb{C}^3)$ which satisfy the L^1 SMB radiation condition (6.3.15) and the L^1 decay property (6.3.16) in the exterior domain. Set its real part $u_0 := \Re v_0$ and assume that $\mathcal{T}(\Sigma, u_0)$ is a (ρ_0, T, δ)-stream tube of the velocity field u_0. Then, for every $\varepsilon_0 > 0$ there exists $\delta_0 > 0$ so that if $\varphi^0 \in C_c^{k+1,\alpha}(\Sigma)$ with $\|\varphi^0\|_{C^{k+1,\alpha}(\Sigma)} < \delta_0$ we have that the real parts u_{n+1} of the solutions $v_{n+1} \in C^{k+1,\alpha}(\overline{\Omega}, \mathbb{C}^3)$ together with the solutions $\varphi_n \in C^{k+1,\alpha}(\overline{\Omega})$ of the modified Grad–Rubin scheme (6.2.4) (see Theorems 6.4.5 and 6.3.14) strongly converge, namely,*

$$u_n \to u \text{ in } C^{k+1,\alpha}(\overline{\Omega}, \mathbb{R}^3) \text{ and } \varphi_n \to \varphi \text{ in } C^{k,\alpha}(\overline{\Omega}),$$

as $n \to +\infty$. In addition, $(u, \lambda + \varphi)$ solves the boundary value problem (6.4.3)*, u has optimal decay $|x|^{-1}$ and $\varphi = \varphi^0$ in Σ. Moreover, $\mathcal{T}(\Sigma, u)$ is a $(\rho_0, T, \delta/2)$-stream tube of u, φ has compact support inside the closure of such stream tube and u is close enough to u_0, specifically*

$$\|u - u_0\|_{C^{k+1,\alpha}(\Omega)} \leq \varepsilon_0 \|u_0\|_{C^{k+1,\alpha}(\Omega)}.$$

Proof. For simplicity of notation, we will denote the stream tubes associated with each vector field u_n which emanates from Σ by $\mathcal{T}_n := \mathcal{T}(\Sigma, u_n)$.

• *Step 1.* Well definition of u_n, φ_n and a priori estimates.

First of all, it is necessary to check whether the hypothesis of Theorems 6.4.5 and 6.3.14 hold and they can be deduced in each step from the corresponding hypotheses in the previous step in the iteration.

∘ *Step 1.1.* Base step of induction. Let us begin with the step $n = 0$:

$$\begin{cases} \nabla\varphi_0 \cdot u_0 = 0, & x \in \Omega, \\ \varphi_0 = \varphi^0, & x \in \Sigma, \end{cases} \qquad \begin{cases} \operatorname{curl} v_1 - \lambda v_1 = \varphi_0 u_0, & x \in \Omega, \\ v_1 \cdot \eta = u_0 \cdot \eta, & x \in S, \\ + L^1 \text{ Decay property (6.2.1)}, & \\ + L^1 \text{ SBM radiation condition (6.2.2)}. & \end{cases}$$

The hypotheses imply that $\mathcal{T}_0$ is a (ρ_0, T, δ)-stream tube of u_0 and $\varphi^0 \in C_c^{k+1,\alpha}(\Sigma)$. Hence, there exists a global solution φ_0 to the transport equation (Theorem 6.4.5). Moreover, $\varphi_0 u_0 \in C_c^{k+1,\alpha}(\overline{\Omega}, \mathbb{R}^3) \subseteq C_c^{k,\alpha}(\overline{\Omega}, \mathbb{R}^3)$ and its compact support is contained in the stream tube $\overline{\mathcal{T}_0}$. In particular, the estimate (6.4.2) ensures that $\operatorname{supp}(\varphi_0 u_0) \subseteq \overline{\mathcal{T}_0} \subseteq \overline{\Omega_R}$, where $\Omega_R := B_R(0) \setminus \overline{G}$ and $R := 2T\|u_0\|_{C^{k+1,\alpha}(\Omega)} + \operatorname{diam}(\Sigma)$. On the other hand, as S is regular enough, so is η and, consequently, $u_0 \cdot \eta \in C^{k+1,\alpha}(S)$. An integration by parts leads to the following expression

$$\int_S (\lambda u_0 \cdot \eta + \varphi_0 u_0 \cdot \eta)\, dS = \lambda \int_S u_0 \cdot \eta\, dS + \int_{\partial B_{R'}(0)} \varphi_0 u_0 \cdot \eta\, dS - \int_{\Omega_{R'}} \operatorname{div}(\varphi_0 u_0)\, dx$$

For $R' > R$, the second term vanishes as a consequence of the previous estimate for the diameter of the initial stream tube. Regarding the third term, notice that the same argument as above leads to

$$\operatorname{div}(\varphi_0 u_0) = \nabla\varphi_0 \cdot u_0 + \varphi_0 \operatorname{div} u_0 = 0.$$

We have $u_0 \cdot \eta = -\frac{1}{\lambda} \operatorname{div}_S(\eta \times u_0)$. Thus, the divergence theorem concludes that the first term vanishes too. Therefore, the hypotheses of Theorem 6.3.14 are satisfied, so there is a unique solution v_1 to the corresponding complex-valued inhomogeneous Beltrami equation in the right hand side of the step $n = 0$.

Let us prove an estimate for $u_1 - u_0$ that will be useful to prove the Cauchy condition in $C^{k+1,\alpha}(\overline{\Omega}, \mathbb{R}^3)$ for the sequence $\{u_n\}_{n\in\mathbb{N}}$. This vector field is the real part of $v_1 - v_0$, which satisfies the complex-valued exterior Neumann problem

$$\left\{\begin{array}{ll}(\operatorname{curl} - \lambda)(v_1 - v_0) = \varphi_0 u_0, & x \in \Omega,\\ (v_1 - v_0)\cdot \eta = 0, & x \in S,\\ + L^1 \text{ decay condition (6.2.1)}, & \\ + L^1 \text{ SMB radiation conditon (6.2.2)}. & \end{array}\right.$$

Therefore, the uniqueness of the solution to this problem (Proposition 6.3.16), the $C^{k+1,\alpha}$ estimates of such solutions (Corollary 6.3.18), and the $C^{k,\alpha}$ estimates for the solution of the steady transport equation (Theorem 6.4.5) allow us to obtain the following estimate for $v_1 - v_0$ and, consequently, for $u_1 - u_0$:

$$\|u_1 - u_0\|_{C^{k+1,\alpha}(\Omega)} = \|\Re(v_1 - v_0)\|_{C^{k+1,\alpha}(\Omega)} \leq \|v_1 - v_0\|_{C^{k+1,\alpha}(\Omega)} \leq C_0 \|\varphi_0 u_0\|_{C^{k,\alpha}(\Omega)}.$$

Here $C_0 > 0$ depends on k, α, λ, G and R. The *Leibniz rule* for the derivative of a product reads

$$D^\gamma(\varphi_0 u_0) = \sum_{\beta\leq\gamma} \binom{\gamma}{\beta} D^\beta \varphi_0 D^{\gamma-\beta} u_0,$$

for any multi-index γ. Therefore, the estimates in Theorem 6.4.5 for the derivatives up to order k of φ_0 and the combination of the mean value theorem and the Calderón's extension theorem (Proposition 6.4.1) to estimate the $C^{0,\alpha}$-norm of the derivatives of u_0 up to order k allow us to arrive at the inequality

$$\|D^\gamma(\varphi_0 u_0)\|_{C^0(\Omega)} \leq C_k \|\varphi^0\|_{C^{k+1,\alpha}(\Sigma)} \kappa\left(\|u_0\|_{C^{k+1,\alpha}(\Omega)}, T\right) \|u_0\|_{C^{k+1,\alpha}(\Omega)},$$

for every multi-index γ with $|\gamma| \leq k$, and

$$\begin{aligned}&\|D^\gamma(\varphi_0 u_0)\|_{C^{0,\alpha}(\Omega)} = \|D^\gamma(\varphi_0 u_0)\|_{C^{0,\alpha}(\mathcal{T}_0)}\\ &\quad\leq C_k C_{\mathcal{P}} \|\varphi^0\|_{C^{k+1,\alpha}(\Sigma)} \kappa\left(\|u_0\|_{C^{k+1,\alpha}}, T\right) \|u_0\|_{C^{k+1,\alpha}(\Omega)} (T\|u_0\|_{C^{k,\alpha}(\Omega)} + \operatorname{diam}\Sigma)^{1-\alpha},\end{aligned}$$

for every multi-index γ so that $|\gamma| = k$ and a nonnegative constant C_k depending on k. To derive the last estimate, we have used that

$$|D^{\gamma-\beta} u_0(x) - D^{\gamma-\beta} u_0(y)| \leq \|D^{\gamma-\beta}\overline{u_0}\|_{C^1(\mathbb{R}^3)} |x-y| \leq C_{\mathcal{P}} \|u_0\|_{C^{k+1,\alpha}(\Omega)} |x-y|^\alpha (\operatorname{diam}\mathcal{T}_0)^{1-\alpha},$$

for every $x, y \in \mathcal{T}_0$ and the estimate (6.4.2) for the diameter of the (ρ_0, T, δ)-stream tube of $\mathcal{T}_0$. Hence the following inequality

$$\begin{aligned}&\|u_1 - u_0\|_{C^{k+1,\alpha}(\Omega)}\\ &\leq K\left\{1 + (T\|u_0\|_{C^{k+1,\alpha}(\Omega)} + \operatorname{diam}\Sigma)^{1-\alpha}\right\} \|\varphi^0\|_{C^{k+1,\alpha}(\Sigma)} \kappa\left(\|u_0\|_{C^{k+1,\alpha}(\Omega)}, T\right) \|u_0\|_{C^{k+1,\alpha}(\Omega)},\end{aligned}$$

holds, with a constant $K = K(k, \alpha, \lambda, G, R)$. Now, we can fix the small parameter δ_0 such that it satisfies the following two conditions:

$$\begin{aligned} & K\left\{1+(4T\|u_0\|_{C^{k+1,\alpha}(\Omega)}+\mathrm{diam}\Sigma)^{1-\alpha}\right\} \\ & \quad\times\left\{\kappa\left(2\|u_0\|_{C^{k+1,\alpha}(\Omega)},T\right)+\|u_0\|_{C^{k+1,\alpha}(\Omega)}\kappa\left(2\|u_0\|_{C^{k+1,\alpha}(\Omega)},T\right)^2\right\}\delta_0<\frac{\min\{\varepsilon_0,1\}}{2}. \\ & K\left\{1+(4T\|u_0\|_{C^{k+1,\alpha}(\Omega)}+\mathrm{diam}\Sigma)^{1-\alpha}\right\} \\ & \quad\times\left\{\kappa\left(2\|u_0\|_{C^{k+1,\alpha}(\Omega)},T\right)+\|u_0\|_{C^{k+1,\alpha}(\Omega)}\kappa\left(2\|u_0\|_{C^{k+1,\alpha}(\Omega)},T\right)^2\right\}\|u_0\|_{C^{k+1,\alpha}(\Omega)}\,\delta_0 \\ & \qquad<\frac{1}{4}\frac{2\delta}{C_{\mathcal{P}}T}e^{-\frac{1}{2}C_{\mathcal{P}}\|u_0\|_{C^{k+1,\alpha}(\Omega)}T}. \end{aligned} \tag{6.4.6}$$

Then we infer

$$\begin{cases} \|u_1-u_0\|_{C^{k+1,\alpha}(\Omega)} & < \min\{\varepsilon_0,1\}\dfrac{1}{2}\|u_0\|_{C^{k+1,\alpha}(\Omega)}, \\ \|u_1-u_0\|_{C^{k+1,\alpha}(\Omega)} & < \dfrac{1}{4}\dfrac{2\delta}{C_{\mathcal{P}}T}e^{-\frac{1}{2}C_{\mathcal{P}}T\|u_0\|_{C^{k+1,\alpha}(\Omega)}}, \\ \|u_1\|_{C^{k+1,\alpha}(\Omega)} & \leq \dfrac{3}{2}\|u_0\|_{C^{k+1,\alpha}(\Omega)}. \end{cases} \tag{6.4.7}$$

∘ *Step 1.2*. Induction hypothesis.

Our goal is to show by induction that the following conditions hold true for any $n\in\mathbb{N}$:

$$\begin{cases} \|u_{n+1}-u_n\|_{C^{k+1,\alpha}(\Omega)} & \leq \dfrac{1}{2^n}\|u_1-u_0\|_{C^{k+1,\alpha}(\Omega)}<\min\{\varepsilon_0,1\}\dfrac{1}{2^{n+1}}\|u_0\|_{C^{k+1,\alpha}(\Omega)}, \\ \|u_{n+1}-u_n\|_{C^{k+1,\alpha}(\Omega)} & < \dfrac{1}{2}\dfrac{1}{2^{n+1}}\dfrac{2\delta}{C_{\mathcal{P}}T}e^{-\frac{1}{2}C_{\mathcal{P}}T\|u_0\|_{C^{k+1,\alpha}(\Omega)}}, \\ \|u_{n+1}-u_0\|_{C^{k+1,\alpha}(\Omega)} & < \min\{\varepsilon_0,1\}\displaystyle\sum_{i=1}^{n+1}\dfrac{1}{2^i}\|u_0\|_{C^{k+1,\alpha}(\Omega)}, \\ \|u_{n+1}-u_0\|_{C^{k+1,\alpha}(\Omega)} & < \dfrac{1}{2}\displaystyle\sum_{i=1}^{n+1}\dfrac{1}{2^i}\dfrac{2\delta}{C_{\mathcal{P}}T}e^{-\frac{1}{2}C_{\mathcal{P}}\|u_0\|_{C^{k+1,\alpha}(\Omega)}}, \\ \|u_{n+1}\|_{C^{k+1,\alpha}(\Omega)} & < \displaystyle\sum_{i=0}^{n+1}\dfrac{1}{2^i}\|u_0\|_{C^{k+1,\alpha}(\Omega)}. \end{cases} \tag{6.4.8}$$

Notice that this is true for $n=0$ due to the above step (6.4.7). Let us assume that the inductive hypotheses holds for all indices less than n. Specifically, we assume that φ_m, v_{m+1} are well defined, i.e., the corresponding problems have a unique solution, that u_{m+1} are divergence-free and (6.4.8) hold for indices $m<n$.

∘ *Step 1.3*. Proof for $m=n$.

The inductive hypotheses imply the existence of a vector field $v_n\in C^{k+1,\alpha}(\overline{\Omega},\mathbb{C}^3)$ and $\varphi_{n-1}\in C^{k,\alpha}(\overline{\Omega})$. Moreover, $\mathcal{T}_n$ is a $\left(\rho_0,T,\left(1-\frac{1}{2}\sum_{i=1}^{n}\frac{1}{2^i}\right)\delta\right)$-stream tube of the real part $u_n=\Re v_n$ because of the third inequality in (6.4.8). Consequently, there exists a unique solution $\varphi_n\in C^{k,\alpha}(\overline{\Omega})$ to the transport problem in the left hand side of (6.2.4) according to Theorem 6.4.5. The last estimate in (6.4.8) along with (6.4.2) lead to $\mathcal{T}_n\subseteq\overline{\Omega_R}$. Therefore, φ_n is compactly supported in $\overline{\Omega_R}\subseteq\overline{\Omega}$ and the same argument as in the step $n=0$ ensures the existence and uniqueness of a solution $v_{n+1}\in C^{k+1,\alpha}(\overline{\Omega},\mathbb{C}^3)$ to the complex-valued exterior Neumann problem for the inhomogeneous Beltrami equation in the right hand side of (6.2.4).

Notice that the vanishing flux hypothesis in Theorem 6.3.14 is satisfied. To check it we get

$$\int_S (\lambda u_0 \cdot \eta + \varphi_n u_n \cdot \eta)\, dS = \lambda \int_S u_0 \cdot \eta\, dS + \int_{\partial B_{R'}(0)} \varphi_n u_n \cdot \eta\, dS - \int_{\Omega_{R'}} \operatorname{div}(\varphi_n u_n)\, dx.$$

The first term is zero as before, the second one also vanishes for a choice $R' > R$ and the last one is zero too because φ_n is a first integral of u_n and u_n is divergence-free according to the induction hypothesis. Consequently, it is easy to verity that u_{n+1} is also divergence-free.

To conclude, let us prove the inductive hypothesis (6.4.8) for $u_{n+1}-u_n$. Taking the difference of the corresponding complex-valued exterior boundary value problems we have that $v_{n+1}-v_n$ solves

$$\begin{cases} (\operatorname{curl}-\lambda)(v_{n+1}-v_n) = \varphi_n u_n - \varphi_{n-1}u_{n-1}, & x \in \Omega, \\ (v_{n+1}-v_n)\cdot \eta = 0, & x \in S, \\ + L^1 \text{ decay conditions (6.2.1)}, \\ + L^1 \text{ SMB radiation condition (6.2.2)}. \end{cases}$$

Again, thanks to the uniqueness property (Proposition 6.3.16), the $C^{k+1,\alpha}$ estimates for these solutions (Corollary 6.3.18) and the $C^{k,\alpha}$ estimates for the solution of the steady transport equation (Theorem 6.4.5), we obtain the following estimate for $v_{n+1}-v_n$ and, consequently, for the real parts $u_{n+1}-u_n$

$$\begin{aligned} \|u_{n+1}-u_n\|_{C^{k+1,\alpha}(\Omega)} &= \|\Re(v_{n+1}-v_n)\|_{C^{k+1,\alpha}(\Omega)} \\ &\leq \|v_{n+1}-v_n\|_{C^{k+1,\alpha}(\Omega)} \leq C_0\|\varphi_n u_n - \varphi_{n-1}u_{n-1}\|_{C^{k,\alpha}(\Omega)}. \end{aligned}$$

Now, observe that the right hand side $\varphi_n u_n - \varphi_{n-1}u_{n-1}$ of the above inhomogeneous Beltrami equation has compact support inside $\overline{\mathcal{T}_n} \cup \overline{\mathcal{T}_{n-1}} \subseteq \overline{\Omega_R}$ (see estimate (6.4.2) and the last inequalities for the $C^{k+1,\alpha}$ norms of u_n and u_{n-1} in the inductive hypothesis). This is a crucial fact because it guarantees that C_0 is independent on the iteration number n. Indeed by the regularity Theorem H.1.10 in Appendix H, the constant C_0 only depends on k, α, λ, G, R because all the supports of the inhomogeneous terms in the complex-valued exterior Neumann problems lie within the same bounded subset $\overline{\Omega_R}$ of the exterior domain. Notice that

$$\|\varphi_n u_n - \varphi_{n-1}u_{n-1}\|_{C^{k,\alpha}(\Omega)} \leq \|(\varphi_n-\varphi_{n-1})u_n\|_{C^{k,\alpha}(\Omega)} + \|\varphi_{n-1}(u_n-u_{n-1})\|_{C^{k,\alpha}(\Omega)}.$$

Since $\mathcal{T}_n$ is a $\left(\rho_0, T, \left(1-\frac{1}{2}\sum_{i=1}^n \frac{1}{2^i}\right)\delta\right)$-stream tube of u_n, $\mathcal{T}_{n-1}$ is a $\left(\rho_0, T, \left(1-\frac{1}{2}\sum_{i=1}^{n-1}\frac{1}{2^i}\right)\delta\right)$-stream tube of u_{n-1} and $u_{n-1}\cdot\eta = u_0\cdot\eta = u_n\cdot\eta$ on S, we can apply both estimates in Theorem 6.4.5 and Corollary 6.4.6 to obtain the inequality

$$\begin{aligned} \|\varphi_n u_n - \varphi_{n-1}u_{n-1}\|_{C^{k+1,\alpha}(\Omega)} &\leq K\|\varphi^0\|_{C^{k+1,\alpha}(\Sigma)}\left\{1+(4T\|u_0\|_{C^{k+1,\alpha}(\Omega)}+\text{diam}\Sigma)^{1-\alpha}\right\} \\ &\times \left\{\kappa(2\|u_0\|_{C^{k+1,\alpha}(\Omega)},T) + \|u_0\|_{C^{k+1,\alpha}(\Omega)}\kappa\left(2\|u_0\|_{C^{k+1,\alpha}(\Omega)},T\right)^2\right\}\|u_n-u_{n-1}\|_{C^{k+1,\alpha}(\Omega)}. \end{aligned}$$

Consequently, the estimate

$$\begin{aligned} \|u_{n+1}-u_n\|_{C^{k+1,\alpha}(\Omega)} &\leq K\|\varphi^0\|_{C^{k+1,\alpha}(\Sigma)}\left\{1+(4T\|u_0\|_{C^{k+1,\alpha}(\Omega)}+\text{diam}\Sigma)^{1-\alpha}\right\} \\ &\times \left\{\kappa(2\|u_0\|_{C^{k+1,\alpha}(\Omega)},T) + \|u_0\|_{C^{k+1,\alpha}(\Omega)}\kappa\left(2\|u_0\|_{C^{k+1,\alpha}(\Omega)},T\right)^2\right\}\|u_n-u_{n-1}\|_{C^{k+1,\alpha}(\Omega)} \end{aligned}$$

holds, with K independent of n. Since $\|\varphi^0\|_{C^{k+1,\alpha}(\Sigma)} < \delta_0$ and δ_0 is small enough to ensure (6.4.6), one has

$$\|u_{n+1} - u_n\|_{C^{k+1,\alpha}(\Omega)} < \frac{1}{2}\|u_n - u_{n-1}\|_{C^{k+1,\alpha}(\Omega)},$$

and the inductive hypothesis for indices less than n leads to the first two inequalities in (6.4.8).

The last three estimates can be obtained as follows. Firstly, the preceding two estimates together with the induction hypotheses lead to

$$\|u_{n+1} - u_0\|_{C^{k+1,\alpha}(\Omega)} \leq \sum_{i=0}^{n} \|u_{i+1} - u_i\|_{C^{k+1,\alpha}(\Omega)} \leq \min\{\varepsilon_0, 1\} \sum_{i=1}^{n+1} \frac{1}{2^i}\|u_0\|_{C^{k+1,\alpha}(\Omega)}.$$

Similarly, we have

$$\|u_{n+1} - u_0\|_{C^{k+1,\alpha}(\Omega)} \leq \sum_{i=0}^{n} \|u_{i+1} - u_i\|_{C^{k+1,\alpha}(\Omega)} \leq \frac{1}{2}\sum_{i=1}^{n+1} \frac{1}{2^i}\frac{2\delta}{C_{\mathcal{P}}T} e^{-\frac{1}{2}C_{\mathcal{P}}T\|u_0\|_{C^{k+1,\alpha}(\Omega)}}.$$

The last inequality in (6.4.8) is obvious by the triangle inequality:

$$\|u_{n+1}\|_{C^{k+1,\alpha}(\Omega)} \leq \|u_0\|_{C^{k+1,\alpha}(\Omega)} + \|u_{n+1} - u_0\|_{C^{k+1,\alpha}(\Omega)} \leq \sum_{i=0}^{n+1} \frac{1}{2^i}\|u_0\|_{C^{k+1,\alpha}(\Omega)}.$$

• *Step 2*. Strong compactness.

Using the above inequalities in (6.4.8) one can show that $\{u_n\}_{n\in\mathbb{N}}$ and $\{\varphi_n\}_{n\in\mathbb{N}}$ are Cauchy sequences in $C^{k+1,\alpha}(\overline{\Omega}, \mathbb{R}^3)$ and $C^{k,\alpha}(\overline{\Omega})$, respectively. On the one hand, we find

$$\|u_{n+m} - u_n\|_{C^{k+1,\alpha}(\Omega)} \leq \sum_{i=n}^{n+m-1} \|u_{i+1} - u_i\|_{C^{k+1,\alpha}(\Omega)} < \sum_{i=n}^{n+m-1} \frac{1}{2^{i+1}}\|u_0\|_{C^{k+1,\alpha}(\Omega)} \leq \frac{1}{2^n}\|u_0\|_{C^{k+1,\alpha}(\Omega)}.$$

Likewise, the third inequality in (6.4.8) along with the property $u_n\cdot\eta = u_0\cdot\eta$ on S, shows that $\mathcal{T}_n$ are $\left(\rho_0, T, \left(1 - \frac{1}{2}\sum_{i=0}^{n}\frac{1}{2^i}\right)\delta\right)$-stream tubes of u_n. Therefore, $\{\varphi_n\}_{n\in\mathbb{N}}$ also satisfies the Cauchy condition in $C^{k,\alpha}(\overline{\Omega})$ due to Corollary 6.4.6. Thus, it converges in $C^{k,\alpha}$ to some $\varphi \in C^{k,\alpha}(\overline{\Omega})$.

• *Step 3*. Identification of the limit and properties.

Let us now take the limit as $n \to +\infty$ in the iterative scheme to deduce

$$\begin{array}{ccccccccc} \operatorname{div} u_{n+1} & = & 0 & \operatorname{curl} u_{n+1} - \lambda u_{n+1} & = & \varphi_n u_n & u_{n+1}\cdot\eta & = & u_0\cdot\eta \\ \downarrow & & \downarrow & \downarrow & & \downarrow & \downarrow & & \downarrow \\ \operatorname{div} u & = & 0 & \operatorname{curl} u - \lambda u & = & \varphi u & u\cdot\eta & = & u_0\cdot\eta. \end{array}$$

Moreover, the L^1 SMB radiation condition (6.3.15) and the decay property (6.3.16) lead to complex-valued solutions v_n to the exterior Neumann problem for the inhomogeneous Beltrami equations in the iterative scheme with the asymptotic behavior $|v_n(x)| \leq C|x|^{-1}$, $x \in \Omega$, for every n and C independent of n. To check it, notice that Theorem 6.3.14 provides a decomposition of v_{n+1} into generalized volume and single layer potentials whose densities are $u_0\cdot\eta$, $\varphi_n u_n$ and the sequence ξ_n of solutions to the boundary integral equations (6.3.23). The single layer potentials and its first order partial derivarives are dominated by the corresponding integral kernels Γ_λ and $\nabla\Gamma_\lambda$ for x far enough from the surface S. This leads to an upper bound $C|x|^{-1}$ where C depends on the C^0 norm of $u_0\cdot\eta$ and ξ_n. Both quantities can be bounded above by $\|u_0\cdot\eta\|_{C^{k,\alpha}(S)}$ and $\|\varphi_n u_n\|_{C^{k,\alpha}(\Omega)}$, which are uniformly bounded with respect to n.

Furthermore, the volume layer potentials and its first order partial derivatives can be bounded by $C|x|^{-1}$ for an n-independent constant thanks to Theorem C.0.4 in Appendix C and the above argument. Consequently, we get the same asymptotic behavior at infinity for the limit vector field u.

Let us show now that $\mathcal{T}(\Sigma, u)$ is a $(\rho_0, T, \delta/2)$-stream tube of u and that the support of φ lies in it. To do so, take limits in the fourth inequality in (6.4.8) and notice that

$$\|u - u_0\|_{C^{k+1,\alpha}(\Omega)} \leq \frac{1}{2}\frac{2\delta}{C_{\mathcal{P}}T}e^{-\frac{1}{2}C_{\mathcal{P}}T\|u_0\|_{C^{k+1,\alpha}(\Omega)}},$$

Then, Corollary 6.4.6 yields the first assertion. The second one is clear by taking into account that $\text{supp}\varphi_n \subseteq \overline{\mathcal{T}_n}$, for every $n \in \mathbb{N}$. Finally, to check that the limit solution is close to the initial strong Beltrami field u_0, it suffices to take limits in the third inequality in (6.4.8) to get

$$\|u - u_0\|_{C^{k+1,\alpha}(\Omega)} \leq \min\{\varepsilon_0, 1\}\sum_{i=1}^{+\infty}\frac{1}{2^i}\|u_0\|_{C^{k+1,\alpha}(\Omega)} \leq \varepsilon_0\|u_0\|_{C^{k+1,\alpha}(\Omega)}. \qquad \square$$

Remark 6.4.8. *The generalized Beltrami field $u \in C^{k+1,\alpha}(\overline{\Omega}, \mathbb{R}^3)$ obtained in the preceding Theorem has proportionality factor $f = \lambda + \varphi$, for some compactly supported perturbation $\varphi \in C^{k,\alpha}(\overline{\Omega})$. Moreover, it decays as $|x|^{-1}$ at infinity. To check that it is optimal note first that $\text{div}(\varphi u) = 0$ and consider any open subset $\Sigma' \subseteq S$ such that $\text{supp}\,\varphi^0 \subseteq \Sigma' \subseteq \overline{\Sigma'} \subseteq \Sigma$. Notice that the preceding proof indeed shows that φ is compactly supported in $\overline{\mathcal{T}(\Sigma', u)}$, that is a $(\rho_0, T, \delta/2)$-stream tube. Take any $x \in \Sigma\setminus\overline{\Sigma'}$ and notice that $u(x)\cdot\eta(x) \geq \rho_0 > 0$. This means that u is transverse to S at some point outside the support of φ^0. Hence,*

$$u = O(|x|^{-1}) \ \text{when} \ |x| \to \infty,$$

is the optimal decay by virtue of Corollary 6.3.20.

A related remark in the harmonic case ($\lambda = 0$) is in order now.

Remark 6.4.9. *Recall that a similar result to that in Theorem 6.4.7 was previously proved in [182] to obtain generalized Beltrami fields $u \in C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^3)$ (nonlinear force-free fields), i.e., solutions to*

$$\text{curl}\, u = fu, \quad x \in \Omega,$$

with compactly supported small proportionality factors $f \in C^{0,\alpha}(\overline{\Omega})$.

- *On the one hand, the low regularity $C^{1,\alpha}$ and $C^{0,\alpha}$ is not a weakness in such result since despite not being directly considered in [182], our regularity results in Appendix H of this thesis provide the necessary background to promote the existence theorem therein to a higher regularity setting.*

- *On the other hand, such generalized Beltrami fields in [182] decay as $|x|^{-2}$ at infinity. There is no contradiction neither with Corollary 6.3.20 (since it holds under the assumption $\lambda \neq 0$) nor with the Liouville theorem in [227] (since it just holds for globally defined generalized Beltrami fields).*

- *On the contrary, the latter can be used to show an interesting property of such generalized Beltrami fields obtained as perturbations of harmonic fields: they cannot be globally extended to the whole space by virtue of the fall-off obstructions in [227]. Unfortunately, the same cannot be directly said for generalized Beltrami fields obtained as perturbations of strong Beltrami fields and it is still an open problem to elucidate if the generalized Beltrami fields in exterior domains that we have constructed by perturbation in Theorem 6.4.7 can be actually extended to entire solutions to the Beltrami equation.*

6.5 Knotted and linked stream lines and tubes in generalized Beltrami fields

Our objective in this section is to apply the convergence result for the modified Grad–Rubin method (6.2.4) that we established in the previous section (Theorem 6.4.7) to show the existence of almost global Beltrami fields of class $C^{k+1,\alpha}$ with a nonconstant factor that realize any given configuration of vortex tubes and vortex lines, modulo a small diffeomorphism. Here k is an arbitrary integer.

6.5.1 Knots and links in almost global generalized Beltrami fields

Our goal here is to show that the partial stability result for almost global Beltrami fields allows us to conclude the existence of Beltrami fields with a non-constant proportionality factor that are defined in all of $\mathbb{R}^3$ but, say, in the complement of an arbitrarily small ball, and which have a collection of vortex tubes and lines of arbitrary topology. Let us recall that, as discussed before, a stream tube (or invariant torus) of a divergence-free velocity field u is structurally stable if any divergence-free field that is close enough to u in $C^{3,\alpha}$ has an invariant torus given by a $C^{0,\alpha}$-small diffeomorphism of the initial tube. Although we shall not state these properties explicitly here, just as in [116] the vortex tubes that we construct are accumulated on by a positive-measure set of invariant tori on which the vortex lines are ergodic, see also the introductory Chapter 1 of the thesis for further information.

Theorem 6.5.1. *Let G be an exterior domain satisfying (6.2.5) and consider any collection of disjoint knotted and linked thin tubes $\mathcal{T}_\varepsilon(\Gamma_1),\ldots,\mathcal{T}_\varepsilon(\Gamma_n)$ whose closure is contained in the exterior domain Ω. Then, for ε small enough and any k,α there exists a nonzero constant λ, an open subset $\Sigma\subseteq S$ and some $\delta_0>0$ with the following property: for any function $\varphi^0\in C_c^{k+1,\alpha}(\Sigma)$ with $\|\varphi^0\|_{C^{k+1,\alpha}(\Sigma)}<\delta_0$ there is a Beltrami field $u\in C^{k+1,\alpha}(\overline{\Omega},\mathbb{R}^3)$ with factor $\lambda+\varphi$, where $\varphi\in C^{k,\alpha}(\overline{\Omega})$ satisfies $\varphi|_\Sigma=\varphi^0$ so that*

$$\begin{cases}\operatorname{curl} u=(\lambda+\varphi)u, & x\in\Omega,\\ \operatorname{div} u=0, & x\in\Omega.\end{cases}$$

Furthermore, $u=O\left(|x|^{-1}\right)$ as $|x|\to+\infty$, the support of φ is compact and lies in the (ρ_0,T,δ)-stream tube $\mathcal{T}(\Sigma,u)$ of u radiating from Σ (with the exception of the endpoints) and $\mathcal{T}_\varepsilon(\Gamma_1),\ldots,\mathcal{T}_\varepsilon(\Gamma_n)$ can be modified (by a diffeomorphism Φ close enough to the identity in any C^m norm) into a collection of structurally stable vortex tubes of u, $\Phi(\mathcal{T}_\varepsilon(\Gamma_1)),\ldots,\Phi(\mathcal{T}_\varepsilon(\Gamma_n))$, (possibly) knotted and linked with the tube $\mathcal{T}(\Sigma,u)$.

Proof. Take a curve Γ_0 intersecting S transversally and such that $\mathcal{T}_\varepsilon(\Gamma_0)\cap\Omega$ has only a connected component. We also assume that Γ_0 does not intersect any of the other curves Γ_j, so that the setup is then as depicted in Figure 6.5. For $\varepsilon>0$ small enough, [116] asserts the existence of some diffeomorphism Φ' arbitrarily close to the identity map in any C^m norm such that $\Phi'(\mathcal{T}_\varepsilon(\Gamma_0)),\ldots,\Phi'(\mathcal{T}_\varepsilon(\Gamma_n))$ are vortex tubes of a strong Beltrami field u_0 which satisfies the equation $\operatorname{curl} u_0=\lambda u_0$ in $\mathbb{R}^3$ for some non-zero constant λ (of order ε^3), see also the introductory Chapter 1. By construction, these tubes are structurally stable and Φ' can be assumed to be arbitrarily close to the identity in any C^m norm, so the new thin tubes enjoy the same geometric features as we had assumed on the initial ones. Let $x_0\in S\cap\Phi'(\Gamma_0)$ be where u_0 points outwards and consider any open and connected neighborhood Σ of x_0 in S such that $\Sigma\subseteq S\cap\Phi'(\mathcal{T}_\varepsilon(\Gamma_0))$.

Recall that u_0 in [116] is of the form

$$u_0=\frac{\operatorname{curl}(\operatorname{curl}+\lambda)}{2\lambda^2}\sum_{l=0}^{L}\sum_{m=-l}^{l}c_l^m j_l(\lambda|x|)Y_l^m\left(\frac{x}{|x|}\right).$$

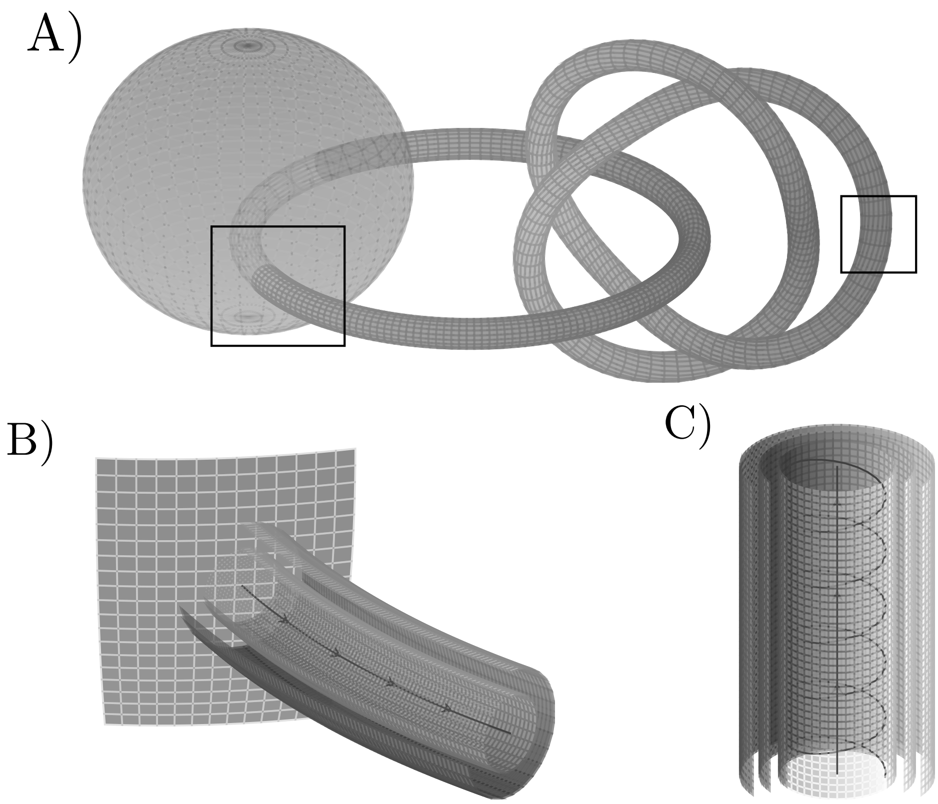


Figure 6.5: A) Collection of knotted and linked vortex tubes $\{\Phi'(\mathcal{T}_\varepsilon(\Gamma_0)), \Phi'(\mathcal{T}_\varepsilon(\Gamma_1))\}$ of the strong Beltrami field u_0, respectively homeomorphic to a ring and a to the trefoil tube. B) Transverse intersection of the vortex tube $\Phi'(\mathcal{T}_\varepsilon(\Gamma_0))$ and the interior domain G. Here we have zoomed in the squared region on the left side of the above figure, showing the smaller outward pointing (ρ_0, T, δ)-stream tube of u_0 that emerges from Σ. The perturbation φ of λ will be supported there. C) Zoom of the vortex tube $\Phi'(\mathcal{T}_\varepsilon(\Gamma_1))$ with trefoil knot. It shows the internal structure of such vortex tube of u_0, which contains uncountably many nested tori and knotted vortex lines.

Since u_0 is obviously real-valued, it is the real part of the vector field

$$v_0 = \frac{\operatorname{curl}(\operatorname{curl} + \lambda)}{2\lambda^2} \sum_{l=0}^{L} \sum_{m=-l}^{l} c_l^m h_l^{(1)}(\lambda|x|) Y_l^m\left(\frac{x}{|x|}\right),$$

where $h_l^{(1)} := j_l + iy_l$ is the spherical Hankel function of l-th order and y_l denotes the spherical Bessel function of the second kind and l-th order. By construction, v_0 satisfies the Beltrami equation (and in particular is smooth) in $\mathbb{R}^3 \setminus \{0\}$, while it diverges at the origin due to the presence of a Bessel function of the second kind. In particular, it is a Beltrami field in Ω.

As the Hankel function $h_l^{(1)}$ has been chosen to satisfy the scalar radiation condition

$$(\partial_r - i\lambda) h_l^{(1)}(\lambda r) = o(r^{-1}),$$

it is straightforward to check that $v_0 \in C^{k+1,\alpha}(\overline{\Omega}, \mathbb{C}^3)$ is a complex-valued solution to the Beltrami equation in the exterior domain Ω, which satisfies the L^1 SMB radiation condition (6.3.15) and the weak L^1 decay property (6.3.16) (see [84, Equation 2.41] along with Remark 6.3.8 and

Figure 6.1). It is also apparent that $\mathcal{T}(\Sigma, u_0) \subseteq \Phi'(\mathcal{T}_\varepsilon(\Gamma_0))$ is a (ρ_0, T, δ)-stream tube of u_0 by construction (see Figure 6.5), and that $\lambda \sim \varepsilon^3$ can be prevented from being a Dirichlet eigenvalue of the Laplace operator in the interior domain G as long as ε is taken small enough. Then, we are ready to apply the convergence Theorem 6.4.7 for the modified Grad–Rubin method starting up with the strong Beltrami field u_0. This result ensures the existence of $\delta_0 > 0$ so that whenever $\|\varphi^0\|_{C^{k+1,\alpha}(\Sigma)} \leq \delta_0$, then there exists a generalized Beltrami field $u \in C^{k+1,\alpha}(\overline{\Omega}, \mathbb{R}^3)$ and a perturbation $\varphi \in C^{k,\alpha}(\overline{\Omega})$ solving the exterior boundary value problem (6.4.3) with $\varphi = \varphi^0$, $x \in \Sigma$. $\mathcal{T}(\Sigma, u)$ is a $(\rho_0, T, \delta/2)$ stream tube of u, φ is compactly supported in the closure of such stream tube and $\|u - u_0\|_{C^{k+1,\alpha}(\Omega)}$ can be made arbitrarily small. In view of the structural stability of the vortex tubes of u_0, the theorem follows. □

6.6 Local stability of generalized Beltrami fields

Our objective in this section is to show that, in fact, any generalized Beltrami field possesses a local partial stability property which can be essentially regarded as a local version of Theorem 6.4.7. We recall that, in view of the results in [117], one cannot prove a full stability result even in arbitrarily small open sets, so we regard this partial stability (where partial is understood in a very precise sense) as a satisfactory counterpart to the results in this paper.

6.6.1 A local stability theorem

We shall next present the local stability result that constitutes the core of this section. The philosophy of this result is that, as one is able to perturb strong Beltrami fields, one should also be able to perturb generalized Beltrami fields in small domains, since in a small region a $C^{k,\alpha}$ function behaves as a constant plus a small perturbation. Somehow, this reduces our effort to estimates similar to the ones that we have already obtained, so our presentation of the proof of this result will be a little sketchier than before. The gist will be to show that, although the strong convergence of the modified Grad–Rubin scheme cannot be granted in $C^{k+1,\alpha}$ for u_n and $C^{k,\alpha}$ for f_n, we can pass to the limit in $C^{1,\alpha}$ and $C^{0,\alpha}$ provided that both the domain and the perturbation of the proportionality factor are small enough. Elliptic regularity will then yield the high order regularity by a bootstrap argument.

In order to support our argument, let us first sketch the effect of the size of the domain on the solutions of the next Neumann boundary value problem associated with the inhomogeneous Beltrami equation in some open ball $B_R(x_0)$

$$\begin{cases} \operatorname{curl} u - \lambda u = w, & x \in B_R(x_0), \\ u \cdot \eta = 0, & x \in \partial B_R(x_0), \end{cases} \tag{6.6.1}$$

where $w \in C^{0,\alpha}(\overline{B}_R(x_0), \mathbb{R}^3)$ has zero flux. We will be interested in the case where R becomes very small.

This problem has being carefully analyzed in [298] for bounded domains and in [182] for exterior unbounded domains in the harmonic case ($\lambda = 0$). The non-harmonic counterpart was studied in [193] and Section 6.3 for the inhomogeneous Beltrami equation in bounded and exterior domains respectively. In the bounded setting, λ has to be assumed "regular" (see [193]). To this end, notice that taking $|\lambda| < c/R$ (for an appropriate universal constant $c > 0$) prevents λ from being an eigenvalue of the Laplacian in $B_R(x_0)$. Hence, $|\lambda| < c/R$ is a sufficient condition ensuring the well-posedness of (6.6.1). All the above results provide an estimate for the unique solution u to (6.6.1) in terms of w of the form

$$\|u\|_{C^{1,\alpha}(B_R(x_0))} \leq C_{\lambda,R} \|w\|_{C^{0,\alpha}(B_R(x_0))},$$

where the dependence of the constant $C_{\lambda,R}$ on λ and R is not explicit. The next technical result aims to provide some explicit R-dependent estimate for u in some space.

Lemma 6.6.1. *Let $u \in C^{1,\alpha}(\overline{B}_R(x_0), \mathbb{R}^3)$ be the unique solution to the Neumann boundary value problem associated with the Beltrami equation (6.6.1) for $|\lambda| < c/R$ and $R \in (0,1)$. Then,*

$$\|u\|_{C^{1,\alpha}(B_R(x_0))} \leq CR^{-\alpha}\|w\|_{C^{0,\alpha}(B_R(x_0))}, \tag{6.6.2}$$

for some positive constant C depending on α but not on u, w, x_0 or R.

Proof. To obtain an explicit R-dependent estimate of u in some space, let us perform the change of variables $y = \frac{x-x_0}{R}$. Then, one obtains the following vector fields in the unit ball centered at the origin:

$$U(y) = u(x), \quad W(y) = w(x),$$

solving the Neumann boundary value problem for the Beltrami equation in $B_1(0)$:

$$\begin{cases} \operatorname{curl} U - \lambda R U = RW, & y \in B_1(0), \\ U \cdot \eta = 0, & y \in \partial B_1(0). \end{cases}$$

Thus, the above-mentioned results yield the following bound for some R-independent $C > 0$

$$\|U\|_{C^{1,\alpha}(B_1(0))} \leq CR\|W\|_{C^{0,\alpha}(B_1(0))},$$

where $|\lambda| < c/R$ has been used to avoid the λ-dependence of the constant C. Note that by definition

$$\|W\|_{C^{0,\alpha}(B_1(0))} = \|w\|_{C^0(B_R(x_0))} + R^\alpha[w]_{\alpha,B_R(x_0)},$$
$$\|U\|_{C^{1,\alpha}(B_1(0))} = \|u\|_{C^0(B_R(x_0))} + R\sum_{i=1}^{3}\|\partial_{x_i}u\|_{C^0(B_R(x_0))} + R^{1+\alpha}\sum_{i=1}^{3}[\partial_{x_i}u]_{\alpha,B_R(x_0)}.$$

Since $R \in (0,1)$, then we are led to (6.6.2). □

Another key ingredient is to show that $C^{1,\alpha}$ vector fields near a non-equilibrium point verify a "structurally stable" flow box theorem, to be understood in the next precise sense.

Lemma 6.6.2. *Let $u \in C^{1,\alpha}(\Omega, \mathbb{R}^3)$ be a (nontrivial) vector field and consider some $x_0 \in \Omega$ such that $u(x_0) \neq 0$. Then, there exist $R_0 > 0$ and $\delta_0 >$ such that $\overline{B}_{2R_0}(x_0) \subseteq \Omega$, u vanishes nowhere in the ball and for every $0 < R < R_0$ and there exists some surface $\Sigma_R \subseteq \partial B_R(x)$ and a positive function $T_R \in C(\Sigma_R)$ such that for every $v \in C^{1,\alpha}(\overline{B}_R(x_0), \mathbb{R}^3)$ with $\|u - v\|_{C^{1,\alpha}(B_R(x_0))} < \delta_0$, then*

$$B_R(x_0) \subseteq \mathcal{T}(\Sigma_R, \overline{v}, T_R) \subseteq B_{2R}(x_0).$$

Here, the above stream tube reads

$$\mathcal{T}(\Sigma_R, \overline{v}, T_R) := \{X^{\overline{v}}(t; 0, x) : x \in \Sigma_R, \ t \in (0, T_R(x))\},$$

$\overline{v}$ is the Calderón extension of v from $B_R(x_0)$ to $\overline{B}_{2R_0}(x_0)$ (see Proposition 6.4.1) and the height T_R of the stream tube is not constant but it continuously depends, stream line by stream line, on the base point $x \in \Sigma_R$ (see Figure 6.6). Furthermore, the parametrizations μ_R of Σ_R can be normalized by choosing

$$\mu_R(s) = R\mu(s), \ s \in D_R,$$

for some open subset $D_R \subseteq D_1(0)$ of the unit disc centered at 0, and some local parametrization of the unit sphere $\mu : D_1(0) \longrightarrow \partial B_1(x_0)$. Since the proof follows the same lines as Lemma 6.4.4 in Section 6.4, we skip it and pass to the central result of this section.

