Geometry of Banach spaces with diameter two properties and octahedral norm

Abraham Rueda Zoca.

UNIVERSIDAD
DE GRANADA

Programa de doctorado en Física y Matemáticas (FisyMat)

Editor: Universidad de Granada. Tesis Doctorales
Autor: Abraham Rueda Zoca
ISBN: 978-84-1306-313-3
URI: http://hdl.handle.net/10481/57262

Geometry of Banach spaces with diameter two properties and octahedral norm

Abraham Rueda Zoca
Correo electrónico: abrahamrueda@ugr.es

Esta memoria ha sido financiada por:

- (1/12/2016-30/9/2017) Contratos predoctorales FPU del Plan Propio del Vicerrectorado de Investigación y Transferencia de la Universidad de Granada.
- (1/10/2017-actualidad) MECD FPU16/00015.
- (02/2017-actualidad) Proyecto MINECO MTM2015-65020-P
- (31/08/2014-07/03/2017) Grupo de Investigación de la Junta de Andalucía FQM-0199.
- (07/03/2017-actualidad) Grupo de Investigación de la Junta de Andalucía FQM-0185.

Además, una estancia realizada en el Laboratoire de Mathèmatiques de Besançon (23/03/2016-22/06/2016), a cargo del profesor Gilles Lancien, fue financiada por Vicerrectorado de Internacionalización y Vicerrectorado de Investigación y Transferencia de la Universidad de Granada, Escuela Internacional de Posgrado de la Universidad de Granada y el Campus de Excelencia Internacional (CEI) BioTic, dentro del programa "Convocatoria de Movilidad Internacional de Estudiantes de Programas de Doctorado. Universidad de Granada".

A María

Agradecimientos

Me gustaría comenzar agradeciendo a los dos directores de esta tesis, Julio Becerra Guerrero y Ginés López Pérez, su guía y paciencia durante todo mi doctorado. En particular, quisiera agradecer a Ginés su consejo desinteresado durante mi formación como investigador y como docente.

También me gustaría dar las gracias a Armando Villena Muñoz por toda la paciencia que ha demostrado tutorizando mi doctorado.

Quisiera seguir dando las gracias a Gilles Lancien por acogerme para realizar mi estancia de tres meses en el Laboratoire de Mathèmatiques de Besançon, donde conocí a gente maravillosa y recibí una excepcional hospitalidad. En especial, me gustaría agradecer a Tony Procházka y a Blanca Corella Abenia, quienes me hicieron sentir como en casa desde que puse los pies en Besançon. Tampoco puedo olvidarme de Luis Carlos García Lirola, gran companẽro y mejor amigo, con quien ha sido un placer trabajar desde que lo conocí y de quien, junto a Paula Juárez Muñoz, he recibido un gran calor humano tanto en mis frecuentes visitas a Murcia como fuera de España.

No puedo dejar fuera a colaboradores como Rafael Chiclana Vega, Miguel Martín Suárez, Tony Procházka o Colin Petitjean. A pesar de que en algunos casos no aparezca investigación conjunta en esta memoria, todo lo que me han transmitido trabajando con ellos hace que estén muy presentes en el estilo de esta tesis.

Además de todo lo anterior, me gustaría mencionar (por orden cronológico) a todos los profesores con quien he compartido docencia y/o experiencias docentes, de quienes no es poco lo que he aprendido: Miguel Martín Suárez, Jerónimo Alaminos Prats, Ginés López Pérez, Manuel Díaz Carrillo, Eduardo Nieto Arco, María Dolores Acosta Vigil, Pilar Muñoz Rivas, José Extremera Lizana, Salvador López Martínez y Javier Merí de la Maza.

Finalmente, tengo que agradecer a mi familia el apoyo que he recibido durante toda mi carrera como matemático (listados por orden alfabético: Antonio, Elena, Isabel, Luis, Magdalena, Manu, María, Mari, Paco y Rocío). Mención especial merecen mis padres, Antonio e Isabel, quienes siempre han creído en mí y me han dado su apoyo e incluso, durante ciertos momentos, el soporte económico necesario para el desarrollo de mi actividad investigadora. Sobre todo, mi pareja María, quien me hizo darme cuenta de que mi verdadera pasión son las matemáticas y sin quien quizá no me hubiera embarcado en su estudio.

Introduction

Motivation

It is fair to say that one of the most studied properties of Banach spaces is the *Radon-Nikodym property (RNP)* because, due to the big amount of characterisations, the RNP has shown to be very useful in several fields of Banach spaces such as representation of bounded linear operators, representation of dual spaces or representation of certain tensor product spaces (see [Bou, DU]). In spite of being a property which is invariant under equivalent renorming, there is a very interesting characterisation of the RNP with a geometric spirit. It is known that a Banach space X has the RNP if, and only if, every bounded subset C of X is *dentable*, i.e. C contains slices of arbitrarily small diameter (we again refer to [Bou, DU] for a proof). Note that, despite the dentability is, at a first glance, a property which depends on the norm of the space, the fact that the characterisation runs over all the bounded subsets of the space makes of the previous characterisation a property which is invariant under equivalent renorming.

In view of the previous characterisation, given a Banach space X failing the RNP, we can find a bounded subset C of X and a positive $\varepsilon \leq \operatorname{diam}(C)$ such that every slice of C has diameter, at least, ε. In order to push further this characterisation of the failure of the RNP, two natural questions arise:

i) Can ε be taken close to $\operatorname{diam}(C)$?

ii) In case that the answer is yes, can C be used to get an equivalent renorming of X such that the diameter of the slices of the new unit ball is (close to) 2?

Both questions were analysed in [SSW]. Making use of the so-called *moduli of non-dentability*, it is shown in [SSW, Theorem 1.1] that if a Banach space X fails the RNP then, for every $\varepsilon > 0$, there exists a bounded, closed, convex and separable subset C of X with $\operatorname{diam}(C) = 1$ and such that the diameter of every slice of C is greater than or equal to $1-\varepsilon$. Concerning ii), it is proved in [SSW, Corollary 3.2] that every Banach space failing the RNP admits, for every $\varepsilon > 0$, an equivalent renorming such that every slice of the new unit ball has diameter, at least, $1-\varepsilon$, and it is posed as an open question whether one can get $2-\varepsilon$ in some equivalent renorming. Concerning the above problem, it is known that the answer is affirmative in the context of Banach lattices [E.Wer]. Furthermore, a positive answer to this problem would yield a more ambitious problem.

Question 1. *Let X be a Banach space failing the RNP. Can X be equivalently renormed with the property that every slice of the new unit ball has diameter exactly 2?*

This question motivates the definition of the *slice diameter two property* (slice-D2P) on a Banach space X as the property that every slice of the unit ball B_X has diameter 2.

Other isomorphic properties of Banach spaces, strongly related to the RNP, are the *point of continuity properties* and the *strong regularity.* A Banach space X:

- has the *point of continuity property* (PCP) if every bounded, closed and non-empty subset A has a point $x \in A$ which is of weak-to-norm continuity, that is, so that the identity mapping $i : (A, w) \to (A, \tau_n)$ is continuous at x, where (A, w) denotes the set A with the restricted weak topology and (A, τ_n) denotes the set A with the restricted norm topology.

- has the *convex point of continuity property* (CPCP) if every bounded, convex, closed and non-empty subset A has a point $x \in A$ which is of weak-to-norm continuity.

- is *strongly regular* (SR) if every closed, convex and bounded subset of X contains convex combinations of slices of arbitrarily small diameter.

We refer to [GMS] for background on the (C)PCP and to [GGMS] for background on the strong regularity. It is not difficult to prove that a Banach space has the PCP (respectively the CPCP) if every non-empty, closed and bounded subset (respectively non-empty, closed, bounded and convex subset) of X contains non-empty relatively weakly open subsets of arbitrarily small diameter. Thus

$$\text{RNP} \Rightarrow \text{PCP} \Rightarrow \text{CPCP} \Rightarrow \text{SR}$$

and no reverse implication holds (see [Bo-Ro],[GMS] and [GMS2] respectively for counter-examples). Note that the reason why CPCP implies SR is a result of J. Bourgain which asserts that, given a non-empty, closed, convex and bounded subset C of a Banach space X, then every non-empty relatively weakly open subset of C contains a convex combination of slices of C [GGMS, Lemma II.1].

In the lines of Question 1 we can pose the following question.

Question 2. *Let X be a Banach space.*

1. *If X fails the CPCP, is there any equivalent renorming such that every non-empty relatively weakly open subset of the unit ball has diameter exactly 2?*

2. *If X fails to be SR, is there any equivalent renorming on X such that all the convex combination of slices of the unit ball has diameter exactly 2?*

This motivates the definition of the *diameter two property* (D2P) (respectively the *strong diameter two property* (SD2P)) as the fact that every non-empty relatively weakly open subset (respectively convex combination of slices) of the unit ball of a Banach space has diameter 2. Note that the slice-D2P (respectively D2P, SD2P) is a natural candidate for a geometric property of Banach spaces to characterise, under an equivalent renorming, the failure of the RNP (respectively CPCP, SR) in connection with Question 1 (respectively Question 2).

Notice also that, taking into account that the dual of a Banach space X is SR if, and only if, X does not contain any isomorphic copy of ℓ_1 [GGMS, Theorem VI. 18], then Question 2.2 can be written in the following way for dual Banach spaces: can every Banach space containing an isomorphic copy of ℓ_1 be equivalently renormed so that the dual has the SD2P? Note that an afirmative answer to this question implies that the SD2P should have a reformulation in terms of a geometric property P verifying

1. A Banach space X admits an equivalent renorming with the property P if, and only if, X contains an isomorphic copy of ℓ_1; and
2. such a property P should imply (or be equivalent to) having big convex combination of slices in the dual unit ball.

A natural candidate for such a property P is, in view of the works [Dev] and [God2], the *octahedral norms*. According to [God2], the norm of a Banach space is said to be *octahedral* if, for every finite-dimensional subspace Y of X and every $\varepsilon > 0$, there exists an element $x \in S_X$ such that

$$\|y + \lambda x\| > (1 - \varepsilon)(\|y\| + |\lambda|)$$

holds for every $y \in Y$ and every $\lambda \in \mathbb{R}$. The connection of octahedral norms with 1 is [God2, Theorem II.4 and Remark II.5], where it is proved that a Banach space X admits an equivalent octahedral norm if, and only if, X contains an isomorphic copy of ℓ_1. The connection with 2 is the paper [Dev] where, by making use of the property of *average rough norms* (see Definition 1.11), R. Deville proves that if the norm of a Banach space is octahedral then every convex combination of weak-star slices of the dual unit ball has diameter exactly 2.

This puts in relation the theory of octahedral norms with the one of the diameter two properties, which will be the central topic of this memory.

Summary of the thesis

The aim of this dissertation is to analyse and solve several problems in the theory of diameter two properties, octahedrality, and to take advantage of the strong connection between the diameter two properties and the octahedrality of the norm to study new examples of Banach spaces whose norm is octahedral. Let us give a detailed description of the content of the thesis:

Chapter 1: Background on the diameter two properties

In this chapter we will exhibit several results about diameter two properties which appeared in the literature before the results exposed in this dissertation.

We will begin in Section 1.1 with preliminaries and basic facts about diameter two properties. In Section 1.2 we point out the relation between the diameter two properties and other geometric properties of Banach spaces which appeared before the starting point of the study of the diameter two properties. Section 1.3 is devoted to exhibiting additional examples of Banach spaces which were studied under the point of view of the diameter two properties. Finally, in Section 1.4 it is analysed the problem of how the diameter two properties are preserved by ℓ_p-sums or by tensor product spaces.

Chapter 2: Diameter two properties

In this chapter we will study several problems related to the diameter two properties and to the octahedrality of the norm.

Section 2.1

In this section we analyse whether the slice-D2P and the D2P are really different or not. Notice that if the answer to Question 1 were affirmative this would imply the existence of an example of a Banach space with the slice-D2P and the PCP (in particular, its unit ball would have all the slices of diameter 2 but it would contain non-empty relatively weakly open subsets of arbitrarily small diameter). In spite of the fact that we do not know the answer to Question 1, we construct an example of Banach space with the slice-D2P and whose unit ball contains non-empty relatively weakly open subsets of arbitrarily small diameter. In fact, we consider a non-empty, bounded, convex and closed subset K of c, the space of all convergent scalar sequences, such that every slice of K has diameter exactly $\operatorname{diam}(K)$ but which contains non-empty relatively weakly open subsets of arbitrarily small diameter (Propositions 2.1 and 2.2). Then, by making use of a renorming technique of those Banach spaces containing an isomorphic copy of c_0 (Lemma 2.3), we get the following result.

Theorem 1 (with Julio Becerra Guerrero and Ginés López-Pérez). *Let X be a Banach space containing an isomorphic copy of c_0. Then there is an equivalent norm on X such that:*

1. *Every slice of the new unit ball of X has diameter 2 for the new equivalent norm.*
2. *There are non-empty relatively weakly open subsets of the new unit ball of X with arbitrarily small diameter.*

The content of the section is based on [BLR1].

Section 2.2

It is known that D2P and the SD2P are actually different properties (an example is $c_0 \oplus_2 c_0$ [ABL]). However, in view of Theorem 1 is it a natural question whether or not the D2P and the SD2P are actually extremely different in the sense that there are Banach spaces with the D2P but whose unit ball contains convex combinations of slices of arbitrarily small diameter. After noticing that the natural example $c_0 \oplus_2 c_0$ does not produce the desired extreme example since every convex combination of slices of the unit ball of $c_0 \oplus_2 c_0$ has diameter, at least, 1, we will follow the underlying ideas of Theorem 1. Namely, we will consider a non-empty, bounded, closed and convex subset K of c_0 where every non-empty relatively weakly open subset has diameter $2 = \operatorname{diam}(K)$ but containing convex combinations of slices of arbitrarily small diameter (Proposition 2.7). After that, making use again of Lemma 2.3, we prove the following theorem.

Theorem 2 (with Julio Becerra Guerrero and Ginés López-Pérez). *Let X be a Banach space containing an isomorphic copy of c_0. Then there is an equivalent norm $|||\cdot|||$ on X such that every non-empty relatively weakly open subset of $B_{(X,|||\cdot|||)}$ has diameter 2 and that $B_{(X,|||\cdot|||)}$ contains convex combinations of slices of arbitrarily small diameter.*

The content of this section is based on [BLR3].

Section 2.3

In this section we consider a strengthening of the SD2P, the so-called *almost squareness (ASQ)*. According to [ALL], a Banach space X is said to be *almost square* if, for every $x_1, \ldots, x_n \in S_X$ and every $\varepsilon > 0$, there exists $y \in S_X$ such that

$$\|x_i \pm y\| \leq 1 + \varepsilon$$

holds for every $i \in \{1, \ldots, n\}$.

It is proved in [ALL] that every ASQ Banach space contains an isomorphic copy of c_0. Conversely, in [ALL] it is proved that every Banach space containing a complemented copy of c_0 can be equivalently renormed to be ASQ. However, the authors of [ALL] ask whether the complementability assumption can be removed. In order to give a positive answer to this question, we prove that ℓ_∞ can be equivalently renormed to be ASQ. Then, making use of the injectivity of ℓ_∞ and of the ASQ renorming on ℓ_∞, we prove the main theorem of the section.

Theorem 3 (with Julio Becerra Guerrero and Ginés López-Pérez)**.** *Let X be a Banach space containing an isomorphic copy of c_0. Then there exists an equivalent norm on X such that X is an ASQ space under the new norm.*

The content of this section is based on [BLR7, Section 2].

Section 2.4

As we have pointed out before, in [Dev] it is proved that, given a Banach space X, if the norm of X is octahedral then the dual space X^* has the w^*-SD2P. However, R. Deville posed in [Dev, Remark (c)] as an open question whether the converse is true or not. The main theorem of the section is the establishment of the converse, namely:

Theorem 4 (with Julio Becerra Guerrero and Ginés López-Pérez)**.** *Let X be a Banach space. Then the following assertions are equivalent:*

1. *The norm of X is octahedral.*

2. *Every convex combination of w^*-slices of B_{X^*} has diameter* 2.

As an application of this result, notice that [God2, Theorem II.4] has the following interpretation in terms of the diameter two properties: A Banach space contains an isomorphic copy of ℓ_1 if, and only if, there exists an equivalent norm on X such that X^* has the w^*-SD2P. A natural question, which is equivalent to question [God2, Remark II.5, 3)], is whether every Banach space containing an isomorphic copy of ℓ_1 can be equivalently renormed such that X^* has the SD2P. The following partial answer is obtained.

Proposition 5 (with Julio Becerra Guerrero and Ginés López-Pérez)**.** *Let X be a separable Banach space containing a subspace isomorphic to ℓ_1. Then, for every $\varepsilon > 0$, there is an equivalent norm on X such that every convex combination of slices of the new unit ball of X^* has diameter, at least, $2 - \varepsilon$.*

Notice that, since a dual Banach space fails to be SR if, and only if, the predual space contains any isomorphic copy of ℓ_1, the previous Proposition can be read as follows: every dual Banach space which fails to be SR and whose predual is separable admits, for every $\varepsilon > 0$, an equivalent dual renorming such that every convex combination of slices of the unit ball has diameter, at least, $2 - \varepsilon$. Thus the previous proposition can be seen as a partial answer to the second question in Question 2 too.

The content of this section is based on [BLR4].

Section 2.5

This section is devoted to pointing out further research, remarks and open questions related to the content of Chapter 2.

Chapter 3: Examples of Banach spaces with an octahedral norm

In this chapter we obtain examples of Banach spaces whose norm is octahedral making use of the characterisation of the octahedrality in terms of the w^*-SD2P in the dual space.

Section 3.1

In this section we analyse the problem of whether the operator norm on a space H such that $X^* \otimes Y \subseteq H \subseteq L(X, Y)$ can or not be octahedral. This problem is closely related to the problem of how the diameter two properties are preserved by taking projective tensor product, a problem which has been analysed in the literature (see [ABR, ALN2]) and explicitly posed as an open question in [ALN2, Question (b)].

In the first half of the section we obtain several sufficient conditions about octahedrality in spaces of operators, being particularly interesting the following stability result.

Theorem 6 (with Julio Becerra Guerrero and Ginés López-Pérez)**.** *Let X and Y be two Banach spaces. If the norm of X^* and Y are octahedral and H is a subspace of $L(X, Y)$ containing $X^* \otimes Y$, then the operator norm on H is octahedral.*

As a consequence, if two Banach spaces X and Y have the SD2P then so does its projective tensor product $X \widehat{\otimes}_\pi Y$.

The previous theorem provides an incomplete answer to [ALN2, Question (b)], where it is asked how the diameter two properties are preserved by taking projective tensor product. In order to give a complete answer, we want to know whether or not the SD2P is actually preserved from just one factor by taking projective tensor product. The answer to this question is given in the following theorem.

Theorem 7 (with Johann Langemets and Vegard Lima)**.** *Let X and Y be Banach spaces and assume that Y^* is uniformly convex. Assume also that there exists a closed subspace H of $L(Y^*, X)$ such that $X \otimes Y \subseteq H$ and that the norm of H is octahedral. Then Y^* is finitely representable in X.*

In particular, given $2 < p < \infty$ and $n \geq 3$, then neither $\ell_\infty \widehat{\otimes}_\pi \ell_p^n$ nor $L_\infty \widehat{\otimes}_\pi \ell_p^n$ enjoy the SD2P.

Now we do have a complete answer to [ALN2, Question (b)] for the SD2P in the projective case: the SD2P is preserved from both factors but not from just one of them by taking projective tensor product. Furthermore we point out that finite-representability is not only a necessary condition for getting octahedrality in certain spaces of operators under uniform convexity assumptions but it is also a sufficient condition in the following special case.

Theorem 8 (with Johann Langemets and Vegard Lima)**.** *Let X be a Banach space. Then:*

1. *If, for all $\varepsilon > 0$, X is $(1+\varepsilon)$ isometric to a subspace of ℓ_1, then the norm of $L(X, \ell_1)$ is octahedral.*

2. *If, for all $\varepsilon > 0$, X is $(1+\varepsilon)$ isometric to a subspace of L_1, then the norm of $L(X, L_1)$ is octahedral.*

The results of this section are based on [BLR5] (from Proposition 3.1 to Corollary 3.12) and on [LLR2, Section 3] (from Lemma 3.17 until the end).

Section 3.2

In this section we consider the vector-valued version of Lipschitz-free spaces and we give sufficient conditions for a vector-valued Lipschitz-free space to have an octahedral norm. In order to do so, we make use of the dual characterisation of octahedrality in terms of the w^*-SD2P and then we study when a space of Lipschitz functions $\mathrm{Lip}_0(M, X^*)$ can have the w^*-SD2P. We will need the assumption that all the Lipschitz functions can be extended without increasing its Lipschitz norm (for details, see the definition of the *contraction-extension property (CEP)* given in Definition 3.31). Now the main theorem of the section is the following:

Theorem 9 (with Julio Becerra Guerrero and Ginés López-Pérez)**.** *Let M be an infinite pointed metric space and let X be a Banach space. Assume that the pair (M, X^*) has the CEP. If M is unbounded or is not uniformly discrete then the norm of $\mathcal{F}(M, X)$ is octahedral. Consequently, the unit ball of $\mathcal{F}(M, X)$ does not have any point of Fréchet differentiability.*

Furthermore, we present an example of metric space M such that, depending on the target Banach space X, the range of possibilities for the space $\mathcal{F}(M, X)$ goes from having points of Fréchet differentiability to being octahedral for its natural norm. This has two important consequences. First, for a Lipschitz-free space it is possible to have points of Fréchet differentiability. Second, the octahedrality of a Lipschitz-free space may depend on the target Banach space.

The results of this section are based on [BLR6].

Section 3.3

In this section we will focus on analysing the octahedrality of a real Lipschitz-free space. Our aim is to find a geometric property of metric spaces which characterises the fact that the norm of corresponding Lipschitz-free space is octahedral. This would make of octahedrality part of the small group of properties of the geometry of Banach spaces which can be checked on a Lipschitz-free space $\mathcal{F}(M)$ looking only at the underlying metric space M. That is what is done in the following theorem, which is the main theorem of Section 3.3.

Theorem 10 (with Antonín Procházka)**.** *For a metric space M it is equivalent:*

1. *The norm of $\mathcal{F}(M)$ is octahedral.*
2. *For each $\varepsilon > 0$ and each finite subset $N \subset M$ there are points $u, v \in M$, $u \neq v$, such that every 1-Lipschitz function $f : N \to \mathbb{R}$ admits an extension $\tilde{f} : M \to \mathbb{R}$ which is $(1+\varepsilon)$-Lipschitz and satisfies $\tilde{f}(u) - \tilde{f}(v) \geq d(u, v)$.*
3. *For each finite subset $N \subseteq M$ and $\varepsilon > 0$, there exist $u, v \in M, u \neq v$, such that*
$$(1-\varepsilon)(d(x, y) + d(u, v)) \leq d(x, u) + d(y, v)$$
holds for all $x, y \in N$.

The property 3. in above theorem, which is a property of metric spaces, is what we define as the *long trapezoid property (LTP)* (see Definition 3.38). The rest of the section is devoted to making use of the previous characterisation to get results about the LTP from octahedrality theory and, vice versa, getting new examples of Lipschitz-free spaces $\mathcal{F}(M)$ whose norm is octahedral. Particularly interesting, among all the consequences of the previous theorem, is the following proposition.

Proposition 11 (with Antonín Procházka)**.** *Let X be a AUC Banach space such that $\delta_X(t) = t$ holds for all $t \geq 0$. Then every infinite subset of X has the LTP.*

In particular, for every infinite subset M of X it follows that the norm of $\mathcal{F}(M)$ is octahedral.

The content of this section is based on [PR].

Section 3.4

This section is devoted to pointing out further research, remarks and open questions related to the content of Chapter 3.

Introducción

Motivación

Es justo decir que una de las propiedades más estudiadas en el contexto de la teoría de los espacios de Banach es la *propiedad de Radon-Nikodym (RNP)* ya que, debido a su gran cantidad de caracterizaciones, la RNP ha demostrado ser muy útil en diversos marcos de los espacios de Banach tales como la representación de operadores lineales y continuos, la representación de espacios duales o la identificación de ciertos productos tensoriales de espacios de Banach (véase por ejemplo [Bou, DU]). A pesar de que se trata de una propiedad que es invariante bajo renormación equivalente, existe una interesante caracterización con un fuerte espíritu geométrico. Más concretamente, es conocido que un espacio de Banach X tiene la RNP si, y solamente si, todo subconjunto acotado C de X es *dentable*, es decir, C contiene rebanadas de diámetro arbitrariamente pequeño (de nuevo referimos a [Bou, DU] para una demostración). Nótese que, a pesar de que la dentabilidad es, en principio, una propiedad que depende de la norma concreta que consideremos en el espacio, el hecho de que la caracterización anterior verse sobre la dentabilidad de todos los subconjuntos acotados del espacio hace de dicha caracterización una propiedad que es invariante bajo renormación equivalente.

En vista de la caracterización anterior, dado un espacio de Banach X fallando la RNP, podemos encontrar un subconjunto acotado C de X y un positivo $\varepsilon \leq \operatorname{diam}(C)$ de manera que cada rebanada de C tiene diámetro, al menos, ε. Con el objetivo de tratar de llevar más lejos el fallo de la RNP, las dos preguntas siguientes parecen naturales:

i) ¿Puede ε ser tomado próximo a $\operatorname{diam}(C)$?

ii) En caso de que la respuesta anterior sea afirmativa, ¿puede usarse dicho conjunto C para obtener una renormación equivalente de X de manera que el diámetro de toda rebanada de la bola unidad sea (próximo a) 2?

Ambas preguntas fueron analizadas en [SSW]. Haciendo uso de los *módulos de no-dentabilidad* introducidos en dicho trabajo, se demostró en [SSW, Theorem 1.1] que si un espacio de Banach X falla la RNP entonces, para cada $\varepsilon > 0$, existe un subconjunto acotado, cerrado, convexo y separable C de X de manera que $\operatorname{diam}(C) = 1$ y de manera que el diámetro de toda rebanada de C es mayor o igual que $1 - \varepsilon$. Respecto a ii), se demostró en [SSW, Corollary 3.2] que todo espacio de Banach que falla la RNP admite, para cada $\varepsilon > 0$, una renormación equivalente de manera que el diámetro de todas las rebanadas de la nueva bola unidad es mayor o igual que $1-\varepsilon$, planteando como problema abierto si se podría obtener el valor $2 - \varepsilon$. Con respecto a esta pregunta, notemos que la respuesta es afirmativa en el contexto de los retículos de Banach [E.Wer]. Además, una respuesta afirmativa en el caso general plantearía el siguiente problema.

Pregunta 1. *Sea X un espacio de Banach que falla la RNP. ¿Puede ser X renormado con la propiedad de que todas las rebanadas de la nueva bola unidad tengan diámetro 2?*

Esta pregunta motiva la definición de la *propiedad de diámetro dos para rebanadas (slice-D2P)* en un espacio de Banach X como la propiedad de que toda rebanada de la bola unidad tenga diámetro 2.

Otras propiedades de la teoría de espacios de Banach estrechamente relacionadas con la RNP son la *propiedad del punto de continuidad, propiedad del punto de continuidad convexa* y la *regularidad fuerte*. Dado un espacio de Banach X, diremos que X:

- tiene la *propiedad del punto de continuidad (PCP)* si todo subconjunto no vacío, acotado y cerrado A de X tiene un punto $x \in A$ que es de débil-norma continuidad, es decir, de manera que la función identidad $i : (A, w) \longrightarrow (A, \tau_n)$ es continua en x, donde (A, w) denota al conjunto A con la topología débil restringida y (A, τ_n) denota a A con la topología de la norma restringida.

- tiene la *propiedad del punto de continuidad convexa (CPCP)* si cada subconjunto no vacío, cerrado, acotado y convexo A de X tiene un punto de débil-norma continuidad.

- es *fuertemente regular (SR)* si todo subconjunto no vacío, cerrado, acotado y convexo de X contiene combinaciones convexas de rebanadas de diámetro arbitrariamente pequeño.

Referimos al lector a [GMS] para más información sobre la (C)PCP y a [GGMS] sobre la SR. No es difícil demostrar que un espacio de Banach tiene la (C)PCP si, y solamente si, todo subconjunto no vacío, cerrado y acotado (respectivamente no vacío, cerrado, acotado y convexo) del espacio contiene abiertos débiles no vacíos de diámetro arbitrariamente pequeño. En consecuencia

$$\text{RNP} \Rightarrow \text{PCP} \Rightarrow \text{CPCP} \Rightarrow \text{SR},$$

y ninguna de las implicaciones recíprocas son ciertas (véase [Bo-Ro], [GMS] y [GGMS] para respectivos contraejemplos). Notemos que la razón por la que CPCP implica SR es un resultado de J. Bourgain que afirma que, dado un conjunto no vacío, cerrado, acotado y convexo C de X, entonces todo abierto débil no vacío de C contiene una combinación convexa de rebanadas de C [GGMS, Lemma II.1].

Siguiendo la línea de la Pregunta 1, es natural preguntarse lo siguiente.

Pregunta 2. *Sea X un espacio de Banach.*

1. *Si X falla la CPCP, ¿puede ser X renormado para que todos los abiertos débiles no vacíos de la bola unidad tengan diámetro 2?*

2. *Si X falla la SR, ¿puede ser X renormado para que todas las combinaciones convexas de rebanadas de la bola unidad tengan diámetro 2?*

Esto motiva la definición de la *propiedad de diámetro dos (D2P)* (respectivamente la *propiedad fuerte de diámetro dos (SD2P)*) como el hecho de que todo abierto débil no vacío de la bola unidad (respectivamente toda combinación convexa de rebanadas de la bola unidad) tenga diámetro 2. Notemos que la slice-D2P (respectivamente la D2P, SD2P) es una candidata natural a propiedad geométrica de espacios de Banach para caracterizar, bajo renormación equivalente, la negaci'on de la RNP (respectivamente CPCP, SR) en conexión con las preguntas anteriores.

Notemos también que, teniendo en cuenta que el dual de un espacio de Banach X falla ser SR si, y solamente si, X contiene una copia isomorfa de ℓ_1 [GGMS, Theorem VI. 18], entonces la Pregunta 2.2. puede reformularse de la siguiente manera: ¿puede renormarse todo espacio de Banach que contenga una copia isomorfa de ℓ_1 para que el dual tenga la SD2P? Notemos que una respuesta afirmativa a esta pregunta implica que la SD2P debería tener una reformulación en términos de una propiedad geométrica, digamos P, satisfaciendo las dos siguientes condiciones:

1. Dado un espacio de Banach X, entonces X admite una renormación equivalente con la propiedad P si, y solamente si, X contiene una copia isomorfa de ℓ_1.

2. Tal propiedad P debería implicar (o ser equivalente a) que las combinaciones convexas de rebanadas de la bola dual tengan diámetro grande.

Un candidato natural para tal propiedad P es, en vista de los trabajos [Dev] y [God2], la octaedralidad de la norma. Dado un espacio de Banach X, diremos que su norma es *octaedral* si, para cada subespacio finito dimensional Y de X y cada $\varepsilon > 0$, existe un elemento $x \in S_X$ de manera que

$$\|y + \lambda x\| > (1 - \varepsilon)(\|y\| + |\lambda|)$$

se cumple para cada $y \in Y$ y cada $\lambda \in \mathbb{R}$. La conexión entre las normas octaedrales y el punto 1. anterior es [God2, Theorem II.4 y Remark II. 5], donde se demostró que un espacio de Banach X admite una renormación equivalente octaedral si, y solamente si, X contiene una copia isomorfa de ℓ_1. Por otro lado, la conexión con el punto 2. viene del trabajo de Deville [Dev] donde, haciendo uso de las *normas rudas en media*, se demuestra que si la norma de un espacio de Banach X es octaedral entonces todas las combinaciones convexas de w^*-rebanadas de la bola unidad dual tienen diámetro 2.

Este hecho pone en relación la teoría de los espacios de Banach con norma octaedral con los espacios de Banach con propiedades de diámetro dos, el cual será uno de los principales ejes de esta memoria.

Resumen de la tesis

El principal objetivo de esta tesis es analizar y resolver diversos problemas enmarcados en el estudio de los espacios de Banach con las propiedades de diámetro dos y de los espacios con norma octaedral, y aprovechar la estrecha relación existente entre las propiedades de diámetro dos y las normas octaedrales para estudiar nuevos ejemplos de espacios de Banach cuya norma es octaedral. A continuación detallamos el contenido de la tesis.

Capítulo 1: Bagaje sobre propiedades de diámetro dos

En este capítulo exhibiremos varios resultados sobre propiedades de diámetro dos que aparecieron en la literatura anteriormente a los expuestos en esta memoria.

Comenzaremos en la Sección 1.1 con preliminares y propiedades básicas sobre las propiedades de diámetro dos. En la Sección 1.2 analizaremos las relaciones entre las propiedades de diámetro dos y otras propiedades de la geometría de los espacios de Banach. La Sección 1.3 está dedicada a proporcionar nuevos ejemplos de espacios de Banach con propiedades de diámetro dos que se obtuvieron cuando dichas propiedades se establecieron como objeto propio de estudio dentro de la geometría de los espacios de Banach. Por último, en la Sección 1.4 se exhibe un análisis sobre cómo se preservan las propiedades de diámetro dos por ℓ_p-sumas o por productos tensoriales de espacios.

Capítulo 2: Propiedades de diámetro dos

En este capítulo estudiaremos varios problemas relativos a las propiedades de diámetro dos y a las normas octaedrales.

Sección 2.1

En esta sección analizamos si la slice-D2P y la D2P son propiedades diferentes o no. Notemos que si la respuesta a la Pregunta 1 fuese afirmativa entonces existiría un ejemplo de espacio de Banach con la slice-D2P y la PCP (en particular, su bola unidad tendría todas las rebanadas de diámetro 2 pero contendría abiertos débiles no vacíos de diámetro arbitrariamente pequeño). A pesar de que no sabemos la respuesta a la Pregunta 1, construimos un ejemplo de espacio de Banach con la slice-D2P pero de manera que su bola unidad contiene abiertos débiles no vacíos de diámetro arbitrariamente pequeño. De hecho, consideramos un conjunto no vacío, cerrado, acotado y convexo K en c, el espacio de las sucesiones convergentes con la norma del supremo, de manera que cada rebanada de K tiene diámetro igual a $\mathrm{diam}(K)$ pero conteniendo abiertos débiles no vacíos de diámetro arbitrariamente pequeño (Propositions 2.1 y 2.2). Después, haciendo uso de una técnica de renormación de los espacios de Banach que contienen una copia isomorfa de c_0 (Lemma 2.3), obtenemos el siguiente resultado.

Teorema 1 (Junto a Julio Becerra Guerrero y Ginés López-Pérez). *Sea X un espacio de Banach que contiene una copia isomorfa de c_0. Entonces existe una norma equivalente en X de manera que:*

1. *Toda rebanada de la nueva bola unidad tiene diámetro 2.*
2. *Su bola unidad contiene abiertos débiles no vacíos de diámetro arbitrariamente pequeño.*

El contenido de esta sección está basado en [BLR1].

Sección 2.2

Es conocido que la D2P y la SD2P son propiedades diferentes (y un contraejemplo es $c_0 \oplus_2 c_0$ [ABL]). Sin embargo, en vista del Teorema 1, una pregunta natural es si la D2P y la SD2P son propiedades diferentes en el mismo sentido en que lo son la slice-D2P y la D2P, es decir, si existe un espacio de Banach con la D2P con la propiedad de que su bola unidad contenga combinaciones convexas de rebanadas de diámetro arbitrariamente pequeño. Después de comprobar que $c_0 \oplus_2 c_0$ no produce dicho contraejemplo extremo, ya que el diámetro de todas las combinaciones convexas de rebanadas es mayor o igual que 1, seguiremos las técnicas del Teorema 1. Más estrictamente, consideramos un subconjunto no vacío, cerrado, acotado y convexo K de c_0 con la propiedad de que todo abierto débil no vacío de K tiene diámetro exactamente $2 = \text{diam}(K)$ y de que K contiene combinaciones convexas de rebanadas de diámetro arbitrariamente pequeño (Proposition 2.7). Después, haciendo uso de nuevo del Lemma 2.3, demostramos el siguiente teorema.

Teorema 2 (Junto a Julio Becerra Guerrero y Ginés López-Pérez)**.** *Sea X un espacio de Banach que contiene una copia isomorfa de c_0. Entonces existe una norma equivalente $|||\cdot|||$ sobre X de manera que todo abierto débil no vacío de $B_{(X,|||\cdot|||)}$ tiene diámetro 2 y de manera que $B_{(X,|||\cdot|||)}$ contiene combinaciones convexas de rebanadas de diámetro arbitrariamente pequeño.*

El contenido de esta sección está basado en [BLR3].

Sección 2.3

En esta sección consideramos un reforzamiento de la SD2P, conocido como *ASQ*. De acuerdo con [ALL], un espacio de Banach X es *casi cuadrado (ASQ)* si, para cada $x_1, \ldots, x_n \in S_X$ y cada $\varepsilon > 0$, existe $y \in S_X$ de manera que

$$\|x_i \pm y\| \leq 1 + \varepsilon$$

se cumple para cada $i \in \{1, \ldots, n\}$.

Se demostró en [ALL] que cada espacio ASQ contiene una copia isomorfa de c_0. Recíprocamente, en [ALL] se demostró también que todo espacio de Banach que contiene una copia complementada de c_0 admite una renormación equivalente ASQ. Sin embargo, los autores de [ALL] plantearon como problema abierto si la complementación podía ser eliminada. Para dar una respuesta afirmativa a esta pregunta, demostramos que ℓ_∞ admite una renormación equivalente para ser ASQ. Después, haciendo uso de la inyectividad del espacio ℓ_∞, demostramos el teorema principal de la sección.

Teorema 3 (Junto a Julio Becerra Guerrero y Ginés López-Pérez)**.** *Sea X un espacio de Banach que contiene una copia isomorfa de c_0. Entonces X admite una renormación equivalente para ser ASQ.*

El contenido de esta sección está basado en [BLR7, Section 2].

Sección 2.4

Tal y como hemos señalado anteriormente, en [Dev] se demuestra que, dado un espacio de Banach X, si la norma de X es octaedral entonces el espacio dual X^* tiene la w^*-SD2P. Sin embargo, R. Deville planteó en [Dev, Remark (c)] como problema abierto si el recíproco es cierto o no. El teorema principal de esta sección da la respuesta afirmativa a esta pregunta:

Teorema 4 (Junto a Julio Becerra Guerrero y Ginés López-Pérez). *Sea X un espacio de Banach. Las siguientes afirmaciones son equivalentes:*

1. *La norma de X es octaedral.*

2. *Toda combinación convexa de w^*-rebanadas de B_{X^*} tiene diámetro 2.*

Como aplicación de este resultado, notemos que [God2, Theorem II.4] tiene la siguiente reformulación en términos de propiedades de diámetro dos: un espacio de Banach X contiene una copia isomorfa de ℓ_1 si, y solamente si, existe una norma equivalente en X de manera que X^* tiene la w^*-SD2P. Una pregunta natural, equivalente a la pregunta [God2, Remark II.5], es si todo espacio de Banach que contiene una copia isomorfa de ℓ_1 admite una renormación equivalente para que X^* tenga la SD2P. En este sentido, una respuesta parcial es la siguiente:

Proposición 1 (Junto a Julio Becerra Guerrero y Ginés López-Pérez). *Sea X un espacio de Banach separable conteniendo una copia isomorfa de ℓ_1. Entonces, para cada $\varepsilon > 0$, existe una norma equivalente en X de manera que todas las combinaciones convexas de rebanadas de la bola unidad de X^* tienen diámetro, al menos, $2 - \varepsilon$.*

Notemos de nuevo que, dado que un espacio dual falla ser SR si, y solamente si, el predual no contiene ninguna copia isomorfa de ℓ_1, la proposición anterior puede reformularse como sigue: todo espacio de Banach dual que falla ser SR cuyo predual es separable admite, para cada $\varepsilon > 0$, una renormación (dual) equivalente de manera que todas las combinaciones convexas de rebanadas de la bola unidad tiene diámetro mayor o igual que $2 - \varepsilon$. En consecuencia, la proposición anterior también puede verse como una respuesta parcial a la segunda cuestión de la Pregunta 2.

El contenido de esta sección está basado en [BLR4].

Sección 2.5

Esta sección está dedicada a mostrar líneas de investigación futuras y derivadas del contenido del Capítulo 2, así como a mostrar comentarios relevantes y a plantear problemas abiertos relacionados.

Capítulo 3: Ejemplos de espacios de Banach con norma octaedral

En este capítulo obtendremos ejemplos de espacios de Banach cuya norma es octaedral haciendo un fuerte uso de la caracterización de la octaedralidad en términos de la w^*-SD2P en el espacio dual.

