# LET'S TEACH GEOMETRY

In this activity for High School students, María Cañadas, Marta Molina, Sandra Gallardo, Manuel Martínez-Santaolalla and María Peñas teach the mathematical concepts of plane and spatial geometry. The final objective of the task is to construct a cube using modular origami.

Making constructions with paper is called *origami* and is considered an art (Royo, 2002). The objective for many fans of origami is to design new figures never constructed before. From the point of view of mathematics education, origami is an interesting didactic activity (Cipoletti & Wilson, 2004; Fairbairn, 2008). Here we propose to help High School students understand new mathematic concepts, and to revise concepts already known, through the origami activity of building a cube.

This activity addresses the following recommendations for the teaching of Geometry in years 9-12:

- To analyze characteristics and properties of two- and three-dimensional geometric shapes and to develop mathematical arguments about geometric relationships,
- To draw and construct representations of two- and three-dimensional geometric objects using a variety of tools.
(NCTM, 2000)

There are many didactic materials for working on Geometry (Alsina, Burgués, & Fortuny, 1988). Here we propose the use of one which it is certainly not new – paper. We focus on the use of paper for the teaching and learning of geometry as it is a manipulative material and, it helps to involve students in mathematical activity and to promote making sense of, and giving meaning to, mathematical concepts (Segovia & Rico, 2001, p.86).

In previous work we have focused on the teaching and learning of plane geometry concepts such as parallelism, perpendicularity, polygons, angles, etc. through the use of paper (Authors, 2003; 2006). This time we decided to address the teaching of three-dimensional geometry that will also inevitably tackle some concepts in plane geometry. In this context, fractions and the measurement of surfaces and lengths are also involved.

## Modular origami

In *modular origami* figures are constructed by joining independent pieces of purposely folded paper. Each piece, called a module, has pockets and flaps which allow them to be assembled. By using these modules, a wide variety of geometric figures can be constructed, particularly polyhedra. These constructions are useful for mathematics teaching because they allow representation and manipulation of geometric figures, and facilitate the study of their properties as these are involved in the construction and design of the modules (Royo, 2002).

Royo classified modules for constructing polyhedra by focusing on the part of the polyhedron which is formed by each module:

a   modules based on vertices
b   modules based on sides
c   modules based on faces.

There are many different ways to construct each type of module. In addition, the same set of modules can be assembled in two different ways:

1   by making the assembly towards the inside of the figure and so, hiding the joins – *Figure 1*
2   by assembling the modules towards outside and so, leaving the join on view – *Figure 2*
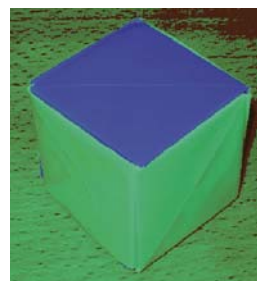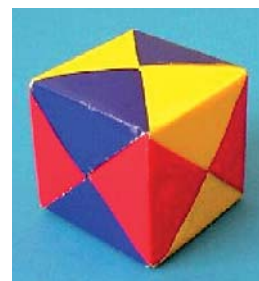


Figure 1. Cube with assembly towards inside

Figure 2. Cube with assembly towards outside

We can see in *Figure 2*, when the assembly is made towards the outside it is not so easy to see that each module becomes a face of the polyhedron.

## The task: to construct a cube by using modular origami

We chose a cube because it is the simplest polyhedra to construct using origami, and the one most familiar to many students. In addition, the construction of a cube using modules based on faces is one of the easiest to achieve. There are other more consistent modules that allow you to construct other polyhedron, however, we selected this module because it enables the construction of a consistent cube without causing difficulties manipulating the paper. In this way, students are able to focus on the mathematics they are involved with through the activity.