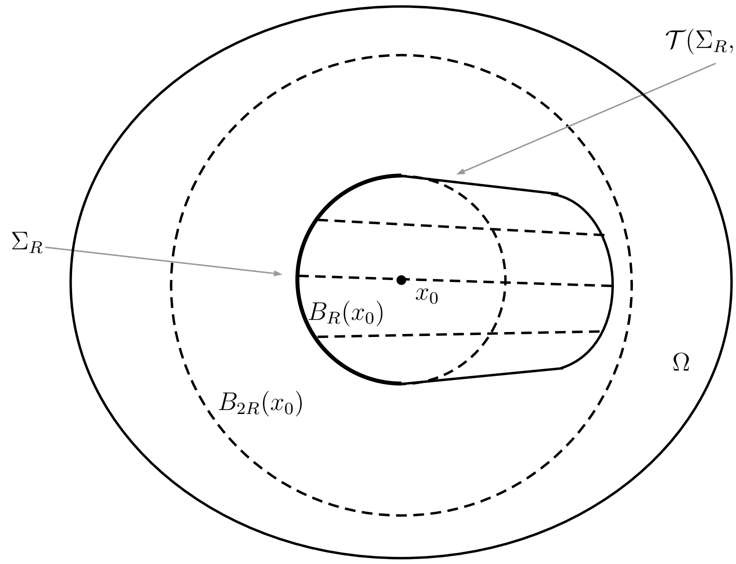


Figure 6.6: Flow box $\mathcal{T}(\Sigma_R,\overline{v},T_R)$ covering the small ball $B_R(x_0)$.

Theorem 6.6.3. *Let u_0 be a nontrivial generalized Beltrami field of class $C^{k+1,\alpha}(\Omega,\mathbb{R}^3)$, where $k\in\mathbb{N}$ and $\alpha\in(0,1)$, and consider its (nonconstant) proportionality factor $f_0\in C^{k,\alpha}(\Omega)$. Take some nonequilibrium point $x_0\in\Omega$ of u_0 and fix some $\varepsilon_0>0$. Then, for each small enough radius $R>0$ there is some surface $\Sigma_R\subseteq\partial B_R(x_0)$ and some constant $\delta_R>0$ so that for every $\varphi^0\in C^{k+1,\alpha}(\Sigma_R)$ with $\|\varphi^0\|_{C^{k+1,\alpha}(\Sigma_R,\mu_R)}<\delta_R$ there exist $\varphi\in C^{k,\alpha}(\overline{B}_R(x_0))$ and $u\in C^{k+1,\alpha}(\overline{B}_R(x_0),\mathbb{R}^3)$ such that $\varphi=\varphi^0$ on Σ_R and u is a strong Beltrami field with proportionality factor $f_0+\varphi$ enjoying the same normal component as u_0 in $\partial B_R(x_0)$:*

$$\begin{cases}\operatorname{curl} u=(f_0+\varphi)u, & x\in B_R(x_0),\\ \operatorname{div} u=0, & x\in B_R(x_0),\\ u\cdot\eta=u_0\cdot\eta, & x\in\partial B_R(x_0).\end{cases}$$

Furthermore,

$$\|u-u_0\|_{C^{k+1,\alpha}(B_R(x_0))}\leq\varepsilon_0\|u_0\|_{C^{k+1,\alpha}(B_R(x_0))}.$$

Proof. The proof has two steps. First, we will prove the theorem for low Hölder exponents and regularity (namely, $\alpha\in(0,1/2)$ and $k=0$). Second, we will show a bootstrap argument based on elliptic gain of regularity that will raise the estimates in the first step to its full strength and will conclude the proof of the theorem for general regularity and Hölder exponents.

Let us first assume that $\alpha\in(0,1/2)$, define $\lambda_0:=f_0(x_0)$ and fix some radius $R_0>0$ so that $\overline{B}_{2R_0}(x_0)\subseteq\Omega$, u_0 vanishes nowhere in $\overline{B}_{2R_0}(x_0)$ and the assertions in Lemma 6.6.2 fulfil. Without loss of generality, we can assume that $R_0<\min\{1,c/|\lambda_0|\}$. Moreover, note that the homogeneous generalized Beltrami equation can be restated as an inhomogeneous Beltrami equation with constant proportionality factor and an inhomogeneous term taking the form of a small remainder, i.e.,

$$\operatorname{curl} u_0-\lambda_0 u_0=\mathcal{R}(x-x_0)u_0,\quad x\in\Omega, \tag{6.6.3}$$

where $f_0(x)=\lambda_0+\mathcal{R}(x-x_0)$ for every $x\in\overline{B}_{2R_0}(x_0)$, i.e.,

$$\mathcal{R}(z):=\left(\int_0^1\nabla f_0(x_0+\theta z)\,d\theta\right)\cdot z,\quad z\in\overline{B}_{2R_0}(0).$$

Next, consider the following modified iterative scheme of Grad–Rubin type. It consists of a sequence of transport equations

$$\begin{cases} \nabla\varphi_n \cdot u_n = -\nabla f_0 \cdot u_n, & x \in B_R(x_0), \\ \varphi_n = \varphi^0, & x \in \Sigma_R., \end{cases} \tag{6.6.4}$$

along with a sequence of boundary value problems associated with the inhomogeneous Beltrami equation

$$\begin{cases} \operatorname{curl} u_{n+1} - \lambda_0 u_{n+1} = \mathcal{R}(x - x_0)u_n + \varphi_n u_n, & x \in B_R(x_0), \\ u_{n+1} \cdot \eta = u_0 \cdot \eta, & x \in \partial B_R(x_0). \end{cases} \tag{6.6.5}$$

Note that they have been chosen in a consistent way so that as long as $\{u_n\}_{n\in\mathbb{N}}$ and $\{\varphi_n\}_{n\in\mathbb{N}}$ have limits (in some sense), then the limits u and φ give rise to a generalized Beltrami field whose proportionality factor is a perturbation $f_0 + \varphi$ of the initial factor f_0. Without loss of generality, we can assume that $\lambda_0 \neq 0$ (in the case $\lambda_0 = 0$ we would need the additional condition $\operatorname{div} u_{n+1} = 0$).

• *Step 1*. Good definition and a priori estimates.

Let us show that both $u_{n+1} \in C^{1,\alpha}(\overline{B}_R(x_0), \mathbb{R}^3)$ and $f_n \in C^{0,\alpha}(\overline{B}_R(x_0))$ are well defined and that the following properties

$$\begin{cases} \|u_{n+1} - u_n\|_{C^{1,\alpha}(B_R(x_0))} \leq \dfrac{1}{2^n}\|u_1 - u_0\|_{C^{1,\alpha}(B_R(x_0))} < \dfrac{\min\{\varepsilon_0, 1\}}{2^{n+1}}\|u_0\|_{C^{1,\alpha}(B_R(x_0))}, \\ \|u_{n+1} - u_0\|_{C^{1,\alpha}(B_R(x_0))} \leq \min\{\varepsilon_0, 1\}\displaystyle\sum_{i=1}^{n+1}\frac{1}{2^i}\|u_0\|_{C^{1,\alpha}(B_R(x_0))}, \\ \|u_{n+1}\|_{C^{1,\alpha}(B_R(x_0))} \leq \min\{\varepsilon_0, 1\}\displaystyle\sum_{i=0}^{n+1}\|u_0\|_{C^{1,\alpha}(B_R(x_0))}, \end{cases} \tag{6.6.6}$$

are fulfilled for every $n \in \mathbb{N}$.

∘ *Step 1.1*. Let us start with the base case $n = 0$.

On the one hand, the transport problem (6.6.4) with $n = 0$ can be solved in $B_R(x_0)$ as $B_R(x_0) \subseteq \mathcal{T}(\Sigma_R, u_0, T_R) \subseteq B_{2R}(x_0)$ by virtue of Lemma 6.6.2. Indeed,

$$\varphi_0(X^{u_0}(t; 0, x)) = \varphi^0(x) - \int_0^t (\nabla f_0 \cdot u_0)(X^{u_0}(\tau, 0, x))\, d\tau, \quad x \in \Sigma_R,\ t \in (0, T_R(x))$$

defines a solution in $\mathcal{T}(\Sigma_R, u_0, T_R)$ and, in particular, in $B_R(x_0)$. Now, notice that

$$\int_{\partial B_R(x_0)} (\mathcal{R}(\cdot - x_0)u_0 + \varphi_0 u_0 + \lambda_0 u_0\ ) \cdot \eta\, dS + \lambda_0 \int_{\partial B_R(x_0)} u_0 \cdot \eta\, dS$$
$$= \int_{B_R(x_0)} (\nabla(f_0 + \varphi_0) \cdot u_0 + (f_0 + \varphi_0) \operatorname{div} u_0)\ dx = 0,$$

and λ is regular (see [193]) with respect to the inhomogeneous problem (6.6.5) with $n = 0$ because $R < R_0 < c/|\lambda_0|$. Hence, (6.6.5) has an unique solution $u_1 \in C^{1,\alpha}(\overline{B}_R(x_0), \mathbb{R}^3)$ by virtue of the existence theorem in [193]. Notice that since $\operatorname{div} u_0 = 0$ and the first integral equations in (6.6.4) hold, then

$$-\lambda_0 \operatorname{div} u_1 = (f_0 + \varphi_0) \operatorname{div} u_0 + \nabla(f_0 + \varphi_0) \cdot u_0 = 0.$$

Furthermore, $u_1 - u_0$ solves the Neumann boundary value problem

$$\begin{cases} (\operatorname{curl} - \lambda_0)(u_1 - u_0) = \mathcal{R}(x - x_0)u_0 + \varphi_0 u_0, & x \in B_R(x_0), \\ (u_1 - u_0) \cdot \eta = 0, & x \in B_R(x_0). \end{cases}$$

Consequently,

$$\|u_1 - u_0\|_{C^{1,\alpha}(B_R(x_0))} \leq \frac{C}{R^\alpha} \left(\|\mathcal{R}(\cdot - x_0)\|_{C^{0,\alpha}(B_R(x_0))} + \|\varphi_0\|_{C^{0,\alpha}(B_R(x_0))} \right) \|u_0\|_{C^{0,\alpha}(B_R(x_0))}.$$

A similar result to that in Theorem 6.4.5 yields the estimate

$$\|\varphi_0\|_{C^{0,\alpha}(B_R(x_0))} \leq \left(\|\varphi^0\|_{C^{1,\alpha}(\Sigma_R,\mu_R)} + R^{1-\alpha} + \|T_R\|_{C^0(\Sigma_R)} \right) \times \kappa \left(\|u_0\|_{C^{1,\alpha}(B_R(x_0))}, \|T_R\|_{C^0(\Sigma_R)}, \|\mu_R\|_{C^{1,\alpha}(B_R(x_0))} \right),$$

for some separately increasing function κ. Regarding the remainder, it is clear that

$$\|\mathcal{R}(\cdot - x_0)\|_{C^{0,\alpha}(B_R(x_0))} \leq C R^{1-\alpha}, \tag{6.6.7}$$

which is indeed the reason behind the estimate for φ_0 that we stated above. Notice that although $\mathcal{R}$ is clearly bounded above by R in $B_R(0)$, the α-Hölder constant is $O(R^{1-\alpha})$. Specifically, take $z_1, z_2 \in B_R(0)$ and split $\mathcal{R}$ as follows

$$\mathcal{R}(z_1) - \mathcal{R}(z_2) = I + II,$$

where each term reads

$$I := \left(\int_0^1 \nabla f_0(x_0 + \theta z_1)\, d\theta \right) \cdot (z_1 - z_2),$$
$$II := \left(\int_0^1 (\nabla f_0(x_0 + \theta z_1) - \nabla f_0(x_0 + \theta z_1))\, d\theta \right) \cdot z_2.$$

By virtue of the α-Hölder continuity of ∇f_0, II can be bounded as follows:

$$|II| \leq \|f_0\|_{C^{1,\alpha}(B_R(x_0))} |z_2| \int_0^1 |z_1 - z_2|^\alpha \theta^\alpha\, d\theta \leq \frac{\|f_0\|_{C^{1,\alpha}(B_R(x_0))}}{\alpha + 1} R |z_1 - z_2|^\alpha.$$

The first term enjoys the bound

$$|I| \leq \|\nabla f_0\|_{C^0(B_R(x_0))} |z_1 - z_2| \leq 2^{1-\alpha} \|\nabla f_0\|_{C^0(B_R(x_0))} R^{1-\alpha} |z_1 - z_2|^\alpha,$$

which then leads to the desired estimate (6.6.7). Notice that one could have raised the $R^{1-\alpha}$ power to R if one assumed that $\nabla f_0(x_0) = 0$.

Also, note that $\|\mu_R\|_{C^{1,\alpha}(D_R)}, \|T_R\|_{C^0(\Sigma_R)} \leq C_0 R$ for some universal constant $C_0 > 0$. Then, the above estimate for $u_1 - u_0$ can be written as

$$\|u_1 - u_0\|_{C^{0,\alpha}(B_R(x_0))} \leq \frac{C}{R^\alpha} \left(\|\varphi^0\|_{C^{1,\alpha}(\Sigma_R,\mu_R)} + 2R^{1-\alpha} \right) \left\{ 1 + \kappa \left(\|u_0\|_{C^{1,\alpha}(B_R(x_0))}, C_0, \|\mu\|_{C^{1,\alpha}(D_1(0))} \right) \right\} \|u_0\|_{C^{0,\alpha}(B_R(x_0))}.$$

Hereafter we will assume that

$$C\left(\frac{\delta_R}{R^\alpha}+2R^{1-2\alpha}\right)\left\{1+\kappa\left(2\|u_0\|_{C^{1,\alpha}(B_R(x_0))},C_0,\|\mu\|_{C^{1,\alpha}(D_1(0))}\right)\cdot\right.$$
$$\left.+\,2\kappa\left(2\|u_0\|_{C^{1,\alpha}(B_R(x_0))},C_0,\|\mu\|_{C^{1,\alpha}(D_1(0))}\right)^2\|u_0\|_{C^{1,\alpha}(B_R(x_0))}\right\}<\frac{\varepsilon_0}{2},\quad (6.6.8)$$

with $\varepsilon_0\in(0,1)$ small enough so that $\varepsilon_0\|u_0\|_{C^{0,\alpha}(B_R(x_0))}<\delta_0$. Since we are considering low Hölder exponents $\alpha\in(0,1/2)$, then we can ensure the existence of small enough $R\in(0,R_0)$ and $\delta_R>0$ enjoying the above property.

∘ *Step 1.2*. Inductive hypothesis.

Assume that we have already defined $f_m\in C^{0,\alpha}(\overline{B}_R(x_0))$ and $u_{m+1}\in C^{1,\alpha}(\overline{B}_R(x_0),\mathbb{R}^3)$ for every $m<n$ such that they verify (6.6.4)–(6.6.6) and $u_m=0$ is divergence-free for every index $m<n$.

∘ *Step 1.3*. We prove the result for $m=n$.

First, the transport problem (6.6.4) can be uniquely solved in $B_R(x_0)$ by virtue of Lemma 6.6.2, the inductive hypothesis (6.6.6) and the assumption on ε_0 since

$$\|u_n-u_0\|_{C^{1,\alpha}(B_R(x_0))}\leq\varepsilon_0\|u_0\|_{C^{1,\alpha}(B_R(x_0))}<\delta_0.$$

Second, the boundary value problem (6.6.5) can also be uniquely solved since

$$\int_{\partial B_R(x_0)}(\mathcal{R}(\cdot-x_0)u_n+\varphi_n u_n)\cdot\eta\,dS+\lambda_0\int_{\partial B_R(x_0)}u_0\cdot\eta\,dS$$
$$=\int_{B_R(x_0)}(\nabla(f_0+\varphi_n)\cdot u_n+(f_0+\varphi_n)\operatorname{div}u_n)\;dx=0,$$

by the inductive hypothesis and λ is assumed to be a regular value. Furthermore, a similar argument to that in the step $n=0$ shows that u_{n+1} is divergence-free again. Let us finally obtain the desired estimates for $u_{n+1}-u_n$. To this end, note that $u_{n+1}-u_n$ solves the boundary value problem

$$\begin{cases}(\operatorname{curl}-\lambda_0)(u_{n+1}-u_n)=w_n, & x\in B_R(x_0),\\ (u_{n+1}-u_n)\cdot\eta=0, & x\in\partial B_R(x_0),\end{cases}$$

where the right hand side is given by

$$w_n:=\mathcal{R}(\cdot-x_0)(u_n-u_{n-1})+(\varphi_n-\varphi_{n-1})u_n+\varphi_{n-1}(u_n-u_{n-1}).$$

Hence, we arrive at the following bound

$$\|u_{n+1}-u_n\|_{C^{1,\alpha}(B_R(x_0))}\leq\frac{C}{R^\alpha}\big(\|\mathcal{R}(\cdot-x_0)\|_{C^{0,\alpha}(B_R(x_0))}\|u_n-u_{n-1}\|_{C^{0,\alpha}(B_R(x_0))}$$
$$+\|\varphi_n-\varphi_{n-1}\|_{C^{0,\alpha}(B_R(x_0))}\|u_n\|_{C^{0,\alpha}(B_R(x_0))}+\|\varphi_{n-1}\|_{C^{0,\alpha}(B_R(x_0))}\|u_n-u_{n-1}\|_{C^{0,\alpha}(B_R(x_0))}\big).$$

On the one hand, the remainder can be bounded above as in (6.6.7). On the other hand, $\|\varphi_n\|_{C^{0,\alpha}(B_R(x_0))}$ and $\|\varphi_n-\varphi_{n-1}\|_{C^{0,\alpha}(B_R(x_0))}$ can be estimated as follows

$$\|\varphi_{n-1}\|_{C^{0,\alpha}(B_R(x_0))}\leq\left(\|\varphi^0\|_{C^{1,\alpha}(\Sigma_R,\mu_R)}+R^{1-\alpha}+\|T_R\|_{C^0(\Sigma_R)}\right)$$
$$\times\,\kappa\left(\|u_{n-1}\|_{C^{1,\alpha}(B_R(x_0))},\|T_R\|_{C^0(\Sigma_R)},\|\mu_R\|_{C^{1,\alpha}(B_R(x_0))}\right),$$
$$\|\varphi_n-\varphi_{n-1}\|_{C^{0,\alpha}(B_R(x_0))}\leq\left(\|\varphi^0\|_{C^{1,\alpha}(\Sigma_R,\mu_R)}+R^{1-\alpha}+\|T_R\|_{C^0(\Sigma_R)}\right)$$

$$\times\,\kappa\left(\|u_n\|_{C^{1,\alpha}(B_R(x_0))},\|T_R\|_{C^0(\Sigma_R)},\|\mu_R\|_{C^{1,\alpha}(B_R(x_0))}\right)$$
$$\times\,\kappa\left(\|u_{n-1}\|_{C^{1,\alpha}(B_R(x_0))},\|T_R\|_{C^0(\Sigma_R)},\|\mu_R\|_{C^{1,\alpha}(B_R(x_0))}\right)\|u_n-u_{n-1}\|_{C^{1,\alpha}(B_R(x_0))}.$$

Consequently, the inductive hypothesis along with our choice (6.6.8) leads to the first inequality in (6.6.6) and the remaining two inequalities obviously follows from the first one by virtue of the triangle inequality.

• *Step 2*. Strong compactness.

As in Section 6.4, the first inequality in (6.6.6) shows that $\{u_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $C^{1,\alpha}(\overline{B}_R(x_0),\mathbb{R}^3)$. By completeness, consider $u\in C^{1,\alpha}(\overline{B}_R(x_0))$ such that $u_n\to u$ in the space $C^{1,\alpha}(\overline{B}_R(x_0,\mathbb{R}^3))$. Moreover, the same reasoning as above yields the estimate

$$\begin{aligned}\|\varphi_n-\varphi_m\|_{C^{0,\alpha}(B_R(x_0))}\leq{}&\left(\|\varphi^0\|_{C^{1,\alpha}(\Sigma_R,\mu_R)}+R^{1-\alpha}+\|T_R\|_{C^0(\Sigma_R)}\right)\\&\times\,\kappa\left(\|u_n\|_{C^{1,\alpha}(B_R(x_0))},\|T_R\|_{C^0(\Sigma_R)},\|\mu_R\|_{C^{1,\alpha}(B_R(x_0))}\right)\\&\times\,\kappa\left(\|u_m\|_{C^{1,\alpha}(B_R(x_0))},\|T_R\|_{C^0(\Sigma_R)},\|\mu_R\|_{C^{1,\alpha}(B_R(x_0))}\right)\|u_n-u_m\|_{C^{1,\alpha}(B_R(x_0))},\end{aligned}$$

for every indices $n,m\in\mathbb{N}$. Then, there exists some constant $K=K(\delta_R,R,\|u_0\|_{C^{0,\alpha}})>0$ so that

$$\|\varphi_n-\varphi_m\|_{C^{0,\alpha}(\overline{B}_R(x_0))}\leq K\|u_n-u_m\|_{C^{1,\alpha}(B_R(x_0))}.$$

Hence, $\{\varphi_n\}_{n\in\mathbb{N}}$ is also a Cauchy sequence in $C^{0,\alpha}(\overline{B}_R(x_0))$ and one can consider $\varphi\in C^{0,\alpha}(\overline{B}_R(x_0))$ such that $\varphi_n\to\varphi\quad$ in $C^{0,\alpha}(\overline{B}_R(x_0))$.

• *Step 3*. Identification of the limit.

Taking limits in (6.6.4)–(6.6.5) we are led to a generalized Beltrami field $u\in C^{1,\alpha}(\overline{B}_R(x_0),\mathbb{R}^3)$ solving

$$\begin{cases}\operatorname{curl}u=(f_0+\varphi)u, & x\in B_R(x_0),\\ \operatorname{div}u=0, & x\in B_R(x_0),\\ u\cdot\eta=u_0\cdot\eta, & x\in\partial B_R(x_0),\end{cases}$$

for a perturbation $\varphi\in C^{0,\alpha}(\overline{B}_R(x_0))$ of the factor such that $\varphi=\varphi^0$ on Σ_R.

• *Step 4*. Gain of regularity.

Let us finally show that $u\in C^{k+1,\alpha}(\overline{B}_R(x_0),\mathbb{R}^3)$ and $\varphi\in C^{k,\alpha}(\overline{B}_R(x_0))$ by a bootstrap argument based on the elliptic gain of regularity. The key observation now is that, by acting with the curl operator on the equation for u, it follows that

$$\begin{cases}\Delta u=-\operatorname{curl}((f_0+\varphi)u), & x\in B_R(x_0),\\ u\cdot\eta=u_0\cdot\eta, & x\in\partial B_R(x_0),\\ \operatorname{curl}u\times\eta=(f_0+\varphi)u\times\eta, & x\in\partial B_R(x_0).\end{cases}$$

Then, the next hierarchy of inequalities

$$\begin{aligned}&\|u\|_{C^{l+1,\alpha}(B_R(x_0))}\\&\quad\leq C(\|(f_0+\varphi_0)u\|_{C^{l,\alpha}(B_R(x_0))}+\|u_0\cdot\eta\|_{C^{l+1,\alpha}(\partial B_R(x_0))}+\|(f_0+\varphi)u\times\eta\|_{C^{l,\alpha}(B_R(x_0))}).\end{aligned}$$

hold for every $l\geq 0$. We then get that the fact that u is of class $C^{1,\alpha}$ implies that φ is of class $C^{0,\alpha}$. In turns, it ensures that u is in $C^{2,\alpha}$ and, repeating the argument as many times as necessary (up to the regularity on φ^0 and u_0, i.e., $C^{k+1,\alpha}$) we derive the desired gain of regularity. Indeed, the estimate

$$\|u-u_0\|_{C^{1,\alpha}(B_R(x_0))}\leq\varepsilon_0\|u_0\|_{C^{1,\alpha}(B_R(x_0))},$$

can be promoted to its $C^{k+1,\alpha}$ version, i.e.,

$$\|u - u_0\|_{C^{k+1,\alpha}(B_R(x_0))} \leq \varepsilon_0 \|u_0\|_{C^{k+1,\alpha}(B_R(x_0))}.$$

• *Step 5*. General exponents $\alpha \in (0, \frac{1}{2})$.

So far, we have only taken low Hölder exponents $\alpha \in (0, 1/2)$. Assume now that $u_0 \in C^{k+1,\alpha'}(\Omega, \mathbb{R}^3)$ and $f_0 \in C^{k,\alpha'}(\Omega)$ for some $\alpha' \in (\alpha, 1)$. In particular, we obtain that $u_0 \in C^{k+1,\alpha}(\overline{B}_{2R_0}(x_0), \mathbb{R}^3)$ and $\varphi_0 \in C^{k,\alpha}(\overline{B}_{2R}(x_0))$. The above argument, yields a strong Beltrami field $u \in C^{k+1,\alpha}(\overline{B}_R(x_0), \mathbb{R}^3)$ with proportionality factor $f_0 + \varphi$ for some perturbation $\varphi \in C^{k,\alpha}(\overline{B}_R(x_0))$ such that $\varphi = \varphi^0$ on Σ_R as long as R is small enough and $\|\varphi^0\|_{C^{k+1,\alpha'}(\Sigma_R)} < \delta_R$. Since

$$\|\varphi^0\|_{C^{k+1,\alpha}(\Sigma_R)} = \|\varphi^0 \circ \mu_R\|_{C^{k+1,\alpha}(D_R)} \leq \|\varphi^0 \circ \mu_R\|_{C^{k+1,\alpha'}(D_R)} = \|\varphi^0\|_{C^{k+1,\alpha'}(\Sigma_R)},$$

then, the above smallness assumption on the $C^{k+1,\alpha}(\Sigma_R)$ norm φ^0 follows from the corresponding assumption on the $C^{k+1,\alpha'}(\Sigma_R)$ norm, i.e., $\|\varphi^0\|_{C^{k+1,\alpha'}(\Sigma_R)} < \delta_R$. Since φ solves

$$\begin{cases} \nabla\varphi \cdot u = -\nabla f_0 \cdot u, & x \in B_R(x_0), \\ \varphi = \varphi^0, & x \in \Sigma_R, \end{cases}$$

then, a similar result to that in Theorem 6.4.5 leads to $\varphi \in C^{1,\alpha}(\overline{B}_R(x_0))$ because so is u, f_0 and φ^0. In particular $\varphi \in C^{0,\alpha'}(\overline{B}_R(x_0))$ and $u \in C^{0,\alpha'}(\overline{B}_R(x_0), \mathbb{R}^3)$. Then, the above bootstrap in the Beltrami equation yields $\varphi \in C^{k,\alpha'}(\overline{B}_R(x_0))$ and $u \in C^{k+1,\alpha'}(\overline{B}_R(x_0))$, thereby concluding the proof. □

Appendices

6.A Sketch of proof of Proposition 6.4.3

We sketch the proof of this result for the reader's convenience. Also, we refer to [182, Lemma 5.1] for the case $k = 0$ and [253, Proposición 2.1.7] for the general proof with $k \geq 1$.

• *Step 1*. Proof of the first three items.

The first assertion is apparent: since u points outwards at any point in Σ, then the stream line of u arising from $\mu(s)$ points towards Ω at $t = 0$. Hence, a small piece of such stream line must stay in Ω. Regarding the second assertion, ϕ is clearly onto by virtue of the definition of $\mathcal{T}(\Sigma, u)$. To check that ϕ is one to one, note that different stream lines cannot touch because of the uniqueness part in Proposition 6.4.2, and that the streamlines of u emerging from Σ cannot be closed loops because u points outwards at Σ.

• *Step 2*. Regularity of $\mathrm{Jac}(\phi)$ and $\mathrm{Jac}(\phi)^{-1}$.

The C^{k+1} regularity of ϕ is clear because so is $X(t; t_0, x_0)$ by *Peano's differentiability theorem* as stated in Proposition 6.4.2. Let us show that its Jacobian matrix is regular at any point in $\mathcal{D}(\Sigma, u)$ to obtain the same regularity of ϕ^{-1} through the inverse mapping theorem. This matrix takes the form

$$\mathrm{Jac}(\phi)(t,s) = \left(\frac{\partial \phi}{\partial t}(t,s) \middle| \frac{\partial \phi}{\partial s_1}(t,s) \middle| \frac{\partial \phi}{\partial s_2}(t,s) \right).$$

For fixed $s \in D$, each column vector is a solution to the linear ODE

$$\dot{x}(t) = \mathrm{Jac}(u)(\phi(t,s))x(t).$$

Thus, $\mathrm{Jac}(\phi)(\cdot, s)$ is a solution matrix to such linear ODE, whose determinant at $t = 0$ equals

$$\det(\mathrm{Jac}(\phi)(0,s)) = \left|\frac{\partial\mu}{\partial s_1}(s) \times \frac{\partial\mu}{\partial s_2}(s)\right| u(\mu(s)) \cdot \eta(\mu(s)) \geq \rho_1\rho_0 > 0. \tag{6.A.1}$$

Here ρ_1 stands for any positive uniform lower bound of the first factor. Thus, $\mathrm{Jac}(\phi)(t,s)$ is regular for all t by the *Jacobi–Liouville formula*. In particular, the derivatives of $\mathrm{Jac}(\phi)$ and $\mathrm{Jac}(\phi)^{-1}$ up to order k can be continuously extended to $\overline{\mathcal{D}(\Sigma,u,T)}$ by the analogous properties of u and μ.

• *Step 3*. Derivation of C^k bounds for $\mathrm{Jac}(\phi)$.

Let us finally recursively show that all of them are bounded and the k-th order ones are α-Hölder continuous indeed. We proceed by induction.

∘ *Step 3.1*. Base step.

First, notice that

$$\left|\frac{\partial\phi}{\partial t}(t,s)\right| \leq \|u\|_{C^0(\Omega)}, \qquad \left|\frac{\partial\phi}{\partial s_i}(t,s)\right| \leq \left|\frac{\partial\mu}{\partial s_i}(s)\right| + \int_0^t \|\mathrm{Jac}(u)\|_{C^0(\Omega)} \left|\frac{\partial\phi}{\partial s_i}(\tau,s)\right| \, d\tau,$$

for every $(t,s) \in \mathcal{D}(\Sigma,u,T)$. As a consequence, Grönwall's lemma amounts to the upper bound

$$\|\mathrm{Jac}(\phi)\|_{C^0(\mathcal{D}(\Sigma,u,T))} \leq \|u\|_{C^0(\Omega)} + \|\mu\|_{C^1(D,\mathbb{R}^3)} e^{T\|\mathrm{Jac}(u)\|_{C^0(\Omega)}} \leq \kappa(\|u\|_{C^{k+1,\alpha}(\Omega)}, T),$$

for some function κ which is separately increasing.

∘ *Step 3.2*. Inductive hypothesis.

Now, assume that

$$\|\mathrm{Jac}(\phi)\|_{C^m(\mathcal{D}(\Sigma,u,T))} \leq \kappa(\|u\|_{C^{k+1,\alpha}(\Omega)}, T), \tag{6.A.2}$$

holds true for some n such that $0 < n \leq k$ and all m with $0 \leq m < n$.

∘ *Step 3.3*. We prove (6.A.2) for $m = n$.

Fix any multi-index γ such that $|\gamma| = n$ and take derivatives on the characteristic system (6.4.1) to arrive at

$$D^\gamma\left(\frac{\partial\phi_i}{\partial t}\right) = D^\gamma(u_i(\phi(t,s))) = \gamma! \sum_{(l,\beta,\delta)\in\mathcal{D}(\gamma)} (D^\delta u_i)(\phi(t,s)) \prod_{r=1}^{l} \frac{1}{\delta_r!}\left(\frac{1}{\beta_r!} D^{\beta_r}\phi(t,s)\right)^{\delta_r}.$$

The above formula is a chain rule for high order partial derivatives that can be found in [206]. Here, $\mathcal{D}(\gamma)$ stands for the set of all the possible decompositions of γ of the form

$$\gamma = \sum_{r=1}^{l} |\delta_r| \beta_r,$$

where δ_r, β_r are multi-indices, $\delta := \sum_{r=1}^{l} \delta_r$ and for every $r = 1, \ldots, l-1$ there exists some $i_r \in \{1,2,3\}$ such that $(\beta_r)_i = (\beta_{r+1})_i$ for every $i \neq i_r$ and $(\beta_r)_{i_r} < (\beta_{r+1})_{i_r}$. Similarly

$$\frac{\partial}{\partial t} D^\gamma\left(\frac{\partial\phi_i}{\partial s_j}\right)$$

$$= \sum_{q=1}^{l}\sum_{\rho\leq\gamma}\sum_{(l,\beta,\delta)\in\mathcal{D}(\rho)}\binom{\gamma}{\rho}\rho!\left(D^{\delta}\frac{\partial u_i}{\partial x_q}\right)(\phi(t,s))\prod_{r=1}^{l}\frac{1}{\delta_r!}\left(\frac{1}{\beta_r!}D^{\beta_r}\phi(t,s)\right)^{\delta_r}D^{\gamma-\rho}\frac{\partial\phi_k}{\partial s_j}(t,s).$$

Notice that the first derivative formula only involves derivatives of $\phi(t,s)$ and u up to order n. Regarding the second formula, the only term involving a derivative of $\phi(t,s)$ of order $n+1$ is the associated with the multi-index $\rho = 0$. Hence, the next estimates hold true by virtue of (6.A.2)

$$\left|D^{\gamma}\left(\frac{\partial\phi_i}{\partial t}\right)(t,s)\right| \leq \kappa(\|u\|_{C^{k+1,\alpha}(\Omega)},T),$$
$$\left|D^{\gamma}\left(\frac{\partial\phi_i}{\partial s_j}\right)(t,s)\right| \leq \kappa(\|u\|_{C^{k+1,\alpha}(\Omega)},T)\sum_{q=1}^{3}\left(1+\int_0^t\left|D^{\gamma}\left(\frac{\partial\phi_q}{\partial s_j}\right)(\tau,s)\right|d\tau\right),$$

for every $(t,s)\in\mathcal{D}(\Sigma,u,T)$. Again, Grönwall's lemma shows that (6.A.2) holds true when $m=n$.

• *Step 4.* Derivation of $C^{k,\alpha}$ bounds for $\mathrm{Jac}(\phi)$.

Finally, let us obtain the aforementioned α-Hölder estimate of the higher order derivatives of $\mathrm{Jac}(\phi)$. To this end, take any column vector $x^j(t,s)$ of the Jacobian matrix $\mathrm{Jac}(\phi)(t,s)$ and note that when $\gamma=(\gamma_1,\gamma_2,\gamma_3)$ is a multi-index of the highest order k, then all the preceding derivative formulas can be added up to obtain the PDE

$$\begin{aligned}\frac{\partial}{\partial t}D^{\gamma}x_i^j(t,s) &= \sum_{q=1}^{3}A_q^{i,j}(\gamma)\frac{\partial u_i}{\partial x_q}(\phi(t,s))D^{\gamma}x_q^j(t,s)+F_i(t,s)\\ &+\sum_{\beta\in\Gamma_\gamma}\sum_{(j_1,\ldots,j_{k+1})\in J_\gamma}\sum_{(i_1,\ldots,i_{k+1})\in I_\gamma}B_{i_1,\ldots,i_{k+1}}^{j_1,\ldots,j_{k+1}}(\beta)(D^{\beta}u_i)(\phi(t,s))x_{i_1}^{j_1}(\phi(t,s))\cdots x_{i_{k+1}}^{j_{k+1}}(\phi(t,s)).\end{aligned} \tag{6.A.3}$$

Here $A_q^{i,j}(\gamma)$ and $B_{i_1,\ldots,i_{k+1}}^{j_1,\ldots,j_{k+1}}(\beta)$ denote nonnegative constant coefficients and $F_i(t,s)$ consists of finitely many sums and products of both derivatives of u up to order k and derivatives of ϕ up to order k. Furthermore, Γ_γ is a set of 3-multi-indices with order $k+1$ depending on γ and I_γ, J_γ are sets of $(k+1)$-multi-indices also depending on γ.

Let us first prove the α-Hölder continuity in the variable s using the integral version of the above equation. Specifically, take $s_1,s_2\in D$, $t\in(0,T)$ and notice that

$$D^{\gamma}x_i^j(t,s_1)-D^{\gamma}x_i^j(t,s_2)=I+II+III+IV,$$

where

$$\begin{aligned}I &:= D^{\gamma}x_i^j(0,s_1)-D^{\gamma}x_i^j(0,s_2),\\ II &:= \int_0^t(F_i(\tau,s_1)-F_i(\tau,s_2))\,d\tau,\\ III &:= \sum_{q=1}^{3}A_q^{i,j}(\gamma)\int_0^t\left(\frac{\partial u_i}{\partial x_q}(\phi(\tau,s_1))D^{\gamma}x_q^j(\tau,s_1)-\frac{\partial u_i}{\partial x_q}(\phi(\tau,s_2))D^{\gamma}x_q^j(\tau,s_2)\right)d\tau,\\ IV &:= \sum_{\substack{\beta\in\Gamma_\gamma\\(i_1,\ldots,i_{k+1})\in I_\gamma\\(j_1,\ldots,j_{k+1})\in J_\gamma}}B_{i_1,\ldots,i_{k+1}}^{j_1,\ldots,j_{k+1}}(\beta)\int_0^t(D^{\beta}u_i)(\phi(\tau,s))x_{i_1}^{j_1}(\phi(\tau,s))\cdots x_{i_{k+1}}^{j_{k+1}}(\phi(\tau,s))\Big|_{s_1}^{s_2}d\tau.\end{aligned}$$

Regarding the terms I, II, one can easily see that

$$I \leq \kappa\left(\|u\|_{C^{k+1,\alpha}(\Omega)}, T\right)|s_1 - s_2|^\alpha, \qquad II \leq T\kappa(\|u\|_{C^{k+1,\alpha}(\Omega)}, T)|s_1 - s_2|^\alpha.$$

In the first case, the estimate obviously follows from the regularity of μ in the particular case when $D^\gamma x_i^j$ involves no derivative of $\phi(t,s)$ with respect to t. A straightforward recursive argument on the order of the derivatives with respect to t yields the general assertion. The second case is obvious by the definition of $F_i(t,s)$ and the mean value theorem. Furthermore, adding and subtracting crossed terms in III, it is clear that it can be bounded by the mean value theorem as follows

$$III \leq \kappa(\|u\|_{C^{k+1,\alpha}(\Omega)}, T)|s_1 - s_2|^\alpha + \kappa\left(\|u\|_{C^{k+1,\alpha}(\Omega)}, T\right)\sum_{q=1}^{3}\int_0^t |D^\gamma x_q^j(\tau, s_1) - D^\gamma x_q^j(\tau, s_2)|\,d\tau.$$

So far, only low order derivatives of u have being involved, and therefore the mean value theorem has sufficed to obtain Lipschitz conditions of such derivatives (terms I, II and III). In contrast, IV contains the derivatives of u of the highest order, $k+1$. Since they cannot be handled again by the mean value theorem, then the α-Hölder continuity of $D^{k+1}u$ must be used. By appropriately adding and subtracting crossed terms, using the above-mentioned Hölder continuity of $D^\beta u_i$ on the first factor and the mean value theorem on the second one, one easily obtains the upper bound

$$IV \leq \sum_{\beta\in\Gamma_\gamma}\ \sum_{(j_1,\ldots,j_{k+1})\in J_\gamma}\ \sum_{(i_1,\ldots,i_{k+1})\in I_\gamma} B_{i_1,\ldots,i_{k+1}}^{j_1,\ldots,j_{k+1}}(\beta)$$
$$\times T\left(\kappa(\|u\|_{C^{k+1,\alpha}(\Omega}, T)^{\alpha+k+1}[D^\beta u_i]_{\alpha,\Omega}|s_1 - s_2|^\alpha + \kappa(\|u\|_{C^{k+1,\alpha}(\Omega}, T)^{k+2}\|D^\beta u_i\|_{C^0(\Omega)}|s_1 - s_2|\right).$$

To conclude, let us combine all the above estimates and use Grönwall's lemma to arrive at

$$|D^\gamma x_i^j(t, s_1) - D^\gamma x_i^j(t, s_2)| \leq \kappa(\|u\|_{C^{k+1,\alpha}(\Omega}, T)|s_1 - s_2|^\alpha,$$

for an appropriately function κ. Regarding the α-Hölder condition in the variable t, one only needs to note that $\partial_t D^\gamma x_i^j(t,s)$ is uniformly bounded by virtue of (6.A.3).

• *Step 5*. Derivation of the bounds for $\mathrm{Jac}(\phi)^{-1}$.
Finally, note that

$$\mathrm{Jac}(\phi)^{-1} = \frac{1}{\det(\mathrm{Jac}(\phi))}\left(\frac{\partial\phi}{\partial s_1}\times\frac{\partial\phi}{\partial s_2}\,\middle|\,\frac{\partial\phi}{\partial s_2}\times\frac{\partial\phi}{\partial t}\,\middle|\,\frac{\partial\phi}{\partial t}\times\frac{\partial\phi}{\partial s_1}\right)^\top,$$

and that the Jacobi–Liouville formula along with the lower bound in (6.A.1) yield a uniform lower bound for the Jacobian determinant:

$$\begin{aligned}\det(\mathrm{Jac}(\phi)(t,s)) &= \det(\mathrm{Jac}(\phi)(0,s))\exp\left(\int_0^t \mathrm{Tr}(\mathrm{Jac}(u)(\phi(\tau,s)))\,d\tau\right)\\ &\geq \rho_0\rho_1\exp\left(-3T\|\,\mathrm{Jac}(u)\|_{C^0(\Omega)}\right).\end{aligned}$$

Hence, the $C^{k+1,\alpha}(\mathcal{D}(\Sigma, u, T))$ estimate for $\mathrm{Jac}(\phi)^{-1}$ easily follows from that of $\mathrm{Jac}(\phi)$.

Part IV

Other works of the thesis and conclusions

CHAPTER 7

Further developments

7.1 Hydrodynamic limits of the thermomechanical Cucker–Smale model with fast and slow temperature relaxation

In this section, we return to the study of collective dynamics models in the spirit of the ones presented in the introductory Chapter 1. Specifically, we shall analyze a new version of the already studied Cuker–Smale model (1.1.7) in Part I, that has recently raised in the literature as an interesting non trivial modification of the classical flocking dynamics.

7.1.1 Introduction to the TCS model

The model under consideration is usually called the *thermomechanical Cucker–Smale model* in the literature and takes the following form:

$$\left\{\begin{array}{l} \dfrac{dx_i}{dt} = v_i, \\ \dfrac{dv_i}{dt} = \dfrac{K_v}{N}\displaystyle\sum_{j=1}^N \phi(|x_i - x_j|)\left(\frac{v_j}{\theta_j} - \frac{v_i}{\theta_i}\right), \\ \dfrac{d\theta_i}{dt} = \dfrac{K_\theta}{N}\displaystyle\sum_{j=1}^N \zeta(|x_i - x_j|)\left(\frac{1}{\theta_i} - \frac{1}{\theta_j}\right), \end{array}\right. \tag{7.1.1}$$

for any $i = 1, 2, \cdots, N$. Here, $x_i, v_i \in \mathbb{R}^d$ represent positions and velocities of agents, whilst the new variable $\theta_i \in \mathbb{R}^+$ is regarded as *temperature or internal energy* of individuals, to be distinguished from phases of oscillators in Part II of this thesis.

On the one hand, notice that the second equation in (7.1.1) is reminiscent of the standard Cucker–Smale model with influence function ϕ, where temperatures have an effect on the alignment of velocities of the individual itself. However, notice that it can be restated as follows

$$
\begin{aligned}
\frac{dv_i}{dt} &= \frac{K_v}{N}\sum_{j=1}^{N}\phi(|x_i-x_j|)\frac{1}{2}\left(\frac{1}{\theta_i}+\frac{1}{\theta_j}\right)(v_j-v_i)\\
&+\frac{K_v}{N}\sum_{j=1}^{N}\phi(|x_i-x_j|)\left(\frac{1}{\theta_j}-\frac{1}{\theta_i}\right)\frac{v_i+v_j}{2}.
\end{aligned}
\tag{7.1.2}
$$

In particular, when all agents' temperatures take the same value, the second term of (7.1.2) vanishes and we readily recover the classical Cucker–Smale model. However, if agents' temperatures are not homogeneous, then the second term of (7.1.2) represents a fully different dynamics. Indeed, in such a case agents are subject to an acceleration (deceleration) in the direction of the mean velocity, what distorts the original flocking dynamics.

On the other hand, by the third equation in (7.1.1) we observe that temperatures are dynamic and we equivalently can restate their evolution in the following way

$$
\frac{d\theta_i}{dt} = \frac{K_\theta}{N}\sum_{j=1}^{N}\zeta(|x_i-x_j|)\frac{\theta_j-\theta_i}{\theta_i\theta_j}. \tag{7.1.3}
$$

This is nothing but relaxation of temperatures towards the values of their neighbors in a weighted way that is mediated by a new influence function ζ that, again, depends on distances between agents.

Then, according to (7.1.3), the expected dynamics for temperatures is that they must converge to a common value. In turns, in such a case (7.1.2) amounts to a simple flocking dynamics for velocities. Then, one expects that they must align to achieve a common value. Consequently, under mild assumptions we eventually expect that the final coupled collective behavior is the full alignment of agent's velocity towards the mean value and the homogenization of all temperatures around the average one, both processes taking place at comparable scales.

The thermomechanical Cucker–Smale model (7.1.1) was proposed by S.-Y. Ha and T. Ruggeri in [161]. This is a thermodynamically consistent particle model motivated by the theory of multi-temperature mixtures of fluids [264, 265] in the case of spatially homogeneous processes. Indeed, the thermomechanical Cucker–Smale model is the unique one in the absence of chemical reactions that is compatible with entropy principle of thermodynamics and exhibits the natural Galilean invariance.

The mathematical analysis of the emergent phenomena was analyzed in [153]. In addition, using the mean field methods that we reviewed in Subsection 1.1.2 of the introductory Chapter 1, the authors also derived in [148] the associated kinetic description:

$$
\frac{\partial f}{\partial t} + v\cdot\nabla_x f + \operatorname{div}_x\left(F[f]f\right) + \frac{\partial}{\partial\theta}\left(G[f]f\right) = 0. \tag{7.1.4}
$$

for the distribution function $f = f(t,x,v,\theta)$ of particles. Here, the functionals $F[f]$ and $G[f]$ are defined as the following integral operators

$$
\begin{aligned}
F[f](t,x,v,\theta) &:= K_v\int_{\mathbb{R}^{2d+1}}\phi(|x-x'|)\left(\frac{v'}{\theta'}-\frac{v}{\theta}\right)f(x',v',\theta')\,dx\,dv\,d\theta,\\
G[f](t,x,\theta) &:= K_\theta\int_{\mathbb{R}^{2d+1}}\zeta(|x-x'|)\left(\frac{1}{\theta}-\frac{1}{\theta'}\right)f(x',v',\theta')\,dx\,dv\,d\theta.
\end{aligned}
\tag{7.1.5}
$$

Later, the monokinetic ansatz of (7.1.4) was proposed in [149] and the authors derived the following macroscopic model

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) = 0, \\ \dfrac{\partial}{\partial t}(\rho u) + \operatorname{div}(\rho u \otimes u) = K_v \left[\phi * \left(\dfrac{\rho u}{e}\right)\rho - (\phi * \rho)\dfrac{\rho u}{e}\right], \\ \dfrac{\partial}{\partial t}(\rho e) + \operatorname{div}(\rho u e) = K_\theta \left[(\zeta * \rho)\dfrac{\rho}{e} - \zeta * \left(\dfrac{\rho}{e}\right)\rho\right]. \end{cases} \tag{7.1.6}$$

Indeed, a well posedness theory was also derived in the same literature. Regarding the rigorous hydrodynamic limit from (7.1.4) to (7.1.6), the first partial result in the literature was recently derived in [146] using analogue methods to those developed in [127] for the simpler Cucker–Smale model. Indeed, strong local alignment was again introduced with the hope that it can provide better control in terms of relative entropy methods. Specifically, in [146] the authors introduced the scaled system

$$\frac{\partial f_\varepsilon}{\partial t} + v \cdot \nabla_x f_\varepsilon + \operatorname{div}_x (F[f_\varepsilon] f_\varepsilon) + \frac{\partial}{\partial \theta}(G[f_\varepsilon] f_\varepsilon) = \frac{1}{\varepsilon} \operatorname{div}_v ((v - u_\varepsilon) f_\varepsilon). \tag{7.1.7}$$

Unfortunately, the complicatedness of the new nonlinearities in (7.1.5) causes serious problems to identify the limit of some macroscopic moments of the distribution function f for general initial data f_0^ε due to the fact that such hierarchy appear in a non closed form. Then, in practice, only the particular case where the initial datum f_0 is isothermal was recovered in such a paper. More specifically, the authors assumed a specific rate of homogenization of initial temperatures as follows

$$\operatorname{diam}(\operatorname{supp}_\theta f_0^\varepsilon) = \mathcal{O}(\varepsilon) \text{ as } \varepsilon \to 0. \tag{7.1.8}$$

In such a particular case, the associated macroscopic model trivially agrees with the classical Euler- alignment system:

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) = 0, \\ \dfrac{\partial}{\partial t}(\rho u) + \operatorname{div}(\rho u \otimes u) = \dfrac{K_v}{\theta_e}\left[\phi * (\rho u)\rho - (\phi * \rho)\rho u\right], \end{cases} \tag{7.1.9}$$

and the dynamics of temperatures is lost. We remark, that most of the analysis has been conducted only for the case of regular weights

$$\phi(r) = \frac{1}{(1+r^2)^{\beta/2}} \text{ and } \zeta(r) = \frac{1}{(1+r^2)^{\gamma/2}}. \tag{7.1.10}$$

The case of singular weights

$$\phi(r) = \frac{1}{r^\beta} \text{ and } \zeta(r) = \frac{1}{r^\gamma}, \tag{7.1.11}$$

remains mostly unexplored.