Sección 3.1

En esta sección analizaremos el problema de si la norma de operadores es o no octaedral en un espacio H de manera que $X^* \otimes Y \subseteq H \subseteq L(X, Y)$. Este problema está estrechamente relacionado con el problema de cómo se preservan, en general, las propiedades de diámetro dos por productos tensoriales, un problema ya considerado en la literatura (véase por ejemplo [ABR, ALN2]) y explícitamente propuesto como problema abierto en [ALN2, Question (b)].

En la primera mitad de la sección trataremos de obtener diversas condiciones suficientes sobre octaedralidad en espacios de operadores, siendo particularmente interesante el siguiente resultado de estabilidad.

Teorema 5 (Junto a Julio Becerra Guerrero y Ginés López-Pérez). *Sean X e Y dos espacios de Banach. Si la norma de X^* e Y son octaedrales y H es un subespacio de $L(X,Y)$ que contiene a $X^* \otimes Y$, entonces la norma de operadores sobre H es octaedral.*

Como consecuencia, si X e Y tienen la SD2P, entonces $X\widehat{\otimes}_\pi Y$ tiene la SD2P.

El teorema anterior proporciona una respuesta incompleta a [ALN2, Question (b)], donde se pregunta cómo se preservan las propiedades de diámetro dos por producto tensor proyectivo. Para tratar de completar la respuesta, nos preguntamos si la SD2P en realidad se preserva desde sólo uno de los dos factores. La respuesta se obtiene como consecuencia del siguiente teorema.

Teorema 6 (Junto a Johann Langemets y Vegard Lima). *Sean X e Y dos espacios de Banach de manera que Y^* es uniformemente convexo. Supongamos que existe un subespacio H de $L(Y^*,X)$ que contiene a $X \otimes Y$ y de manera que la norma de operadores sobre H es octaedral. Entonces, Y^* es finitamente representable en X.*

Con el teorema anterior ya tenemos una respuesta completa para [ALN2, Question (b)] en el caso de la SD2P y la norma proyectiva: la SD2P se preserva desde los dos factores por producto tensor proyectivo pero no desde uno solo de ellos. Además, notemos que la representabilidad finita no es solamente una condición necesaria para octaedralidad en ciertos espacios de operadores en presencia de la convexidad uniforme, sino que también es una condición suficiente en el siguiente caso particular.

Teorema 7 (Junto a Johann Langemets y Vegard Lima). *Sea X un espacio de Banach. Entonces:*

1. *Si, para cada $\varepsilon > 0$, X es $(1+\varepsilon)$-isométrico a un subespacio de ℓ_1, entonces la norma de $L(X,\ell_1)$ es octaedral.*

2. *Si, para cada $\varepsilon > 0$, X es $(1+\varepsilon)$-isométrico a un subespacio de L_1, entonces la norma de $L(X,L_1)$ es octaedral.*

Los resultados de esta sección están basados en [BLR5] (desde Proposition 3.1 hasta Corollary 3.12) y en [LLR2, Section 3] (desde Lemma 3.17 hasta el final).

Sección 3.2

En esta sección consideramos los espacios Lipschitz libres vector valuados y damos condiciones suficientes para que su norma sea octaedral. Para ello, usaremos la caracterización de octaedralidad en términos de la w^*-SD2P en el dual. En consecuencia, nos centramos en analizar cuándo un espacio de funciones Lipschitzianas $\mathrm{Lip}_0(M,X^*)$ puede tener la w^*-SD2P. Para ello será esencial la hipótesis de que todas las funciones Lipschitzianas pueden extenderse sin incrementar su norma Lipschitz (para más detalles referimos a la definición de *contraction-extension property (CEP)* dada en la Definition 3.31). En estos términos, el teorema principal de la sección reza como sigue:

Teorema 8 (Junto a Julio Becerra Guerrero y Ginés López-Pérez). *Sea M un espacio métrico infinito y sea X un espacio de Banach. Supongamos que el par (M, X^*) tiene la CEP. Si M no está acotado o si M no es uniformemente discreto, entonces la norma de $\mathcal{F}(M, X)$ es octaedral. En consecuencia, la bola unidad de $\mathcal{F}(M, X)$ no tiene ningún punto de diferenciabilidad Fréchet.*

Además, presentamos un ejemplo de espacio métrico infinito M de manera que, dependiendo del espacio de llegada X, el rango de posibilidades para el espacio $\mathcal{F}(M, X)$ varía desde tener puntos de diferenciabilidad Fréchet hasta que su norma sea octaedral. Esto tiene dos consecuencias importantes. La primera es que para un espacio de Banach Lipschitz libre, es posible que su bola unidad tenga puntos de diferenciabilidad Fréchet. Segundo, este ejemplo prueba que la octaedralidad de la norma de $\mathcal{F}(M, X)$ depende tanto del espacio métrico subyacente M como del espacio de Banach de llegada X.

Los resultados de esta sección están basados en [BLR6].

Section 3.3

En esta sección nos centraremos en analizar la octaedralidad de un espacio Lipschitz libre real. Nuestro objetivo es encontrar una propiedad (geométrica) de espacios métricos que caracterice el hecho de que el espacio Lipschitz libre correspondiente tenga norma octaedral, lo cual haría de la octaedralidad de la norma parte del pequeño grupo de propiedades de espacios de Banach que podría comprobarse en un espacio Lipschitz libre $\mathcal{F}(M)$ analizando solo el espacio métrico subyacente M. Esto es justamente lo que hacemos en el siguiente teorema, el cual es el resultado principal de la sección.

Teorema 9 (Junto a Antonín Procházka). *Sea M un espacio métrico. Son equivalentes:*

1. *La norma de $\mathcal{F}(M)$ es octaedral.*

2. *Para cada $\varepsilon > 0$ y cada subconjunto finito $N \subset M$ existen puntos $u, v \in M$, $u \neq v$, de manera que cada función Lipschitziana $f : N \to \mathbb{R}$ de norma 1 admite una extensión $\tilde{f} : M \to \mathbb{R}$ cuya norma Lipschitziana es menor o igual que $1 + \varepsilon$ y satisface que $\tilde{f}(u) - \tilde{f}(v) \geq d(u, v)$.*

3. *Para cada subconjunto finito $N \subseteq M$ y cada $\varepsilon > 0$, existen $u, v \in M, u \neq v$, de manera que*
$$(1 - \varepsilon)(d(x, y) + d(u, v)) \leq d(x, u) + d(y, v)$$
se cumple para cada $x, y \in N$.

La propiedad 3. en el teorema anterior, que es una propiedad de espacios métricos, es lo que se define como *propiedad de trapecios largos (LTP)* (véase Definition 3.38). El resto de la sección se dedica a hacer uso de la caracterización anterior para obtener resultados sobre LTP a partir de resultados de octaedralidad en espacios Lipschitz libres y, recíprocamente, para obtener nuevos resultados de espacios Lipschitz libres cuya norma es octaedral. Particularmente interesante, sobre todas las consecuencias obtenidas del teorema anterior, es la siguiente proposición.

Proposición 2 (Junto a Antonín Procházka)**.** *Sea X un espacio de Banach AUC de manera que $\delta_X(t) = t$ se verifica para cada $t \geq 0$. Entonces, todo subconjunto infinito de X tiene la LTP.*

En particular, para todo subconjunto infinito M de X se sigue que la norma de $\mathcal{F}(M)$ es octaedral.

El contenido de esta sección está basado en [PR].

Sección 3.4

Esta sección está dedicada a mostrar líneas de investigación futuras y derivadas del contenido del Capítulo 3, así como a mostrar comentarios relevantes y a plantear problemas abiertos relacionados.

Notation

We will follow standard notation and will usually follow the books [AK] and [FHHMPZ]. We will consider real Banach spaces. Given a Banach space X then B_X (respectively S_X) stands for the closed unit ball (respectively the unit sphere) of X. We will denote by X^* the topological dual of X. Given a Banach space Y, $L(X,Y)$ denotes the space of all bounded linear operators from X to Y. Given a convex subset D of X we will denote by $\text{ext}(D)$ the set of all extreme points of D. Given a subset C of X, we will denote by $\text{conv}(C)$ the convex hull of C and by $\text{span}(C)$ the linear hull of C. If C is bounded then by a *slice* of C we will mean a set of the following form

$$S(C, f, \alpha) := \{x \in C : f(x) > \sup f(C) - \alpha\}$$

where $f \in X^*$ and $\alpha > 0$. Notice that a slice is nothing but the intersection of a half-space with the bounded (and not necessarily convex) set C. Furthermore, if X is itself a dual Banach space (say X is dual of X_*), the previous set will be a *w^*-slice* when $f \in X_*$. If C is assumed to be convex we will mean by a *convex combination of slices* a set of the following form

$$\sum_{i=1}^{n} \lambda_i S_i,$$

where $\lambda_1, \ldots, \lambda_n \in]0,1]$ are such that $\sum_{i=1}^{n} \lambda_i = 1$ and S_i is a slice of C for every $i \in \{1, \ldots, n\}$. Again, in the particular case that X is a dual space, the previous set will be a *convex combination of w^*-slices* whenever each slice S_i is actually a w^*-slice.

Given a non-empty set I and a ultrafilter $\mathcal{U}$ on I, we say that $\mathcal{U}$ is a *principal ultrafilter* if there exists $i \in I$ such that $\mathcal{U} = \{Y \subseteq I : i \in Y\}$. Otherwise, we will say that $\mathcal{U}$ is *non-principal*. We refer to [Wil] for background on ultrafilters. If I is an infinite set, it is known that there are non-principal ultrafilter on I. Given a ultrafilter $\mathcal{U}$ on I and a bounded function $f : I \longrightarrow \mathbb{R}$ we will denote by $\lim_{\mathcal{U}} f$ the limit of f by the ultrafilter $\mathcal{U}$, which is the unique real number α with the property that

$$\{i \in I : |f(i) - \alpha| < \varepsilon\} \subseteq \mathcal{U}$$

holds for every $\varepsilon > 0$. It is known that, given a ultrafilter $\mathcal{U}$ on $\mathbb{N}$, then $\mathcal{U}$ is a non-principal ultrafilter if, and only if,

$$\lim_{\mathcal{U}} x = \lim_{n\to\infty} x(n)$$

holds for every convergent sequence x.

Let X and Y be two infinite-dimensional Banach spaces. We recall that X is *finitely representable in Y* if, given any finite-dimensional subspace E of X and $\varepsilon > 0$, then there exists a bounded linear isomorphism $T : E \longrightarrow T(E) \subseteq Y$ such that

$$\|T\|\|T^{-1}\| \leq 1 + \varepsilon.$$

It is known that, given $1 < p < \infty$ then ℓ_p is finitely representable in ℓ_1 if, and only if, ℓ_p is isometric to a subspace of L_1 if, and only if, $1 \leq p \leq 2$ (see [AK, Section 11.1]). We will freely use this fact without any explicit reference in Section 3.1.

Given a metric space M, a point $x \in M$ and a positive r, we will denote by $B(x, r)$ the open ball of center x and radius r, that is, the open set $\{y \in M : d(x, y) < r\}$.

Contents

Chapter 1

Background on diameter two properties

In this chapter we will introduce basic results about the dimeter two properties. Though the starting point of the diameter two property is probably the paper [NW], they actually appeared earlier in relation to other geometric properties of Banach spaces such as the roughness of the norm or the Daugavet property. Consequently, after the introduction of definitions and very basic facts about the diameter two properties in Section 1.1, we will devote Section 1.2 to exhibit some of such properties which will give us the first examples of Banach spaces enjoying the diameter two properties. We will exhibit in Section 1.3 further classical examples of Banach spaces with the diameter two properties. Finally, in Section 1.4, we will introduce an exhaustive study of how the diameter two properties are preserved by considering ℓ_p-sums and by considering spaces of Bochner integrable functions.

1.1 Preliminaries

We will begin with one of the main definitions of this dissertation.

Definition 1.1. Let X be a Banach space.

1. X has the *slice diameter two property (slice-D2P)* if every slice of B_X has diameter two.

2. X has the *diameter two property* (D2P) if every non-empty relatively weakly open subset of B_X has diameter two.

3. X has the *strong diameter two property* (SD2P) if every convex combination of slices of B_X has diameter two.

Remark 1.2. The slice-D2P is also known as the *local diameter two property* (see, e.g. [ALN2]). We prefer the term of slice diameter two property (c.f. [ABL, BLR1]) since we think it is more descriptive.

It is clear that D2P implies the slice-D2P. Furthermore, the SD2P implies the D2P since every non-empty relatively weakly open subset of B_X contains a convex combination of slices of B_X [GGMS, Lemma II.1]. Furthermore, notice that in dual Banach spaces it does make sense to replace the concept of slice (respectively weakly open set, convex combination of slices) with the one of weak-star slice (respectively weakly-star open set, convex combination of weak-star slices). Thus, we will consider the following definition.

Definition 1.3. Let X be a dual Banach space.

1. X has the *weak-star slice diameter two property (w^*-slice-D2P)* if every w^*-slice of B_X has diameter two.

2. X has the *weak-star diameter two property* (w^*-D2P) if every non-empty relatively weakly-star open subset of B_X has diameter two.

3. X has the *weak-star strong diameter two property* (w^*-SD2P) if every convex combination of w^*-slices of B_X has diameter two.

Let us consider the following diagram

$$\begin{array}{ccccc} \text{SD2P} & \overset{(1)}{\Longrightarrow} & \text{D2P} & \overset{(2)}{\Longrightarrow} & DLD2P \\ \Downarrow (3) & & \Downarrow (4) & & \Downarrow (5) \\ w^*\text{-SD2P} & \overset{(6)}{\Longrightarrow} & w^*\text{-D2P} & \overset{(7)}{\Longrightarrow} & w^*\text{slice-D2P} \end{array} \tag{1.1}$$

where the last row only makes sense in dual Banach spaces. We will see in Sections 2.1 and 2.2 that none of the reverse implications hold. For all the vertical arrows we have the following example.

Example 1.4. $\mathcal{C}([0,1])^*$ has the w^*-SD2P since $\mathcal{C}([0,1])$ has the Daugavet property (see Proposition 1.18). However, $B_{\mathcal{C}([0,1])^*}$ contains slices of arbitrarily small diameter. To see this, consider $\varphi : \mathcal{C}([0,1])^* \longrightarrow \mathbb{R}$ given by

$$\varphi(\mu) := \mu(\{0\}).$$

It follows that the family of slices $\{S(B_{\mathcal{C}([0,1])^*}, \varphi, \alpha) : \alpha > 0\}$ produces slices of arbitrarily small diameter.

Let us finish the section with the following relation between the diameter two properties and their corresponding weak-star version in the bidual space, which is a consequence of the weak-star denseness of a Banach space into the bidual and the weak-star lower semicontinuity of dual norms (see e.g. [Lan, Proposition 2.14] for details).

Proposition 1.5. *Let X be a Banach space. Then X has the slice-D2P (respectively D2P, SD2P) if, and only if, X^{**} has the w^*-slice-D2P (respectively w^*-D2P, w^*-SD2P).*

Remark 1.6. As a consequence of Proposition 1.5 we get that, given a Banach space X, then the diameter two properties are inherited from X^{**} to X, and an easy proof follows from the Principle of Local Reflexivity. A generalisation of this fact can be found in [ALN], where it is studied how the diameter two properties are inherited by *almost isometric ideals* (see [ALN] for formal definitions and background).

1.2 Geometric properties implying the diameter two properties

The relations between the diameter two properties and different geometric properties of Banach space probably appeared first in connection with the following property.

Definition 1.7. Let X be a Banach space and let $\varepsilon > 0$. It is said that the norm of X is *ε-rough* if

$$\limsup_{\|h\|\to 0} \frac{\|x+h\|+\|x-h\|-2\|x\|}{\|h\|} \geq \varepsilon$$

holds for every $x \in X$. It is said that the norm of X is *rough* if there exists a positive ε such that the norm of X is ε-rough.

On the one hand, notice that the property of a norm of being rough is a condition of uniform non Fréchet differentiability (see [DGZ, Lemma I.1.3]). On the other hand, it is obvious from the triangle inequality that if the norm of a Banach space is ε-rough then $\varepsilon \leq 2$. This restriction on ε is also clear from the following dual characterisation of the roughness of a norm, proved in [JZ].

Theorem 1.8. *Let X be a Banach space and let $\varepsilon > 0$. The following assertions are equivalent:*

1. *The norm of X is ε-rough.*
2. *The diameter of every weak-star slice of B_{X^*} is greater than or equal to ε.*

As a consequence of the previous Theorem 1.8 and Proposition 1.5 we have a characterisation of the slice-D2P.

Corollary 1.9. *A Banach space X has the slice-D2P if, and only if, the norm of X^* is 2-rough.*

Example 1.10. Given an infinite set Γ then the norm of $\ell_1(\Gamma)$ is 2-rough. Indeed, given $x \in S_{\ell_1(\Gamma)}$ and $t > 0$, we can find a finite set $F \subseteq \Gamma$ such that $\sum_{\gamma\in F} |x(\gamma)| > 1 - t^2$. Now, for any $\gamma \in \Gamma \setminus F$, it follows that

$$\frac{\|x+te_\gamma\|+\|x-te_\gamma\|-2}{\|te_\gamma\|} \geq \frac{(2+2t)(1-t^2)-2}{t} = 2(1-t^2)-2t,$$

which tends to 2 when $t \to 0$. Consequently, the norm of ℓ_1 is 2-rough, so c_0 has the slice-D2P by Corollary 1.9.

A strengthening of the concept of roughness was introduced in [Dev].

Definition 1.11. Let X be a Banach space and $\varepsilon > 0$. It is said that the norm of X is *ε-average rough* if, for every $n \in \mathbb{N}$ and every $x_1, \ldots, x_n \in S_X$, it follows

$$\limsup_{\|h\|\to 0} \frac{1}{n}\sum_{i=1}^{n} \frac{\|x_i+h\|+\|x_i-h\|-2}{\|h\|} \geq \varepsilon.$$

It is said that the norm of X is *average rough* if there exists a positive ε such that the norm of X is ε-average rough.

We have considered the previous reformulation of average roughness [Dev, Theorem 1] to make clear that average roughness implies roughness. Furthermore, a characterisation of average roughness in connection with the diameter two properties is given in [Dev, Theorem 1].

Theorem 1.12. *Let X be a Banach space and $\varepsilon > 0$. Then the following assertions are equivalent:*

1. *The norm of X is ε-average rough.*
2. *Every convex combination of weak-star slices of B_{X^*} has diameter greater than or equal to ε.*

Again, the previous theorem joint with Proposition 1.5 yields the following dual characterisation of the SD2P.

Corollary 1.13. *A Banach space X has the SD2P if, and only if, the norm of X^* is average 2-rough.*

A class of Banach spaces having 2-average rough norm considered in [Dev] connects with another central concept in this dissertation: the one of octahedral norm.

Definition 1.14. Let X be a Banach space. We say that the norm of X is *octahedral* if, for every finite-dimensional subspace Y of X and every $\varepsilon > 0$, there exists an element $x \in S_X$ such that

$$\|y + \lambda x\| \geq (1 - \varepsilon)(\|y\| + |\lambda|)$$

holds for every $y \in Y$ and every $\lambda \in \mathbb{R}$.

The concept of octahedral norm was introduced by G. Godefroy and B. Maurey in the unpublished work [GM], where the authors proved that a separable Banach space X admits an equivalent octahedral norm if, and only if, X contains an isomorphic copy of ℓ_1. Later, G. Godefroy proved in [God2] that separability can be removed in the previous statement, that is, a Banach space X can be equivalently renormed to have an octahedral norm if, and only if, X contains an isomorphic copy of ℓ_1.

The connection between octahedral norms and 2-average roughness is the following proposition coming from [Dev, Proposition 3]

Proposition 1.15. *Every octahedral norm is 2-average rough.*

In the language of the diameter two properties, the previous proposition reads as follows: the dual space of a Banach space with an octahedral norm has the w^*-SD2P. At this point, a very natural question is whether the converse is true or not. In fact, R. Deville posed as an open question [Dev, Remark c)] whether every 2-average rough norm is octahedral. An explicit positive answer will be given in Section 2.4.

We will end with another geometric property of Banach spaces, which is due to I. Daugavet.

Definition 1.16. Let X be a Banach space. We say that X has the *Daugavet property* if every rank-one linear and bounded operator $T : X \longrightarrow X$ satisfies

$$\|T + \mathrm{Id}\,\| = 1 + \|T\| \tag{DE}$$

(DE) is known as the *Daugavet equation* since I. Daugavet proved that $\mathcal{C}([0,1])$ has the Daugavet property [Dau]. More examples of Banach spaces with the Daugavet property are $L_1(\mu)$ and $L_\infty(\mu)$ when μ is a non-atomic measure or $\mathcal{C}(K)$ when K is a compact, Hausdorff and perfect (i.e. does not have any isolated point) topological space. We refer the reader to [AA, KSSW, D.Wer] for background on the Daugavet property and further examples.

The relation between the diameter two properties and the Daugavet property comes from the following geometric characterisation, appearing in [KSSW, Lemma 2.1].

Theorem 1.17. *Let X be a Banach space. The following assertions are equivalent:*

1. *X has the Daugavet property.*
2. *For every $y_0 \in S_X$ and every slice $S(B_X, f, \alpha_0)$ there is another slice $S(B_X, g, \alpha_1) \subseteq S(B_X, f, \alpha_0)$ such that*
$$\|y_0 + x\| > 2 - \alpha_0$$
holds for all $x \in S(B_X, g, \alpha_1)$.
3. *For every $f_0 \in S_{X^*}$ and every w^*-slice $S(B_{X^*}, y_0, \alpha_0)$ there is another w^*-slice $S(B_{X^*}, x_0, \alpha_1) \subseteq S(B_{X^*}, y_0, \alpha_0)$ such that*
$$\|f_0 + f\| > 2 - \alpha_0$$
holds for all $f \in S(B_{X^*}, x_0, \alpha_1)$.

An analysis in the proof of [Shv, Lemma 2.2] yields that slices in (2) (respectively w^*-slices in (3)) can be replaced with convex combinations of slices (respectively convex combinations of w^*-slices). This fact was exploited in [ALN2] to get the following result.

Proposition 1.18. *If a Banach space X has the Daugavet property then X has the SD2P. Furthermore, X^* has the w^*-SD2P.*

1.3 Examples

In this section we will exhibit further examples of Banach spaces enjoying the diameter two property, which probably established the diameter two properties as a researchline in the geometry of Banach spaces.

1.3.1 Uniform algebras

We begin with the uniform algebras because they were the starting point of the diameter two properties (though they were defined more than ten years later in [ALN2]). Recall that a *uniform algebra* is a closed subalgebra of some $\mathcal{C}(K)$ space which separates the points of K and that contain the constant functions. We refer the reader to [Gam] for background on uniform algebras.

Motivated by a study of non-dentability on infinite-dimensional uniform algebras, O. Nygaard and D. Werner proved in [NW, Theorem 2] that every infinite-dimensional uniform algebra has the D2P. However, having a closer look to the proof, it is actually proved that every convex combination of slices of the unit ball has diameter two, a fact which was pointed out in [ALN2, Theorem 4.2]. Consequently, the following result follows.

Proposition 1.19. *Every infinite-dimensional uniform algebra has the SD2P.*

1.3.2 C^*-algebras and JB^*-triples

After the paper of [NW], new examples of Banach spaces with the diameter two properties appeared in [BLRo] and [BLPR], where extra algebraic assumptions on the spaces were assumed. In order to exhibit such examples we need to introduce notation. We recall that a complex JB^**-triple* is a complex Banach space X with a continuous triple product $\{...\} : X \times X \times X \to X$ which is linear and symmetric in the outer variables, and conjugate-linear in the middle variable, and satisfies:

1. For all x in X, the mapping $y \to \{xxy\}$ from X to X is a Hermitian operator on X and has nonnegative spectrum.

2. The main identity

$$\{ab\{xyz\}\} = \{\{abx\}yz\} - \{x\{bay\}z\} + \{xy\{abz\}\}$$

   holds for all a, b, x, y, z in X.

3. $\|\{xxx\}\| = \|x\|^3$ for every x in X.

Concerning the condition (1) above, we also recall that a bounded linear operator T on a complex Banach space X is said to be Hermitian if $\|\exp(irT)\| = 1$ for every r in $\mathbb{R}$. Examples of complex JB^*-triples are all C^*-algebras under the triple product

$$\{xyz\} := \frac{1}{2}(xy^*z + zy^*x).$$

Following [IKR], we define real JB^*-triples as norm-closed real subtriples of complex JB^*-triples. Here, by a subtriple we mean a subspace which is closed under triple products of its elements. Real JBW^*-triples where first introduced as those real JB^*-triples which are dual Banach spaces in such a way that the triple product becomes separately w^*-continuous (see [IKR, Definition 4.1 and Theorem 4.4]). Later, it has been shown in [MP] that the requirement of separate w^*-continuity of the triple product is superabundant. The bidual of every real (respectively, complex) JB^*-triple X is a JBW^*-triple under a suitable triple product which extends the one of X [IKR, Lemma 4.2] (respectively, [Din]).

Now we can state the following result coming from [BLRo, Proposition 2.4] and [BLPR, Theorem 3.3].

Proposition 1.20. *Let X be a real or complex JB^*-triple. Then X fails the D2P if, and only if, X is reflexive.*

By making use of the results of Subsection 1.3.4 one can get the SD2P in Proposition 1.20 (see Example 1.27).

The previous proposition shows that, for a given real or complex JB^*-triple X, then the (extremely opposite) possibilities for X are either having the RNP or the SD2P.

1.3.3 M-ideals and L-embedded Banach spaces

In [Lop] new examples of spaces with the diameter two properties appeared in connection with the theory of M-ideals and L-embedded spaces. Let us begin with considering necessary definitions.

Definition 1.21. Let X be a Banach space.

(1) An *L-projection* on X is a linear projection $p : X \longrightarrow X$ such that

$$\|x\| = \|p(x)\| + \|x - p(x)\|$$

holds for every $x \in X$. A subspace $M \subseteq X$ is said to be an *L-summand* if M is the rank of an L-projection. We say that X is an *L-embedded space* if X is an L-summand in X^{**}.

(2) An *M-projection* on X is a linear projection $p : X \longrightarrow X$ such that

$$\|x\| = \max\{\|p(x)\|, \|x - p(x)\|\}$$

holds for every $x \in X$. A subspace $M \subseteq X$ is said to be an *M-summand* if M is the rank of an M-projection. M is said to be an *M-ideal* in X if $M^{\perp}$, the annihilator of M, is an L-summand in X^*. It is said that X is an *M-embedded space* if X is an M-ideal in X^{**}.

For a detailed treatment of M-embedded and L-embedded spaces we refer the reader to [HWW].

The main results of [Lop] are the following.

Proposition 1.22. *1. If X is an M-embedded space then both X and X^{**} have the D2P.*

2. Let X be an L-embedded space, say $X^{**} = X \oplus_1 Z$. If B_Z is weak-star dense in $B_{X^{**}}$ then X has the D2P.

Remark 1.23. On the one hand, (1) in Proposition 1.22 was improved in [ALN2, Theorem 4.10] where, making use of the same techniques as those of [Lop, Theorem 2.4], the authors obtained the SD2P under the same assumptions. On the other hand, (2) in Proposition 1.22 was improved in [BM, Theorem 2.2], where it is obtained the Daugavet property under the same assumptions.

As a consequence of Proposition 1.22, the following renorming result was proved in [Lop, Proposition 2.6] (see also [Iva2, Theorem 6]).

Proposition 1.24. *Every Banach space containing an isomorphic copy of c_0 can be equivalently renormed to have the D2P.*

Later, in [ALN2, Proposition 4.7], the authors proved that D2P can be replaced with SD2P in the above proposition.

1.3.4 Banach spaces with infinite-dimensional centralizer

In this section we will get new examples of Banach spaces with the diameter two properties in presence of extra algebraic assumptions. In order to present the results of [BR2] we need to introduce notation. Following the notation of [BR2], given a Banach space X over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, a *multiplier on* X is a bounded linear operator $T : X \longrightarrow X$ satisfying that every extreme point of B_{X^*} is an eigenvector of T^*. Given a multiplier T on X, then for every $p \in \text{ext}(B_{X^*})$ there exists a scalar $a_T(p)$ such that

$$p \circ T = T^*(p) = a_T(p)p.$$

We define the *centralizer of* X, denoted by $Z(X)$, as the set of those multipliers T for which there exists another multiplier S such that

$$a_T(p) = \overline{a_S(p)}$$

holds for all $p \in \text{ext}(B_{X^*})$. It is obvious that when X is a real Banach space then $Z(X)$ coincides with the set of all multipliers on X.

Making use of theory or representability of Banach spaces, in [BR2, Proposition 2.4] it is proved that a Banach space X has the D2P whenever $Z(X)$ is infinite-dimensional and its unit ball contains any extreme point. In order to show a sharper result we introduce some notation, coming from [BR2]. Notice that, by the canonical isometric injection of a Banach space in its bidual, we have the following chain of Banach spaces

$$X \subseteq X^{**} \subseteq X^{(4} \subseteq \ldots \subseteq X^{(2n} \subseteq \ldots$$

Thus we have that $\bigcup\limits_{n=0}^{\infty} X^{(2n}$ is a vector space and, for a given $x \in \bigcup\limits_{n=0}^{\infty} X^{(2n}$, the formula

$$\|x\| := \|x\|_{(2n} \Leftrightarrow x \in X^{(2n}$$

defines a norm on $\bigcup\limits_{n=0}^{\infty} X^{(2n}$. We denote by $X^{(\infty}$ the completion of $\bigcup\limits_{n=0}^{\infty} X^{(2n}$ under the above norm.

It is known that $T \in Z(X)$ implies $T^{**} \in Z(X^{**})$ [HWW, Corollary I.3.15]. This fact allows us to embed $Z(X)$ into $Z(X^{(\infty})$ in the natural way. Indeed, given $T \in Z(X)$, we can consider the action of T on the elements of $X^{(2n}$ as the operator $T^{(2n}$ (the $2n$-th adjoint of T) for every $n \in \mathbb{N}$. Indeed, under this point of view, we can actually see $T \in Z(X^{(\infty})$ [BR2, Proposition 4.3].

Now the main result of [BR2] is the following.

Proposition 1.25. *Let X be a Banach space. If $Z(X^{(\infty})$ is infinite-dimensional then X has the D2P.*

Remark 1.26. The previous result was improved in [ABL, Theorem 3.3], by using the same techniques as the original ones of [BR2, Corollary 4.2], where the SD2P was obtained under the same assumptions.

Example 1.27. Proposition 1.25 applies for the following class of Banach spaces:

1. Non-reflexive JB^*-triples [BR2, Theorem 5.3].

2. Every non-reflexive Banach space such that X^* is L-embedded [AB, Proposition 3.3].

3. $\mathcal{C}(K,X)$ for every infinite compact Hausdorff topological space K and for every Banach space X [BR2, Proposition 3.2].

4. $L(X,Y)$ for every Banach spaces X and Y satisfying that either $Z(X^*)$ or $Z(Y)$ are infinite-dimensional [HWW, Lemma VI.1.1].

1.4 Stability results of diameter two properties

In this section we will introduce the problem of how the diameter two properties are preserved in two different ways. On the one hand, we will exhibit how the diameter two properties are preserved by taking ℓ_p-sums, which will lead us the difference between the D2P and the SD2P. On the other hand, we will present some results about the diameter two properties in tensor product spaces, a study which will be completed in Section 3.1.

1.4.1 Cartesian products

Given two Banach spaces X and Y, a natural question is when $X\oplus_p Y$ enjoys any diameter two property. Several works have dealt with this problem [ABL, ALN2, BL, Lop] because it is connected with different problems about diameter two properties. For instance, the stability of the D2P by ℓ_p-sums is applied in [BL] to analyse the D2P in spaces of Bochner integrable functions and in [Lop] to study the D2P in M-embedded Banach spaces.

Let us consider the case of the ℓ_∞-sum first. This case appears in [Lop], where it is proved that the D2P is preserved from just one factor by taking ℓ_∞-sum. A similar statement with the same ideas is established in [ALN2] for the SD2P. They are summarised in the following proposition.

Proposition 1.28. *Let X and Y be two Banach spaces. If X has the slice-D2P (respectively D2P, SD2P), then $X\oplus_\infty Y$ has the slice-D2P (respectively D2P, SD2P).*

Now we turn to analyse the case of the ℓ_1-sum. The first results in this line appeared in [BL], where it was proved that the D2P is preserved from **both** factors by taking ℓ_1-sums. Later in [ABL], where a very deep study of stability results of diameter two properties in Cartesian products was made, it was proved that all the diameter two properties in an ℓ_1-sum depend on both factors of the sum. More precisely, the following result follows.

Proposition 1.29. *Let X and Y be two Banach spaces. Then $X\oplus_1 Y$ has the slice-D2P (respectively D2P, SD2P) if, and only if, X and Y have the slice-D2P (respectively D2P, SD2P).*

The study of the ℓ_p-sum is quite more delicate. The first results in this line appeared in [ALN2], where it was proved the the slice-D2P and the D2P are stable by taking ℓ_p-sums $1<p<\infty$. Later, in [ABL] this result was improved to get the following.

Proposition 1.30. *Let X and Y be Banach spaces and $1 < p < \infty$. Then $X \oplus_p Y$ has the slice-D2P (respectively the D2P) if, and only if, X and Y have the slice-D2P (respectively the D2P).*

In contrast with the slice-D2P and the D2P, the SD2P turns out to have a bad behaviour with respect to the ℓ_p-sum of spaces. Indeed, the following result was proved in [ABL]. See [HL, Oja] for different proofs about this result.

Proposition 1.31. *Given two Banach spaces X and Y and $1 < p < \infty$ then $X \oplus_p Y$ fails to have the SD2P.*

Notice that this result points out the difference between the case $p = 1$ and $p \neq 1$ concerning stability results of the diameter two properties by classical ℓ_p-sums. However, the most interesting consequence of the above results is that the D2P and the SD2P are not equivalent, as the following example coming from [ABL] shows.

Example 1.32. Let $X := c_0 \oplus_2 c_0$. Then X has the D2P but not the SD2P.

By Proposition 1.5 it follows that the bidual of the previous space has the w^*-D2P but not the w^*-SD2P. Consequently, implications (1) and (6) of (1.1) do not reverse.

As a consequence of the study of the diameter two properties in ℓ_p-sums of Banach spaces we present the following result, coming from [BL, Theorem 2.13], concerning spaces $L_1(\mu, X)$ of Bochner integrable X-valued functions and $L_\infty(\mu, X)$ of essentially bounded X-valued functions.

Proposition 1.33. *Let (Ω, Σ, μ) a finite measure space and X be a Banach space. Then:*

1. *$L_1(\mu, X)$ fails the SD2P if, and only if, μ contains any atom or X fails the SD2P.*
2. *$L_\infty(\mu, X)$ fails the SD2P if, and only if, μ is purely atomic and X fails the SD2P.*

1.4.2 Tensor product spaces

Another line of stability results considered about the diameter two properties is the stability by taking tensor product. In order to show them, we need to introduce standard definitions about tensor product spaces. For background on tensor product theory we refer the reader to [Ryan]. Let X and Y be Banach spaces. Given $x \in X$ and $y \in Y$ denote by $x \otimes y$ the evaluation functional acting on elements $T \in L(X, Y^*)$ as follows:

$$(x \otimes y)(T) = T(x)(y).$$

Notice that $x \otimes y \in L(X, Y^*)^\#$. Now we consider the *(algebraic) tensor product of X and Y*, denoted by $X \otimes Y$, as the vector space spanned by $\{x \otimes y : x \in X, y \in Y\}$. Notice that every element u of $X \otimes Y$ is of the form

$$u = \sum_{i=1}^{n} x_i \otimes y_i,$$

where $n \in \mathbb{N}, x_1, \ldots, x_n \in X$ and $y_1, \ldots, y_n \in Y$, and that the representation above is not unique in general. Notice that, under the above point of view, every element of the form $x \otimes y$ can be seen as an operator $x \otimes y : X^* \longrightarrow Y$.

Now we will introduce two different norms on $X \otimes Y$. First, we will consider the *injective norm* defined by the equation

$$\|u\|_\varepsilon = \sup\left\{\sum_{i=1}^n |x^*(x_i)y^*(y_i)| : x^* \in S_{X^*}, y^* \in S_{Y^*}\right\},$$

where $u := \sum_{i=1}^n x_i \otimes y_i$. Notice that this norm is nothing but the operator norm on $X\otimes Y \subseteq L(X^*, Y)$. Now we will define the *injective tensor product of* X *and* Y, denoted as $X\widehat{\otimes}_\varepsilon Y$, as the completion of $X\otimes Y$ in the above norm. Note that every element of $X\widehat{\otimes}_\varepsilon Y \subseteq L(X^*, Y)$ is a compact operator from X^* to Y which is weak*-to-weak continuous.

Also, we will consider the *projective norm on* $X\otimes Y$ which is defined, given an element $u \in X \otimes Y$, by the equation

$$\|u\| := \inf\left\{\sum_{i=1}^n \|x_i\|\|y_i\| : u = \sum_{i=1}^n x_i \otimes y_i\right\}.$$

As in the case of the injective tensor product, we define the *projective tensor product of* X *and* Y, denoted by $X\widehat{\otimes}_\pi Y$, as the completion of $X \otimes Y$ under the above norm. It is known that $B_{X\widehat{\otimes}_\pi Y} = \overline{\text{conv}}(B_X \otimes B_Y) = \overline{\text{conv}}(S_X \otimes S_Y)$ and that $(X\widehat{\otimes}_\pi Y)^* = L(X, Y^*)$.

Concerning stability results of the diameter two properties by taking injective tensor product, the first result is probably [BL, Corollary 2.9] where it is proved, using an explicit description of the injective tensor product of a $\mathcal{C}(K)$ space [Ryan, Section 3.2], that if X is an infinite-dimensional L_1 predual and Y is a non-zero Banach space, then $X\widehat{\otimes}_\varepsilon Y$ has the D2P. This result was improved in [ABR, Theorem 5.3], where the following result was proved.

Proposition 1.34. *Let X be a Banach space such that $\sup\limits_{n\in\mathbb{N}} \dim(Z(X^{(2n)})) = \infty$. If Y is a non-zero Banach space then $X\widehat{\otimes}_\varepsilon Y$ has the D2P.*

Concerning the projective tensor product, it has been pointed out several times with no proof that the slice-D2P is preserved by just one factor by taking projective tensor product (see [ABR, Section 1] or [ALN2, Theorem 2.7]). Let us include a short proof of this result because it will shed light on the techniques used in Section 3.1.

Proposition 1.35. *Let X and Y be non-zero Banach spaces. If X has the slice-D2P, then so does $X\widehat{\otimes}_\pi Y$.*

Proof. Pick a slice $S := S(B_{X\widehat{\otimes}_\pi Y}, T, \alpha)$, where $\alpha > 0$ and $T \in S_{L(X,Y^*)}$. Let us prove that $\text{diam}(S) = 2$. To this end pick $u \in S_X, y \in S_Y$ such that $T(u)(y) > 1 - \alpha$, that is, $u \otimes y \in S$. Notice that $u \otimes y \in S$ if, and only if, $T(u)(y) = (y \circ T)(u) > 1 - \alpha$, which is in turn equivalent to $u \in S(B_X, y \circ T, \alpha)$. Since X has the slice-D2P then we can find $x, z \in S(B_X, y\circ T, \alpha)$ such that $\|x - z\| > 2 - \varepsilon$. Now $(y \circ T)(x) = T(x)(y) > 1 - \alpha$ means that $x \otimes y \in S$. Similarly, is obvious that $z \otimes y \in S$. So

$$\text{diam}(S) \geq \|x \otimes y - z \otimes y\| = \|(x - z) \otimes y\| = \|x - z\|\|y\| = \|x - z\| > 2 - \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary we conclude the desired result. †

Concerning the D2P and the SD2P, such stability result was not so clear, and it was even posed as an open question in [ALN2, Question (b)]. We recollect the known results, coming from [ABR], where a strong use of techniques involving the framework of Banach spaces with infinite-dimensional centralizer is made.