Manipulation brings the geometric elements closer to students through their exploration, analysis, identification, construction and description. Also involved is the need to compute measurements of surfaces and lengths in the different figures obtained through the construction. These questions aim to make students think about different partitions of a shape, and possibly work with fractions and irrational numbers. It will depend on the unit chosen to measure: length of the initial squared piece of paper versus length of the side of the small square that it is obtained when the initial squared piece of paper is folded as shown in *Figure 1*. The only formulæ needed to find these measurements are the formula for the area of a square, and the formula of the area of a triangle. Most of the proposed questions do not necessary imply computations. We aim to encourage students to take advantage of the relationships they have already discovered, or the computations made.
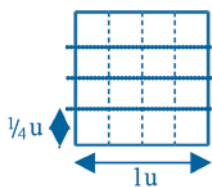


*Figure 3. Paper for working on fractions*

Through the construction of the module, students:
- Recognise geometric shapes and some of their properties
- Apply geometric concepts such as parallelism and intersecting lines
- Obtain and relate geometric figures by decomposing other previously constructed shapes
- Use the theorem of Pythagoras
- Apply the property of the sum of the angles of a triangle
- Establish proportional relations between different measurements of surface area and length
- Work with fractions and irrational numbers
- Recognize 30º, 45º, 60º, 90º, and 270º angles.

Constructing the cube, with the previously constructed modules, allows students to explore the characteristics of this polyhedron tackling simple concepts such as the number of sides, vertices, and other more complex concepts such as the measurement of the surface area, and the volume. This can lead to consideration of proportionality relationships that exist between the measurements of the cube, and the measurements of the initial squared piece of paper.

## Reflections and conclusions

Our objective is to facilitate the students' understanding of some mathematical concepts related to measurement, fractions and geometry and to bring some geometrical concepts closer to High School students through an engaging, entertaining, and meaningful mathematics activity. As we have shown, using paper allows students to manipulate representations, and to analyse the geometric objects which are produced. Moreover, it enables students to approach plane and spatial geometry intuitively, through the logical, efficient and economic process of construction.

Using this activity will demonstrate the potential of using paper as a didactic resource for teaching Geometry, together with the learning and consolidation of a variety of mathematical concepts, not uniquely geometrical.

The methodology used in this origami construction can be applied to many other constructions. Teachers can also vary this task to match the needs of their student by focusing on appropriate concepts, adding, supplementing, or removing some of the questions, removing the diagrams or by choosing different discussion points and outcomes.

## Activity

### Construction of the module

1   Take a rectangular sheet of paper. What are the properties of a rectangle? *Hints such as: Look at its sides, angles, diagonals…*
2   Make a square with the paper. How do you know that it is a square? *Justify your answer to a friend, a group, or maybe the teacher.*
3   Once you have the square, fold it to mark both its diagonals and cut it out – then: **i)** calculate the length of the diagonals in relation to the length of side of the square, **ii)** What is the size of the angles in the square, after folding?
4   Fold the square into two equal rectangles by tracing a line parallel to one of its sides. Then, fold each rectangle again parallel to the

previous folding. Then, unfold the square.
(Hint: See figures 4a and 4b)

*Figures 4a
and 4b*
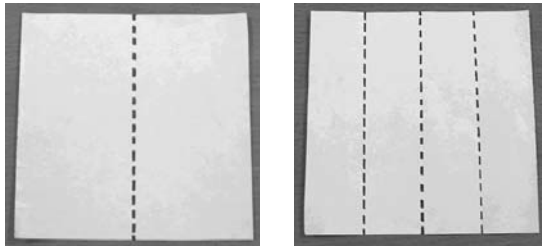


Consider the four equal rectangles obtained:
What is the relationship between the lengths of the sides in the rectangle, and the lengths of the sides in the square?
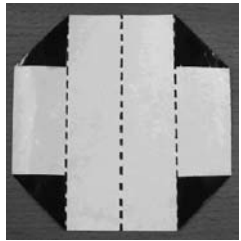
5   Now fold the square by moving each corner to the intersection of each diagonal with the closest parallel line to that corner (See result in *figure 4c.*)
   a   What shape is formed by the external borders of the paper?
   b   What is the relationship between the measurement of its sides and the length of the side of the square?
   c   What is the relationship between the perimeter of the rectangle, and the perimeter of the square?
   d   Compare the area of this shape to the area of the square
   e   What is the size of each angle of the shape? – are there some you don't know? – cannot calculate? …

*Figure 4c*



6   Fold the paper along the two external parallel lines to the sides of the square previously marked. In this way, we hide the corners previously folded (See result in *figure 4d.*)
   a   What shapes can you find in this figure?
   b   Which angles do you distinguish in these shapes?
   c   Which is the measurement of their sides in relation to the measurement of the side of the square? And their perimeter?
   d   Which is the measurement of their surface in relation to area of the square?