In this section we propose some new versions of the agent-based model (7.1.1) and new scalings of the kinetic equation (7.1.4) so that new dynamics for temperatures can be preserved in the hydrodynamic limit (7.1.9). Our scaling is inspired in that of Chapter 2 and can be regarded as vanishing inertia limits. We introduce them in the subsequent subsection.

7.1.2 TCS particles that interact with environment

The main intuition behind the new models that we propose is that, in general, agents are not isolated but they are affected by the environment that they occupy. Indeed, those individuals can feel physical stress induced by medium, which can be modelled in many different ways. Along this section, the main modification of (7.1.1) arises after we account for the well known *Newton's law of cooling*. Specifically: *if the heat transfer coefficient is relatively independent of the temperature difference between objects and environment, then the rate of heat loss of a body is proportional to the difference in temperatures between such a body and its surroundings.* We here propose two main variants of (7.1.1) that bear those new effects in mind.

Uniform ambient temperature

On the one hand, when the ambient space has uniform temperature, then we shall consider the following agent-based model described in terms of a coupled system of SODEs:

$$\begin{cases} dx_i = v_i\,dt, \\ dv_i = \dfrac{K_v}{N}\displaystyle\sum_{j=1}^N \phi(|x_i-x_j|)\left(\frac{v_j}{\theta_j}-\frac{v_i}{\theta_i}\right)dt - \mu_v v_i\,dt - \nabla_x\psi\,dt + \sqrt{2\sigma}W_t^i, \\ d\theta_i = \dfrac{K_\theta}{N}\displaystyle\sum_{j=1}^N \zeta(|x_i-x_j|)\left(\frac{1}{\theta_i}-\frac{1}{\theta_j}\right)dt + \mu_\theta(\theta^\infty-\theta_i)\,dt. \end{cases} \tag{7.1.12}$$

Regarding the second equation, notice that we have introduced some linear damping of velocities due to friction with medium with coefficient μ_v, the effect of an external force $\nabla_x\psi$ and stochastic movement in terms of some Brownian motion W_t^i. For the second equation, we have only adjusted a new term, that can be though of relaxation of temperatures towards the uniform value θ^∞ of the temperatures of the environment with heat transfer coefficient μ_θ.

Non-uniform ambient temperature

A second possibility is that the environment is not in isothermal equilibrium and, consequently, ambient temperature is not uniform, but it is described in terms of some temperature field $T = T(t,x)$, that can be either stationary or time-dependent. In such a case, we propose the following modification of (7.1.12).

$$\begin{cases} dx_i = v_i\,dt, \\ dv_i = \dfrac{K_v}{N}\displaystyle\sum_{j=1}^N \phi(|x_i-x_j|)\left(\frac{v_j}{\theta_j}-\frac{v_i}{\theta_i}\right)dt - \mu_v v_i\,dt - \nabla_x\psi\,dt + \sqrt{2\sigma}W_t^i, \\ d\theta_i = \dfrac{K_\theta}{N}\displaystyle\sum_{j=1}^N \zeta(|x_i-x_j|)\left(\frac{1}{\theta_i}-\frac{1}{\theta_j}\right)dt + \mu_\theta(T(t,x_i)-\theta_i)\,dt. \end{cases} \tag{7.1.13}$$

Notice that the main change appears in the third equation of (7.1.13), where the temperature field $T(t,x)$ plays a role in Newton's law of cooling. Let us remark that if $T(t,x)$ is a uniform temperature field, then (7.1.13) reduces to (7.1.12). However, $T(t,x)$ is not necessarily uniform, but may arise from different physical laws, e.g. *Fourier's law*. In such case, the dynamics of $T(t,x)$ is reduced to the *heat equation*:

$$\frac{\partial T}{\partial t} = D\Delta T, \tag{7.1.14}$$

for $t \geq 0,\ x \in \mathbb{R}^d$, where D is the thermal conductivity of the medium.

Since mathematical technicalities are similar to those in Chapter 2, we do not stick to many rigorous arguments along this section. Instead, we simply state the main results and macroscopic limits that we can derive with such methods and the fundamental ideas behind them.

7.1.3 The case of uniform ambient temperature

In this part, we focus on the study of (7.1.12). In such a case we formally can repeat the mean-field techniques that we reviewed in Subsection 1.1.2 to derive the corresponding Vlasov–McKean equations. We introduce two possible scaling of the system that we respectively call fast temperature relaxation and slow temperature relaxation regimes:

Fast temperature relaxation

$$\begin{aligned}\frac{\partial f_\varepsilon}{\partial t} + v\cdot\nabla_x f_\varepsilon + \frac{1}{\varepsilon}\operatorname{div}_v\left(F[f_\varepsilon]f_\varepsilon - vf_\varepsilon - \nabla_x\psi f_\varepsilon - \nabla_v f_\varepsilon\right)&\\ + \frac{1}{\varepsilon}\frac{\partial}{\partial\theta}\left(G[f_\varepsilon]f_\varepsilon + (\theta^\infty - \theta)f_\varepsilon\right) = 0.&\end{aligned} \tag{7.1.15}$$

On the one hand, notice that we are assuming that collective alignment of velocities, friction with medium, collective alignment of temperatures and relaxation towards the ambient uniform temperature take all place very fast. This is a new overdamped limit or Smoluchoski dynamics, where we expect inertia terms to disappear as $\varepsilon \to 0$ and also temperature variable should degenerate to a single value θ^∞ without the extra hypothesis (7.1.8), the reason being that relaxation towards ambient temperatures is large enough in this regime independently of ϕ and ζ being regular of singular functions.

For convenience, hereafter we will assume that the initial configurations have compact temperature support away from zero, namely,

$$\operatorname{supp}_\theta f_\varepsilon^0 \subseteq [\theta_m, \theta_M] \subseteq \mathbb{R}^+.$$

In addition, we define the temperature diameter by

$$D_\theta^\varepsilon(t) := \operatorname{diam}(\operatorname{supp}_\theta f_\varepsilon),$$

for all $t \geq 0$ and $\varepsilon > 0$.

Let us now compute the hierarchy of equations associated to the following velocity and temperature moments:

$$\begin{aligned}
\rho_\varepsilon(t,x) &:= \int_{\mathbb{R}^d\times\mathbb{R}^+} f(t,x,v,\theta)\,dv\,d\theta,\\
j_\varepsilon(t,x) &:= \int_{\mathbb{R}^d\times\mathbb{R}^+} v f_\varepsilon(t,x,v,\theta)\,dv\,d\theta,\\
h_\varepsilon(t,x) &:= \int_{\mathbb{R}^d\times\mathbb{R}^+} \theta f_\varepsilon(t,x,v,\theta)\,dv\,d\theta,\\
A_\varepsilon(t,x) &:= \int_{\mathbb{R}^d\times\mathbb{R}^+} \frac{v}{\theta} f_\varepsilon(t,x,v,\theta)\,dv\,d\theta,\\
B_\varepsilon(t,x) &:= \int_{\mathbb{R}^d\times\mathbb{R}^+} \frac{1}{\theta} f_\varepsilon\,dv\,d\theta,\\
E_\varepsilon(t,x) &:= \int_{\mathbb{R}^d\times\mathbb{R}^+} \frac{1}{2}|v|^2 f_\varepsilon\,dv\,d\theta.
\end{aligned}\tag{7.1.16}$$

Then, the first three macroscopic quantities obey the following equations

$$\begin{aligned}
&\frac{\partial\rho_\varepsilon}{\partial t} + \operatorname{div}_x j_\varepsilon = 0,\\
&\varepsilon\frac{\partial j_\varepsilon}{\partial t} + \varepsilon\operatorname{div}_x\left(\int_{\mathbb{R}^d\times\mathbb{R}^+} v\otimes v f_\varepsilon\,dv\,d\theta\right) + \rho_\varepsilon\nabla_x\psi_\varepsilon + j_\varepsilon - \rho_\varepsilon(\phi * A_\varepsilon) + A_\varepsilon(\phi*\rho_\varepsilon) = 0,\\
&\varepsilon\frac{\partial h_\varepsilon}{\partial t} + \varepsilon\operatorname{div}_x\left(\int_{\mathbb{R}^d\times\mathbb{R}^+} v\theta f_\varepsilon\,dv\,d\theta\right) + h_\varepsilon - \theta^\infty\rho_\varepsilon + \rho_\varepsilon(\zeta * B_\varepsilon) - B_\varepsilon(\zeta*\rho_\varepsilon) = 0,
\end{aligned}\tag{7.1.17}$$

Notice that the hierarchy is not closed. Also, it is not apparent how to identify the limits of A_ε and B_ε as $\varepsilon \to 0$ in terms of ρ, j and h, that is the main issue in [146] for the TCS as well. In order the provide some answer to our problem, we need to control three main aspects: diameters $D^\varepsilon_\theta(t)$, total kinetic energy $\int_{\mathbb{R}^d} E_\varepsilon\,dx$, limits of $A_\varepsilon, B_\varepsilon$ and h_ε.

• *Step 1.* On the one hand, by inspection we obtain that

$$\frac{d}{dt}D^\varepsilon_\theta(t) \le -\frac{1}{\varepsilon}D^\varepsilon_\theta(t),\ \text{ for all } t>0,$$

thanks to the scaling of Newton's rule of cooling. Then, no matter (7.1.8) is assumed or not, we readily recover it instantaneously after a small time layer, that is,

$$D^\varepsilon_\theta(t) \approx \mathcal{O}(\varepsilon^\kappa),\ \text{ for all }\ t > \varepsilon^\delta,\tag{7.1.18}$$

where $\kappa > 0$ and $0 < \delta < 1$ are any couple of parameters.

• *Step 2.* Looking at the equation of E_ε we obtain

$$\begin{aligned}
E_\varepsilon &+ \int_{\mathbb{R}^d\times\mathbb{R}^+} F(f_\varepsilon)\cdot\frac{v}{2} f_\varepsilon\,dv\,d\theta\\
&= \varepsilon\frac{\partial E_\varepsilon}{\partial t} + \varepsilon\operatorname{div}_x\left(\int_{\mathbb{R}^2\times\mathbb{R}^+}\frac{|v|^2}{2} v f_\varepsilon\,dv\,d\theta\right) - \int_{\mathbb{R}^d\times\mathbb{R}^+}\nabla\psi_\varepsilon\cdot v f_\varepsilon\,dv\,d\theta + d\rho_\varepsilon.
\end{aligned}\tag{7.1.19}$$

Taking integrals with respect to x and t and splitting the second term in the left hand side of (7.1.19) into negative and positive parts (use the ideas in (7.1.2)), we observe that its negative part is bounded as long as we assume

$$\sup_{\varepsilon>0} D_\theta^\varepsilon(0) < \theta_m^2.$$

Then, we derive uniform-in-ε bounds of E_ε. Notice that this argument is clear for regular weights (7.1.10), but requires some explanation for the singular case (7.1.11). Indeed, in order to ensure that the negative part of that integral term is bounded, notice that we require the hypothesis (7.1.8) because the influence function ζ is no longer bounded at the origin in that later case.

In any case, we derive a bound of E_ε, that in turns, amounts to weak compactness of all the terms of the hierarchy (7.1.16). Then, we can pass to the limit $\varepsilon \to 0$ in the equations (7.1.17) of ρ_ε, j_ε and h_ε. However, it is not clear yet how we can identify the limits of A_ε and B_ε in terms of the above quantities, so that we can close the hierarchy of the limiting system.

• *Step 3.* This is where (7.1.18) comes to play and shows that

$$\begin{aligned}
A_\varepsilon - \frac{j}{\theta(t)} &\overset{*}{\rightharpoonup} 0, \text{ in } L^2_w(\varepsilon_0, T; \mathcal{M}(\mathbb{R}^d)),\\
B_\varepsilon - \frac{\rho}{\theta(t)} &\overset{*}{\rightharpoonup} 0, \text{ in } L^\infty_w(\varepsilon_0, T; \mathcal{M}(\mathbb{R}^d)),\\
h_\varepsilon - \theta(t)\rho &\overset{*}{\rightharpoonup}, \text{ in } L^\infty_w(\varepsilon_0, T; \mathcal{M}(\mathbb{R}^d)),
\end{aligned}$$

for each $0 < \varepsilon_0 < T$, where $\theta(t)$ is the only value on the temperature support of f. Taking special care of the nonlinear terms in the singular cases in the same lines as in Chapter 2, we obtain that both for regular (7.1.10) and singular (7.1.11) (with $\beta, \gamma \in (0, 1]$) influence function, the macroscopic dynamic is governed by the following system

$$\left\{\begin{array}{l}
\dfrac{\partial \rho}{\partial t} + \operatorname{div}_x(\rho u) = 0,\\
\nabla_x \psi + u = \dfrac{1}{\theta(t)} \left[\phi * (\rho u) - (\phi * \rho) u\right],\\
\theta(t) = \theta^\infty,
\end{array}\right. \tag{7.1.20}$$

that is reminiscent of the model (2.1.1) in Chapter 2 with isothermal configuration of temperatures. Notice that, like in (7.1.9), the system has lost all the dynamics of temperatures that are constant and agrees with the value of the environment temperature.

Slow temperature relaxation

Instead of imposing such a strong relaxation of temperatures in Newton's law of cooling, we can let it evolve at the same scale as inertia of the system. Then, a plausible scaling of the Vlasov–McKean equation of (7.1.12) is:

$$\begin{aligned}
\frac{\partial f_\varepsilon}{\partial t} + v \cdot \nabla_x f_\varepsilon + \frac{1}{\varepsilon} \operatorname{div}_v \left(F[f_\varepsilon] f_\varepsilon - v f_\varepsilon - \nabla_x \psi f_\varepsilon - \nabla_v f_\varepsilon\right)&\\
+ \frac{\partial}{\partial \theta} \left(\frac{1}{\varepsilon} G[f_\varepsilon] f_\varepsilon + (\theta^\infty - \theta) f_\varepsilon\right) = 0.&
\end{aligned} \tag{7.1.21}$$

This time, the hierarchy of equations for moments takes the form:

$$\begin{aligned}
&\frac{\partial \rho_\varepsilon}{\partial t} + \operatorname{div}_x j_\varepsilon = 0,\\
&\varepsilon \frac{\partial j_\varepsilon}{\partial t} + \varepsilon \operatorname{div}_x \left(\int_{\mathbb{R}^d \times \mathbb{R}^+} v \otimes v f_\varepsilon \, dv \, d\theta \right) + \rho_\varepsilon \nabla_x \psi_\varepsilon + j_\varepsilon - \rho_\varepsilon (\phi * A_\varepsilon) + A_\varepsilon (\phi * \rho_\varepsilon) = 0,\\
&\varepsilon \frac{\partial h_\varepsilon}{\partial t} + \varepsilon \operatorname{div}_x \left(\int_{\mathbb{R}^d \times \mathbb{R}^+} v \theta f_\varepsilon \, dv \, d\theta \right) + \varepsilon (h_\varepsilon - \theta^\infty \rho_\varepsilon) + \rho_\varepsilon (\zeta * B_\varepsilon) - B_\varepsilon (\zeta * \rho_\varepsilon) = 0.
\end{aligned} \tag{7.1.22}$$

Notice that the main difference is that the dynamics of $D_\theta^\varepsilon(t)$ has a substantially different scale. Specifically, we obtain that

$$\frac{d}{dt} D_\theta^\varepsilon(t) \leq -\frac{1}{\varepsilon} \zeta(D_x^\varepsilon(t)) D_\theta^\varepsilon(t) - D_\theta^\varepsilon(t),$$

for every $t > 0$.

∘ *Regular case*. If influence functions are regular (7.1.10), then we can achieve bounds for the position diameter $D_x^\varepsilon(t)$. Indeed, by disregarding the second term, that is non-positive, we obtain

$$\frac{d}{dt} D_\theta^\varepsilon(t) \leq -\frac{1}{\varepsilon} \zeta(D_x^\varepsilon(t)) D_\theta^\varepsilon(t),$$

for every $t > 0$. Then, an analogue argument as in the above fast relaxation case allows obtaining a unique temperature after a finite time layer, that is, (7.1.18) holds true again. We remark that this time it is not Newton's rule of cooling, but collective alignment of temperatures which guarantees an isothermal distribution of temperatures.

∘ *Singular case*. In such a case, we can no longer control the first term of the above decay rate of $D_\theta^\varepsilon(t)$. Fortunately, since such a term is non-positive, we can neglect it and achieve the inequality

$$\frac{d}{dt} D_\theta^\varepsilon(t) \leq -D_\theta^\varepsilon(t),$$

for every $t > 0$. Then, in order to recover a uniform temperature, we require the initial hypothesis (7.1.8). This time, we are using again Newton's rule of cooling as main mechanism of isothermality, but since the scale is not fast enough, we are setting isothermal initial data via the above hypothesis.

In any case, we obtain that (7.1.18) takes place. *Steps* 2 and 3 remain the same and produce weak limits ρ, j and h. Indeed, if we call $\theta(t)$ the uniform temperature of the limiting f, we can again identify limits as follows

$$\begin{aligned}
A_\varepsilon - \frac{j}{\theta(t)} &\overset{*}{\rightharpoonup} 0, \text{ in } L_w^2(\varepsilon_0, T; \mathcal{M}(\mathbb{R}^d)),\\
B_\varepsilon - \frac{\rho}{\theta(t)} &\overset{*}{\rightharpoonup} 0, \text{ in } L_w^\infty(\varepsilon_0, T; \mathcal{M}(\mathbb{R}^d)),\\
h_\varepsilon - \theta(t)\rho &\overset{*}{\rightharpoonup} 0, \text{ in } L_w^\infty(\varepsilon_0, T; \mathcal{M}(\mathbb{R}^d)),
\end{aligned}$$

for each $0 < \varepsilon_0 < T$. Taking limits in the hierarchy (7.1.22), we now obtain a non trivial dynamics for the uniform temperature $\theta(t)$ of the population:

$$\left\{\begin{array}{l}\dfrac{\partial\rho}{\partial t}+\operatorname{div}_x(\rho u)=0,\\ \nabla_x\psi+u=\dfrac{1}{\theta(t)}\left[\phi*(\rho u)-(\phi*\rho)u\right],\\ \dfrac{d}{dt}\theta(t)=\theta^\infty-\theta(t).\end{array}\right. \tag{7.1.23}$$

This means that population's temperatures are uniform but relax slowly via Newton's rule of cooling towards the uniform constant value of the ambient's temperature.

7.1.4 The case of non-uniform ambient temperature

Now, we return to the more general case (7.1.13), where Newton's rule of cooling has been modified to account for a non-uniform temperature field of the environment. In this case, we do not provide further technical details. Instead, we restrict to providing the associated macroscopic equations for each of the fast and slow temperature relaxation scalings. For more details we refer to a forthcoming publication.

Fast temperature relaxation

In this case, we propose the following analogue fast temperature relaxation scaling for the Vlasov–McKean equation associated with (7.1.13)

$$\begin{split}\frac{\partial f_\varepsilon}{\partial t}+v\cdot\nabla_x f_\varepsilon+\frac{1}{\varepsilon}\operatorname{div}_v\left(F[f_\varepsilon]f_\varepsilon-vf_\varepsilon-\nabla_x\psi f_\varepsilon-\nabla_v f_\varepsilon\right)\\ +\frac{1}{\varepsilon}\frac{\partial}{\partial\theta}\left(G[f_\varepsilon]f_\varepsilon+(T(t,x)-\theta)f_\varepsilon\right)=0.\end{split} \tag{7.1.24}$$

Under appropriate hypothesis, its macroscopic dynamics is governed by the following hydrodynamic model:

$$\left\{\begin{array}{l}\dfrac{\partial\rho}{\partial t}+\operatorname{div}_x(\rho u)=0,\\ \nabla_x\psi+u=\dfrac{1}{\theta(t)}\left[\phi*(\rho u)-(\phi*\rho)u\right],\\ \theta(t)=\displaystyle\int_{\mathbb{R}^d}T(t,x)\rho(t,x)\,dx.\end{array}\right. \tag{7.1.25}$$

Hence, the population is isothermal again and the value of the temperature is instantaneously described by the average value of the temperature of the environment. However, in this case we have lost the dynamics of agents' temperatures and it passively evolves according to the distribution $T(t,x)$ itself.

Slow temperature relaxation

$$\begin{split}\frac{\partial f_\varepsilon}{\partial t}+v\cdot\nabla_x f_\varepsilon+\frac{1}{\varepsilon}\operatorname{div}_v\left(F[f_\varepsilon]f_\varepsilon-vf_\varepsilon-\nabla_x\psi f_\varepsilon-\nabla_v f_\varepsilon\right)\\ +\frac{1}{\varepsilon}\frac{\partial}{\partial\theta}\left(G[f_\varepsilon]f_\varepsilon+(T(t,x)-\theta)f_\varepsilon\right)=0.\end{split} \tag{7.1.26}$$

This time, we obtain the following macroscopic equation:

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \mathrm{div}_x(\rho u) = 0, \\ \nabla_x \psi + u = \dfrac{1}{\theta(t)} \left[\phi * (\rho u) - (\phi * \rho) u \right], \\ \dfrac{d}{dt}\theta(t) = \displaystyle\int_{\mathbb{R}^d} T(t,x)\rho(t,x)\, dx - \theta(t). \end{cases} \tag{7.1.27}$$

In this case the dynamics is richer due to the fact that the uniform temperature of the population is dynamic and evolves, via Newton's rule of cooling towards the average value of the temperature of the environment. In turns, it is dynamic. In particular, recall that an acceptable choice for its dynamics is given by the heat equation (7.1.14).

7.2 Modeling of morphogen transportation along moving cytonemes in Droshophila melanogaster

In this section, we introduce another work in process. This is an interdisciplinary collaboration with biologists from Centro de Biología Molecular Severo Ochoa (Madrid) in the laboratory of Isabel Guerrero. Our objective is to derive a mathematical model of cytonemes in *Drosophila melanogaster*, as main responsible for cellular communication and direct transportation of *Hh* morphogen from cell to cell. First, we will introduce the biological background of the problem. Later, we formulate our model, that consists of several different processes, all of them taking place simultaneously during the real complex dynamics. For an easier readability, we have decided to present each of the components separately.

7.2.1 Introduction to cytonemes and morphogen transportation

Since A. Turing [289] proposed a pioneering mathematical model of morphogenesis, the principal mechanism of the cell communication processes that explain the transport of morphogens in a living organism has always been associated with free diffusion processes, see [199, 300, 274]. According to this old conception, morphogenes are proteins that spread througout the extracellular matrix. In its path, they determine a concentration gradient that reach cells within the forming tissues. It has been experimentally verified that the expossure of cells to different levels of morphogen has the ability to activate the corresponding target genes. In turns, these genes are responsible for the appropriate specialization of cells and other biological mechanisms involved in patterning and the proper formation of organs and tissues, see [303]. Our goal along this part is to propose an alternative description to the above one according to the more recent experimental evidence.

Morphogenesis does not only occur at the early stages of the formation living of organisms (that is called *embryogenesis*), but it also plays a determining role throughout its whole life, e.g., correct maintenance of organs, tissue renewal, etc. In particular, a deregulation of the correct communication mechanisms is often associated with the formation and growth of cancerous cells. Thereby, determining the precise formation of morphogen concentration gradients has an incalculable value from the point of view of real medical applications. In particular, it may be use to determine the mechanisms of duplication of stem cancerous cells and their differentiation into tumoral tissues.

Due to the relative facilities to design *in vivo* experiments, Drosophila melanogaster is the most studied organism in this context. Here, we will mostly focus on a specific process that has been deeply analyzed during the last years: the formation of wings from the primordial *imaginal disc* of lavae. A closely related process that, for simplicity, we will not address here is the formation of abdomen after migration of histoblast cells. Both processes are qualitatively equivalent, and the methods developed for the former, can be adapted to the latter.

In the morphogenetic proces of wing formation, the main signalling mechanism is mediated by Hedgehog (Hh) morphogen, that activates its target gene Cubitus interruptus (*ci*) (analogue for vertebrates are Shh and *gli*). In this mechanism, the geometrical disposition of each component plays a role. Wing imaginal discs are bags of epithelial cell. They represent the primordial structures of larvae that, during metamorphosis of pupae, will become the final wings of an adult fly. It consists of nearly 60.000 undifferentiated cells that arranged into two distinguished compartments: *anterior* (A) and *posterior* (P), separated by a boundary that is often called $A\backslash P$ *border*, see Figure 7.1. Cells of P-compartment secrete Hh proteins. These Hh proteins determine a gradient of concentration, that spreads through the extracellular matrix, crosses the

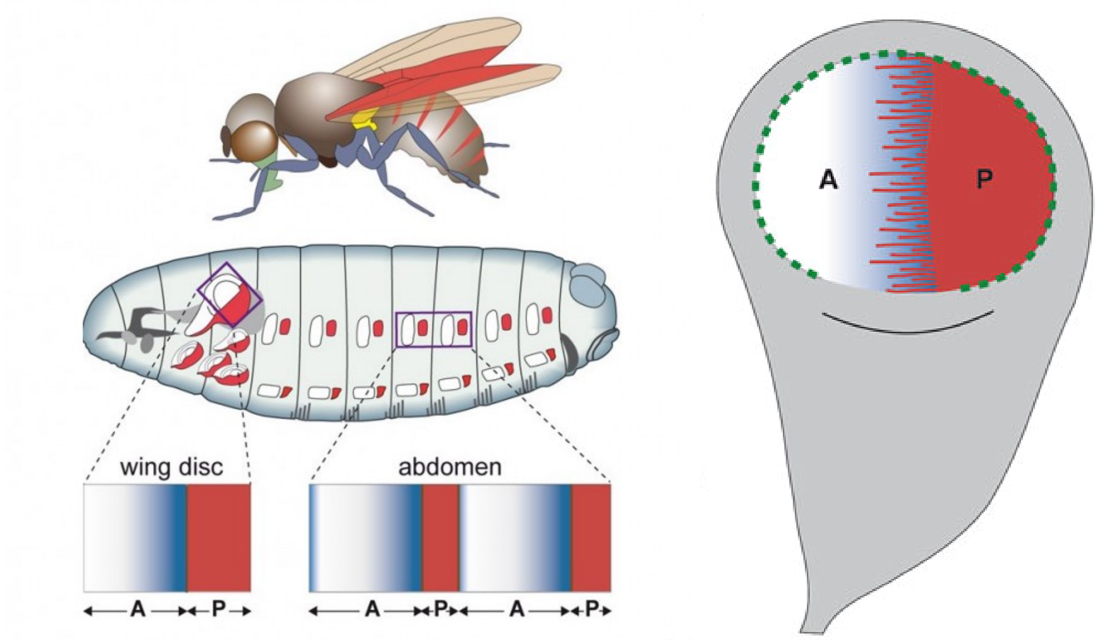


Figure 7.1: On the left, location of wing discs on larvae of Drosophila melanogaster. On the right, zoom of an imaginal disc and division into A\P compartments. Pictures are taken from and the website http://guerreroslab.cbm.uam.es/wp/ and from [271].

A\ P border and invades the cells in the A compartment. In a very simplified manner, it is known that in the reception process, Ptc receptors protein and glipicans like Dally and Dlp are involved, as well as the adhesion molecule Ihog.

Recently, it has been experimentally observed [34, 141, 192, 260, 267] that Hh proteins do not spread randomly through the medium, unlike the initial idea of Turing. Specifically, Hh proteins travels inside vesicles along specialized small actin-based filament structures that are called signalling filipodia or cytonemes. Those filopodia emerge from cell membranes of both Hh emitting and receiving cells. They have been identified as dynamic structures with the ability to grow, retract and orient, driven by some mechanisms that we shall describe later. As a consequence of the dynamics, cytonemes from emitting and receiving cells stablish a sort of synapse (connection) and vesicles with Hh proteins (among others) are transferred. Once cargo has been absorbed by the receiving filopodium, the connection is broken, proteins travel back along the new cytoneme towards the receiving cell and the filament-like structures retract towards the corresponding cell membranes, see Figure 7.2. This direct transportation mechanism clearly contradicts the preceding literature based on linear diffusion and represents a new paradigm with regard to morphogene spreading. Our goal here is to introduce a solid mathematical model that accounts for all these microscopic phenomena based on experimental evidence.

Some attempts to provide a more realistic mathematical description of such a communication mechanism has already been developed in the literature in terms of macroscopic PDE models based on flux-saturated and/or porous-media mechanisms [53, 54, 55, 266, 294]. Nevertheless, any of those models takes into account the abovementioned microscopic description. Very recently, the modelling of cytonemes has been addressed by several authors in terms of both deterministic and stochastic models [46, 47, 187, 188, 285], but none of them provide a description of the underlying microscopic dynamics of such structures: growth, orientation,

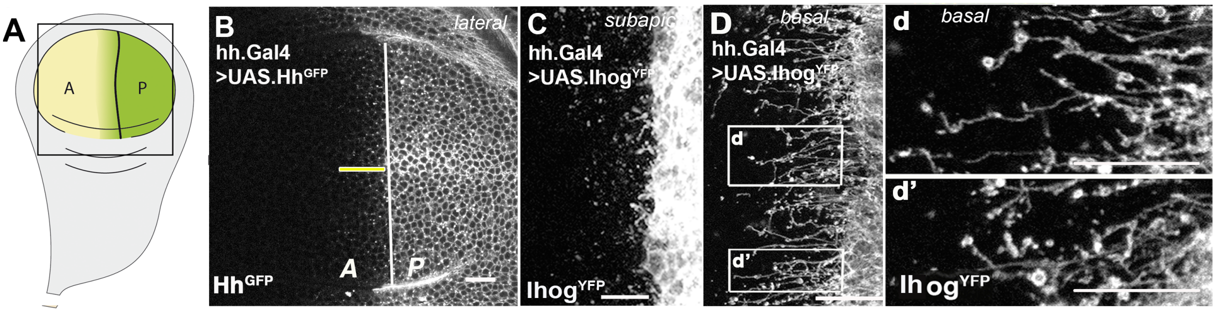


Figure 7.2: Zoom at Hh emitting cells in P compartment near the A\ B border. We observe that cytonemes emanate from cell membranes towards the opposite A compartment to stablish connections with Hh receiving cells. Indeed, concentration of Hh is higher in P-cells that A-cells. Also, we observe that vesicles are transported along the filament-like structures. This picture has been taken from [266].

synapses, transportation and transference of morphogene, detachment, contraction, etc. This is the goal of this section. Since it is a work in progress, we just aim at modelling the early stages of the process, namely: dynamics of cytonemes, orientation mechanisms and propagation of proteins along such moving structures. All the others step of the modeling will be provided later in a forthcoming publication, where we will also contrasts the existing experimental data with predictions coming from numerical simulations of this model.

7.2.2 Mechanical description of cytonemes dynamics

There are several types of filament-type structures playing a role on the mechanics of cell membranes and cytoplasms of cells, see [35]. For instance, the *cortex* is a specialized layer of the cytoplasm on the inner face of the cell membrane that functions as a modulator of membrane behavior. *Stress fibers* play an important role in cellular contractivity, providing force for a number of functions such as cell adhesion or migration. A *lamellipodium* is a cytoskeletal projection on the leading edge of the cell that determines a mesh helping cells to propel across a substrate, see Figure 7.3. All of those structures are made of actin proteins that grow and break via polimerization or depolimerization of actin dimers. In our case, we are interested in *filopodia*, that are thin cytoplasmic projections extending from cell membranes of many sort of migrating cells. They consist in actin filaments cross-linked into bundles by actin-binding proteins and they help on the cell adhesion to substrate. Many types of migrating cells exhibit some type filopodia. In our case, our filipodia have signaling functions and they are called *cytonemes* for distinction.

Discrete cytonemes with finite length

A first approach to model a cytoneme is as a chain of finitely many connected actin molecules at positions

$$x_0(t), x_1(t), \ldots, x_N(t) \in \mathbb{R}^3,$$

where the first one $x_0(t)$ is always attached to a fixed point x_0 in the cell membrane and all of them are subject to the effect of some orientation force $F = -\nabla\phi$ (to be described later) as displayed in Figure 7.4. Since actin filaments are plastic, but non-elastic, then we assume that $x_i(t)$ and $x_{i+1}(t)$ are linked but separated by a fixed distance l. This produces a simple problem from Lagrangian mechanics where we minimize action subject to the holonomic constraints:

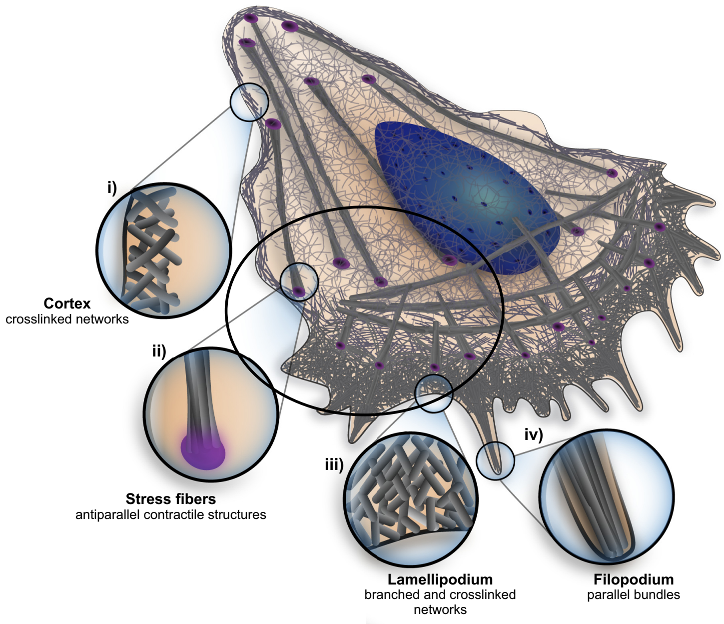


Figure 7.3: Different types of actin-based structures in the membrane of a cell: cortex, stress fibers, lammelipodium and filipodium. Cartoon taken from [35].

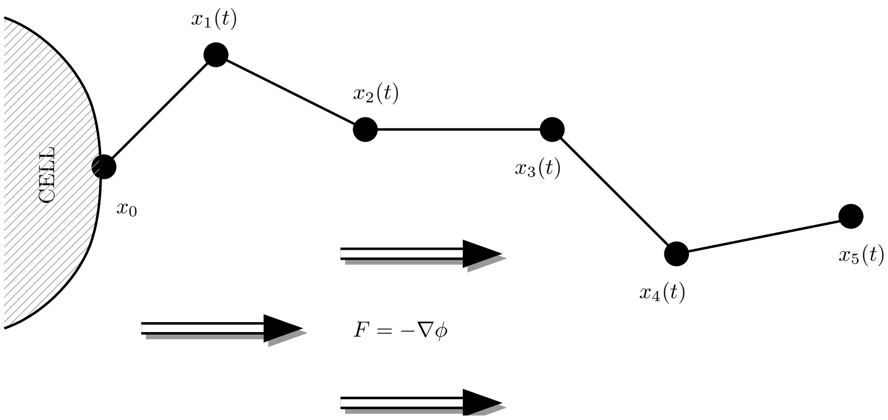


Figure 7.4: Cytoneme regarded as a chain of actin molecules.

$$\begin{cases} \min \quad \mathcal{L}[x_1, \dots, x_N] := \displaystyle\int_0^T \left(\frac{1}{2} m |\dot{x}_i|^2 - \phi(t, x_i) \right) dt, \\ \text{s.t.} \quad C_i(x_1, \dots, x_N) := |x_i - x_{i-1}|^2 - l^2 = 0, \quad i = 1, \dots, N. \end{cases} \tag{7.2.1}$$

To solve it, we use the Euler–Lagrange equations with Lagrange multipliers $\lambda_1(t), \dots, \lambda_N(t)$, that yields the equations of motion:

$$\begin{cases} m\dfrac{d^2 x_i}{dt^2} = -\nabla\phi(t, x_i) + 2\lambda_i(x_i - x_{i-1}) - 2\lambda_{i+1}(x_{i+1} - x_i), & 1 \le i \le N-1, \\ m\dfrac{d^2 x_N}{dt^2} = -\nabla\phi(t, x_N) + 2\lambda_N(x_N - x_{N-1}), & i = N, \\ |x_i(t) - x_{i-1}(t)| = l, & 1 \le i \le N, \\ x_0(t) = x_0. \end{cases} \tag{7.2.2}$$

This is nothing but Newton's second law for each note $x_i(t)$, whose acceleration is determined both by they orientation force and a constraint fictitious force due to the inelastic link. However, this is a too ideal situation due to three different reasons:

1. Cytonemes are not discrete structures, but continuous ones.
2. The discrete cytoneme has length $N \cdot l$ but, in reality, cytonemes have variable lengths as they grow and retract by (de)polimerization of actin dimers.
3. The extracellular matrix is viscous and it imposes friction with substrate.

We will address the first two questions later. Regarding the third one, the solution is easy because we can actually modify system (7.2.2) to account for such friction with medium. For instance, we can introduce linear velocity damping in the spirit of the overdamped dynamics that we presented in the introductory Chapter 1, namely,

$$\begin{cases} m\dfrac{d^2 x_i}{dt^2} = \dfrac{m}{\tau}\dfrac{dx_i}{dt} - \nabla\phi(t, x_i) + 2\lambda_i(x_i - x_{i-1}) - 2\lambda_{i+1}(x_{i+1} - x_i), & 1 \le i \le N-1, \\ m\dfrac{d^2 x_N}{dt^2} = \dfrac{m}{\tau}\dfrac{dx_N}{dt} - \nabla\phi(t, x_N) + 2\lambda_N(x_N - x_{N-1}), & i = N, \\ |x_i(t) - x_{i-1}(t)| = l, & 1 \le i \le N, \\ x_0(t) = x_0, \end{cases} \tag{7.2.3}$$

where τ stands for relaxation time under friction.

Discrete generalized variables

Before we solve the other two issues of this approach, we recall that we can often get rid of multipliers if we introduce appropriate *generalized variables*. For simplicity, we shall only do this in a two-dimensional setting. Specifically, we will restrict to a single z-layer of the imaginal

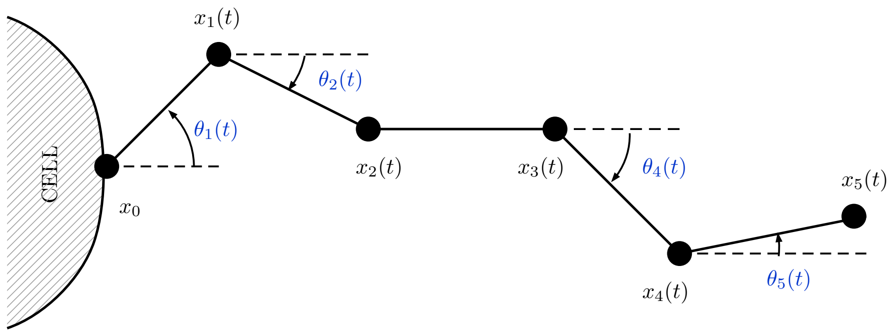


Figure 7.5: Representation of generalized coordinates

disc. This suggests that in our case, the natural choice is given by the corresponding angles between any two link of the chain, like in Figure 7.5, that is,

$$|x_i - x_{i-1}| = l \Longleftrightarrow x_i = x_{i-1} + l(\cos\theta_i, \sin\theta_i),$$

for all $i = 1, \ldots, N$. By inspection in (7.2.3), observe that we obtain the following equivalent representation without multipliers

$$\left(\sum_{i=k}^{N} m\ddot{x}_i\right) \cdot (x_k - x_{k-1})^{\perp} = \left(-\sum_{i=k}^{N} \nabla\phi(t, x_i) - \frac{m}{\tau}\sum_{i=k}^{N} \dot{x}_i\right) \cdot (x_k - x_{k-1})^{\perp},$$

for any $k = 1, \ldots, N$. This allows restating System (7.2.3) in the following equivalent form:

$$\begin{aligned} m\sum_{i=k}^{N}\sum_{j=1}^{i} &\left(\cos(\theta_j - \theta_k)\ddot{\theta}_j - \sin(\theta_j - \theta_k)\dot{\theta}_j^2\right) \\ &= -\frac{1}{l}\sum_{i=k}^{N}((-\sin\theta_k)\partial_x\phi(t, x_i) + (\cos\theta_k)\partial_y\phi(t, x_i)) - \frac{m}{\tau}\sum_{i=k}^{N}\sum_{j=1}^{i}\cos(\theta_j - \theta_k)\dot{\theta}_j, \end{aligned}$$

for any $k = 1, \ldots, N$. This is a system of implicit second order equations for the angles $\theta_1 \ldots, \theta_N$ that we can write in a more explicit way as follows. First, consider the vector of angles $\Theta := (\theta_1, \ldots, \theta_N)$ and define the matrix $M(\Theta)$ and vector $b(\Theta)$ described by the following components

$$\begin{aligned} M_{ij}(\Theta) =& m(N + 1 - \max\{i, j\})\cos(\theta_i - \theta_j), \\ b_k(\Theta, \dot{\Theta}) =& -\frac{1}{l}\sum_{i=k}^{N}((-\sin\theta_k)\partial_x\phi(t, x_i) + (\cos\theta_k)\partial_y\phi(t, x_i)) \\ &+ m\sum_{i=k}^{N}\sum_{j=1}^{i}\sin(\theta_j - \theta_k)\dot{\theta}_j^2 - \frac{m}{\tau}\sum_{i=k}^{N}\sum_{j=1}^{i}\cos(\theta_j - \theta_k)\dot{\theta}_j, \end{aligned}$$

for all $1 \le i, j, k \le N$. Then, the system can be identified with a coupled system of explicit nonlinear ODEs

$$\begin{pmatrix} \Theta \\ \dot{\Theta} \end{pmatrix}' = \begin{pmatrix} \dot{\Theta} \\ M(\Theta)^{-1} b(\Theta, \dot{\Theta}) \end{pmatrix}.$$

The main relevance of the above expression for the dynamics of the angles of each cytoneme is from the numerical point of view. Specifically, after we use an inversion numerical method for the matrices M, we can readily obtain simulations based on simple Euler or Runge–Kutta methods.

We remark that once angles θ_i have been computed, real positions are recovered from the following formula:

$$x_i = x_0 + l \sum_{j=1}^{i} (\cos\theta_j, \sin\theta_j),$$

for any $i = 1, \dots, N$.

Continuous cytonemes with fixed length

We now focus on the derivation of a continuous description for the discrete model (7.2.3). We start with the case of constant-in-time length L. First, we define the following continuous objects associated with the discrete ones:

$$\begin{aligned}
\gamma_N(t,\xi) &:= \sum_{i=1}^{N} \chi_{(\xi_{i-1}^N, \xi_i^N]}(\xi)\, x_i^N(t),\\
\lambda_N(t,\xi) &:= l^2 \sum_{i=1}^{N} \chi_{(\xi_{i-1}^N, \xi_i^N]}(\xi)\, \lambda_i^N(t).\\
\rho_N(t,\xi) &:= \sum_{i=1}^{N} \chi_{(\xi_{i-1}^N, \xi_i^N]}(\xi)\, \frac{m}{L} \equiv \frac{Nm}{L} = \frac{m}{l},
\end{aligned}$$

for $\xi \in (0, L_N)$, where the nodes read $\xi_i^N := li$ for all $i = 1, \dots, N$, and $L_N := Nl$ determines the total length of the chain of N molecules.

By definition, γ_N and λ_N are continuous descriptions of positions and multipliers in a parametrized way in terms of ξ. Indeed, as we move such a parameter from subinterval $(\xi_{i-1}^N, \xi_i^N]$ to next one $(\xi_i^N, \xi_{i+1}^N]$, the parametrization jumps from particle x_i^N to particle x_{i+1}^N and so on. Intuitively, we will make the link l between nodes small and we will take the amount N of nodes large enough so that we smoothly fill a whole curve on the plane. Also, notice that all masses had been set to m, which suggest that the continuous cytoneme will be homogeneous and density of actin is constant as well. Indeed, notice that the above associated continuous density ρ_N is a constant functions of ξ.

In order to prove such intuition, let us observe that γ_N and λ_N fulfil the following system of coupled equations in a weak sense:

$$\begin{aligned}
\rho(t,\xi)\ddot{\gamma}_N(t,\xi) = &-\frac{1}{l}\nabla\phi(t,\gamma_N(t,\xi)) - \frac{\rho(t,\xi)}{\tau}\dot{\gamma}_N(t,\xi)\\
&+ \left\{ \frac{1}{l}\lambda_N(t,\xi)\frac{\gamma_N(t,\xi)-\gamma_N(t,\xi-l)}{l} \right.\\
&\left. -\frac{1}{l}\lambda_N(t,\xi+l)\frac{\gamma_N(t,\xi+l)-\gamma_N(t,\xi)}{l} \right\} \chi_{(\xi_1^N, \xi_{N-1}^N]}(\xi)\\
&+ \frac{1}{l^2}\lambda_N(t,\xi)\frac{\gamma_N(t,\xi)-\gamma_N(t,\xi-l)}{l}\chi_{(\xi_{N-1}^N, \xi_N^N]}(\xi),
\end{aligned} \tag{7.2.4}$$

for any $\xi \in (0, L_N)$. In addition, we also obtain the corresponding constraints

$$\begin{aligned} \frac{|\gamma_N(t,\xi) - \gamma_N(t,\xi - l)|}{l} \chi_{(\xi_1^N, L_N]}(\xi) &= 1, \\ \frac{|x_0 - \gamma_N(t,l)|}{l} &= 1. \end{aligned} \tag{7.2.5}$$

Let us set the following natural hypothesis during the continuous limit

$$L = Nl, \; Nm = L\rho, \; \text{ and } \; \phi = l\psi,$$

that sets the total length and density of the cytoneme to L and ρ respectively and appropriately rescales the potential ϕ. Then, taking limits formally as $N \to \infty$ (thus $l \to 0$) in (7.2.4)-(7.2.5), we obtain the following continuous system:

$$\begin{cases} \rho\ddot{\gamma} + \dfrac{\partial}{\partial \xi}(\lambda\,\gamma') + \dfrac{\rho}{\tau}\dot{\gamma} = -\nabla\psi(t,\gamma), \\ \gamma(t,0) = x_0, \\ \gamma(0,\xi) = \gamma_0(\xi) \; \text{ and } \; \dot{\gamma}(0,\xi) = \dot{\gamma}_0(\xi), \\ |\gamma'(t,\xi)| = 1, \\ \lambda(t,L) = 0, \end{cases} \tag{7.2.6}$$

for every $t \geq 0$ and $\xi \in (0, L)$, where we denote

$$\dot{\gamma} = \frac{\partial \gamma}{\partial t} \; \text{ and } \; \gamma' = \frac{\partial \gamma}{\partial \xi}.$$

Notice that constraints $C_i = 0$ in the discrete system translates into the condition that the limiting curve γ is parametrized by arc length. In turns, this guarantees that length is preserved for all times and equals L. On the other hand, the last condition of the multiplier is natural due to the fact that the last term of (7.2.4) has a different scale with regards to l. This suggests that the endpoint $\gamma(t, L)$ of the cytoneme moves freely and is not affected by the fictitious force.