Proposition 1.36. *Let X and Y be non-zero Banach spaces.*

1. *If $X^{(\infty}$ and $Y^{(\infty}$ have infinite-dimensional centralizer, then $X\widehat{\otimes}_\pi Y$ has the D2P.*
2. *If $X^{(\infty}$ has an infinite-dimensional centralizer and there exists an element $y^* \in S_{Y^*}$ such that the set $\{y \in S_Y : y^*(y) = 1\}$ is norming for Y^*, then $X\widehat{\otimes}_\pi Y$ has the D2P.*
3. *If K is an infinite compact Hausdorff topological space and $X = \mathcal{C}(K)$, then $X\widehat{\otimes}_\pi Y$ has the D2P.*

Chapter 2

Diameter two properties

This chapter is devoted to analysing several problems related to the diameter two properties. In Section 2.1 we deal with the question whether the slice-D2P and the D2P are or not equivalent properties. As a consequence of our study, we deduce that a Banach space X which contains an isomorphic copy of c_0 can be equivalently renormed to have the slice-D2P but its unit ball contains non-empty relatively weakly open subsets of arbitrarily small diameter. The content of this section is based on [BLR1]. In view of the content of Section 2.1, in Section 2.2 we analyse the possibility that the D2P and the SD2P, which are known to be different properties, are actually different in the extreme way that the slice-D2P and the D2P are. As a consequence, we prove that every Banach space containing an isomorphic copy of c_0 can be equivalently renormed to have the D2P but whose unit ball contains convex combinations of slices of arbitrarily small diameter, which shows that the D2P and the SD2P are too different in the extreme way that slice-D2P and D2P are. The content of this section is based on [BLR3]. In Section 2.3 we consider *almost square Banach spaces* (see Definition 2.11), a geometric property of Banach spaces that implies the SD2P and that is introduced in [ALL]. We show that a Banach space X admits an equivalent renorming to be almost square if, and only if, X contains an isomorphic copy of c_0, which solves an open problem coming from [ALL]. The content of this section is based on [BLR7, Section 2]. Finally, Section 2.4 is devoted to proving that the norm of a Banach space X is octahedral if, and only if, X^* has the w^*-SD2P which, as we announced in Section 1.2, solves an open problem posed by R. Deville [Dev, Remark (c)]. The main application of this equivalence in the present chapter is to prove that every Banach space which contains an isomorphic copy of ℓ_1 satisfies that, for every $\varepsilon > 0$, there exists an equivalent renorming on X such that every convex combination of slices of B_{X^*} has diameter, at least, $2 - \varepsilon$. This can be seen as a kind of partial answer to the open problem posed by G. Godefroy of whether every Banach space containing an isomorphic copy of ℓ_1 admits an equivalent renorming so that the bidual norm is octahedral [God2]. The content of this section is based on [BLR4]. We end the chapter with Section 2.5, where we exhibit further research, remarks and open questions related to the content of the present chapter.

2.1 Slice diameter two property versus diameter two property

At this point, all the Banach spaces with the slice-D2P which have appeared actually satisfy the D2P. In order to get an example of a Banach space with the slice-D2P and failing the D2P, a natural idea is to consider examples of Banach spaces with the CPCP and failing the RNP, and analyse whether such examples have the slice-D2P or, at least, can be equivalently renormed to have the slice-D2P since CPCP is invariant under equivalent renorming. A well-known example of Banach space with the PCP (and hence with the CPCP) and failing the RNP is the space B, the predual of the James tree space JT, constructed in [Jam] (see [FG] for background on this space). However, it was proved in [SSW, Theorem 5.1] that there exists a constant $\beta < 2$ such that every closed and convex subset C of the unit ball of B contains a slice of diameter less than or equal to β so, in particular, B fails the slice-D2P. Thus this natural candidate does not produce the desired counterexample. Furthermore, since [SSW, Theorem 5.1] deals with every closed and convex subset of the unit ball of B, we are not optimistic with the possibility of renorming B (though we do not actually know whether this technique does the trick). Because of this reason, our strategy will be a bit different. Our aim will be to find a Banach space X and a closed, convex and bounded subset K of X such that every slice of K has diameter equal to diam(K) and that K contains non-empty relatively weakly open subsets of arbitrarily small diameter and, after that, trying to get an equivalent renorming on X involving K which transfers the properties of K to the new unit ball to get the desired counterexample. If our renorming technique were sharp enough, we would get an example of a Banach space with the slice-D2P but whose unit ball contains non-empty relatively weakly open subsets of arbitrarily small diameter, a statement which is by far stronger than failing the D2P. In order to get the construction of such set K, we will introduce some notation. $\mathbb{N}^{<\omega}$ stands for the set of all ordered finite sequences of positive integers including the empty sequence denoted by $\emptyset$. If $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^{<\omega}$, we define the length of α by $|\alpha| = n$ and $|\emptyset| = 0$. Also we use the natural order in $\mathbb{N}^{<\omega}$ given by:

$$\alpha \leq \beta \text{ if } |\alpha| \leq |\beta| \text{ and } \alpha_i = \beta_i \ \forall i \in \{1, \cdots, |\alpha|\}.$$

We also define $\emptyset \leq \alpha$ for all $\alpha \in \mathbb{N}^{<\omega}$. In order to avoid possible confusions with the notation, we remark that the finite sequence (p) with only one element $p \in \mathbb{N}$ will be denoted by $\emptyset \frown p$ which is an element of $\mathbb{N}^{<\omega}$. Also, the resulting finite sequence from the concatenation of an element $\alpha \in \mathbb{N}^{<\omega}$ with the sequence (p) will be denoted by $\alpha \frown p$.

As $\mathbb{N}^{<\omega}$ is a countable set we can construct a bijective map $\phi : \mathbb{N}^{<\omega} \to \mathbb{N}$ so that $\phi(\emptyset) = 1$ and $\phi(\alpha) \leq \phi(\beta)$ whenever $\alpha \leq \beta \in \mathbb{N}^{<\omega}$ and $\phi(\alpha \frown j) \leq \phi(\alpha \frown k)$ for every $\alpha \in \mathbb{N}^{<\omega}$ and $j \leq k \in \mathbb{N}$. Indeed, consider $\{p_n\}$ to be an enumeration of prime positive integers numbers and define the one to one map $\phi_0 : \mathbb{N}^{<\omega} \to \mathbb{N}$ given by $\phi_0(\alpha_1, \ldots, \alpha_k) = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$. Now take a strictly increasing and bijective map $\phi_1 : \phi_0(\mathbb{N}^{<\omega}) \to \mathbb{N}$ and put $\phi = \phi_1 \circ \phi_0$. Then ϕ satisfies the desired properties. Observe that, from the above construction, $\{\phi(\alpha \frown j)\}_j$ is a strictly increasing sequence for every $\alpha \in \mathbb{N}^{<\omega}$.

We begin the construction of a subset A of c, the space of convergent scalar sequences with the sup norm. To this end, $\{e_n\}$ and $\{e_n^*\}$ stand for the usual basis and the sequence of biorthogonal functionals of c_0, the space of null scalar sequences with the sup norm.

Define, for every $\alpha \in \mathbb{N}^{<\omega}$, $e_\alpha := e_{\phi(\alpha)} \in c$, $e^*_\alpha := e^*_{\phi(\alpha)} \in c^*$ and $x_\alpha \in c$ by $x_\alpha(i) = 1$ if $\phi^{-1}(i) \leq \alpha$ and $x_\alpha(i) = -1$ otherwise. It is clear that $x_\alpha \in S_c$ for every $\alpha \in \mathbb{N}^{<\omega}$. Note that if $\alpha \neq \beta \in \mathbb{N}^{<\omega}$ then $\|x_\alpha - x_\beta\|_\infty = 2$.

Define $A = \{x_\alpha : \alpha \in \mathbb{N}^{<\omega}\}$, which is a subset of the unit sphere of c and $K = \overline{\text{conv}}(A \cup -A)$ which is a closed, convex and symmetric subset of B_c whose diameter is 2. Let us see that such set K satisfies the desired properties. Note that this set is a modification of a set appearing in [AOR] which was also used in [LS2] in order to characterise the failure of PCP for subsets not containing sequences equivalent to the ℓ_1 basis.

To begin with, in the following proposition we will prove that every slice of K has diameter 2.

Proposition 2.1. *Every slice of K has diameter 2.*

Proof. Pick $x^* \in S_{c^*}$, $\lambda < \sup x^*(K)$ and put $S = \{x \in K : x^*(x) > \sup x^*(K) - \lambda\}$. As S is a slice of K and $K = \overline{\text{conv}}(A \cup -A)$, we deduce that S intersects either to A or to $-A$. From the symmetry of K we can assume that $S \cap A \neq \emptyset$. Then there is $\alpha \in \mathbb{N}^{<\omega}$ such that $x_\alpha \in S$. Pick $j \in \mathbb{N}$. Then $x_{\alpha\frown j}$ is an element in A given by $x_{\alpha\frown j}(i) = 1$ if $\phi^{-1}(i) \leq \alpha \frown j$ and $x_{\alpha\frown j}(i) = -1$ in otherwise. Hence $\{x_{\alpha\frown j}\}_j$ is a sequence in $A \subset K$ weakly convergent to x_α. So there is $j \in \mathbb{N}$ such that $x_{\alpha\frown j} \in S$, and hence

$$\text{diam}(S) \geq \|x_{\alpha\frown j} - x_\alpha\| \geq |x_{\alpha\frown j}(\phi(\alpha \frown j)) - x_\alpha(\phi(\alpha \frown j))| = |1 - (-1)| = 2.$$

Taking into account that K has diameter 2, we deduce that S has diameter 2, so the proposition is proved from the arbitrariness of S. †

Recall that the unit ball of the sequence space c contains many extreme points, unlike c_0. It is known that a point which is both extreme and continuity point is a denting point from [LLT]. Moreover, the extreme points of B_c are extreme in $B_{c^{**}}$, and therefore each extreme point has a base of weakly relatively open neighborhoods made by slices from [GMZ, Proposition 9.1]. Then, in order to prove that K has non-empty relatively weakly open subsets with arbitrarily small diameter, our choice of such relatively weakly open subsets has to avoid extreme points. Taking into account the above comments, we prove now that K, as a subset of c, has relatively weakly open subsets with arbitrarily small diameter.

Proposition 2.2. *Given $n \in \mathbb{N}$ and $\rho > 0$ with $\rho < \frac{2}{n(24n-9)}$, it follows that* $\text{diam}(W_n) < \frac{5}{n}$*, where W_n is the non-empty relatively weakly open subset of $K \subset c$ given by*

$$W_n = \left\{ x \in K : e^*_{\emptyset\frown i}(x) > \frac{2}{n} - 1 - 2\rho,\ 1 \leq i \leq n,\ \lim_k x(k) < -1 + \rho \right\}.$$

Proof. First of all, note that $x_0 = \sum_{i=1}^n \frac{x_{\emptyset\frown i}}{n} \in W_n$. For this note that $\lim_k x_{\emptyset\frown i}(k) = -1$ and then $\lim_k x_0(k) = -1 < -1 + \rho$. Furthermore x_0 is a convex combination of elements of $A \subset K$ and so $x_0 \in K$. Finally, for every $1 \leq j \leq n$, it follows that

$$e^*_{\emptyset\frown j}(x_0) = \sum_{i=1\ i\neq j}^{n} \frac{1}{n} e^*_{\emptyset\frown j}(x_{\emptyset\frown i}) + \frac{1}{n} e^*_{\emptyset\frown j}(x_{\emptyset\frown j}) = -\frac{n-1}{n} + \frac{1}{n} > \frac{2}{n} - 1 - 2\rho.$$

Then W_n is non-empty. In order to prove that $\operatorname{diam}(W_n) < \frac{5}{n}$, it is enough to prove that $\operatorname{diam}(W_n \cap \operatorname{conv}(A \cup -A)) < \frac{5}{n}$. For this, pick any $x, x' \in \operatorname{conv}(A \cup -A)$, hence there are $\lambda, \lambda' \in (0,1]$, $a, a', -b, -b' \in \operatorname{conv}(A)$ such that $x = \lambda a + (1-\lambda)b$ and $x' = \lambda' a' + (1-\lambda')b'$. Now $\lim_k a(k) = \lim_k a'(k) = -1$ and $\lim_k b(k) = \lim_k b'(k) = 1$. As $x \in W_n$, we have that $\lim_k \lambda a(k) + (1-\lambda)b(k) < -1 + \rho$ and then we get that

$$2(1-\lambda) < \rho. \tag{2.1}$$

Similarly we get that

$$2(1-\lambda') < \rho, \tag{2.2}$$

and so

$$|\lambda - \lambda'| < \rho/2. \tag{2.3}$$

For $i \in \{1, \cdots, n\}$ we get, taking into account (2.1) and the fact $x = \lambda a + (1-\lambda)b \in W_n$, that

$$e^*_{\emptyset\frown i}(a) > \frac{\frac{2}{n} - 1 - 2\rho - (1-\lambda)e^*_{\emptyset\frown i}(b)}{\lambda} > \frac{\frac{2}{n} - 1 - 2\rho - \rho/2}{\lambda}.$$

It follows that

$$e^*_{\emptyset\frown i}(a) > \frac{2}{n} - 1 - \frac{5\rho}{2}. \tag{2.4}$$

Similarly, from (2.2) we get that

$$e^*_{\emptyset\frown i}(a') > \frac{2}{n} - 1 - \frac{5\rho}{2}. \tag{2.5}$$

Now we have that

$$\begin{aligned} \|x - x'\| &\leq \|\lambda a - \lambda' a'\| + \|(1-\lambda)b - (1-\lambda')b'\| \\ &\overset{(2.2),(2.3)}{\leq} \|\lambda a - \lambda' a'\| + \rho \\ &\leq \lambda\|a - a'\| + |\lambda - \lambda'| + \rho \\ &\overset{(2.3)}{\leq} \|a - a'\| + \frac{3\rho}{2}. \end{aligned} \tag{2.6}$$

Now, our goal is estimate $\|a - a'\|$. For this put $a = \sum_{j=1}^p \lambda_j x_{\alpha_j}$ and $a' = \sum_{j=1}^q \beta_j x_{\alpha'_j}$, where $p, q \in \mathbb{N}$, $\lambda_j, \beta_j > 0$, $\sum_{j=1}^p \lambda_j = \sum_{j=1}^q \beta_j = 1$ and $x_{\alpha_j}, x_{\alpha'_j} \in A$.

We denote by $\mathbf{1}$ the sequence in c whose all its coordinates are equal to 1. Obviously $\|a - a'\| = \|a + \mathbf{1} - (a' + \mathbf{1})\|$. Now

$$a + \mathbf{1} = \sum_{j=1}^p \lambda_j x_{\alpha_j} + \mathbf{1} = \sum_{j=1}^p \lambda_j (x_{\alpha_j} + \mathbf{1}) = \sum_{j=1}^p \lambda_j \hat{x}_{\alpha_j},$$

where $\hat{x}_{\alpha_j}$ is the element in c given by $\hat{x}_{\alpha_j}(i) = 2$ if $\phi^{-1}(i) \leq \alpha_j$ and $\hat{x}_{\alpha_j}(i) = 0$ in otherwise. Similarly $a' + \mathbf{1} = \sum_{j=1}^q \beta_j \hat{x}_{\alpha'_j}$, where $\hat{x}_{\alpha'_j}$ is the element in c given by $\hat{x}_{\alpha'_j}(i) = 2$ if $\phi^{-1}(i) \leq \alpha'_j$ and $\hat{x}_{\alpha'_j}(i) = 0$ otherwise.

Now (2.4) and (2.5) imply that

$$e^*_{\emptyset\frown i}(\hat{a}),\ e^*_{\emptyset\frown i}(\hat{a}') > \frac{2}{n} - \frac{5\rho}{2} \quad \text{holds for all} \quad 1 \leq i \leq n, \tag{2.7}$$

where $\hat{a} = a + \mathbf{1}$ and $\hat{a}' = a' + \mathbf{1}$.

For every $i \in \{1, \cdots, n\}$ we define

$$A_i = \{j \in \{1, \cdots, p\} : \alpha_j \geq \varnothing \frown i\}, \; A_i^{'} = \{j \in \{1, \cdots, q\} : \alpha_j^{'} \geq \varnothing \frown i\}.$$

If $i \neq k$ then $A_i \cap A_k = \emptyset$ since the elements $\varnothing \frown i$ and $\varnothing \frown k$ are incomparable in $\mathbb{N}^{<\omega}$. Similarly, $A_i^{'} \cap A_k^{'} = \emptyset$.

Now we have that from (2.7) that

$$\begin{aligned} \sum_{j \in A_i} \lambda_j &\geq \frac{e^*_{\varnothing \frown i}(\hat{a})}{2} \overset{(2.7)}{>} \frac{1}{n} - \frac{5\rho}{4} > \frac{1}{n} - \frac{3\rho}{2}, \\ \sum_{j \in A_i^{'}} \beta_j &\geq \frac{e^*_{\varnothing \frown i}(\hat{a}')}{2} \overset{(2.7)}{>} \frac{1}{n} - \frac{5\rho}{4} > \frac{1}{n} - \frac{3\rho}{2}. \end{aligned} \tag{2.8}$$

Then

$$1 = \sum_{j=1}^{p} \lambda_j = \sum_{j \in \cup_{i=1}^n A_i} \lambda_j + \sum_{j \in (\cup_{i=1}^n A_i)^c} \lambda_j = \sum_{i=1}^{n} \sum_{j \in A_i} \lambda_j + \sum_{j \in (\cup_{i=1}^n A_i)^c} \lambda_j,$$

and we deduce from (2.8) that , for every $k \in \{1, \cdots, n\}$, the following holds

$$\begin{aligned} \sum_{j \in A_k} \lambda_j &= 1 - \sum_{i=1 \; i \neq k} \sum_{j \in A_i} \lambda_j - \sum_{j \in (\cup_{i=1}^n A_i)^c} \lambda_j \\ &\overset{(2.8)}{<} 1 - \sum_{i=1 \; i \neq k}^{n} \frac{1}{n} - \frac{3\rho}{2} \\ &= 1 - (n-1)\left(\frac{1}{n} - \frac{3\rho}{2}\right). \end{aligned} \tag{2.9}$$

Similarly, we get

$$\sum_{j \in A_k^{'}} \beta_j < 1 - (n-1)\left(\frac{1}{n} - \frac{3\rho}{2}\right). \tag{2.10}$$

Also, from (2.8)

$$\sum_{j \in (\cup_{i=1}^n A_i)^c} \lambda_j, \quad \sum_{j \in (\cup_{i=1}^n A_i^{'})^c} \beta_j < \frac{3n\rho}{2}. \tag{2.11}$$

Observe that the vectors $\sum_{j \in A_i} \lambda_j \hat{x}_{\alpha_j} - \sum_{j \in A_i^{'}} \beta_j \hat{x}_{\alpha_j^{'}}$ have disjoint supports for coordinates $k > 1$ and $1 \leq i \leq n$, and so

$$\max_{k>1} \left| \left(\sum_{i=1}^{n} \left(\sum_{j \in A_i} \lambda_j \hat{x}_{\alpha_j} - \sum_{j \in A_i^{'}} \beta_j \hat{x}_{\alpha_j^{'}} \right) \right) (k) \right| \leq 2 \max_{1 \leq i \leq n} \left\{ \sum_{j \in A_i} \lambda_j + \sum_{j \in A_i^{'}} \beta_j \right\}, \tag{2.12}$$

since $\|\hat{x}_{\alpha_j}\| = \|\hat{x}_{\alpha_j'}\| = 2$. Now, applying (2.11) and (2.12), we get

$$\begin{aligned}
\|\hat{a} - \hat{a}'\| &= \left\| \sum_{j=1}^{p} \lambda_j \hat{x}_{\alpha_j} - \sum_{j=1}^{q} \beta_j \hat{x}_{\alpha_j'} \right\| \\
&= \max \left\{ \left| \left(\sum_{j=1}^{p} \lambda_j \hat{x}_{\alpha_j} - \sum_{j=1}^{q} \beta_j \hat{x}_{\alpha_j'} \right)(k) \right| : k \in \mathbb{N} \right\} \\
&= \max \left\{ \left| \left(\sum_{j=1}^{p} \lambda_j \hat{x}_{\alpha_j} - \sum_{j=1}^{q} \beta_j \hat{x}_{\alpha_j'} \right)(\phi(0)) \right|, \right. \\
&\left. \max_{k>1} \left| \left(\sum_{j=1}^{p} \lambda_j \hat{x}_{\alpha_j} - \sum_{j=1}^{q} \beta_j \hat{x}_{\alpha_j'} \right)(k) \right| \right\} \\
&= \max \left\{ \left| \left(\sum_{j=1}^{p} \lambda_j \hat{x}_{\alpha_j} - \sum_{j=1}^{q} \beta_j \hat{x}_{\alpha_j'} \right)(\phi(0)) \right|, \right. \\
&\left. \max_{k>1} \left| \left(\sum_{j=1}^{p} \lambda_j \hat{x}_{\alpha_j} - \sum_{j=1}^{q} \beta_j \hat{x}_{\alpha_j'} \right)(k) \right| \right\} \\
&= \max \left\{ 0, \max_{k>1} \left| \left(\sum_{j=1}^{p} \lambda_j \hat{x}_{\alpha_j} - \sum_{j=1}^{q} \beta_j \hat{x}_{\alpha_j'} \right)(k) \right| \right\} \\
&\leq \max_{k>1} \left\{ \left| \left(\sum_{i=1}^{n} \left(\sum_{j\in A_i} \lambda_j \hat{x}_{\alpha_j} - \sum_{j\in A_i'} \beta_j \hat{x}_{\alpha_j'} \right) \right)(k) + \right. \right. \\
&\left. \left(\sum_{j\in(\cup_{i=1}^{n} A_i)^c} \lambda_j \hat{x}_{\alpha_j} \right)(k) - \left(\sum_{j\in(\cup_{i=1}^{n} A_i')^c} \beta_j \hat{x}_{\alpha_j'} \right)(k) | \right\} \\
&\leq \max_{k>1} \left\{ \left| \left(\sum_{i=1}^{n} \left(\sum_{j\in A_i} \lambda_j \hat{x}_{\alpha_j} - \sum_{j\in A_i'} \beta_j \hat{x}_{\alpha_j'} \right) \right)(k) \right| \right\} \\
&+ 2 \left(\sum_{j\in(\cup_{i=1}^{n} A_i)^c} \lambda_j + \sum_{j\in(\cup_{i=1}^{n} A_i')^c} \beta_j \right) \\
&\leq 2 \max_{1\leq i\leq n} \left\{ \sum_{j\in A_i} \lambda_j + \sum_{j\in A_i'} \beta_j \right\} + 6n\rho \\
&\leq 4(1-(n-1)\left(\frac{1}{n} - \frac{3\rho}{2}\right) + 6n\rho = \frac{4}{n} + 12n\rho - 6\rho
\end{aligned}$$

Finally, we conclude from (2.6) and the above estimate that

$$\begin{aligned}
\|x - x'\| \leq \|a - a'\| + 3\rho/2 &\leq \frac{4}{n} + 12n\rho - 6\rho + \frac{3\rho}{2} \\
&= \frac{4}{n} + 12n\rho - \frac{9\rho}{2} = \frac{4}{n} + \frac{(24n-9)\rho}{2} < \frac{5}{n}
\end{aligned}$$

since $\rho < \frac{2}{n(24n-9)}$. Hence we have proved that $\operatorname{diam}(W_n) < \frac{5}{n}$, as desired. †

The above results provide a closed, bounded, convex and symmetric subset K of B_c satisfying that every slice of K has diameter 2 and K contains non-empty relatively weakly open sets with arbitrarily small diameter. Our next goal is getting a Banach space whose unit ball behaves like K with respect to the size of slices and of relatively weakly open subsets. To this end, we need the following lemma.

Lemma 2.3. *Let X be a Banach space containing an isomorphic copy of c_0. Then there is an equivalent norm $|||\cdot|||$ on X satisfying that $(X, |||\cdot|||)$ contains an isometric copy of c and that for every $x \in B_{(X,|||\cdot|||)}$ there are sequences $\{x_n\}$, $\{y_n\} \in B_{(X,|||\cdot|||)}$ which are weakly convergent to x and such that $|||x_n - y_n||| = 2$ holds for every $n \in \mathbb{N}$. In fact, $x_n = x + (1-\alpha_n)e_n$ and $y_n = x - (1+\alpha_n)e_n$ for some scalars sequence $\{\alpha_n\}$ satisfying that $|\alpha_n| \leq 1$ holds for every n.*

Proof. As X contains isomorphic copies of c we can assume, up considering an equivalent renorming, that c is in fact an isometric subspace of X. Then, for every separable subspace Y of X containing c, there is a linear and continuous projection $P_Y : Y \longrightarrow c$ with $\|P\| \leq 8$. Indeed, let us consider the onto linear isomorphism $T : c \longrightarrow c_0$ given by $T(x)(1) = \frac{1}{2}\lim_n x(n)$ and $T(x)(n) = \frac{1}{2}(x(n) - \lim_n x(n))$ for every $n > 1$. Note that $\|T\| = 1$ and $\|T^{-1}\| = 4$. Now, following [FHHMPZ, Theorem 5.11], we get the desired projection P_Y with $\|P_Y\| \leq 2\|T^{-1}\| = 8$.

Let Υ be the family of subspaces Y of X containing c such that c has finite codimension in Y. Consider the filter basis Υ given by $\{Y \in \Upsilon : Y_0 \subset Y\}$, where $Y_0 \in \Upsilon$ is arbitrary and call $\mathcal{U}$ a ultrafilter containing the above filter basis.

For every $Y \in \Upsilon$, we define a new norm in X given by

$$\|x\|_Y := \max\{\|P_Y(x)\|, \|x - P_Y(x)\|\}.$$

Finally, we define the norm on X given by $|||x||| := \lim_{\mathcal{U}} \|x\|_Y$. Observe that $\frac{1}{2}\|x\| \leq |||x||| \leq 9\|x\|$ for every $x \in X$ and so $|||\cdot|||$ is an equivalent norm in X such that $|||x||| = \|x\|_\infty$ for every $x \in c$, where $\|\cdot\|_\infty$ is the sup norm in c. Hence $(X, |||\cdot|||)$ contains an isometric copy of c.

Pick $x_0 \in B_{(X,|||\cdot|||)}$. In order to prove the remaining statement let $\{e_n\}$ and $\{e_n^*\}$ the usual basis of c_0 and the biorthogonal functionals sequence, respectively.

Choose $\lambda \in \mathbb{R}$ and $n \in \mathbb{N}$. For every $Y \in \Upsilon$ with $x_0 \in Y$ we have that

$$\|x_0 + \lambda e_n\|_Y = \max\{\|P_Y(x_0) + \lambda e_n\|, \|x_0 - P_Y(x_0)\|\} =$$

$$\max\{|\lambda + e_n^*(P_Y(x_0))|, \|P_Y(x_0) - e_n^*(P_Y(x_0))e_n\|, \|x_0 - P_Y(x_0)\|\}.$$

Call $\beta_n = \lim_{\mathcal{U}} \max\{\|P_Y(x_0) - e_n^*(P_Y(x_0))e_n\|, \|x_0 - P_Y(x_0)\|\}$ and $\alpha_n = \lim_{\mathcal{U}} e_n^*(P_Y(x_0))$. Then $|||x_0 + \lambda e_n||| = \max\{|\lambda + \alpha_n|, \beta_n\}$. Note that $|\alpha_n| \leq 1$ and $\beta_n \leq 1$ just considering $\lambda = 0$ and taking into account that $|||x_0||| \leq 1$.

Define $x_n := x_0 + (1-\alpha_n)e_n$ and $y_n := x_0 - (1+\alpha_n)e_n$ for every n, and note that $x_n, y_n \in B_{(X,|||\cdot|||)}$ because of the previous computations. Finally, it is clear that $\{x_n\}$ and $\{y_n\}$ are weakly convergent sequences to x_0 and $|||x_n - y_n||| = 2$ for every $n \in \mathbb{N}$. †

Now we are ready to announce the main theorem of the section.

Theorem 2.4. *Let X be a Banach space containing an isomorphic copy of c_0. Then there is an equivalent norm on X such that:*

1. *Every slice of the new unit ball of X has diameter 2 for the new equivalent norm.*
2. *There are non-empty relatively weakly open subsets of the new unit ball of X with arbitrarily small diameter.*

Proof. From Lemma 2.3, we can assume that X contains an isometric copy of c and that, for every $x \in B_X$, there are sequences $\{x_n\}$, $\{y_n\} \in B_X$ which are weakly convergent to x such that $\|x_n - y_n\| = 2$ for every $n \in \mathbb{N}$.

Fix $0 < \varepsilon < 1$ and consider in X the equivalent norm $\|\cdot\|_\varepsilon$ whose unit ball is $B_\varepsilon = \overline{\text{conv}}(A \cup -A \cup [(1-\varepsilon)B_X + \varepsilon B_{c_0}])$. Then we have $\|x\| \leq \|x\|_\varepsilon \leq \frac{1}{1-\varepsilon}\|x\|$ for every $x \in X$ and $\|x\| = \|x\|_\infty$ for every $x \in c$.

In order to prove (2), fix $\gamma > 0$. Pick $n \in \mathbb{N}$ with $10 < n(1-\varepsilon)\gamma$ and choose ρ such that $0 < \rho < \frac{2}{n(24n-9)}$, $2\rho < \gamma$ and $2\rho < \varepsilon$. Consider the relatively weakly open subset of K given by

$$W_n = \left\{ x \in K : e^*_{\emptyset\frown i}(x) > \frac{2}{n} - 1 - 2\rho,\ 1 \leq i \leq n,\ \lim_k x(k) < -1 + \rho \right\}.$$

From Proposition 2.2, $W_n \neq \emptyset$ and $\text{diam}_{\|\cdot\|_\infty}(W_n) \leq 5/n$.

Now, we define

$$W = \left\{ x \in B_\varepsilon : e^*_{\emptyset\frown i}(x) > \frac{2}{n} - 1 - \rho,\ 1 \leq i \leq n,\ \lim_k(x) < -1 + \rho^2 \right\},$$

where e^*_n and $\lim_k$ denote to Hanh-Banach extensions to the whole X of the corresponding functionals defined on c. It is clear that $\|e^*_{\emptyset\frown i}\|_\varepsilon = \|e^*_{\emptyset\frown i}\| = 1$ for every $i \in \{1, \ldots, n\}$ and $\|\lim_k\|_\varepsilon = \|\lim_k\| = 1$.

We prove that $x_0 = \sum_{i=1}^n \frac{x_{\emptyset\frown i}}{n} \in W$. For this note that $\lim_k x_{\emptyset\frown i}(k) = -1$ and then $\lim_k x_0(k) = -1 < -1 + \rho^2$. Furthermore x_0 is a convex combination of elements of A and so $x_0 \in B_\varepsilon$. Finally, for $1 \leq j \leq n$, we get that

$$e^*_{\emptyset\frown j}(x_0) = \sum_{i=1\ i\neq j}^n \frac{1}{n} e^*_{\emptyset\frown j}(x_{\emptyset\frown i}) + \frac{1}{n} e^*_{\emptyset\frown j}(x_{\emptyset\frown j}) = -\frac{n-1}{n} + \frac{1}{n} > \frac{2}{n} - 1 - \rho.$$

Then W is a non-empty relatively weakly open subset of B_ε. In order to estimate the diameter of W, it is enough compute the diameter of $W \cap \text{conv}(A \cup -A \cup [(1-\varepsilon)B_X + \varepsilon B_{c_0}])$. Furthermore, $\text{conv}(A \cup -A \cup [(1-\varepsilon)B_X + \varepsilon B_{c_0}]) = \text{conv}(\text{conv}(A) \cup \text{conv}(-A) \cup [(1-\varepsilon)B_X + \varepsilon B_{c_0}])$. So, given $x \in W$, we can assume that $x = \lambda_1 a + \lambda_2(-b) + \lambda_3[(1-\varepsilon)x_0 + \varepsilon y_0]$, where $\lambda_i \in [0,1]$ with $\sum_{i=1}^3 \lambda_i = 1$ and $a, b \in \text{conv}(A)$, $x_0 \in B_X$, and $y_0 \in B_{c_o}$. Since $x \in W$, we have that $\lim_k(x) < -1 + \rho^2$, and hence,

$$-\lambda_1 + \lambda_2 + \lambda_3(1-\varepsilon)\lim_k(x_0) = -\lambda_1 + \lambda_2 + \lambda_3 \lim_k[(1-\varepsilon)x_0 + \varepsilon y_0] < -1 + \rho^2.$$

Note that $-1 \leq \lim_k(x_0)$. This implies that

$$2\lambda_2 + \lambda_3\varepsilon - 1 = -\lambda_1 + \lambda_2 - \lambda_3(1-\varepsilon) < -1 + \rho^2.$$

Since $2\rho < \varepsilon$, then $\lambda_2 + \lambda_3 < \frac{1}{2}\rho$. As a consequence we get that $\lambda_1 > 1 - \frac{\rho}{2}$.

Given $i \in \{1, \dots, n\}$ then

$$\frac{2}{n} - 1 - \rho < e^*_{\emptyset\frown i}(x) = \lambda_1 e^*_{\emptyset\frown i}(a) + \lambda_2 e^*_{\emptyset\frown i}(-b) + \lambda_3 e^*_{\emptyset\frown i}[(1-\varepsilon)x_0 + \varepsilon y_0].$$

Since $\|e^*_{\emptyset\frown i}\|_\varepsilon = 1$ and $-b, (1-\varepsilon)x_0 + \varepsilon y_0 \in B_\varepsilon$, we have that $e^*_{\emptyset\frown i}(-b) \leq 1$ and $e^*_{\emptyset\frown i}[(1-\varepsilon)x_0 + \varepsilon y_0] \leq 1$. It follow that

$$\lambda_1 e^*_{\emptyset\frown i}(a) + \lambda_2 e^*_{\emptyset\frown i}(-b) + \lambda_3 e^*_{\emptyset\frown i}[(1-\varepsilon)x_0 + \varepsilon y_0] \leq$$

$$\lambda_1 e^*_{\emptyset\frown i}(a) + \lambda_2 + \lambda_3 < \lambda_1 e^*_{\emptyset\frown i}(a) + \frac{1}{2}\rho.$$

We deduce that

$$e^*_{\emptyset\frown i}(\lambda_1 a) > \frac{2}{n} - 1 - \frac{3\rho}{2} > \frac{2}{n} - 1 - 2\rho$$

for $1 \leq i \leq n$. On the other hand, we have that

$$\lim_k(\lambda_1 a) = -\lambda_1 < -1 + \frac{\rho}{2} < -1 + \rho,$$

and we conclude that $\lambda_1 a \in W_n$.

Finally, given $x, x' \in W$, we can assume that

$$x = \lambda_1 a + \lambda_2(-b) + \lambda_3[(1-\varepsilon)x_0 + \varepsilon y_0], \; x' = \lambda'_1 a' + \lambda'_2(-b') + \lambda'_3[(1-\varepsilon)x'_0 + \varepsilon y'_0],$$

where $\lambda_i, \lambda'_i \in [0,1]$ with $\sum_{i=1}^3 \lambda_i = \sum_{i=1}^3 \lambda'_i = 1$, and $a, b, a', b' \in \operatorname{conv}(A)$, $x_0, x'_0 \in B_X$ and $y_0, y'_0 \in B_{c_0}$. We have that

$$\|x - x'\|_\varepsilon \leq \|\lambda_1 a - \lambda'_1 a'\|_\varepsilon + \lambda_2 + \lambda_3 + \lambda'_2 + \lambda'_3 < \|\lambda_1 a - \lambda'_1 a'\|_\varepsilon + \rho.$$

Since $\|x\|_\varepsilon \leq \frac{1}{1-\varepsilon}\|x\|$ holds for every $x \in X$, $\|x\| = \|x\|_\infty$ holds for every $x \in c$ and $\lambda_1 a, \lambda'_1 a' \in W_n$, we get

$$\|x - x'\|_\varepsilon \leq \frac{1}{1-\varepsilon}\|\lambda_1 a - \lambda'_1 a'\|_\infty + \rho \leq \frac{5}{n(1-\varepsilon)} + \rho \leq \gamma.$$

Since $x, x' \in W$ were arbitrary we deduce that $\operatorname{diam}_{\|\cdot\|_\varepsilon}(W) \leq \gamma$.

In order to prove (1), note that $B_\varepsilon \subset B_X$ and so $\|x\|_\varepsilon \geq \|x\|$ holds for every $x \in X$.

Pick $f \in X^*$, $\|f\|^*_\varepsilon = 1$ and $\beta > 0$, and consider the slice

$$S = \{x \in B_\varepsilon : f(x) > 1 - \beta\}.$$

From the properties of slices then either there exists an element $a \in (A \cup -A) \cap S$ or there exists an element $(1-\varepsilon)x_0 + \varepsilon y_0 \in (1-\varepsilon)B_X + \varepsilon B_{c_0}$ such that $(1-\varepsilon)x_0 + \varepsilon y_0 \in S$.

In the first case, from the symmetry of $A \cup -A$, we can assume with no loss of generality that $a \in A$, so there is $\alpha \in \mathbb{N}^{<\omega}$ such that $a = x_\alpha$. We recall that $x_{\alpha\frown j}(k) = 1$ if $\phi^{-1}(k) \leq \alpha \frown j$ and $x_{\alpha\frown j}(k) = -1$ in otherwise, then $\{x_{\alpha\frown j}\}_j$ is a weakly convergent sequence to x_α. Hence we can choose j so that $x_{\alpha\frown j} \in S$. Note that $x_{\alpha\frown j} - x_\alpha = 2e_{\alpha\frown j}$, then $2 = \|2e_{\alpha\frown j}\|_\infty = \|x_{\alpha\frown j} - x_\alpha\| \leq \|x_{\alpha\frown j} - x_\alpha\|_\varepsilon$. It follow that $\operatorname{diam}_{\|\cdot\|_\varepsilon}(S) = 2$.

In the second case, that is, if there are $x_0 \in B_X$ and $y_0 \in B_{c_0}$ such that

$$(1-\varepsilon)x_0 + \varepsilon y_0 \in S,$$

then, since S is a norm open set, we can assume that the support of y_0 is finite. From Lemma 2.3, there is a sequence of scalars $\{t_j\}$ with $|t_j| \leq 1$ for every j such that, if we define $x_j = x_0 + (1-t_j)e_j$ and $y_j = x_0 - (1+t_j)e_j$ for every j, we have that $\{x_j\}$ and $\{y_j\}$ are weakly convergent sequences to x_0 in B_X. We put j_0 such that $e_j^*(y_0) = 0$ for every $j \geq j_0$, then $y_0 + e_j, y_0 - e_j \in B_{c_0}$ for every $j \geq j_0$.

So it follows that $\{(1-\varepsilon)x_j + \varepsilon(y_0 + e_j)\}_{j \geq j_0}$ and $\{(1-\varepsilon)y_j + \varepsilon(y_0 - e_j)\}_{j \geq j_0}$ are sequences in $(1-\varepsilon)B_\varepsilon + \varepsilon B_{c_0} \subset B_\varepsilon$ which are weakly convergent to $(1-\varepsilon)x_0 + \varepsilon y_0$. Hence we can choose j large enough so that $(1-\varepsilon)x_j + \varepsilon(y_0 + e_j)$, $(1-\varepsilon)y_j + \varepsilon(y_0 - e_j) \in S$. Then

$$\begin{aligned}
&\|[(1-\varepsilon)x_j + \varepsilon(y_0 + e_j)] - [(1-\varepsilon)y_j + \varepsilon(y_0 - e_j)]\|_\varepsilon \\
&= \|2(1-\varepsilon)e_j + 2\varepsilon e_j\|_\varepsilon = \|2e_j\|_\varepsilon \geq \|2e_j\| = \|2e_j\|_\infty \\
&= 2,
\end{aligned}$$

so $\operatorname{diam}_{\|\cdot\|_\varepsilon}(S) = 2$. †

As a consequence of the above result we have the following corollary, which answers by the negative the problem about the equivalence between slice-D2P and D2P, posed in [ALN2, Section 5].

Corollary 2.5. *Every Banach space containing an isomorphic copy of c_0 can be equivalently renormed satisfying the slice-D2P and failing the D2P.*

Notice that the previous corollary not only gives an example of a Banach space with the slice-D2P and failing the D2P, but also shows a large class of (non-isomorphic) examples satisfying our requirements. On the other hand, notice that Theorem 2.4 points out that the slice-D2P and the D2P are different in the extreme way that the unit ball of a Banach space can have all its slices of diameter two but contain non-empty relatively weakly open subsets of arbitrarily small diameter.

2.2 Diameter two property versus strong diameter two property

In Example 1.32 it is exhibited an example of a Banach space with the D2P and failing the SD2P. This example, together with Corollary 2.5 and Example 1.4, shows that no reverse implication in the diagram (1.1) hold. However, in view of Theorem 2.4, a natural question is whether the D2P and the SD2P are different in an extreme way. More precisely, we can wonder whether there exists any Banach space with the D2P and whose unit ball contains convex combinations of slices of arbitrarily small diameter. In the search of this extreme example, the first natural idea is to analyse the size of the convex combinations of slices of the example exhibited in Example 1.32. Unfortunately, the following proposition shows that the space $c_0 \oplus_p c_0$, is far from satisfying that its unit ball contains convex combinations of slices with arbitrarily small diameter.