7   Fold the piece in half so that two equal squares are produced. Then, unfold and take two opposite corners and fold them by moving them to the intersection of the border of the figure with the last fold made. You just overlapped both sides of the figure (see result in *figures 4e and 4f.*)
   a   Which shape did you make?
   b   What is the relationship between the length of the sides in your shape, and the length the side of the big square? – What is the relationship between the perimeters of the
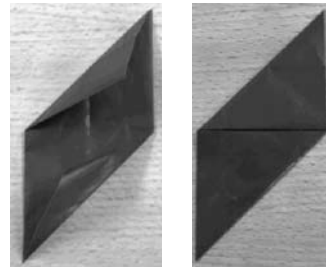
*Figure 4d*



*Figures 4e
and 4f*

   two shapes?
   c   can you find a relationship between the areas of the two shapes?
   d   Can you give the size of some of the angles of the shape?

8   Take the corners toward inside of the figure. This stops the paper from opening. This is the module you need to build a cube. Try to guess which parts are pockets and flaps, and which part of the module will become the face of the cube.

9   Turn down the figure. Fold the module so that you get two equal triangles and a square. This square will be the face of the cube (see *figures 4g and 4h*).
   a   What types of triangles did you use or make?
   b   What is the relationship between the length of the side of this square, and the square you started with?
   c   What is the relationship between the area of this square and the area of the square you started with?
   d   What is the relationship between the area of each triangle and the area of the shape you started with?
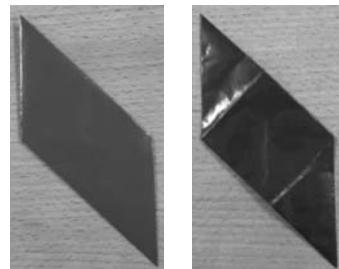


*Figures 4g
and 4h*

## Construction of the cube

10  How many modules like the one you built do you need?
11  When you have all the modules you need, build the cube using the flaps in the pockets.
12  Now describe the figure you have just constructed.
   a   How many sides does it have?
   b   How many vertices does it have?
   c   Find the area of its faces?
   d   Calculate is its volume?
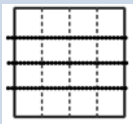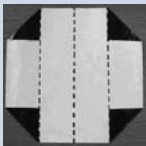   e   Find the length of its diagonals?

## Solutions

Here we include some solutions, however these are not the only valid solutions.

In Table 1 we provide the responses for tasks 1 to 9, with the exception of measurement questions. In Table 2 and 3 we present responses to measurement questions: surfaces in Table 2 and lengths and angles in Table 3. In both tables two different measurement units are considered to avoid, or force working with, fractions.
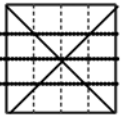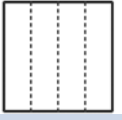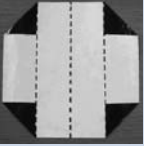
| Table 1: Responses to non-measurement questions | |
|---|---|
| Question | Response |
| 1 | Polygon; Irregular quadrilateral; Parallel sides two by two; Equal opposite angles; Four 90º angles; Perpendicular diagonals; Two symmetry exes. |
| 2 | Fold by moving the short side of the rectangle over the longer side of the rectangle. The fold obtained is one diagonal of the square. If you fold again by moving the same short side over the other longer side of the rectangle you get the other diagonal of the square. Now you have the 4 vertexes marked on the paper. Fold and cut the piece of paper to get the square. It is a quadrilateral with four equal sides/angles (90º). |
| 5 | a) Irregular Octagon |
| 6 | a) Right-angle triangle, isosceles triangle, isosceles trapezoid and concave irregular hexagon |
| 7 | a) Rhomboid |
| 9 | a) Right-angle isosceles triangles |

### Table 2: Measuring surfaces

| Picture | Shape | Unit of measurement surface of small square | surface of big square |
|---|---|---|---|
|  | Big square | 16 u² | 1 u² |
| | Small square | 1 u² | $^1/_{16}$ u² |
|  | Rectangle within a square | 4 u² | $^1/_4$ u² |
|  | Non regular Octagon | 14 u² | $^7/_8$ u² |
|  | Rectangle | 8 u² | $^1/_2$ u² |
| | Trapezium | 3 u² | $^3/_{16}$ u² |
| | Big Isosceles Triangle | 1 u² | $^1/_{16}$ u² |
| | Small Isosceles Triangle | $^1/_2$ u² | $^1/_{32}$ u² |
| | Concave Hexagon | 6 u² | $^3/_8$ u² |
|  | Right angle Triangle | 2 u² | $^3/_8$ u² |
|  | Rhomboid | 4 u² | $^1/_4$ u² |
| | Right-angle Triangle | 1 u² | $^1/_{16}$ u² |
| | Square | 2 u² | $^1/_8$ u² |