Continuous description of cytonemes with variable length

In this part, we address the case where cytoneme γ can grow and reduce its length by polimerization or depolimerization of actin dimers at the free endpoint $\gamma(t, L)$. This means that the above value $L = L(t)$ depends on time and $\dot{L}(t)$ is described by the corresponding polimerization rates of actin, that has to be set according to experimental data.

Let us formulate an analogue continuous minimization problem to the discrete one in (7.2.1). This time, curves $\gamma = \gamma(t, \xi)$ are defined for $(t, \xi) \in \Omega_T$, where the domain is

$$\Omega_T := \{(t,\xi) : \; t \in [0,T], \; \xi \in [0, L(t)]\},$$

in order to account for variable lengths of the cytoneme, see Figure 7.6. Then, the continuous problem of minimization of action takes the following form

$$\begin{cases} \min \quad \mathcal{L}[\gamma] = \displaystyle\iint_{\Omega_T} \left(\frac{1}{2}\rho(t,\xi)|\dot{\gamma}(t,\xi)|^2 - \psi(t,\gamma(t,\xi)) \right) dt\, d\xi, \\ \text{s.t.} \quad C(\gamma') = |\gamma'| = 1. \end{cases} \tag{7.2.7}$$

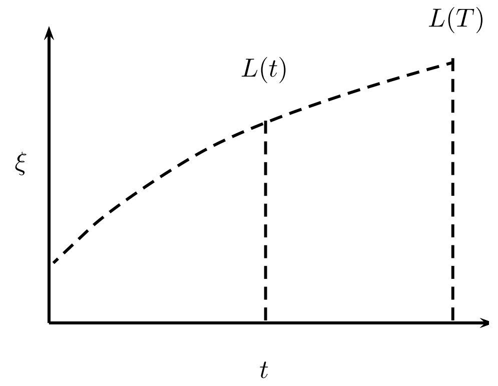


Figure 7.6: Domain Ω_T for the case of variable length

Again, we can solve it in terms of the Euler–Lagrange equations. Before we state it, notice that, according to the shape of Ω_T in Figure 7.6, we have prescribed conditions on the piece of boundary with $\xi = 0$ and $t \geq 0$, namely,

$$\gamma(t, 0) = 0, \text{ for all } t \geq 0.$$

Also, for $t = 0$ and $\xi \in (0, L(0))$, we need to set the initial shape and velocity configurations of the cytoneme, that is,

$$\gamma(0, \xi) = \gamma_0(\xi) \text{ and } \dot{\gamma}(0, \xi) = \dot{\gamma}_0(\xi), \text{ for all } \xi \in (0, L(0)).$$

However, we do not have any clear constraint on the upper piece of boundary with $\xi = L(t)$ and $t \geq 0$. Using standard tools from calculus of variations, we shall consider the natural *free-boundary condition* that arises from the minimization problem (7.2.7). Putting everything together, we achieve the following system:

$$\begin{cases} \dfrac{\partial}{\partial t}(\rho\dot{\gamma}) + \dfrac{\partial}{\partial \xi}(\lambda\gamma') + \dfrac{\rho}{\tau}\dot{\gamma} = -\nabla\psi(t, \gamma), \\ \gamma(t, 0) = x_0, \\ \gamma(0, \xi) = \gamma_0(\xi) \text{ and } \dot{\gamma}(0, \xi) = \dot{\gamma}_0(\xi), \\ |\gamma'(t, \xi)| = 1, \\ \rho(t, L(t))\dot{\gamma}(t, L(t))\dot{L}(t) = 2\lambda(t, L(t))\gamma'(t, L(t)), \end{cases} \tag{7.2.8}$$

for any $t \geq 0$ and $\xi \in (0, L(t))$. Notice that for constant length, i.e., $\dot{L} \equiv 0$, the free boundary condition reduces to $\lambda(t, L) = 0$ and we recover the above case (7.2.6). Also notice that actin density ρ along the cytoneme does not need to be uniform in this case. However, experiments suggest that it is indeed nearly uniform.

Continuous generalized variables

By using similar ideas to the discrete case, we can eliminate the multiplier from de the continuous equations of motion (7.2.8) in terms of *appropriate continuous generalized variables*. By analogy with the discrete case, we set those variables $\theta(t, \xi)$ to be the angles of the tangent vector $\gamma'(t, \xi)$ to the curve, that is,

$$\gamma'(t, \xi) = (\cos\theta(t, \xi), \sin\theta(t, \xi)),$$

for any $t \geq 0$ and $\xi \in (0, L(t))$. Using them, we can deduce the following implicit integro-differential equation for $\theta(t, \xi)$:

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_{\xi}^{L(t)}\int_{0}^{\xi'} \cos(\theta(t,\xi'') - \theta(t,\xi))\dot{\theta}(t,\xi'')\rho(t,\xi')\,d\xi''\,d\xi' \\
&\qquad - \int_{\xi}^{L(t)}\int_{0}^{\xi'} \sin(\theta(t,\xi'') - \theta(t,\xi))\dot{\theta}(t,\xi'')\dot{\theta}(t,\xi)\rho(t,\xi')\,d\xi''\,d\xi' \\
&\qquad = -\frac{1}{\tau}\int_{\xi}^{L(t)}\int_{0}^{\xi'} \cos(\theta(t,\xi'') - \theta(t,\xi))\dot{\theta}(t,\xi'')\rho(t,\xi')\,d\xi''\,d\xi' \\
&- \int_{\xi}^{L(t)}\int_{0}^{\xi'} \big((-\sin\theta(t,\xi))\partial_x\psi(t,\gamma(t,\xi'')) + (\cos\theta(t,\xi))\partial_y\psi(t,\gamma(t,\xi''))\big)\,d\xi''\,d\xi'. \quad (7.2.9)
\end{aligned}$$

Such an equivalent equation arises from an analogue argument like in the above discrete case, where sums have to be replaced by the corresponding integrals, and the free-boundary condition is used to cancel boundary terms during integration by parts. The interest of such a formula is again from a numerical point of view. Indeed, we can again discretize such equation and solve it with appropriate numerical methods, adapted to the fact that L is no longer constant, but dynamic.

7.2.3 Orientation forces

Once we have described the equations of motion (7.2.8) for each cytoneme, along this section we will focus on describing the orientation forces F that we assumed in the above part. It will be responsible for the guidance of both families of cytonemes (Hh emitting and receiving) towards the appropriate contact sites where synapses will take place.

The current experimental evidence [34, 141, 271] supports the hypothesis that the main proteins involved in the communication mechanism are Hh, Ihog, Ptc and gliplicans Dally and Dlp. For the sake of clarity, we introduce a brief summary of the main relations between such proteins. This outline has been set according to recent experiments.

1. Hh level in P compartment is larger than in A compartment.

2. Ihog level in P compartment is larger than in A compartment.

3. Hh and Ptc do not directly affect orientation of cytonemes.

4. Ihog and glipicans are the main responsible for the orientation mechanism.

5. Cytonemes with high levels of Ihog repel cytonemes with high levels of Ihog.

6. Cytonemes with low levels of Ihog repel cytonemes with low levels of Ihog.

7. Cytonemes with high levels of Ihog attract cytonemes with low levels of Ihog.

8. Cytonemes at large distances do not interact.

9. A single cytoneme does not interact with itself.

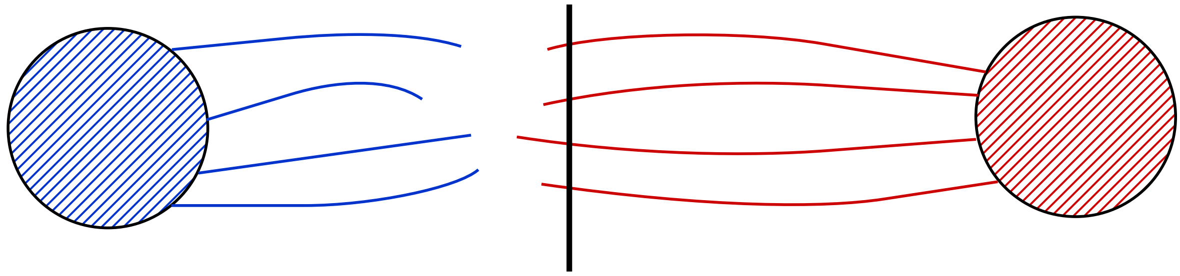


Figure 7.7: In red, emitting cytonemes that emanate from the cell membranes of the P comparment. In blue, receiving cytonemes emerging from cell membranes of the A compartment. Emitting cytonemes grow and invade the A compartment to stablish synapses with receiving cytonemes near the A\ P border.

Short range interactions: cytonemes potentials

We first introduce the orientation forces that are self-generated by the complete ensemble of cytonemes themselves. According to the above experimental insight, those interactions must be of short range since far apart filopodia do not feel the presence of each other. Also, the above heuristics suggests that cytonemes can be split into two different families according to the levels of Ihog that they carry, see Figure 7.7. The family of cytonemes with larger levels of Ihog corresponds to cytonemes that emerge from emitting cells in the P compartment, whilst the other family of cytonemes with lower Ihog corresponds to receiving cells located in the A compartment. For simplicity, we call them Hh emitting and receiving cytonemes. Here on, we shall use the following notation to denote the main quantities associated to cytonemes of each family:

Quantities	Receiving	Emitting
Parametrization	$\gamma_j^r(t,\xi)$	$\gamma_i^e(t,\xi)$
Lengths	$L_j^r(t)$	$L_i^e(t)$
Charge	$q_j^r(t,\xi)$	$q_i^e(t,\xi)$
Orientation forces	$F_j^r(t,x)$	$F_i^e(t,x)$
Signature	$\sigma^r=-1$	$\sigma^e=+1$

Here superscripts denote “receiving” or “emitting” cytonemes, whilst subscripts j and i range along the total amount $n^r, n^e \in \mathbb{N}$ of receiving and emitting filopodia. By charges q_j^r and q_i^e, we refer to an appropriate weighted sum of the total densities of all the proteins inside each cytoneme that are involved in the orientation mechanism: namely, Ihog, Dally and Dlp both inside vesicles and over the filaments’ membranes. Specifically, we shall define

$$
\begin{aligned}
q = \;& \beta_{\text{Ihog}}[\text{Ihog}]_{\text{ves}} - \beta_{\text{Dally}}[\text{Dally}]_{\text{ves}} - \beta_{\text{Dlp}}[\text{Dlp}]_{\text{ves}} \\
&+ \beta_{\text{Ihog}}[\text{Ihog}]_{\text{cyt}} - \beta_{\text{Dally}}[\text{Dally}]_{\text{cyt}} - \beta_{\text{Dlp}}[\text{Dlp}]_{\text{cyt}},
\end{aligned}
$$

for some non-negative weights β_{Ihog}, β_{Dally} and β_{Dlp}. Interestingly, we observe that, for emitting cytonemes, the chage q_i^e has positive sign, whilst for receiving cytonemes q_j^r is negative. This allow us to distinguish both families by using different “signatures” (or labels) $+1$ and -1 for each of them. Then σ^r and σ^e are reminiscent of the sign of charges in electrodynamics. Like in that setting, the repelling or attractive character of interactions is determined by the product of signatures. Indeed, as mentioned before, cytonemes belonging to the same family interact in a repulsive way whilst those belonging to opposite families exhibit attractive interactions.

Finally, $F_j^r(t,x)$ and $F_i^r(t,x)$ represent the corresponding orientation forces for cytonemes $\gamma_j^r(t,\xi)$ and $\gamma_i^r(t,\xi)$, which will determine the final equations of motion (7.2.8) for each cytoneme. Our final goal along this part is to describe those self-generated forces by accounting for the above attractive and repulsive relations between both families of filopodia. With all the above information in mind, we first describe the following single layer potentials:

$$\begin{aligned}
\psi_i^e(t,x) &:= -\sum_{\substack{l=1\\ l\neq i}}^{n^e} \int_0^{L_l^e(t)} W_R(|x-\gamma_l^e(t,\xi)|)q_l^e(t,\xi)\,d\xi \\
&\quad -\sum_{m=1}^{n^r} \int_0^{L_m^r(t)} W_A(|x-\gamma_m^r(t,\xi)|)q_m^r(t,\xi)\,d\xi, \\
\psi_j^r(t,x) &:= -\sum_{l=1}^{n^e} \int_0^{L_l^e(t)} W_A(|x-\gamma_l^e(t,\xi)|)q_l^e(t,\xi)\,d\xi \\
&\quad -\sum_{\substack{m=1\\ m\neq j}}^{n^r} \int_0^{L_m^r(t)} W_R(|x-\gamma_m^r(t,\xi)|)q_m^r(t,\xi)\,d\xi,
\end{aligned} \tag{7.2.10}$$

for $i=1,\ldots,n^e$ and $j=1,\ldots,n^r$. Notice that we have omitted the interaction of any cytoneme with itself according to experiments. These corresponds to the potentials of orientation fields inspired by electrodynamic. Indeed, the corresponding orientation forces on each cytoneme can by obtained from the fields after multiplication by sources, namely,

$$\begin{aligned}
F_i^e(t,\gamma_i^e(t,\xi)) &:= q_i^e(t,\xi)\nabla\psi_i^e(t,\gamma_i^e(t,\xi)), \\
F_j^r(t,\gamma_j^r(t,\xi)) &:= q_j^r(t,\xi)\nabla\psi_j^r(t,\gamma_j^r(t,\xi)),
\end{aligned} \tag{7.2.11}$$

for each $i=1,\ldots,n^e$ and $j=1,\ldots,n^r$. In (7.2.10), $W_R=W_R(r)$ and $W_A=W_A(r)$ stand for the precise interaction potentials that mediate attractive and repulsive interactions. We shall assume that they take the following specific form, see Figure 7.8

$$W_M'(r) := \begin{cases} \dfrac{K\delta_0^\alpha}{r^\alpha}, & r\in(0,\delta_0), \\ K, & r\in[\delta_0,\delta_1), \\ \dfrac{K}{1+e^{\frac{2r-(\delta_1+\delta_2)}{(\delta_2-r)(r-\delta_1)}}}, & r\in[\delta_1,\delta_2), \\ 0, & r\in[\delta_2,\infty). \end{cases} \tag{7.2.12}$$

K is the coupling strength, $\alpha>0$ is a parameter that measures a singular interaction at the origin and $0<\delta_0<\delta_1<\delta_2$ are small distances that we shall describe in the sequel. The reason behind that choice is supported by experimental evidence. Specifically,

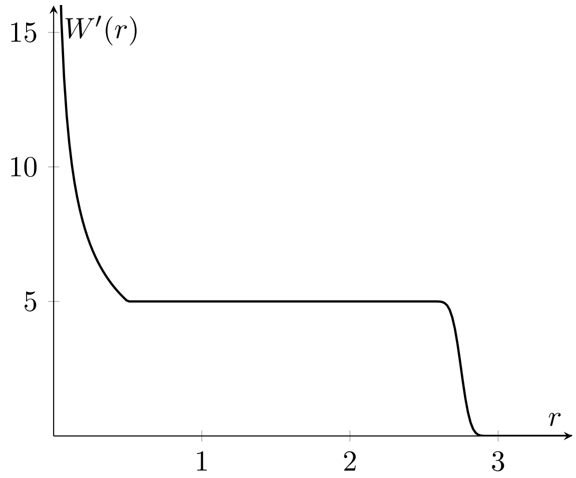


Figure 7.8: Plot of the interaction potential $W(r)$ for $K = 5$, $\alpha = \frac{1}{2}$, $\delta_0 = 0.5$, $\delta_1 = 2.5$ and $\delta_2 = 3$.

1. Cytoneme-cytoneme interactions have short range. Indeed, according to experimental evidence, there exists a minimal size δ_2 so that cytonemes located at larger distances do no feel the presence of each other.

2. Once inside that range, interactions do not seem to depend on distances between cytonemes, but just on the concentration itself. This has been modelized in terms of a flat part of the forcing kernel W' for distances ranging from a small δ_0 to δ_1.

3. Below that small range δ_0, cytonemes must feel an intense force that either repels them, to guarantee the separation property by Reynolds in Chapter 1, or attracts them, depending on whether they belong to same or opposite families. Indeed, attractive interactions need to be strong enough to allow for finite-time contacts. Bearing in mind our results in Chapter 3, we have decided to set an inverse power low near the origin, that is compatible with finite-time sticking behavior.

However, we still need to experimentally verify whether the repulsive interactions between both families of emitting and receiving cytonemes are comparable and can represented in terms of a unique kernel W_R or require different coefficients. Also, we do not know if the attractive interactions are given by a similar interaction potential W_A. We are working on the description of the involved constants to verify whether they are functionally or computationally similar.

Long-range interactions: the membrane potential

As mentioned before, cytoneme to cytoneme interactions have short range. Notice that this may cause a severe problem since A and P compartments might stay unconnected when, initially, cytonemes lengths are too short. In that case, we do not expect any contact between emitting and receiving cytonemes. However, we experimentally observe that despite cytonemes being far apart initially, they already feel a sort of attraction towards the membrane of the A\P border, see Figure 7.9. To find it, let us compute the similar charge associated to the background concentration of Ihog, Dally and Dlp throughout the extracellular matrix, namely,

$$q_{\text{back}} = \beta_{\text{Ihog}}[\text{Ihog}]_{\text{back}} - \beta_{\text{Dally}}[\text{Dally}]_{\text{back}} - \beta_{\text{Dlp}}[\text{Dlp}]_{\text{back}}.$$

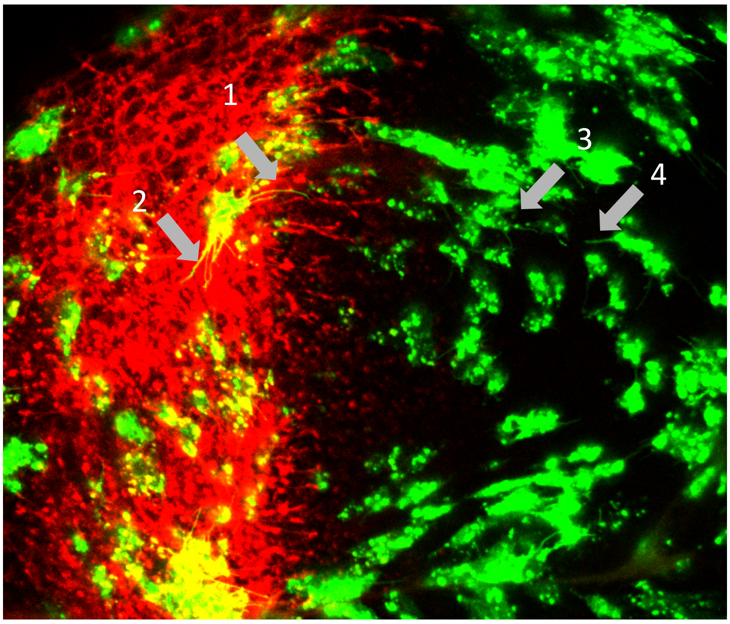


Figure 7.9: Both cytonemes emanating from cells in the A and P compartment orient towards the membrane of the A\P border.

If we plot that quantity, then we find an interesting change of signs of the charge q_{back} so that both families get attracted by the cell membrane via a similar mechanism. Then, we shall accompany the above sort range orientation forces F_i^e and F_j^r in (7.2.10)-(7.2.11) with a longer range orientation force towards the membrane generated by such background charges. Indeed, we first describe the orientation potential towards the membrane in terms of the following volume potentials

$$\widehat{\psi}(t,x) := -\int_D W_M(|x-y|)q_{\text{back}}(y)\,dy, \tag{7.2.13}$$

where $D \subseteq \mathbb{R}^2$ represents the domain occupied by the wing imaginal disc. Then, the orientation force will be described in an analogue way

$$\begin{aligned} \widehat{F}_i^e(t,\gamma_i^e(t,\xi)) &:= q_i^e(t,\xi)\nabla\hat{\psi}(t,\gamma_i^e(t,\xi)),\\ \widehat{F}_j^r(t,\gamma_j^r(t,\xi)) &:= q_j^r(t,\xi)\nabla\hat{\psi}(t,\gamma_j^r(t,\xi)), \end{aligned} \tag{7.2.14}$$

for each $i =, 1, \ldots, n^e$ and $j = 1, \ldots, n^r$. This time, the interaction potential W_M has longer range interactions. However, at short ranges W_R and W_A dominate, that is, cytonemes do not stick to the cell membrane. A plausible way of modelling such fact is by eliminating the singular sticking force at the origin in (7.2.12), that is,

$$W_M'(r) := \begin{cases} K, & r \in [0,\delta_1),\\ \dfrac{K}{1+e^{\frac{2r-(\delta_1+\delta_2)}{(\delta_2-r)(r-\delta_1)}}}, & r \in [\delta_1,\delta_2),\\ 0, & r \in [\delta_2,\infty), \end{cases} \tag{7.2.15}$$

see Figure 7.10 for a comparison of the type of kernels. Finally, to conclude, let us summarize the final global dynamics of cytonemes

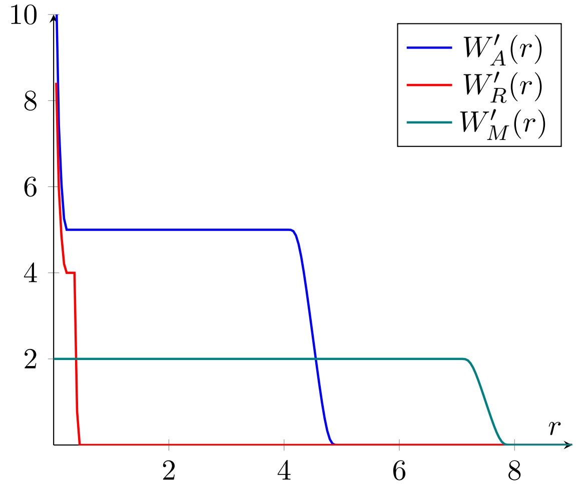


Figure 7.10: Plot of the interaction potentials W_A, W_R and W_M for the values $K = 5, \alpha = \frac{1}{2}, \delta_0 = 0.2, \delta_1 = 4, \delta_2 = 5$ (for W_A), $K = 4, \alpha = \frac{1}{2}, \delta_0 = 0.2, \delta_1 = 0.3, \delta_2 = 0.5$ (for W_R) and $K = 2, \delta_1 = 7, \delta_2 = 8$ (for W_M).

$$\begin{cases} \frac{\partial}{\partial t}(\rho_i^e \dot{\gamma}_i^e) + \frac{\partial}{\partial \xi}(\lambda_i^e \gamma_i^{e\prime}) + \frac{\rho_i^e}{\tau}\dot{\gamma}_i^e = F_i^e(t, \gamma_i^e) + \widehat{F}_i^e(t, \gamma_i^e), \\ \gamma_i^e(t, 0) = x_i^e, \text{ and } |\gamma_i^{e\prime}(t, \xi)| = 1, \\ \rho(t, L_i^e(t))\dot{\gamma}_i^e(t, L_i^e(t))\dot{L}_i^e(t) = 2\lambda_i^e(t, L_i^e(t))\gamma_i^{e\prime}(t, L_i^e(t)), \end{cases}$$

for any $i = 1, \ldots, n^e$, and

$$\begin{cases} \frac{\partial}{\partial t}(\rho_j^r \dot{\gamma}_j^r) + \frac{\partial}{\partial \xi}(\lambda_j^r \gamma_j^{r\prime}) + \frac{\rho_j^r}{\tau}\dot{\gamma}_j^r = F_j^r(t, \gamma_j^r) + \widehat{F}_j^r(t, \gamma_j^r), \\ \gamma_j^r(t, 0) = x_j^r, \text{ and } |\gamma_j^{r\prime}(t, \xi)| = 1, \\ \rho(t, L_j^r(t))\dot{\gamma}_j^r(t, L_j^r(t))\dot{L}_j^r(t) = 2\lambda_j^r(t, L_j^r(t))\gamma_j^{r\prime}(t, L_j^r(t)), \end{cases}$$

for each $j = 1, \ldots, n^r$, where the orientation forces are given by (7.2.11) and (7.2.14).

7.2.4 Transportation of proteins along moving cytonemes

In this last part, we shall derive appropriate transport equations for the propagation of proteins along the above dynamical cytonemes. In particular, we shall focus on the propagation of Hh morphogen, but a similar approach can be applied to other types of proteins (e.g., Ihog, Dally, Dlp, Ptc, etc). As observed in experiments, proteins travel inside vesicles that move along filopodia, recall Figure 7.2. However, the underlying mechanical description for such transportation has not been clarified experimentally. It seems that it should involve some internal miosin-based motor of filopodia to carry vesicles along actin wires. For simplicity, along

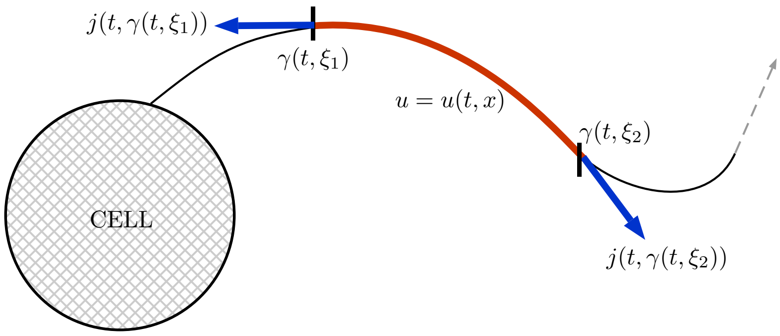


Figure 7.11: Conservation of Hh morphogen along the cytoneme

this first stage, we have preferred to simplify the microscopic dynamics. Instead of a description vesicle by vesicle, we shall introduce a macroscopic continuous approach in terms of the concentration gradient of proteins along cytonemes.

Fix any emitting cytoneme $\gamma = \gamma_i^e$ and denote the concentration of Hh morphogen along it by $u = u(t,x)$. Now, take any arch of the cytoneme as in Figure 7.11, i.e.,

$$A(t,\xi_1,\xi_2) = \{\gamma(t,\xi) : \xi \in [\xi_1,\xi_2]\},$$

for $0 < \xi_1 < \xi_2 < L(t)$, and observe that by the conservation of mass, we can claim that the rate of decrease of protein concentration on such an arch is determined by the outwards flux at the endpoints. Specifically, we obtain the following integral form of the continuity equation

$$\frac{\partial}{\partial t}\int_{A(t,\xi_1,\xi_2)} u\, d_x S = -\, j(t,\gamma(t,\xi))\cdot\gamma'(t,\xi)\Big|_{\xi=\xi_1}^{\xi=\xi_2},$$

where j denotes such an outwards flux of proteins. To simplify, let us define

$$\overline{u}(t,\xi) = u(t,\gamma(t,\xi)),$$

and notice that the above integral formulation, that is valid for any arch, is equivalent to the following PDE

$$\begin{cases} \dfrac{\partial\bar{u}}{\partial t} + \dfrac{\partial}{\partial\xi}(j(t,\gamma)\cdot\gamma') = 0, & \xi \in [0,L(t)], \\ \bar{u}(0,\xi) = \bar{u}_0(\xi), & \xi \in [0,L(0)], \\ -j(t,x_0)\cdot\gamma'(t,0) = \beta(t), & t \geq 0. \end{cases} \tag{7.2.16}$$

Here, $\beta = \beta(t)$ represent the inwards flux of Hh morphogen at the base endpoint $\gamma(t,0) = x_0$ that the corresponding emitting cell transfers to its cytoneme. In order to determine a final closed form of (7.2.16), we need to set the relation between the outwards flux j and the concentration of proteins u itself.

On the one hand, we recall that, according to previous literature [199, 300, 274], the classical choice would be to set j according to Fick's law

$$j = \nu\nabla_\gamma u. \tag{7.2.17}$$

This represents that proteins spread towards the areas of the cytoneme with lowest concentration with diffusion coefficient $\nu > 0$. Here, ∇_γ represents the Riemannian gradient of $u(t,\cdot)$ along the curve $\gamma(t,\cdot)$. i.e.,

$$(\nabla_\gamma u)(t,\gamma(t,\xi)) = \overline{u}'(t,\xi)\gamma'(t,\xi).$$

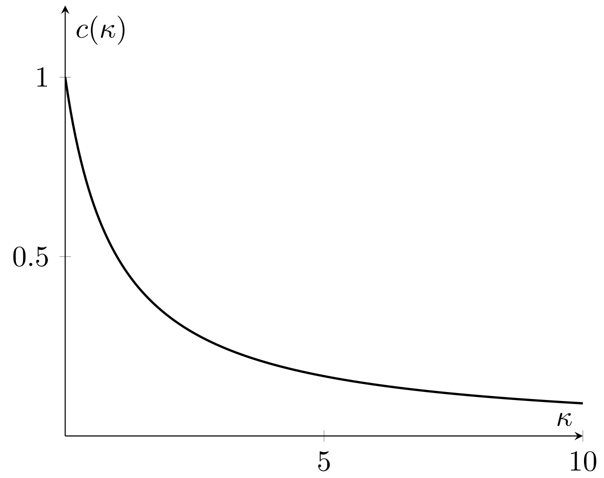


Figure 7.12: Inverse dependence of speed of propagation on curvature of cytonemes for the choice $\omega = 1$ and $c_{\max} = 1$.

Notice that in such a case, transport of proteins would be governed by a linear heat equation along the cytoneme. However, this choice is not acceptable from a biological point of view because it yields infinite speed of propagation.

Instead of (7.2.17) and, inspired by the flux-saturated mechanisms in [53, 54, 55, 266, 294], we can consider a porous-medium flux-saturated type description of j as follows

$$j = -\nu \frac{u^m \nabla_\gamma u}{\sqrt{u^2 + \frac{\nu^2}{c^2}|\nabla_\gamma u|^2}}, \tag{7.2.18}$$

where the coefficient m represents porosity of the cytoneme, ν is called the kinematic viscosity and c is a bound for the maximum speed of propagation along the cytoneme. Such a choice amounts to the following nonlinear conservation law for the concentration of Hh morphogen

$$\begin{cases} \dfrac{\partial \bar{u}}{\partial t} = \nu \dfrac{\partial}{\partial \xi} \left(\dfrac{\bar{u}^m \bar{u}'}{\sqrt{\bar{u}^2 + \frac{\nu^2}{c^2}(\bar{u}')^2}} \right), & \xi \in [0, L(t)], \\ \bar{u}(0, \xi) = \bar{u}_0(\xi), & \xi \in [0, L(0)], \\ -\dfrac{\bar{u}^m(t,0)\bar{u}'(t,0)}{\sqrt{\bar{u}^2(t,0) + \frac{\nu^2}{c^2}(\bar{u}'(t,0))^2}} = \beta(t), & t \geq 0. \end{cases}$$

Regarding the speed of proteins propagation, it still must be determines according to experimental data. However, we have observed an interesting functional dependence: near very curved points of the cytoneme, there is congestion of vesicles that slow down their movement. A possible way of modelling it is though inverse dependence of speed of propagation on cytoneme's curvature $\kappa(t, \xi) = |\gamma''(t, \xi)|$. In particular, we can set

$$c(t, \xi) = c(|\kappa(t, \xi)|) = \frac{\omega c_{max}}{\omega + c_{max}|\gamma''(t, \xi)|}.$$

Notice that for that choice, the speed of propagation $c = c(\kappa)$ is always bounded by $c_{\max}$. On the one hand, if the cytoneme is very flat (that is $\kappa \simeq 0$) then speed is close to $c_{\max}$. On the

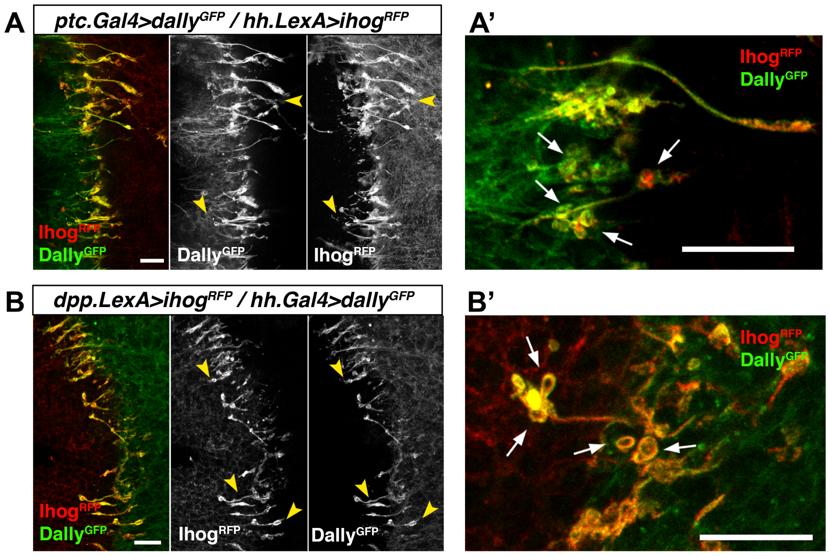


Figure 7.13: Formation of synaptic buttons at contacts sites. Taken from [141].

other hand, if the cytoneme is very curved (that is $\kappa \simeq \infty$), then speed is closed to 0, see Figure 7.12.

7.2.5 Further modeling steps and related works

In future works, we would like to address the remaining steps of the dynamics: synapsis, transference of proteins and contraction of cytonemes. Indeed, it has been experimentally observed [141] that synapses take place in a polarized way so that emitting cytonemes stay below receiving cytonemes, see [143]. Also, contacts occur at very specific sites of cytonemes, where a sort of synaptic button is formed, see [141] and Figure 7.13. In addition, there are plenty of free parameters that we need to set according to current experiments (e.g., δ_i, $\dot{L}$, c, α, etc).

From a mathematical point of view, some relevant question is the rigorous mean field limit (see Subsection 1.1.2 of Chapter 1) of the above microscopic models when the amount of cytonemes (n^e and n^r) is large enough. This question also has biological implications since normally there are so many cytonemes that continuous descriptions may be of interest. In turns, that would allow us to compare this new dynamics with previous PDE models in the literature.

Conclusions and perspectives

In this part we present some ongoing projects and future works that have arisen as a consequence of the results developed during this dissertation. Some of them have already been mentioned along this pages. Here, we will focus on a specific short list containing some of the most challenging problems from our point view.

In Chapter 2 we derived a hyperbolic hydrodynamic limit of vanishing inertia type for the kinetic Cucker–Smale model towards singular influence functions. In doing so, we obtained the limiting macroscopic system (2.1.1) for all the values of the singularity $\alpha \in (0, \frac{1}{2}]$. Indeed, the dissipation inequality (2.2.19) in Corollary 2.2.8 proved useful to derive appropriate non-concentration estimates. In particular, such bounds allowed us to identity the limit of the commutators $(\phi_\varepsilon * \rho_\varepsilon)j_\varepsilon - (\phi_\varepsilon * j_\varepsilon)\rho_\varepsilon$ in (2.2.9) for the critical case $\alpha = \frac{1}{2}$. It was done via an appropriate cancellation of the kernel's singularity in a symmetrized weak form of such a nonlinear term. However, notice that the dissipation inequality (2.2.19) is actually true for all the values of the exponent $\alpha > 0$. In particular, as discussed in Remark 2.2.19, it can be used to show that the above weak formulation of the commutators admits a uniform-in-ε bound of the following form

$$\left| \int_0^T \int_{\mathbb{R}^{2d}} \phi_\varepsilon(|x-y|)(\varphi(t,x) - \varphi(t,y)) \, (\rho_\varepsilon \otimes j_\varepsilon - j_\varepsilon \otimes \rho_\varepsilon) \, dx \, dy \, dt \right| \leq C,$$

for any smooth test function φ. Here, the valid range of parameters turns to be larger (namely, $\alpha \in (0,1)$) that the smaller range $\alpha \in (0, \frac{1}{2}]$ where we could identify the limit. This suggests that it should be also possible to identify the limit for the whole of range of parameters $\alpha \in (0,1)$ despite the fact that the above expression does not makes a clear sense for $\alpha \in (\frac{1}{2}, 1)$ (notice that in such a case, no extra cancellation of the kernel can be achieved and the limiting ρ and j are merely measure-valued solutions). Then, it is an interesting problem to elucidate if the dissipation inequality (2.2.19) actually hides a more important cancellation property that has not appeared along our approach.

In Chapter 3 we introduced a new type of agent-based model of Kuramoto type for coupled oscillators arising from neuroscience. It emerged from a fast learning regime of a suitable adaptive rule for the weights that is governed by a Hebbian-type plasticity functions with singularities. We presented a well-posedness theory of classical solutions (respectively Filippov solutions) for the subcritical an critical regimes $\alpha \in (0, \frac{1}{2}]$ and derived complete phase synchronization estimates for identical oscillators that are confined to a half circle. Interestingly, in such a case phase synchronization takes place in finite time. Regarding the supercritical regime

$\alpha \in (\frac{1}{2}, 1)$, we developed two different existence results of Filippov solutions: 1) singular limit of regular interactions and 2) continuation criterion after collisions. However, we could not prove uniqueness of Filippov solutions for general amount of oscillators $N \geq 3$. Then, it may be possible that both methods provide different type of solutions. Only when they agree, we obtained analogue finite-time phase synchronization of identical oscillators within a half circle. There are several open problems that we would like to solve related to this topic. We list some of them:

1. Recall that for the subcritical case $\alpha \in (0, \frac{1}{2})$ we achieved emergence of phase-locked states in the large coupling regime for small enough initial diameter of the system. Like in the case of the classical Kuramoto model [147], we are interested in eliminating the last constraint to recover general convergence for diameters beyond the basin of attraction. A similar question arises in the critical and supercritical regimes $\alpha \in [\frac{1}{2}, 1)$. However, such cases must be harder to tackle according to Theorem 3.5.10 because collision-less phase-locked states are unstable.

2. If coupling strength is not large enough, the scenario is much more convoluted. Indeed, recall that for the Cucker–Smale model in one dimension, the complete cluster predictability was characterized in [151] and the authors computed the final amount of emergent flocks. In that case, regular influence functions were assumed, thus discarding sticking behavior of particles. We recall that in our case, we could characterized the explicit necessary and sufficient conditions of natural frequencies so that oscillators stick together after collisions. When those conditions are violated, then oscillators instantaneously detach. We are interested in characterizing the final amount of clusters in our model by extending the above literature to singular coupling weights.

3. Like for the original Kuramoto model, one can augment the first-order system towards a second-order model under the effect of inertia (recall the introductory Chapter 1). The presence of an inertia term introduces a kind of time-delay in the dynamics that has proved relevant in certain real situations. We are interested in characterizing the type of solutions along with the expected dynamics when we include the effects of singular weights like in Chapter 3.

These are joint works with Juan Soler (University of Granada) and Jinyeong Park (Hanyang University, Seoul).

In Chapter 4, we introduced the kinetic equation associated with the above singular Kuramoto model. Indeed, we showed the local-in-time mean field limit of the particles system as the amount N of oscillators tends to infinity. This helped us to derive the emergence of finite-time phase synchronization of identical oscillators for measure-valued (Filippov) solutions in the subcritical and critical regimes. By accordance with the above open problems of the particle system, we also propose related questions associated with the kinetic model. First, for the subcritical and critical regimes we are interested in deriving uniform contractivity estimates for the Wasserstein distance in the spirit of [58] (for the Kuramoto–Sakaguchi equation). Such result restricted again to phase supports confined to a half circle but allows obtaining uniform mean field limits for such restricted regime. Our next step would be to get rid of such hypothesis and show the convergence of solutions towards a global equilibrium like we did in Chapter 5 for the Kuramoto–Sakaguchi equation. Indeed, we seek an analogue uniform contractivity estimate of the fiberwise Wasserstein distance $W_{2,g}$ and similar explicit concentration rates for general initial data. As mentioned before, the critical regime must be a special case due to Theorem 3.5.10 because collision-less phase-locked states are unstable. Our intuition is that some type of continuous sticking conditions must select the type of final equilibrium (that is not necessarily a collision-less state).

In Chapter 5, we proved the convergence towards a global equilibrium for the Kuramoto–Sakaguchi equation issued at general initial data with phase support not necessarily contained within a half circle. Recall that the Kuramoto–Sakaguchi equation for identical oscillators is a Wasserstein gradient flow in the sense of Otto calculus. However, if oscillators are not identical, the equation loses such a gradient flow structure and convergence towards a global equilibrium cannot be conducted through extension of the standard ideas in [238]. Fortunately, we found a key dissipation functional $\mathcal{I}[f]$ and an appropriate transportation distance $W_{2,g}$ so that, like for real gradient flows, the same relations take place in terms of analogue logarithmic Sobolev and Talagrand inequalities. This has suggested to us that there must exist an abstract infinite-dimensional Riemannian structure on the space of probability measures $(\mathcal{P}_g(\mathbb{T}\times\mathbb{R}), W_{2,g})$, so that the Kuramoto–Sakaguchi equation can be regarded as a $W_{2,g}$-gradient flow. Our goal is to make such argument rigorous and apply it to more general models in the literature, where the main obstructions to apply the classical theory of gradient flows in the presence of some type of heterogeneity in the model. This is a joint project with J. Morales (CSCAMM, College Park) and J. Peszek (IMPAN, Warsaw).

Let us recall that, in Section 7.2, we proposed a new mathematical model to explain the propagation of morphogene in Drosophila melanogaster. It consists in a microscopic coupled description for the orientation mechanism of cytonemes and the transportation of proteins along such signaling filopodia. We are interested in achieving a macroscopic approximation of the model using the scaling limits introduced in Chapter 1, i.e., mean field and hydrodynamic limits. Notice that in this problem there are two different objects to look at: filaments and morphogene. First, we are interested in obtaining the mean-field limit of the system of interacting filaments as the amount of them tends to infinity. We emphasize that, to the best of our knowledge, there are only two related results that have recently raised in that literature [30, 31]. In those works, the authors tackled the particular case of first order dynamics governed either by regular forcing terms or by Euler-type interactions described by a mollified Biot-Savart kernel. Our case is second order dynamics with singular interactions and, in addition, dynamics is coupled to morphogene propagation. Second, after the mean field description of cytonemes is achieved, we would like to accordingly obtain a unique equation for morphogene propagation, that must be subordinated to the above kinetic description of filopodia. Notice that the interest of such problems is not only from a mathematical point of view, but it also has important implications in biology in order to describe the formation and evolution of morphogene concentration gradients from first principles. This is a joint proyect with Manuel Cambón, Juan Soler (Universidad de Granada), Adrián Aguirre-Tamaral and Isabel Guerrero (CBMSO, Madrid).

Finally, but not less challenging, let us mention another project closely related to the preceding one that has emerged during a research stay of the candidate in CSCAMM (University of Maryland, USA), under the supervision of Prof. Pierre-Emanuel Jabin. The heart of the matter is the study of mean-field methods and nonstandard propagation of chaos for many-particle models with new effects in the dynamics. We are specially interested in the particular type of dynamics that gave rise to the Kuramoto model with singular coupling weights in Chapters 3 and 4. Specifically, we propose the following system:

$$\begin{cases} \dfrac{dx_i}{dt} = \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} a_{ij}F(x_i, x_j), \\ \dfrac{da_{ij}}{dt} = G(x_i, x_j, a_{ij}), \\ x_i(0) = x_{i,0},\ a_{ij}(0) = a_{ij,0}, \end{cases}$$

for any $i, j = 1, \ldots, N$. Here, F describes the forcing term between particles and G determines the specific mechanisms for the adaptation of weights. It stands to reason that the second equation for the evolution of weights imposes a strong time-correlation on agents positions. This is the main reason why the standard propagation of chaos is not expected to work for this system. As mentioned in Subsection 1.1.2, these methods are very sensitive to eventual loss of the standard symmetries ans structures in the equation. In particular, notice that the standard mean-field approach using empirical measures fails. Indeed, let us denote the empirical measure of positions by

$$\mu_1^N(y_1) := \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i(t)}(y_1).$$

Then, it verifies the following transport equation in distributional sense

$$\frac{\partial \mu_1^N}{\partial t} + \operatorname{div}_{y_1}\left(\int_{\mathbb{R}^d} F(y_1, z)\frac{1}{N^2}\sum_{i,j=1}^{N} a_{ij}\, \delta_{x_i}(y_1)\delta_{x_j}(z)\, dz\right) = 0.$$

Obviously, the presence of weights prevents from writing such equation in closed form in terms of μ_1^N, like for the simpler agent-based systems in Chapter 1. This suggests that we should define the following (generalized) empirical measure of two particles

$$\mu_2^N(y_1, y_2) := \frac{1}{N^2}\sum_{i,j=1}^{N} a_{ij}\delta_{x_i}(y_1)\delta_{x_j}(y_2),$$

and restate the above equation as follows

$$\frac{\partial \mu_1^N}{\partial t} + \operatorname{div}_{y_1}\left(\int_{\mathbb{R}^d} F(y_1, z)\mu_2^N(y_1, z)\, dz\right) = 0.$$

We emphasize that it is not a closed equation in terms of μ_1^N. By looking at the equation of μ_2^N, we observe that we require two new (generalized) empirical measures of three particles. It is apparent that, by repeating this process, we end up with an infinite family of generalized empirical measures that can be constructed in a recursive way. On the one hand, notice that passing to the limit in the whole hierarchy as $N \to \infty$ is not an appropriate description for the macroscopic dynamics as it depends on an infinite hierarchy of coupled PDEs. Our intuition is that those terms of the hierarchy must be finite-dimensional projections of the real (probably infinite-dimensional) object that describes the macroscopic dynamics. The objective of this project is to explore such idea and describe conditions on weights so that we can find such an infinite-dimensional representation of the dynamics. Indeed, a extremely important related question is to compute the order of complexity during the approximation of the real complex macroscopic dynamics by that simpler dynamics of N particles. Such a question has a clear answer for standard systems without weights and it represents a nice way to compare the increase of complexity implied by the inclusion of adaptive coupling weights. This is a joint project with Pierre-Emmanuel Jabin (CSCAMM, College Park) and Juan Soler (University of Granada).

APPENDIX A

Lebesgue–Bochner-type spaces and duality

In this appendix, we recall some notation and basic concepts that are used systematically along this thesis regarding Banach-valued L^p-type spaces and its duals representability. Specifically, we shall first recall the concept of *Bochner integral* of Banach-valued functions and *Lebesgue–Bochner spaces* $L^p(0,T;X)$ for any Banach space X. The main goal here is to recall the corresponding representability of their topological dual spaces $L^p(0,T;X)^*$. This is an intriguing topic where one has to be specially careful as the standard representability through Riesz-type theorems (that is, $L^p(0,T;X)^* = L^{p'}(0,T;X^*)$) is governed by the well known *Radon–Nikodym property (RNP)* of the topological dual X^* (that is known to fulfil if X is reflexive or X^* is separable). Hence, we shall introduce here two different branches of result that can be applied to substantially different situations: namely, the classical result for spaces with the RNP and the *Dinculeanu–Foias theorem* when the RNP is failing. Of course, the space representing the topological dual of $L^p(0,T;X)$ will be much more exotic if RNP is missing and, to that end, we shall introduce the *weak-* Bochner-Lebesgue spaces* $L^{p'}_w(0,T;X^*)$.