Proposition 2.6. *If $p \geq 1$, then every convex combination of slices of $B_{c_0 \oplus_p c_0}$ has diameter, at least, 1.*

Proof. Put $X = c_0 \oplus_p c_0$ and let $\sum_{i=1}^n \lambda_i S(B_X, (x_i^*, y_i^*), \alpha_i)$ be a convex combination of slices of B_X, where $n \in \mathbb{N}$, $0 < \alpha_i < 1$ holds for every i, $(x_i^*, y_i^*) \in S_{X^*}$ and $\lambda_i > 0$ for every i with $\sum_{i=1}^n \lambda_i = 1$. If $\alpha = \min_i \alpha_i$, then $S_i \subset S(B_X, (x_i^*, y_i^*), \alpha_i)$, where $S_i = S(B_X, (x_i^*, y_i^*), \alpha)$ for every i. Now, given an arbitrary $\varepsilon > 0$, for every $1 \leq i \leq n$ we choose $(x_i, y_i) \in S_i$ such that $\|(x_i, y_i)\|_X > 1 - \varepsilon$ with $A_i := \operatorname{supp}(x_i)$ and $B_i := \operatorname{supp}(y_i)$ being finite, where $\operatorname{supp}(z) = \{n \in \mathbb{N} : z(n) \neq 0\}$ for every $z \in c_0$. Pick $k_0 \geq \max \cup_{i=1}^n A_i \cup \cup_{i=1}^n B_i$ and $k > k_0$ such that $x_i \pm \|x_i\|_\infty e_k$, $y_i \pm \|y_i\|_\infty e_k \in S_i$ holds for every i. From here we have that

$$\begin{aligned}
\operatorname{diam}\left(\sum_{i=1}^n \lambda_i S(B_X, (x_i^*, y_i^*), \alpha_i)\right) &\geq \operatorname{diam}\left(\sum_{i=1}^n \lambda_i S_i\right) \\
&\geq 2\left\|\sum_{i=1}^n \lambda_i (\|x_i\|_\infty e_k, \|y_i\|_\infty e_k)\right\|.
\end{aligned}$$

As $\|x_i\|_\infty^p + \|y_i\|_\infty^p > 1 - \varepsilon$ one has that for every i either $\|x_i\|_\infty \geq (\frac{1-\varepsilon}{2})^{1/p}$ or $\|y_i\|_\infty \geq (\frac{1-\varepsilon}{2})^{1/p}$. Put $I = \{i : \|x_i\|_\infty \geq (\frac{1-\varepsilon}{2})^{1/p}\}$ and $t = \sum_{i \in I} \lambda_i$ ($t = 0$ if $I = \emptyset$). Then $t \in [0, 1]$ and $1 - t = \sum_{i \notin I} \lambda_i$. Now we have that

$$\begin{aligned}
\operatorname{diam}\left(\sum_{i=1}^n \lambda_i S(B_X, (x_i^*, y_i^*), \alpha_i)\right) &\geq \operatorname{diam}\left(\sum_{i=1}^n \lambda_i S_i\right) \\
&\geq 2\left\|\sum_{i=1}^n \lambda_i (\|x_i\|_\infty e_k, \|y_i\|_\infty e_k)\right\| \\
&\geq 2\left(\left(\frac{t(1-\varepsilon)^{1/p}}{2^{1/p}}\right)^p + \left(\frac{(1-t)(1-\varepsilon)^{1/p}}{2^{1/p}}\right)^p\right)^{1/p} \\
&= \frac{2(1-\varepsilon)^{1/p}}{2^{1/p}}(t^p + (1-t)^p)^{1/p} \\
&\geq \frac{2(1-\varepsilon)^{1/p}}{2^{1/p}}\left(\frac{1}{2^p} + \frac{1}{2^p}\right)^{1/p} = (1-\varepsilon)^{1/p}.
\end{aligned}$$

Since ε is arbitrary we get that $\operatorname{diam}(\sum_{i=1}^n \lambda_i S(B_X, (x_i^*, y_i^*), \alpha_i)) \geq 1$ and we are done. †

The previous proposition shows that the space $c_0 \oplus_2 c_0$ does not produce an example of Banach space with the D2P and whose unit ball contains convex combination of slices of arbitrarily small diameter. In order to get such example we will follow the line of ideas in the construction of Theorem 2.4, that is, we will find a Banach space X with a closed, convex and bounded subset $K \subseteq X$ satisfying our desired properties, that is, that every non-empty relatively weakly open subset of K has diameter equal to $\operatorname{diam}(K)$ and that K contains convex combinations of slices of arbitrarly small diameter, and then we will try to get a renorming of X, involving the set K, so that the unit ball satisfies the same properties than the set K.

In order to find such X and K, let us consider a family of closed, bounded and convex subsets of c_0 whose diameter is 1 satisfying that every non-empty relatively weakly open subset has diameter 1 and containing convex combinations of slices of arbitrarily small diameter.

Pick a nonincreasing null sequence $\{\varepsilon_n\}$ in $\mathbb{R}^+$. We construct an increasing sequence of closed, bounded and convex subsets $\{K_n\}$ in c_0 and a sequence $\{g_n\}$ in c_0 as follows: First define $K_1 = \{e_1\}$, $g_1 = e_1$ and $K_2 = \text{conv}(e_1, e_1 + e_2)$. Choose $l_2 > 1$ and $g_2, \ldots, g_{l_2} \in K_2$ an ε_2-net in K_2. Assume that $n \geq 2$ and that m_n, l_n, K_n and $\{g_1, \ldots, g_{l_n}\}$ have been constructed, with $K_n \in B_{\text{span}\{e_1,\ldots,e_{m_n}\}}$ and $g_i \in K_n$ for every $1 \leq i \leq l_n$. Define K_{n+1} as

$$K_{n+1} = \text{conv}(K_n \cup \{g_i + e_{m_n+i} : 1 \leq i \leq l_n\}).$$

Consider $l_{n+1} = m_n + l_n$ and choose $\{g_{l_n+1}, \ldots, g_{l_{n+1}}\} \in K_{n+1}$ so that $\{g_1, \ldots, g_{l_{n+1}}\}$ is an ε_{n+1}-net in K_{n+1}. Finally we define $K_0 = \overline{\cup_n K_n}$. Then it follows that K_0 is a non-empty closed, bounded and convex subset of c_0 such that $x(n) \geq 0$ for every $n \in \mathbb{N}$ and $\|x\|_\infty = 1$ for every $x \in K_0$ and so $\text{diam}(K_0) \leq 1$.

Now, fixed i, we have from the construction that $\{g_i + e_{m_n+i}\}_n$ is a sequence in K_0 which is weakly convergent to g_i and $\|(g_i - e_{m_n+i}) - g_i\| = \|e_{m_n+i}\| = 1$ holds for every n. Then $\text{diam}(K_0) = 1$. We will freely use the set K_0 and the above construction throughout the section. Observe that, from the above construction, it follows that

$$K_0 = \overline{\{g_i : i \in \mathbb{N}\}}^w = \overline{\{g_i : i \in \mathbb{N}\}}.$$

Note that the construction of K_0 follows word by word the definition of Poulsen simplex in ℓ_2 [Pou], that is, the unique up to homeomorphism Choquet simplex with a dense subset of extreme points [LOS]. In fact, it is known [AOR] that the weak-star closure of K_0 in ℓ_∞ is affinely weak-star homeohorphic to the Poulsen simplex. However K_0 is not a Choquet simplex, because it is not weakly compact, so K_0 is a simplex in a more general sense than Choquet simplex. Notice that this set was used in [LS1] in order to characterise the failure of CPCP in closed, bounded and convex subsets which do not contain any sequence equivalent to the ℓ_1 basis.

Let us see that K_0 satisfies the requirements that we are looking for.

Proposition 2.7. *K_0 is a closed, bounded and convex subset of c_0 with* $\text{diam}(K_0) = 1$ *satisfying that every non-empty relatively weakly open subset of K_0 has diameter 1 and K_0 contains convex combinations of slices of arbitrarily small diameter.*

Proof. The fact that K_0 is a closed, bounded and convex subset of c_0 with $\text{diam}(K_0) = 1$ has been proved after the construction of K_0. From [AOR, Theorem 1.2], we deduce that K_0 has convex combinations of slices of arbitrarily small diameter. Now pick a non-empty relatively weakly open subset U of K_0. From the construction of K_0 we note that $K_0 = \overline{\{g_i : i \in \mathbb{N}\}}^w$ and so there is $i \in \mathbb{N}$ such that $g_i \in U$. Now, again from the construction of K_0, $g_i + e_{m_n+i} \in K_0$ for every n. Thus, $g_i + e_{m_n+i} \in U$ for every n greater than some n_0, since $\{g_i + e_{m_n+i}\}_n$ is weakly convergent to g_i. Therefore, $\text{diam}(U) \geq \|e_{m_n+i}\| = 1$. †

Now our aim is to get from K_0 a closed, absolutely convex, bounded subset with diameter 2, containing convex combinations of slices with arbitrarily small diameter and so that every non-empty relatively weakly open subset has diameter 2. To this end let us consider K_0 as a subset of c, the space of scalars convergent sequences with the sup norm and define

$$K := 2\overline{\mathrm{conv}}\left(\left(K_0 - \frac{\mathbf{1}}{2}\right) \cup \left(-K_0 + \frac{\mathbf{1}}{2}\right)\right),$$

where $\mathbf{1}$ is the sequence of c with every coordinate equal 1. Now, it is clear that K is a closed, absolutely convex and bounded subset of c with $\mathrm{diam}(K) = 2$.

Now, following the spirit of Theorem 2.4, we will construct an equivalent renorming of a Banach space containing an isomorphic copy of c_0 by making use of the set K and Lemma 2.3 to get that such renorming satisfies the required properties, i.e. the new unit ball has all its non-empty relatively weakly open subsets of diameter two but contains convex combinations of slices of arbitrarily small diameter. That is exactly what will be done in the following theorem.

Theorem 2.8. *Let X be a Banach space containing an isomorphic copy of c_0. Then there is an equivalent norm $|||\cdot|||$ on X such that every non-empty relatively weakly open subset of $B_{(X,|||\cdot|||)}$ has diameter 2 and that $B_{(X,|||\cdot|||)}$ contains convex combinations of slices of arbitrarily small diameter.*

For the proof we will need the following auxiliary lemma.

Lemma 2.9. *Let X be a vector space and A, B be two convex subsets of X such that $\frac{A-A}{2} \subseteq B$. Then*

$$\mathrm{conv}(A \cup -A \cup B) = \mathrm{conv}(A \cup B) \cup \mathrm{conv}(-A \cup B).$$

Proof. Since the inclusion $\supseteq$ is obvious we will only prove that

$$\mathrm{conv}(A \cup -A \cup B) \subseteq \mathrm{conv}(A \cup B) \cup \mathrm{conv}(-A \cup B).$$

To this end, take $x \in \mathrm{conv}(A \cup -A \cup B)$. Since A and B are convex we can find $a_1, a_2 \in A, b \in B$ and $\lambda_1, \lambda_2, \lambda_3 \in [0, 1]$ such that $\lambda_1 + \lambda_2 + \lambda_3 = 1$ and such that $x = \lambda_1 a_1 - \lambda_2 a_2 + \lambda_3 b$. If we assume that $\lambda_1 \geq \lambda_2$ then we write x as follows

$$x = (\lambda_1 - \lambda_2)a_1 + 2\lambda_2 \frac{a_1 - a_2}{2} + \lambda_3 b.$$

Notice that the previous expression is a convex combination of $a_1, \frac{a_1-a_2}{2}, b$ from the inequality $\lambda_1 \geq \lambda_2$. In fact, since $\frac{a_1-a_2}{2} \in \frac{A-A}{2} \subseteq B$, it follows that the previous is a convex combination of elements of $A \cup B$, so $x \in \mathrm{conv}(A \cup B)$. In the case that $\lambda_1 \leq \lambda_2$ similarly follows that $x \in \mathrm{conv}(-A \cup B)$. In any case, $x \in \mathrm{conv}(A \cup B) \cup \mathrm{conv}(-A \cup B)$, as desired. †

Proof of Theorem 2.8. From Lemma 2.3, we can assume that X contains an isometric copy of c and that, for every $x \in B_X$, there are sequences $\{x_n\}$, $\{y_n\} \in B_X$ which are weakly convergent to x and such that $\|x_n - y_n\| = 2$ holds for every $n \in \mathbb{N}$.

Fix $0 < \varepsilon < 1$ and consider on X the equivalent norm $\|\cdot\|_\varepsilon$ whose unit ball is $B_\varepsilon = \overline{\mathrm{conv}}(2(K_0 - \frac{\mathbf{1}}{2}) \cup 2(-K_0 + \frac{\mathbf{1}}{2}) \cup [(1-\varepsilon)B_X + \varepsilon B_{c_0}])$. Then we have $\|x\| \leq \|x\|_\varepsilon \leq \frac{1}{1-\varepsilon}\|x\|$ for every $x \in X$ and $\|x\| = \|x\|_\infty$ for every $x \in c$.

Fix $\gamma > 0$. From Proposition 2.7, there exist slices $S_1, \cdots, S_n$ of K_0 such that

$$\operatorname{diam}\left(\frac{1}{n}\sum_{i=1}^{n} S_i\right) < \frac{1}{4}(1-\varepsilon)\gamma.$$

We can assume that $S_i = \{x \in K : x_i^*(x) > 1 - \widetilde{\delta}\}$ where $x_i^* \in c^*$ and $\sup x_i^*(K_0) = 1$ holds for every $i = 1, \ldots, n$ and $0 < \widetilde{\delta} < 1$. Denote by $\mathbf{1}$ the sequence in c with all its coordinates equal 1. It is clear that $\sup x_i^*(2(K_0 - \frac{\mathbf{1}}{2})) = 2(1 - x_i^*(\frac{\mathbf{1}}{2}))$, for all $i = 1, \cdots, n$. We put $\rho, \delta > 0$ such that $\frac{1}{2}\rho\|x_i^*\| + \delta < \widetilde{\delta}$, $2\rho < \varepsilon$, $\rho\|x_i^*\| < 4\delta$, and $\frac{(7-2\varepsilon)\rho}{(1-\varepsilon)} < \gamma$, for all $i = 1, \ldots, n$. We consider the relatively weakly open set of B_ε given by

$$U_i := \left\{x \in B_\varepsilon : x_i^*(x) > 2\left(1 - \delta - x_i^*\left(\frac{\mathbf{1}}{2}\right)\right) + \frac{1}{2}\rho\|x_i^*\|,\ \lim_k x(k) < -1 + \rho^2\right\}$$

for every $i = 1, \ldots, n$, where x_i^* and $\lim_n$ denote to Hahn-Banach extensions to the whole X of the corresponding functionals defined on c. It is clear that $\|x_i^*\|_\varepsilon = \|x_i^*\|$ for every $i = 1, \ldots, n$ and $\|\lim_n\|_\varepsilon = \|\lim_n\| = 1$.

Since $\rho\|x_i^*\| < 4\delta$, we have that $2(1 - x_i^*(\frac{\mathbf{1}}{2})) > 2(1 - \delta - x_i^*(\frac{\mathbf{1}}{2})) + \frac{1}{2}\rho\|x_i^*\|$. Now, we have that $\sup x_i^*(2(K_0 - \frac{\mathbf{1}}{2})) = 2(1 - x_i^*(\frac{\mathbf{1}}{2}))$, then there exists $x \in K_0$ such that $x_i^*(2(x - \frac{\mathbf{1}}{2})) > 2(1 - \delta - x_i^*(\frac{\mathbf{1}}{2})) + \frac{1}{2}\rho\|x_i^*\|$ and $\lim_k 2(x(k) - \frac{1}{2}) = -1 < -1 + \rho^2$. This implies that $U_i \neq \emptyset$ for every $i = 1, \ldots, n$. In order to estimate the diameter of $\frac{1}{n}\sum_{i=1}^{n} U_i$, it is enough to compute the diameter of

$$\frac{1}{n}\sum_{i=1}^{n} U_i \cap \operatorname{conv}\left(2\left(K_0 - \frac{\mathbf{1}}{2}\right) \cup -2\left(K_0 - \frac{\mathbf{1}}{2}\right) \cup [(1-\varepsilon)B_X + \varepsilon B_{c_0}]\right).$$

Since $2(K_0 - \frac{\mathbf{1}}{2})$ and $(1-\varepsilon)B_X + \varepsilon B_{c_0}$ are a convex subsets of B_ε, given $x \in B_\varepsilon$, we can assume that $x = \lambda_1 2(a - \frac{\mathbf{1}}{2}) + \lambda_2 2(-b + \frac{\mathbf{1}}{2}) + \lambda_3[(1-\varepsilon)x_0 + \varepsilon y_0]$, where $\lambda_i \in [0,1]$ with $\sum_{i=1}^{3}\lambda_i = 1$ and $a, b \in K_0$, $x_0 \in B_X$, and $y_0 \in B_{c_0}$.

So given $x, y \in \frac{1}{n}\sum_{i=1}^{n} U_i$, for $i = 1, \cdots, n$, there exist $a_i, a_i', b_i, b_i' \in K_0$, $\lambda_{(i,j)}, \lambda'_{(i,j)} \in [0,1]$ with $j = 1, 2, 3$ and, $x_i, x_i' \in B_X$, and $y_i, y_i' \in B_{c_0}$, such that

$$2\lambda_{(i,1)}\left(a_i - \frac{\mathbf{1}}{2}\right) + 2\lambda_{(i,2)}\left(-b_i + \frac{\mathbf{1}}{2}\right) + \lambda_{(i,3)}[(1-\varepsilon)x_i + \varepsilon y_i]$$

$$2\lambda'_{(i,1)}\left(a_i' - \frac{\mathbf{1}}{2}\right) + 2\lambda'_{(i,2)}\left(-b_i' + \frac{\mathbf{1}}{2}\right) + \lambda'_{(i,3)}[(1-\varepsilon)x_i' + \varepsilon y_i']$$

belong to U_i for every $i \in \{1, \ldots, n\}$ and that

$$x = \frac{1}{n}\sum_{i=1}^{n} 2\lambda_{(i,1)}\left(a_i - \frac{\mathbf{1}}{2}\right) + 2\lambda_{(i,2)}\left(-b_i + \frac{\mathbf{1}}{2}\right) + \lambda_{(i,3)}[(1-\varepsilon)x_i + \varepsilon y_i]$$

and

$$y = \frac{1}{n}\sum_{i=1}^{n} 2\lambda'_{(i,1)}\left(a_i' - \frac{\mathbf{1}}{2}\right) + 2\lambda'_{(i,2)}\left(-b_i' + \frac{\mathbf{1}}{2}\right) + \lambda'_{(i,3)}[(1-\varepsilon)x_i' + \varepsilon y_i'].$$

For $i \in \{1, \ldots, n\}$ we have that

$$2\lambda_{(i,1)}\left(a_i - \frac{\mathbf{1}}{2}\right) + 2\lambda_{(i,2)}\left(-b_i + \frac{\mathbf{1}}{2}\right) + \lambda_{(i,3)}[(1-\varepsilon)x_i + \varepsilon y_i] \in U_i,$$

then

$$\lim_k \left(2\lambda_{(i,1)}\left(a_i - \frac{\mathbf{1}}{2}\right) + 2\lambda_{(i,2)}\left(-b_i + \frac{\mathbf{1}}{2}\right) + \lambda_{(i,3)}[(1-\varepsilon)x_i + \varepsilon y_i]\right) < -1 + \rho^2.$$

This implies that

$$2\lambda_{(i,2)} + \lambda_{(i,3)}\varepsilon - 1 = -\lambda_{(i,1)} + \lambda_{(i,2)} - \lambda_{(i,3)}(1-\varepsilon) < -1 + \rho^2.$$

Since $2\rho < \varepsilon$, we deduce that $\lambda_{(i,2)} + \lambda_{(i,3)} < \frac{1}{2}\rho$. As a consequence we get that

$$\lambda_{(i,1)} > 1 - \frac{1}{2}\rho, \tag{2.13}$$

and similarly we get that

$$\lambda'_{(i,1)} > 1 - \frac{1}{2}\rho, \tag{2.14}$$

for every $i = 1, \ldots, n$. Now the previous inequalities imply that

$$\begin{aligned}
\|x - y\|_\varepsilon \le\ & \frac{1}{n}\left\|\sum_{i=1}^n 2\lambda_{(i,1)}\left(a_i - \frac{\mathbf{1}}{2}\right) - 2\lambda'_{(i,1)}\left(a'_i - \frac{\mathbf{1}}{2}\right)\right\|_\varepsilon \\
& + \frac{1}{n}\sum_{i=1}^n \left\|2\lambda_{(i,2)}\left(-b_i + \frac{\mathbf{1}}{2}\right)\right\|_\varepsilon + \frac{1}{n}\sum_{i=1}^n \left\|2\lambda'_{(i,2)}\left(-b'_i + \frac{\mathbf{1}}{2}\right)\right\|_\varepsilon \\
& + \frac{1}{n}\sum_{i=1}^n \|\lambda_{(i,3)}[(1-\varepsilon)x_i + \varepsilon y_i]\|_\varepsilon + \frac{1}{n}\sum_{i=1}^n \|\lambda'_{(i,3)}[(1-\varepsilon)x'_i + \varepsilon y'_i]\|_\varepsilon \\
\le\ & \frac{1}{n}\left\|\sum_{i=1}^n 2\lambda_{(i,1)}\left(a_i - \frac{\mathbf{1}}{2}\right) - 2\lambda'_{(i,1)}\left(a'_i - \frac{\mathbf{1}}{2}\right)\right\|_\varepsilon \\
& + \frac{1}{n}\sum_{i=1}^n \left(\lambda_{(i,2)} + \lambda_{(i,3)}\right) + \frac{1}{n}\sum_{i=1}^n \left(\lambda'_{(i,2)} + \lambda'_{(i,3)}\right) \\
\overset{(2.13),(2.14)}{\le}\ & \frac{1}{n}\left\|\sum_{i=1}^n 2\lambda_{(i,1)}\left(a_i - \frac{\mathbf{1}}{2}\right) - 2\lambda'_{(i,1)}\left(a'_i - \frac{\mathbf{1}}{2}\right)\right\|_\varepsilon + \rho \\
\le\ & \frac{2}{n}\left\|\sum_{i=1}^n \lambda_{(i,1)}a_i - \lambda'_{(i,1)}a'_i\right\|_\varepsilon + \frac{1}{n}\sum_{i=1}^n |\lambda_{(i,1)} - \lambda'_{(i,1)}|\|\mathbf{1}\|_\varepsilon + \rho \\
\le\ & \frac{2}{n}\left\|\sum_{i=1}^n \lambda_{(i,1)}a_i - \lambda'_{(i,1)}a'_i\right\|_\varepsilon + \frac{(3-2\varepsilon)}{2(1-\varepsilon)}\rho.
\end{aligned}$$

Now

$$\left\|\sum_{i=1}^{n}\lambda_{(i,1)}a_i-\lambda'_{(i,1)}a'_i\right\|_{\varepsilon}$$

$$\leq \left\|\sum_{i=1}^{n}(\lambda_{(i,1)}-1)a_i\right\|_{\varepsilon}+\left\|\sum_{i=1}^{n}a_i-a'_i\right\|_{\varepsilon}+\left\|\sum_{i=1}^{n}(\lambda'_{(i,1)}-1)a'_i\right\|_{\varepsilon}$$
$$\leq \frac{1}{1-\varepsilon}\left\|\sum_{i=1}^{n}a_i-a'_i\right\|+\sum_{i=1}^{n}\frac{1}{1-\varepsilon}|\lambda_{(i,1)}-1|\|a_i\|+\sum_{i=1}^{n}\frac{1}{1-\varepsilon}|\lambda'_{(i,1)}-1|\|a'_i\|$$
$$\leq \frac{1}{1-\varepsilon}\left\|\sum_{i=1}^{n}a_i-a'_i\right\|+\frac{1}{1-\varepsilon}n\rho.$$

Hence

$$\|x-y\|_{\varepsilon}\leq \frac{2}{1-\varepsilon}\left\|\frac{1}{n}\sum_{i=1}^{n}a_i-a'_i\right\|+\frac{(7-2\varepsilon)}{2(1-\varepsilon)}\rho. \tag{2.15}$$

Now in order to prove that the previous norm is small we will prove that both elements $\frac{1}{n}\sum_{i=1}^{n}a_i, \frac{1}{n}\sum_{i=1}^{n}a'_i$ are elements of $\frac{1}{n}\sum_{i=1}^{n}S_i$, which has small diameter. To this end note that, for every $i\in\{1,\ldots,n\}$, it follows

$$x_i^*\left(2\lambda_{(i,1)}\left(a_i-\frac{\mathbf{1}}{2}\right)+2\lambda_{(i,2)}\left(-b_i+\frac{\mathbf{1}}{2}\right)+\lambda_{(i,3)}[(1-\varepsilon)x_i+\varepsilon y_i]\right)$$
$$>2\left(1-\delta-x_i^*\left(\frac{\mathbf{1}}{2}\right)\right)+\ \rho\|x_i^*\|,$$

then

$$x_i^*\left(2\lambda_{(i,1)}\left(a_i-\frac{\mathbf{1}}{2}\right)\right)+\frac{1}{2}\rho\|x_i^*\|$$
$$\geq x_i^*\left(2\lambda_{(i,1)}\left(a_i-\frac{\mathbf{1}}{2}\right)\right)+\lambda_{(i,2)}\|x_i^*\|_{\varepsilon}+\lambda_{(i,3)}\|x_i^*\|_{\varepsilon}$$
$$\geq x_i^*\left(2\lambda_{(i,1)}\left(a_i-\frac{\mathbf{1}}{2}\right)+2\lambda_{(i,2)}\left(-b_i+\frac{\mathbf{1}}{2}\right)+\lambda_{(i,3)}[(1-\varepsilon)x_i+\varepsilon y_i]\right).$$

We have that

$$x_i^*\left(2\lambda_{(i,1)}\left(a_i-\frac{\mathbf{1}}{2}\right)\right)>2\left(1-\delta-x_i^*\left(\frac{\mathbf{1}}{2}\right)\right),$$

and hence

$$x_i^*(\lambda_{(i,1)}a_i)>1-\delta-(1-\lambda_{(i,1)})x_i^*\left(\frac{\mathbf{1}}{2}\right)\geq 1-\delta-\frac{1}{2}\rho\|x_i^*\|.$$

We recall that $\delta+\frac{1}{2}\rho\|x_i^*\|<\widetilde{\delta}$, then $x_i^*(\lambda_{(i,1)}a_i)>1-\widetilde{\delta}$. It follows that $x_i^*(a_i)>1-\widetilde{\delta}$. Now $a_i\in K_0\cap S_i$, and similarly we get that $a'_i\in K_0\cap S_i$, for every $i=1,\ldots,n$, and $\frac{1}{n}\sum_{i=1}^{n}a_i,\ \frac{1}{n}\sum_{i=1}^{n}a'_i\in\frac{1}{n}\sum_{i=1}^{n}S_i$. Since the diameter of $\frac{1}{n}\sum_{i=1}^{n}S_i$ is less than $\frac{1}{4}(1-\varepsilon)\gamma$, we deduce that $\frac{1}{n}\|\sum_{i=1}^n a_i - a_i'\| < \frac{1}{4}(1-\varepsilon)\gamma$. Finally, we conclude from (2.15) and the above estimate that

$$\|x-y\|_\varepsilon \leq \gamma.$$

Hence the set $\frac{1}{n}\sum_{i=1}^n U_i$ has diameter, at most γ, for the norm $\|\cdot\|_\varepsilon$. We recall now that every relatively weakly open subset of B_ε contains a convex combination of slices [GGMS, Lemma II.1]. So we conclude that B_ε has convex combinations of slices with arbitrarily small diameter.

In order to prove that every non-empty relatively weakly open subset of B_ε has diameter 2, we recall that $K_0 = \overline{\{g_i : i \in \mathbb{N}\}}$.

Denote by $A := 2(K_0 - \frac{\mathbf{1}}{2})$ and $B := (1-\varepsilon)B_X + \varepsilon B_{c_0}$. Observe that $B_\varepsilon = \overline{\text{conv}}(A \cup -A \cup B)$ so, since $\frac{A-A}{2} = K_0 - K_0 \subseteq B_{c_0} \subseteq B$ we get fron Lemma 2.9 that

$$B_\varepsilon = \overline{\text{conv}(A \cup B) \cup \text{conv}(-A \cup B)}.$$

Thus, in order to prove that every non-empty relatively weakly open subset of B_ε has $\|\cdot\|_\varepsilon$-diameter 2 it is enough to prove that every non-empty relatively weakly open subset of $\overline{\text{conv}}((2K_0 - \mathbf{1}) \cup [(1-\varepsilon)B_X + \varepsilon B_{c_0}])$ has $\|\cdot\|_\varepsilon$-diameter 2.

Pick a weakly open subset U of X such that

$$U \cap \overline{\text{conv}}((2K_0 - \mathbf{1}) \cup [(1-\varepsilon)B_X + \varepsilon B_{c_0}]) \neq \emptyset.$$

Then there is $g_i \in K_0$, $x_0 \in B_X$, $y_0 \in B_{c_0}$ and $\lambda \in [0,1]$ such that $\lambda(2g_i - \mathbf{1}) + (1-\lambda)[(1-\varepsilon)x_0 + \varepsilon y_0]$ belong to U.

As U is a norm open set, we can assume that the support of y_0 is finite. From the Lemma 2.3, there is a scalar sequence $\{t_j\}$ with $|t_j| \leq 1$ for every j such that, if we put $x_j = x_0 + (1-t_j)e_j$ and $y_j = x_0 - (1+t_j)e_j$ for every j, we have that $\{x_j\}$ and $\{y_j\}$ are weakly convergent sequences to x_0 in B_X. Choose $j_0 \in \mathbb{N}$ such that $e_j^*(y_0) = 0$ for every $j \geq j_0$, then $y_0 + e_j, y_0 - e_j \in B_{c_0}$ for every $j \geq j_0$. Now, again from the construction of K_0, $g_i + e_{m_n+i} \in K_0$ for every n, and hence, $\{g_i + e_{m_n+i}\}_n$ is weakly convergent to g_i.

Therefore we get for n big enough that

$$x := \lambda(2(g_i + e_{m_n+i}) - \mathbf{1}) + (1-\lambda)[(1-\varepsilon)x_{m_n+i} + \varepsilon(y_0 + e_{m_n+i})]$$

and

$$y := \lambda(2(g_i - \mathbf{1}) + (1-\lambda)[(1-\varepsilon)y_{m_n+i} + \varepsilon(y_0 - e_{m_n+i})]$$

belong to U. Therefore

$$\begin{aligned}\text{diam}_{\|\cdot\|_\varepsilon}(U) &\geq \|x-y\|_\varepsilon \\ &= \|2\lambda e_{m_n+i} + (1-\lambda)[2(1-\varepsilon)e_{m_n+i} + 2\varepsilon e_{m_n+i}]\|_\varepsilon \\ &= 2\|e_{m_n+i}\|_\varepsilon \geq 2\|e_{m_n+i}\| = 2\|e_{m_n+i}\|_\infty = 2.\end{aligned}$$

We conclude that $\text{diam}_{\|\cdot\|_\varepsilon}(U) = 2$ and the theorem is proved. †

The following corollary shows that there are many spaces satisfying D2P and failing SD2P.

Corollary 2.10. *Every Banach space containing an isomorphic copy of c_0 can be equivalently renormed satisfying D2P and failing SD2P.*

Notice that the previous corollary not only gives an example of a Banach space with the D2P and failing the SD2P, but also shows a large class of (non-isomorphic) examples satisfying our requirements. On the other hand, notice that Theorem 2.8 points out that the D2P and the SD2P are different in the extreme way that the unit ball of a Banach space can have all its non-empty relatively weakly open subsets of diameter two but it contains convex combinations of slices of arbitrarily small diameter.

2.3 Almost square Banach spaces

In Section 1.2 the interplay between the diameter two properties and earlier geometric properties of Banach spaces is exhibited. However, after the appearance of the diameter two properties, new properties related to them have been considered in the literature. That is the case of the *almost square Banach spaces*, considered in [ALL].

Definition 2.11. Let X be a Banach space. X is said to be

1. *locally almost square (LASQ)* if for every $x \in S_X$ there exists a sequence $\{y_n\}$ in B_X such that $\|x \pm y_n\| \to 1$ and $\|y_n\| \to 1$.
2. *weakly almost square (WASQ)* if for every $x \in S_X$ there exists a sequence $\{y_n\}$ in B_X such that $\|x \pm y_n\| \to 1$, $\|y_n\| \to 1$ and $\{y_n\} \to 0$ weakly.
3. *almost square (ASQ)* if for every $x_1, \ldots, x_k$ elements of S_X there exists a sequence $\{y_n\}$ in S_X such that $\|y_n\| \to 1$ and $\|x_i \pm y_n\| \to 1$ for every $i \in \{1, \ldots, k\}$.

It is obvious from the very definition that WASQ Banach spaces are LASQ. Furthermore, it is known that the sequence involved in the definition of ASQ can be taken $(1+\varepsilon)$-equivalent to the c_0 basis [ALL, Theorem 2.8], so ASQ actually implies WASQ.

In [ALL] it is pointed out the nice relation between almost square Banach spaces and diameter two properties. Indeed, LASQ (respectively WASQ, ASQ) Banach spaces enjoy the slice-D2P (respectively the D2P, SD2P). Notice that the corresponding implications for the LASQ and the WASQ were proved in [Kub], where the diameter two properties are analysed in the Cesàro functions spaces. However, in [ALL] not only do the authors study almost square Banach spaces from a geometric point of view but also from an isomorphic one.

Indeed, as we have pointed out, it is proved in [ALL, Lemma 2.6] that every ASQ Banach space contains an isomorphic copy of c_0. In the reverse direction, using the fact that almost squareness is preserved by taking ℓ_∞-sums from one of the factors [ALL, Proposition 5.7], it was proved in [ALL, Theorem 2.9] that every Banach space containing a complemented copy of c_0 can be equivalently renormed to be ASQ. In particular every separable Banach space containing an isomorphic copy of c_0 can be equivalently renormed to be ASQ, as an immediate consequence of the well-known Soboczyk theorem [FHHMPZ, Theorem 5.11]. However, if one tries to get rid of the separability condition, it seems clear that the first non-separable example to analyse is ℓ_∞, because c_0 is not complemented in ℓ_∞ [FHHMPZ, Theorem 5.6].

If one wants to construct an equivalent renorming of ℓ_∞ to be ASQ, it is natural to follow the spirit of the renorming of [ALL, Theorem 2.9] and try to find a closed and complemented subspace X of ℓ_∞ such that X is ASQ under the ℓ_∞ norm. So, assume that such an X exists. Consider $x_1, \ldots, x_n \in S_X$ and $\varepsilon > 0$. As X is assumed to be ASQ then there exists $y \in S_X$ such that

$$\|x_i \pm y\| \leq 1 + \frac{\varepsilon}{2}.$$

holds for every $i \in \{1, \ldots, n\}$. Now consider $k \in \mathbb{N}$ such that $|y(k)| > 1 - \frac{\varepsilon}{2}$. Then $|x_i(k)| < \varepsilon$ holds for every $i \in \{1, \ldots, n\}$. Now, if we define

$$\mathcal{B} := \{U(x, \varepsilon) := \{n \in \mathbb{N} : |x(n)| < \varepsilon\} : x \in S_X, \varepsilon > 0\},$$

the previous estimates prove that $\mathcal{B}$ is a filter basis on $\mathbb{N}$. If we extend $\mathcal{B}$ to a ultrafiler $\mathcal{U}$ over $\mathbb{N}$, it follows

$$X \subseteq \ker(\lim_{\mathcal{U}}).$$

As we also want X to be complemented in ℓ_∞ in order to apply [ALL, Proposition 5.7] to get a renorming on ℓ_∞, we can additionally assume that X has codimension 1 in ℓ_∞, from where we get that $X = \ker(\lim_{\mathcal{U}})$. Finally, since $\ker(\lim_{\mathcal{U}})$ is isometric to ℓ_∞ whenever $\mathcal{U}$ is a principal ultrafilter, we conclude that $\mathcal{U}$ has to be non-principal.

Bearing all the above comments in mind, consider a non-principal ultrafilter $\mathcal{U}$ over $\mathbb{N}$ and define $\lim : \ell_\infty \longrightarrow \mathbb{R}$ to be the linear and continuous functional given by

$$\lim(x) = \lim_{\mathcal{U}}(x)$$

for every $x \in \ell_\infty$. Then a natural candidate to an equivalent ASQ norm on ℓ_∞ is the following:

$$|||x||| := \max\left\{|\lim(x)|, \sup_{n\in\mathbb{N}} |x(n) - \lim(x)|\right\} \; x \in \ell_\infty. \tag{2.16}$$

Notice that this expression comes from the decomposition $\ell_\infty = \ker(\lim) \oplus \operatorname{span}\{\mathbf{1}\}$ and considering ℓ_∞ sum in the above direct sum. Notice also that this expression, when restricted to the space of convergent sequences c, defines an equivalent renorming on c which is isometrically isomorphic to (the ASQ Banach space) c_0. Because of all the above reasons, (2.16) defines a natural equivalent norm on ℓ_∞ under which ℓ_∞ should be ASQ. Let us now prove the following theorem.

Theorem 2.12. *There exists an equivalent norm on ℓ_∞, say $|||\cdot|||$, such that the Banach space $(\ell_\infty, |||\cdot|||)$ is an ASQ Banach space.*

Proof. Consider on ℓ_∞ the norm given by

$$|||x||| := \max\left\{|\lim(x)|, \sup_n |x(n) - \lim(x)|\right\},$$

Let us prove that the norm defined above is equivalent to the classical one on ℓ_∞. To this end consider $x \in \ell_\infty$. Now, on the one hand

$$|||x||| \leq \max\left\{|\lim(x)|, \sup_n |x(n)| + |\lim(x)|\right\} \leq \|x\|_\infty + \|x\|_\infty = 2\|x\|_\infty,$$

since $\|\lim\| = 1$. On the other hand

$$|||x||| \geq \sup_n |x(n) - \lim(x)| \geq \sup_n |x(n)| - |\lim(x)|.$$

Hence

$$\|x\|_\infty \leq |||x||| + |\lim(x)| \leq |||x||| + |||x||| = 2|||x|||.$$

So $|||\cdot|||$ and $\|\cdot\|_\infty$ are equivalent norms. Let us now prove that $(\ell_\infty, |||\cdot|||)$ is an ASQ Banach space. To this aim pick $x_1, \ldots, x_n \in S_{\ell_\infty}$ and $\varepsilon > 0$. Given $i \in \{1, \ldots, n\}$ consider the sets

$$A_i := \{n \in \mathbb{N} \ : \ |x_i(n) - \lim x_i| < \varepsilon\}.$$

Then $A_1, \ldots, A_n \in \mathcal{U}$ by the definition of limit by ultrafilter, so $A := \bigcap_{i=1}^{n} A_i \in \mathcal{U}$ as a finite intersection of elements of $\mathcal{U}$. Since $\mathcal{U}$ is a ultrafilter we have that $A \neq \emptyset$, so pick $n \in A$. Let us now estimate $|||x_i \pm e_n|||$ for every $i \in \{1, \ldots, n\}$. To this aim pick $i \in \{1, \ldots, n\}$. Then, on the one hand,

$$|\lim(x_i \pm e_n)| = |\lim(x_i)|$$

since $\lim e_n = 0$. On the other hand

$$\begin{aligned} \sup_k |x_i(k) \pm e_n(k) - \lim(x_i \pm e_n)| &= \max\left\{\sup_{k \neq n} |x_i(k) - \lim(x_i)|, |x_i(n) - \lim(x_i) \pm 1|\right\} \\ &\leq \max\left\{\sup_{k \neq n} |x_i(k) - \lim(x_i)|, |x_i(n) - \lim(x_i)| + 1\right\} \\ &\leq \max\left\{\sup_n |x_i(n) - \lim(x_i)|, 1 + \varepsilon\right\} \end{aligned}$$

Consequently, by definition of the norm $|||\cdot|||$, we get

$$|||x_i \pm e_n||| \leq \max\{|||x_i|||, 1 + \varepsilon\} = 1 + \varepsilon.$$

Moreover, note that $|||\cdot|||$ agrees with the original norm on c_0. Consequently, $e_n \in S_{\ell_\infty}$. From [ALL, Proposition 2.1] we get that $(\ell_\infty, |||\cdot|||)$ is ASQ Banach space, as desired. †

Remark 2.13. From the above proof it follows that, given $x_1, \ldots, x_n \in S_{\ell_\infty}$ and $\varepsilon > 0$ we can find $y \in S_{c_0}$ such that $|||x_i \pm y||| \leq 1 + \varepsilon$. Roughly speaking we can say that the fact that ℓ_∞ under the norm of Theorem 2.12 is ASQ relies on the subspace c_0. This simple observation will be the key to proving the general renorming result.

As we have pointed out above, the Banach space ℓ_∞ plays an important role as an example of Banach space containing an isomorphic copy of c_0 which can be equivalently renormed to be ASQ. Furthermore, as dual Banach spaces containing an isomorphic copy of c_0 actually contain a complemented copy of ℓ_∞ [LT, Proposition 2.e.8], we can deduce our general result from this particular example by giving a suitable renorming in the bidual space.

Theorem 2.14. *Let X be a Banach space containing an isomorphic copy of c_0. Then there exists an equivalent norm on X such that X is an ASQ space under the new norm.*

Proof. Assume that X contains a subspace Y which is isometric to c_0.

As $Y^{**} \subseteq X^{**}$ is linearly isometric to ℓ_∞, then Y^{**} is complemented in X^{**} [FHHMPZ, Proposition 5.10].