## Table 3: Measuring lengths and angles

| Figure | Shape | Unit of measurement | | Angles |
|---|---|---|---|---|
| | | Length of side of small square | length of side of big square | |
|  | Big square | Sides: 4 u <br> Diagonal: $4\sqrt{2}$ u | Sides: 1 u <br> Diagonal: $\sqrt{2}$ u | 90°, 45° |
| | Small square | Sides: 1 u | Sides: $\frac{1}{4}$ u | |
|  | Rectangles within a square | Longer side: 4 u <br> Smaller side: 1 u | Longer side: 1 u <br> Smaller side: $\frac{1}{4}$ u | 90° |
|  | Non regular Octagon | Longer side: 2 u <br> Smaller side: $\sqrt{2}$ u <br> Perimeter: $8 + 4\sqrt{2}$ u | Longer side: $\frac{1}{2}$ u <br> Smaller side: $\frac{\sqrt{2}}{4}$ u <br> Perimeter: $2 + \sqrt{2}$ u | 135° |
|  | Rectangle | Longer side: 4 u <br> Smaller side: 2 u <br> Perimeter: 12 u | Longer side: 1 u <br> Smaller side: $\frac{1}{2}$ u <br> Perimeter: 3 u | 90° |
| | Isosceles Trapezium | Longer side: 4 u <br> Medium side: 2 u <br> Smaller side: $\sqrt{2}$ u <br> Perimeter: $6 + 2\sqrt{2}$ u | Longer side: 1 u <br> Medium side: $\frac{1}{2}$ u <br> Smaller side: $\frac{\sqrt{2}}{4}$ u <br> Perimeter: $\frac{3+\sqrt{2}}{2}$ u | 45°, 270° |
|  | Big Isosceles (right-angle) Triangle | Longer side: 2 u <br> Smaller side: $\sqrt{2}$ u <br> Perimeter: $2 + 2\sqrt{2}$ u | Longer side: $\frac{1}{2}$ u <br> Smaller side: $\frac{\sqrt{2}}{4}$ u <br> Perimeter: $\frac{1+\sqrt{2}}{2}$ u | 45°, 90° |
| | Small Isosceles (right-angle) Triangle | Longer side: $\sqrt{2}$ u <br> Smaller sides: 1 u <br> Perimeter: $2 + \sqrt{2}$ u | Longer side: $\frac{\sqrt{2}}{4}$ u <br> Smaller sides: $\frac{1}{4}$ u <br> Perimeter: $\frac{\sqrt{2}}{4} + \frac{1}{2}$ u | 45°, 90° |
| | Concave Hexagon | Longer sides: 4 u <br> Smaller sides: $\sqrt{2}$ u <br> Perimeter: $8 + 4\sqrt{2}$ u | Longer sides: 1 u <br> Smaller sides: $\frac{\sqrt{2}}{4}$ u <br> Perimeter: $2 + \sqrt{2}$ u | 45°, 270° |
|  | Right angle Triangle | Longer side: $2\sqrt{2}$ u <br> Smaller sides: 2 u <br> Perimeter: $4 + 2\sqrt{2}$ u | Longer side: $\frac{\sqrt{2}}{2}$ u <br> Smaller sides: $\frac{1}{2}$ u <br> Perimeter: $1 + \frac{\sqrt{2}}{2}$ u | 45°, 90° |
|  | Rhomboid | Longer sides: $2\sqrt{2}$ u <br> Smaller sides: 2 u <br> Perimeter: $4 + 4\sqrt{2}$ u | Longer sides: $\frac{\sqrt{2}}{2}$ u <br> Smaller sides: $\frac{1}{2}$ u <br> Perimeter: $1 + \sqrt{2}$ u | 45°, 135° |
| | Right-angle Triangle | Longer side: 2 u <br> Smaller sides: $\sqrt{2}$ u <br> Perimeter: $2 + 2\sqrt{2}$ u | Longer side: $\frac{1}{2}$ u <br> Smaller sides: $\frac{\sqrt{2}}{4}$ u <br> Perimeter: $\frac{1}{2} + \frac{\sqrt{2}}{2}$ u | 45°, 90° |
| | Square | Side: $\sqrt{2}$ u <br> Perimeter: $4\sqrt{2}$ u | Side: $\frac{\sqrt{2}}{4}$ u <br> Perimeter: $\sqrt{2}$ u | 90° |