Definition A.0.1 ([105]). *Let X be a Banach space and consider $f:[0,T]\longrightarrow X$.*

1. *f is called simple if there exists $x_1,\ldots,x_k\in X$ and measurable subsets $E_1,\ldots,E_k\subseteq[0,T]$ such that f can be expressed as the piece-wise constant function*

$$f(t)=\sum_{i=1}^{k}\chi_{E_i}(t)x_i,\ \textit{ for all }\ t\in[0,T].$$

2. *f is called (strongly) measurable if there is a sequence $\{f_n\}_{n\in\mathbb{N}}$ of simple functions so that*

$$\lim_{n\to\infty}\|f_n(t)-f(t)\|_X=0,\ \textit{for a.e. } t\in[0,T].$$

3. *f is called weakly measurable if the following function is measurable*

$$t\in[0,T]\longmapsto\langle x^*,f(t)\rangle\,,$$

for any $x^*\in X^*$.

4. *Assume that $X = Y^*$ is the dual space of some Banach space Y. Then, f is called weak-* measurable if the following function is measurable*

$$t \in [0,T] \longrightarrow \langle f(t), y \rangle\,,$$

for any $y \in Y$.

Of course, measurable functions are weakly measurable, but the reverse statement is generally false. Indeed, the following theorem clarifies the relation between both definitions.

Theorem A.0.2 (Pettis' measurability theorem, [105]). *Let X be a Banach space and consider a Banach-valued function $f : [0,T] \longrightarrow X$. Then, f is measurable if, and only if, the following conditions hold true:*

1. *f is weakly measurable.*
2. *f is essentially separably valued, i.e., there exists a negligible set $N \subseteq [0,T]$ such that $f([0,T]\backslash N)$ is a (norm) separable subset of X.*

In particular, when X is a separable Banach space the second condition trivially fulfils and, consequently, measurable and weakly measurable functions agree. For the dual case

$$\text{measurable} \quad \Longrightarrow \quad \text{weakly measurable} \quad \Longrightarrow \quad \text{weak-* measurable},$$

but none of the reverse is necessarily true as depicted in the examples in [105, p. 43]. Once this concepts are sets, one is able to extend the notion of Lebesgue integral of scalar functions to a Banach-valued version via the use of simple functions. In that way we arrive at the concept of Bochner integrable function and Bochner integral.

Definition A.0.3 (Bochner integral [105]). *Let X be a Banach space and consider a Banach-valued function and consider a simple function g like in Definition A.0.1 for associated to $E_1, \ldots, E_k \subseteq [0,T]$ pairwise disjoint measurable subsets and $x_1, \ldots, x_k \in X$. We define its Bochner integral via the formula*

$$\int_{[0,T]} g(t)\,dt = \sum_{i=1}^{k} |E_i| x_i.$$

For a general measurable function $f : [0,T] \longrightarrow X$, we will say that it is Bochner integrable if the following scalar function $t \in [0,T] \longmapsto \|f(t)\|_X$ is Lebesgue-integrable. Equivalently, there is a sequence of simple functions $\{f_n\}_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} \int_{[0,T]} \|f_n(t) - f(t)\|_X\,dt = 0.$$

Its Bochner integral is then defined by

$$\int_{[0,T]} f(t)\,dt = \lim_{n\to\infty} \int_{[0,T]} f_n(t)\,dt.$$

Bochner integral allows extending most of the ideas in classical measure theory for scalar functions (dominated convergence theorem, absolute continuity of integral, Lebesgue points theorem, etc). More interestingly, one can define the analogue of Lebesgue spaces.

Definition A.0.4 (Lebesgue–Bochner spaces [105]). *Consider a Banach space X. We define*

$$L^p(0,T;X) := \{f : [0,T] \longrightarrow X : f \text{ is measurable and } \|f\|_X \in L^p(0,T)\},$$

for any $1 \leq p \leq \infty$. In this way, $L^p(0,T;X)$ becomes a Banach endowed with the norms

$$\|f\|_{L^p(0,T;X)} = \|\|f\|_X\|_{L^p(0,T)}.$$

Notice that as it is the case for the classical Lebesgue spaces of scalar functions, we always identify functions that agree almost everywhere in $[0,T]$, that is, we quotient such space by the relation

$$f \sim g \Longleftrightarrow f(t) = g(t) \text{ for almost every } t \in [0,T].$$

Our next goal is to recall the duality properties in Lebesgue–Bochner spaces.

Lemma A.0.5 (Isometrical embedding [105]). *Let X be a Banach space, $1 \leq p < \infty$ and define*

$$\begin{array}{rccc} \Phi_p : & L^{p'}(0,T;X^*) & \longrightarrow & L^p(0,T;X)^*, \\ & f & \longmapsto & \Phi_p[f], \end{array} \qquad \langle \Phi_p[f], g\rangle := \int_{[0,T]} \langle f(t), g(t)\rangle \, dt,$$

for any $g \in L^p(0,T;X)$. Then Φ_p is a linear isometry.

The above result allows embedding $L^{p'}(0,T;X^*)$ isometrically into $L^p(0,T;X^*)$. However, it is not clear whether Φ_p is onto like in the scalar case. This is the content of the following result.

Theorem A.0.6 (Riesz representation [36, 105]). *Let X be a Banach space. Then, Φ_p is surjective if, and only if, X^* verifies the Radon–Nikodym property (RNP) with respect to Lebesgue measure in $[0,T]$.*

The spaces characterizing the above representability property were firstly called *Gelf'and spaces* and can be characterized as those spaces so that any absolutely continuous functions $f : [0,T] \longrightarrow X$ are differentiable almost everywhere, see [105, Definition IV.3.1]. Later, RNP was proved as the key point for Banach-valued meausures to verify the Radon-Nikodym theorem and both concepts were unified. Specifically, Gelf'and spaces agree with the spaces that verify the RNP with respect to Lebesgue space, see [105, Theorem IV.3.2]. In addition, as shown in [105, Corollary V.3.8], a space verifies RNP with respect to Lebesgue measure in $[0,T]$ if, and only if, it verifies RNP with respect to every finite measure space. As a consequence, we simply say RNP independently on the fixed measure space. Several classical results attempt to characterize the spaces with RNP but a comprehensive classification has not been achieved so far. The most important examples of such Banach spaces were found by Philips, Dunford and Pettis and can be summarized in the following result, [105, Corollary III.2.13 and Theorem III.3.1].

Proposition A.0.7. *Let X be a Banach space:*

1. *(Philips) If X is reflexive, then X has the RNP.*

2. *(Dunford-Pettis) If $X = Y^*$ is a separable dual, then X has the RNP.*

Remark A.0.8. *In this thesis, we are interested in applying the duality result in Theorem B.0.1 to several situations in order to endow the corresponding Lebesgue-Bochner space with a weak-* topology so that weak-* compactness can be derived from Alaouglu-Bourbaki theorem. Let us illustrate a few examples:*

1. *If $X = L^q(\mathbb{R}^d)$ with $1 < q < \infty$, then reflexivity guarantees*

$$L^{p'}(0,T;L^{q'}(\mathbb{R}^d)) \equiv L^p(0,T;L^q(\mathbb{R}^d))^*,$$

for any $1 \leq p < \infty$. Similar comments follow for $X = W^{k,q}(\mathbb{R}^d)$ with $k \in \mathbb{N}$ and $1 < q < \infty$.

2. *If $X = C_0(\mathbb{R}^d)$, then $X^* = \mathcal{M}(\mathbb{R}^d)$ is the space of finite Radon measures in $\mathbb{R}^d$. Of course, reflexivity is not true and first criterion by Philips does not work. Also, notice that the map*

$$\begin{array}{rccc} \Phi : & \mathbb{R}^d & \longrightarrow & \mathcal{M}(\mathbb{R}^d), \\ & x & \longmapsto & \delta_x, \end{array}$$

allows embedding $\mathbb{R}^d$ as an uncountable and discrete subset of $\mathcal{M}(\mathbb{R}^d)$. Of course, this prevents $\mathcal{M}(\mathbb{R}^d)$ from being separable and, in turns, we cannot expect that such space verifies RNP.

The second example in last Remark shows that the classical representation in Theorem B.0.1 fails for $L^p(0,T;C_0(\mathbb{R}^d))$ and one has to find an alternative representation for its dual space. Although this idea was explored in the seminal paper [36], the authors did not account for a simple representation in terms of nice spaces, as they based their results in the theory of vector-valued measures. However, another approach was later obtained through the existence of lifting, see [173, 174]. To that end, we shall introduce the following short of *weak-* Lebesgue-Bochner spaces*.

Definition A.0.9 ([124, 173, 174, 239]). *Consider a Banach space X. We will define*

$$L^p_w(0,T;X^*) := \left\{ f : [0,T] \longrightarrow X^* : \begin{array}{c} \langle f, x\rangle \in L^p(0,T) \text{ for all } x \in X, \\ \text{and } \sup\limits_{\|x\|_X \leq 1} \|\langle f, x\rangle\|_{L^p(0,T)} < \infty, \end{array} \right\}$$

for any $1 \leq p \leq \infty$. Notice that the above just implies that f is weak- measurable but not (strongly) measurable. In this way, $L^p_w(0,T;X^*)$ becomes a Banach space endowed with the norm*

$$\|f\|_{L^p_w(0,T;X^*)} = \sup_{\|x\|_X \leq 1} \|\langle f, x\rangle\|_{L^p(0,T)}.$$

Again, we actually identify it with its quotient by another (different) relation

$$f \approx g \Longleftrightarrow \langle f(t), x\rangle = \langle g(t), x\rangle \text{ a.e. } t \in [0,T], \text{ for any } x \in X.$$

Notice that for $\approx$, the negligible subset of $[0,T]$ depends on $x \in X$, as opposed to $\sim$ in Definition A.0.4.

Notice that, by definition, the weak-* Lebesgue–Bochner spaces are larger than their stronger versions, i.e.,

$$L^p(0,T;X^*) \subseteq L^p_w(0,T;X^*),$$

although the identity is not necessarily true. See more details about this relation in the following remark.

Remark A.0.10. 1. *When X is separable, we can span the unit ball B_X with a countable subset and prove that relations $\sim$ and $\approx$ do agree. In addition, for any $f \in L^p_w(0,T;X^*)$, $\|f\|_X \in L^p$ and*

$$\|f\|_{L^p_w(0,T;X^*)} = \|\|f\|_X\|_{L^p(0,T)}.$$

Nevertheless, it does not amount to saying that $f \in L^p(0,T;X^*)$ because f is merely weak- measurable, but not (strongly) measurable.*

2. *However, when X^* is separable not only we recover all the above (notice that it implies that X is separable), bur also, a similar result to Pettis measurability theorem like in A.0.2 ensures that weak-* and strong measurability agree and we can them conclude that*

$$L^p(0,T;X^*) = L^p_w(0,T;X^*),$$

with equality of norms $\|\cdot\|_{L^p(0,T;X^*)} = \|\cdot\|_{L^p_w(0,T;X^*)}$.

These spaces have proved useful to complete the representability properties of the dual space of Lebesgue–Bochner spaces when X^* fails the RNP. This is the main result in this part, whose proof can be found in [173] and [174, p. 95 and 99] and a more simplified exposition has been given in [239, Theorem 10.1.16] [124, Theorems 12.2.11 and 12.9.2].

Theorem A.0.11. *Let X be a Banach space, set $1 \leq p < \infty$ and consider the mapping*

$$\begin{array}{rccc} \widetilde{\Phi}_p: & L^{p'}_w(0,T;X^*) & \longrightarrow & L^p(0,T;X)^*, \\ & f & \longmapsto & \widetilde{\Phi}_p[f], \end{array} \qquad \left\langle \widetilde{\Phi}_p[f], g \right\rangle = \int_{[0,T]} \langle f(t), g(t)\rangle \, dt,$$

for any $g \in L^p(0,T;X)$. Then, $\widetilde{\Phi_p}$ is a surjective isometry.

Corollary A.0.12. *Consider any $1 \leq p < \infty$. Then,*

$$L^{p'}_w(0,T;\mathcal{M}(\mathbb{R}^d)) \equiv L^p(0,T;C_0(\mathbb{R}^d))^*.$$

Finally, observe that the above ideas can be actually extended to Sobolev-type spaces Bochner and weak Bochner spaces.

Definition A.0.13 (Sobolev–Bochner spaces)**.** *Consider a Banach space X. We define*

$$W^{1,p}(0,T;X) := \left\{ f:[0,T] \longrightarrow X : \ f \in L^p(0,T;X) \ \text{and} \ f' \in L^p(0,T;X) \right\},$$

for any $1 \leq p \leq \infty$, where f' denotes the distributional derivative in the space of Banach-valued distributions $\mathcal{D}'((0,T),X)$. In this way, the vector space $W^{1,p}(0,T;X)$ becomes a Banach spaces endowed with the norms

$$\|f\|_{W^{1,p}(0,T;X)} := \left(\|f\|^p_{L^p(0,T;X)} + \|f'\|^p_{L^p(0,T;X)} \right)^{\frac{1}{p}}.$$

Definition A.0.14 (Weak-* Sobolev–Bochner spaces)**.** *Consider a Banach space X. We define*

$$W^{1,p}_w(0,T;X^*) := \left\{ f:[0,T] \longrightarrow X^* : \begin{array}{c} \langle f, x\rangle \in W^{1,p}(0,T) \ \text{for all} \ x \in X, \\ \text{and} \ \sup_{\|x\|_X \leq 1} \|\langle f, x\rangle\|_{W^{1,p}(0,T)} < \infty, \end{array} \right\}$$

for any $1 \leq p \leq \infty$. Again, notice that it does not imply that f and the distributional derivative f' are (strongly) measurable, but just weak- measurable. In this way, the vector space $W^{1,p}_w(0,T;X^*)$ becomes a Banach spaces endowed with the norms*

$$\|f\|_{W^{1,p}(0,T;X)} := \sup_{\|x\|_X \leq 1} \|\langle f, x\rangle\|_{W^{1,p}(0,T)}.$$

Again, by definition we have the relation

$$W^{1,p}(0,T;X^*) \subseteq W^{1,p}_w(0,T;X^*),$$

and similar comments to those in Remark A.0.10 follow. Since we will not use it in this thesis, we will not address here the duality properties of the Sobolev–Bochner and their weak versions.

APPENDIX B

Weak versions of the Banach-valued Ascoli–Arzelà theorem

In this appendix, we will continue the lines in the preceding part and we shall now recall some weak version of the Ascoli-Arzelà theorem for Banach-valued functions. Although the proof is often referred as folklore, we will give a simple derivation supported by classical arguments inspired in Cantor diagonalization method as in the standard proof of the scalar case of Ascoli–Arzelà theorem. The main idea is to replace Weierstrass theorem in the scalar case, by other more sophisticated (weak or weak-*) compactness results originated in the classical Alaoglu-Bourbaki theorem.

Theorem B.0.1. *Let X be a Banach space and consider the associated space $C([0,T],X^*)$, where X^* represents the topological dual of X endowed with its dual norm. Consider some sequence $\{f_n\}_{n\in\mathbb{N}}$ in $C([0,T],X^*)$ and assume that the following condition are fulfilled*

$$(\textit{Uniform boundedness})\quad \sup_{n\in\mathbb{N}}\max_{t\in[0,T]}\|f_n(t)\|_{X^*}=:M<+\infty, \tag{B.0.1}$$

$$(\textit{Uniform equicontinuity})\quad \forall\,\varepsilon>0,\,\exists\,\delta>0\ \ s.t.\ \sup_{n\in\mathbb{N}}\max_{|t_1-t_2|<\delta}\|f_n(t_1)-f_n(t_2)\|_{X^*}\leq\varepsilon. \tag{B.0.2}$$

Then, there is a subsequence $\{f_{\sigma(n)}\}_{n\in\mathbb{N}}$ and some limiting $f\in C([0,T],X^*)$ such that $f_{\sigma(n)}\to f$ in $C([0,T],X^*-weak*)$, that is to say,

$$\lim_{n\to\infty}\max_{t\in[0,T]}|\left\langle f_{\sigma(n)}(t)-f(t),x\right\rangle|=0,\ \forall x\in X.$$

Here $\langle\cdot,\cdot\rangle$ represents the duality pairing of the Banach space X and its topological dual X^*.

Proof. Let us consider the intervals

$$I_n(t):=\left(t-\frac{1}{n},t+\frac{1}{n}\right)\cap[0,T],$$

for any $t\in[0,T]$ and each $n\in\mathbb{N}$ and notice that $\{I_n(t)\,:\,t\in[0,T]\}$ defines an open covering of $[0,T]$ for each fixed $n\in\mathbb{N}$. By compactness, let us select some finite subset $S_n\subseteq[0,T]$ so that $\{I_n(t)\,:\,t\in S_n\}$ is still a covering for each fixed $n\in\mathbb{N}$ and define the countable set $S:=\cup_{n\in\mathbb{N}}S_n$. Since it is countable we can enumerate all their elements

$$S=\{s_i:i\in\mathbb{N}\}.$$

• *Step 1*: Cantor diagonal argument.

For $t = s_1$ (the first item) notice that $\{f_n(s_1)\}_{n\in\mathbb{N}}$ is a bounded sequence of X^* by virtue of (B.0.1). Then, the Alaouglu-Bourbaki theorem allows extracting a subsequence $\{f_{\sigma_1(n)}(s_1)\}_{n\in\mathbb{N}}$ that converges weakly * in X^*. For $t = s_2$ (the second item) notice that $\{f_{\sigma_1(n)}(s_2)\}_{n\in\mathbb{N}}$ is again a bounded sequence of X^* and the Alaouglu-Bourbaki theorem allows finding a new subsequence $\{f_{\sigma_2(n)}(s_2)\}_{n\in\mathbb{N}}$ that converges weakly *. Obviously, $\{f_{\sigma_2(n)}(s_1)\}_{n\in\mathbb{N}}$ also converges weakly * to the above limit in the preceding step by construction. By induction, we obtain a family of nested subsequences $\{f_{\sigma_k(n)}\}$ that verify that $\{f_{\sigma_k(n)}(s_i)\}_{n\in\mathbb{N}}$ weakly * converges for every $i = 1, \ldots, k$ and each $k \in \mathbb{N}$. Consider the Cantor diagonal subsequence corresponding to the choice $\sigma(n) := \sigma_n(n)$, that is, $\{f_{\sigma(n)}\}_{n\in\mathbb{N}} \equiv \{f_{\sigma_n(n)}\}_{n\in\mathbb{N}}$ and notice that all the above imply that

$$\{f_{\sigma(n)}(s_i)\}_{n\in\mathbb{N}} \text{ converges weakly } *, \tag{B.0.3}$$

for every $i \in \mathbb{N}$.

• *Step 2*: Weak-* Cauchy condition.

Let us now fix $x \in X\backslash\{0\}$ and $\varepsilon > 0$. Assumptions (B.0.2) imply that there exists $\delta = \delta_{x,\varepsilon} > 0$ so that the following inequality holds true

$$\|f_{\sigma(n)}(t_1) - f_{\sigma(n)}(t_2)\|_{X^*} \leq \frac{\varepsilon}{3\|x\|_{X^*}}, \tag{B.0.4}$$

for every $|t_1 - t_2| \leq \delta$ and any $n \in \mathbb{N}$. Consider $k = k_{x,\varepsilon} \in \mathbb{N}$ large enough so that $\frac{1}{k} < \delta$. Since S_k consists of finitely many terms, it is clear that the convergence (B.0.3) implies the existence of some $N = N_{x,\varepsilon} \in \mathbb{N}$ so that

$$|\left\langle f_{\sigma(n)}(s) - f_{\sigma(m)}(s), x\right\rangle| \leq \frac{\varepsilon}{3}, \tag{B.0.5}$$

for each $n, m \geq N$ and for each $s \in S_k$. Recall that $\{I_k(s) : s \in S_k\}$ is a finite open covering of $[0, T]$ and consider any $t \in [0, T]$. Then, we take $s_i \in S_k$ for some $i \in \mathbb{N}$ so that $t \in I_k(s_i)$, that is, $|t - s_i| < \frac{1}{k} < \delta$. Consequently for any $n, m \geq N$ we obtain

$$\begin{aligned}
&|\left\langle f_{\sigma(n)}(t) - f_{\sigma(m)}(t), x\right\rangle| \\
&\quad\leq |\left\langle f_{\sigma(n)}(t) - f_{\sigma(n)}(s_i), x\right\rangle| + |\left\langle f_{\sigma(n)}(s_i) - f_{\sigma(m)}(s_i), x\right\rangle| + |\left\langle f_{\sigma(m)}(s_i) - f_{\sigma(m)}(t), x\right\rangle| \\
&\quad\leq \frac{2\varepsilon}{3} + |\left\langle f_{\sigma(n)}(s_i) - f_{\sigma(m)}(s_i), x\right\rangle| \\
&\quad\leq \varepsilon,
\end{aligned}$$

where we have applied (B.0.4) in the third line and (B.0.5) in the last one. This amounts to the weak * Cauchy-type condition

$$\max_{t\in[0,T]} |\langle f_{\sigma_n}(t) - f_{\sigma_m}(t), x\rangle| \leq \varepsilon, \tag{B.0.6}$$

for all $n, m \geq N$.

• *Step 3*: Convergence in $C([0, T], X^* - \text{weak}*)$.

In particular (B.0.6) allows showing that $\{\left\langle f_{\sigma(n)}(t), x\right\rangle\}_{n\in\mathbb{N}}$ converges as $n \to \infty$ for every $t \in [0, T]$ and $x \in X$. Then, Banach–Steinhauss' theorem about uniform converges guarantees that for every $t \in [0, T]$ there exists $f(t) \in X^*$ so that

$$\{f_{\sigma(n)}(t)\}_{n\in\mathbb{N}} \overset{*}{\rightharpoonup} f(t) \text{ in } X^*.$$

Taking limits as $m \to \infty$ in (B.0.6) implies

$$\max_{t\in[0,T]} |\left\langle f_{\sigma(n)}(t) - f(t), x \right\rangle| \leq \varepsilon,$$

for all $n \geq N$, what proves the convergence part of the thesis. The only that remains is to show the continuity of the limit $f \in C([0,T], X^*)$ but it clearly follows by taking limits $n \to \infty$ in the uniform equicontinuity condition (B.0.2). □

Corollary B.0.2. *Let X be a reflexive Banach space and consider the associated space $C([0,T], X)$. Consider some sequence $\{f_n\}_{n\in\mathbb{N}}$ in $C([0,T], X)$ and assume that the following condition are fulfilled*

$$(\textit{Uniform boundedness}) \quad \sup_{n\in\mathbb{N}} \max_{t\in[0,T]} \|f_n(t)\|_X =: M < +\infty, \tag{B.0.7}$$

$$(\textit{Uniform equicontinuity}) \quad \forall\, \varepsilon > 0,\ \exists\, \delta > 0 \ \ s.t. \ \sup_{n\in\mathbb{N}} \max_{|t_1-t_2|<\delta} \|f_n(t_1) - f_n(t_2)\|_X \leq \varepsilon. \tag{B.0.8}$$

Then, there is a subsequence $\{f_{\sigma(n)}\}_{n\in\mathbb{N}}$ and some limiting $f \in C([0,T], X)$ such that $f_{\sigma(n)} \to f$ in $C([0,T], X - weak)$, that is to say,

$$\lim_{n\to\infty} \max_{t\in[0,T]} |\left\langle x^*, f_{\sigma(n)}(t) - f(t) \right\rangle| = 0,\ \forall x^* \in X^*.$$

Here $\langle \cdot, \cdot \rangle$ represents the duality pairing of the Banach space X and its topological dual X^*.

Proof. Consider the canonical isometry of X into its bidual X^{**}

$$J : X \longrightarrow X^{**},$$

where $\langle J(x), x^* \rangle = \langle x^*, x \rangle$, for every $x \in X$ and $x^* \in X^*$. Recall that by reflexivity we can identify X-weak with X^{**}-weak*. Then, the corollary follows from a simple application of Theorem B.0.1 to the sequence $\{J(f_n)\} \in C([0,T], X^{**})$. □

APPENDIX C

The Hardy–Littlewood–Sobolev inequality

Since it is used along this thesis, we briefly recall the Hardy–Littlewood–Sobolev inequality for the reader's convenience. We split the classical result into three distinguished parts. Firstly, we recall the L^q integrability of fractional integrals $I_\beta f := |\cdot|^{-(d-\beta)} * f$ associated with a Riesz kernel $|x|^{-(d-\beta)}$ and a L^p density f. Secondly, we shall address the case of L^∞ bounds for $I_\beta f$. Finally, we address the fall-off of fractional integrals in terms of the corresponding decay of the corresponding densities.

Definition C.0.1 (Weak L^p spaces)**.** *For any $1 \leq p \leq \infty$ we define the weak L^p space as follows*

$$L^{p,\infty}(\mathbb{R}^d) := \{f : \mathbb{R}^d \longrightarrow \mathbb{R} : \ f \text{ is measurable and } \sup_{\lambda>0} \lambda\, \mu_f(\lambda)^{1/p} < \infty\},$$

where μ_f represents the distribution function of f, that is,

$$\mu_f(\lambda) = \left|\{x \in \mathbb{R}^d : \ |f(x)| > \lambda\}\right|, \ \text{ for all } \lambda \geq 0.$$

So defined, $L^{p,\infty}(\mathbb{R}^d)$ becomes a quasinormed space with the quasinorm

$$\|f\|_{L^{p,\infty}(\mathbb{R}^d)} := \sup_{\lambda>0} \mu_f(\lambda)^{1/p}.$$

Theorem C.0.2 (Hardy–Littlewood–Sobolev inequality)**.** *Consider any exponent $\beta \in (0, d)$ and set $1 \leq p < q < \infty$ so that the following identity fulfils*

$$\frac{1}{q} = \frac{1}{p} - \frac{\beta}{d}.$$

Then:

1. *If $p > 1$ then there exists $C > 0$ so that*

$$\|I_\beta f\|_{L^q(\mathbb{R}^d)} \leq C\|f\|_{L^p(\mathbb{R}^d)},$$

for every $f \in L^p(\mathbb{R}^d)$.

2. *If $p = 1$, then there exists $C > 0$ so that*

$$\|I_\beta f\|_{L^{q,\infty}(\mathbb{R}^d)} \leq C\|f\|_{L^1(\mathbb{R}^d)},$$

for every $f \in L^1(\mathbb{R}^d)$.

We refer to [278, Theorem 1.2.1] for the standard proof, that follows from *Marcinkiewicz' interpolation theorem*, see [279, Theorem 2.4]. Notice that in the above Theorem C.0.2, we must exclude $q = \infty$ the case. Hence, simple L^p integrability of f is not enough to ensure that the factional integral can be bounded. In the following result, we address such later case by requiring further assumptions on f.

Theorem C.0.3. *Consider any exponent $\beta \in (0, d)$ and set $1 \leq p < q \leq \infty$ so that*

$$\frac{1}{q} < \frac{\beta}{d} < \frac{1}{p}.$$

Then, there exists a constant $C > 0$ so that

$$\|I_\beta f\|_{L^\infty(\mathbb{R}^d)} \leq C\|f\|_{L^p(\mathbb{R}^d)}^{(\frac{\beta}{d}-\frac{1}{q})/(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^q(\mathbb{R}^d)}^{(\frac{1}{p}-\frac{\beta}{d})/(\frac{1}{p}-\frac{1}{q})},$$

for every $f \in L^p(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$.

We omit the proof and refer to [137, 278]. The above Theorem C.0.2 quantifies the decay of fractional integrals $I_\beta f$ in L^p spaces. However, it is possible to derive some pointwise decay of $I_\beta f$ from pointwise fall-off of f at infinity. This is the content of our last result.

Theorem C.0.4. *Consider any exponent $\beta \in (0, d)$ and set any measurable function $f : \mathbb{R}^N \longrightarrow \mathbb{R}$. Then, we have:*

1. *If $f = O(|x|^{-\rho})$ as $|x| \to +\infty$, where $\beta < \rho < d$ then,*

$$|I_\beta f(x)| \leq C\||x|^\rho f\|_{L^\infty(\mathbb{R}^d)}|x|^{-(\rho-\beta)},$$

holds for every $x \in \mathbb{R}^d$. Here, C stands for a positive constant that depends on d, β and ρ but does not depend on f.

2. *The optimal decay $|x|^{-(d-\beta)}$ is obtained in the compactly supported case, i.e.,*

$$|I_\beta(x)\| \leq C\|f\|_{L^\infty(\mathbb{R}^d)}|x|^{-(d-\beta)},$$

for every $x \in \mathbb{R}^d$, as long as $f \in L^\infty(\mathbb{R}^d)$ has compact support inside some ball $B_{R_0}(0)$. Now, not only does C depend on d and β but also on the size $R_0 > 0$ of the support.

Proof. • *Step 1.* First item.

Fix any constant $0 < R < 1$ (e.g., $R = 1/2$) and split the integral we are interested in into the next two parts

$$\int_{\mathbb{R}^d} \frac{1}{|x-y|^{d-\beta}} f(y)\,dy = I_1 + I_2,$$

where

$$I_1 = \int_{B_{R|x|}(0)} \frac{1}{|x-y|^{d-\beta}} f(y)\,dy, \qquad I_2 = \int_{B^c_{R|x|}} \frac{1}{|x-y|^{d-\beta}} f(y)\,dy.$$

In order to estimate I_1, notice that

$$y \in B_{R|x|}(0) \Longrightarrow |x-y| \geq (1-R)|x|.$$

Therefore, I_1 is bounded by

$$\begin{aligned}
\int_{B_{R|x|}(0)} \frac{1}{|x-y|^{d-\beta}} |f(y)|\, dy &\leq \frac{K}{(1-R)^{d-\beta}} \frac{1}{|x|^{d-\beta}} \int_{B_{R|x|}(0)} \frac{1}{|y|^{\rho}}\, dy \\
&= \frac{K\omega_d}{(1-R)^{d-\beta}} \frac{1}{|x|^d - \beta} \int_0^{R|x|} r^{d-1} \frac{1}{r^{\rho}}\, dr \\
&= \frac{K\omega_d}{d-\rho} \frac{R^{d-\rho}}{(1-R)^{d-\beta}} \frac{1}{|x|^{\rho-\beta}}.
\end{aligned}$$

Here $K := \||x|^{-\rho} f\|_{L^\infty(\mathbb{R}^d)}$ and ω_d stands for the $(d-1)$-dimensional area of the unit sphere in $\mathbb{R}^d$. It is worth remarking that we are dealing with finite integrals as a consequence of the hypothesis $\rho < d$. Similarly, the second integral, I_2, can also be split as follows

$$\begin{aligned}
&\int_{B_{R|x|}(0)^c} \frac{1}{|x-y|^{d-\beta}} |f(y)|\, dy \\
&\quad = \int_{B_{R|x|}(x) \setminus B_{R|x|}(0)} \frac{1}{|x-y|^{d-\beta}} |f(y)|\, dy + \int_{(B_{R|x|}(0) \cup B_{R|x|}(x))^c} \frac{1}{|x-y|^{d-\beta}} |f(y)|\, dy.
\end{aligned}$$

An analogous argument can be used to obtain the next upper bound of the first term

$$\begin{aligned}
&\int_{B_{R|x|}(x) \setminus B_{R|x|}(0)} \frac{1}{|x-y|^{d-\beta}} |f(y)|\, dy \\
&\qquad \leq \int_{B_{R|x|}(x)} \frac{1}{|x-y|^{d-\beta}} |f(y)|\, dy \leq K \int_{B_{R|x|}(x)} \frac{1}{|x-y|^{d-\beta}} \frac{1}{|y|^{\rho}}\, dy \\
&\qquad = K \int_{B_{R|x|}(0)} \frac{1}{|x-y|^{\rho}} \frac{1}{|y|^{d-\beta}}\, dy \leq \frac{K\omega_d}{\beta} \frac{R^{-\beta}}{(1-R)^{\rho}} \frac{1}{|x|^{\rho-\beta}}.
\end{aligned}$$

This time, integrals are finite due to the hypothesis $\beta < \rho$. Regarding the second term, let us decompose the integral into two parts once more. The appropriate subdomains to be considered are

$$\begin{aligned}
A &= \{y \in (B_{R|x|}(0) \cup B_{R|x|}(x))^c :\ |x-y| \leq |y|\}, \\
B &= \{y \in (B_{R|x|}(0) \cup B_{R|x|}(x))^c :\ |x-y| > |y|\}.
\end{aligned}$$

Let us complete the proof of the first inequality with the following estimates for the integrals over A and B, which follow from the same reasoning involving the hypothesis $\beta < \rho$:

$$\begin{aligned}
&\int_A \frac{1}{|x-y|^{d-\beta}} |f(y)|\, dy \\
&\quad \leq K \int_A \frac{1}{|x-y|^{d-\beta}} \frac{1}{|y|^{\rho}}\, dy \leq K \int_A \frac{1}{|x-y|^{d-\beta+\rho}}\, dy \leq K \int_{B_{R|x|}(x)^c} \frac{1}{|x-y|^{d-\beta+\rho}}\, dy \\
&\quad = K\omega_d \int_{R|x|}^{+\infty} r^{d-1} \frac{1}{r^{d-\beta+\rho}}\, dr = \frac{K\omega_d}{\rho-\beta} \frac{1}{R^{\rho-\beta}} \frac{1}{|x|^{\rho-\beta}}, \\
&\int_B \frac{1}{|x-y|^{d-\beta}} |f(y)|\, dy
\end{aligned}$$

$$\leq K\int_B \frac{1}{|x-y|^{d-\beta}}\frac{1}{|y|^\rho}\,dy \leq K\int_B \frac{1}{|y|^{d-\beta+\rho}}\,dy \leq K\int_{B_{R|x|}(0)^c} \frac{1}{|y|^{d-\beta+\rho}}\,dy$$
$$= K\omega_d \int_{R|x|}^{+\infty} r^{d-1}\frac{1}{r^{d-\beta+\rho}}\,dr = \frac{K\omega_d}{\rho-\beta}\frac{1}{R^{\rho-\beta}}\frac{1}{|x|^{\rho-\beta}}.$$

• *Step 2*. Second item.
Let us start with $|x| > 2R_0$, so that

$$\left|\left(\frac{1}{|x|^{d-\beta}} * f\right)(x)\right| \leq \int_{B_{R_0}(0)} \frac{1}{|x-y|^{d-\beta}}|f(y)|\,dy.$$

Notice that whenever $y \in B_{R_0}(0)$, then one has

$$|x-y| \geq |x|-|y| \geq |x|-R_0 = \left(1-\frac{R_0}{|x|}\right)|x| \geq \frac{1}{2}|x|.$$

Therefore

$$\left|\left(\frac{1}{|x|^{d-\beta}} * f\right)(x)\right| \leq \frac{2^{d-\beta}}{|x|^{d-\beta}}\|f\|_{L^1(\mathbb{R}^d)} \leq 2^{d-\beta}|B_{R_0}(0)|\frac{\|f\|_{L^\infty(\mathbb{R}^d)}}{|x|^{d-\beta}}.$$

The case $|x| \leq 2R_0$ is easier since

$$y \in B_{R_0}(0) \Longrightarrow |x-y| \leq |x|+|y| < 3R_0,$$

and consenquently, Young's inequality for the convolution of L^p functions leads to

$$\left|\left(\frac{1}{|x|^{d-\beta}} * f\right)(x)\right| \leq \int_{B_{3R_0}(x)} \frac{1}{|x-y|^{d-\beta}}|f(y)|\,dy = \int_{B_{3R_0}(0)} |f(x-y)|\frac{1}{|y|^{d-\beta}}\,dy$$
$$= |f| * \left(\chi_{B_{3R_0}(0)}\frac{1}{|x|^{d-\beta}}\right)(x) \leq \left\|\frac{1}{|x|^{d-\beta}}\right\|_{L^1(B_{3R_0}(0))}\|f\|_{L^\infty(\mathbb{R}^d)}$$
$$\leq (2R_0)^{d-\beta}\left\|\frac{1}{|x|^{d-\beta}}\right\|_{L^1(B_{3R_0}(0))}\frac{\|f\|_{L^\infty(\mathbb{R}^d)}}{|x|^{d-\beta}},$$

where $1 \leq \frac{2R_0}{|x|}$ has been used in the last inequality. □

APPENDIX D

Set-valued maps and Filippov existence theory

D.1 Filippov theory in the autonomous case

This appendix is devoted to summarize the main basis and tools in the so called *Filippov existence theory*. This theory is applicable to an *autonomous* ODE

$$\begin{cases} \dot{x} = F(x), \\ x(0) = x_0, \end{cases}$$

where the right hand side $F = F(x)$ is a discontinuous vector field. An analogue theory can be developed for *non-autonomous* right hand side $F = F(t,x)$ with a explicit dependence on time. For simplicity, we shall address first the time-independent case and will postpone the non-autonomous case to the following section. The heuristic idea to solve such system for a general discontinuous $F = F(x)$ is to replace the above dynamical system by an appropriate differential inclusion

$$\begin{cases} \dot{x} \in \mathcal{F}(x), \\ x(0) = x_0, \end{cases}$$

where $\mathcal{F} : \mathbb{R}^d \longrightarrow 2^{\mathbb{R}^d}$ is a set-valued map with nice properties that extends the single-valued map F in the sense

$$F(x) \in \mathcal{F}(x), \text{ for all } x \in \mathbb{R}^d.$$

For more information about this topic, we refer to the textbooks [14, 130]. Let us first introduce the necessary notation that will be used here on: $2^{\mathbb{R}^d}$ stands for the power set of $\mathbb{R}^d$, $|\mathcal{N}|$ for the Lebesgue measure of any measurable set $\mathcal{N} \subseteq \mathbb{R}^d$, $\mathrm{co}(A)$ is the convex hull of A and $\overline{\mathrm{co}}(A) = \overline{\mathrm{co}(A)}$ is its closure. For every convex set C we denote

$$m(C) := \{x \in C : |x| \leq |y| \text{ for all } y \in C\},$$

that is the element of minimal norm of C, i.e. $m(C) = \pi_C(0)$, where π_C is the orthogonal projection operator over the convex set C. The main ingredient will be the *Fillipov set-valued map* of *Filippov's convexification* of a given single-valued measurable map.

Definition D.1.1. *Let $F : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be any measurable map. The Filippov set-valued map $\mathcal{F} : \mathbb{R}^d \longrightarrow 2^{\mathbb{R}^d}$ is defined for any $x \in \mathbb{R}^d$ as follows*

$$\mathcal{F}(x) := \bigcap_{\delta>0} \bigcap_{|\mathcal{N}|=0} \overline{\mathrm{co}}\,(F(B_\delta(x) \setminus \mathcal{N}))\,.$$

Before we discuss why the Filippov set-valued map in Definition D.1.1 is well-behaved, let us recall some useful extension of standard definitions for single-valued functions case to the set-valued setting that will be used along this thesis.

Definition D.1.2. *Let $\mathcal{F} : \mathbb{R}^d \longrightarrow 2^{\mathbb{R}^d}$ be a set-valued map.*

1. ***(Lower and upper inverse)***
   The lower and upper inverses of $\mathcal{F}$ are respectively defined as follows

$$\mathcal{F}^-(B) := \{x \in \mathbb{R}^d :\, \mathcal{F}(x) \cap B \neq \emptyset\},$$
$$\mathcal{F}^+(B) := \{x \in \mathbb{R}^d :\, \mathcal{F}(x) \subseteq B\},$$

   for any subset $B \subseteq \mathbb{R}^d$.

2. ***(Lower and upper semicontinuity)***

   - *$\mathcal{F}$ is called lower semicontinuous when*

$$O \subseteq \mathbb{R}^d \text{ open } \implies \mathcal{F}^-(O) \text{ is open}.$$

     A sequencial characterization of such property is that for any sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^d$, any $x \in \mathbb{R}^d$ such that $x_n \to x$ and any $X \in \mathcal{F}(x)$ there exists $\{X_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^d$ with $X_n \to X$.

   - *$\mathcal{F}$ is called upper semicontinuous when*

$$O \subseteq \mathbb{R}^d \text{ open } \implies \mathcal{F}^+(O) \text{ is open}.$$

     A sequencial characterization of such property is that for any sequence $\{x_n\}_{n\in\mathbb{N}}$, any $x \in \mathbb{R}^d$ such that $x_n \to x$, and any open set $O \supseteq \mathcal{F}(x)$ there exists $n_0 \in \mathbb{N}$ such that $O \supseteq \mathcal{F}(x_n)$ for every $n \geq n_0$.

3. ***(Graph of a set-valued map)***
   The graph of the set-valued map $\mathcal{F}$ is defined as follows

$$\mathrm{Graph}(\mathcal{F}) := \{(x, X) :\, X \in \mathcal{F}(x),\, x \in \mathbb{R}^d\}.$$

   In particular, the graph of $\mathcal{F}$ is closed if, and only if, for any couple of sequences $\{x_n\}_{n\in\mathbb{N}}, \{X_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^d$ and any couple $x, X \in \mathbb{R}^d$ such that $X_n \in \mathcal{F}(x_n)$ for any $n \in \mathbb{N}$ and $x_n \to x$, we have that $X \in \mathcal{F}(x)$.

4. ***(One-sided Lipschitz-continuity)***
   The set-valued map $\mathcal{F}$ is called one-sided Lipschitz with constant $L > 0$ when for every couple $x, y \in \mathbb{R}^d$ and any $X \in \mathcal{F}(x)$, $Y \in \mathcal{F}(y)$ the following inequality fulfils

$$|(X - Y) \cdot (x - y)| \leq L|x - y|^2.$$

The main interest in considering such map can be summarized in the next couple of results, see for instance [14, Theorem 2.1.3, Theorem 2.1.4, Proposition 2.1.1].

Lemma D.1.3. *Let $F : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be any measurable map and consider its Filippov set-valued map $\mathcal{F}$. Then,*

1. *$\mathcal{F}(x)$ is a closed and convex set for every $x \in \mathbb{R}^d$.*
2. *$F(x) \in \mathcal{F}(x)$ for almost every $x \in \mathbb{R}^d$.*
3. *If F is continuous at $x \in \mathbb{R}^n$, then $\mathcal{F}(x) = \{F(x)\}$.*
4. *If $\mathcal{F}$ takes non-empty values, then $\mathcal{F}$ is has closed graph.*
5. *If $\mathcal{F}$ has closed graph and $m(\mathcal{F})(U_x)$ lies in a compact set for some neighborhood U_x of each $x \in \mathbb{R}^d$, then $\mathcal{F}$ is upper semicontinuous.*
6. *If F is locally essentially bounded, then $\mathcal{F}$ is upper semicontinuous, it takes non-empty values and $m(\mathcal{F})(U_x)$ lies in a compact set for some neighborhood U_x of each $x \in \mathbb{R}^d$.*
7. *If F is essentially bounded, then $\mathcal{F}$ is upper semicontinuous, it takes non-empty values and $m(\mathcal{F})(\mathbb{R}^d)$ lies in a compact set.*

Here $m(\mathcal{F})$ stands for the map $m(\mathcal{F})(x) := m(\mathcal{F}(x))$ for every $x \in \mathbb{R}^d$.

Lemma D.1.4. *Let $\mathcal{F} : \mathbb{R}^d \longrightarrow 2^{\mathbb{R}^d}$ be any set valued-map with non-empty closed and convex values. Assume that $\mathcal{F}$ is upper semicontinuous and consider the following initial value problem (IVP) associated with any given initial datum $x_0 \in \mathbb{R}^N$:*

$$\begin{cases} \dot{x} \in \mathcal{F}(x), \\ x(0) = x_0. \end{cases}$$

1. *If $m(\mathcal{F})(U_x)$ lies in a compact set for some neighborhood U_x of any $x \in \mathbb{R}^d$, then (IVP) has an absolutely continuous local-in-time solution.*
2. *If $m(\mathcal{F})(\mathbb{R}^d)$ lies in a compact set, then (IVP) has an absolutely continuous global-in-time solution.*

Putting together Lemmas D.1.3 and D.1.5 we arrive at the next result.

Lemma D.1.5. *Let $F : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be any measurable map and consider its Filippov set-valued map $\mathcal{F}$. Consider the following initial value problem (IVP) associated with any given initial datum $x_0 \in \mathbb{R}^d$:*

$$\begin{cases} \dot{x} \in \mathcal{F}(x), \\ x(0) = x_0. \end{cases}$$

1. *If F is locally essentially bounded, then (IVP) has an absolutely continuous local-in-time solution.*
2. *If, in addition, F is globally essentially bounded, then such a solution is indeed global.*

The solutions to such differential inclusion are called *solutions in Filippov's sense* to the original discontinuous dynamical system. To deal with uniqueness we first introduce the next technical result.

Lemma D.1.6. *Let $F : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be a measurable and locally essentially bounded map and consider its associated Filippov set-valued map $\mathcal{F} : \mathbb{R}^d \longrightarrow 2^{\mathbb{R}^d}$. If F verifies the one-sided Lipschitz-condition a.e., then $\mathcal{F}$ also verifies it in the set-valued sense, see Definition D.1.2.*

Proof. Consider any couple $x, y \in \mathbb{R}^d$ and fix $X \in \mathcal{F}(x), Y \in \mathcal{F}(y)$. Also fix any $\delta > 0$ (assume $\delta < 1$ without loss of generality) and any negligible set $\mathcal{N}$. Using the definition of $\mathcal{H}$, the following properties hold true

$$X \in \overline{\text{co}}\big(F(B_\delta(x) \setminus \mathcal{N})\big) \text{ and } Y \in \overline{\text{co}}\big(F(B_\delta(y) \setminus \mathcal{N})\big).$$

Then, one can take a couple of sequences $\{X_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^d$ and $\{Y_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^d$ such that $X_n \to X$, $Y_n \to Y$ and

$$X_n \in \text{co}\left(F(B_\delta(x) \setminus \mathcal{N})\right) \text{ and } Y_n \in \text{co}\left(F(B_\delta(y) \setminus \mathcal{N})\right),$$

for every $n \in \mathbb{N}$. Therefore, the Caratheodory theorem from convex analysis allows restating X_n and Y_n as a convex combination

$$X_n = \sum_{i=1}^{d+1} \alpha_i^n F(x_i^n) \quad \text{and} \quad Y_n = \sum_{j=1}^{d+1} \beta_j^n F(y_j^n),$$

where $x_i^n \in B_\delta(x) \setminus \mathcal{N}$, $y_j^n \in B_\delta(y) \setminus \mathcal{N}$ and the coefficients $\alpha_i^n, \beta_j^n \in [0,1]$ verify

$$\sum_{i=1}^{d+1} \alpha_i^n = 1 = \sum_{j=1}^{d+1} \beta_j^n.$$

Note that

$$X_n = \sum_{i=1}^{d+1}\sum_{j=1}^{d+1} \alpha_i^n \beta_j^n F(x_i^n) \quad \text{and} \quad Y_n = \sum_{i=1}^{d+1}\sum_{j=1}^{d+1} \alpha_i^n \beta_j^n F(y_j^n).$$

By defining the constants

$$M_x := \operatorname*{ess\,sup}_{z\in B_1(x)} |F(z)| \text{ and } M_y := \operatorname*{ess\,sup}_{z\in B_1(y)} |F(z)|,$$

we have

$$\begin{aligned}
(X_n - Y_n)\cdot(x-y) &= \Big(\sum_{i,j=1}^{d+1} \alpha_i^n \beta_j^n \big(F(x_i^n) - F(y_j^n)\big)\Big)\cdot(x-y)\\
&= \sum_{i,j=1}^{d+1} \alpha_i^n \beta_j^n \Big(\big(F(x_i^n) - F(y_j^n)\big)\cdot(x-y)\Big)\\
&= \sum_{i,j=1}^{d+1} \alpha_i^n \beta_j^n \Big(\big(F(x_i^n) - F(y_j^n)\big)\cdot(x_i^n - y_j^n)\\
&\qquad + \big(F(x_i^n) - F(y_j^n)\big)\cdot\big((x - x_i^n) - (y - y_j^n)\big)\Big)\\
&\le \sum_{i,j=1}^{d+1} \alpha_i^n \beta_j^n \Big(M|x_i^n - y_j^n|^2 + 2(M_x + M_y)\delta\Big)\\
&\le \sum_{i,j=1}^{d+1} \alpha_i^n \beta_j^n \Big(M(|x-y| + 2\delta)^2 + 2(M_x + M_y)\delta\Big)\\
&= M(|x-y| + 2\delta)^2 + 2(M_x + M_y)\delta.
\end{aligned}$$

Since the above property holds for arbitrary $n \in \mathbb{N}$ and $0 < \delta < 1$, we obtain

$$(X - Y)\cdot(x-y) \le M|x-y|^2.$$

□

Lemma D.1.7. *Let $F:\mathbb{R}^d\longrightarrow\mathbb{R}^d$ be a measurable and essentially bounded vector field and consider the Filippov set–valued map $\mathcal{F}:\mathbb{R}^d\longrightarrow 2^{\mathbb{R}^d}$. In addition, assume that F verifies the local one-sided Lipschitz condition. Then, the following initial value problem (IVP) associated with any initial configuration $x_0\in\mathbb{R}^N$ enjoys one global-in-time absolutely continuous solution, that is unique forwards in time*

$$\begin{cases}\dot{x}\in\mathcal{F}(x),\\ x(0)=x_0.\end{cases}$$

Proof. The existence of global-in-time Filippov's solutions follows from Lemma D.1.5. Let us just discuss the uniqueness of solution. We consider two Filippov solutions $x_1=x_1(t)$ and $x_2=x_2(t)$ with the same initial datum x_0 and define

$$T:=\inf\{t>0:x_1(t)\neq x_2(t)\}.$$

Our main goal is to prove that $T=+\infty$ by contradiction. We assume that $T<+\infty$. Let us define $x^*:=x_1(T)=x_2(T)$ and take a small enough neighborhood $\mathcal{V}$ of x^* so that F verifies the one-sided Lipschitz condition in it. By continuity there is some $\varepsilon>0$ so that $x_1(t),x_2(t)\in\mathcal{V}$ for every $t\in[T,T+\varepsilon]$. Consequently,

$$\frac{d}{dt}\frac{1}{2}|x_1-x_2|^2\in(\mathcal{F}(x_1(t))-\mathcal{F}(x_2(t)))\cdot(x_1(t)-x_2(t)).$$

By the one-sided Lipschitz condition, there exists some constant M depending on x^* such that

$$\frac{d}{dt}|x_1-x_2|^2\leq M|x_1-x_2|^2$$

for every $t\in[T,T+\varepsilon]$. By Grönwall's lemma, one then obtains $x_1(t)=x_2(t)$, for every $t\in[T,T+\varepsilon]$, and this contradicts the assumption on $T<+\infty$. □

D.2 Some remarks in the non-autonomous case

In Chapter 4, we need no apply a similar Filippov theory for non-autonomous vector fields. The following Lemmas summarize the main results regarding the existence of absolutely continuous solution to differential inclusions, see [14, 130, 205, 249].