Then we can consider on X^{**} an equivalent norm so that

$$X^{**} = Z \oplus_\infty Y^{**},$$

and such norm agrees with the original one of Y^{**}.

Now we can consider on Y^{**} the norm defined in Theorem 2.12, so Y^{**} becomes into an ASQ space and the new norm agrees with the original one on $Y \subseteq X$. This defines an equivalent norm on X^{**} which we will denote by $\|\cdot\|$. Clearly X^{**} is an ASQ space [ALL, Proposition 5.7]. Our aim is to prove that X is an ASQ space following similar ideas to [ALL, Proposition 5.7] and Remark 2.13.

To this end pick $x_1, \ldots, x_n \in S_X$ and $\varepsilon > 0$. Now $x_i \in X^{**} = Z \oplus_\infty Y^{**}$ for each $i \in \{1, \ldots, n\}$, so we can find $z_i \in Z$ and $y_i \in Y^{**}$ such that $x_i = (z_i, y_i)$ for all $i \in \{1, \ldots, n\}$. We can assume, making a perturbation argument if necessary, that $y_i \neq 0$ holds for all $i \in \{1, \ldots, n\}$. From Remark 2.13 we can find $y \in S_{c_0}$ such that

$$\left\| \frac{y_i}{\|y_i\|} \pm y \right\| \leq 1 + \varepsilon. \tag{2.17}$$

Define $z := (0, y) \in S_{c_0} \subseteq X$. Then

$$\begin{aligned} \|x_i \pm z\| &= \max\{\|z_i\|, \|y_i \pm y\|\} \leq \max\{1, \|y_i \pm y\|\} \\ &= \left\{1, \left\| \|y_i\| \left(\frac{y_i}{\|y_i\|} \pm y \right) \pm (1 - \|y_i\|) y \right\| \right\} \\ &\overset{(2.17)}{\leq} \max\left\{1, \|y_i\|(1+\varepsilon) + (1 - \|y_i\|)\|y\|\right\} \leq 1 + \varepsilon. \end{aligned}$$

To sum up we have proved that, given $x_1, \ldots, x_n \in S_X$ and $\varepsilon > 0$, we can find $z \in S_X$ such that

$$\|x_i \pm z\| \leq 1 + \varepsilon$$

holds for every $i \in \{1, \ldots, n\}$. Thus X is an ASQ space under the new equivalent norm, so we are done.

†

Note that the above theorem allows us to strengthen Proposition 1.24.

Moreover, from Theorem 2.14 and [ALL, Theorem 2.4] we get the following corollary, which improves [ALL, Corollary 2.10].

Corollary 2.15. *Let X be a Banach space. Then there exists an equivalent norm on X such that X is an ASQ Banach space under the new norm if, and only if, X contains an isomorphic copy of c_0.*

In [ALL] the relation between ASQ Banach spaces and the intersection property is pointed out. Recall that a Banach space X has the *intersection property* if for every $\varepsilon > 0$ there exist $x_1, \ldots, x_n \in X$ such that $\|x_i\| < 1$ and such that if $y \in X$ verifies that

$\|x_i - y\| \leq 1$ for every $i \in \{1, \ldots, n\}$ then $\|y\| \leq \varepsilon$. Given $0 < \varepsilon < 1$, X is said to *ε-fail the intersection property* if $\gamma(\varepsilon) = 1$, where

$$\gamma(\varepsilon) := \sup_{x_1,\ldots,x_n \in B_{[0,1)}} \inf_{y \in B_{(\varepsilon,1]}} \max_{1 \leq i \leq n} \|x_i - y\|$$

and $B_I := \{x \in X \ / \ \|x\| \in I\}$ for each $I \subseteq \mathbb{R}^+$. Finally, a Banach space is said to *fail the intersection property* if X ε-fails the intersection property for some $0 < \varepsilon < 1$.

On the one hand, it is known that a Banach space X is ASQ if, and only if, X ε-fails the intersection property for every $0 < \varepsilon < 1$ [ALL, Proposition 6.1]. On the other hand, it is known that a Banach space admits an equivalent norm which fails the intersection property if, and only if, X contains an isomorphic copy of c_0 [HR, Theorem 1.7]. Now we can improve the above theorem as a straightforward application of Corollary 2.15.

Theorem 2.16. *Let X be a Banach space. Then X admits an equivalent norm which ε-fails the intersection property for each $0 < \varepsilon < 1$ if, and only if, X contains an isomorphic copy of c_0.*

2.4 Octahedral norms and the strong diameter two property

As we have pointed out in Section 1.2, from Proposition 1.15 joint with Corollary 1.13 it follows that if the norm of a Banach space X is octahedral then X^* has the w^*-SD2P. The main aim of this section is to prove the converse statement, which gives a positive answer to the open question [Dev, Remark (c)]. We will prove the complete equivalence for the sake of completeness.

Theorem 2.17. *Let X be a Banach space. Then the following assertions are equivalent:*

1. *The norm of X is octahedral.*

2. *Every convex combination of w^*-slices of B_{X^*} has diameter 2.*

Proof. (1) $\Rightarrow$ (2). Pick $N \in \mathbb{N}, x_1, \ldots, x_N \in S_X, \rho_1, \ldots, \rho_N \in (0,1)$ and $\alpha_1, \ldots, \alpha_N > 0$ such that $\sum_{i=1}^N \alpha_i = 1$. Let $\rho := \min\limits_{1 \leq i \leq N} \rho_i$. Since

$$\sum_{i=1}^N \alpha_i S(B_{X^*}, x_i, \rho) \subseteq \sum_{i=1}^N \alpha_i S(B_{X^*}, x_i, \rho_i),$$

it is enough to prove that $\operatorname{diam}\left(\sum_{i=1}^N \alpha_i S(B_{X^*}, x_i, \rho)\right) = 2$.

Put $Y = \operatorname{span}(\{x_1, \ldots, x_N\})$ and fix $n \in \mathbb{N}$. As the norm of X is octahedral there exists $x_n \in S_X$ such that

$$\|y + \alpha x_n\| \geq \left(1 - \frac{1}{n}\right)(\|y\| + |\alpha|)$$

holds for every $y \in Y$ and every $\alpha \in \mathbb{R}$. In particular

$$\|x_i \pm x_n\| \geq 2\left(1 - \frac{1}{n}\right) \quad i \in \{1, \dots, N\} \tag{2.18}$$

For $i \in \{1, \dots, N\}$, by (2.18) and Hahn-Banach theorem there exist $f_{in}, g_{in} \in S_{X^*}$ such that

$$f_{in}(x_i + x_n) = \|x_i + x_n\| \overset{(2.18)}{\geq} 2\left(1 - \frac{1}{n}\right)$$
$$g_{in}(x_i - x_n) = \|x_i - x_n\| \overset{(2.18)}{\geq} 2\left(1 - \frac{1}{n}\right)$$

As a consequence, for $i \in \{1, \dots, N\}$ and $n \in \mathbb{N}$, we have that

$$f_{in}(x_i) > 1 - \frac{2}{n}, f_{in}(x_n) > 1 - \frac{2}{n}$$
$$g_{in}(x_i) > 1 - \frac{2}{n}, g_{in}(x_n) < -\left(1 - \frac{2}{n}\right)$$

Pick $T \in \mathbb{N}$ such that $1 - \frac{2}{T} > 1 - \rho$. Then, for $k \geq T$, it follow that $f_{ik}, g_{ik} \in S(B_{X^*}, x_i, \rho)$ and so

$$\sum_{i=1}^{N} \alpha_i f_{ik}, \sum_{i=1}^{N} \alpha_i g_{ik} \in \sum_{i=1}^{N} \alpha_i S(B_{X^*}, x_i, \rho).$$

Moreover

$$\begin{aligned}
\left\|\sum_{i=1}^{N} \alpha_i f_{ik} - \sum_{i=1}^{N} \alpha_i g_{ik}\right\| &\geq \left|\sum_{i=1}^{N} \alpha_i f_{ik}(x_k) - \sum_{i=1}^{N} \alpha_i g_{ik}(x_k)\right| \\
&= \sum_{i=1}^{N} \alpha_i f_{ik}(x_k) - \sum_{i=1}^{N} \alpha_i g_{ik}(x_k) \\
&> \sum_{i=1}^{N} \alpha_i \left(1 - \frac{2}{k}\right) + \sum_{i=1}^{N} \alpha_i \left(1 - \frac{2}{k}\right) \\
&= 2\left(1 - \frac{2}{k}\right) \sum_{i=1}^{n} \alpha_i = 2 - \frac{4}{k}.
\end{aligned}$$

Since $k \geq T$ was arbitrary we deduce that $\operatorname{diam}\left(\sum_{i=1}^{n} S(B_{X^*}, x_i, \rho)\right) = 2$, as desired.

(2) $\Rightarrow$ (1). Let $Y \subseteq X$ be a finite-dimensional subspace, $\varepsilon \in \mathbb{R}^+$ and $\delta \in \mathbb{R}^+$ such that $2\delta < \varepsilon$. By compactness of S_Y pick a $\delta-$net $\{y_1, \dots, y_n\}$ in S_Y. Let us consider the convex combination of w^*-slices

$$\sum_{i=1}^{n} \frac{1}{n} S(B_{X^*}, y_i, \rho),$$

where $0 < \rho < \delta$ and pick $0 < \widehat{\rho} < \frac{\rho}{n}$.

By the assumption, $\operatorname{diam}\left(\sum_{i=1}^n \frac{1}{n} S(B_{X^*}, y_i, \rho)\right) = 2$, hence there exist

$$\sum_{i=1}^n \frac{1}{n} f_i, \sum_{i=1}^n \frac{1}{n} g_i \in \sum_{i=1}^n \frac{1}{n} S(B_{X^*}, y_i, \rho)$$

such that

$$\left\| \sum_{i=1}^n \frac{1}{n} f_i - \sum_{i=1}^n \frac{1}{n} g_i \right\| > 2 - \widehat{\rho}\ .$$

We put $x \in S_X$ such that $\sum_{i=1}^n \frac{1}{n}(f_i(x) - g_i(x)) > 2 - \widehat{\rho}$. A convexity argument yields that

$$f_i(x) - g_i(x) > 2 - \rho$$

holds for every $i \in \{1, \ldots, n\}$. This implies that, for every $i \in \{1, \ldots, n\}$, then

$$f_i(x) > 1 - \rho, g_i(x) < -(1 - \rho).$$

Furthermore, for every $i \in \{1, \ldots, n\}$, we get, since $f_i, g_i \in S(B_{X^*}, y_i, \rho)$, that

$$f_i(y_i) > 1 - \rho, g_i(y_i) > 1 - \rho.$$

Take any $t \in \mathbb{R}_0^+$. Now given an arbitrary $\alpha \geq 0$ we get

$$\|ty_i + \alpha x\| \geq f_i(ty_i + \alpha x) \geq t(1 - \rho) + \alpha(1 - \rho) = (1 - \rho)(t\|y_i\| + |\alpha|).$$

On the other hand, if $\alpha \leq 0$ it follows

$$\begin{aligned} \|ty_i + \alpha x\| \geq g_i(ty_i + \alpha x) &= tg_i(y_i) + (-\alpha)(-g_i(x)) \\ &\geq t(1 - \rho) + (-\alpha)(1 - \rho) \\ &= (1 - \rho)(t\|y_i\| + |\alpha|). \end{aligned}$$

In any case, we have that

$$\|ty_i + \alpha x\| \geq (1 - \rho)(t\|y_i\| + |\alpha|)$$

holds for every $\alpha \in \mathbb{R}$ and every $t \geq 0$. Bearing in mind that $\{y_1, \ldots, y_n\}$ is a δ-net of S_Y it is not difficut to prove from the above inequality that

$$\|y + \alpha x\| \geq (1 - \varepsilon)(\|y\| + |\alpha|)$$

holds for every $y \in Y$ and every $\alpha \in \mathbb{R}$, which means that the norm of X is octahedral, and the theorem is proved. †

If we combine Theorem 2.17 with Proposition 1.5 we get the following characterisation of the SD2P.

Corollary 2.18. *Let X be a Banach space. Then, X has the SD2P if, and only if, the norm of X^* is octahedral.*

From Proposition 1.18 and Theorem 2.17 it follows that if a Banach space X has the Daugavet property then the norms of X and X^* are octahedral. However, the converse is no longer true, as the following example shows.

Remark 2.19. Take $X = L_1([0,1]) \oplus_\infty \ell_1$. Now, $L_1([0,1])$ has the Daugavet property and so, $L_1([0,1])$ has the SD2P. Then X has the SD2P by Proposition 1.28, and so the norm of X^* is octahedral by Corollary 2.18. On the other hand, $X^* = L_\infty([0,1]) \oplus_1 \ell_\infty$, and notice that L_∞ and ℓ_∞ have the SD2P. As a consequence of Proposition 1.29 then X^* has the SD2P, so the norm of X is octahedral by Theorem 2.17. Finally it is easy to see that X fails the Daugavet property, essentially because ℓ_1 fails the Daugavet property (see the comment preceding [KSSW, Proposition 2.16]).

The following corollary characterises the octahedral norms for real JB^*-triples (see Subsection 1.3.2 for the definitions).

Corollary 2.20. *Let X be a real JB^*-triple. Then X has the Daugavet property if, and only if, the norm of X is octahedral.*

Proof. If X has the Daugavet property, then the norm of X is octahedral. Conversely, assume that the norm of X is octahedral. From Theorem 2.17, every w^*-slice of B_{X^*} has diameter 2 and so, by [DGZ, Proposition I.1.11], X has no Fréchet differentiability points. From [BM, Theorem 3.10] X has the Daugavet property. †

For the dual of a JB^*-triple, having octahedral norm is automatic.

Corollary 2.21. *Let X be a nonreflexive real JB^*-triple. Then the norm of X^* is octahedral.*

Proof. Let us recall that X^* is a nonreflexive L-embedded Banach space [BLPR, Propo sition 2.2]. In particular, there exists $u \in X^{***} \setminus \{0\}$ such that

$$\|x^* + u\| = \|x\| + \|u\|$$

holds for every $x^* \in X^*$. This implies that the norm of X^* is octahedral [GK, Lemma 9.1], so we are done. †

From Corollary 2.18 it follows that the norm of every Banach space with the Daugavet property is octahedral, so every convex combination of w^*-slices in B_{X^*} has diameter 2. On the other hand, if X is a real JB^*-triple, every extreme point of B_{X^*} is actually a strongly exposed point. Indeed, given $f \in \mathrm{ext}(B_{X^*})$, by [PS, Corollary 2.1] and [BM, Lemma 3.1], we can ensure the existence of $u \in S_{X^{**}}$ such that $u(f) = 1$, and u is a point of Fréchet-differentiability of the norm of X^{**}. This implies that f is strongly exposed by u (see [DGZ, Corollary I.1.5]). Consequently, the next corollary follows.

Corollary 2.22. *Let X be a real JB^*-triple with the Daugavet property. Then every convex combination of w^*-slices in B_{X^*} has diameter 2, but there are convex combinations of slices in B_{X^*} with arbitrarily small diameter.*

As we said in Section 1.2, a Banach space X contains an isomorphic copy of ℓ_1 if, and only if, X admits an equivalent octahedral norm [God2, Theorem II.4]. Notice that, thanks to Theorem 2.17 and [Dev, Theorem 1 and Proposition 3], this is equivalent to the fact that there exists an equivalent renorming of X such that X^* has the w^*-SD2P. It is then a natural question whether a Banach space X containing an isomorphic copy of ℓ_1 can be equivalently renormed such that X^* has the SD2P. In view of Theorem 2.17 this question is equivalent to the question of whether a Banach space X containing an isomorphic copy of ℓ_1 can be equivalently renormed such that the norm of X^{**} is octahedral, a question which is explicitly stated in [God2, Remark II.5, 3)].

Our next result can be seen as a partial answer to the above question.

Proposition 2.23. *Let X be a separable Banach space containing a subspace isomorphic to ℓ_1. Then, for every $\varepsilon > 0$, there is an equivalent norm on X such that every convex combination of slices of the new unit ball of X^* has diameter, at least, $2-\varepsilon$.*

In order to prove the above proposition we need a couple of lemmata.

Lemma 2.24. *Let X be a Banach space and C be a w^*-compact and convex subset of B_{X^*} such that every convex combination of slices of C has diameter 2. Then the set $K = \operatorname{conv}(C \cup -C)$ is a w^*-compact convex subset of B_{X^*} such that every convex combination of slices of K has diameter 2.*

Proof. As C is a w^*-compact and convex set, then K is also w^*-compact and convex. This is a consequence from the fact that

$$K = \{\lambda a - (1-\lambda)b : \lambda \in [0,1],\ a, b \in C\}.$$

Pick slices $S_1, \ldots, S_n$ of K and $\lambda_1, \ldots, \lambda_n > 0$ with $\sum_{i=1}^n \lambda_i = 1$. Let $A = \{i \in \{1, \ldots, n\} : S_i \cap C \neq \emptyset\}$ and $B := \{1, \ldots, n\} \setminus A$. Let us observe that every slice of K has non-empty intersection either with C or with $-C$.

Now we have that

$$\Lambda := \sum_{i\in A} \lambda_i (S_i \cap C) + \sum_{i\in B} \lambda_i (S_i \cap (-C)) \subset \sum_{i=1}^n \lambda_i S_i,$$

and then

$$\begin{aligned}\Lambda - \Lambda &= \sum_{i\in A} \lambda_i(S_i \cap C) + \sum_{i\in B} \lambda_i(S_i \cap (-C)) - \sum_{i\in A} \lambda_i(S_i \cap C) - \sum_{i\in B} \lambda_i(S_i \cap (-C)) \\ &= \sum_{i\in A} \lambda_i(S_i \cap C) + \sum_{i\in B} \lambda_i(-S_i \cap C) - \left(\sum_{i\in A} \lambda_i(S_i \cap C) + \sum_{i\in B} \lambda_i(-S_i \cap C)\right) \\ &= D - D,\end{aligned}$$

where $D = \sum_{i\in A} \lambda_i(S_i \cap C) + \sum_{i\in B} \lambda_i(-S_i \cap C)$ is a convex combination of slices in C. From the hypothesis, we have that diam(D)=2, hence we get that $\operatorname{diam}(\Lambda) = 2$ and so $\operatorname{diam}(\sum_{i=1}^n \lambda_i S_i) = 2$. †

Now let us consider the following renorming lemma, which is based on the renorming technique of [SSW, Theorem 3.1]

Lemma 2.25. *Let $(X, \|\cdot\|)$ be a Banach space and $C \subset B_X$ be an absolutely convex and closed subset satisfying that every convex combination of slices has $\|\cdot\|$-diameter 2. Then for every $\varepsilon > 0$ there is an equivalent norm $|\cdot|$ on X such that every convex combination of slices of $B_{(X,|\cdot|)}$ has $|\cdot|$-diameter, at least, $2-\varepsilon$.*

Proof. Pick an arbitrary $\varepsilon > 0$ and we put $\eta \in \mathbb{R}^+$ such that $\frac{2-2\eta}{1+\eta} > 2-\varepsilon$. Consider $|\cdot|$ the equivalent norm in X whose unit ball is

$$B_{|\cdot|} := \overline{C + \eta B_X} \ .$$

Now choose $n \in \mathbb{N}$, $\beta_1, \ldots, \beta_n \in (0,1)$, $\lambda_1, \ldots, \lambda_n > 0$ with $\sum_{i=1}^n \lambda_i = 1$ and $f_1, \ldots, f_n \in S_{(X,|\cdot|)^*}$. Let us see that the convex combination of slices $\sum_{i=1}^n \lambda_i S(B_{|\cdot|}, f_i, \beta_i)$ has diameter $2-\varepsilon$. We put, for $i \in \{1, \ldots, n\}$, $\gamma_i := \sup_C f_i$ and $\delta_i := \sup_{B_X} f_i$, then we have that $\gamma_i + \eta\delta_i = 1$. We consider $\rho \in \mathbb{R}$ such that $0 < \rho < \min\{\beta_i, \gamma_i, \delta_i, \beta_i\eta, \gamma_i\eta, \delta_i\eta : i = 1, \ldots, n\}$. As a consequence we have that, for every $1 \leq i \leq n$, we have

$$S\left(C, f_i, \frac{\rho}{2}\right) + \eta S\left(B_X, f_i, \frac{\rho}{2\eta}\right) \subset S(B_{|\cdot|}, f_i, \rho).$$

So $\sum_{i=1}^n \lambda_i S(C, f_i, \frac{\rho}{2}) + \lambda_i \eta S(B_X, f_i, \frac{\rho}{2\eta})$ is contained in $\sum_{i=1}^n \lambda_i S(B_{|\cdot|}, f_i, \beta_i)$. Now, as

$$\Delta := \sum_{i=1}^n \lambda_i S\left(C, f_i, \frac{\rho}{2}\right)$$

is a convex combination of slices of C, we get that $\|\cdot\| - \operatorname{diam}(\Delta) = 2$. Moreover

$$\Gamma := \sum_{i=1}^n \lambda_i S\left(B_X, f_i, \frac{\rho}{2\eta}\right)$$

is a subset of B_X, and hence $\|\cdot\|-$diameter is at most 2. Hence

$$\|\cdot\| - \operatorname{diam}(\Delta + \eta\Gamma) \geq 2 - 2\eta$$

and so

$$\|\cdot\| - \operatorname{diam}\left(\sum_{i=1}^n \lambda_i S(B_{|\cdot|}, f_i, \beta_i)\right) \geq 2 - 2\eta.$$

Finally, from $B_{|\cdot|} \subset (1+\eta)B_X$ we deduce that

$$|\cdot| - \operatorname{diam}\left(\sum_{i=1}^n \alpha_i S(B_{|\cdot|}, x_i^*, \beta_i)\right) \geq \frac{2-2\eta}{1+\eta} > 2 - \varepsilon.$$

†

Proof of Proposition 2.23. Assume that X contains a subspace isometric to ℓ_1 and fix $\varepsilon > 0$. From [DGH, Theorem 2] we know $\mathcal{C}(\Delta)$ is isometrically isomorphic to a quotient space of X, where $\Delta = \{0,1\}^{\mathbb{N}}$ is the Cantor set. Now X^* contains a subspace Z isometric to $\mathcal{C}(\Delta)^*$. Furthermore, Z is w^*-closed in X^* and the weak-star topology of X^* on Z is the weak-star topology of $\mathcal{C}(\Delta)^*$ on Z. Now, from [SSW, Theorem 4.6], there is a w^*-compact and convex subset C of S_Z so that every convex combination of slices of C has diameter 2. From Lemma 2.24 we get that $K := \operatorname{conv}(C \cup (-C))$ is a w^*-compact and absolutely convex subset of B_{X^*} such that every convex combination of slices of K has diameter 2. Finally, from Lemma 2.25 we get an equivalent norm on X^* such that the new unit ball B in X^* satisfies that every convex combination of slices of B has diameter $2-\varepsilon$. As we have, for some $\eta > 0$, that $B = \operatorname{conv}(K + \eta B_{X^*})$, which is w^*-closed, the new norm in X^* is a dual norm and the proof is complete. †

2.5 Remarks and open questions

In this section we will recollect some research lines, remarks and open questions related to the present chapter.

2.5.1 Section 2.1

As far as we are concerned, it is not known any example of Banach space with the slice-D2P and failing the D2P which can be constructed with different techniques that those coming from Theorem 2.4. Also, the following question is pertinent:

Question 3. *Is there any Banach space X with the slice-D2P and the CPCP?*

Note that if the answer to Question 1 were positive, such a space would exist.

2.5.2 Section 2.2

In contrast with what happened with the case of the slice-D2P and the D2P, a new example of Banach space X with the D2P whose unit ball contains convex combination of slices of arbitrarily small diameter appeared in [AHNTT]. Indeed, in [AHNTT, Proposition 2.11] it is proved that, for every $\varepsilon > 0$, there exists a Banach space X_ε which is isomorphic to $\mathcal{C}([0,1])$ with the following properties:

1. X_ε has the D2P.
2. The unit ball of X_ε contains convex combination of slices whose diameter is smaller than ε.
3. X is *midpoint locally uniformly rotund (MLUR)*, i.e. whenever $x \in S_X$ and $\{x_n\}$ is a sequence in S_X with $\{\|x \pm x_n\|\} \to 1$ then $\{x_n\} \to 0$ in the norm topology.

Then, if one considers $X = \ell_2 - \oplus_{n\in\mathbb{N}} X_{\frac{1}{n}}$, then X has the D2P and its unit ball contains convex combinations of slices of arbitrarily small diameter (see the comment previous to [BLR3, Proposition 2.8]). See [ALNT] for further connections between the diameter two properties and convexity and smoothness properties. In an opposite way, see [LR] for the connections between the diameter two properties and polyhedral Banach spaces.

An open question related to Question 3 is the following.

Question 4. *Is there any Banach space with the D2P and being SR?*

Note that if the answer to first question in Question 2 were positive then such a Banach space would exist.

2.5.3 Section 2.3

We have introduced in Definition 2.11 the concepts of LASQ, WASQ and ASQ Banach spaces. From [ALL, Proposition 3.5] it is deduced that $L_1([0,1])$ is an example of WASQ Banach spaces which is not ASQ because it does not contain any isomorphic copy of c_0. However, the following question is posed in [ALL, Question 3.4]

Question 5. *Is there any LASQ Banach space X which is not WASQ?*

Another natural question about ASQ spaces, also posed in [ALL], is the following:

Question 6. *Is there any dual ASQ Banach space?*

The first natural attempt is to analyse whether the ASQ renorming of ℓ_∞ given in Theorem 2.12 does the work. However, such renorming is far from being a dual Banach space, as the following observation shows.

Proposition 2.26. *Let $|||\cdot|||$ be the ASQ norm on ℓ_∞. Then $\operatorname{ext}(B_{\ell_\infty}) = \emptyset$. As a consequence, $(\ell_\infty, |||\cdot|||)$ is not isometric to any dual Banach space.*

Proof. Consider $x \in S_{\ell_\infty}$. Then we have the following considerations:

1. If $|\lim(x)| < 1$ then there exists $\varepsilon > 0$ such that $|\lim(x) \pm \varepsilon| < 1$. Now consider

$$y := x + \varepsilon\mathbf{1} \quad z := x - \varepsilon\mathbf{1}.$$

Then clearly $|\lim(y)| \leq 1$ and $|\lim(z)| \leq 1$. On the other hand, given $n \in \mathbb{N}$, one has

$$|y(n) - \lim(y)| = |x(n) + \varepsilon - \lim(x) - \varepsilon| = |x(n) - \lim(x)| \leq |||x||| \leq 1.$$

So $y \in B_{\ell_\infty}$. By a similar argument we have that $z \in B_{\ell_\infty}$. Since $x = \frac{y+z}{2}$ we get that $x \notin \operatorname{ext}(B_{\ell_\infty})$ in this case.

2. If $|\lim(x)| = 1$, we shall assume with no loss of generality that $\lim(x) = 1$. Then $\sup_n |x(n) - 1| \leq 1$ from where we conclude that $x(n) \geq 0$ $\forall n \in \mathbb{N}$. Moreover as $\lim(x) = 1$ then we can find $\varepsilon > 0$ and $n \in \mathbb{N}$ such that $x(n) \geq \varepsilon$. We claim that

$$x \pm \varepsilon e_n \in B_{\ell_\infty}.$$

Indeed,

$$\lim(x \pm \varepsilon e_n) = \lim(x).$$

Moreover, given $k \in \mathbb{N}$,

$$|x(k) \pm \varepsilon e_n(k) - \lim(x \pm \varepsilon e_n)| = |x(k) \pm \varepsilon \delta_{kn} - \lim(x)|$$

If $k \neq n$ clearly last quantity is less than or equal to $|||x||| \leq 1$. Moreover, if $k = n$ then we have

$$|x(n) \pm \varepsilon - \lim(x)| = |x(n) \pm \varepsilon - 1| \leq |x(n) - 1| + \varepsilon = 1 - x(n) + \varepsilon \leq 1.$$

Thus $x \pm \varepsilon e_n \in B_{\ell_\infty}$, so $x \notin \operatorname{ext}(B_{\ell_\infty})$.

This proves that $\operatorname{ext}(B_{\ell_\infty}) = \emptyset$. Now ℓ_∞ is not a dual Banach space as an easy consequence of Krein-Milman theorem. †

Related to Question 6, in [GR] the following strengthening of almost squareness is introduced.

Definition 2.27. Let X be a Banach space. We will say that X is *unconditionally almost square* (UASQ) if, for each $\varepsilon > 0$, there exists a subset $\{x_\gamma\}_{\gamma\in\Gamma} \subseteq S_X$ (depending on ε) such that:

1. for each $y_1, \ldots, y_k \in S_X$ and $\delta > 0$ there exists $\gamma \in \Gamma$ such that
$$\|y_i \pm x_\gamma\| \leq 1 + \delta$$
holds for every $i \in \{1, \ldots, k\}$,
2. for every finite subset F of Γ and every choice of signs $\xi_\gamma \in \{-1, 1\}$, $\gamma \in F$, it follows $\|\sum_{\gamma\in F} \xi_\gamma x_\gamma\| \leq 1 + \varepsilon$.

In some sense, we can very roughly say that this concept encodes that X is ASQ "through the c_0-orthogonal" set $\{x_\gamma : \gamma \in \Gamma\}$. Despite this concept seems to be quite stronger than ASQ, it is proved that ASQ and UASQ agree for a large class of Banach spaces as is the separable Banach spaces [GR, Corollary 2.4]. The interest in UASQ Banach spaces is the following result [GR, Theorem 2.5].

Theorem 2.28. *Let X be a Banach space. Then X^* is not UASQ.*

Though the previous theorem could not be applied to solve Question 6, it was applied to prove that certain spaces of Lipschitz functions are not isometric to any dual Banach space (see [GR, Sections 3,4 and 5] and [GPR, Proposition 2.11 and Remark 2.12]).

Apart from the previous application, the ASQ Banach spaces were used in [BLR7] in order to provide new examples of symmetric projective tensor products spaces with the diameter two properties.

Finally, according to [Flo], for a Banach space X and $N \in \mathbb{N}$, we will denote by $\widehat{\otimes}_{\pi,s,N} X$ the *symmetric projective N-tensor product of X*. This space is the completion of the linear space generated by $\left\{x^N := x \otimes \overset{N}{\ldots} \otimes x : x \in X\right\}$ under the norm given by

$$\|z\| := \inf\left\{\sum_{i=1}^{k} |\lambda_i| : z = \sum_{i=1}^{k} \lambda_i x_i^N, \lambda_i \in \mathbb{R}, x_i \in S_X\ \forall i \in \{1, \ldots, k\}\right\}.$$

It is well known that its topological dual space is identified with the space of all N-homogeneous and bounded polynomials on X by the action

$$P\left(x^N\right) := P(x) \ \ \forall x \in X$$

for each N-homogeneous and bounded polynomial P (see [Flo] for background).

Related to the diameter two properties in symmetric projective tensor products it is known that if X is a Banach space such that $X^{(\infty}$ has an infinite-dimensional centralizer then $\widehat{\otimes}_{\pi,s,N} X$ has the D2P for every natural number N [AB, Theorem 3.2]. Making a strong use of the c_0 behaviour of ASQ spaces the following result was proved in [BLR7, Theorem 3.3]

Theorem 2.29. *Let X be an ASQ space and $N \in \mathbb{N}$. Then $\widehat{\otimes}_{\pi,s,N} X$ has the SD2P.*

Notice that we have introduced the LASQ, WASQ and ASQ Banach spaces as natural stronger properties than the diameter two properties. For further reference about new strenghtening of the diameter two properties, we refer the reader to [ANP] for the definition of the *symmetric strong diameter two property*, an intermediate property between SD2P and the ASQ, and to [BLR2] for the definition of the *diametral diameter two properties*, which are intermediate between the diameter two properties and the Daugavet property.

2.5.4 Section 2.4

After the paper [BLR4], R. Haller, J. Langemets and M. Põldvere considered in [HLP1] the following octahedral-kind properties:

Definition 2.30. Let X be a Banach space. It is said that X is:

1. *locally octahedral* (LOH) if for every $x \in S_X$ and $\varepsilon > 0$ there exists $y \in S_X$ such that $\|x \pm y\| > 2 - \varepsilon$.

2. *weakly octahedral* (WOH) if for every $x_1, \ldots, x_n \in S_X$, $x^* \in B_{X^*}$ and $\varepsilon > 0$ there exists $y \in S_X$ such that $\|x_i \pm ty\| \geq (1 - \varepsilon)(|x^*(x_i)| + t)$ for all $i \in \{1, \ldots, n\}$ and $t > 0$.

It is obvious from the definitions that octahedrality implies weak octahedrality, which in turn implies local octahedrality. None of the converse of the above implications holds because no reverse implication in the diagram 1.1 hold and because of the following result, proved in [HLP1].

Proposition 2.31. *Let X be a Banach space. Then:*

1. *X is LOH if, and only if, X^* has the w^*-slice-D2P.*

2. *X is WOH if, and only if, X^* has the w^*-D2P.*

See [HLP1] for further nice reformulations of LOH, WOH and octahedral norms as well as for a re-proof of Theorem 2.17.

In view of Proposition 2.23, the natural question is the following.

Question 7. *Let X be a Banach space which contains an isomorphic copy of ℓ_1. Can X be equivalently renormed such that X^* has the SD2P?*

First of all, notice that it has been recently proved in [LL] that the answer to the previous question is positive if X is separable.

Moreover, from the fact that the dual of a Banach space X is strongly regular if, and only if, X does not contain any isomorphic copy of ℓ_1 [GGMS, Corollary VI.18], this question is equivalent to the second question in Question 2 in the context of dual Banach spaces.

Finally, we want to point out the existing connection between octahedral norms and different properties of Banach spaces as are the *almost Daugavet property* (see [KSW]) and with the *thickness of the unit ball* (see [God2, GK]).

Chapter 3

Examples of Banach spaces with an octahedral norm

In this chapter we will analyse the octahedrality of the norm of two different kind of Banach spaces. First, in Section 3.1 we will analyse the octahedrality of the operator norm in spaces of operators. We get in Theorem 3.9 that, given two Banach spaces X and Y, then the operator norm on every subspace $H \subseteq L(X, Y)$ containing the finite rank operators is octahedral as soon as the norms of X^* and Y are octahedral. We also prove in Lemma 3.17 that the assumption of octahedrality on just one factor is not sufficient. As a consequence, Corollary 3.10 joint with Theorem 3.18 imply that the SD2P is preserved from both factors by taking projective tensor product but not from just one of them, which gives a complete answer to [ALN2, Question (b)] in the case of the projective tensor product and the SD2P. Further, we give in Theorem 3.27 a characterisation of when $L(\ell_p, X)$ and $L(\ell_p^n, X)$, where X is either ℓ_1 or L_1, have an octahedral norm in terms of $1 \leq p \leq \infty$ and of $n \in \mathbb{N}$. The results of this section are based on [BLR5] (from Proposition 3.1 to Corollary 3.12) and on [LLR2, Section 3] (from Lemma 3.17 until the end). We will also analyse the octahedrality of Lipschitz-free spaces in Sections 3.2 and 3.3. In Section 3.2 we consider the vector-valued Lipschitz-free spaces and prove in Theorem 3.32 that the norm of a Lipschitz-free space $\mathcal{F}(M, X)$ is octahedral whenever M is unbounded or it is bounded but it is not uniformly discrete under the additional assumption of extensions of X^*-valued Lipschitz functions (see Definition 3.31). We also construct vector-valued Lipschitz-free spaces $\mathcal{F}(M, X)$ where not only its norm fails to be octahedral but also its unit ball contains points of Fréchet differentiability. The content of the section is based on [BLR6]. In Section 3.3 we focus on octahedrality of the norm of real Lipschtz free spaces, where we introduce a geometric property of metric spaces, the *long trapezoid property*, which characterises the octahedrality of Lipschitz-free spaces in the sense that a metric space M has the long trapezoid property if, and only if, the norm of $\mathcal{F}(M)$ is octahedral (Theorem 3.35). By making use of this characterisation we prove, for instance, that the norm of $\mathcal{F}(M)$ is octahedral if M is an infinite subset of ℓ_1 (see Proposition 3.49). The content of the section is based on [PR]. We end the chapter with Section 3.4, where we exhibit further research, remarks and open questions related to the content of the present chapter.

3.1 Octahedrality in spaces of operators

Let X and Y be two Banach spaces. In this section we will analyse when the operator norm of a subspace H of $L(X,Y)$ is octahedral. Apart from being a natural question to analyse the octahedrality of the norm in a classical Banach space as $L(X,Y)$ is, the strong connection with SD2P in the projective tensor product thanks to Theorem 2.17 makes of such study a useful tool to answer the question of how is the SD2P preserved by taking projective tensor product [ALN2, Question (b)].

Our starting point is the next proposition, which will be used in order to deal with spaces satisfying the SD2P and which is somehow encoded in the proof of Theorem 2.17.

Proposition 3.1. *Let X be a Banach space and $C := \sum_{i=1}^{n} \lambda_i S_i$ be a convex combination of slices of B_X such that*

$$\operatorname{diam}(C) = 2.$$

Then for every $\varepsilon > 0$ there exist $x_i, y_i \in S_i$ for all $i \in \{1, \ldots, n\}$ and $f \in S_{X^*}$ such that

$$f(x_i - y_i) > 2 - \varepsilon$$

holds for all $i \in \{1, \ldots, n\}$. Thus

$$f(x_i),\ f(-y_i) > 1 - \varepsilon$$

holds for all $i \in \{1, \ldots, n\}$.

Proof. Fix an arbitrary $\varepsilon > 0$. Choose $\delta > 0$ such that $\delta < m\varepsilon$, for $m := \min\limits_{1 \leq i \leq n} \lambda_i$ (notice that we can assume that $\lambda_i \neq 0$ for every $i \in \{1, \ldots, n\}$).

Since $\operatorname{diam}(C) = 2$ then, for every $1 \leq i \leq n$, there exist $x_i,\ y_i \in S_i$ such that

$$\left\| \sum_{i=1}^{n} \lambda_i x_i - \sum_{i=1}^{n} \lambda_i y_i \right\| > 2 - \delta.$$

Hence there exists $f \in S_{X^*}$ satisfying

$$f\left(\sum_{i=1}^{n} \lambda_i x_i - \sum_{i=1}^{n} \lambda_i y_i \right) = \sum_{i=1}^{n} \lambda_i f(x_i - y_i) > 2 - \delta.$$

As a consequence, we have

$$f(x_i - y_i) > 2 - \varepsilon \tag{3.1}$$

holds for all $i \in \{1, \ldots, n\}$. Indeed, assume that there exists $i \in \{1, \ldots, n\}$ such that $f(x_i - y_i) \leq 2 - \varepsilon$. Then

$$\begin{aligned} 2 - \delta < \sum_{j=1}^{n} \lambda_j f(x_j - y_j) &= \lambda_i f(x_i - y_i) + \sum_{j \neq i} \lambda_i f(x_j - y_j) \\ &\leq \lambda_i (2 - \varepsilon) + 2(1 - \lambda_i) \\ &= 2 - \lambda_i \varepsilon < 2 - \delta \end{aligned}$$

a contradiction. So (3.1) holds. †

We omit the proof of next proposition which is the dual version of the above one.

Proposition 3.2. *Let X be a Banach space and $C := \sum_{i=1}^{n} \lambda_i S_i$ be a convex combination of w^*-slices of B_{X^*} such that*

$$\operatorname{diam}(C) = 2.$$

Then for every $\varepsilon > 0$ there exist $f_i, g_i \in S_i$ for every $i \in \{1, \ldots, n\}$ and $x \in S_X$ such that

$$(f_i - g_i)(x) > 2 - \varepsilon$$

holds for all $i \in \{1, \ldots, n\}$. Thus

$$f_i(x),\ g_i(-x) > 1 - \varepsilon$$

holds for all $i \in \{1, \ldots, n\}$.

Our last preliminary lemma allows us to write the unit ball of the dual of certain spaces of operators in terms of basic tensor of functionals in the following sense.

Lemma 3.3. *Let X, Y be Banach spaces. Let H be a closed subspace of $L(X, Y)$ such that $X^* \otimes Y \subseteq H$. Then we have that $B_{H^*} = \overline{\operatorname{conv}}^{w^*}(S_X \otimes S_{Y^*})$.*

Proof. Given $T \in H$, it is clear that

$$\|T\| = \sup\{y^*(T(x)) : x \in S_X,\ y^* \in S_{Y^*}\}.$$

This implies that the set of continuous linear functionals $x \otimes y^* \in H^*$ given by $(x \otimes y^*)(T) := y^*(T(x))$, for $x \in S_X$ and $y^* \in S_{Y^*}$, is a norming subset of H^*. Since $X^* \otimes Y \subseteq H$, we have that $\|x \otimes y^*\| = 1$ for every $x \in S_X$ and $y^* \in S_{Y^*}$. By a separation argument, we get that $B_{H^*} = \overline{\operatorname{conv}}^{w^*}(S_X \otimes S_{Y^*})$. †

Remark 3.4. Notice that we can replace S_X (respectively S_{Y^*}) with a norming set $A \subseteq S_X$ for X^* (respectively a norming set $B \subseteq S_{Y^*}$ for Y) in Lemma 3.3.