| Table 4: Responses to task 12 | |
|---|---|
| **Big Square 16 u² area** | **Big Square 1 u² area** |
| A) 6 sides | A) 6 sides |
| B) 6 vertexes of grade 3 | B) 6 vertexes of grade 3 |
| C) 2 u² | C) $\frac{1}{8}$ u² |
| D) $2\sqrt{2}u^3$ | D) $\frac{\sqrt{2}}{32}u^3$ |
| E) diagonal of the face: 2 u<br>diagonal of the cube: $\sqrt{6}$ u | F) diagonal of the face: $\frac{1}{2}$ u<br>diagonal of the cube: $\frac{\sqrt{6}}{4}$ |

The tasks related to the perimeter are associated to the length of the sides of the particular shapes. In Table 4 we provide the responses to task 12.

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# A DANTZOGRAMMATIC FRACTION WALL
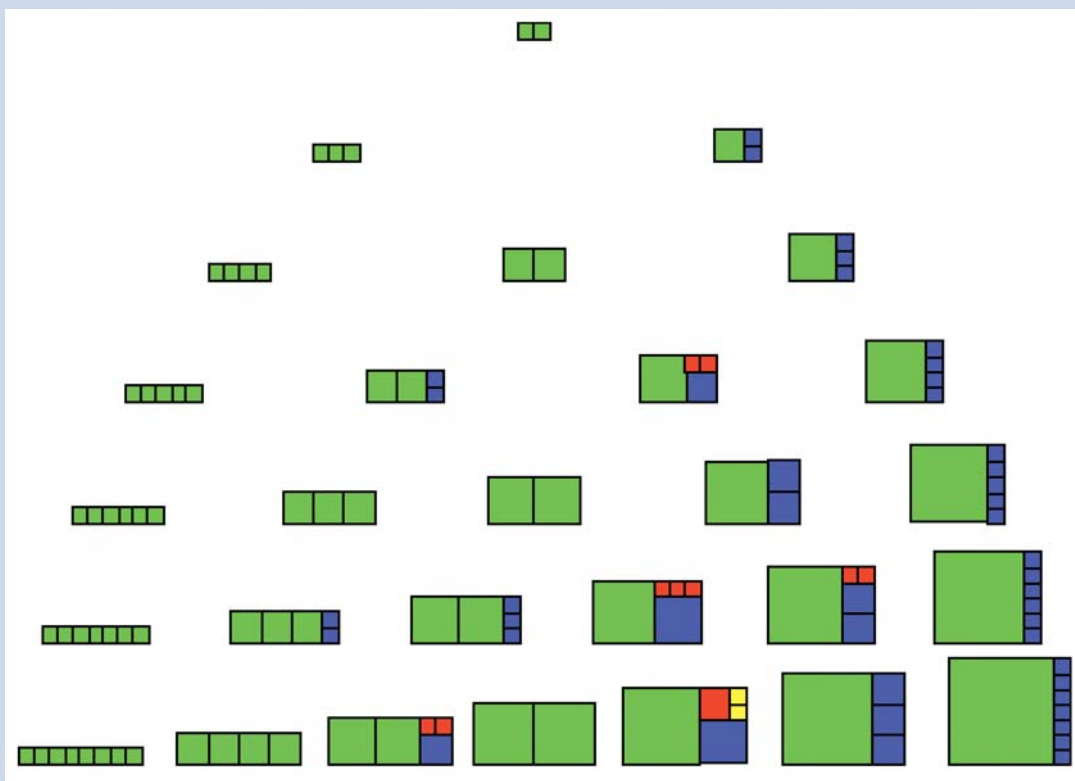


Identify the rule used to make these figures, then go to **http://nrich.maths.org/1397** to see what Toni and Alan Beardon have to say about them. (I first saw them in Tobias Dantzig's book **Number**, which is why I call them 'dantzograms'.)

Paul Stephenson is operations director of The Magic Mathworks Travelling Circus.