Lemma D.2.1. *Let $F:\mathbb{R}_0^+\times\mathbb{R}^d\longrightarrow\mathbb{R}^d$ be any measurable map. Assume that for every compact subset $K\subseteq\mathbb{R}_0^+\times\mathbb{R}^d$ there exists a nonnegative function $m_K\in L^1_{loc}(\mathbb{R}_0^+)$ such that*

$$|F(t,x)|\leq m_K(t),\ \forall\,(t,x)\in K. \tag{D.2.1}$$

Consider its associated Filippov set-valued map with respect to the variable x, namely,

$$\mathcal{K}[F(t,\cdot)](x)=\bigcap_{\delta>0}\bigcap_{|\mathcal{N}|=0}\overline{\text{co}}(F(t,B_\delta(x)\setminus\mathcal{N})), \tag{D.2.2}$$

for all $(t,x)\in\mathbb{R}_0^+\times\mathbb{R}^N$. Then, it satisfies the following properties:

1. *$\mathcal{K}[F(t,\cdot)](x)$ is non-empty, convex and compact for a.e. $t\geq0$ and every $x\in\mathbb{R}^d$.*
2. *The set-valued map $t\in\mathbb{R}_0^+\longmapsto\mathcal{K}[F(t,\cdot)](x)$ is Effros-measurable for every $x\in\mathbb{R}^d$.*
3. *$\mathcal{K}[F(t,\cdot)]$ is upper semicontinuous for a.e. $t\geq0$.*

4. *For every compact subset* $K \subseteq \mathbb{R}_0^+ \times \mathbb{R}^d$ *and the above* $m_K \in L^1_{loc}(\mathbb{R}_0^+)$ *one has*

$$|m(\mathcal{K}[F(t,\cdot)](x)| \leq m_K(t),\ \forall\,(t,x) \in K. \tag{D.2.3}$$

Here, $m(C) = \pi_C(0)$ *and* π_C *is the orthogonal projection operator onto the convex set* C*. In other words, the above condition* (D.2.3) *equivalently reads*

$$\mathrm{dist}(0, \mathcal{K}[F(t,\cdot)](x))) \leq m_K(t),\ \forall\,(t,x) \in K.$$

The proof follows the same ideas as Lemma D.1.1 for autonomous fields and we omit it. Our next result shows the appropriate conditions on the non-autonomous set valued map $\mathcal{F}$ that guarantee the existence of absolutely continuous solutions to the differential inclusion.

Lemma D.2.2. *Let* $\mathcal{F} : \mathbb{R}_0^+ \times \mathbb{R}^d \longrightarrow 2^{\mathbb{R}^d}$ *be any set-valued map. Assume that it fulfills the following assumptions:*

1. $\mathcal{F}$ *takes non-empty, convex and compact values.*
2. $\mathcal{F}(\cdot, x)$ *is Effros-measurable for every* $x \in \mathbb{R}^d$*.*
3. $\mathcal{F}(t,\cdot)$ *is upper semicontinuous for a.e.* $t \geq 0$*.*
4. *For every compact subset* $K \subseteq \mathbb{R}_0^+ \times \mathbb{R}^d$ *there exists a nonnegative* $m_K \in L^1_{loc}(\mathbb{R}_0^+)$ *such that*

$$|m(\mathcal{F}(t,x))| \leq m_K(t),\ \forall\,(t,x) \in K. \tag{D.2.4}$$

Consider the following initial value problem (IVP) issued at any initial datum $x_0 \in \mathbb{R}^d$

$$\begin{cases} \dot{x} \in \mathcal{F}(t,x), \\ x(0) = x_0. \end{cases}$$

Then, it has a local-in-time absolutely continuous solution. If the locally integably boundedness condition (D.2.4) *holds true globally, i.e. with* K *replaced with the whole* $\mathbb{R}_0^+ \times \mathbb{R}^d$*, then a global-in-time absolutely continuous solution exists.*

A similar result can be found in [14, Theorems 2.1.3, 2.1.4] when $\mathcal{F}$ satisfies a stronger assumption, namely, $\mathcal{F}$ is upper semicontinuous in the joint variables (t,x). However, for our purposes in Chapter 4 (where time upper semicontinuity is missing) such result does not longer apply. Fortunately, Lema D.2.2 provides a solution when there is not time upper semicontinuouty, see [130, Theorem 2.7.5] and [205, 249] for the detailed proofs. Note that such result becomes a literal translation to the multivalued case of the classical Caratheodory's existence theorem for single-valued dynamical system. Finally, putting Lemmas D.2.1 and D.2.2 together, we arrive at the next result (see [130, Theorem 2.7.8]).

Lemma D.2.3. *Let* $F : \mathbb{R}_0^+ \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ *be any measurable map and assume that the local integrably boundedness condition* (D.2.1) *is satisfied. Let* $\mathcal{K}[F(t,\cdot)]$ *be the Filippov set-valued map with respect to* x *according to* (D.2.2)*. Consider the following initial value problem (IVP) issued at any initial datum* $x_0 \in \mathbb{R}^d$

$$\begin{cases} \dot{x} \in \mathcal{K}[F(t,\cdot)](x), \\ x(0) = x_0. \end{cases}$$

Then, it has a local-in-time absolutely continuous solution. If the locally integably boundedness condition (D.2.1) *holds true globally, i.e. with* K *replaced with the whole* $\mathbb{R}_0^+ \times \mathbb{R}^d$*, then a global-in-time absolutely continuous solution exists.*

Similarly, solutions to such differential inclusion are also called *Filippov solutions* to the original discontinuous non-autonomous dynamical system associated with $F = F(t,x)$.

APPENDIX E

Measurable selections and Castaign representations of set-valued maps

In this part, we recall to the reader all the main concepts to deal with *measurable selections Castaign-type representations* of set-valued maps. For further details, we refer to [69, 197]. Such techniques prove useful tools, that we use in Chapter 3. In particular, it allows identifying Filippov solutions of systems with discontinuous right hand side (recall Appendix D) as singular limit of regularized systems when the convergence is very weak, but the structure of the Filippov set-valued map is more explicit and is given in terms of an appropriate H-representation. Compared to the results in the preceding Appendix D, the required regularity of the set-valued map is much lower. Indeed, the main concept is a set-valued adaptation of classical measurability condition for single-valued maps, namely, *Effros measurability*.

Definition E.0.1 (Effros measurability). *Consider $n, m \in \mathbb{N}$ and any set-valued map $\mathcal{F} : \mathbb{R}^n \longrightarrow 2^{\mathbb{R}^m}$. $\mathcal{F}$ is called Effros-measurable when the following property holds true (recall Definition D.1.2):*

$$O \subseteq \mathbb{R}^m \text{ open } \implies \mathcal{F}^-(O) \text{ is measurable.}$$

The following result constitutes the basis in the existence theory of measurable selections for set-valued maps. See [197] for the original proof or the textbook [69, Theorem III.6] for this and other related selection theorems.

Lemma E.0.2 (Kuratowski–Ryll–Nardzewski). *Consider $n, m \in \mathbb{N}$ and any set-valued map $\mathcal{F} : \mathbb{R}^n \longrightarrow 2^{\mathbb{R}^m}$ with values in the non-empty and closed subsets of $\mathbb{R}^m$. Assume that $\mathcal{F}$ is Effros-measurable. Then, $\mathcal{F}$ has a measurable selection, i.e., there exists a measurable function $F : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that*

$$F(x) \in \mathcal{F}(x), \text{ a.e. } x \in \mathbb{R}^n.$$

Sometimes, it is helpful to control how many of these single-valued measurable selections of the Effros-measurable set-valued map do we need in order to essentially have the whole set-valued map "represented" in some sense. This is the content of an intimately related result: the *Castaing representation theorem*, see [69, Theorem III.30].

Lemma E.0.3 (Castaing). *Consider any $n, m \in \mathbb{N}$ and any set-valued map $\mathcal{F} : \mathbb{R}^n \longrightarrow 2^{\mathbb{R}^m}$ with values in the non-empty and closed subsets of $\mathbb{R}^m$. Assume that $\mathcal{F}$ is Effros-measurable. Then $\mathcal{F}$ has* *a Castaing representation, i.e., there exists a sequence $\{F^k\}_{k\in\mathbb{N}}$ of measurable maps $F^k:\mathbb{R}^n\longrightarrow\mathbb{R}^m$ such that*

$$\mathcal{F}(x)=\overline{\{F^k(x):\,k\in\mathbb{N}\}},\ \ \text{a.e. } x\in\mathbb{R}^n.$$

For its applications when the right hand side of the differential inclusion is not only discontinuous but also unbounded (see Subsection 3.4.2), we require an adaptation of the above theorem to allow for integrable representations of the set-valued map. The key observation is that the Effros-measurability has to be improved to some extra integrability condition for set-valued maps.

Lemma E.0.4. *Consider $n,m\in\mathbb{N}$ and any set-valued map $\mathcal{F}:\mathbb{R}^n\longrightarrow 2^{\mathbb{R}^m}$ with values in the non-empty and closed subsets of $\mathbb{R}^m$. Assume that $\mathcal{F}$ is Effros-measurable and strongly integrable, that is, the single-valued map $|\mathcal{F}|$ is integrable, where $|\mathcal{F}|$ is defined by*

$$|\mathcal{F}|(x):=\sup\{|X|:\,X\in\mathcal{F}(x)\},\ \ \text{a.e. } x\in\mathbb{R}^n.$$

Then, every measurable selection of $\mathcal{F}$ is integrable. In particular, $\mathcal{F}$ enjoys a Castaing representation consisting of integrable selections.

Proof. Let us take any measurable selection F of the set-valued $\mathcal{F}$, that exists by Lemma E.0.2. Then, by definition of $|\mathcal{F}|$ we obtain

$$|F(x)|\leq|\mathcal{F}|(x),\ \ \text{a.e. } x\in\mathbb{R}^n.$$

Since $|\mathcal{F}|$ is integrable, the first part of the result holds true. The second one is a simple consequence of the first one along with Lemma E.0.3. □

Remark E.0.5. *Notice that the same ideas as in the above result in Lemma E.0.4 also yield similar statements for the spaces $L^1_{loc}(\mathbb{R}^n)$ and $L^\infty(\mathbb{R}^n)$. Namely,*

1. *When $\mathcal{F}$ is locally strongly integrable, i.e., $|\mathcal{F}|\in L^1_{loc}(\mathbb{R}^n)$, then every measurable selection belongs to the space $L^1_{loc}(\mathbb{R}^n)$.*
2. *When $\mathcal{F}$ is strongly essentially bounded, i.e., $|\mathcal{F}|\in L^\infty(\mathbb{R}^n)$, then each measurable selection belongs to the space $L^\infty(\mathbb{R}^n)$.*

APPENDIX F

Optimal transport theory and Wasserstein distances

This appendix is devoted to introduce a brief presentation of the main tools of optimal transport theory that will be used along the thesis. For a detailed presentation of these topics and its applications we refer to the well known textbooks [9, 268, 296]. We split the contents into several distinguished sections. The first ones focus on the classical theory of optimal transport and classical Wasserstein distances in the space of probability measures whilst the last one introduces a new Wasserstein-type distance that we will call the fiberwise quadratic Wasserstein distance and will be of great use in Chapter 4, specifically in Section 4.4 to derive uniqueness of weak measure-valued solutions to (4.1.3) with $\alpha \in (0, \frac{1}{2})$ for generic initial data without further assumptions on the Ω-moments. Also, see Section 4.6 for analogue derivations in the critical regime $\alpha = \frac{1}{2}$.

F.1 The Monge–Kantorovich problem

Unless otherwise stated, in this thesis we shall restrict measurable spaces to the class of *Polish spaces*, that is, complete separable metric spaces $\mathcal{X}$ endowed with its Borel σ-algebra $\mathcal{B}(\mathcal{X})$.

Definition F.1.1 (Transference plans). *Let $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ and $(\mathcal{Y}, \mathcal{B}(\mathcal{Y}))$ be measurable spaces and consider two probability measures $\mu \in \mathcal{P}(\mathcal{X})$ and $\nu \in \mathcal{P}(\mathcal{Y})$. A transference plan (transport plan or coupling) for μ and ν is any probability measure $\gamma \in \mathcal{P}(\mathcal{X} \times \mathcal{Y})$ so that μ and ν are their marginals, that is,*

$$(\pi_x)_{\#}\gamma = \mu \ \text{ and } \ (\pi_y)_{\#}\gamma = \nu,$$

where π_x and π_y are the projections on $\mathcal{X}$ and $\mathcal{Y}$ respectively. We will denote the family of all those transference plans by $\Pi(\mu, \nu)$.

Definition F.1.2 (Transference plans and deterministic couplings). *Let $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ and $(\mathcal{Y}, \mathcal{B}(\mathcal{Y}))$ be measurable spaces and consider two probability measures $\mu \in \mathcal{P}(\mathcal{X})$ and $\nu \in \mathcal{P}(\mathcal{Y})$. A measurable map $T : \mathcal{X} \to \mathcal{Y}$ is a called a transference map (or transport map) if $T_{\#}\mu = \nu$. Any transference map has the associated transference plan $\gamma_T = (I, T)_{\#}\mu$, that is,*

$$\int_{\mathcal{X} \times \mathcal{Y}} \varphi(x, y)\, d_{(x,y)}\gamma_T = \int_{\mathcal{X}} \varphi(x, T(x))\, d_x\mu,$$

for any $\varphi \in C_b(\mathcal{X} \times \mathcal{Y})$. Those special transference plans are called deterministic coupling.

On the one hand, it is clear that transference plans always exist (just take the product measure $\mu\otimes\nu$). On the other hand, transference maps do not necessarily exist, but it depends on the properties of the measure μ. Indeed, the presence of atoms is fateful for them to exist. Notice that the transport map T associated with a deterministic coupling is defined μ-a.e. in $\mathcal{X}$ and, consequently we call it the transport map.

The main objective of the theory of optimal transport is to solve the *Monge–Kantorovich problem* associated with a cost functional $c : \mathcal{X} \times \mathcal{Y} \longrightarrow [0, +\infty]$. That is, given $\mu \in \mathcal{P}(\mathcal{X})$ and $\nu \in \mathcal{P}(\mathcal{Y})$, we wonder about the solvability of

$$\mathcal{C}(\mu,\nu) := \inf_{\gamma\in\Pi(\mu,\nu)} \int_{\mathcal{X}\times\mathcal{Y}} c(x,y)\, d_{(x,y)}\gamma. \tag{F.1.1}$$

Specifically, one wants to find a transference plan $\gamma \in \Pi(\mu,\nu)$ at which the optimal value of the transportation cost is achieved. Additionally, one may be interested in understanding when the solution is unique and whether it is a deterministic coupling for an appropriate transport map, that is, *Monge's problem*. In the following, we state the main results about the existence of solutions to both problems.

Theorem F.1.3 (Existence of optimal coupling)**.** *Let $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ and $(\mathcal{Y}, \mathcal{B}(\mathcal{Y}))$ be Polish spaces and consider two probability measures $\mu \in \mathcal{P}(\mathcal{X})$ and $\nu \in \mathcal{P}(\mathcal{Y})$. Consider a cost functional $c : \mathcal{X} \times \mathcal{Y} \longrightarrow (-\infty, +\infty]$ that is lower semicontinuous so that there exist a couple of functions $a : \mathcal{X} \longrightarrow [-\infty, +\infty)$ and $b : \mathcal{Y} \longrightarrow [-\infty, +\infty)$ with $a \in L^1(\mathcal{X}, \mu)$ and $b \in L^1(\mathcal{X}, \nu)$ and verifying*

$$c(x,y) \geq a(x) + b(x),$$

for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. Then, the Monge–Kantorovich problem (F.1.1) *has a solution.*

The proof is a clever application of *Prokhorov's compactness theorem* to the subspace of transference plans $\Pi(\mu,\nu)$ along with the lower semicontinuity with respect to the narrow topology that holds for the transportation functional associated to the cost c, see [296, Theorem 4.1]. Such abstract result is very often applied to nonnegative cost functionals that guarantee the lower bound condition by just taking $a, b = 0$, see [268, Theorem 1.7]. More specifically, a typical scenario is when $\mathcal{X} = \mathcal{Y}$ and $c = \frac{d^2}{2}$ is the squared distance of the Polish space $\mathcal{X}$. Indeed, in such particular case the solvability of the Monge–Kantorovich problem via transport map (Monge's problem) is easier to analyse. The classical result is due to Y. Brenier, that first solved it in the Euclidean space.

Theorem F.1.4. *Let $c(x,y) = \frac{1}{2}|x-y|^2$ in $\mathbb{R}^d$ and consider $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$ so that*

$$\int_{\mathbb{R}^d} |x|^2\, d\mu + \int_{\mathbb{R}^d} |x|^2\, d\nu < +\infty.$$

Assume that μ does not give mass to $d-1$ surfaces of class C^2. Then, the solution to the Monge–Kantorovich problem (F.1.3) *is unique and it is the deterministic coupling γ_T associated with a transport map of the form*

$$T(x) = \nabla u(x), \;\; \textit{for}\;\; \mu\textit{-a.e. } x \in \mathbb{R}^d,$$

for some convex and lower semicontinuous function $u : \mathbb{R}^d \longrightarrow \mathbb{R} \cup \{\pm\infty\}$.

See [268, Theorem 1.22] and [296, Theorem 9.4] for some more recent proofs. This result has several generalization to more general Polish spaces $\mathcal{X}$ and $\mathcal{Y}$ than the Euclidean space and more general cost functionals than the squared distance, see [296, Theorems 5.30, 10.28, 10.38 and Corollary 9.3]. For our purposes we can skip all those details and will simply state the analogue of Theorem F.1.4 in more general Riemannian manifolds, see [296, Theorem 10.41].

Theorem F.1.5. *Let $(M, \langle\cdot,\cdot\rangle)$ be a Riemannian manifold with nonnegative sectional curvature and consider its Riemannian distance $d(\cdot,\cdot)$. Consider the cost functional $c(x,y) = \frac{1}{2}d(x,y)^2$ and set $\mu,\nu \in \mathcal{P}(M)$ so that $\mathcal{C}(\mu,\nu) < +\infty$. Assume that μ is absolutely continuous with respect to the volume measure. Then the solution to the Monge–Kantovich problem is unique and it is the deterministic coupling γ_T associated with a transport map of the form*

$$T(x) = \exp_x(-\nabla\psi(x)), \text{ for } \mu\text{-a.e. } x \in M.$$

Here $\exp_x$ represents the exponential map of the Riemannian manifold and ψ is $\frac{d^2}{2}$-concave, that is,

$$\psi(x) = \inf_{y\in M}\left(\zeta(y) - \frac{1}{2}d(x,y)^2\right),$$

for some function ζ on M.

Notice that Theorem F.1.5 reduces to the classical Theorem F.1.4 if $M = \mathbb{R}^d$. Indeed, notice that in such case

$$\exp_x(-\nabla\psi) = x - \nabla\psi = \nabla\left(\frac{1}{2}|x|^2 - \psi(x)\right),$$

and $u = \frac{1}{2}|x|^2 - \psi(x)$ is convex and lower semicontinuous because ψ is $\frac{d^2}{2}$-concave, see [268, Proposition 1.21].

F.2 The classical Wasserstein distances

Definition F.2.1 (Wasserstein space). *Consider a Polish space $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ and $p \in [1,\infty)$. The p-th Wasserstein space is defined by*

$$\mathcal{P}_p(\mathcal{X}) := \left\{\mu \in \mathcal{P}_2(\mathcal{X}) : \int_{\mathcal{X}} d(x,x_0)^2\, d_x\mu < \infty\right\},$$

where $x_0 \in \mathcal{X}$ is any arbitrary point and the definition does not depend on the specific x_0. For any $\mu,\nu \in \mathcal{P}_p(\mathcal{X})$ we define the p-th Wasserstein distance between μ and ν by

$$W_p(\mu,\nu) := \left(\inf_{\gamma\in\Pi(\mu,\nu)} \int_{\mathcal{X}\times\mathcal{Y}} d(x,y)^p\, d_{(x,y)}\gamma\right)^{\frac{1}{p}}.$$

The case corresponding to $p = 1$ has a special representation via Kantorovich–Rubinstein duality formula, that asserts

$$W_1(\mu,\nu) = \sup_{[\varphi]_{C^{0,1}}\leq 1} \int_{\mathcal{X}} \varphi\, d(\mu - \nu),$$

where $[\,\cdot\,]_{C^{0,1}}$ denotes the Lipschitz seminorm, that is,

$$[\varphi]_{C^{0,1}} = \sup_{x\neq y} \frac{|\varphi(x) - \varphi(y)|}{d(x,y)},$$

see [296, Theorem 5.10 and Remark 6.5]. Notice that all the above distances are achieved at some transference plans $\gamma \in \Pi(\mu,\nu)$ thanks to Theorem F.1.3. Indeed, Theorems F.1.4 and F.1.5 can be used to characterize when there is an optimal transport map. Specifically, if $\mathcal{X} = M$ is a Riemannian manifold with nonnegative sectional curvature, $p = 2$ and the first measure μ is absolutely continuous, the optimal transport map exists. We summarize the main metric properties of the Wasserstein space in the following result.

Proposition F.2.2. *The Wasserstein space $(\mathcal{P}_p(\mathcal{X}), \mathcal{B}(\mathcal{X}))$ with $p \in [1,+\infty)$ is again a Polish space and the following conditions are equivalent for any sequence $\{\mu_n\}_{n\in\mathbb{N}} \subseteq \mathcal{P}_p(\mathcal{X})$ and any $\mu \in \mathcal{P}_p(\mathcal{X})$:*

1. $\lim_{n\to\infty} W_p(\mu_n, \mu) = 0$.
2. *$\mu_n \to \mu$ narrow in $\mathcal{P}(\mathcal{X})$ and* $\int_{\mathcal{X}} d(x,x_0)^p\, d_x\mu^n \to \int_{\mathcal{X}} d(x,x_0)^p\, d_x\mu$.
3. $\int_{\mathcal{X}} \varphi\, d\mu_n \to \int_{\mathcal{X}} \varphi$*, for any continuous function $\varphi : \mathcal{X} \longrightarrow \mathbb{R}$ so that there exists $C > 0$ with $|\varphi(x)| \leq C(1 + d(x,x_0)^p)$ for all $x \in \mathcal{X}$.*

F.3 Riemannian structure of $\mathcal{P}_2(M)$ and Benamou–Brenier formula

When $(M, \langle\cdot,\cdot\rangle)$ is a complete Riemannian manifold, the Wasserstein space $(\mathcal{P}_2(M), W_2)$ can be regarded as an infinite dimensional Riemannian manifold. This comes back to F. Otto [237] that used it to describe from a geometrical point of view the porous medium equation as a gradient flow. Here, we shall briefly introduce a naive introduction to this topic that can be useful in some parts of the thesis. For further details, see [9, 268, 296]

The starting observation is that round any point $\mu \in \mathcal{P}_2(M)$ we can take a *pointed Gromov–Hausdorff limit* for a family of rescaled spaces $\mathcal{P}_2(M)$ to define the tangent space $T_\mu\mathcal{P}_2(M)$ of $\mathcal{P}_2(M)$ at μ. Interestingly, such tangent space defined via the limiting process can be identified with the following closed vector space

$$\widetilde{T}_\mu\mathcal{P}_2(M) := \overline{\text{Span}(\{\nabla\psi :\, \psi \in C_c^1(M)\})}^{L^2(\mu;TM)}$$

with respect to the following norm of tangent vector fields

$$\|\nabla\psi\|_{L^2(\mu;TM)} := \left(\int_M |\nabla\psi|^2\, d\mu\right)^{1/2}.$$

Indeed, we can endow $\widetilde{T}_\mu\mathcal{P}_2(M)$ (thus also $T_\mu\mathcal{P}_2(M)$) with the associated inner product

$$\langle\nabla\psi_1, \nabla\psi_2\rangle_{L^2(\mu;TM)} := \int_M \langle\nabla\psi_1, \nabla\psi_2\rangle\, d\mu.$$

In this way, $\mathcal{P}_2(M)$ acquires an infinite-dimensional Riemannian manifold structure. Although this construction looks complicated, we can gain some intuition via the following result that relates tangent vectors to curves of measures with velocity fields in the continuity equation for such curves of measures.

Theorem F.3.1. *Let $(M, \langle\cdot,\cdot\rangle)$ be a complete Riemannian manifold and consider any family of measures $\{\nu_\tau\}_{\tau\in[0,1]} \subseteq \mathcal{P}_2(M)$. Then, the following two properties take place:*

1. *Assume that the family $\{\nu_\tau\}_{\tau\in[0,1]}$ is absolutely continuous in the Wasserstein space, that is, there exists some nonnegative function $\delta \in L^1(0,1)$ so that*

$$W_2(\nu_{\tau_1}, \nu_{\tau_2}) \leq \int_{t_1}^{t_2} \delta(\tau)\, d\tau, \ \textit{for any } \ 0 \leq \tau_1 < \tau_2 \leq 1.$$

Then, there exists another family $\{v_\tau\}_{\tau\in[0,1]}$ of tangent fields with $v_\tau \in \widetilde{T}_{\nu_\tau}\mathcal{P}_2(M)$, so that $\|v_\tau\|_{L^2(\nu;TM)} = \left|\frac{d\nu_\tau}{d\tau}\right|$ for a.e. $\tau \in [0,1]$ and the following continuity equation fulfils weakly in the sense of measures

$$\frac{\partial\mu_\tau}{\partial\tau} + \text{div}(v_\tau\mu_\tau) = 0. \tag{F.3.1}$$

2. *Conversely, assume that there exists a family of tangent fields $\{v_\tau\}_{\tau\in[0,1]}$ such that $v_\tau \in \widetilde{T}_{\nu_\tau}\mathcal{P}_2(M)$ for a.e. $\tau \in [0,1]$ and assume that $\int_0^1 \|v_\tau\|_{L^2(\nu_\tau;TM)}\,d\tau < \infty$ and the above continuity equation holds true. Then, $\{\nu_\tau\}_{\tau\in[0,1]}$ is absolutely continuous and $\left|\frac{d\nu_\tau}{d\tau}\right| \leq \|v_\tau\|_{L^2(\nu_\tau;TM)}$ for a.e. $\tau \in [0,1]$.*

Here $|\frac{d\nu_\tau}{d\tau}|$ represent the metric derivative of the curve in the Wasserstein space, that is

$$\left|\frac{d\nu_\tau}{d\tau}\right| = \lim_{\varepsilon\to 0}\frac{W_2(\nu_{\tau+\varepsilon},\nu_\tau)}{\varepsilon}.$$

This suggest that $v_\tau \in \widetilde{T}_{\nu_\tau}\mathcal{P}_2(M)$ and the abstract tangent vectors $\frac{d\nu_\tau}{d\tau} \in T_{\nu_\tau}\mathcal{P}(M)$ can be identified as long as they are related though the continuity equation (F.3.1). Notice that in particular, this allows regarding $\mathcal{P}_2(M)$ both as a metric space with distance W_2 and a Riemmanian manifold with metric $\langle\cdot,\cdot\rangle_{L^2(\mu;TM)}$. Fortunately, both structures are compatible and the Wasserstein distance is nothing but the associated Riemmanian distance of the infinite dimensional Riemannian manifold. This is a consequence of the following result.

Theorem F.3.2. *Let $(M,\langle\cdot,\cdot\rangle)$ be a complete Riemannian manifold and consider $\mu^1,\mu^2 \in \mathcal{P}_2(M)$. Then, there exists an absolutely continuous minimizing geodesic $\{\nu_\tau\}_{\tau\in[0,1]} \subseteq \mathcal{P}_2(M)$. In addition, if both μ^1 and μ^2 are absolutely continuous with respect to the volume measure, then the minimizing geodesic is unique.*

See [296, Corollaries 7.22, 7.23] for a proof and its relation with *displacement interpolation* (we will not address such topic here). In particular, such result asserts that the Wasserstein space $(\mathcal{P}_2(M), W_2)$ is a geodesic space as every couple of measures can be joint via a (Wasserstein) minimizing geodesic. Recall that geodesic spaces are metric spaces so that its distance between two points agrees with the $\inf$ of lengths of absolutely continuous curves curves joining both points and it is achieved at some curve, that we call minimizing geodesic. This justifies the following well known *Benamou–Brenier formula* for the Wasserstein distance.

Corollary F.3.3 (Benamou–Brenier)**.** *Let $(M,\langle\cdot,\cdot\rangle)$ be a complete Riemannian manifold and consider $\mu^1,\mu^2 \in \mathcal{P}_2(M)$. Then,*

$$W_2(\mu^1,\mu^2)^2 = \min\left\{\int_0^1 \|v_\tau\|^2_{L^2(\mu_\tau;TM)}\,d\tau\,:\,\text{(F.3.1) \textit{holds}},\ \nu_{\tau=0}=\mu^1\ \textit{and}\ \nu_{\tau=1}=\mu^2\right\}.$$

Proof. By virtue of Theorem F.3.2, we have that

$$W_2(\mu^1,\mu^2) = \min\left\{\int_0^1 \left|\frac{d\nu_\tau}{d\tau}\right| d\tau\,:\,\{\nu_\tau\}_{\tau\in[0,1]}\subseteq\mathcal{P}_2(M)\text{ is a.c.},\ \nu_{\tau=0}=\mu^1\ \text{and}\ \nu_{\tau=1}=\mu^2\right\}. \tag{F.3.2}$$

In addition, it is easy to verify that minimizers of the above length functional can be taken with constant-speed, that is, $\left|\frac{d\nu_\tau}{d\tau}\right| = W_p(\mu^1,\mu^2)$ for a.e. $\tau\in[0,1]$. A similar argument shows that those constant-speed geodesics also solve the minimization problem of the "kinetic energy" functional, that is,

$$\min\left\{\int_0^1 \left|\frac{d\nu_\tau}{d\tau}\right|^2 d\tau\,:\,\{\nu_\tau\}_{\tau\in[0,1]}\subseteq\mathcal{P}_2(M)\text{ is a.c.},\ \nu_{\tau=0}=\mu^1\ \text{and}\ \nu_{\tau=1}=\mu^2\right\}.$$

This implies that $W_2(\mu^1,\mu^2)^2$ has to agree with such minimum of the kinetic energy and, by applying Theorem F.3.1, we end the proof. □

See [268, Theorem 5.28] or [23] for the original derivation in the Euclidean space. Finally, recall that geodesic have very particular tangent vectors. Indeed, in classical Riemannian geometry we known that those tangent vectors are parallel along the geodesic itself. In the particular case of Wasserstein geodesics, it must appear as a particular choice of the vector fields $\{v_\tau\}_{\tau\in[0,1]}$ in Theorem F.3.1. This is the content of the following result:

Theorem F.3.4. *Let $(M, \langle\cdot,\cdot\rangle)$ be a complete Riemannian manifold, consider $\mu^1, \mu^2 \in \mathcal{P}_2(M)$ and a minimizing geodesic $\{\nu_\tau\}_{\tau\in[0,1]}$ so that $\nu_{\tau=0} = \mu^1$ and $\nu_{\tau=1} = \mu^2$. Then, the following equation holds true*

$$\left\{\begin{array}{l} \dfrac{\partial \nu_\tau}{\partial \tau} + \operatorname{div}(\nabla\psi_\tau \nu_\tau) = 0, \\ \dfrac{\partial \psi_\tau}{\partial \tau} + \dfrac{1}{2}|\nabla \psi_\tau|^2 = 0,\ \psi_{\tau=0} = \psi_0, \end{array}\right. \tag{F.3.3}$$

for some initial datum ψ_0 so that $-\psi_0$ is $\frac{d^2}{2}$-concave. In particular,

$$W_2(\mu^1,\mu^2)^2 = \int_0^1 \|\nabla\psi_\tau\|^2_{L^2(\nu_\tau;TM)}\,d\tau = \int_0^1\int_M |\nabla\psi_\tau|^2\,d\nu_\tau\,d\tau. \tag{F.3.4}$$

See [296, p. 339].

F.4 The fiberwise quadratic Wasserstein distance in $\mathbb{T}\times\mathbb{R}$

To start, we recall some useful tool coming form measure theory that allows describing the *disintegrations* or *conditional probabilities* of probability measures defined in a product space.

Theorem F.4.1 (Disintegration)**.** *Let X and Y be separable complete metric spaces and define the projection mapping*

$$\begin{array}{rccc} \pi_Y: & X\times Y & \longrightarrow & Y, \\ & (x,y) & \longmapsto & y. \end{array}$$

Consider any Borel probability measure $\mu \in \mathcal{P}(X\times Y, \mathcal{B}(X\times Y))$ and the Y-marginal probability measure $\nu := (\pi_Y)_\#\mu$. Then, there exits a Borel measurable map

$$\begin{array}{ccc} (Y,\mathcal{B}(Y)) & \longrightarrow & \mathcal{P}(X,\mathcal{B}(X)), \\ y & \longmapsto & \mu(\cdot|y), \end{array}$$

such that the following formula holds true

$$\int_{X\times Y}\varphi(x,y)d_{(x,y)}\mu = \int_Y\left(\int_X \varphi(x,y)\,d_x\mu(\cdot|y)\right)d_y\nu, \tag{F.4.1}$$

for every Borel-measurable map $\varphi : X\times Y \longrightarrow \mathbb{R}$.

Such a family $\{\mu(\cdot|y)\}_{y\in Y}$ is called a disintegration of μ or conditional probabilities with respect to y. It is uniquely defined ν-a.e. in Y, see [9, Theorem 5.3.1] and [102, III.70] for more details.

In the following result, a new Wasserstein-type distance is introduce in a subspace of $\mathcal{P}(\mathbb{T}\times\mathbb{R})$. This will be the cornerstone in our uniqueness result to avoid the assumption of bounded Ω-moments of the solutions.

Proposition F.4.2 (Fiberwise quadratic Wasserstein distance)**.** *Consider any probability measure $g \in \mathcal{P}(\mathbb{R})$ and define the subspace of probability measures in $\mathcal{P}(\mathbb{T}\times\mathbb{R})$ that enjoy the same distribution g of natural frequencies, namely,*

$$\mathcal{P}_g(\mathbb{T}\times\mathbb{R}) := \{\mu \in \mathcal{P}(\mathbb{T}\times\mathbb{R}) :\ (\pi_\Omega)_{\#}\mu = g\},$$

where π_Ω is the projection (N.1)*. Also, let us define the functional $W_{2,g}$ as follows*

$$W_{2,g}(\mu^1,\mu^2) = \left(\int_{\mathbb{R}} W_2(\mu^1(\cdot|\Omega),\mu^2(\cdot|\Omega))^2\, d_\Omega g\right)^{1/2}, \tag{F.4.2}$$

for every $\mu^1,\mu^2 \in \mathcal{P}_g(\mathbb{T}\times\mathbb{R})$, where $\{\mu^1(\cdot|\Omega)\}_{\Omega\in\mathbb{R}}$ and $\{\mu^2(\cdot|\Omega)\}_{\Omega\in\mathbb{R}}$ stand for their associated families of disintegrations with respect to Ω. Then, $(\mathcal{P}_g(\mathbb{T}\times\mathbb{R}), W_{2,g})$ is a metric space.

The proof is clear and is a consequence of (disintegration) Theorem F.4.1 along with the fact that $W_2(\cdot,\cdot)$ is a distance in $\mathcal{P}(\mathbb{T})$ and $\|\cdot\|_{L^2(\mathbb{R},dg)}$ is a norm in $L^2(\mathbb{R},dg)$. Based on the aforementioned Benamou–Brenier formulation of optimal transportation, we can restate the fiberwise quadratic Wasserstein distance in terms of *fiberwise Wasserstein geodesics*. This is the content of the following result.

Proposition F.4.3. *Consider any probability measure $g \in \mathcal{P}(\mathbb{R})$ and $\mu^1,\mu^2 \in \mathcal{P}_g(\mathbb{T}\times\mathbb{R})$. For g-a.e. $\Omega\in\mathbb{R}$ consider a minimizing geodesic $\{\nu_\tau(\cdot|\Omega)\}_{\tau\in[0,1]}$ so that*

$$\nu_{\tau=0}(\cdot|\Omega) = \mu^1(\cdot|\Omega) \ \text{ and } \ \nu_{\tau=1}(\cdot|\Omega) = \mu^2(\cdot|\Omega).$$

Consider the associated family of functions $\{\psi_\tau(\cdot|\Omega)\}_{\tau\in[0,1]}$ so that according to Theorem F.3.4 we have

$$\left\{\begin{array}{l} \dfrac{\partial \nu_\tau(\cdot|\Omega)}{d\tau} + \operatorname{div}_z\left(\nabla\psi_\tau(\cdot,\Omega)\nu_\tau(\cdot|\Omega)\right) = 0,\\ \dfrac{\partial\psi_\tau(\cdot,\Omega)}{\partial\tau} + \dfrac{1}{2}|\nabla_z\psi_\tau(\cdot,\Omega)|^2 = 0,\ \psi_{\tau=0}(\cdot,\Omega) = \psi_0(\cdot,\Omega), \end{array}\right. \tag{F.4.3}$$

for a $\frac{d^2}{2}$-concave function $-\psi_0$ with respect to z. Then, the following identity holds true

$$W_{2,g}(\mu^1,\mu^2)^2 = \int_0^1\int_{\mathbb{T}\times\mathbb{R}} |\nabla_z\psi_\tau|^2\, d_{(z,\Omega)}\mu^1\, d\tau \tag{F.4.4}$$

$$= \int_{\mathbb{T}\times\mathbb{R}} |\nabla_z\psi_\tau|^2\, d_{(z,\Omega)}\mu^1, \quad \forall\,\tau\in[0,1]. \tag{F.4.5}$$

The proof is a clear application of Theorem F.3.4 in $M = \mathbb{T}$ to restate the Wasserstein distance between the conditional probabilities $W_2(\mu^1(\cdot|\Omega),\mu^2(\cdot|\Omega))$ in terms of the Hamilton–Jacobi equation (F.3.3), along with definition (F.4.2) to glue all the fiberwise information together in a weighted g-dependent manner. We then omit it.

The relation between the classical quadratic Wasserstein distance W_2 in $\mathcal{P}(\mathbb{T}\times\mathbb{R})$ and the fiberwise version $W_{2,g}$ in $\mathcal{P}_g(\mathbb{T}\times\mathbb{R})$ is not completely apparent. We sketch the main relation in the following result.

Proposition F.4.4. *Consider $g\in\mathcal{P}_2(\mathbb{T})$ and $\mu^1,\mu^2\in\mathcal{P}_g(\mathbb{T}\times\mathbb{R})$. Then,*

$$W_2(\mu^1,\mu^2) \le W_{2,g}(\mu^1,\mu^2).$$

The identity is true when the optimal transference plan $\gamma_0\in\Pi(\mu^1,\mu^2)$ for the quadratic Wasserstein distance W_2 is symmetric with respect to the variables ω and ω'.

Proof. • *Step 1*: Proof of the inequality.

Consider for g-a.e. $\Omega \in \mathbb{R}$ the optimal coupling $\gamma_{0,\Omega} \in \Pi(\mu^1(\cdot|\Omega), \mu^2(\cdot|\Omega))$ for $W_{2,g}$ between the conditional probabilities $\mu^1(\cdot|\Omega)$ and $\mu^2(\cdot|\Omega)$. Then, we can construct the probability measure $\gamma \in \mathcal{P}(\mathbb{T}^2 \times \mathbb{R}^2)$ given by

$$\gamma := \gamma_{0,\Omega}(z, z') \otimes \delta_\Omega(\Omega') \otimes g(\Omega). \tag{F.4.6}$$

Let us see first that $\gamma \in \Pi(\mu^1, \mu^2)$. To such end, consider $\varphi \in C_b(\mathbb{T} \times \mathbb{R})$ and note that

$$\begin{aligned}\int_{\mathbb{T}\times\mathbb{R}} \varphi \, d_{(z,\Omega)}(\pi_{(z,\Omega)\,\#}\gamma) &= \int_{\mathbb{T}^2\times\mathbb{R}^2} \varphi(z,\Omega)\, d_{(z,z')}\gamma_{0,\Omega}\, d_{\Omega'}(\delta_\Omega)\, d_\Omega g \\ &= \int_{\mathbb{T}^2\times\mathbb{R}} \varphi(z,\Omega)\, d_{(z,z')}\gamma_{0,\Omega}\, d_\Omega g = \int_{\mathbb{T}\times\mathbb{R}} \varphi(z,\Omega)\, d_z(\pi_{z\,\#}\gamma_{0,\Omega})\, d_\Omega g \\ &= \int_{\mathbb{T}\times\mathbb{R}} \varphi(z,\Omega)\, d_z\mu^1(\cdot|\Omega)\, d_\Omega g = \int_{\mathbb{T}\times\mathbb{R}} \varphi\, d_{(z,\Omega)}\mu^1.\end{aligned}$$

Then, $\pi_{(z,\Omega)\,\#}\gamma = \mu^1$. Similarly note that

$$\begin{aligned}\int_{\mathbb{T}\times\mathbb{R}} \varphi \, d_{(z',\Omega')}(\pi_{(z',\Omega')\,\#}\gamma) &= \int_{\mathbb{T}^2\times\mathbb{R}^2} \varphi(z',\Omega')\, d_{(z,z')}\gamma_{0,\Omega}\, d_{\Omega'}(\delta_\Omega)\, d_\Omega g \\ &= \int_{\mathbb{T}^2\times\mathbb{R}} \varphi(z',\Omega)\, d_{(z,z')}\gamma_{0,\Omega}\, d_\Omega g = \int_{\mathbb{T}\times\mathbb{R}} \varphi(z',\Omega)\, d_{z'}(\pi_{z'\,\#}\gamma_{0,\Omega})\, d_\Omega g \\ &= \int_{\mathbb{T}\times\mathbb{R}} \varphi(z',\Omega)\, d_{z'}\mu^2(\cdot|\Omega)\, d_\Omega g = \int_{\mathbb{T}\times\mathbb{R}} \varphi\, d_{(z',\Omega')}\mu^2.\end{aligned}$$

Then we also recover $\pi_{(z',\Omega')\,\#}\gamma = \mu^2$. Also note that by definition

$$\begin{aligned}W_{2,g}(\mu^1,\mu^2)^2 &= \int_{\mathbb{R}\times\mathbb{T}^2} d(z,z')^2\, d_{(z,z')}\gamma_{0,\Omega}\, d_\Omega g = \int_{\mathbb{T}^2\times\mathbb{R}^2} d(z,z')^2\, d_{((z,\Omega),(z',\Omega'))}\gamma \\ &= \int_{\mathbb{T}^2\times\mathbb{R}^2} (d(z,z')^2 + (\Omega-\Omega')^2)\, d_{((z,\Omega),(z',\Omega'))}\gamma \geq W_2(\mu^1,\mu^2)^2,\end{aligned}$$

where the extra term that has been added in the second line vanishes because of the presence of $\delta_\Omega(\Omega')$ in (F.4.6).

• *Step 2*: Proof of the identity.