Now we are ready to get sufficient conditions on octahedrality in spaces of operators. The first result is the following.

Theorem 3.5. *Let X, Y be Banach spaces. Assume that the norm of Y is octahedral and that there exists $f \in S_{X^*}$ such that $\{x \in S_X : x^*(x) = 1\}$ is norming for X^*. Let be H a closed subspace of $L(X, Y)$ such that $X^* \otimes Y \subseteq H$. Then the norm of H is octahedral.*

Proof. By Theorem 2.17, the norm of H is octahedral, if and only if, every convex combination of w^*-slices of B_{H^*} has diameter 2. So, in order to prove the theorem, let $C := \sum_{i=1}^{n} \lambda_i S_i$ be a convex combination of w^*-slices in B_{H^*}. We can assume that there exist $\varepsilon \in \mathbb{R}^+$ and $A_i \in S_H$ such that $S_i = S(B_{H^*}, A_i, \varepsilon)$ for every $i \in \{1, \ldots, n\}$.

Fix $\delta \in \mathbb{R}^+$. By Lemma 3.3 we can assume that there exist $x_i \in S_X$, $y_i^* \in S_{Y^*}$ such that $x_i \otimes y_i^* \in S_i$ for all $i \in \{1, \ldots, n\}$. Thus $\sum_{i=1}^{n} \lambda_i x_i \otimes y_i^* \in C$.

Since the set $\{x \in S_X : f(x) = 1\}$ is norming for X^* then, by Remark 3.4, we can assume with no loss of generality that $f(x_i) = 1$ holds for every $i \in \{1, \ldots, n\}$.

Fix $i \in \{1, \ldots, n\}$ and $y^* \in B_{Y^*}$. Then the following equivalences hold

$$x_i \otimes y^* \in S_i \Leftrightarrow A_i(x_i \otimes y^*) > 1 - \varepsilon \Leftrightarrow A_i(x_i)(y^*) > 1 - \varepsilon.$$

Thus

$$x_i \otimes y^* \in S_i \Leftrightarrow y^* \in S(B_{Y^*}, A_i(x_i), \varepsilon).$$

Since $\sum_{i=1}^n \lambda_i S(B_{Y^*}, A_i(x_i), \varepsilon)$ is a convex combination of w^*-slices in B_{Y^*} we deduce, by Proposition 3.2 and since the norm of Y is octahedral, the existence of $u_i^*, v_i^* \in B_{Y^*}$ such that $x_i \otimes u_i^*, x_i \otimes v_i^* \in S_i$ for $i \in \{1, \ldots, n\}$, and $y \in S_Y$ such that

$$y(u_i^* - v_i^*) > 2 - \delta$$

holds for every $i \in \{1, \ldots, n\}$.

Then $\sum_{i=1}^n \lambda_i x_i \otimes u_i^*, \sum_{i=1}^n \lambda_i x_i \otimes v_i^* \in C$.

Define $T : X \to Y$ by the equation $T(x) := f(x)y$ for $x \in X$. Clearly $\|T\| = 1$ and $T \in H$. Thus

$$\begin{aligned}
\operatorname{diam}(C) &\geq \left\| \sum_{i=1}^n \lambda_i x_i \otimes u_i^* - \sum_{i=1}^n \lambda_i x_i \otimes v_i^* \right\| = \left\| \sum_{i=1}^n \lambda_i x_i \otimes (u_i^* - v_i^*) \right\| \\
&\geq \sum_{i=1}^n \lambda_i T(x_i)(u_i^* - v_i^*) = \sum_{i=1}^n \lambda_i f(x_i) y(u_i^* - v_i^*) \\
&> (2 - \delta) \sum_{i=1}^n \lambda_i = 2 - \delta.
\end{aligned}$$

From the arbitrariness of δ we deduce that $\operatorname{diam}(C) = 2$ and the theorem is proved. †

From the symmetry in the proof of the above theorem we get the following theorem.

Theorem 3.6. *Let X, Y be Banach spaces. Assume that the norm of X^* is octahedral and that there exists $f \in S_Y$ such that $\{y^* \in S_{Y^*} : y^*(f) = 1\}$ is norming for Y. Let H be a closed subspace of $L(X, Y)$ such that $X^* \otimes Y \subseteq H$. Then the norm of H is octahedral.*

The sufficient condition obtained in the above theorem is satisfied when it is assumed the following infinite-dimensional centralizer condition.

Corollary 3.7. *Let X, Y be Banach spaces. Assume that $Z((Y^{(\infty)})^*)$ is infinite-dimensional and that there exists $f \in S_{X^*}$ such that $\{x \in S_X : f(x) = 1\}$ is norming for X^*. Let be H a closed subspace of $L(X, Y)$ such that $X^* \otimes Y \subseteq H$. Then the norm of H is octahedral.*

Proof. By Remark 1.26, it follows that Y^* has the SD2P, so Theorem 3.5 applies. †

Bearing in mind the projective tensor product, we improve 2 in Proposition 1.36 invoking Theorem 3.5.

Corollary 3.8. *Let X, Y be Banach spaces. Assume that $Z(Y^{(\infty)})$ is infinite-dimensional and that there exists $f \in S_{X^*}$ such that $\{x \in S_X : f(x) = 1\}$ is norming for X^*. Then $X \widehat{\otimes}_\pi Y$ has the SD2P.*

The next result follows the lines of the Theorem 3.5, but for another kind of spaces. As a consequence, we will get the stability of octahedral norms in spaces of operators.

Theorem 3.9. *Let X, Y be Banach spaces. Assume that the norms of X^* and Y are octahedral. Let be H a closed subspace of $L(X, Y)$ such that $X^* \otimes Y \subseteq H$. Then the norm of H is octahedral.*

Proof. Again by Theorem 2.17 the norm of H is octahedral if and only if every convex combination of w^*-slices of B_{H^*} has diameter 2. Let $C := \sum_{i=1}^n \lambda_i S_i$ be a convex combination of w^*-slices in B_{H^*}. Hence there exist $\varepsilon \in \mathbb{R}^+$ and $A_i \in S_H$ such that $S_i = S(B_{H^*}, A_i, \varepsilon)$ for every $i \in \{1, \ldots, n\}$. Fix $\delta \in \mathbb{R}^+$.

By Lemma 3.3 we can ensure the existence of $x_i \in S_X$, $y_i^* \in S_{Y^*}$ such that $x_i \otimes y_i^* \in S_i$ for all $i \in \{1, \ldots, n\}$. Thus $\sum_{i=1}^n \lambda_i x_i \otimes y_i^* \in C$. For every $i \in \{1, \ldots, n\}$ we consider $A_i^*(y_i^*) : X \to \mathbb{R}$ defined by $A_i^*(y_i^*)(x) = y_i^*(A_i(x))$ for every $x \in X$. Then we have that

$$x \otimes y_i^* \in S_i \Leftrightarrow x \in S(B_X, A_i^*(y_i^*), \varepsilon).$$

Now, the norm of X^* is octahedral, so every convex combination of slices of B_X has diameter 2.

Applying Proposition 3.2, there exists $w_i \in S(B_X, A_i^*(y_i^*), \varepsilon)$ and $f \in S_{X^*}$ such that, for all $i \in \{1, \ldots, n\}$, we get that

$$f(w_i) > 1 - \delta,$$

and that

$$\sum_{i=1}^n \lambda_i w_i \otimes y_i \in C.$$

Following as in the proof of Theorem 3.5, we deduce that $\operatorname{diam}(C) > (2-\delta)(1-\delta)$. Due to the arbitrariness of δ we deduce that $\operatorname{diam}(C) = 2$ and the theorem is proved. †

The first consequence of the above theorem is the stability of the SD2P for projective tensor products of Banach spaces. Indeed, taking into account the duality $(X\widehat{\otimes}_\pi Y)^* = L(X, Y^*)$ and the duality between octahedrality and the SD2P exhibited in Section 2.4, the following corollary easily follows.

Corollary 3.10. *Let X and Y be Banach spaces. If X and Y have the SD2P, then so does $X\widehat{\otimes}_\pi Y$.*

This last corollary gives a stability result of the SD2P for projective tensor products, which partially answers to [ALN2, Question (b)] for the case of the projective tensor product and the SD2P.

Furthermore, recall that given Banach spaces X and Y such that $Z(X^{(\infty})$ and $Z((Y^{(\infty)})^*)$ are infinite-dimensional, then the norms of X^* and Y are octahedral. Consequently, the following corollary holds.

Corollary 3.11. *Let X, Y be Banach spaces such that $Z(X^{(\infty})$ and $Z((Y^{(\infty)})^*)$ are infinite-dimensional. Let be H a closed subspace of $L(X, Y)$ such that $X^* \otimes Y \subseteq H$. Then the norm of H is octahedral.*

As a new consequence, we get a result improving 1. in Proposition 1.36 in the following sense.

Corollary 3.12. *Let X, Y be Banach spaces such that $Z(X^{(\infty})$ and $Z(Y^{(\infty})$ are infinite-dimensional. Then the space $X\widehat{\otimes}_\pi Y$ has the SD2P.*

Notice that a natural question is whether the hypothesis on one of the spaces in Corollary 3.10, can be removed; that is, whether the SD2P is stable from just one factor by taking projective tensor product. Notice that an answer to this question would completely answer how the SD2P is preserved by taking projective tensor product, explicitly posed in [ALN2, Question (b)]. In order to do so, it is natural to look for necessary conditions for a projective tensor product to have the SD2P. That is what is obtained in the following proposition in the language of octahedrality in spaces of operators.

Proposition 3.13. *Let X, Y be Banach spaces and let H be a closed subspace of $L(X, Y)$ whose norm is octahedral such that $X^* \otimes Y \subseteq H$. Assume that the norm of X^* is non-rough. Then the norm of Y is octahedral.*

Proof. Let us prove that every convex combination of w^*-slices of B_{Y^*} has diameter 2. We put $y_1, \ldots, y_n \in S_Y$, $\delta > 0$ and $\lambda_1, \ldots, \lambda_n \in (0, 1)$ with $\sum_{i=1}^n \lambda_i = 1$, and consider the convex combination of w^*-slices

$$\sum_{i=1}^n \lambda_i S(B_{Y^*}, y_i, \delta).$$

Let $\varepsilon > 0$. Since the norm of X^* is non-rough we have from Theorem 1.8 that there exist $x^* \in S_{X^*}$ and $\alpha > 0$ such that $\operatorname{diam}(S(B_X, x^*, \alpha)) < \varepsilon$. Put $\rho := \min\{\delta, \alpha\}$ and $x_0 \in S_X \cap S(B_X, x^*, \alpha)$. Consider the convex combination of w^*-slices of B_{H^*} given by

$$\sum_{i=1}^n \lambda_i S(B_{H^*}, x^* \otimes y_i, \rho^2).$$

Now, since H has octahedral norm, then for $i \in \{1, \ldots, n\}$ there exist $f_i, g_i \in S_{H^*} \cap S(B_{H^*}, x^* \otimes y_i, \rho^2)$ such that

$$\left\| \sum_{i=1}^n \lambda_i f_i - \sum_{i=1}^n \lambda_i g_i \right\| > 2 - \varepsilon.$$

By Lemma 3.3 we can assume that $f_i = \sum_{k=1}^{m_i} \gamma_{(k,i)} x_{(k,i)} \otimes y^*_{(k,i)}$ and $g_i = \sum_{k=1}^{m_i} \gamma'_{(k,i)} u_{(k,i)} \otimes v^*_{(k,i)}$, where $x_{(k,i)}, u_{(k,i)} \in S_X$, $y^*_{(k,i)}, v^*_{(k,i)} \in S_{Y^*}$, and $\sum_{k=1}^{m_i} \gamma_{(k,i)} = 1 = \sum_{k=1}^{m_i} \gamma'_{(k,i)}$ which $\gamma_{(k,i)}, \gamma'_{(k,i)} \in [0, 1]$ for all $(k, i), k \in \{1, \ldots, m_i\}$ and $i \in \{1, \ldots, n\}$. For $i \in \{1, \ldots, n\}$, we consider the sets $P_i := \{(k, i) \in \{1, \ldots, m_i\} \times \{i\} : (x^* \otimes y_i)(x_{(k,i)} \otimes y^*_{(k,i)}) > 1 - \rho\}$ and $Q_i := \{(k, i) \in \{1, \ldots, m_i\} \times \{i\} : (x^* \otimes y_i)(u_{(k,i)} \otimes v^*_{(k,i)}) > 1 - \rho\}$. Then we have that

$$\begin{aligned}
1 - \rho^2 &< (x^* \otimes y_i) \left(\sum_{k=1}^{m_i} \gamma_{(k,i)} x_{(k,i)} \otimes y^*_{(k,i)} \right) \\
&= \sum_{i \in P_i} \gamma_{(k,i)} (x^* \otimes y_i)(x_{(k,i)} \otimes y^*_{(k,i)}) + \sum_{i \notin P_i} \gamma_{(k,i)} (x^* \otimes y_i)(x_{(k,i)} \otimes y^*_{(k,i)}) \\
&\leq \sum_{i \in P_i} \gamma_{(k,i)} + (1 - \rho) \sum_{i \notin P_i} \gamma_{(k,i)} = 1 - \sum_{i \notin P_i} \gamma_{(k,i)} + (1 - \rho) \sum_{i \notin P_i} \gamma_{(k,i)}.
\end{aligned}$$

We conclude that $\sum_{i\notin P_i}\gamma_{(k,i)}<\rho$, and hence we have that

$$(x^*\otimes y_i)\left(\sum_{i\in P_i}\gamma_{(k,i)}x_{(k,i)}\otimes y^*_{(k,i)}\right)>1-\rho.$$

It follows that

$$y_i\left(\sum_{i\in P_i}\gamma_{(k,i)}y^*_{(k,i)}\right)\geq (x^*\otimes y_i)\left(\sum_{i\in P_i}\gamma_{(k,i)}x_{(k,i)}\otimes y^*_{(k,i)}\right)>1-\rho,$$

and $\sum_{i\in P_i}\gamma_{(k,i)}y^*_{(k,i)}\in B_{Y^*}$, so $\varphi_i:=\sum_{i\in P_i}\gamma_{(k,i)}y^*_{(k,i)}\in S(B_{Y^*},y_i,\delta)$.

For $(k,i)\in P_i$, we have that $(x^*\otimes y_i)(x_{(k,i)}\otimes y^*_{(k,i)})>1-\rho$. This implies that $x^*(x_{(k,i)})>1-\rho$, and as a consequence $\|x_{(k,i)}-x_0\|<\varepsilon$. In a similar way, we have that

$$y_i\left(\sum_{i\in Q_i}\gamma'_{(k,i)}y^*_{(k,i)}\right)\geq (u^*\otimes v_i)\left(\sum_{i\in Q_i}\gamma'_{(k,i)}u_{(k,i)}\otimes v^*_{(k,i)}\right)>1-\rho,$$

and $\sum_{i\in Q_i}\gamma'_{(k,i)}v^*_{(k,i)}\in B_{Y^*}$, so $\psi_i:=\sum_{i\in Q_i}\gamma_{(k,i)}v^*_{(k,i)}\in S(B_{Y^*},y_i,\delta)$.

For $(k,i)\in Q_i$, we have that $(x^*\otimes y_i)(u_{(k,i)}\otimes v^*_{(k,i)})>1-\rho$. This implies that $x^*(u_{(k,i)})>1-\rho$, and as a consequence $\|u_{(k,i)}-x_0\|<\varepsilon$. It follows that

$$\begin{aligned}\|f_i-x_0\otimes\varphi_i\| &\leq \left\|f_i-\sum_{i\in P_i}\gamma_{(k,i)}x_{(k,i)}\otimes y^*_{(k,i)}\right\|+\left\|\sum_{i\in P_i}\gamma_{(k,i)}x_{(k,i)}\otimes y^*_{(k,i)}-x_0\otimes\varphi_i\right\|\\ &=\left\|\sum_{i\notin P_i}\gamma_{(k,i)}x_{(k,i)}\otimes y^*_{(k,i)}\right\|+\left\|\sum_{i\in P_i}\gamma_{(k,i)}(x_{(k,i)}-x_0)\otimes y^*_{(k,i)}\right\|\\ &\leq\sum_{i\notin P_i}\gamma_{(k,i)}+\sum_{i\in P_i}\gamma_{(k,i)}\|x_{(k,i)}-x_0\|\|y^*_{(k,i)}\|\leq\rho+\varepsilon.\end{aligned}$$

In a similar way, we have that

$$\|g_i-x_0\otimes\psi_i\|\leq\rho+\varepsilon.$$

As a consequence we have that

$$\begin{aligned}2-\varepsilon &<\left\|\sum_{i=1}^n\lambda_i f_i-\sum_{i=1}^n\lambda_i g_i\right\|\\ &\leq\left\|\sum_{i=1}^n\lambda_i(f_i-x_0\otimes\varphi_i)\right\|+\left\|\sum_{i=1}^n\lambda_i x_0\otimes(\varphi_i-\psi_i)\right\|+\left\|\sum_{i=1}^n\lambda_i(g_i-x_0\otimes\psi_i)\right\|\\ &\leq 2(\rho+\varepsilon)+\left\|\sum_{i=1}^n\lambda_i x_0\otimes(\varphi_i-\psi_i)\right\|=2(\rho+\varepsilon)+\left\|x_0\otimes\sum_{i=1}^n\lambda_i(\varphi_i-\psi_i)\right\|\\ &\leq 2(\rho+\varepsilon)+\|x_0\|\left\|\sum_{i=1}^n\lambda_i(\varphi_i-\psi_i)\right\|\leq 2(\rho+\varepsilon)+\left\|\sum_{i=1}^n\lambda_i(\varphi_i-\psi_i)\right\|.\end{aligned}$$

It follows that
$$\left\|\sum_{i=1}^{n} \lambda_i \varphi_i - \sum_{i=1}^{n} \lambda_i \psi_i\right\| > 2 - 2\rho - 3\varepsilon.$$
We recall that $\varphi_i, \psi_i \in S(B_{Y^*}, y_i, \delta)$, and hence
$$\operatorname{diam}\left(\sum_{i=1}^{n} \lambda_i S(B_{Y^*}, y_i, \delta)\right) \geq 2 - 2\rho - 3\varepsilon \geq 2 - 2\delta - 3\varepsilon.$$

Since ε is arbitrary, we conclude that
$$\operatorname{diam}\left(\sum_{i=1}^{n} \lambda_i S(B_{Y^*}, y_i, \delta)\right) \geq 2 - 2\delta.$$

Hence, for $0 < \eta < \delta$ we have
$$\sum_{i=1}^{n} \lambda_i S(B_{Y^*}, y_i, \eta) \subseteq \sum_{i=1}^{n} \lambda_i S(B_{Y^*}, y_i, \delta),$$

and $\operatorname{diam}\left(\sum_{i=1}^{n} \lambda_i S(B_{Y^*}, y_i, \eta)\right) \geq 2 - 2\eta$ by using a similar argument. Hence
$$\operatorname{diam}\left(\sum_{i=1}^{n} \lambda_i S(B_{Y^*}, y_i, \delta)\right) \geq 2 - 2\eta.$$

Since $\eta \in (0, \delta)$ is arbitrary we deduce that
$$\operatorname{diam}\left(\sum_{i=1}^{n} \lambda_i S(B_{Y^*}, y_i, \delta)\right) = 2,$$

and we are done. †

From the symmetry of the spaces X and Y in the proof of the above result, this one can also be written in the following way.

Corollary 3.14. *Let X, Y be Banach spaces and let be H a closed subspace of $L(X,Y)$ whose norm is octahedral and such that $X^* \otimes Y \subseteq H$. Assume that Y has non-rough norm. Then the norm of X^* is octahedral.*

As a consequence of Proposition 3.13 and Theorem 3.5 we get the following equivalence.

Corollary 3.15. *Let X, Y be Banach spaces. Assume that the norm of X^* is non-rough and that there exists $f \in S_{X^*}$ such that $\{x \in S_X : f(x) = 1\}$ is norming for X^*. Then, for every closed subspace H of $L(X,Y)$ such that $X^* \otimes Y \subseteq H$, the following assertion are equivalent:*

i) The norm of H is octahedral.

ii) The norm of Y is octahedral.

Again, using the duality between having octahedral norm and the SD2P joint with the duality $(X\widehat{\otimes}_\pi Y)^* = L(X, Y^*)$, we get from Proposition 3.13, a necessary condition for a projective tensor product to have the SD2P.

Corollary 3.16. *Assume that X and Y are Banach spaces such that $X\widehat{\otimes}_\pi Y$ has SD2P and X^* has non-rough norm. Then Y has the SD2P.*

Although the previous corollary gives a necessary condition for a projective tensor product to have the SD2P, as far as we are concerned it does not produce any counterexample to the question whether the SD2P is preserved from one factor by taking projective tensor product. Because of this reason, we will present another necessary condition for a space of operators to have an octahedral norm.

Lemma 3.17. *Let X and Y be Banach spaces and assume that Y^* is uniformly convex. Assume also that there exists a closed subspace H of $L(Y^*, X)$ such that $X \otimes Y \subseteq H$ and that the norm of H is octahedral. Then Y^* is finitely representable in X.*

Proof. Recall that the modulus of uniform convexity of Y^* is defined by

$$\delta(\varepsilon) = \inf \left\{ 1 - \left\| \frac{f+g}{2} \right\| : f, g \in B_{Y^*}, \|f - g\| \geq \varepsilon \right\}.$$

Note that if $f, g \in B_{Y^*}$ satisfy $f(y) > 1 - \delta(\varepsilon)$ and $g(y) > 1 - \delta(\varepsilon)$, for some $y \in S_Y$, then $\|f - g\| < \varepsilon$.

Let $\varepsilon > 0$ and choose $\nu > 0$ so small that $(1+\nu)(1-3\nu)^{-1} < 1 + \varepsilon$. Pick $0 < \eta < \nu/2$ such that $\delta(\eta) < \nu/2$.

Let $F \subseteq Y^*$ be a finite-dimensional subspace. Pick a ν-net $(f_i)_{i=1}^n$ for S_F. Choose $y_i \in S_Y$ such that $f_i(y_i) = 1$.

Let $x \in S_X$. By assumption the norm of H is octahedral, so there exists a $T \in S_H$ such that

$$\|y_i \otimes x + T\| > 2 - \delta(\eta)$$

holds for every $i \in \{1, ..., n\}$.

We want to show that F is $(1+\varepsilon)$ isometric to a subspace of X. We have $\|T(f)\| \leq \|f\|$ since T has norm one. For every $i \in \{1, \ldots, n\}$ we choose $\varphi_i \in S_{Y^*}$ such that

$$\|\varphi_i(y_i)x + T(\varphi_i)\| > 2 - \delta(\eta).$$

By the triangle inequality $|\varphi_i(y_i)| > 1 - \delta(\eta)$ and $\|T(\varphi_i)\| > 1 - \delta(\eta)$. We may assume that $\varphi_i(y_i) > 1 - \delta(\eta)$. Since $f_i(y_i) = 1$ we get from the uniform convexity of Y^* that $\|f_i - \varphi_i\| < \eta < \nu/2$. We also get

$$\|T(f_i)\| \geq \|T(\varphi_i)\| - \|T\|\|f_i - \varphi_i\| > 1 - \delta(\eta) - \frac{\nu}{2} > 1 - \nu.$$

From [AK, Lemma 11.1.11] we see that T restricted to F is a $(1+\varepsilon)$ isometry. †

From the previous lemma we get the desired counterexample.

Theorem 3.18. *Let $2 < p < \infty$ and $n \geq 3$. Then neither $\ell_\infty \widehat{\otimes}_\pi \ell_p^n$ nor $L_\infty \widehat{\otimes}_\pi \ell_p^n$ enjoy the SD2P.*

Proof. Notice that $(\ell_p^n)^* = \ell_{p^*}^n$, where $\frac{1}{p} + \frac{1}{p^*} = 1$, is not finitely representable in L_1 nor in ℓ_1. This means that the norms of $\ell_{p^*}^n \widehat{\otimes}_\varepsilon \ell_1$ and $\ell_{p^*}^n \widehat{\otimes}_\varepsilon L_1$ are not octahedral by Lemma 3.17. Consequently, their dual spaces, which are $\ell_p^n \widehat{\otimes}_\pi \ell_\infty$ and $\ell_p^n \widehat{\otimes}_\pi L_\infty$ [Ryan, Theorem 5.33], fail the w^*-SD2P, and the theorem follows. †

Remark 3.19. In [ALN2, Question (b)] it is asked how the diameter two properties are preserved by tensor products. We can now provide a complete answer to this question for the SD2P in the projective case. The SD2P is preserved from both factors, by Corollary 3.10, but not in general from one of them, by Theorem 3.18.

Remark 3.20. Note that L_∞ as well as ℓ_∞ have an infinite-dimensional centralizer [HWW, Example I.3.4.(h)]. From Theorem 3.18 we see that, given two Banach spaces X and Y, it is not enough to assume that X has an infinite-dimensional centralizer to ensure that $X \widehat{\otimes}_\pi Y$ has the SD2P. But both L_∞ and ℓ_∞ are isometric to $C(K)$ spaces so $L_\infty \widehat{\otimes}_\pi Y$ and $\ell_\infty \widehat{\otimes}_\pi Y$ do have the D2P for any Y by 3 in Proposition 1.36.

Remark 3.21. Our results also give natural examples of tensor products failing the Daugavet property.

By Theorem 4.2 and Corollary 4.3 in [KKW] there exists a two dimensional complex Banach space E such that both $L_1^{\mathbb{C}} \widehat{\otimes}_\varepsilon E$ and $L_\infty^{\mathbb{C}} \widehat{\otimes}_\pi E^*$ fail the Daugavet property.

Note that both real and complex L_1 and L_∞ have the Daugavet property.

However, from our techniques we improve the above-mentioned results of [KKW] by giving examples of (real) Daugavet spaces such that their projective tensor product fail to have the SD2P, as the example $\ell_p^n \widehat{\otimes}_\pi L_\infty$ shows in Theorem 3.18.

Let X and Y be two Banach spaces and H be a subspace of $L(Y^*, X)$ containing $X \otimes Y$. Notice that Lemma 3.17 shows that, in presence of uniform convexity assumptions, finite representability of Y^* in X is a necessary condition in order to get that the norm of H is octahedral. In the case that $X = \ell_1$ or L_1, it turns out to be a sufficient condition too.

Theorem 3.22. *Let X be a Banach space. Then:*

1. *If, for all $\varepsilon > 0$, X is $(1+\varepsilon)$ isometric to a subspace of ℓ_1, then the norm of $L(X, \ell_1)$ is octahedral.*
2. *If, for all $\varepsilon > 0$, X is $(1+\varepsilon)$ isometric to a subspace of L_1, then the norm of $L(X, L_1)$ is octahedral.*

Proof. (1). Let $\varepsilon > 0$ and $\psi : X \to \ell_1$ be a $(1+\varepsilon)$ isometry. Let $T_1, \ldots, T_n \in S_{L(X,\ell_1)}$ and, for every $i \in \{1, \ldots, n\}$, pick $x_i \in S_X$ such that $\|T_i(x_i)\| > 1 - \varepsilon$.

Let P_k be the projection on ℓ_1 onto the first k coordinates. Choose $k \in \mathbb{N}$ so that $\|P_k(T_i(x_i)) - T_i(x_i)\| < \varepsilon$ and $\|P_k(\psi(x_i)) - \psi(x_i)\| < \varepsilon$ for every $i \in \{1, \ldots, n\}$.

Let $\varphi_k : \ell_1 \to \ell_1$ be the shift operator defined by

$$\varphi_k(x)(n) := \begin{cases} 0 & \text{if } n \leq k, \\ x(n-k) & \text{if } n > k. \end{cases}$$

Define $S := \varphi_k \circ P_k \circ \psi$. Now, as $P_k(T_i(x_i))$ and $S(x_i)$ have disjoint support, we have that

$$\begin{aligned} \|T_i + S\| &\geq \|P_k T_i(x_i)\| - \varepsilon + \|P_k(\psi(x_i))\| \\ &\geq \|T_i(x_i)\| + \|\psi(x_i)\| - 3\varepsilon > 2 - 5\varepsilon, \end{aligned}$$

so we are done.

(2). Define $A := [0,1]$. Let $T_1, \ldots, T_n \in S_{L(X,L_1)}$ and $\varepsilon > 0$. By assumption there exists $x_i \in S_X$ such that $\|T_i(x_i)\| = \int_A |T_i(x_i)| > 1 - \frac{\varepsilon}{2}$ for all $i \in \{1, \ldots, n\}$. Pick a non-empty closed interval $I \subseteq A$ such that $\int_I |T_i(x_i)| < \frac{\varepsilon}{2}$ holds for each $i \in \{1, \ldots, n\}$.

By assumption there exists a $(1+\varepsilon)$ linear isometry $T : X \to L_1$. Let $\phi : I \to A$ be an increasing and affine bijection. Define $S_I : L_1 \to L_1$ by the equation

$$S_I(f) = (f \circ \phi)\phi' \chi_I \quad \text{for all } f \in L_1,$$

where χ_I denotes the characteristic function on the interval I. Note that S_I is a linear isometry because of the change of variable theorem. Indeed

$$\|S_I(f)\| = \int_I |(f \circ \phi)\phi'| = \int_{\phi(I)} |f| = \int_A |f| = \|f\| \text{ for all } f \in L_1.$$

Define $G := S_I \circ T$, which is a $(1+\varepsilon)$ linear isometry such that $\operatorname{supp}(G(f)) \subseteq I$ holds for all $f \in L_1$. Given $i \in \{1, \ldots, n\}$, we have

$$\|T_i + G\| \geq \|T_i(x_i) + G(x_i)\| = \int_{A \setminus I} |T_i(x_i)| + \int_I |T_i(x_i) + G(x_i)|.$$

Now

$$\int_{A \setminus I} |T_i(x_i)| = \|T_i(x_i)\| - \int_I |T_i(x_i)| > 1 - \varepsilon.$$

Moreover

$$\int_I |T_i(x_i) + G(x_i)| \geq \int_I |G(x_i)| - |T_i(x_i)| > \int_I |G(x_i)| - \frac{\varepsilon}{2}.$$

Finally note that, as $\operatorname{supp}(G(x_i)) \subseteq I$, we have $\int_I |G(x_i)| = \|G(x_i)\| > (1-\varepsilon)\|x_i\| = 1-\varepsilon$. Consequently

$$\|T_i + G\| > 2 - \frac{5\varepsilon}{2}.$$

As ε was arbitrary we conclude that the norm of $L(X, L_1)$ is octahedral, as desired. †

From here we can conclude the following result.

Corollary 3.23. *If X is a 2-dimensional Banach space, then the norms of both $\ell_1 \widehat{\otimes}_\varepsilon X = L(c_0, X)$ and $L_1 \widehat{\otimes}_\varepsilon X$ are octahedral.*

Proof. We have that X^* is isometric to a subspace of L_1 [Dor, Corollary 1.4], so in particular Theorem 3.22 applies. †

Note that the above corollary improves [HLP1, Proposition 2.3], where the authors show that the norm of $L(c_0, \ell_p^2)$ is octahedral for every $1 \leq p \leq \infty$. Dualising we get the following result, which improves [LLR1, Proposition 2.10] for two dimensional Banach spaces.

Corollary 3.24. *If X is a 2-dimensional Banach space, then $c_0 \widehat{\otimes}_\pi X$ has the SD2P.*

Next we give more examples of finite-dimensional Banach spaces for which the norm of its projective tensor product with ℓ_∞ and L_∞ have the SD2P.

Proposition 3.25. *Let $n \geq 3$ be a natural number and $2 \leq p \leq \infty$. Then the norms of both $L(\ell_{p^*}^n, \ell_1)$ and $L(\ell_{p^*}^n, L_1)$ are octahedral.*

Proof. We know that ℓ_{p^*} is isometric to a subspace of L_1 [AK, Theorem 6.4.19] which in turn contains $\ell_{p^*}^n$ isometrically. In particular, $\ell_{p^*}^n$ is finitely-representable in ℓ_1 and in L_1, so Theorem 3.22 applies, which shows that the norm of $L(\ell_{p^*}^n, Y)$ is octahedral for $Y = \ell_1$ and $Y = L_1$. †

In fact, an infinite-dimensional version of the previous result also works.

Proposition 3.26. *Let $2 \leq p < \infty$. Then:*

1. *Given a closed subspace H of $L(\ell_{p^*}, \ell_1)$ containing $\ell_p \otimes \ell_1$, then the norm of H is octahedral.*

2. *Given a closed subspace H of $L(\ell_{p^*}, L_1)$ containing $\ell_p \otimes L_1$, then the norm of H is octahedral.*

Proof. (1). We proceed as in Theorem 3.22. Given $T_1, \ldots, T_n \in S_H$ and $\varepsilon > 0$ we start by choosing, for every $i \in \{1, \ldots, n\}$, an element $x_i \in S_{\ell_{p^*}}$ such that $\|T_i(x_i)\| > 1 - \varepsilon$. Find $m \in \mathbb{N}$ such that $\|P_m(x_i) - x_i\| < \varepsilon$, where P_m is the projection onto the first m coordinates. Since ℓ_{p^*} is finitely representable in ℓ_1 there exists a $(1+\varepsilon)$ isometry $T : P_m(\ell_{p^*}) \to \ell_1$. The operator $\psi := T \circ P_m$ is then well-defined and using this ψ we define $S := \varphi_k \circ P_k \circ \psi$ as in the proof of Theorem 3.22. Note that $S \in \ell_p \otimes \ell_1 \subseteq H$ since P_m has finite rank. Similar estimates to the ones in Theorem 3.22 conclude the proof.

The proof of (2) is similar. †

Let us end the section with a summary of the behaviour of the octahedrality of the subspaces of H of $L(X, Y)$ containing the finite rank operators when $X = \ell_1$ or L_1 and Y is an ℓ_p space.

Theorem 3.27. *Let $1 \leq p \leq \infty$ and let X be either L_1 or ℓ_1. Then:*

1. *If H is a closed subspace of $L(\ell_{p^*}, X)$ which contains $\ell_p \otimes X$, then the norm of H is octahedral if, and only if, $2 \leq p$ or $p = 1$.*

2. *If H is a closed subspace of $L(\ell_1, X)$ which contains $c_0 \otimes X$, then the norm of H is octahedral.*

3. *If n is a natural number and H is a closed subspace of $L(\ell_{p^*}^n, X)$ which contains $\ell_p^n \otimes X$, then the norm of H is octahedral if, and only if, either $n \leq 2$ or if $n > 2$ and $2 \leq p$ or $p = 1$.*

The case $p = \infty$ follows since ℓ_∞^n is *alternatively octahedral* and so the octahedrality follows (see [HLP2, Theorem 2.1] for details).

3.2 Octahedrality in vector-valued Lipchitz-free Banach spaces

The following two sections will be devoted to studying octahedrality of the norm of Lipschitz-free spaces (see formal definitions below). Our motivation for this study is that, roughly speaking, there is a strong ℓ_1 behaviour in these spaces (e.g. every Lipschitz-free space contains a complemented copy of ℓ_1 whenever it is infinite-dimensional [CDW]). This means, according to [God2, Theorem II.4], that every infinite-dimensional Lipschitz-free space admits an equivalent octahedral norm. In view of such result, it is natural the question whether the norm of Lipschitz-free spaces is or not octahedral. This is the aim of the following two sections.

In order to do so, we will introduce the Lipschitz-free spaces. Given a metric space M with a designated origin 0 and a Banach space X, we will denote by $\mathrm{Lip}_0(M,X)$ the Banach space of all X-valued Lipschitz functions on M which vanish at 0 under the standard Lipschitz norm

$$\|f\| := \sup\left\{\frac{\|f(x)-f(y)\|}{d(x,y)} : \ x,y\in M, x\neq y\right\}.$$

First of all, notice that we can consider every point of M as the origin with no loss of generality. Indeed, given $x,y\in M$, let $\mathrm{Lip}_x(M,X)$ ($\mathrm{Lip}_y(M,X)$) be the space of X-valued Lipschitz functions which vanish at x (respectively at y). Then the map

$$\begin{array}{ccc}\mathrm{Lip}_x(M,X) & \longrightarrow & \mathrm{Lip}_y(M,X)\\ f & \longmapsto & f-f(y),\end{array}$$

defines an onto linear isometry. So the designated origin will be freely chosen.

From a straightforward application of Ascoli-Arzela theorem it can be checked that $B_{\mathrm{Lip}_0(M,X^*)}$ is a compact set for the pointwise topology. Hence $\mathrm{Lip}_0(M,X^*)$ is itself a dual Banach space. In fact, the map

$$\begin{array}{rcc}\delta_{m,x}:\mathrm{Lip}_0(M,X^*) & \longrightarrow & \mathbb{R}\\ f & \longmapsto & f(m)(x)\end{array}$$

defines a linear and bounded map for each $m\in M$ and $x\in X$. In other words, $\delta_{m,x}\in \mathrm{Lip}_0(M,X^*)^*$. Then if we define

$$\mathcal{F}(M,X) := \overline{\mathrm{span}}(\{\delta_{m,x} : m\in M, x\in X\})$$

then we have that $\mathcal{F}(M,X)^* = \mathrm{Lip}_0(M,X^*)$ by [Kai, Theorem 1]. We will refer to the previous space as the *X-valued Lipschitz-free space over M*. The reference to the Banach space X will be omitted when $X=\mathbb{R}$.

Lipschitz-free Banach spaces have been intensively studied in the last 20 years. We refer to [God1, GK, Kal, Wea] for background about Lipschitz-free spaces. Concerning the vector-valued versions, we refer the reader to [Joh, Section 4].

Notice that, given a metric space M, then the mapping $\delta : M \longrightarrow \mathcal{F}(M)$ such that $\delta(m)=\delta_m$ defines an isometry. The fundamental linearisation property of Lipschitz-free spaces is the following: given a metric space M and a Banach space X, then for every

Lipschitz mapping $f : M \longrightarrow X$ then there exists a bounded operator $T_f : \mathcal{F}(M) \longrightarrow X$ such that the following diagram commutes

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, hook, "\delta"'] & X \\ \mathcal{F}(M) \arrow[ur, "T_f"'] & \end{tikzcd}$$

The commutativity of the previous diagram says nothing but $T_f(\delta_m) := f(m)$. This defines a linear isometry

$$\begin{array}{rcl} \Phi : \mathrm{Lip}_0(M, X) & \longrightarrow & L(\mathcal{F}(M), X) \\ f & \longmapsto & T_f. \end{array} \tag{3.2}$$

We claim that the previous isometry is onto. Indeed, given a bounded operator $T : \mathcal{F}(M) \longrightarrow X$ then define $f(m) := T(\delta(m))$ for every $m \in M$. It is clear, from the boundedness of T, that f is Lipschitz. Furthermore T_f and T agree on $\{\delta_m : m \in M\} \subseteq \mathcal{F}(M)$. So $T = T_f$, which proves the surjectivity of Φ.

In the dual setting, notice that Φ is an isometric isomorphism between $\mathrm{Lip}_0(M, X^*) = \mathcal{F}(M, X)^*$ and $L(\mathcal{F}(M), X^*) = (\mathcal{F}(M)\widehat{\otimes}_\pi X)^*$. We can wonder whether the isometry Φ is, in such a case, $w^* - w^*$ continuous, in order to guarantee that $\mathcal{F}(M, X)$ and $\mathcal{F}(M)\widehat{\otimes}_\pi X$ are isometrically isomorphic. That is what will be done in the next result. Before the statement of such result, notice that a bounded net $\{f_s\}$ in $\mathrm{Lip}_0(M, X^*)$ converges in the weak-star topology to a function $f \in \mathrm{Lip}_0(M, X^*)$ if, and only if, $\{f_s(m)\} \to f(m)$ for each $m \in M$, where the last convergence is in the weak-star topology of X^*. Now we can prove the desired result.

Proposition 3.28. *$\mathcal{F}(M, X)$ and $\mathcal{F}(M)\widehat{\otimes}_\pi X$ are isometrically isomorphic Banach spaces for every metric space M and for every Banach space X.*

Proof. It is enough to prove that Φ is $w^* - w^*$ continuous, where the weak-star topologies are respectively induced by $\mathcal{F}(M, X)$ on $\mathrm{Lip}_0(M, X^*)$ and by $\mathcal{F}(M)\widehat{\otimes}_\pi X$ on $L(\mathcal{F}(M), X^*)$.