Let us consider an optimal coupling $\gamma_0 \in \Pi(\mu^1, \mu^2)$ for $W_2(\mu^1, \mu^2)$ and assume that it is symmetric with respect to the variables Ω and Ω'. Specifically, consider the map that swaps those variables, i.e., $\mathcal{S}(z, z', \Omega, \Omega') := (z, z', \Omega', \Omega)$ and assume that

$$\mathcal{S}_\#\gamma_0 = \gamma_0. \tag{F.4.7}$$

Let us define the measure $\widetilde{\gamma}_0 := \pi_{(z,z',\Omega)\,\#}\gamma_0$. Then, we can consider its family of conditional probabilities with respect to g-a.e. $\Omega \in \mathbb{R}$, that is $\widetilde{\gamma}_0(\cdot|\Omega) \in \mathcal{P}(\mathbb{T}^2)$. Let us see that $\widetilde{\gamma}_0(\cdot|\Omega)$ is a transference plan between conditional probabilities, that is, $\widetilde{\gamma}_0(\cdot|\Omega) \in \Pi(\mu^1\widetilde{\gamma}_0(\cdot|\Omega), \mu^2\widetilde{\gamma}_0(\cdot|\Omega))$ for g-a.e. $\Omega \in \mathbb{R}$. Indeed, take $\varphi \in C(\mathbb{T})$ and $\psi \in C_b(\mathbb{R})$ and note that

$$\begin{aligned}\int_{\mathbb{R}} \psi(\Omega) \int_{\mathbb{T}} \varphi(z)\, d_z(\pi_{z\,\#}\widetilde{\gamma}_0\widetilde{\gamma}_0(\cdot|\Omega))\, d_\Omega g &= \int_{\mathbb{T}^2\times\mathbb{R}} \psi(\Omega)\varphi(z)\, d_{(z,z')}\widetilde{\gamma}_0\widetilde{\gamma}_0(\cdot|\Omega)\, d_\Omega g \\ &= \int_{\mathbb{T}^2\times\mathbb{R}} \psi(\Omega)\varphi(z)\, d_{(z,z',\Omega)}\widetilde{\gamma}_0\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{T}^2\times\mathbb{R}^2} \psi(\Omega)\varphi(z)\, d_{((z,\Omega),(z',\Omega'))}\gamma_0 \\
&= \int_{\mathbb{R}} \psi(\Omega) \int_{\mathbb{T}} \varphi(\Omega)\, d_z \mu^1 \widetilde{\gamma}_0(\cdot|\Omega)\, d_\Omega g.
\end{aligned}
$$

Then, $\pi_{z\,\#}\widetilde{\gamma}_0^{\Omega} = \mu^1\widetilde{\gamma}_0(\cdot|\Omega)$. Similarly we obtain

$$
\begin{aligned}
\int_{\mathbb{R}} \psi(\Omega) \int_{\mathbb{T}} \varphi(z')\, d_{z'}(\pi_{z'\,\#}\widetilde{\gamma}_0\widetilde{\gamma}_0(\cdot|\Omega))\, d_\Omega g &= \int_{\mathbb{T}^2\times\mathbb{R}} \psi(\Omega)\varphi(z')\, d_{(z,z')}\widetilde{\gamma}_0\widetilde{\gamma}_0(\cdot|\Omega)\, d_\Omega g \\
&= \int_{\mathbb{T}^2\times\mathbb{R}} \psi(\Omega)\varphi(z')\, d_{(z,z',\Omega)}\widetilde{\gamma}_0 \\
&= \int_{\mathbb{T}^2\times\mathbb{R}^2} \psi(\Omega)\varphi(z')\, d_{((z,\Omega),(z',\Omega'))}\gamma_0 \\
&= \int_{\mathbb{T}^2\times\mathbb{R}^2} \psi(\Omega')\varphi(z')\, d_{((z,\Omega),(z',\Omega'))}\gamma_0 \\
&= \int_{\mathbb{R}} \psi(\Omega') \int_{\mathbb{T}} \varphi(\Omega')\, d_{z'}\mu^2\widetilde{\gamma}_0(\cdot|\Omega')\, d_{\Omega'} g,
\end{aligned}
$$

where (F.4.7) has been used in the fourth line. Then, $\pi_{z'\,\#}\widetilde{\gamma}_0\widetilde{\gamma}_0(\cdot|\Omega) = \mu^2\widetilde{\gamma}_0(\cdot|\Omega)$ g-a.e. $\Omega \in \mathbb{R}$. In addition, notice that

$$
\begin{aligned}
W_{2,g}(\mu^1,\mu^2)^2 &\le \int_{\mathbb{R}\times\mathbb{T}^2} d(z,z')^2\, d_{(z,z')}\widetilde{\gamma}_0\widetilde{\gamma}_0(\cdot|\Omega)\, d_\Omega g = \int_{\mathbb{R}\times\mathbb{T}^2} d(z,z')^2\, d_{(z,z',\Omega)}\widetilde{\gamma}_0 \\
&= \int_{\mathbb{T}^2\times\mathbb{R}^2} d(z,z')^2\, d_{((z,\Omega),(z',\Omega'))}\gamma_0 = W_2(\mu^1,\mu^2)^2,
\end{aligned}
$$

thus ending the proof. □

Remark F.4.5. *Consider the empirical measures*

$$
\mu^1 := \frac{1}{2}\left(\delta_{(z_1,\Omega_1)} + \delta_{(z_2,\Omega_2)}\right) \;\; \textit{and} \;\; \mu^2 := \frac{1}{2}\left(\delta_{(z_2,\Omega_1)} + \delta_{(z_1,\Omega_2)}\right),
$$

for some $z_1, z_2 \in \mathbb{T}$ and $\Omega_1, \Omega_2 \in \mathbb{R}$ and define $\varepsilon_z := d(z_1,z_2)$ and $\varepsilon_\Omega := |\Omega_1 - \Omega_2|$. Then,

$$
\pi_{\Omega\,\#}\mu^1 = \pi_{\Omega\,\#}\mu^2 = \frac{1}{2}(\delta_{\Omega_1} + \delta_{\Omega_2}) =: g,
$$

and, consequently, $\mu^1, \mu^2 \in \mathcal{P}_g(\mathbb{T}\times\mathbb{R})$. In addition

$$
W_{2,g}(\mu^1,\mu^2)^2 = \varepsilon_z^2 \;\; \textit{and} \;\; W_2(\mu^1,\mu^2)^2 = \min\{\varepsilon_z^2, \varepsilon_\Omega^2\}.
$$

Therefore,

$$
\begin{aligned}
W_2(\mu^1,\mu^2) &< W_{2,g}(\mu^1,\mu^2), \quad \textit{if } \varepsilon_\Omega < \varepsilon_z, \\
W_2(\mu^1,\mu^2) &= W_{2,g}(\mu^1,\mu^2), \quad \textit{if } \varepsilon_\Omega \ge \varepsilon_z.
\end{aligned}
$$

APPENDIX G

Vector calculus on hypersurfaces of the Euclidean space

In this Appendix, we recall some well known properties of differential operators on hypersurfaces of the Euclidean space $\mathbb{R}^d$. For convenience, whe shall restrict to the particular case $d = 3$ and will introduce some useful formulas for the gradient, curl and divergence operators on compact surfaces $S \subseteq \mathbb{R}^3$ that enclose a bounded comain of $\mathbb{R}^3$. These formulas are necessary at some technical points in Chapter 6 to study boundary integrals. We refer to the textbook [299] for proofs and further details.

G.1 Musical isomorphisms and Riemannian gradient

Let us consider the vector spaces of smooth tangent vector fields along S and smooth 1-forms, i.e., $\mathfrak{X}(S)$ and $\Omega^1(S)$ respectively. It is well known that these vector spaces can be identified using the Riemannian metric on S by virtue of the musical isomorphisms

$$\begin{array}{rccc} \flat: & \mathfrak{X}(S) & \longrightarrow & \Omega^1(S), \\ & V & \longmapsto & V^\flat, \end{array} \quad \begin{array}{rccc} \sharp: & \Omega^1(S) & \longrightarrow & \mathfrak{X}(S) \\ & \alpha & \longmapsto & \alpha^\sharp. \end{array}$$

These are defined as

$$V^\flat(W) = V \cdot W, \qquad \alpha^\sharp \cdot V = \alpha(V).$$

for any given $V, W \in \mathfrak{X}(S)$ and $\alpha \in \Omega^1(S)$. The *gradient vector field* over S of any function $f \in C^1(S)$ can be identified with the exterior differential 1-form over S through the musical isomorphisms:

$$\nabla_S f := (d_S f)^\sharp.$$

If $\overline{f} \in C^1(\mathbb{R}^3)$ is any extension of f, it turns out that $\nabla_S f$ is the tangential component to the surface of the $\mathbb{R}^3$ gradient field $\nabla\overline{f}$, that is,

$$\nabla_S f = -\eta \times (\eta \times \nabla\overline{f}) \text{ on } S,$$

where η represents the outwards unitary normal vector field of S.

G.2 Hodge star operator and codifferential

Recall that Hodge star operator $*$ acts on each k-forms space $\Omega^k(S)$ as the bijection

$$* : \Omega^k(S) \longrightarrow \Omega^{2-k}(S),$$

given by

$$\alpha \wedge *\beta = \alpha \cdot \beta \, dS,$$

where dS stands for the Riemannian area 2-form on S and $\alpha, \beta \in \Omega^1(S)$. The dot symbol here is the pointwise inner product of k-forms induced by the musical isomorphisms. Its inverse can be computed thought the next classical formula

$$** = (-1)^{k(2-k)} I \quad \text{in} \ \ \Omega^k(S).$$

Analogously,

$$\delta_S : \Omega^k(S) \longrightarrow \Omega^{k-1}(S),$$

acts on each k-forms space $\Omega^k(S)$ as

$$\delta_S \alpha := (-1)^{2k-1} (*d_S*)\alpha.$$

Recall that δ_S is the adjoint of d_S. Specifically, for any $\alpha \in \Omega^1(S)$ and $\varphi \in C^1(S)$ one has

$$\int_S \varphi \, \delta_S \alpha \, dS = \int_S d_S \varphi \cdot \alpha \, dS,$$

where the above pointwise inner product is the one induced by the Riemannian metric in S through the musical isomorphisms, i.e.,

$$\int_S \varphi \, \delta_S \alpha \, dS = \int_S (d_S \varphi)^\sharp \cdot \alpha^\sharp \, dS = \int_S \nabla_S \varphi \cdot \alpha^\sharp \, dS. \tag{G.2.1}$$

G.3 Intrinsic divergence and curl of tangent fields

The *divergence* and *curl* of a tangent vector $V \in \mathfrak{X}(S)$ are defined by the rules

$$\begin{aligned} \operatorname{div}_S V &= -\delta_S(V^\flat) = (*d_S*)(V^\flat), \\ \operatorname{curl}_S V &= (*d_S)(V^\flat). \end{aligned}$$

Using formula (G.2.1), we can show that for any $V \in \mathfrak{X}(S)$ and $\varphi \in C^1(S)$ we have

$$\int_S \operatorname{div}_S(V) \, \varphi \, dS = -\int_S \delta_S(V^\flat) \varphi \, dS = -\int_S \nabla_S \varphi \cdot V \, dS.$$

A similar formula can be found for curl_S and it implies the well known integration by parts formulas

$$\begin{aligned} \int_S \operatorname{div}_S(V) \varphi \, dS &= -\int_S V \cdot \nabla_S \varphi \, dS, & \forall \, \varphi \in C^\infty(S), \\ \int_S \operatorname{curl}_S(V) \varphi \, dS &= -\int_S V \cdot (\eta \times \nabla_S \varphi) \, dS, & \forall \, \varphi \in C^\infty(S). \end{aligned}$$

G.4 A short list of useful formulas

For the reader convenience, we recall here some useful identities and properties involving the operators ∇_S, div_S and curl_S.

- $\operatorname{curl}_S(V) = -\operatorname{div}_S(\eta \times V)$, for all $V \in \mathfrak{X}(S)$.
- $\operatorname{curl}_S(-\eta \times (\eta \times F)) = \eta \cdot \operatorname{curl} F$, for all $F \in C^1(\overline{\Omega})$.
- $\operatorname{curl}_S(\nabla_S f) = 0$, for all $f \in C^2(S)$.
- $\operatorname{div}_S(\eta \times \nabla_S f) = 0$, for all $f \in C^2(S)$.
- (Poincaré's lemma) Assume that S is simply connected and consider any tangent vector field $V \in \mathfrak{X}(S)$ such that $\operatorname{curl}_S(v) = 0$. Then, there exists some $f \in C^2(S)$ such that $V = \nabla_S f$.

APPENDIX H

Potential theory for inhomogeneous integral kernels

Our goal here is to extend some results of classical potential theory to inhomogeneous kernels like the fundamental solution of the Helmholtz equation $\Gamma_\lambda(x)$ (see e.g. [83, 96, 138, 139, 201, 215, 214, 273, 278] in the case of homogeneous kernels). While there are some previous results concerning the inhomogeneous case (see [84, 85, 228] for a study of $\Gamma_\lambda(x)$ with non-zero λ), only low order Hölder estimates have been obtained. Our approach roughly follows the treatment of [167, 229] for the harmonic case (i.e., $\lambda = 0$), and we will introduce nontrivial modifications to derive higher order Hölder estimates of generalized volume and single layer potentials in the inhomogeneous setting. These regularity estimates are necessary in Section 6.3 of Chapter 6 to stablish existence and regularity of solutions to the inhomogeneous Beltrami equation with Neumann boundary conditions (6.3.23). We divide our presentation into two parts:

- In the first section, we provide the main regularity properties of the inhomogeneous volume and single layer potentials associated with $\Gamma_\lambda(x)$.
- In the second section, we used them to show that the boundary integral operator T_λ in Theorem 6.3.14 of Section 6.3 enjoy appropriate regularity properties.

We emphasize that our results apply to exterior (unbounded) domains.

H.1 Inhomogeneous volume and single layer potentials

In our context, all the integral kernels to be considered come from the fundamental solution of the 3-dimensional Helmholtz equation (6.3.5)

$$\Gamma_\lambda(z) = \frac{e^{i\lambda|z|}}{4\pi|z|} = \frac{1}{4\pi}\left(\frac{\cos(\lambda|z|)}{|z|} + i\frac{\sin(\lambda|z|)}{|z|}\right),\ z \in \mathbb{R}^3 \setminus \{0\}.$$

For $\lambda = 0$ we recover the Newtonian potential associated with the Laplace equation in $\mathbb{R}^3$, [138, 139, 215, 214]. As for $\lambda \neq 0$, it is no longer homogeneous, the classical theory cannot be directly applied.

Fortunately, this kernel can be though to be "almost homogeneous" in the following sense. Let us consider the functions

$$\begin{cases} \phi_\lambda(r) := \dfrac{e^{i\lambda r}}{4\pi r}, & r > 0, \\ \psi_\lambda(r) := \phi_\lambda(r) - \dfrac{1}{4\pi r} \equiv \dfrac{e^{i\lambda r} - 1}{4\pi r}, & r > 0. \end{cases} \tag{H.1.1}$$

From the definition one has the following split

$$\Gamma_\lambda(z) = \phi_\lambda(|z|) = \frac{1}{4\pi|z|} + \psi_\lambda(|z|) =: \Gamma_0(z) + R_\lambda(z), \tag{H.1.2}$$

that amounts to a decomposition of the inhomogeneous kernel $\Gamma_\lambda(z)$ into the homogeneous part $\Gamma_0(z)$ and an inhomogeneous remainder $R_\lambda(z)$ exhibiting a lower order singularity at the origin. The main argument supporting our subsequent results is that we do not need our whole kernel to be purely homogeneous, but only the principal (or more singular) part. While high order derivatives of harmonic potentials can be directly controlled through the harmonic kernel $\Gamma_0(z)$ and the classical results [138, 139, 215, 214], it is also important to control the behavior of the higher order derivatives of the remainder $R_\lambda(z)$.

To this end, let us compute the derivative of $\psi_\lambda(r)$

$$\psi'_\lambda(r) = i\lambda\frac{1}{4\pi r} + \left(i\lambda - \frac{1}{r}\right)\psi_\lambda(r).$$

and note that since $\psi_\lambda(r)$ is locally bounded near $r = 0$ and decay as r^{-1} at infinity, it is globally bounded. Hence, a recursive reasoning leads to estimates for high order derivatives:

$$|\psi_\lambda^{(m)}(r)| \leq C\left(1 + \frac{1}{r^m}\right), \quad r > 0, \tag{H.1.3}$$

for a nonnegative constant $C = C(\lambda, m)$. It obviously turns into

$$|D^\gamma R_\lambda(z)| \leq C\left(1 + \frac{1}{|z|^{|\gamma|}}\right), \tag{H.1.4}$$

for every $z \in \mathbb{R}^3 \setminus \{0\}$ and each multi-index γ, in contrast with the analogous bounds for $\Gamma_0(z)$:

$$|D^\gamma \Gamma_0(z)| \leq C\frac{1}{|z|^{|\gamma|+1}}. \tag{H.1.5}$$

A basic fact is that, being inhomogeneous, $R_\lambda(z)$ is one degree less singular than $\Gamma_0(z)$. Thus, we will combine results of Calderón–Zygmund type for singular integrals (e.g. $D^2\Gamma_0(z)$) with a treatment in the spirit of Hardy–Littlewood–Sobolev theorem for weakly singular integral kernels (e.g. $D^2R_\lambda(z)$) in the preceding Appendix C. See also [228] for a treatment of pseudo-homogeneous kernels.

For the sake of completeness, we shall next introduce the kind of kernels to deal with in this section. Let us consider a bounded domain $D \subseteq \mathbb{R}^d$. A continuous function $K = K(x, z),\ x \in \overline{D},\ z \in \mathbb{R}^d \setminus \{0\}$ is a *weakly singular kernel of exponent* β if there is $C > 0$ such that

$$|K(x, z)| \leq \frac{C}{|z|^\beta},\ x \in \overline{D},\ z \in \mathbb{R}^d \setminus \{0\},$$

for a given $0 \leq \beta \leq d-1$. In this thesis, the singular kernels that will appear are first order partial derivatives of positively homogeneous kernel of degree $-(d-1)$, i.e.,

$$\frac{\partial}{\partial z_i} K(x,z),\ x \in \overline{D},\ z \in \mathbb{R}^d \setminus \{0\},$$

where $K(x,z)$ satisfies

$$K(x,\lambda z) = \lambda^{-(d-1)} K(x,z),$$

for all $x \in \overline{D},\ z \in \mathbb{R}^d \setminus \{0\},\ \lambda > 0$ and $K(x,\sigma)$ is continuous for $x \in \overline{D}$ and $\sigma \in \partial B_1(0)$.

The same lines as in the classical result [214, Teorema 2.I] can be used to achieve bounds for the single layer potential associated with $\Gamma_\lambda(z)$ both in bounded and unbounded domains:

Theorem H.1.1 (Generalized single layer potential). *Let $G \subseteq \mathbb{R}^3$ be a bounded domain with regularity $C^{k+1,\alpha}$, $\Omega := \mathbb{R}^3 \setminus \overline{G}$ its outer domain and $S = \partial G$ the boundary surface. Consider the generalized single layer potential associated with the Helmholtz equation and generated by a density ζ along S,*

$$(\mathcal{S}_\lambda \zeta)(x) := \int_S \Gamma_\lambda(x-y)\zeta(y)\, d_yS,\ \ x \in \mathbb{R}^3 \setminus S.$$

Then, the restrictions of $\mathcal{S}_\lambda\zeta$ to the interior and exterior domain defines bounded linear operators

$$\mathcal{S}_\lambda^- : C^{k,\alpha}(S) \longrightarrow C^{k+1,\alpha}(\overline{G}), \qquad \mathcal{S}_\lambda^+ : C^{k,\alpha}(S) \longrightarrow C^{k+1,\alpha}(\overline{\Omega}).$$

We omit the proof of this theorem since we are interested in a more singular regularity result that follows similar ideas. Specifically, we will study the regularity along the boundary surface S of these generalized single layer potentials and other related potentials with inhomogeneous kernels via similar arguments to those in [214, Teorema 2.I]. The main goal of the next results is to derive the classical *Hölder–Korn–Lichtenstein–Giraud inequality* for high order estimates of Hölder type in the inhomogeneous case, i.e., the regularity of generalized volume (or Newtonian) potentials with compactly supported densities both for interior and exterior domains.

Lemma H.1.2. *Let $G \subseteq \mathbb{R}^3$ be a bounded domain with regularity $C^{k+1,\alpha}$ and $S = \partial G$ the boundary surface. The generalized volume potential on G associated with the Helmholtz equation and a density ζ in G,*

$$(\mathcal{N}_\lambda^- \zeta)(x) = \int_G \Gamma_\lambda(x-y)\zeta(y)\, dy,\ \ x \in G,$$

defines a bounded linear map $\mathcal{N}_\lambda^- : C^{k,\alpha}(\overline{G}) \longrightarrow C^{k+2,\alpha}(\overline{G})$.

Proof. The proof follows the lines of [214, Teorema 3.II] for the harmonic case $\lambda = 0$, that we extend to the inhomogeneous case.

A C^1 estimate of $\mathcal{N}_\lambda^-\zeta$ can be achieve by taking derivatives under the integral sign

$$\frac{\partial}{\partial x_i}(\mathcal{N}_\lambda^- \zeta)(x) = \int_G \frac{\partial}{\partial x_i}\Gamma_\lambda(x-y)\zeta(y)\, dy,\ \ x \in G,$$

and using the local integrability of $\Gamma_\lambda(z)$, $\nabla\Gamma_\lambda(z)$, along with the boundedness of G and the fact that $\zeta \in C^0(G)$:

$$\|\mathcal{N}_\lambda^- \zeta\|_{C^1(G)} \leq C\|\zeta\|_{C^0(G)} \leq C\|\zeta\|_{C^{k,\alpha}(G)}.$$

Now, fix any multi-index γ with $|\gamma| \leq k$ and takes derivatives again under the integral sign to get

$$D^\gamma \frac{\partial}{\partial x_i}(\mathcal{N}_\lambda^- \zeta)(x) = \int_G D_x^\gamma \frac{\partial}{\partial x_i}\Gamma_\lambda(x-y)\zeta(y)\, dy.$$

A recursive reasoning supported by some chained integrations by parts leads to

$$
\begin{aligned}
D^\gamma \frac{\partial}{\partial x_i}(\mathcal{N}_\lambda^- \zeta)(x) = &- \sum_{m_1=1}^{\gamma_1} \int_S D_x^{\gamma - m_1 e_1} \frac{\partial}{\partial x_i}\Gamma_\lambda(x-y) D^{(m_1-1)e_1}\zeta(y)\eta_1(y)\, d_yS \\
&- \sum_{m_2=1}^{\gamma_2} \int_S D_x^{\gamma - \gamma_1 e_1 - \gamma_2 e_2} \frac{\partial}{\partial x_i}\Gamma_\lambda(x-y) D^{\gamma_1 e_1 + (m_2-1)e_2}\zeta(y)\eta_2(y)\, d_yS \\
&- \sum_{m_3=1}^{\alpha_3} \int_S D_x^{\gamma - \gamma_1 e_1 - \gamma_2 e_2 - m_3 e_3} \frac{\partial}{\partial x_i}\Gamma_\lambda(x-y) D^{\gamma_1 e_1 + \gamma_2 e_2 + (m_3-1)e_3}\zeta(y)\eta_3(y)\, d_yS \\
&+ \int_G \frac{\partial}{\partial x_i}\Gamma_\lambda(x-y) D^\gamma \zeta(y)\, dy.
\end{aligned}
$$

Combining the preceding arguments with Theorem H.1.1 we arrive at

$$
\left\| D^\gamma \frac{\partial}{\partial x_i}(\mathcal{N}_\lambda^- \zeta) \right\|_{C^0(G)} \leq K \|\zeta\|_{C^{k,\alpha}(G)}.
$$

To complete the proof, we consider the derivatives of order $k+2$. For $1 \leq j \leq 3$ we then have

$$
\begin{aligned}
D^\gamma \frac{\partial^2}{\partial x_i \partial x_j}(\mathcal{N}_\lambda^- \zeta)(x) = &- \sum_{m_1=1}^{\gamma_1} \int_S D_x^{\gamma - m_1 e_1 + e_j} \frac{\partial}{\partial x_i}\Gamma_\lambda(x-y) D^{(m_1-1)e_1}\zeta(y)\eta_1(y)\, d_yS \\
&- \sum_{m_2=1}^{\gamma_2} \int_S D_x^{\gamma - \gamma_1 e_1 - \gamma_2 e_2 + e_j} \frac{\partial}{\partial x_i}\Gamma_\lambda(x-y) D^{\gamma_1 e_1 + (m_2-1)e_2}\zeta(y)\eta_2(y)\, d_yS \\
&- \sum_{m_3=1}^{\alpha_3} \int_S D_x^{\gamma - \gamma_1 e_1 - \gamma_2 e_2 - m_3 e_3 + e_j} \frac{\partial}{\partial x_i}\Gamma_\lambda(x-y) D^{\gamma_1 e_1 + \gamma_2 e_2 + (m_3-1)e_3}\zeta(y)\eta_3(y)\, d_yS \\
&+ \int_G \frac{\partial^2}{\partial x_i \partial x_j}\Gamma_\lambda(x-y) D^\gamma \zeta(y)\, dy.
\end{aligned}
$$

Similar estimates for the boundary terms can be obtained in $C^{0,\alpha}(G)$ by virtue of Theorem H.1.1, while the last term requires an adaptation of the ideas in the harmonic case [214, Teorema 3.II]. We first split it into two parts and use again integration by parts in the second term

$$
\begin{aligned}
&\int_G \frac{\partial^2}{\partial x_i \partial x_j}\Gamma_\lambda(x-y) D^\gamma \zeta(y)\, dy \\
&\quad = \int_G \frac{\partial^2}{\partial x_i \partial x_j}\Gamma_\lambda(x-y)(D^\gamma\zeta(y) - D^\gamma\zeta(x))\, dy + D^\gamma\zeta(x) \int_G \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}\Gamma_\lambda(x-y)\, dy \\
&\quad = \int_G \frac{\partial^2}{\partial x_i \partial x_j}\Gamma_\lambda(x-y)(D^\gamma\zeta(y) - D^\gamma\zeta(x))\, dy - D^\gamma\zeta(x) \int_S \frac{\partial}{\partial x_i}\Gamma_\lambda(x-y)\eta_j(y)\, d_yS \\
&\quad =: F(x) - H(x).
\end{aligned}
$$

The idea behind such decomposition is that Theorem H.1.1 yields

$$
\|H\|_{C^{0,\alpha}(G)} \leq K \|\eta\|_{C^{0,\alpha}(G)} \|\zeta\|_{C^{k,\alpha}(S)}
$$

and we can cancel an α power of the singularity in $F(x)$:

$$
|F(x)| \leq [D^\gamma \zeta]_{\alpha,G} \int_G \left| \frac{\partial^2}{\partial x_i \partial x_j}\Gamma_\lambda(x-y) \right| |x-y|^\alpha\, dy.
$$

Bearing the estimates (H.1.4) and (H.1.5) in mind, along with the local integrability of $|z|^{\alpha-3}$ and the boundedness of G, we find the following C^0 estimate

$$\|F\|_{C^0(G)} \leq K\|\zeta\|_{C^{k,\alpha}(G)}.$$

Let us finally show the local α-Hölder property for F, i.e.,

$$|F(x^1) - F(x^2)| \leq C\|\zeta\|_{C^{k,\alpha}(G)}|x^1 - x^2|^\alpha,$$

for every $x^1, x^2 \in G$ such that $|x^1 - x^2| < \delta$ and some small $\delta > 0$ (the global one follows from the boundedness of F). To this end, consider a neighborhood U of x^1 with $B_{2d}(x^1) \subseteq U \subseteq B_{7d}(x^1)$ an taking Euclidean norms, we finally arrive at

$$\begin{aligned} |F(x^1) - F(x^2)| \leq &\int_{G\cap B_{7d}(x^1)} \left|\frac{\partial^2\Gamma_\lambda(x^1-y)}{\partial x_i \partial x_j}\right| |(D^\gamma\zeta(y) - D^\gamma\zeta(x^1))|\,dy \\ &+ \int_{G\cap B_{8d}(x^2)} \left|\frac{\partial^2\Gamma_\lambda(x^2-y)}{\partial x_i \partial x_j}\right| |(D^\gamma\zeta(y) - D^\gamma\zeta(x^2))|\,dy \\ &+ \int_{G\setminus B_{2d}(x^1)} \left|\frac{\partial^2\Gamma_\lambda(x^1-y)}{\partial x_i \partial x_j} - \frac{\partial^2\Gamma_\lambda(x^2-y)}{\partial x_i \partial x_j}\right| |D^\gamma\zeta(y) - D^\gamma\zeta(x^1)|\,dy \\ &+ |D^\gamma\zeta(x^1) - D^\gamma\zeta(x^2)| \int_{G\setminus U} \left|\frac{\partial^2\Gamma_\lambda(x^2-y)}{\partial x_i \partial x_j}\right| dy, \end{aligned} \tag{H.1.6}$$

where in the last three terms we have respectively used that $G \cap B_{7d}(x^1) \subseteq G \cap B_{8d}(x^2)$, $G \setminus B_{7d}(x^1) \subseteq G \setminus B_{2d}(x^1)$ and $G \setminus B_{7d}(x^1) \subseteq G \setminus U$.

The first and second terms in (H.1.6) can be bounded as desired by virtue of the α-Hölder property for $D^\gamma\zeta$ and the fact that $D^2\Gamma_\lambda(z) = O\left(|z|^{-3}\right)$. For both cases, note that the underlying kernel $|z|^{-(3-\alpha)}$ is integrable "near the origin". Regarding the third term in (H.1.6), the mean value theorem shows

$$\left|\frac{\partial^2\Gamma_\lambda}{\partial z_i \partial z_j}(x^1-y) - \frac{\partial^2\Gamma_\lambda}{\partial z_i \partial z_j}(x^2-y)\right| \leq C\frac{|x^1-x^2|}{|x^1-y|^4}, \quad \forall\, y \in G \setminus B_{2d}(x^1).$$

In this case, the same ideas bring to light the underlying kernel $|z|^{-(4-\alpha)}$ that is "integrable at infinity" and gives rise to the desired estimate for the third term. Concerning the last term in (H.1.6), we are done as long as one notices that $D^\gamma\zeta \in C^{0,\alpha}(G)$ and shows

$$\int_{G\setminus U} \left|\frac{\partial^2\Gamma_\lambda(x^2-y)}{\partial x_i \partial x_j}\right| dy \leq C,$$

for some positive constant C depending on δ but not on $d = |x^1 - x^2|$. There are two different situations to be analyzed, either $2d \leq \text{dist}(x^1, S)$ or $2d > \text{dist}(x^1, S)$. See Figure H.1 for a sketch of the geometrical disposition of the different components.

First, if $2d \leq \text{dist}(x^1, S)$, let us define $U := \overline{B}_{2d}(x^1)$. In such case, $\partial(G \setminus U) = S \cup \partial B_{2d}(x^1)$. Then, we obtain that

$$\begin{aligned} &\int_{G\setminus U} \frac{\partial^2\Gamma_\lambda(x^2-y)}{\partial x_i \partial x_j}\,dy \\ &\qquad = \int_S \frac{\partial\Gamma_\lambda(x^2-y)}{\partial x_i}\eta_j(y)\,d_yS - \int_{\partial B_{2d}(x^1)} \frac{\partial\Gamma_\lambda(x^2-y)}{\partial x_i}\frac{(y-x^1)_j}{|y-x^1|}\,d_yS. \end{aligned} \tag{H.1.7}$$

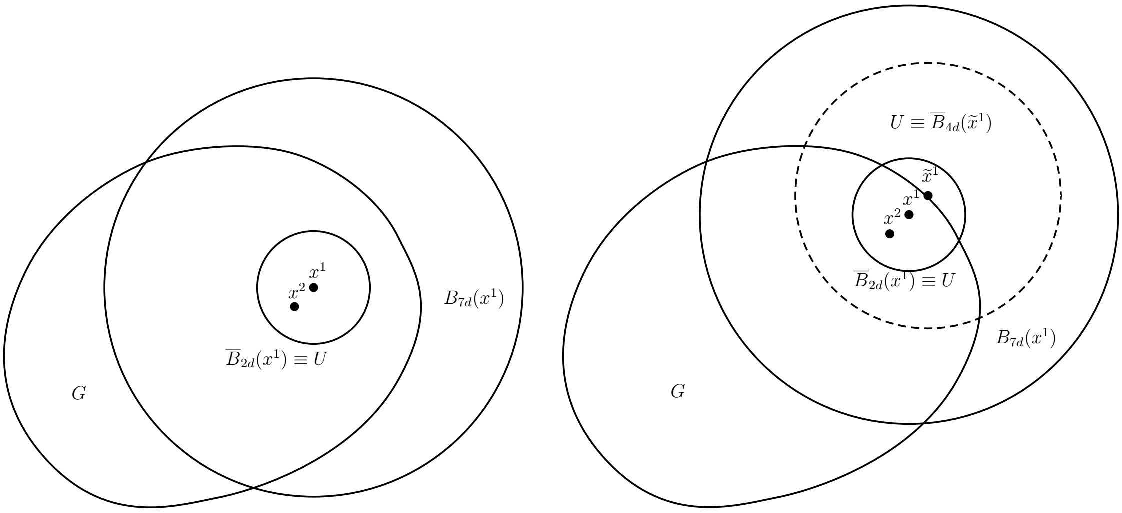


Figure H.1: Right: $U = \overline{B}_{2d}(x^1)$ in the first case. Left: $U = \overline{B}_{4d}(\widetilde{x}^1)$ in the second case.

On the one hand, Theorem H.1.1 provides a bound for the first term of (H.1.7). On the other hand, note that a combination of the control of $\nabla\Gamma_\lambda(z)$ at infinity (see Equations (H.1.4)–(H.1.5)) along with the estimate

$$\sup_{y\in\partial B_{2d}(x^1)} \frac{1}{|x^2-y|^2} \leq \frac{1}{|x^1-x^2|^2},$$

entail the aforementioned bound for the second term in (H.1.7).

Secondly, let us consider the opposite case $2d > \text{dist}(x^1, S)$. Now the configuration is slightly different. Let us fix some $\widetilde{x}^1 \in S$ so that $|x^1 - \widetilde{x}^1| = \text{dist}(x^1, S)$ and define $U := \overline{B}_{4d}(\widetilde{x}^1)$. Since $x^2 \in B_{3d}(\widetilde{x}^1)$ then, $B_{2d}(x^1) \subseteq U \subseteq B_{7d}(x^1)$. This time,

$$\int_{G\setminus U} \frac{\partial^2\Gamma_\lambda(x^2-y)}{\partial x_i \partial x_j}\,dy = \int_S \frac{\partial\Gamma_\lambda(x^2-y)}{\partial x_i}\eta_j(y)\,d_yS - \int_{\partial(U\cap G)} \frac{\partial\Gamma_\lambda(x^2-y)}{\partial x_i}\nu_j(y)\,d_yS. \tag{H.1.8}$$

The first term in (H.1.8) can be bounded through the same reasonings as above, thus we focus on the second term that will follow the idea in [214, Lemma 2.IV]. To this end, define some cut-off function $\xi\left(\frac{|y-\widetilde{x}^1|}{d}\right)$ for $\xi \in C_c^\infty(\mathbb{R}_0^+)$ such that

$$\begin{cases} \xi(r) = 1, & r \in \left[0, \frac{7}{2}\right], \\ \xi(r) \in (0,1), & r \in \left(\frac{7}{2}, 4\right), \\ \xi(r) = 0, & r \geq 4, \end{cases}$$

and consider the split

$$\begin{aligned} &\int_{\partial(G\cap B_{4d}(\widetilde{x}^1))} \frac{\partial\Gamma_\lambda(x^2-y)}{\partial x_i}\nu_j(y)\,d_yS \\ &\quad= \int_{G\cap\partial B_{4d}(\widetilde{x}^1)} \frac{\partial\Gamma_\lambda(x^2-y)}{\partial x_i}\nu_j(y)\,d_yS \\ &\quad+ \int_{S\cap B_{4d}(\widetilde{x}^1)} \frac{\partial\Gamma_\lambda(x^2-y)}{\partial x_i}\left[1-\xi\left(\frac{|y-\widetilde{x}^1|}{d}\right)\right]\nu_j(y)\,d_ySr \\ &\quad+ \int_{S\cap B_{4d}(\widetilde{x}^1)} \frac{\partial\Gamma_\lambda(x^2-y)}{\partial x_i}\xi\left(\frac{|y-\widetilde{x}^1|}{d}\right)\nu_j(y)\,d_yS. \end{aligned} \tag{H.1.9}$$

Bear in mind again that $x^2 \in B_{3d}(\widetilde{x}^1)$ and $\nabla\Gamma_\lambda(z) = O(|z|^{-2})$ when $|z| \to +\infty$. In the first term, note that $y \in G \cap \partial B_{4d}(\widetilde{x}^1)$ and consequently, $|y - x^2| \geq d$, what shows the boundedness of such term. For the second term, we have that $y \in S \cap B_{4d}(\widetilde{x}^1)$ but, in order that y belongs to the support of the cut-off function, one has to assume $|y - \widetilde{x}^1| \geq \frac{7}{2}d$. Thus, $|y - x^2| \geq \frac{d}{2}$ and a similar reasoning now yields

$$\left|\int_{S\cap B_{4d}(\widetilde{x}^1)} \frac{\partial\Gamma_\lambda(x^2 - y)}{\partial x_i}\nu_j(y)\, d_yS\right| \leq \frac{\widetilde{C}}{|x^1 - x^2|^2}|S \cap B_{4d}(\widetilde{x}^1)|.$$

The upper bound for the second term is done once we note that for regular surfaces $|S \cap B_{4d}(\widetilde{x}^1)| \leq Cd^2$. To prove the corresponding bound for the third term in (H.1.9), we consider the potential

$$\mathcal{S}(x) = \int_S \frac{\partial\Gamma_\lambda(x - y)}{\partial x_i}\xi\left(\frac{|y - \widetilde{x}^1|}{d}\right)\nu_j(y)\, d_yS, \quad x \in G,$$

whose $C^{0,\alpha}$ estimate follows again from Lemma H.1.1:

$$\|\mathcal{S}\|_{C^{0,\alpha}(\overline{G})} \leq C\left\|\xi\left(\frac{|\cdot - \widetilde{x}^1|}{d}\right)\nu_j\right\|_{C^{0,\alpha}(S)} \leq C\left(1 + \frac{1}{d^\alpha}\right).$$

Let us now fix $0 < \delta < 1$ small enough so that $x - \theta\eta(x) \in G$ for every couple $x \in S$ and $0 < \theta < 4\delta$. Thus, $\mathcal{S}(\widetilde{x}^1 - 4d\eta(\widetilde{x}^1)) = 0$ and consequently,

$$|\mathcal{S}(x^2)| = |\mathcal{S}(x^2) - \mathcal{S}(\widetilde{x}^1 - 4d\eta(\widetilde{x}^1))| \leq \frac{C}{d^\alpha}|x^2 - \widetilde{x}^1 + 4d\eta(\widetilde{x}^1)|^\alpha \leq \frac{C}{d^\alpha}(3d + 4d)^\alpha \leq \widetilde{C}.$$

□

Remark H.1.3. *In the above reasoning, the property $|S \cap B_{4d}(\widetilde{x}^1)| \leq Cd^2$ for regular surfaces was essential. This is a regularity property intimately connected with deep issues in harmonic analysis. Indeed, a surface S is said to be Alhfolrs–David regular when*

$$|S \cap B_R(x)| \leq CR^2,$$

for every couple $x \in S$, $R > 0$ and some nonnegative constant C. These surfaces (originally curves) arise from the study of singular integrals along curves [96], and had already appeared in the works of A. Calderón (e.g., [52]) on L^2 estimates for the Cauchy integral along Lipschitz curves. His results were improved by G. David [96] to the more general setting of Alhfors–David curves and S. Semmes [273] generalized these results to the d-dimensional framework. Specifically, Ahlfors–David regularity was shown to control singular integral operators that are much more general than the Cauchy integral. Of course, $C^{k,\alpha}$ surfaces are Ahlfors–David regular.

Lemma H.1.4. *Let $G \subseteq \mathbb{R}^3$ be a bounded domain with regularity $C^{k+1,\alpha}$, $\Omega := \mathbb{R}^3 \setminus \overline{G}$ its exterior domain and $S = \partial G$ the boundary surface. The generalized volume potential on Ω associated with the Helmholtz equation and generated by a density ζ in G*

$$(\mathcal{N}_\lambda^+\zeta)(x) = \int_G \Gamma_\lambda(x - y)\zeta(y)\, dy, \quad x \in \Omega,$$

defines a bounded linear map $\mathcal{N}_\lambda^+ : C^{k,\alpha}(\overline{G}) \longrightarrow C^{k+2,\alpha}(\overline{\Omega})$.

Proof. Our argument is based on some ideas of [214, Teorema 3.II]. Consider $R > 0$ such that $\overline{G} \subseteq B_R(0)$ and let us estimate $\|\mathcal{N}_\lambda^+\zeta\|_{C^{k+2,\alpha}(\Omega)}$ in $\mathbb{R}^3 \setminus \overline{B}_R(0)$ and $\overline{\Omega_{2R}}$ where $\Omega_{2R} := B_{2R}(0) \setminus \overline{G}$. Set

$$d_R := \min\{|x-y| : x \in \mathbb{R}^3 \setminus B_R(0),\, y \in \overline{G}\} > 0,$$

and assume that $d_R < 1$.

Equations (H.1.2), (H.1.4) and (H.1.5) yield

$$|D_x^\gamma \Gamma_\lambda(x-y)| \leq \widetilde{C}\frac{1}{d_R^{|\gamma|+1}},$$

for every multi-index γ and every $x \in \mathbb{R}^3 \setminus B_R(0)$ and $y \in G$. One can then take derivatives under the integral sign and obtain the desired estimate for the $C^{k+2,\alpha}$ norm in $\mathbb{R}^3 \setminus \overline{B}_R(0)$. On the other hand, consider $\overline{\zeta} \in C^{k,\alpha}(\mathbb{R}^3)$ some extension though Calderón's theorem (see Proposition 6.4.1 in Chapter 6). Then,

$$(\mathcal{N}_\lambda^+\zeta)(x) = \int_{B_{2R}(0)} \Gamma_\lambda(x-y)\overline{\zeta}(y)\,dy - \int_{\Omega_{2R}} \Gamma_\lambda(x-y)\overline{\zeta}(y)\,dy,$$

for every $x \in \Omega_{2R}$. Since $\Omega_{2R} \subseteq B_{2R}(0)$, the triangle inequality yields:

$$\|\mathcal{N}_\lambda^+\zeta\|_{C^{k+2,\alpha}(\overline{\Omega_{2R}})} \leq \left\|\int_{B_{2R}(0)} \Gamma_\lambda(\cdot-y)\overline{\zeta}(y)\,dy\right\|_{C^{k+2,\alpha}(\overline{B}_{2R}(0))} + \left\|\int_{\Omega_{2R}} \Gamma_\lambda(\cdot-y)\overline{\zeta}(y)\,dy\right\|_{C^{k+2,\alpha}(\overline{\Omega_{2R}})}.$$

Finally, note that both domains are bounded and, consequently, Lemma H.1.2 and Proposition 6.4.1 apply and yield the desired estimate

$$\|\mathcal{N}_\lambda^+\zeta\|_{C^{k+2,\alpha}(\overline{\Omega_{2R}})} \leq M\|\zeta\|_{C^{k,\alpha}(B_{2R}(0))} \leq MC_{\mathcal{P}}\|\zeta\|_{C^{k,\alpha}(\overline{G})}.$$

□

Now, we focus on similar bounds for singular and weakly singular kernels in the whole space $\mathbb{R}^d$. This results are classical in the homogeneous harmonic case, $\Gamma_0(z)$, and can be found in [139, 215, 214]. However, not only will we need harmonic potentials, but we will also deal with general singular and weakly singular kernels. To this end, we remind [167, Satz 3.4, Satz 5.4].

Theorem H.1.5 (Weakly singular kernels)**.** *Let us consider* $0 \leq \beta \leq d-1$, $0 < \alpha < 1$ *and* $K(x,z)$, $x \in \overline{D}, z \in \mathbb{R}^N \setminus \{0\}$ *a weakly singular integral kernel of exponent* β *satisfying the following three hypothesis:*

1. *For each* $x \in \overline{D}$
$$K(x,\cdot) \in C^1(\mathbb{R}^d \setminus \{0\}).$$
2. *For each* $x \in \overline{D}$ *and* $z \in \mathbb{R}^d \setminus \{0\}$
$$|\nabla_z K(x,z)| \leq \frac{C}{|z|^{\beta+1}}.$$
3. *For all* $x_1, x_2 \in \overline{D}$ *and* $z \in \mathbb{R}^d \setminus \{0\}$ *one has*
$$|K(x_1,z) - K(x_2,z)| \leq C\frac{|x_1-x_2|^\alpha}{|z|^\beta}.$$

Then, the generalized volume potential generated by a density ζ in $\mathbb{R}^d$,

$$(\mathcal{N}_K\zeta)(x) := \int_{\mathbb{R}^d} K(x, x-y)\zeta(y)\,dy,\ x \in \overline{D},$$

defines a bounded linear map for each positive radius R

$$\mathcal{N}_K : C_c^{0,\alpha}(B_R(0)) \longrightarrow C^{0,\alpha}(\overline{D}).$$

Theorem H.1.6 (Singular kernels). *Consider $0 < \alpha < 1$ and $K(x,z),\ x \in \overline{D},\ z \in \mathbb{R}^d \setminus \{0\}$ a kernel satisfying the following hypotheses:*

1. *$K(x,z)$ is positively homogeneous of degree $-(d-1)$ with respect to the second variable, i.e.,*

$$K(x, \lambda z) = \lambda^{-(d-1)} K(x,z),$$

for all $x \in \overline{D},\ z \in \mathbb{R}^d \setminus \{0\}$ and $\lambda > 0$.

2. *$K(x,z)$ has the following regularity properties for every $x \in \overline{D}$ and each indices $1 \le i,j \le d$:*

$$\begin{aligned}
K &\in C^1(\overline{D} \times (\mathbb{R}^d \setminus \{0\})), & K(x,\cdot) &\in C^2(\mathbb{R}^d \setminus \{0\}),\\
\frac{\partial K}{\partial x_i} &\in C(\overline{D} \times (\mathbb{R}^d \setminus \{0\})), & \frac{\partial K}{\partial x_i}(x,\cdot) &\in C^1(\mathbb{R}^d \setminus \{0\}),\\
\frac{\partial K}{\partial z_i} &\in C(\overline{D} \times (\mathbb{R}^d \setminus \{0\})), & &\\
\frac{\partial^2 K}{\partial z_i \partial x_j} &\in C(\overline{D} \times (\mathbb{R}^d \setminus \{0\})), & \frac{\partial^2 K}{\partial z_i \partial z_j} &\in C(\overline{D} \times (\mathbb{R}^d \setminus \{0\})).
\end{aligned}$$

3. *The first derivatives of $K(x,z)$ are Hölder-continuous with exponent α with respect to x in the sense that, for each $x_1, x_2 \in \overline{D},\ z \in \mathbb{R}^d \setminus \{0\}$ and for all index $1 \le i \le d$,*

$$\begin{aligned}
\left|\frac{\partial K}{\partial x_i}(x_1,z) - \frac{\partial K}{\partial x_i}(x_2,z)\right| &\le C\frac{|x_1-x_2|^\alpha}{|z|^{d-1}},\\
\left|\frac{\partial K}{\partial z_i}(x_1,z) - \frac{\partial K}{\partial z_i}(x_2,z)\right| &\le C\frac{|x_1-x_2|^\alpha}{|z|^{d}}.
\end{aligned}$$

Then, the generalized volume potential defines a bounded linear map for every positive radius $R > 0$

$$\mathcal{N}_K : C_c^{0,\alpha}(B_R(0)) \longrightarrow C^{1,\alpha}(\overline{D}).$$

Moreover, for every $1 \le i \le d$,

$$\frac{\partial}{\partial x_i}(\mathcal{N}_K\zeta) = \mathcal{N}_{\frac{\partial K}{\partial x_i}}\zeta + \mathcal{N}_{\frac{\partial K}{\partial z_i}}\zeta.$$

Notice that the singular integral kernel $\frac{\partial K}{\partial z_i}$ has an associated singular integral operator $\mathcal{N}_{\frac{\partial K}{\partial z_i}}$, where the integrals require to be understood in the sense of Cauchy principal values by virtue of the cancellation properties arising from the homogeneity in z of the original kernel $K(x,z)$. Another interesting remark, that explains some differences between volume potentials in the whole $\mathbb{R}^d$ and volume potentials in a bounded domain, is the change of variables formula

$$(\mathcal{N}_K\zeta)(x) = \int_{\mathbb{R}^d} K(x,x-y)\zeta(y)\,dy = \int_{\mathbb{R}^d} K(x,z)\zeta(x-z)\,dz, \tag{H.1.10}$$

which lets us take derivatives in any of the fwo factors. When the kernel is not sufficiently well behaved, we can put the derivatives on the density, or the other way round. Obviously, it is no longer valid for densities on G, where the integration by part argument in the proof of Lemma H.1.2 is required, producing new boundary term that must be studied via Theorem H.1.1.

As a consequence, one can prove the next two corollaries, where higher order derivatives of these generalized volume potentials can be considered.

Corollary H.1.7. *Let us consider* $0 \leq \beta \leq d-1$, $0 < \alpha < 1$, $k, m \in \mathbb{N}$ *so that* $\beta + m \leq d-1$ *and* $K(x,z)$, $x \in \overline{D}, z \in \mathbb{R}^d \setminus \{0\}$, *a weakly singular integral kernel of exponent* β *verifying the next hypothesis for each couple of multi-indices* γ_1, γ_2 *with* $|\gamma_1| \leq k$ *and* $|\gamma_2| \leq m$*:*

1. $D_x^{\gamma_1+\gamma_2} K(x,z)$ *is weakly singular with exponent* β *and* $D_x^{\gamma_1} D_z^{\gamma_2} K(x,z)$ *is the sum of weakly singular integral kernels with exponents raging from* β *to* $\beta + |\gamma_2|$, *i.e.,*

$$\left|D_x^{\gamma_1+\gamma_2} K(x,z)\right| \leq \frac{C}{|z|^\beta},$$
$$|D_x^{\gamma_1} D_z^{\gamma_2} K(x,z)| \leq C\left(\frac{1}{|z|^\beta} + \frac{1}{|z|^{\beta+|\gamma_2|}}\right).$$

2. *For every* $x \in \overline{D}$,

$$(D_x^{\gamma_1+\gamma_2} K)(x,\cdot),\ (D_x^{\gamma_1} D_z^{\gamma_2} K)(x,\cdot) \in C^1(\mathbb{R}^d \setminus \{0\}).$$

3. *For all* $x \in \overline{D}$, $z \in \mathbb{R}^d \setminus \{0\}$,

$$\left|\nabla_z D_x^{\gamma_1+\gamma_2} K(x,z)\right| \leq C\left(\frac{1}{|z|^\beta} + \frac{1}{|z|^{\beta+1}}\right),$$
$$|\nabla_z D_x^{\gamma_1} D_z^{\gamma_2} K(x,z)| \leq C\left(\frac{1}{|z|^\beta} + \frac{1}{|z|^{\beta+|\gamma_2|+1}}\right).$$

4. *For any* $x_1, x_2 \in \overline{D}$, $z \in \mathbb{R}^N \setminus \{0\}$,

$$|D_x^{\gamma_1+\gamma_2} K(x_1,z) - D_x^{\gamma_1+\gamma_2} K(x_2,z)| \leq \frac{C}{|z|^\beta}|x_1 - x_2|^\alpha,$$
$$|D_x^{\gamma_1} D_z^{\gamma_2} K(x_1,z) - D_x^{\gamma_1} D_z^{\gamma_2} K(x_2,z)| \leq C\left(\frac{1}{|z|^\beta} + \frac{1}{|z|^{\beta+|\gamma_2|}}\right)|x_1 - x_2|^\alpha.$$

Then, the generalized volume potential defines a bounded linear operator for every positive radius R

$$\mathcal{N}_K : C_c^{k,\alpha}(B_R(0)) \longrightarrow C^{k+m,\alpha}(\overline{D}).$$

Moreover, for every multi-index $\gamma = \gamma_1 + \gamma_2$ *so that* $|\gamma_1| \leq k$ *and* $|\gamma_2| \leq m$

$$D^\gamma(\mathcal{N}_K \zeta) = \sum_{\delta \leq \gamma_1} \binom{\gamma_1}{\delta} \left(\mathcal{N}_{D_x^{\delta+\gamma_2} K} D^{\gamma_1-\delta}\zeta + \mathcal{N}_{D_x^\delta D_z^{\gamma_2} K} D^{\gamma_1-\delta}\zeta\right).$$

Corollary H.1.8. *Let* $0 < \alpha < 1$, $k \in \mathbb{N}$, $x \in \overline{D}$, $z \in \mathbb{R}^d \setminus \{0\}$ *and* $K(x,z)$ *be a weakly singular kernel, which has the following properties:*

1. $K(x,z)$ *is positively homogeneous of degree* $-(d-1)$ *in the second variable.*

2. *$K(x,z)$ has the regularity properties*

$$D_x^\gamma K \in C^1(\overline{D} \times (\mathbb{R}^d \setminus \{0\})), \qquad (D_x^\gamma K)(x,\cdot) \in C^2(\mathbb{R}^d \setminus \{0\}),$$
$$\frac{\partial}{\partial x_i} D_x^\gamma K \in C(\overline{D} \times (\mathbb{R}^d \setminus \{0\})), \qquad \left(\frac{\partial}{\partial x_i} D_x^\gamma K\right)(x,\cdot) \in C^1(\mathbb{R}^d \setminus \{0\}),$$
$$\frac{\partial}{\partial z_i} D_x^\gamma K \in C(\overline{D} \times (\mathbb{R}^d \setminus \{0\})),$$
$$\frac{\partial^2}{\partial z_i \partial x_j} D_x^\gamma K \in C(\overline{D} \times (\mathbb{R}^d \setminus \{0\})), \qquad \frac{\partial^2}{\partial z_i \partial z_j} D_x^\gamma K \in C(\overline{D} \times (\mathbb{R}^d \setminus \{0\})),$$

for each couple of indices $1 \le i,j \le d$ and each multi-index γ with $|\gamma| \le k$.

3. *The derivatives of $K(x,z)$ with respect to x up to order k are α-Hölder continuous,*

$$\left|\left(\frac{\partial}{\partial x_i} D_x^\gamma K\right)(x_1,z) - \left(\frac{\partial}{\partial x_i} D_x^\gamma K\right)(x_2,z)\right| \le C \frac{|x_1-x_2|^\alpha}{|z|^{d-1}},$$
$$\left|\left(\frac{\partial}{\partial z_i} D_x^\gamma K\right)(x_1,z) - \left(\frac{\partial}{\partial z_i} D_x^\gamma K\right)(x_2,z)\right| \le C \frac{|x_1-x_2|^\alpha}{|z|^{d}},$$

for each $x_1, x_2 \in \overline{D}$, $z \in \mathbb{R}^d \setminus \{0\}$, each index $1 \le i \le d$ and $|\gamma| \le k$.

Then, the generalized volume potential defines a bounded linear operator for every positive radius R

$$\mathcal{N}_K : C_c^{k,\alpha}(B_R(0)) \longrightarrow C^{k+1,\alpha}(\overline{D}).$$

Moreover, for every multi-index γ with $|\gamma| \le k$ and any index $1 \le i \le d$:

$$\frac{\partial}{\partial x_i} D_x^\gamma (\mathcal{N}_K \zeta) = \sum_{\delta \le \gamma} \binom{\gamma}{\delta} \left(\mathcal{N}_{\frac{\partial}{\partial x_i} D_x^\gamma K} D^{\delta-\gamma} \zeta + \mathcal{N}_{\frac{\partial}{\partial z_i} D_x^\gamma K} D^{\delta-\gamma} \zeta \right).$$

When the constants C appearing in the statements of the above results do not depend on the chosen bounded domain D, the above estimates can be extended from Hölder estimates over $\overline{D}$, to global estimates in $\mathbb{R}^d$. This is the case for the integral kernels which do not depend on the variable x (e.g., $\Gamma_0(z)$, $R_\lambda(z)$ and $\Gamma_\lambda(z)$). In this way, we get the next result in the spirit of Lemmas H.1.2 and H.1.4.

Lemma H.1.9. *The generalized volume potential in $\mathbb{R}^3$ associated with the Helmholtz equation*

$$(\mathcal{N}_\lambda \zeta)(x) := \int_{\mathbb{R}^3} \Gamma_\lambda(x-y)\zeta(y)\,dy, \; x \in \mathbb{R}^3,$$

defines a bounded linear operator for every positive radius R

$$\mathcal{N}_\lambda : C_c^{k,\alpha}(B_R(0)) \longrightarrow C^{k+2,\alpha}(\mathbb{R}^3).$$

Combining the above results, we can estimate generalized volume potentials in Ω whose densities have compact support in $\overline{\Omega}$ by means of an appropriate splitting. Using Calderón's extension theorem (Proposition 6.4.1), for every $\zeta \in C_c^{k,\alpha}(\overline{\Omega})$ there exists an extension $\overline{\zeta} \in C_c^{k,\alpha}(\mathbb{R}^3)$, so

$$\mathcal{N}_\lambda^+ \zeta = \left(\mathcal{N}_\lambda \overline{\zeta}\right)\big|_\Omega - \mathcal{N}_\lambda^+ \left(\overline{\zeta}\big|_G\right) \text{ in } \Omega.$$

Then, Lemmas H.1.4 and H.1.9 lead to the following result:

Theorem H.1.10 (Generalized volume potential). *Let $G \subseteq \mathbb{R}^3$ be a bounded domain with regularity $C^{k+1,\alpha}$, $\Omega := \mathbb{R}^3 \setminus \overline{G}$ its exterior domain and $S = \partial G$ the boundary surface. The generalized volume potential associated with the Helmholtz equation and generated by a density ζ in Ω,*

$$(\mathcal{N}_\lambda^+ \zeta)(x) = \int_\Omega \Gamma_\lambda(x-y)\zeta(y)\,dy,\ x \in \Omega,$$

defines a bounded linear operator for every positive radius R

$$\mathcal{N}_\lambda^+ : C_c^{k,\alpha}(B_R(0) \setminus G) \longrightarrow C^{k+2,\alpha}(\overline{\Omega}).$$

H.2 Regularity of the boundary integral operator T_λ

The next step is to analyze the regularity properties of the boundary integral operator T_λ (6.3.24) arising in the boundary integral equation (6.3.23) associated with the boundary data $\eta \times u$ in Theorem 6.3.14 of Chapter 6. Firstly, we split the operator T_λ into

$$T_\lambda = \mathcal{M}_\lambda^T + \lambda \mathcal{S}_\lambda^T.$$

$\mathcal{M}_\lambda^T \zeta$ is known as the *magnetic dipole operator*, which is the tangent component of the electric field generated by a dipole distribution with density $\zeta \in \mathfrak{X}(S)$, i.e.,

$$(\mathcal{M}_\lambda^T \zeta)(x) := \int_S \eta(x) \times \operatorname{curl}_x \left(\Gamma_\lambda(x-y)\zeta(y)\right)\, d_yS,\ \ x \in S.$$

$\mathcal{S}_\lambda^T \zeta$ is the tangential component of the generalized single layer potential generated by ζ,

$$(\mathcal{S}_\lambda^T \zeta)(x) = \int_S \Gamma_\lambda(x-y)\eta(x) \times \zeta(y)\, d_yS,\ \ x \in S.$$

The integral kernel of $\mathcal{S}_\lambda^T$ is weakly singular over S, so this integral is absolutely convergent under suitable hypotheses for ζ. The integral in $\mathcal{M}_\lambda^T$ is absolutely convergent under minimal assumption on ζ. Indeed, although the integral kernel looks singular over S let us see this it is again weakly singular when ζ is a tangent vector field on S. Notice that, given any tangent field $\zeta \in \mathfrak{X}^{k,\alpha}(S)$ along S, one can split

$$\eta(x) \times (\nabla_x \Gamma_\lambda(x-y) \times \zeta(y))\ = (\eta(x) - \eta(y)) \cdot \zeta(y)\, \nabla_x \Gamma_\lambda(x-y) - \eta(x) \cdot \nabla_x \Gamma_\lambda(x-y)\, \zeta(y).$$

Consequently, the j-th coordinate of the integrand in $(\mathcal{M}_\lambda^T \zeta)(x)$ read

$$(\eta(x) \times (\nabla_x \Gamma_\lambda(x-y) \times \zeta(y)))_j \\ = \sum_{i=1}^3 (\eta_i(x) - \eta_i(y))\zeta_i(y)\, \partial_{x_j} \Gamma_\lambda(x-y) - \eta(x) \cdot \nabla_x \Gamma_\lambda(x-y)\, \zeta_j(y),$$

and, the corresponding the j-th coordinate of the integrand of $(\mathcal{S}_\lambda^T \zeta)(x)$ takes the form

$$(\Gamma_\lambda(x-y)\eta(x) \times \zeta(y))_j = \sum_{i=1}^3 \Gamma_\lambda(x-y)(e_i \times e_j) \cdot \eta(x)\zeta_i(y),$$

respectively. Consider any extension $\widetilde{\eta} \in C_c^{k+4,\alpha}(\mathbb{R}^3)$ of the outward unit normal vector field η to the compact surface S and define the kernels

$$\begin{gathered} K_\lambda^{\mathcal{D}}(x,z) = \widetilde{\eta}(x) \cdot \nabla \Gamma_\lambda(z), \\ K_\lambda^{i,j}(x,z) = (\widetilde{\eta}_i(x) - \widetilde{\eta}_i(x-z))\, \partial_{z_j} \Gamma_\lambda(z), \qquad \widetilde{K}_\lambda^{i,j}(x,z) = (e_i \times e_j) \cdot \widetilde{\eta}(x)\, \Gamma_\lambda(z). \end{gathered} \tag{H.2.1}$$

Then, we have the associated splitting of the operators $\mathcal{M}_\lambda^T$ and $\mathcal{S}_\lambda^T$

$$(\mathcal{M}_\lambda^T\zeta)_j(x)=\sum_{i=1}^3 T_{K_\lambda^{i,j}}\zeta_i-T_{K_\lambda^{\mathcal{D}}}\zeta_j,\qquad (\mathcal{S}_\lambda^T\zeta)_j(x)=\sum_{i=1}^3 T_{\widetilde{K}_\lambda^{i,j}}\zeta_i, \tag{H.2.2}$$

where the integral operators in the above decomposition are

$$(T_{K_\lambda^{\mathcal{D}}}\zeta_j)(x)=\int_S K_\lambda^{\mathcal{D}}(x,x-y)\zeta_j(y)\,d_yS,$$
$$(T_{K_\lambda^{i,j}}\zeta_i)(x)=\int_S K_\lambda^{i,j}(x,x-y)\zeta_i(y)\,d_yS,\qquad (T_{\widetilde{K}_\lambda^{i,j}}\zeta_i)(x)=\int_S \widetilde{K}_\lambda^{i,j}(x,x-y)\zeta_i(y)\,d_yS. \tag{H.2.3}$$

Since every C^2 compact surface satisfies

$$|\eta(x)\cdot(x-y)|\leq L|x-y|^2,\qquad |\eta(x)-\eta(y)|\leq L|x-y|,$$

for each $x,y\in S$, then all the preceding integral kernels are weakly singular. In particular, we do not need to regard these integrals in the Cauchy principal value sense.

The study of Hölder estimates for all these potentials can be performed along the same lines as in [167, Satz 4.3, Satz 4.4]. In that work, the author dealt with the homogeneous harmonic case $\lambda=0$, where the kernels have a simpler form. In our case $\lambda\neq 0$, we will decompose the 3-dimensional kernels into a homogeneous part and an inhomogeneous but less singular part as in (H.1.2). Then, we will consider a coordinate system over S which allows transforming the integrals over S into integrals over planar domains by means of a change of variables. The homogeneous and more singular parts will satisfy the hypothesis in Corollary H.1.8 and the terms in the remainder will verify those in Corollary H.1.7. We remark here that C^{k+5} boundaries are needed precisely for the operators in (H.2.3) of first and second type to be bounded from the space $C^{k,\alpha}(S)$ into the space $C^{k+1,\alpha}(S)$. However, we only need C^{k+4} boundaries to ensure the corresponding result for only the third kind of operators in (H.2.3) (see [167, Satz 4.3, Satz 4.4] for the homogeneous harmonic case $\lambda=0$). Our regularity result then reads as follows:

Theorem H.2.1. *Let G be a bounded domain of class C^{k+5}, $S=\partial G$ the boundary surface, $\eta\in C^{k+4}(S,\mathbb{R}^3)$ the outward unit normal vector field along S and any extension $\widetilde{\eta}\in C_c^{k+4}(\mathbb{R}^3,\mathbb{R}^3)$ of η. Let $K_\lambda^{\mathcal{D}}(x,z)$, $K_\lambda^{i,j}(x,z)$ and $\widetilde{K}_\lambda^{i,j}(x,z)$ be the kernels given by (H.2.1) Then, the associated boundary operators $T_{K_\lambda^{\mathcal{D}}}$, $T_{K_\lambda^{i,j}}$ and $T_{\widetilde{K}_\lambda^{i,j}}$ given by (H.2.3) are bounded*

$$T_{K_\lambda^{\mathcal{D}}}:C^{k,\alpha}(S)\longrightarrow C^{k+1,\alpha}(S),$$
$$T_{K_\lambda^{i,j}}:C^{k,\alpha}(S)\longrightarrow C^{k+1,\alpha}(S),\qquad T_{\widetilde{K}_\lambda^{i,j}}:C^{k,\alpha}(S)\longrightarrow C^{k+1,\alpha}(S).$$

As a consequence, the next linear operators are also bounded

$$\mathcal{M}_\lambda^T:\mathfrak{X}^{k,\alpha}(S)\longrightarrow \mathfrak{X}^{k+1,\alpha}(S),\qquad \mathcal{S}_\lambda^T:\mathfrak{X}^{k,\alpha}(S)\longrightarrow \mathfrak{X}^{k+1,\alpha}(S).$$

Proof. Since the kernel $\widetilde{K}_\lambda^{i,j}(x,z)$ can be analyzed through a similar reasoning (as shown in [167] for the case $\lambda=0$), we will restrict our analysis to the kernels $K_\lambda^{i,j}(x,z)$ and $K_\lambda^{\mathcal{D}}(x,z)$, which were not explictely studied in [167]. Let us then split these inhomogeneous kernels into a homogeneous part and some less singular part (see the decomposition (H.1.2) and the functions ϕ_λ and ψ_λ in (H.1.1)). To this end, notice that

$$K_\lambda^{\mathcal{D}}(x,z)=\frac{\phi_\lambda'(|z|)}{|z|}\widetilde{\eta}(x)\cdot z,\qquad K_\lambda^{i,j}(x,z)=(\widetilde{\eta}_i(x)-\widetilde{\eta}_i(x-z))\,\frac{\phi_\lambda'(|z|)}{|z|}z_j, \tag{H.2.4}$$

Consequently, one can decompose,

$$K_\lambda^{i,j}(x,z) = K_{\lambda,0}^{i,j} + K_{\lambda,1}^{i,j}, \quad K_\lambda^{\mathcal{D}}(x,z) = K_{\lambda,0}^{\mathcal{D}} + K_{\lambda,1}^{\mathcal{D}}, \tag{H.2.5}$$

where each factor take the form,

$$\begin{aligned} K_{\lambda,0}^{i,j}(x,z) &:= -\frac{1}{4\pi}(\widetilde{\eta}_i(x) - \widetilde{\eta}_i(x-z))\frac{z_j}{|z|^3}, \; K_{\lambda,1}^{i,j}(x,z) &&:= (\widetilde{\eta}_i(x) - \widetilde{\eta}_i(x-z))\frac{\psi_\lambda'(|z|)}{|z|}z_j, \\ K_{\lambda,0}^{\mathcal{D}}(x,z) &:= -\frac{1}{4\pi}\widetilde{\eta}(x)\cdot\frac{z}{|z|^3}, \; K_{\lambda,1}^{\mathcal{D}}(x,z) &&:= \widetilde{\eta}(x)\cdot\frac{\psi_\lambda'(|z|)z}{|z|}. \end{aligned} \tag{H.2.6}$$

Notice that the associated integral operators only involve values $x, y \in S$. Define

$$d_S := 2\max_{x,y\in S}|x-y|,$$

and take $x \in S$, $z \in B_{d_S}(0)$. Thus, an easy computation yields the following relations, that will be steadily used along the proof,

$$|z|^{-\beta_1} + |z|^{-\beta_2} \leq \left(1 + d_S^{M-m}\right)|z|^{-M}, \quad |z|^{\beta_1} + |z|^{\beta_2} \leq \left(1 + d_S^{M-m}\right)|z|^{m}, \tag{H.2.7}$$

for any couple of exponents $\beta_1, \beta_2 \geq 0$ and any $z \in B_{d_S}(0)$. Here m and M stand for the minimum and maximum values i.e.,

$$m := \min\{\beta_1, \beta_2\} \text{ and } M := \max\{\beta_1, \beta_2\}.$$

Another useful remark is that $f_\lambda(r) := \psi_\lambda'(r)/r$, arising in (H.2.6), can be controlled by (H.1.3) as follows

$$\begin{aligned} |f_\lambda^{(m)}(r)| &\leq C\left(\frac{1}{r} + \frac{1}{r^{m+2}}\right), && r > 0, \\ |f_\lambda^{(m)}(r)| &\leq \widetilde{C}\frac{1}{r^{m+2}}, && r \in (0, d_S). \end{aligned} \tag{H.2.8}$$

for some $C > 0$ that does not depend on m and some $\widetilde{C}$ depending on m and d_S.

Let us study the boundedness of the integral operators associated with the integral kernels $K_{\lambda,n}^{i,j}$ and $K_{\lambda,n}^{\mathcal{D}}$ for $n = 0, 1$. To this end, let us consider a finite covering of S by M coordinate neighborhoods $\Sigma_1, \ldots, \Sigma_M \subseteq S$ endowed with the associated local charts $\mu_m \in C^{k+5}(\overline{D}_m, \Sigma_m)$ that enjoy homeomorphic extensions up to the boundary of the planar disks $D_m \subseteq \mathbb{R}^2$. Also consider the associated partition of unity of class C^{k+5}, $\{\varphi_m\}_{m=1}^M \subseteq C^{k+5}(S)$, subordinated to the above open covering. The Jacobian of each chart will be denoted by

$$J_m(s) := \left|\left(\frac{\partial\mu_m}{\partial s_1} \times \frac{\partial\mu_m}{\partial s_2}\right)(s)\right|, \quad s \in D_m.$$

All the above notation then yields the decompositions

$$(T_{K_{\lambda,n}^{i,j}}\zeta)(\mu_m(s)) = \sum_{m'=1}^M \int_{D_{m'}} K_{\lambda,n}^{i,j}(\mu_m(s), \mu_m(s) - \mu_{m'}(t))\varphi_{m'}(\mu_{m'}(t))\zeta(\mu_{m'}(t))J_{m'}(t)\,dt, \tag{H.2.9}$$

$$(T_{K_{\lambda,n}^{\mathcal{D}}}\zeta)(\mu_m(s)) = \sum_{m'=1}^M \int_{D_{m'}} K_{\lambda,n}^{\mathcal{D}}(\mu_m(s), \mu_m(s) - \mu_{m'}(t))\varphi_{m'}(\mu_{m'}(t))\zeta(\mu_{m'}(t))J_{m'}(t)\,dt. \tag{H.2.10}$$

We will study the most singular case $m'=m$ and then show how the case $m'\neq m$ follows from it. An important fact is that we will extract the most singular homogeneous parts of $K^{i,j}_{\lambda,0}(x,z)$ and $K^{\mathcal{D}}_{\lambda,0}(x,z)$ by virtue of the split (H.2.5). However, the change of variables in the coordinate neighborhoods Σ_m gives rise to new inhomogeneous planar kernels, $K^{i,j}_{\lambda,0}(\mu_m(s),\mu_m(s)-\mu_m(t))$ and $K^{\mathcal{D}}_{\lambda,0}(\mu_m(s),\mu_m(s)-\mu_m(t))$. To solve this difficulty, we will decompose them again into the more sigular homogeneous part, which stands for a planar homogeneous kernel of degree -1, and some inhomogeneous but less singular term. Then, we will prove the corresponding regularity results for each term through Corollaries H.1.7 and H.1.8.

Since both $K^{i,j}_{\lambda,0}(x,z)$ and $K^{\mathcal{D}}_{\lambda,0}(x,z)$ can be studied by means of a similar reasoning, we will just analyze one of them, e.g. $K^{i,j}_{\lambda,0}(x,z)$. In fact, $K^{\mathcal{D}}_{\lambda,0}(x,z)$ stands for the integral kernel of the adjoint operator of the harmonic Neumann–Poincaré operator, that was studied in [167, Satz 4.4]. Inspired by [167, Lemma 4.2], let us expand $\mu_m(s)-\mu_m(t)$ though Taylor's theorem up to second order; that is,

$$|\mu_m(s)-\mu_m(t)|=(P_m(s,s-t)+Q_m(s,s-t))^{1/2}\,, \tag{H.2.11}$$

where,

$$P_m(s,u):=\sum_{p,q=1}^{2}\frac{\partial\mu_m}{\partial s_p}(s)\cdot\frac{\partial\mu_m}{\partial s_q}(s)u_pu_q=\sum_{p,q=1}^{2}g_m^{pq}(s)u_pu_q=((g_m^{pq}(s))u)\cdot u, \tag{H.2.12}$$

$$\begin{aligned}Q_m(s,u)&:=-2\sum_{p,q,r=1}^{2}\frac{\partial\mu_m}{\partial s_p}(s)\cdot\left(\int_0^1(1-\theta)\frac{\partial^2\mu_m}{\partial s_q\partial s_r}(s-\theta u)\,d\theta\right)u_pu_qu_r\\&+\sum_{p,q,r,l=1}^{2}\left(\int_0^1(1-\theta)\frac{\partial^2\mu_m}{\partial s_p\partial s_q}(s-\theta u)\,d\theta\right)\cdot\left(\int_0^1(1-\theta)\frac{\partial^2\mu_m}{\partial s_r\partial s_l}(s-\theta u)\,d\theta\right)u_pu_qu_ru_l.\end{aligned} \tag{H.2.13}$$

First, $P_m(s,u)$ is positively homogeneous on u of degree 2 with respect to u and the following control is achieved (see [167, Satz 4.2])

$$\begin{array}{rcl}\frac{1}{C}|u|^2\leq & |P_m(s,u)| & \leq C|u|^2,\\ & |Q_m(s,u)| & \leq C|u|^3,\\ \frac{1}{C}|u|^2\leq & |P_m(s,u)+Q_m(s,u)| & \leq C|u|^2.\end{array}\qquad \begin{array}{rl}|D_s^\gamma P_m(s,u)| & \leq C|u|^2,\\ |D_s^\gamma Q_m(s,u)| & \leq C|u|^3,\\ |D_s^\gamma(P(s,u)+Q(s,u))| & \leq C|u|^2,\\ \left|\frac{\partial}{\partial u_i}D_s^\gamma P_m(s,u)\right| & \leq C|u|,\\ \left|\frac{\partial}{\partial u_i}D_s^\gamma Q_m(s,u)\right| & \leq C|u|^2,\\ \left|\frac{\partial}{\partial u_i}D_s^\gamma(P(s,u)+Q(s,u))\right| & \leq C|u|,\\ \left|\frac{\partial^2}{\partial u_i\partial u_j}D_s^\gamma P_m(s,u)\right| & \leq C|u|^0,\\ \left|\frac{\partial^2}{\partial u_i\partial u_j}D_s^\gamma Q_m(s,u)\right| & \leq C|u|,\\ \left|\frac{\partial^2}{\partial u_i\partial u_j}D_s^\gamma(P(s,u)+Q(s,u))\right| & \leq C|u|^0.\end{array} \tag{H.2.14}$$

hold for each $s\in D_m$, $u\in\mathbb{R}^2$ such that $s-u\in D_m$ and every multi-index with $|\gamma|\leq k$.

Our homogenization procedure follows from the next decomposition

$$K^{i,j}_{\lambda,0}(\mu_m(s),\mu_m(s)-\mu_m(t))=H^{i,j}_{\lambda,0}(s,s-t)+R^{i,j}_{\lambda,0}(s,s-t),$$

where the homogeneous part $H^{i,j}_{\lambda,0}(s,u)$ and the remainder $R^{i,j}_{\lambda,0}(s,u)$ take the form

$$\begin{aligned} H^{i,j}_{\lambda,0}(s,u) &:= -\frac{1}{4\pi} P_m(s,u)^{-3/2} \sum_{p,q=1}^{2} \frac{\partial(\widetilde{\eta}\circ\mu_m)_i}{\partial s_p}(s)\frac{\partial(\mu_m)_j}{\partial s_q}(s) u_p u_q,\\ R^{i,j}_{\lambda,0}(s,u) &:= \widetilde{R}^{i,j}_{\lambda,0}(s,u) + \widehat{R}^{i,j}_{\lambda,0}(s,u). \end{aligned}$$

Above, the remainder is split into

$$\begin{aligned} \widetilde{R}^{i,j}_{\lambda,0}(s,u) &:= -\frac{1}{4\pi}\left((P_m(s,u)+Q_m(s,u))^{-3/2} - P_m(s,u)^{-3/2}\right)\\ &\quad\times\left\{\sum_{p,q=1}^{2}\frac{\partial(\widetilde{\eta}\circ\mu_m)_i}{\partial s_p}(s)\frac{\partial(\mu_m)_j}{\partial s_q}(s)u_pu_q\right\},\\ \widehat{R}^{i,j}_{\lambda,0}(s,u) &:= -\frac{1}{4\pi}\left(P_m(s,u)+Q_m(s,u)\right)^{-3/2}\\ \times&\left\{-\sum_{p,q,r=1}^{2}\left(\int_0^1(1-\theta)\frac{\partial^2(\widetilde{\eta}\circ\mu_m)_i}{\partial s_p\partial s_q}(s-\theta u)\,d\theta\right)\frac{\partial(\mu_m)_j}{\partial s_r}(s)u_pu_qu_r\right.\\ &-\sum_{p,q,r=1}^{2}\frac{\partial(\widetilde{\eta}\circ\mu_m)_i}{\partial s_p}(s)\left(\int_0^1(1-\theta)\frac{\partial(\mu_m)_j}{\partial s_q\partial s_r}(s-\theta u)\,d\theta\right)u_pu_qu_r\\ &\left.+\sum_{p,q,r,l=1}^{2}\left(\int_0^1(1-\theta)\frac{\partial^2(\widetilde{\eta}\circ\mu_m)_i}{\partial s_p\partial s_q}(s-\theta u)\,d\theta\right)\left(\int_0^1(1-\theta)\frac{\partial^2(\mu_m)_j}{\partial s_r\partial s_l}(s-\theta u)\,d\theta\right)u_pu_qu_ru_l\right\}. \end{aligned}$$

Note again that small values of $u=s-t$ are involved here, thus leading to estimates like (H.2.7) for u.

Let us next analyze each term in the above decomposition for $K^{i,j}_{\lambda,0}(\mu_m(s),\mu_m(s)-\mu_m(t))$. Firstly, since $P_m(s,u)$ is positively homogeneous on u with degree 2, then $H^{i,j}_{\lambda,0}(s,u)$ is positively homogeneous on u with degree -1. The regularity properties in the second part in Corollary H.1.8 can be straighforwardly checked. Let us then concentrate on the regularity properties in the third part of such corollary and, to this end, let us compute the next partial derivative

$$D^\gamma_s H^{i,j}_{\lambda,0}(s,u) = -\frac{1}{4\pi}\sum_{\sigma\leq\gamma}\binom{\gamma}{\sigma}D^\sigma_s\left(P_m(s,u)^{-3/2}\right)\left[\sum_{p,q=1}^{2}D^{\gamma-\sigma}_s\left(\frac{\partial(\widetilde{\eta}\circ\mu_m)_i}{\partial s_p}(s)\frac{\partial(\mu_m)_j}{\partial s_q}(s)\right)u_pu_q\right].$$

Define the homogeneous function $h(t):=t^{-3/2}$ and use the chain rule to arrive at

$$D^\sigma_s\left(P_m(s,u)^{-3/2}\right) = \sum_{(l,\beta,\delta)\in\mathcal{D}(\sigma)}(D^\delta h)(P_m(s,u))\prod_{r=1}^{l}\frac{1}{\delta_r!}\left(\frac{1}{\beta_r!}D^{\beta_r}_s P_m(s,u)\right)^{\delta_r}.$$

(Recall the definition of $\mathcal{D}(\sigma)$ in (N.3) for the above chain rule). By virtue of (H.2.14),

$$\left|D^\gamma_s H^{i,j}_{\lambda,0}(s,u)\right| \leq C|u|^{-1}.$$

Let us take derivatives with respect to u and arrive at

$$\nabla_u D^\gamma_s H^{i,j}_{\lambda,0}(s,u)$$

$$
\begin{aligned}
= & -\frac{1}{4\pi}\sum_{\sigma\leq\gamma}\binom{\gamma}{\sigma}\nabla_u D_s^\eta\left(P_m(s,u)^{-3/2}\right)\left[\sum_{p,q=1}^2 D_s^{\sigma-\gamma}\left(\frac{\partial(\widetilde{\eta}\circ\mu_m)_i}{\partial s_p}(s)\frac{\partial(\mu_m)_j}{\partial s_q}(s)\right)u_pu_q\right]\\
& -\frac{1}{4\pi}\sum_{\sigma\leq\gamma}\binom{\gamma}{\sigma}D_s^\sigma\left(P_m(s,u)^{-3/2}\right)\left[\sum_{p,q=1}^2 D_s^{\sigma-\gamma}\left(\frac{\partial(\widetilde{\eta}\circ\mu_m)_i}{\partial s_p}(s)\frac{\partial(\mu_m)_j}{\partial s_q}(s)\right)\nabla_u(u_pu_q)\right],
\end{aligned}
$$

that can be similarly estimated by means of (H.2.14):

$$\left|D_s^\gamma\nabla_u H_{\lambda,0}^{i,j}(s,u)\right|\leq C|u|^{-2}.$$

Thus, $H_{\lambda,0}^{i,j}$ has the regularity properties required in Corollary H.1.8, so

$$\left\|\int_{D_m}H_{\lambda,0}^{i,j}(\cdot,\cdot-t)\varphi_m(\mu_m(t))\zeta(\mu_m(t))J_m(t)\,dt\right\|_{C^{k+1,\alpha}(D_m)}\leq M\|\zeta\|_{C^{k,\alpha}(\Sigma_m)}.$$

Let us now move to the remainder $R_{\lambda,0}^{i,j}(s,u)$ and show that the hypothesis in Corollary H.1.7 are satisfied too. On the one hand, in the first term $\widetilde{R}_{\lambda,0}^{i,j}(s,u)$ in $R_{\lambda,0}^{i,j}(s,u)$ one can arrange terms by Barrow's rule as

$$(P_m(s,u)+Q_m(s,u))^{-3/2}-P_m(s,u)^{-3/2}=-\frac{3}{2}Q_m(s,u)\int_0^1(P_m(s,u)+\theta Q_m(s,u))^{-5/2}\,d\theta.$$

Therefore, a D_s^γ derivative of $\widetilde{R}_{\lambda,0}^{i,j}(s,u)$ takes the form

$$
\begin{aligned}
D_s^\gamma\widetilde{R}_{\lambda,0}^{i,j}(s,u)=\frac{1}{4\pi}\frac{3}{2}\sum_{\sigma\leq\gamma}\binom{\gamma}{\sigma}D_s^\sigma&\left(Q_m(s,u)\int_0^1(P_m(s,u)+\theta Q_m(s,u))^{-5/2}\,d\theta\right)\\
&\times\sum_{p,q=1}^2 D^{\gamma-\sigma}\left(\frac{\partial(\widetilde{\eta}\circ\mu_m)_i}{\partial s_p}(s)\frac{\partial(\mu_m)_j}{\partial s_q}(s)\right)u_pu_q.
\end{aligned}
$$

Define the homogeneous function $\widetilde{h}(t)=t^{-5/2}$, a similar argument shows that

$$
\begin{aligned}
&D_s^\sigma\left(Q_m(s,u)\int_0^1(P_m(s,u)+\theta Q_m(s,u))^{-5/2}\,d\theta\right)\\
&\quad=\sum_{\rho\leq\sigma}\binom{\sigma}{\rho}D_s^\rho(Q_m(s,u))\int_0^1 D_s^{\sigma-\rho}\left((P_m(s,u)+\theta Q_m(s,u))^{-5/2}\right)\,d\theta\\
&\quad=\sum_{\rho\leq\sigma}\binom{\sigma}{\rho}D_s^\rho(Q_m(s,u))\int_0^1\sum_{(l,\beta,\delta)\in\mathcal{D}(\sigma-\rho)}(D^\delta\widetilde{h})(P_m(s,u)+\theta Q_m(s,u))\\
&\qquad\qquad\times\prod_{r=1}^l\frac{1}{\delta_r!}\left(\frac{1}{\beta_r!}D_s^{\beta_r}(P_m(s,u)+\theta Q_m(s,u))\right)^{\delta_r}\,d\theta.
\end{aligned}
$$

Now, the estimates in (H.2.14) yields

$$
\begin{gathered}
\left|D_s^\gamma\widetilde{R}_{\lambda,0}^{i,j}(s,u)\right|\leq C|u|^0,\\
\left|\partial_{u_l}D_s^\gamma\widetilde{R}_{\lambda,0}^{i,j}(s,u)\right|\leq C|u|^{-1},\qquad\left|\partial_{u_{l_1}u_{l_2}}D_s^\gamma\widetilde{R}_{\lambda,0}^{i,j}(s,u)\right|\leq C|u|^{-2}.
\end{gathered}
$$

These estimates ensure that all the hypotheses in Corollary H.1.7 are satisfied, so

$$\left\|\int_{D_m}\widetilde{R}^{i,j}_{\lambda,0}(\cdot,\cdot-t)\varphi_m(\mu_m(t))\zeta(\mu_m(t))J_m(t)\,dt\right\|_{C^{k+1,\alpha}(D_m)}\leq M\|\zeta\|_{C^{k,\alpha}(\Sigma_m)}.$$

Regarding the second term $\widehat{R}^{i,j}_{\lambda,0}(s,u)$ of $R^{i,j}_{\lambda,0}(s,u)$ we can use a similar argument. First,

$$D_s^\gamma\widehat{R}^{i,j}_{\lambda,0}(s,u)=\frac{1}{4\pi}\sum_{\sigma\leq\gamma}\binom{\gamma}{\sigma}D_s^\sigma\left((P_m(s,u)+Q_m(s,u))^{-3/2}\right)$$
$$\times\left\{\sum_{p,q,r=1}^{2}D_s^{\gamma-\sigma}\left(\left(\int_0^1(1-\theta)\frac{\partial^2(\widetilde{\eta}\circ\mu_m)_i}{\partial s_p\partial s_q}(s-\theta u)\,d\theta\right)\frac{\partial(\mu_m)_j}{\partial s_r}(s)\right)u_pu_qu_r\right.$$
$$+\sum_{p,q,r=1}^{2}D_s^{\gamma-\sigma}\left(\frac{\partial(\widetilde{\eta}\circ\mu_m)_i}{\partial s_p}(s)\left(\int_0^1(1-\theta)\frac{\partial(\mu_m)_j}{\partial s_q\partial s_r}(s-\theta u)\,d\theta\right)\right)u_pu_qu_r$$
$$\left.-\sum_{p,q,r,l=1}^{2}D_s^{\gamma-\sigma}\left(\left(\int_0^1(1-\theta)\frac{\partial^2(\widetilde{\eta}\circ\mu_m)_i}{\partial s_p\partial s_q}(s-\theta u)\,d\theta\right)\left(\int_0^1(1-\theta)\frac{\partial^2(\mu_m)_j}{\partial s_r\partial s_l}(s-\theta u)\,d\theta\right)\right)u_pu_qu_ru_l\right\},$$

and the higher-order chain formula leads again to

$$D_s^\sigma\left((P_m(s,u)+Q_m(s,u))^{-3/2}\right)$$
$$=\sum_{(l,\beta,\delta)\in\mathcal{D}(\sigma)}(D^\delta h)(P_m(s,u)+Q_m(s,u))\prod_{r=1}^{l}\frac{1}{\delta_r!}\left(\frac{1}{\beta_r!}D_s^{\beta_r}(P_m(s,u)+Q_m(s,u))\right)^{\delta_r}.$$

Consequently, the estimates in (H.2.14) show that

$$\left|D_s^\gamma\widehat{R}^{i,j}_{\lambda,0}(s,u)\right|\leq C|u|^0,$$
$$\left|\partial_{u_l}D_s^\gamma\widehat{R}^{i,j}_{\lambda,0}(s,u)\right|\leq C|u|^{-1},\qquad\left|\partial_{u_{l_1}u_{l_2}}D_s^\gamma\widehat{R}^{i,j}_{\lambda,0}(s,u)\right|\leq C|u|^{-2},$$

and Corollary H.1.7 yields

$$\left\|\int_{D_m}\widehat{R}^{i,j}_{\lambda,0}(\cdot,\cdot-t)\varphi_m(\mu_m(t))\zeta(\mu_m(t))J_m(t)\,dt\right\|_{C^{k+1,\alpha}(D_m)}\leq M\|\zeta\|_{C^{k,\alpha}(\Sigma_m)}$$

Now we move to $K^{i,j}_{\lambda,1}(x,z)$ that can be expanded as

$$K^{i,j}_{\lambda,1}(\mu_m(s),\mu_m(s)-\mu_m(s-u))$$
$$=f_\lambda(|P_m(s,u)+Q_m(s,u)|^{1/2})\sum_{p,q=1}^{2}\left(\int_0^1\frac{\partial(\widetilde{\eta}_i\circ\mu_m)}{\partial s_q}(s-\theta u)\,d\theta\right)\left(\int_0^1\frac{\partial(\mu_m)_j}{\partial s_q}(s-\theta u)\,d\theta\right)u_pu_q.$$

Then, the D_s^γ derivative of $K^{i,j}_{\lambda,1}(\mu_m(s),\mu_m(s)-\mu_m(t))$ takes the form

$$D_s^\gamma K^{i,j}_{\lambda,1}(\mu_m(s),\mu_m(s)-\mu_m(s-u))$$
$$=\sum_{\sigma\leq\gamma}\binom{\gamma}{\sigma}D_s^\sigma\left(f_\lambda(|P_m(s,u)+Q_m(s,u)|^{1/2})\right)$$

$$\times \left[\sum_{p,q=1}^{2} D_s^{\gamma-\sigma} \left(\left(\int_0^1 \frac{\partial(\widetilde{\eta}_i \circ \mu_m)}{\partial s_q}(s-\theta u)\, d\theta \right) \left(\int_0^1 \frac{\partial(\mu_m)_j}{\partial s_q}(s-\theta u)\, d\theta \right) \right) u_p u_q \right].$$

Again, by the chain derivative formula we obtain

$$D_s^\sigma \left(f_\lambda((P_m(s,u)+Q_m(s,u))^{1/2}) \right)$$
$$= \sum_{(l,\beta,\delta)\in\mathcal{D}(\sigma)} D^\delta (f_\lambda(\cdot^{1/2}))\Big|_{P_m(s,u)+Q_m(s,u)} \prod_{r=1}^{l} \frac{1}{\delta_r!} \left(\frac{1}{\beta_r!} D_s^{\beta_r}(P_m(s,u)+Q_m(s,u)) \right)^{\delta_r}.$$

Notice that (H.2.8) leads to

$$\left| \frac{d^k}{dr^k} \left(f_\lambda(r^{1/2}) \right) \right| \leq \widetilde{C} \frac{1}{r^{k+1}},\ \forall\, r \in (0, d_S^m).$$

Consequently, (H.2.14) proves the upper bounds

$$\left| D_s^\gamma K_{\lambda,1}^{i,j}(\mu_m(s), \mu_m(s)-\mu_m(s-u)) \right| \leq C|u|^0,$$
$$\left| \partial_{u_{l_1}} D_s^\gamma K_{\lambda,1}^{i,j}(\mu_m(s), \mu_m(s)-\mu_m(s-u)) \right| \leq C|u|^{-1},$$
$$\left| \partial_{u_{l_1} u_{l_2}} D_s^\gamma K_{\lambda,1}^{i,j}(\mu_m(s), \mu_m(s)-\mu_m(s-u)) \right| \leq C|u|^{-2},$$

so the hypotheses in Corollary H.1.7 are satisfied and

$$\left\| \int_{D_m} K_{\lambda,1}^{i,j}(\mu_m(\cdot), \mu_m(\cdot)-\mu_m(t)) \varphi_m(\mu_m(t)) \zeta(\mu_m(t)) J_m(t)\, dt \right\|_{C^{k+1,\alpha}(D_m)} \leq M \|\zeta\|_{C^{k,\alpha}(\Sigma_m)}.$$

In order to complete the proof of the theorem, let us show how to deal with the terms $m' \neq m$ in (H.2.9) and (H.2.10). The idea is to obtain estimates over $\Sigma_m \cap \Sigma_{m'}$ and $\Sigma_m \setminus \overline{\Sigma_{m'}}$ separately. First,

$$\left\| \int_{D_{m'}} K_\lambda^{i,j}(\mu_m(\cdot), \mu_m(\cdot)-\mu_{m'}(t)) \varphi_{m'}(\mu_{m'}(t)) \zeta(\mu_{m'}(t)) J_{m'}(t)\, dt \right\|_{C^{k+1,\alpha}(\mu_m^{-1}(\Sigma_m\cap\Sigma_{m'}))}$$
$$\leq C \left\| \int_{D_{m'}} K_\lambda^{i,j}(\mu_{m'}(\cdot), \mu_{m'}(\cdot)-\mu_{m'}(t)) \varphi_{m'}(\mu_{m'}(t)) \zeta(\mu_{m'}(t)) J_{m'}(t)\, dt \right\|_{C^{k+1,\alpha}(D_{m'})}$$
$$\leq \widetilde{C} \|\zeta\|_{C^{k,\alpha}(\Sigma_{m'})}.$$

Second, define $C_{m'} := \mu_{m'}^{-1}(\operatorname{supp}\varphi_{m'})$, $K_{m'} := \mu_{m'}(C_{m'})$ and $d_{m,m'} := \operatorname{dist}\left(\Sigma_m \setminus \overline{\Sigma_{m'}}, K_{m'}\right) > 0$ as in Figure H.2. This avoids the singularity near $z = 0$ in the preceding kernels. Hence,

$$D_s^\gamma \int_{D_{m'}} K_\lambda^{i,j}(\mu_m(s), \mu_m(s)-\mu_{m'}(t)) \varphi_{m'}(\mu_{m'}(t)) \zeta(\mu_{m'}(t)) J_{m'}(t)\, dt$$
$$= \sum_{(l,(\delta^1,\delta^2),\beta)\in\mathcal{D}(\gamma)} \int_{C_{m'}} \left(D_x^{\delta^1} D_z^{\delta^2} K_\lambda^{i,j} \right) (\mu_m(s), \mu_m(s)-\mu_{m'}(t))$$
$$\times \left[\prod_{r=1}^{l} \frac{1}{\delta_r^1! \delta_r^2!} \left(\frac{1}{\beta_r!} D^{\beta_r} \mu_m(s) \right)^{\delta_r^1} \left(\frac{1}{\beta_r!} D^{\beta_r} \mu_m(s) \right)^{\delta_r^2} \right] \varphi_{m'}(\mu_{m'}(t)) \zeta(\mu_{m'}(t)) J_{m'}(t)\, dt.$$

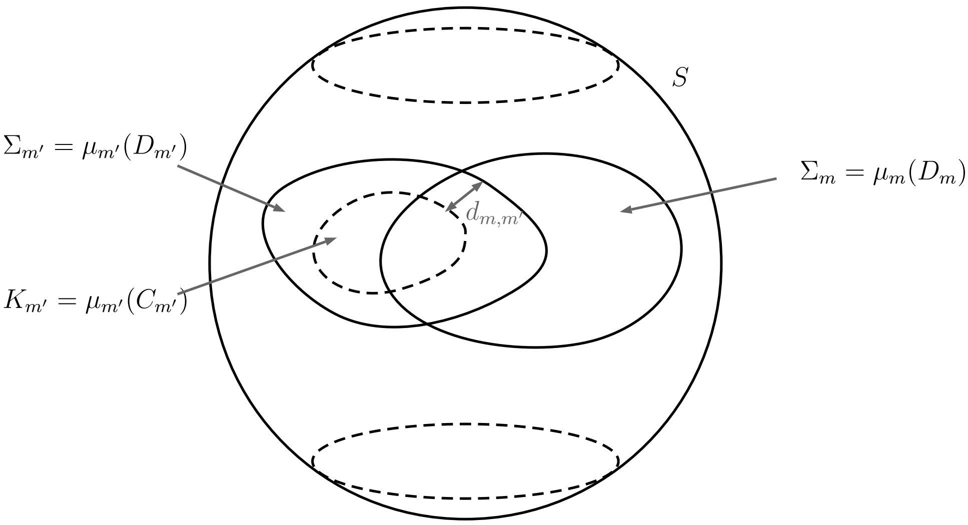


Figure H.2: Overlapping coordinate neighborhoods Σ_m and $\Sigma_{m'}$.

for each $s \in \mu_m^{-1}(\Sigma_m \setminus \overline{\Sigma_{m'}})$. Since $|D_x^{\delta^1} D_z^{\delta^2} K_\lambda^{i,j}(x,z)| \leq \widetilde{C}|z|^{-|\delta^2|}$ for every $z \in B_{d_{m,m'}}(0)$, then

$$\left| D_s^\gamma \int_{D_{m'}} K_\lambda^{i,j}(\mu_m(s), \mu_m(s) - \mu_{m'}(t))\varphi_{m'}(\mu_{m'}(t))\zeta(\mu_{m'}(t))J_{m'}(t)\,dt \right| \leq \frac{C}{d^{|\gamma|}}|C_{m'}|\|\zeta\|_{C^0(\Sigma_{m'})},$$

Here, $0 < d < 1$ is such that $d < d_{m,m'}$ for every $m' \neq m$. Since one can take any $|\gamma| \leq k+2$ by the regularity of S, then we obtain the desired estimate for $m' \neq m$ and the result follows. $\square$

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