Note that Φ is $w^* - w^*$ continuous if, and only if, for every $z \in \mathcal{F}(M)\widehat{\otimes}_\pi X$ one has that $z \circ \Phi$ is a weak-star continuous functional. By [FHHMPZ, Corollary 3.94] it is enough to prove that, given $z \in \mathcal{F}(M)\widehat{\otimes}_\pi X$, we have that $\ker(z \circ \Phi) \cap B_{\mathrm{Lip}_0(M,X^*)}$ is weak-star closed. So, pick $z \in \mathcal{F}(M)\widehat{\otimes}_\pi X$ and consider $\{f_s\}$ a net in $\ker(z \circ \Phi) \cap B_{\mathrm{Lip}_0(M,X^*)}$ which is weak-star convergent to f, and let us prove that $(z \circ \Phi)(f) = 0$. To this end, pick $\varepsilon > 0$. Note that z can be expressed as

$$z := \sum_{n=1}^{\infty} \gamma_n \otimes x_n$$

where $\gamma_n \in \mathcal{F}(M)$ and $x_n \in X$ verify that $\|z\| \leq \sum_{n=1}^{\infty} \|\gamma_n\|\|x_n\| < \infty$ [Ryan, Proposition 2.8]. Now, consider a sequence $\{\varepsilon_n\}$ in $\mathbb{R}^+$ such that $\sum_{n=1}^{\infty} \varepsilon_n < \frac{\varepsilon}{3}$ and consider, for each $n \in \mathbb{N}$, an element $\psi_n \in \mathrm{span}\{\delta_m : m \in M\}$ verifying $\|\gamma_n - \psi_n\|\|x_n\| < \frac{\varepsilon_n}{2}$ for each $n \in \mathbb{N}$. As it is clear that $\sum_{n=1}^{\infty} \|\psi_n\|\|x_n\| < \infty$, consider $k \in \mathbb{N}$ such that $\sum_{n=k+1}^{\infty} \|\psi_n\|\|x_n\| < \frac{\varepsilon}{6}$. Finally, in view of weak-star topology of $\mathrm{Lip}_0(M, X^*)$, it is obvious that $\{f_s(\psi_n)(x_n)\} \to$

$f(\psi_n)(x_n)$ for each $n \in \mathbb{N}$, hence we can find s such that $|(f - f_s)(\psi_n)(x_n)| < \frac{\varepsilon}{3k}$ for each $n \in \{1, \ldots, k\}$. Now, bearing in mind that $\|f - f_s\| \leq 2$, we get

$$|(z \circ \Phi)(f)| = |(z \circ \Phi)(f - f_s)| = \left| \sum_{n=1}^{\infty} T_{f-f_s}(\gamma_n)(x_n) \right| \leq$$

$$\left| \sum_{n=1}^{k} T_{f-f_s}(\psi_n)(x_n) \right| + \|f - f_s\| \sum_{n=k+1}^{\infty} \|\gamma_n - \psi_n\| \|x_n\| \leq$$

$$\sum_{n=1}^{k} |(f - f_s)(\psi_n)(x_n)| + \|f - f_s\| \sum_{n=k+1}^{\infty} \|\psi_n\| \|x_n\| + \frac{\varepsilon}{3}$$

$$< \sum_{n=1}^{k} \frac{\varepsilon}{3k} + \frac{2\varepsilon}{3} = \varepsilon.$$

As $\varepsilon > 0$ was arbitrary we conclude that $(z \circ \Phi)(f) = 0$, so we are done. †

Let M be a metric space and X be a Banach space. Notice that we have a useful description of $\mathcal{F}(M, X)$ because we know a dense subspace of it. This fact will play an important role in the following because diameter two properties actually rely on dense subspaces in the following sense.

Proposition 3.29. *Let X be a Banach space. Let $Y \subseteq X^*$ be a norm dense subspace. Then:*

1. *If for each $f \in S_Y$ and $\alpha \in \mathbb{R}^+$ the slice $S(B_X, f, \alpha)$ has diameter two, then X has the slice-D2P.*
2. *If for each $f_1, \ldots, f_n \in S_Y$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{R}^+$ such that $W := \bigcap_{i=1}^{n} S(B_X, f_i, \alpha_i) \neq \emptyset$ it follows that W has diameter two, then X has the D2P.*
3. *If for each $f_1, \ldots, f_n \in S_Y, \alpha_1, \ldots, \alpha_n \in \mathbb{R}^+$ and $\lambda_1, \ldots, \lambda_n \in]0, 1]$ with $\sum_{i=1}^{n} \lambda_i = 1$, the convex combination of slices $\sum_{i=1}^{n} \lambda_i S(B_X, f_i, \alpha_i)$ has diameter two, then X satisfies the SD2P.*

Proof. We will prove statement (1) because the proofs of (2) and (3) are completely similar.

Pick a slice $S := S(B_X, f, \alpha)$ of B_X. Since Y is norm dense in X^* we can find $\varphi \in S_Y$ such that $\|f - \varphi\| < \frac{\alpha}{2}$.

By hypothesis, given an arbitrary $\delta \in \mathbb{R}^+$ we can find $x, y \in S(B_X, \varphi, \frac{\alpha}{2})$ such that $\|x - y\| > 2 - \delta$. Let us prove that $x \in S$, being the proof of $y \in S$ similar. Bearing in mind that $\varphi(x) > 1 - \frac{\alpha}{2}$ and that $\|f - \varphi\| < \frac{\alpha}{2}$ we deduce

$$f(x) = \varphi(x) + (f - \varphi)(x) \geq \varphi(x) - \|f - \varphi\| > 1 - \alpha.$$

On the other hand, as $x, y \in S$, we conclude

$$2 - \delta < \|x - y\| \leq \operatorname{diam}(S).$$

As $\delta \in \mathbb{R}^+$ was arbitrary we conclude that X has the slice-D2P, as desired. †

Now we consider the weak-star version of proposition above.

Proposition 3.30. *Let X be a Banach space and $Y \subseteq X$ be a dense subspace. Then:*

1. *If for each $y \in S_Y$ and $\alpha \in \mathbb{R}^+$ the slice $S(B_{X^*}, y, \alpha)$ has diameter two, then X has the w^*-slice-D2P.*
2. *If for each $y_1, \ldots, y_n \in S_Y$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{R}^+$ such that $W := \bigcap_{i=1}^{n} S(B_{X^*}, y_i, \alpha_i) \neq \emptyset$ one has that W has diameter two, then X has the w^*-D2P.*
3. *If for $y_1, \ldots, y_n \in S_Y, \alpha_1, \ldots, \alpha_n \in \mathbb{R}^+$ and $\lambda_1, \ldots, \lambda_n \in]0, 1]$ with $\sum_{i=1}^n \lambda_i = 1$ the convex combination of weak-star slices $\sum_{i=1}^n \lambda_i S(B_{X^*}, y_i, \alpha_i)$ has diameter two, then X satisfies the w^*-SD2P.*

Recall that one of the central results in the theory of Lipschitz functions is the classical McShane's extension theorem. It says that if $N \subseteq M$ and $f : N \longrightarrow \mathbb{R}$ is a Lipschitz function, then there is an extension to a Lipschitz function $F : M \longrightarrow \mathbb{R}$ with the same Lipschitz constant (see, e.g. [Wea, Theorem 1.5.6]). It is known that such result is false in the vector-valued setting (see below for counterexamples). Because of the implications of the previous theorem, we will consider the following definition.

Definition 3.31. Let M be a metric space and let X be a Banach space.

We will say that the pair (M, X) satisfies the *contraction-extension property* (CEP) if given a set $N \subseteq M$ and a Lipschitz function $f : N \longrightarrow X$ then there exists a Lipschitz function $F : M \longrightarrow X$ which extends f such that

$$\|F\|_{\mathrm{Lip}_0(M,X)} = \|f\|_{\mathrm{Lip}_0(N,X)}.$$

On the one hand note that, in the particular case of M being a Banach space, the definition given above agrees with the one given in [BeLi].

On the other hand, let us give some examples of pairs which have the CEP. First of all, given a metric space M, the pair $(M, \mathbb{R})$ has the CEP by McShane extension theorem. In addition, in [BeLi, Chapter 2] we can find some examples of Banach spaces X such that the pair (X, X) satisfies the CEP such as Hilbert spaces and ℓ_∞^n. Finally, if Y is a strictly convex Banach space such that there exists a Banach space X with $\dim(X) \geq 2$ and verifying that the pair (X, Y) has the CEP, then Y is a Hilbert space [BeLi, Theorem 2.11].

Let us explain in a rough way the key idea of the main result of the section which proves, for every unbounded or not uniformly discrete metric space M, that the norm of $\mathcal{F}(M, X)$ is octahedral, whenever the pair (M, X^*) has the CEP, where X is any Banach space. For this, it is enough to show that, given a convex combination of w^*-slices C in the unit ball of $\mathrm{Lip}_0(M, X^*)$, then C has diameter exactly 2. What is done first is to observe that it is enough to consider w^*-slices given by elements in $\mathrm{span}\{\delta_{m,x} : m \in M, x \in X\}$, which is based on Proposition 3.30. Now, depending on the assumptions on the metric space, we construct a pair of Lipschitz functions in every w^*-slice defining C. Each pair of these Lipschitz functions are defined on different finite metric subspaces so that, when extended to the whole metric space by making use of the CEP assumption, each pair of these norm preserving extensions are in the corresponding w^*-slice of those defining C.

All these pairs define two elements of C which we can guarantee that are far enough to get that C has diameter 2 from the construction too. The detailed construction will be made discussing by cases depending on the topological structure of the metric space, but the existence of a unified idea motivates us to present the following result in a joint way.

Theorem 3.32. *Let M be an infinite pointed metric space and let X be a Banach space. Assume that the pair (M, X^*) has the CEP. If M is unbounded or is not uniformly discrete then the norm of $\mathcal{F}(M,X)$ is octahedral. Consequently, the unit ball of $\mathcal{F}(M,X)$ can not have any point of Fréchet differentiability.*

Proof. We will prove, by Theorem 2.17, that $\text{Lip}_0(M,X^*)$ has the w^*-SD2P. Let $C = \sum_{i=1}^k \lambda_i S(B_{\text{Lip}_0(M,X^*)}, \varphi_i, \alpha)$ be a convex combination of weak-star slices in $B_{\text{Lip}_0(M,X^*)}$ and let us prove that C has diameter exactly 2. From Proposition 3.30 we can assume that $\varphi_i \in \text{span}\{\delta_{m,x} : m \in M, x \in X\}$ for each $i \in \{1,\ldots,k\}$. So assume that

$$\varphi_i = \sum_{j=1}^{n_i} \lambda_j^i \delta_{m_{i,j},x_{i,j}},$$

for suitable $n_i \in \mathbb{N}$, $m_{i,j} \in M\backslash\{0\}$, $x_{i,j} \in X\backslash\{0\}$, $\lambda_j^i \in \mathbb{R}$ for $i \in \{1,\ldots,k\}, j \in \{1,\ldots,n_i\}$.

Pick $g_i \in S(B_{\text{Lip}_0(M,X^*)}, \varphi_i, \alpha)$ and $\delta_0 \in \mathbb{R}^+$ verifying

$$\frac{\varphi_i(g_i)}{1+\delta_0} > 1-\alpha$$

holds for all $i \in \{1,\ldots,k\}$. Fix $0 < \delta < \delta_0$. Now we will divide the proof in several steps.

Step 1: we will define, for every $i \in \{1,\ldots,k\}$, a subspace $M_i \subset M$ and functions F_i and G_i in $\text{Lip}_0(M_i, X^*)$.

We will do this depending on the following cases: M is unbounded, M is bounded and discrete but not uniformly discrete or M is bounded and $0 \in M'$. It is clear that when M is unbounded or not uniformly discrete, it is enough to study each of these three cases.

Assume that M is unbounded. Then there exists a sequence $\{m_n\} \subseteq M$ verifying

$$\{d(m_n,0)\} \to \infty.$$

Hence

$$\{d(m_n,m)\} \to \infty$$

for each $m \in M$ in view of triangle inequality. Now pick a positive integer N so that

$$\frac{d(m_{i,j},0)}{d(m_N,m_{i,j})} + \frac{\|g_i(m_{i,j})\|}{d(m_N,m_{i,j})} < \delta \tag{3.3}$$

holds for every $i \in \{1,\ldots,k\}, j \in \{1,\ldots,n_i\}$. Choose $x^* \in S_{X^*}$ and define $M_i := \{0\} \cup \bigcup_{i=1}^k \bigcup_{j=1}^{n_i} \{m_{i,j}\} \cup \{m_N\}$ for every $i \in \{1,\ldots,k\}$. (In this case M_i does not depend on the index i). We also define $F_i, G_i : M_i \longrightarrow X^*$ given by

$$F_i(m_{i,j}) = G_i(m_{i,j}) = g_i(m_{i,j}) \quad i \in \{1,\ldots,k\}, j \in \{1,\ldots,n_i\},$$

$$F_i(0) = G_i(0) = 0, F_i(m_N) = -G_i(m_N) = d(m_N,0)x^*.$$

Assume now that M is bounded and discrete, but not uniformly discrete. As M is discrete we can find $r > 0$ such that

$$B(0,r) = \{0\}, B(m_{i,j}, r) = \{m_{i,j}\}$$

holds for all $i \in \{1, \ldots, k\}, j \in \{1, \ldots, n_i\}$. Furthermore, since M is not uniformly discrete we can find a pair of sequences $\{x_n\}, \{y_n\}$ in M such that $0 < d(x_n, y_n) \to 0$. Pick $n \in \mathbb{N}$ big enough so that $d(x_n, y_n) < \delta$ and that

$$\frac{1 + \frac{d(x_n,y_n)}{d(x_n,v)}}{1 - \frac{d(x_n,y_n)}{d(x_n,v)}} < 1 + \delta \tag{3.4}$$

holds for all $v \in \{m_{i,j} : 1 \leq i \leq k, 1 \leq j \leq n_i\} \cup \{0\}$. Note that such an n exists since $\{d(x_n, v)^{-1}\}$ is a well defined bounded sequence because M is discrete and bounded in this case. Given $i \in \{1, \ldots, k\}$ and $x^* \in S_{X^*}$ define $M_i := \{0\} \cup \bigcup_{j=1}^{n_i} \{m_{i,j}\} \cup \{x_n, y_n\}$ and $F_i, G_i := M_i \longrightarrow \mathbb{R}$ given by

$$F_i(0) = g_i(0) = 0, F_i(m_{i,j}) = G_i(m_{i,j}) = g_i(m_{i,j}), j \in \{1, \ldots, n_i\},$$

and

$$F_i(x_n) = G_i(x_n) = g_i(x_n), F_i(y_n) = g_i(x_n) + d(y_n, x_n)x^*,$$
$$G_i(y_n) = g_i(x_n) - d(y_n, x_n)x^*.$$

Finally, we assume that M is bounded and $0 \in M'$. Then we can find a sequence $\{m_n\}$ in $M \setminus \{0\}$ such that $\{m_n\} \to 0$. So there exists a positive integer m such that $m_n \notin \{m_{i,j} : i \in \{1, \ldots, k\}, j \in \{1, \ldots, n_i\}\}$ for every $n \geq m$. Now pick $x^* \in S_{X^*}$ and, for each $i \in \{1, \ldots, k\}$, we define $M_i := \{0, m_n\} \bigcup \cup_{j=1}^{n_i} \{m_{i,j}\}$ and $F_i, G_i : M_i \longrightarrow X^*$ by the equations

$$F_i(m_{i,j}) = G_i(m_{i,j}) = g_i(m_{i,j}) \quad i \in \{1, \ldots, k\}, j \in \{1, \ldots, N_i\}$$

and

$$F_i(m_n) = -G_i(m_n) = d(m_n, 0)x^*, F_i(0) = G_i(0) = 0.$$

Now, for each unbounded or not uniformly discrete metric space M we have defined the desired subspaces M_i and functions F_i and G_i in $\mathrm{Lip}_0(M_i, X^*)$ for every $i \in \{1, \ldots, k\}$.

Step 2: we claim that $\|F_i\|_{\mathrm{Lip}_0(M_i,X^*)} \leq 1 + \delta$ holds for all $i \in \{1, \ldots, k\}$. To this end we have three cases again: M is unbounded, M is bounded, discrete but not uniformly discrete or M is bounded and $0 \in M'$.

Assume that M is unbounded. Given $i \in \{1, \ldots, k\}$ and $u, v \in M_i$, we have two different possibilities:

a) If $u, v \notin \{m_N\}$ then

$$\frac{\|F_i(u) - F_i(v)\|}{d(u,v)} = \frac{\|g_i(u) - g_i(v)\|}{d(u,v)} \leq \|g_i\| \leq 1.$$

b) If $u = m_N$ then

$$\begin{aligned}\frac{\|F_i(u) - F_i(v)\|}{d(u,v)} &= \frac{\|d(m_N,0)x^* - F_i(v)\|}{d(m_N,v)} \\ &\leq \frac{d(m_N,0)}{d(m_N,v)} + \frac{\|g_i(v)\|}{d(m_N,v)} \\ &\leq 1 + \frac{d(v,0)}{d(m_N,v)} + \frac{\|g_i(v)\|}{d(m_N,v)} \\ &\overset{(3.3)}{<} 1 + \delta.\end{aligned}$$

Now, taking supremum in u and v, we get

$$\|F_i\|_{\mathrm{Lip}_0(M_i,X^*)} \leq 1 + \delta.$$

Assume now that M is bounded, discrete but not uniformly discrete. Again given $u, v \in M_i, u \neq v$ and $i \in \{1, \ldots, k\}$ we have different possibilities:

a) If $u \neq y_n$ and $v \neq y_n$ then we have

$$\frac{\|F_i(u) - F_i(v)\|}{d(u,v)} = \frac{\|g_i(u) - g_i(v)\|}{d(u,v)} \leq \|g_i\| \leq 1.$$

b) If $u = y_n$, $v \neq x_n$ then

$$\begin{aligned}\frac{\|F_i(u) - F_i(v)\|}{d(u,v)} &= \frac{\|g_i(x_n) + d(x_n,y_n)x^* - g_i(v)\|}{d(y_n,v)} \\ &\leq \frac{\|g_i(x_n) - g_i(v)\| + d(x_n,y_n)}{d(y_n,v)} \\ &< \frac{d(x_n,v) + d(x_n,y_n)}{d(x_n,v) - d(y_n,x_n)} \\ &= \frac{1 + \frac{d(x_n,y_n)}{d(x_n,v)}}{1 - \frac{d(x_n,y_n)}{d(x_n,v)}} \\ &\overset{(3.4)}{<} 1 + \delta.\end{aligned}$$

c) If $u = y_n$ and $v = x_n$ then

$$\frac{\|F_i(u) - G_i(v)\|}{d(u,v)} = \frac{d(x_n,y_n)\|x^*\|}{d(x_n,y_n)} = 1$$

Then, taking supremum in u and v, it follows

$$\|F_i\|_{\mathrm{Lip}_0(M_i,X^*)} \leq 1 + \delta.$$

If M is bounded and $0 \in M'$ we can also get that

$$\|F_i\|_{\mathrm{Lip}_0(M_i,X^*)} \leq 1 + \delta$$

using similar arguments to the ones of the above cases taking large enough n.

Similar computations also arise that

$$\|G_i\|_{\mathrm{Lip}_0(M_i,X^*)} \leq 1+\delta$$

holds for every $i \in \{1,\ldots,k\}$.

Now, we have defined subspaces $M_i \subset M$ and functions $F_i,\ G_i \in \mathrm{Lip}_0(M_i,X^*)$ such that

$$\max_{1\leq i\leq k}\{\|F_i\|_{\mathrm{Lip}_0(M_i,X^*)}, \|G_i\|_{\mathrm{Lip}_0(M_i,X^*)}\} \leq 1+\delta.$$

Since the pair (M,X^*) has the CEP then, for each $i \in \{1,\ldots,k\}$, we can find an extension of F_i and G_i to the whole M respectively, which we will call again F_i and G_i, respectively, such that

$$\|F_i\|_{\mathrm{Lip}_0(M,X^*)} \leq 1+\delta, \|G_i\|_{\mathrm{Lip}_0(M,X^*)} \leq 1+\delta.$$

So $\frac{F_i}{1+\delta}, \frac{G_i}{1+\delta} \in B_{\mathrm{Lip}_0(M,X^*)}$ for each $i \in \{1,\ldots,k\}$.

Step 3: we prove that $\sum_{i=1}^k \lambda_i \frac{F_i}{1+\delta} \in C$, $\sum_{i=1}^k \lambda_i \frac{G_i}{1+\delta} \in C$ and conclude from here that C has diameter 2. We prove this fact in the case M is unbounded. For the other cases, the arguments and estimates are similar. Then, assume M is unbounded. Given $i \in \{1,\ldots,k\}$ it follows

$$\varphi_i\left(\frac{F_i}{1+\delta}\right) = \frac{\sum_{j=1}^{n_i}\lambda_j^i F_i(m_{i,j})(x_{i,j})}{1+\delta} = \frac{\sum_{j=1}^{n_i}\lambda_j^i g_i(m_{i,j})(x_{i,j})}{1+\delta} = \frac{g_i(\varphi_i)}{1+\delta} > 1-\alpha.$$

So $\sum_{i=1}^k \lambda_i \frac{F_i}{1+\delta} \in C$. Similarly one has $\sum_{i=1}^k \lambda_i \frac{G_i}{1+\delta} \in C$. Hence

$$\begin{aligned}\mathrm{diam}(C) &\geq \left\|\sum_{i=1}^k \lambda_i \frac{F_i}{1+\delta} - \sum_{i=1}^k \lambda_i \frac{G_i}{1+\delta}\right\| \\ &\geq \frac{\left\|\sum_{i=1}^k \lambda_i \frac{F_i(m_N)}{1+\delta} - \sum_{i=1}^k \lambda_i \frac{G_i(m_N)}{1+\delta}\right\|}{d(m_N,0)} \\ &= \frac{\left\|\sum_{i=1}^k 2\lambda_i \frac{d(m_N,0)x^*}{1+\delta}\right\|}{d(m_N,0)} = \frac{2}{1+\delta}.\end{aligned}$$

From the above estimate and the arbitrariness of $0<\delta<\delta_0$ we deduce that $\mathrm{diam}(C) = 2$, and we are done. †

Now let us end the section by analysing the vector-valued Lipschitz-free Banach space over a concrete metric space. From here, we will get two interesting consequences: on the one hand, we will get examples of vector-valued Lipschitz-free Banach spaces which not only fail to have an octahedral norm but also its unit ball contains points of Fréchet differentiability. On the other hand, we will prove that the construction of the points of Fréchet differentiability does depend on the underlying target Banach space.

For the construction of such a metric space consider an infinite set Γ. Define $M := \Gamma \cup \{0\} \cup \{z\}$. Consider on M the following distance:

$$d(x,y) := \begin{cases} 1 & \text{if } x,y \in \Gamma\cup\{0\}, x \neq y, \\ 1 & \text{if } x = z, y \in \Gamma \text{ or } x \in \Gamma, y = z, \\ 2 & \text{if } x = z, y = 0 \text{ or } x = 0, y = z, \\ 0 & \text{Otherwise.} \end{cases}$$

This is obviously an infinite, bounded and uniformly discrete metric space. Moreover, it is not difficult to prove that the pair (M, X) has the CEP for every Banach space X. Consider a Banach space X, pick $y \in S_X$ and notice that $\delta_{z,y}$ is a 2-norm functional, so define $\varphi := \frac{\delta_{z,y}}{2} \in S_{\mathcal{F}(M,X)}$. Given $\alpha \in \mathbb{R}^+$ consider

$$S_\alpha := S\left(B_{\mathrm{Lip}_0(M,X^*)}, \varphi, \frac{\alpha}{2}\right) = \{f \in B_{\mathrm{Lip}_0(M,X^*)} \ / \ f(z)(y) > 2-\alpha\}.$$

Consider $x \in \Gamma$ and $f \in S_\alpha$. We claim that

$$f(x)(y) > 1-\alpha.$$

Indeed, assume by contradiction that $f(x)(y) \leq 1-\alpha$. Then

$$1 < f(z)(y) - f(x)(y) = (f(z)-f(x))(y) \leq \|f(z)-f(x)\| \leq d(z,x) = 1,$$

a contradiction.

We will prove that $\inf_\alpha \operatorname{diam}(S_\alpha)$ depends on the target space X^*.

Proposition 3.33. *If y is a point of Fréchet differentiability of B_X, then* $\inf_\alpha \operatorname{diam}(S_\alpha) = 0$.

Proof. Notice that, as y is a point of Fréchet differentiability, then there exists (Šmulyan lemma) $\delta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that $\delta(\varepsilon) \overset{\varepsilon\to 0}{\longrightarrow} 0$ and such that

$$\left.\begin{array}{r} x^*, y^* \in B_{X^*} \\ x^*(y) > 1-\alpha \\ y^*(y) > 1-\alpha \end{array}\right\} \Rightarrow \|x^*-y^*\| < \delta(\alpha). \tag{3.5}$$

Pick $f, g \in S\left(B_{\mathrm{Lip}_0(M,X^*)}, \varphi, \frac{\alpha}{2}\right)$ and $u, v \in M \setminus \{0\}, u \neq v$. Our aim is to estimate

$$\begin{aligned}\frac{\|f(u)-g(u)-(f(v)-g(v))\|}{d(u,v)} &\leq \|f(u)-g(u)-(f(v)-g(v))\| \leq \\ &\leq \|f(u)-g(u)\| + \|f(v)-g(v)\| =: K.\end{aligned}$$

If $u = z$ then we have

$$\frac{f(u)(y)}{2} > 1-\frac{\alpha}{2}, \frac{g(u)(y)}{2} > 1-\frac{\alpha}{2} \overset{(3.5)}{\Longrightarrow} \|f(u)-g(u)\| \leq 2\delta\left(\frac{\alpha}{2}\right).$$

Similarly, if $u \in \Gamma$ then

$$f(u)(y) > 1-\alpha, g(u)(y) > 1-\alpha \overset{(3.5)}{\Longrightarrow} \|f(u)-g(u)\| \leq \delta\left(\alpha\right).$$

Hence $K \leq \delta(\alpha) + \max\left\{\delta(\alpha), 2\delta\left(\frac{\alpha}{2}\right)\right\}$.

From the arbitrariness of $f, g \in S\left(B_{\mathrm{Lip}_0(M)}, \varphi, \frac{\alpha}{2}\right)$ we conclude that

$$\operatorname{diam}\left(S\left(B_{\mathrm{Lip}_0(M)}, \varphi, \frac{\alpha}{2}\right)\right) \leq \delta(\alpha) + \max\left\{\delta(\alpha), 2\delta\left(\frac{\alpha}{2}\right)\right\}.$$

Finally, taking infimum in $\alpha \in \mathbb{R}^+$, from the hypothesis on δ and the continuity of the map max we conclude the desired result. †

Despite the obtained result in Proposition 3.33, we will prove that $\mathcal{F}(M,X)$ has a dramatically different behaviour whenever X^* has the w^*-slice-D2P.

Proposition 3.34. *If X^* has the w^*-slice-D2P, then* $\inf_\alpha S_\alpha = 2$.

Proof. Pick two arbitrary numbers $\alpha > 0$ and $\varepsilon > 0$. As X^* has the w^*-slice-D2P we can find $x^*, y^* \in S\left(B_{X^*}, x, \frac{\alpha}{2}\right)$ such that $\|x^* - y^*\| > 2 - \varepsilon$. Now define $f, g : M \longrightarrow X^*$ by the equations

$$f(t) := d(t,0)x^* \quad g(t) := d(t,0)y^* \quad \forall t \in M.$$

Now f, g are clearly norm one Lipschitz functions. Moreover

$$\varphi(f) = \frac{f(z)(x)}{2} = x^*(x) > 1 - \frac{\alpha}{2}.$$

So $f \in S_\alpha$. Analogously $g \in S_\alpha$. Consequently

$$\operatorname{diam}(S_\alpha) \geq \|f - g\| \geq \frac{\|f(z) - g(z)\|}{2} = \|x^* - y^*\| > 2 - \varepsilon.$$

As ε and α were arbitrary we conclude that $\operatorname{diam}(S_\alpha) = 2$ for every α, so we are done. †

From the two propositions above we can get the desired consequences. From Proposition 3.33 we get vector-valued Lipschitz-free Banach spaces with points of Fréchet differentiability which, keeping in mind that the pair (M, X^*) has the CEP for every Banach space X, proves that the assumptions on the metric space in Theorem 3.32 cannot be removed. However, from Proposition 3.34 we conclude that the existence of such Fréchet differentiability point depends on the target space. Indeed, we can even get octahedrality for suitable choices of X in the above example. For instance, the norm of $\mathcal{F}(M, \ell_1) = \mathcal{F}(M)\widehat{\otimes}_\pi \ell_1 = \ell_1(\mathcal{F}(M))$ is octahedral.

3.3 Octahedrality in real Lipschitz-free spaces

In this section we will focus on the octahedrality in real Lipschitz-free Banach spaces in order to characterise when the norm of a Lipschitz-free space $\mathcal{F}(M)$ is octahedral in terms of a metric property of the underlying metric space M. That is what will be done in the following theorem, which is the main result of the section.

Theorem 3.35. *For a metric space M it is equivalent:*

1. *The norm of $\mathcal{F}(M)$ is octahedral.*

2. *For each $\varepsilon > 0$ and each finite subset $N \subset M$ there are points $u, v \in M$, $u \neq v$, such that every 1-Lipschitz function $f : N \to \mathbb{R}$ admits an extension $\tilde{f} : M \to \mathbb{R}$ which is $(1+\varepsilon)$-Lipschitz and satisfies $\tilde{f}(u) - \tilde{f}(v) \geq d(u,v)$.*

3. *For each finite subset $N \subseteq M$ and $\varepsilon > 0$, there exist $u, v \in M, u \neq v$, such that*

$$(1-\varepsilon)(d(x,y) + d(u,v)) \leq d(x,u) + d(y,v)$$

holds for all $x, y \in N$.

To prove this theorem we will need the following result, which brings to light the importance of norming subsets of Banach spaces with an octahedral norm.

Proposition 3.36. *Let X be a Banach space whose norm is octahedral and consider a norming subset $V \subseteq S_X$ for X^*. Then, given $x_1, \ldots, x_n \in S_X$ and $\varepsilon > 0$, there exists $v \in V$ such that*

$$\|x_i + v\| > 2 - \varepsilon$$

holds for every $i \in \{1, \ldots, n\}$.

Proof. The proof will strongly rely on the ideas of Theorem 2.17. Pick $x_1, \ldots, x_n \in S_X$ and $\varepsilon > 0$. Consider the convex combination of weak-star slices of B_{X^*} defined by

$$C := \frac{1}{n} \sum_{i=1}^{n} S(B_{X^*}, x_i, \varepsilon).$$

Since $\operatorname{diam}(C) = 2$ by Theorem 2.17 then there are $\frac{1}{n}\sum_{i=1}^n f_i, \frac{1}{n}\sum_{i=1}^n g_i \in C$ such that

$$\left\| \frac{1}{n} \sum_{i=1}^{n} (f_i - g_i) \right\| > 2 - \frac{\varepsilon}{n}.$$

Since V is norming for X^* we can find $v \in V$ such that

$$\frac{1}{n} \sum_{i=1}^{n} (f_i - g_i)(v) > 2 - \frac{\varepsilon}{n}.$$

It follows that $f_i(v) - g_i(v) > 2 - \varepsilon$ and consequently $f_i(v) > 1 - \varepsilon$ holds for every $i \in \{1, \ldots, n\}$. With this and since $f_i \in S(B_{X^*}, x_i, \varepsilon)$ we have

$$\|x_i + v\| \geq f_i(x_i) + f_i(v) > 1 - \varepsilon + 1 - \varepsilon = 2 - 2\varepsilon$$

for every $i \in \{1, \ldots, n\}$, and the result follows. †

Proof of Theorem 3.35. (2) $\Rightarrow$ (1): Pick finitely-supported measures $\mu_1, \ldots, \mu_n \in S_{\mathcal{F}(M)}$ and $\varepsilon > 0$. Define $N := \{0\} \cup \bigcup_{i=1}^{n} \operatorname{supp}(\mu_i)$, which is a finite subset of M. For each $i \in \{1, \ldots, n\}$ we can find $g_i \in S_{\mathrm{Lip}_0(N)}$ such that $g_i(\mu_i) = \|\mu_i\|$. By (2) we can find $u, v \in M, u \neq v$ such that, for each $i \in \{1, \ldots, n\}$, there exists $f_i \in \mathrm{Lip}_0(M)$ such that $f_i = g_i$ on N, $f_i(u) - f_i(v) \geq d(u, v)$ and $\|f_i\| \leq 1 + \varepsilon$. Pick $i \in \{1, \ldots, n\}$. Now

$$\left\| \mu_i + \frac{\delta_u - \delta_v}{d(u, v)} \right\| \geq \frac{f_i(\mu_i) + \frac{f_i(u) - f_i(v)}{d(u,v)}}{1 + \varepsilon} \geq \frac{g_i(\mu_i) + 1}{1 + \varepsilon} = \frac{\|\mu_i\| + 1}{1 + \varepsilon}.$$

Consequently, the norm of $\mathcal{F}(M)$ is octahedral, as desired.
(1) $\Rightarrow$ (3): Pick a finite subset $N \subseteq M$ and $\varepsilon > 0$. Since the norm of $\mathcal{F}(M)$ is octahedral we can find, making use of Proposition 3.36, two elements $u \neq v \in M$ such that

$$\left\| \frac{\delta_x - \delta_y}{d(x, y)} + \frac{\delta_u - \delta_v}{d(u, v)} \right\| > 2 - \varepsilon,$$

holds for every $x \neq y \in N$. Hence, given $x \neq y \in N$, there exists $f \in S_{\mathrm{Lip}_0(M)}$ such that

$$\frac{f(x)-f(y)}{d(x,y)} + \frac{f(u)-f(v)}{d(u,v)} > 2-\varepsilon.$$

This implies the following two conditions

$$\frac{f(x)-f(y)}{d(x,y)} > 1-\varepsilon, \quad \text{and} \quad \frac{f(u)-f(v)}{d(u,v)} > 1-\varepsilon.$$

Now, we have the following chain of inequalities:

$$\begin{aligned} 1 \geq \frac{f(x)-f(v)}{d(x,v)} &= \frac{f(x)-f(y)+f(u)-f(v)+f(y)-f(u)}{d(x,v)} \\ &> \frac{(1-\varepsilon)d(x,y)+(1-\varepsilon)d(u,v)-d(u,y)}{d(x,v)}. \end{aligned}$$

Consequently

$$(1-\varepsilon)(d(x,y)+d(u,v)) < d(x,v)+d(u,y).$$

Since $x \neq y \in N$ were arbitrary we conclude (3).
(3) $\Rightarrow$ (2): Let $N \subset M$ finite and $\varepsilon > 0$ be given. By the assumptions, there are $u, v \in M$, $u \neq v$, such that

$$\frac{1}{1+\varepsilon}(d(x,y)+d(u,v)) \leq d(x,u)+d(y,v)$$

for all $x, y \in N$. Given a 1-Lipschitz function f on N we define $\tilde{f}(u) = \inf\limits_{x\in N} f(x) + (1+\varepsilon)d(x,u)$, $\tilde{f}(v) = \sup\limits_{x\in N\cup\{u\}} \tilde{f}(x) - (1+\varepsilon)d(x,v)$. Here we have just used twice the classical "infimal-convolution" extension of f (see Theorem 1.5.6 in [Wea] and comments after the proof), so $\tilde{f} = f$ on N and $\tilde{f}$ is $(1+\varepsilon)$-Lipschitz on $N \cup \{u, v\}$. Hence $\tilde{f}$ admits an $(1+\varepsilon)$-Lipschitz extension to the whole of M (for example using again the infimal convolution). Since N is finite, there exist $z \in N$ and $z' \in N \cup \{u\}$ such that $\tilde{f}(u) = f(z) + (1+\varepsilon)d(z,u)$ and $\tilde{f}(v) = \tilde{f}(z') - (1+\varepsilon)d(z',v)$. If $z' = u$, we have $\tilde{f}(u) - \tilde{f}(v) = (1+\varepsilon)d(u,v)$. If $z' \neq u$, we have

$$\begin{aligned} \tilde{f}(u) - \tilde{f}(v) &= f(z) - f(z') + (1+\varepsilon)(d(z,u)+d(z',v)) \\ &\geq f(z) - f(z') + \frac{1+\varepsilon}{1+\varepsilon}(d(z,z')+d(u,v)) \geq d(u,v) \end{aligned}$$

which finishes the proof. †

Remark 3.37. Let M be a metric space and $0 \leq r < 1$. Note that, adapting the proof of Theorem 3.35, it can be proved that each of the following assertions implies the next one:

1. For every $\mu_1, \ldots, \mu_n \in S_{\mathcal{F}(M)}$ and every $\varepsilon > 0$ there exist $u \neq v \in M$ such that

$$\left\Vert \mu_i + \frac{\delta_u - \delta_v}{d(u,v)} \right\Vert \geq 2 - r - \varepsilon$$

holds for every $i \in \{1, \ldots, n\}$.

2. For each finite subset $N \subseteq M$ and $\varepsilon > 0$, there exist $u, v \in M, u \neq v$, such that
$$(1 - r - \varepsilon)(d(x, y) + d(u, v)) \leq d(x, u) + d(y, v)$$
holds for all $x, y \in N$.

3. For each $\varepsilon > 0$ and each finite subset $N \subset M$ there are points $u, v \in M$, $u \neq v$, such that every 1-Lipschitz function $f : N \to \mathbb{R}$ admits an extension $\tilde{f} : M \to \mathbb{R}$ which is $\frac{1}{1-r-\varepsilon}$-Lipschitz and satisfies $\tilde{f}(u) - \tilde{f}(v) \geq d(u, v)$.

4. For every $\mu_1, \ldots, \mu_n \in S_{\mathcal{F}(M)}$ and every $\varepsilon > 0$ there exists $u \neq v \in M$ such that
$$\left\| \mu_i + \frac{\delta_u - \delta_v}{d(u, v)} \right\| \geq 2 - 2r - \varepsilon$$
holds for every $i \in \{1, \ldots, n\}$.

We do not know whether (3) actually implies (1). Moreover, notice that Theorem 3.35 is the particular case of the above implications whenever $r = 0$. Finally, notice that assertion (1) is equivalent to the fact that the *Whitley's thickness index of* $\mathcal{F}(M)$ is greater than or equal to $2 - r$ (we refer to [CPS] and references therein for formal definitions and background on such index).

Theorem 3.35 motivates the following definition.

Definition 3.38. Let M be a metric space. We will say that M has the *long trapezoid property (LTP)* if, for each finite subset $N \subseteq M$ and $\varepsilon > 0$, there exist $u, v \in M, u \neq v$, such that
$$(1 - \varepsilon)(d(x, y) + d(u, v)) \leq d(x, u) + d(y, v)$$
holds for all $x, y \in N$.

Note that, in terms of the LTP, Theorem 3.35 reads as follows: a metric space M has the LTP if, and only if, the norm of $\mathcal{F}(M)$ is octahedral.

To begin with, let us reprove the scalar version of Theorem 3.32 in terms of the LTP condition.

Example 3.39. Any of the following properties implies that a metric space M has the LTP.

1. M is unbounded.

2. $\inf\limits_{x \neq y} d(x, y) = 0$.

Proof. Pick a finite subset $N \subseteq M$ and $\varepsilon > 0$. In order to prove (1), consider $v = 0$. Then, if $d(0, u)$ is large enough, we have for every $x, y \in N$ that
$$\frac{d(x, y) + d(u, 0)}{d(x, u) + d(y, 0)} \leq \frac{1 + \frac{d(x,y)+d(x,0)}{d(x,u)}}{1 + \frac{d(y,0)}{d(x,u)}} < \frac{1}{1 - \varepsilon}.$$

In order to prove (2), let $\theta = \inf_{x\neq y\in N} d(x,y)$ and find $u, v \in M$, $u \neq v$, such that $d(u,v) < \frac{\varepsilon\theta}{2}$. Then, for every $x, y \in N$, we have

$$\begin{aligned} d(x,y) + d(u,v) &\leq d(x,u) + d(y,v) + 2d(u,v) \\ &\leq d(x,u) + d(y,v) + \varepsilon(d(x,y) + d(u,v)). \end{aligned}$$

This proves (2). †

We will apply Theorem 3.35 to prove two stability results for the LTP. But first, we have to state a preliminary result concerning the octahedrality in ℓ_1-sums of Banach spaces. Though this proposition may be well known for specialist, we will include a proof for easy reference.

Proposition 3.40. *Let X and Y be Banach spaces. Then the norm of $X\oplus_1 Y$ is octahedral if and only if the norm of X or the norm of Y is octahedral.*

Proof. The sufficiency is proved in [HLP1, Proposition 3.10]. Let us prove the necessity. We will assume that the norms of X and Y both fail to be octahedral and we will prove that the norm of $Z := X \oplus_1 Y$ is not octahedral. In order to do that we will prove that $Z^* = X^* \oplus_\infty Y^*$ fails the w^*-SD2P. By assumptions both X^* and Y^* fail the w^*-SD2P, hence there are two convex combinations of weak-star slices of the following form

$$C_1 := \tfrac{1}{m}\textstyle\sum_{i=1}^m S(B_{X^*}, \hat{x}_i, \alpha), \quad C_2 := \tfrac{1}{n}\textstyle\sum_{i=1}^n S(B_{Y^*}, \hat{y}_i, \alpha)$$

such that $\operatorname{diam}(C_1) < 2$ and $\operatorname{diam}(C_2) < 2$. Assume, with no loss of generality, that $n \geq m$, and define

$$C := \frac{1}{n}\left(\sum_{i=1}^m S(B_{X^*}, \hat{x}_i, \alpha) \times S(B_{Y^*}, \hat{y}_i, \alpha) + \sum_{i=m+1}^n B_{X^*} \times S(B_{Y^*}, \hat{y}_i, \alpha)\right).$$

Notice that C is a convex combination of non-empty relatively weakly-star open subsets of B_{Z^*}. Since each non-empty relatively weakly-star open subset of B_{Z^*} contains a convex combination of weak-star slices of B_{Z^*} (see the proof of [GGMS, Lemma II.1]), it is enough to prove that $\operatorname{diam}(C) < 2$. To this aim pick $\frac{1}{n}\sum_{i=1}^n (x_i, y_i), \frac{1}{n}\sum_{i=1}^n (x_i', y_i') \in C$. Now

$$\left\| \frac{1}{n}\sum_{i=1}^n ((x_i, y_i) - (x_i', y_i'))\right\| = \max\left\{ \left\| \frac{1}{n}\sum_{i=1}^n (x_i - x_i')\right\|, \left\| \frac{1}{n}\sum_{i=1}^n (y_i - y_i')\right\| \right\}.$$

Let us prove that both members of the above maximum are strictly smaller than 2. On the one hand, notice that $\frac{1}{n}\sum_{i=1}^n y_i, \frac{1}{n}\sum_{i=1}^n y_i' \in C_2$, hence $\left\|\frac{1}{n}\sum_{i=1}^n (y_i - y_i')\right\| \leq \operatorname{diam}(C_2) < 2$. On the other hand

$$\left\| \frac{1}{n}\sum_{i=1}^n (x_i - x_i')\right\| \leq \frac{1}{n}\left(\left\| \sum_{i=1}^m (x_i - x_i')\right\| + \sum_{i=m+1}^n \|x_i - x_i'\| \right).$$

Again, since $\frac{1}{m}\sum_{i=1}^m x_i, \frac{1}{m}\sum_{i=1}^m x_i' \in C_1$ we get that $\|\sum_{i=1}^m (x_i - x_i')\| \leq m \operatorname{diam}(C_1)$. So

$$\left\| \frac{1}{n}\sum_{i=1}^n (x_i - x_i')\right\| \leq \frac{1}{n}\left(m \operatorname{diam}(C_1) + (n-m)2\right) < 2$$

which finishes the proof. †

Now we will exhibit the announced stability result for the LTP.

Proposition 3.41. *Let M be a metric space. Then:*

1. *Assume that M is the ℓ_1 sum of its two subsets, say T_1, T_2, i.e. $M = T_1 \cup T_2$, $T_1 \cap T_2 = \{0\}$ and*
$$d(x,y) = d(x,0) + d(0,y)$$
for every $x \in T_1$ and every $y \in T_2$. Then, M has the LTP if, and only if, T_1 or T_2 has the LTP.

2. *If M has the LTP and N_1 is a subset of M such that $M \setminus N_1$ is finite, then N_1 has the LTP.*

Proof. (1) Notice that the assumptions imply that $\mathcal{F}(M) = \mathcal{F}(T_1) \oplus_1 \mathcal{F}(T_2)$ [KauP, Proposition 5.1]. Now the result follows applying Theorem 3.35 twice and Proposition 3.40 once in between.

(2) We assume without loss of generality that $0 \in N_1$. Let us denote $N_2 := \{0\} \cup M \setminus N_1$. Notice that if M is either unbounded or non-uniformly discrete then so is N_1. So we will assume that M is a bounded and uniformly discrete metric space. In this case the following retractions will be Lipschitz:

$$r_1(x) = \begin{cases} x & \text{if } x \in N_1 \\ 0 & \text{if } x \in N_2 \end{cases} \quad \text{and } r_2(x) = \begin{cases} 0 & \text{if } x \in N_1 \\ x & \text{if } x \in N_2 \end{cases}$$

Clearly $r_1 \circ r_2(x) = r_2 \circ r_1(x) = 0$ and so the unique linear extensions $\bar{r_i} : \mathcal{F}(M) \to \mathcal{F}(N_i)$ of r_i, $i = 1, 2$, are continuous linear projections such that $\ker r_1 = \mathcal{F}(N_2)$ and vice versa. It follows that $\mathcal{F}(M) = \mathcal{F}(N_1) \oplus \mathcal{F}(N_2)$. The norm on $\mathcal{F}(M)$ is octahedral by the hypothesis and Theorem 3.35. Since $\dim \mathcal{F}(N_2) < \infty$, [Abr, Theorem 3.9] implies that $\mathcal{F}(N_1)$ is octahedral. Now another application of Theorem 3.35 shows that N_1 has the LTP. †

Remark 3.42. The assumption in Proposition 3.41 (2) of $M \setminus N_1$ being finite can not be removed. This can be easily seen by taking the ℓ_1 sum of two infinite metric spaces, one enjoying and the other one failing the LTP, and applying Proposition 3.41 (1).

Let us now turn to an analysis of the failure of the LTP. When a metric space M fails the LTP, one might wonder whether this can be checked on a subset N consisting of mere 2 points. The next example provides a negative answer.

Example 3.43. Consider $M := \{\alpha, \beta, 0, z\} \cup \{x_n : n \in \mathbb{N}\}$ whose distance is defined in the following way:

$$d(0, x_n) = d(x_n, z) = 1, d(0, z) = 2, d(\alpha, 0) = d(\beta, 0) = 1, d(\alpha, \beta) = 2,$$
$$d(\alpha, x_n) = d(\beta, x_n) = 2, d(\alpha, z) = d(\beta, z) = 3 \text{ and } d(x_n, x_m) = 1.$$

Denote by $T := \{0, z\} \cup \{x_n : n \in \mathbb{N}\}$. Then it follows by [KauP, Proposition 5.1] that

$$\mathcal{F}(M) = \mathcal{F}(T) \oplus_1 \mathcal{F}(\{0, \alpha\}) \oplus_1 \mathcal{F}(\{0, \beta\}),$$

so the norm of $\mathcal{F}(M)$ fails to be octahedral (note that the norm of $\mathcal{F}(T)$ is not octahedral since T is the metric space considered in Proposition 3.33) and, consequently, M fails the LTP. We will prove, however, that the condition of LTP holds for every subset of M of cardinality 2. To this aim, pick $a, b \in M$. Then we have three possibilities for a and b:

1. $d(a,b)=3$. Then, up to re-labeling a and b, $b=z$ and a is either α or β. We will assume, with no loss of generality, that $a=\alpha$. Then, the choice $u=\beta$ and $v=0$ does the work.

2. $d(a,b)=2$. In this case, we still have two more possibilities:

   (a) $a=0$ and $b=z$. In this case it is enough to choose $u=\alpha, v=\beta$.

   (b) $b=x_n$ for certain $n\in\mathbb{N}$ and a is either α or β. We assume, with no loss of generality, that $a=\alpha$. In this case $u=\beta$ and $v=0$ yields the desired condition.

3. $d(a,b)=1$. In this case, choose $u\neq v$ points such that $d(u,v)=1$ and u,v being different from a and b, and the inequality trivially holds.

In spite of the previous example, the failure of the LTP can be checked on subsets of two points when we restrict our attention to a suitable metric subspace. More precisely, we get the following result.

Proposition 3.44. *Let M be a metric space failing the LTP. Then there exists an infinite subspace $A\subset M$ such that, for some $\varepsilon>0$ and some $x,y\in A$, we have*

$$(1-\varepsilon)(d(x,y)+d(u,v))>\min\{d(x,u)+d(y,v),d(x,v)+d(y,u)\}$$

for all $u,v\in A$.

Proof. There is $\varepsilon>0$ and a finite $N\subset M$ such that, for every pair $u\neq v\in M\setminus N$, we have

$$(1-\varepsilon)(d(x,y)+d(u,v))>\min\{d(x,u)+d(y,v),d(x,v)+d(y,u)\}$$

for some pair $x\neq y\in N$. Since there are only finitely many pairs $x\neq y\in N$, a direct application of Ramsey's theorem gives that there exist $x_0\neq y_0\in N$ and an infinite $A'\subset M$ such that

$$(1-\varepsilon)(d(x_0,y_0)+d(u,v))>\min\{d(x_0,u)+d(y_0,v),d(x_0,v)+d(y_0,u)\} \tag{3.6}$$

for every $u\neq v\in A'$. If $\{x_0,y_0\}\subset A'$, the result is true for $A:=A'$. If not, we denote

$$A(x):=\{z\in A':\ (3.6)\text{ fails for } u=x, v=z\}.$$

Note that

$$A(x_0):=\{z\in A':d(y_0,z)\geq(1-\varepsilon)(d(x_0,y_0)+d(x_0,z))\},$$

that $A(x_0)\cap\{x_0,y_0\}=\emptyset$, and that similar properties hold for $A(y_0)$. We put

$$A:=\{x_0,y_0\}\cup A'\setminus(A(x_0)\cup A(y_0)).$$

We claim that $A(x_0)$, resp. $A(y_0)$, is a singleton at most. In order to get a contradiction assume that $z\neq w\in A(x_0)$. We have, without loss of generality, the following inequality

$$\begin{aligned}(1-\varepsilon)(d(x_0,y_0)+d(z,w))&>d(x_0,z)+d(y_0,w)\\&\geq d(x_0,z)+(1-\varepsilon)(d(x_0,y_0)+d(x_0,w))\\&=(1-\varepsilon)(d(x_0,y_0)+d(x_0,z)+d(x_0,w))+\varepsilon d(x_0,z)\\&\geq(1-\varepsilon)(d(x_0,y_0)+d(z,w))+\varepsilon d(x_0,z),\end{aligned}$$

which is absurd. Hence $|A(x_0)|\leq 1$. An identical proof shows that $|A(y_0)|\leq 1$. It follows that A is infinite and the proposition follows. †

A prominent class of non-octahedral norms are the norms that admit a point of Fréchet differentiability. Theorem 3.32 implies that for the norm of $\mathcal{F}(M)$ this can happen only when M is uniformly discrete and bounded. Though this is not the general case as we will see in Proposition 3.49, in Proposition 3.33 an example of a metric space M such that $\mathcal{F}(M)$ admits a point of Fréchet differentiability appears. In the following result we will take a closer look at this phenomenon.

Theorem 3.45. *Let M be a uniformly discrete bounded metric space. Consider $x_1, \ldots, x_n, y \in M$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}^+$ such that $\sum_{i=1}^n \lambda_i = 1$. Define $\varphi := \sum_{i=1}^n \lambda_i \frac{\delta_{x_i} - \delta_y}{d(x_i, y)}$. The following are equivalent:*

1. *φ is a Fréchet differentiability point of $\mathcal{F}(M)$.*
2. *Given $z \in M$ there exists $i \in \{1, \ldots, n\}$ such that*
$$d(x_i, y) = d(x_i, z) + d(z, y).$$
3. *φ is a Gâteaux differentiability point of $\mathcal{F}(M)$.*

Proof. We will assume with no loss of generality that $y = 0$.

(2)⇒(1). Pick $\varepsilon > 0$ and $f \in B_{\mathrm{Lip}_0(M)}$ such that $\varphi(f) = \sum_{i=1}^n \lambda_i \frac{f(x_i)}{d(x_i,0)} > 1 - \frac{\varepsilon}{\min\limits_{1\leq i\leq n} \lambda_i}$. An easy convexity argument yields that $f(x_i) > (1-\varepsilon)d(x_i, 0)$ for each $i \in \{1, \ldots, n\}$. Pick an element $z \in M$. By assumptions there exists $i \in \{1, \ldots, n\}$ such that $d(x_i, 0) = d(x_i, z) + d(z, 0)$. Now

$$\begin{aligned} d(z,0) \geq f(z) &\geq f(x_i) - |f(z) - f(x_i)| > (1-\varepsilon)d(x_i,0) - d(x_i,z) \\ &= (1-\varepsilon)(d(x_i,z) + d(z,0)) - d(x_i,z) = d(0,z) - \varepsilon d(x_i,0) \end{aligned}$$

We thus have $|f(z) - d(z,0)| < \varepsilon d(x_i, 0) < \varepsilon \operatorname{diam}(M)$. Consequently, one has

$$\|f - d(\cdot, 0)\| \leq C \|f - d(\cdot, 0)\|_\infty \leq \varepsilon C \operatorname{diam}(M)$$

where $C \geq 1$ is the constant of equivalence between the Lipschitz and the uniform norm on $\mathrm{Lip}_0(M)$ (we recall that $\mathrm{Lip}_0(M)$ is isomorphic to $\ell_\infty(M \setminus \{0\})$ as M is uniformly discrete and bounded). According to Šmulyan lemma, φ is a point of Fréchet differentiability (with $d(\cdot, 0) \in \mathcal{F}(M)^*$ being the differential).

(1)⇒(3) is obvious.

(3)⇒(2). Assume that for some $z \in M$, (2) does not hold for any x_j and let us prove that (3) does not hold either. To see that define $f_i : \{0, x_1, \ldots, x_n, z\} \to \mathbb{R}$ for $i = 1, 2$ as follows: $f_i(0) = 0$, $f_i(x_j) = d(0, x_j)$ for every $j \in \{1, \ldots, n\}$, $f_1(z) = d(0, z)$ and $f_2(z) = \max\{-d(0, z), \max\limits_{1\leq i\leq n} d(x_i, 0) - d(z, x_i)\}$. We clearly have that $\|f_i\| = 1$, $f_i(\varphi) = 1$ and $f_1 \neq f_2$. Indeed, by assumptions $f_2(z) < d(z, 0) = f_1(z)$. Now the respective norm-one extensions $\tilde{f}_i$ of f_i, $i = 1, 2$ show that φ is not a point of Gâteaux differentiability. †

Let X be a Banach space whose norm $\|\cdot\|$ is not octahedral. It is easily seen that there exists $\varepsilon > 0$ such that every norm $|\cdot|$ which satisfies

$$\frac{1}{1+\varepsilon} \|x\| \leq |x| \leq (1+\varepsilon) \|x\|$$

is non-octahedral. Let now (M, d) be a bounded uniformly discrete metric space which fails the LTP. Then it follows from the above and from Theorem 3.35 that there exists $\varepsilon > 0$ such that every metric d' on M which satisfies

$$\frac{1}{1+\varepsilon} d(x,y) \leq d'(x,y) \leq (1+\varepsilon) d(x,y)$$

fails the LTP too.

We single out a particular example of this fact. In what follows we will work with the metric graph $M = \{0, z\} \cup \{x_i : i \in \mathbb{N}\}$ where the edges are the pairs of the form $\{0, x_i\}$ or $\{x_i, z\}$ and the metric d is the shortest path distance.

Lemma 3.46. *Let d' be a metric on M such that (M, d) and (M, d') are Lipschitz equivalent with distortion $D < 2$. Then (M, d') fails the LTP.*

Notice that the countable equilateral space is 2-Lipschitz equivalent to (M, d) so the above lemma is optimal.

Proof. By the hypothesis there are $D < 2$ and $s > 0$ such that

$$\frac{s}{D} d(x,y) \leq d'(x,y) \leq s d(x,y)$$

for all $x, y \in M$. Since the LTP is invariant under scaling of the metric, we may assume that $s = 1$. We are going to show that for $N = \{0, z\}$, $0 < \varepsilon < 1 - \frac{D}{2}$ and all $u, v \in M$ we have

$$A := (1-\varepsilon)(d'(0,z) + d'(u,v)) > \min\{d'(0,u) + d'(z,v), d'(0,v) + d'(z,u)\} =: B.$$

When $(u, v) = (x_n, x_m)$ we have $A > 2 \geq B$. When $(u, v) = (0, x_n)$ we have $A > \frac{3}{2} > 1 \geq B$. The same relation holds when $(u, v) = (z, x_n)$. †

Proposition 3.47. *For every $1 < p < \infty$, the above space (M, d) embeds into ℓ_p with distortion $D < 2$. Consequently, ℓ_p contains a subset A failing the LTP.*

Proof. Let $1 < p < \infty$ be fixed. We define $\phi : M \to \ell_p$ as $\phi(0) = -e_1$, $\phi(z) = e_1$ and $\phi(x_i) = 2^{\frac{p-1}{p}} e_i$, where (e_i) is the canonical basis of ℓ_p. A routine computation shows that the distortion of ϕ is $\sqrt[p]{1 + 2^{p-1}}$ which is strictly less than 2 for $p > 1$. It follows from Lemma 3.46 that $(\phi(M), \|\cdot\|_p)$ fails the LTP. †

It is shown in [Yag, Proposition 3.4] that if X is a separable L-embedded space, then the norm of every non-reflexive subspace Y of X is octahedral. We thus get the following.

Corollary 3.48. *Let M be a separable metric space which contains an infinite subset without the LTP. Then $\mathcal{F}(M)$ is not L-embedded. In particular, $\mathcal{F}(\ell_p)$ is not L-embedded when $1 < p < \infty$.*

Even though we prove below that ℓ_1 has no subset without the LTP, we do not know whether $\mathcal{F}(\ell_1)$ is L-embedded. In fact, it is a famous open problem whether $\mathcal{F}(\ell_1)$ is even complemented in its bidual. Similarly, it is not known whether $\mathcal{F}(\mathbb{R}^n)$, $n \geq 2$, is L-embedded for some norm on $\mathbb{R}^n$. Nevertheless, it has been recently shown in [CKK] that the spaces $\mathcal{F}(\mathbb{R}^n)$ are complemented in their biduals.

The distortion of the embedding in Proposition 3.47 tends to 2 when $p \to \infty$ or $p \to 1$. In the case of $p \to \infty$, this is not of fundamental importance. Indeed, one can easily embed isometrically (M, d) into c, the space of convergent sequences. Similarly, one can easily embed isometrically the space considered in Proposition 3.33 into c_0. Thus both c and c_0 contain subsets failing the LTP.

On the other hand the behaviour of the distortion when $p \to 1$ is a manifestation of a fundamental fact that we will present next.

We need to introduce the following concepts. Given a Banach space $(X, \|\cdot\|)$ it is said that X is *asymptotically uniformly convex (AUC)* if, for every $t > 0$, the following inequality holds

$$\overline{\delta}_X(t) := \inf_{x \in S_X} \sup_{codim(Y) < \infty} \inf_{y \in S_Y} \|x + ty\| - 1 > 0.$$

The function $\overline{\delta}_X$ is called the *modulus of asymptotic uniform convexity* of X and it was introduced in [Mil] (see also [JLPS1] for some further properties of this modulus). It is clear that $\overline{\delta}_X(t) \leq t$ holds for every $t > 0$. Moreover, $X = \ell_1$ satisfies that $\overline{\delta}_X(t) = t$ for all $t \in \mathbb{R}^+$.

Proposition 3.49. *Let X be an AUC Banach space such that $\overline{\delta}_X(t) = t$ holds for all $t \geq 0$. Then every infinite subset of X has the LTP.*

In particular, for every infinite subset M of X it follows that the norm of $\mathcal{F}(M)$ is octahedral.

In particular, the previous proposition applies for every metric space which is an $\mathbb{R}$-*tree* (see [Go] for formal definition of $\mathbb{R}$-trees and their importance in the theory of Lipschitz-free spaces), since those spaces embed isometrically embeds into ℓ_1 of the corresponding density. Even though the last claim seems to be quite natural, the only proof we know of is in [JLPS2] where it is proved in Proposition 4.1 that the notion of a separable $\mathbb{R}$-tree coincides with the notion "SMT" introduced in that paper. It is proved in [JLPS2, Corollary 2.1] that every SMT embeds isometrically into ℓ_1. The non-separable case follows the same lines, using transfinite induction.

In the proof of Proposition 3.49 we shall need the following lemma.

Lemma 3.50. *Let X be a Banach space such that $\overline{\delta}_X(1) = 1$. Then, for every $x \in X$ and every $\varepsilon > 0$ there exists a finite-codimensional subspace $Y \subseteq X$ such that, for every $y \in Y$, it follows*

$$\|x + y\| \geq (1 - \varepsilon)(\|x\| + \|y\|) \text{ for all } y \in Y.$$

In particular, $\overline{\delta}_X(t) = t$ holds for every $t > 0$.

Proof. Pick $x \in X \setminus \{0\}$ and $\varepsilon > 0$. Since $\overline{\delta}_X(1) = 1$, then there exists a finite-codimensional subspace Y of X such that, for every $y \in S_Y$, it follows

$$\left\| \frac{x}{\|x\|} + y \right\| \geq 2 - \varepsilon.$$

Call $z = \frac{x}{\|x\|}$. Consider $t_1, t_2 \in [0, 1]$ such that $t_1 + t_2 = 1$ and assume, with no loss of generality, that $t_1 \geq t_2$. Then

$$\begin{aligned}\|t_1 y + t_2 z\| = \|t_1(z + y) + (t_2 - t_1)z\| &\geq t_1\|z + y\| - (t_1 - t_2) \\ &> t_1(2 - \varepsilon) + t_2 - t_1 \geq t_1 + t_2 - \varepsilon = 1 - \varepsilon.\end{aligned}$$

Finally, given $y \in Y$, from the previous estimates we get

$$\frac{\|x+y\|}{\|x\|+\|y\|} = \left\| \frac{\|x\|}{\|x\|+\|y\|} \frac{x}{\|x\|} + \frac{\|y\|}{\|x\|+\|y\|} \frac{y}{\|y\|} \right\| \geq 1 - \varepsilon,$$

and the lemma follows. †

Proof of Proposition 3.49. In order to get a contradiction assume that there exists an infinite subset $A \subseteq X$ failing the LTP. By Proposition 3.44 we can assume, with no loss of generality, that there are $\varepsilon_0 > 0$ and $x \neq y \in A$ such that, for every $u \neq v \in A$, we get

$$(1-\varepsilon_0)(\|x-y\| + \|u-v\|) > \min\{\|x-u\| + \|y-v\|, \|x-v\| + \|y-u\|\}.$$

Since $\overline{\delta}_X(1) = 1$ we conclude the existence of a finite-codimensional subspace $Y \subseteq X$ such that, for all $z \in Y$, it follows

$$\|x-y+z\| \geq (1-\varepsilon)(\|x-y\| + \|z\|),$$

where $0 < \varepsilon < \varepsilon_0$. Since Y is finite-codimensional in X we can find a finite-dimensional subspace $F \subseteq X$ such that $X = Y \oplus F$. Consider P and Q to be the corresponding linear and continuous projections onto Y and F respectively. Note that, since F is finite-dimensional, Q is bounded and A is bounded then we can find $B \subseteq A$ such that, for every $u \neq v \in B$, we have that $\|Q(u-v)\| < \frac{\varepsilon_0 - \varepsilon}{4}\|x-y\|$. Now, for fixed $u \neq v$ in B, we have

$$(1-\varepsilon_0)(\|x-y\| + \|u-v\|) > \min\{\|x-u\| + \|y-v\|, \|x-v\| + \|y-u\|\}.$$

We can assume, with no loss of generality, that the following inequality holds:

$$(1-\varepsilon_0)(\|x-y\| + \|u-v\|) > \|x-u\| + \|y-v\|. \tag{3.7}$$

Now

$$\begin{aligned} \|x-u\| + \|y-v\| &\geq \|x-y-(u-v)\| = \|x-y-P(u-v)-Q(u-v)\| \\ &\geq \|x-y-P(u-v)\| - \|Q(u-v)\|. \end{aligned}$$

Since $P(u-v) \in Y$ we conclude that $\|x-y-P(u-v)\| > (1-\varepsilon)(\|x-y\| + \|P(u-v)\|)$. Consequently, using this joint to (3.7), we get

$$(1-\varepsilon_0)(\|x-y\| + \|u-v\|) > (1-\varepsilon)(\|x-y\| + \|P(u-v)\|) - \|Q(u-v)\|.$$

Now, the triangle inequality implies that $\|u-v\| \leq \|P(u-v)\| + \|Q(u-v)\|$. Consequently, the previous inequalities imply

$$0 \geq (\varepsilon_0 - \varepsilon)(\|x-y\| + \|P(u-v)\|) - 2\|Q(u-v)\| > \frac{\varepsilon_0 - \varepsilon}{2}\|x-y\|,$$

which is a contradiction. Consequently, we conclude that there exists no subset A of X failing the LTP, so we are done. †

Remark 3.51. It is well known and easily seen, that if a Banach space X satisfies that for every $\varepsilon > 0$ there are finite dimensional spaces E_n, $n \in \mathbb{N}$, such that X is $(1+\varepsilon)$-isomorphic to a subspace of $(\oplus_{n=1}^{\infty} E_n)_{\ell_1}$ then $\overline{\delta}_X(t) = t$. Up to our knowledge it is an open problem whether every Banach space such that $\overline{\delta}_X(t) = t$ must be of this form.

On the other hand, many (Lipschitz-free) Banach spaces satisfying this hypothesis have been pinned down by C. Petitjean in [Pet]. Among others, one such example is the space $\mathcal{F}(M)$ whenever M is a countable compact space.

3.4 Remarks and open questions

In this section we will recollect some research lines, remarks and open questions related to the present chapter.

3.4.1 Section 3.1

Theorems 3.5 and 3.9 were generalised to spaces of weak∗-to-weak continuous operators in [LLR1, Theorem 2.2]. Indeed, the following result was proved there.

Proposition 3.52. *Let X and Y be Banach spaces and let $H \subseteq L(X^*, Y)$ be a closed subspace such that $X \otimes Y \subseteq H$. Assume that each $T \in H$ is weak*-to-weakly continuous.*

1. *If the norms of X and Y are octahedral, then the norm of H is octahedral.*
2. *If the norm of X is octahedral and there exists $y \in S_Y$ such that the set $\{y^* \in S_{Y^*} : y^*(y) = 1\}$ is norming for Y, then the norm of H is octahedral..*

The main interest in the generalisation given in Proposition 3.52 is to cover the case of H being the injective tensor product of X and Y. As a consequence of Proposition 3.52 and Theorem 3.18 the preservation of the octahedrality by injective tensor product is completely obtained: octahedrality is preserved by taking injective tensor product from both factors but not from one of them (and a counterexample is $\ell_1 \widehat{\otimes}_\varepsilon \ell_p^n$ for every $n \geq 3$ and $1 < p < 2$, according to Lemma 3.17). Finally, concerning further generalisations of the results of Section 3.1, in relation to Theorem 3.22, the connection between octahedrality in spaces of operators and finite representability in ℓ_1 was put further in [Rue2] where the following result, which generalises Theorem 3.22, was proved.

Proposition 3.53. *Let Y be a Banach space which embeds isometrically in L_1 and has a monotone basis. Let X be a Banach space whose norm is octahedral and consider a subspace $H \subseteq L(Y, X)$ containing the space of finite-rank operators. Then the operator norm on H is octahedral.*

As a consequence of this result and since every 2-dimensional real Banach space is isometrically isomorphic to a subspace of L_1 [Dor, Corollary 1.4], it follows that $X \widehat{\otimes}_\pi Y$ has the SD2P whenever X has the SD2P and $\dim(Y) = 2$.

Further generalisations of results of Section 3.1 were considered in [HLP2].

Let us end with some open problems related with the content of Section 3.1. First, we point out the following problem posed in [D.Wer2, Section 6. (3)].

Question 8. *If X and Y have the Daugavet property, does $X \widehat{\otimes}_\pi Y$ and $X \widehat{\otimes}_\varepsilon Y$ have the Daugavet property?*

First of all, notice that a partial answer has been given in [RTV].

Furthermore, in this direction, in [KSW] it was shown that there exists a complex 2-dimensional Banach space E such that $L_1^{\mathbb{C}}([0,1]) \widehat{\otimes}_\varepsilon E$ and $L_\infty^{\mathbb{C}}([0,1]) \widehat{\otimes}_\pi E^*$ fail the Daugavet property, answering by the negative the question also posed in [D.Wer2] whether the Daugavet property is preserved by projective or injective tensor product from just one of the factors. Apart from this result, the study of the Daugavet property in tensor product spaces have just appeared by showing new particular examples (see [BR1] for examples coming from representation theory of Banach spaces, [Rue1] for examples coming from an additional L-embedded structure and [GPrR] for examples where one factor is a Lipschitz-free space).

A very related problem is the following.

Question 9. *When is the norm of $X\widehat{\otimes}_\pi Y$ octahedral?*

Note that in [LLR1, Question 4.4] it is particularly asked whether octahedrality is preserved by projective tensor product from just one factor. Two partial positive answers to this problem are the following:

Proposition 3.54. *Let X and Y be two Banach spaces.*

1. *[LLR1, Corollary 2.9] If X is ASQ, Y is Asplund and if either X^* or Y^* has the approximation property, then the norm of $X^*\widehat{\otimes}_\pi Y^*$ is octahedral.*

2. *[LLR2, Theorem 4.3] If X is a non-reflexive L-embedded space and either X^{**} or Y has the metric approximation property, then the norm of $X\widehat{\otimes}_\pi Y$ is octahedral.*

3.4.2 Section 3.2

Different ways in which the space $\mathrm{Lip}_0(M,\mathbb{R})$ enjoy the diameter two properties have been studied in the literature. In [Iva] it is proved that $\mathrm{Lip}_0(M)$ has the slice-D2P whenever M is unbounded or bounded but not uniformly discrete. Furthermore, in [IKW] it is proved that $\mathrm{Lip}_0(M)$ has the Daugavet property whenever M is an *almost metrically convex metric space*, spaces where the distance of each pair of points can be approximated by the length of rectifiable curves joining them (see [IKW] for formal definitions). See also the recent paper [HLLN] where it is analysed the so-called *w^*-symmetric strong diameter two property.*

Under the assumptions of Theorem 3.32 we have that $\mathrm{Lip}_0(M,X^*)$ has the w^*-SD2P. This rises a natural question.

Question 10. *Let M and X under the hypothesis of Theorem 3.32. Does $\mathrm{Lip}_0(M,X^*)$ satisfy the SD2P?*

Note that in the recent paper [CCGMR] it has been proved that this is the case if M' is infinite and if M is discrete but not uniformly discrete. Furthermore, it turns out to be also the case when M is infinite and compact [CCGMR, Section 5].

Furthermore, Propositions 3.33 and 3.34 show that geometry of vector-valued Lipschitz-free Banach spaces does not only depend on underlying scalar Lipschitz-free space but also on the target Banach space. This fact makes natural to pose the following question.

Question 11. *Let M be a pointed metric space and let X be a non-zero Banach space.*

1. *Does Theorem 3.32 hold without assuming that the pair (M,X^*) has the CEP?*

2. *Does $\mathcal{F}(M,X)$ have an octahedral norm whenever $\mathcal{F}(M)$ does?*

Bearing in mind the identification $\mathcal{F}(M,X) = \mathcal{F}(M)\widehat{\otimes}_\pi X$, the above question is related to Question 9. Furthermore, particular examples where no CEP assumption is needed are given in [GR, Remark 3.2] and [GPR, Remark 2.12].

Finally, we have analysed octahedrality in $\mathcal{F}(M,X)$ whenever M is a metric space and X is a Banach space. However, we did not get any result about the dual properties (i.e. diameter two properties). More precisely.

Question 12. *Given M a metric space and X a non-zero Banach space.*

Which assumptions do we need over M and X in order to ensure that $\mathcal{F}(M,X)$ has the slice-D2P (respectively D2P, SD2P)?

Note that a complete answer to this question has been recently given if $X=\mathbb{R}$. Indeed, putting all together [AM, GPrR, IKW], it follows that given a complete metric space M then either $\mathcal{F}(M)$ has the Daugavet property or its unit ball is dentable (actually $B_{\mathcal{F}(M)}$ has a strongly exposed point).

Also, taking into account the equality $\mathcal{F}(M,X) = \mathcal{F}(M)\widehat{\otimes}_\pi X$, general results in the real case joint with the results of Section 3.1 yield examples of vector-valued free spaces with the SD2P.

3.4.3 Section 3.3

Note that the fact that 1 and 2 are equivalent in Theorem 3.35 is not a special case of Lipschitz-free spaces but a manifestation of a general equivalence which holds for a general Banach space. Indeed, in [Lan, Theorem 3.21] the following result is proved (which should be compared with the first two assertions of Theorem 3.35).

Proposition 3.55. *Let X be a Banach space. The following assertions are equivalent:*

1. *The norm of X is octahedral.*
2. *For every finite-dimensional subspace E of X, every $n\in\mathbb{N}$, every $x_1^*,\ldots,x_n^*\in B_{X^*}$, every $\varepsilon>0$ and $0<\varepsilon_0<\varepsilon$ then there exists $y\in S_X$ with the property that given $|\gamma_i|\leq 1+\varepsilon_0$, $i\in\{1,\ldots,n\}$, we can find $y_i^*\in X^*$ for $i\in\{1,\ldots,n\}$ such that, for every $i\in\{1,\ldots,n\}$, it follows:*
   (a) *y_i^* and x_i^* agree on E;*
   (b) *$y_i^*(y)=\gamma_i$;*
   (c) *$\|y_i^*\|\leq 1+\varepsilon$.*

In Proposition 3.41 we have cited the preprint version of [Kau] since [KauP, Proposition 5.1] does not appear in the final published version.

In his thesis, L. García-Lirola proved that every *asymptotically uniformly smooth* Banach space contains an infinite subset failing the LTP [Gar, Proposition 5.1.17], which extends Proposition 3.47.

Conclusions

In this thesis we have dealt with diameter two properties and related isometric properties of Banach spaces such as octahedrality of the norms and almost square Banach spaces.

After an introductory chapter about basic aspects of diameter two properties, first examples, stability results in the literature and relations with other geometric properties of Banach spaces (Chapter 1), we can obtain the following consequences:

- All the diameter two properties are different in an extreme way. In fact, from Theorems 2.4 and 2.8 we get that every Banach space containing an isomorphic copy of c_0 can be equivalently renormed to have the slice-D2P (respectively the D2P) and its new unit ball contains non-empty relatively weakly open subsets (respectively convex combination of slices) of arbitrarily small diameter. The isomorphic nature of the above results also implies the abundance of such extreme examples.

- In Section 2.3 we have considered almost square Banach spaces, a geometric property of Banach spaces which is stronger than the SD2P. In Theorem 2.14 we have shown that a Banach space X admits an equivalent renorming to be ASQ if, and only if, X contains an isomorphic copy of c_0. An interesting feature of that result is that almost squareness is a geometric property of Banach spaces strongly related to the diameter two properties which characterises, up to considering an equivalent renorming, the fact that a Banach space contains an isomorphic copy of c_0.

- The main consequence of Section 2.4 is that the SD2P and the octahedrality of a norm are dual properties, in the sense that the norm of a Banach space X is octahedral if, and only if, X^* has the w^*-SD2P (Theorem 2.17). This characterisation establishes a bridge of information which allows to get consequences about octahedrality of the norm by studying the w^*-SD2P in the dual, and vice versa. This bridge of information is exploited until the end of the thesis. Indeed, Proposition 2.23 is the first example of this fact where the duality between octahedrality and the SD2P is used in order to get a partial answer to the question whether every Banach space containing an isomorphic copy of ℓ_1 can be equivalently renormed so that the bidual norm is octahedral, a problem coming from [God2, Remark II.5].

- In Section 3.1, a deep analysis of octahedrality in $L(X, Y^*)$ or, equivalently, of the SD2P in $X\widehat{\otimes}_\pi Y$ is considered. One of the main consequences obtained there is that if X and Y have the SD2P then so does $X\widehat{\otimes}_\pi Y$, but in the above sentence we can not replace "and" with "or". Furthermore, we conclude from Lemma 3.17 and Theorem 3.22 that there is a strong relation between octahedrality of the operator norm in spaces of operators and the theory of finite representability.

- The techniques involving the results of Section 3.2 reveal two central consequences. First, the use of the duality between octahedral norms and the w^*-SD2P exposed in Section 2.4 is fundamental to derive consequences about octahedrality in spaces $\mathcal{F}(M,X)$. Also, it is essential, in view of those techniques, the assumption of the CEP. Further, the two final results of Section 3.2 derive two important conclusions. On the one hand, in spite of the abundance of Lipschitz-free spaces whose norm is octahedral, it is possible for these spaces to contain points of Fréchet differentiability. On the other hand, the fact that the norm of $\mathcal{F}(M,X)$ is octahedral depends on the underlying metric space M as well as on the target Banach space X.

- From the results of Section 3.2 it follows that if the norm of $\mathcal{F}(M)$ is not octahedral then M has to be uniformly discrete and bounded. However, thanks to the characterisation given in Theorem 3.35, it follows that there are a lot of uniformly discrete bounded metric spaces M such that the norm of $\mathcal{F}(M)$ is octahedral, and an example of the abundance of such metric spaces is given in Proposition 3.49. Further, Lemma 3.36 reveals, again making a strong use of the duality between octahedral norms and the w^*-SD2P, that norming subsets of the sphere play an important role in the octahedrality of a given norm.

Conclusiones

En esta tesis hemos abordado el estudio de las propiedades de diámetro dos y propiedades geométricas relacionadas tales como la octaedralidad de una norma o los espacios de Banach casi cuadrados.

Después de un capítulo introductorio sobre resultados básicos relativos a propiedades de diámetro dos, ejemplos, resultados de estabilidad y relaciones con otras propiedades geométricas de espacios de Banach (Capítulo 1), podemos obtener las siguientes consecuencias.

- Todas las propiedades de diámetro dos son diferentes entre sí en un sentido extremo. Más concretamente, de los Teoremas 2.4 y 2.8 obtenemos que todo espacio de Banach que contenga una copia isomorfa de c_0 puede renormarse equivalentemente para tener la slice-D2P (respectivamente la D2P) y para que su bola unidad contenga abiertos débiles no vacíos (respectivamente combinaciones convexas de rebanadas) de diámetro arbitrariamente pequeño. Además, la naturaleza isomórfica de este resultado nos indica la abundancia de tales contraejemplos.

- En la Sección 2.3 hemos considerado los espacios casi cuadrados, una propiedad geométrica que es (estrictamente) más fuerte que la SD2P. En el Teorema 2.14 demostramos que un espacio de Banach X admite una renormación equivalente para ser ASQ si, y solamente si, el espacio contiene una copia isomorfa de c_0. Como consecuencia obtenemos que la propiedad de ser ASQ, que es una propiedad estrechamente relacionada con las propiedades de diámetro dos, caracteriza, bajo renormación equivalente, el hecho de que un espacio contenga una copia isomorfa de c_0.

- La principal consecuencia de la Sección 2.4 es que la SD2P y la octaedralidad de la norma son propiedades duales, en el sentido de que la norma de un espacio de Banach es octaedral si, y solamente si, X^* tiene la w^*-SD2P (Teorema 2.17). Esta caracterización establece un puente de información que permite obtener consecuencias sobre octaedralidad de normas en términos de la w^*-SD2P en el dual, y vice versa. De hecho, la Proposición 2.23 es el primer lugar donde esta dualidad entre octaedralidad y SD2P es empleada para dar una respuesta parcial a la pregunta de si todo espacio de Banach que contiene una copia isomorfa de ℓ_1 admite una renormación equivalente de manera que la norma bidual es octaedral, un problema que proviene de [God2, Remark II.5]

- En la Sección 3.1 hacemos un profundo análisis sobre la octaedralidad de la norma de operadores en $L(X, Y^*)$ o, equivalentemente, de la SD2P en $X\widehat{\otimes}_\pi Y$. Una de las principales consecuencias que obtenemos es que si X e Y tienen la SD2P, entonces $X\widehat{\otimes}_\pi Y$

también tiene la SD2P, pero en el enunciado anterior no podemos reemplazar "e" por "o". Además, del Lema 3.17 y del Teorema 3.22 concluimos que, bajo hipótesis de convexidad uniforme, existe una fuerte relación entre la octaedralidad de la norma de operadores en espacios de operadores acotados y la teoría de representabilidad finita.

- Las técnicas empleadas en los resultados de la Sección 3.2 revelan dos hechos fundamentales. En primer lugar, el uso de la dualidad entre la octaedralidad de normas y la w^*-SD2P expuesta en la Sección 2.4 es esencial para obtener condiciones suficientes sobre la octaedralidad de la norma de $\mathcal{F}(M, X)$. Es también esencial, en vista de dichas técnicas, la hipótesis de la CEP. Por último, los dos resultados finales de la sección arrojan dos importantes conclusiones. Por un lado, a pesar de la abundancia de espacios Lipschitz libres cuya norma es octaedral, es posible que la bola de estos espacios puedan contener puntos de diferenciabilidad Fréchet. Por otra parte, el hecho de que la norma de $\mathcal{F}(M, X)$ sea octaedral depende tanto del espacio métrico subyacente como del espacio de llegada X.

- De los resultados de la Sección 3.2 se sigue que si la norma de $\mathcal{F}(M)$ no es octaedral entonces M tiene que ser uniformemente discreto y acotado. Sin embargo, gracias a la caracterización expuesta en Teorema 3.35, se sigue que existen muchos espacios uniformemente discretos y acotados M de manera que la norma de $\mathcal{F}(M)$ es octaedral, y un ejemplo de la abundancia de tales espacios métricos lo proporciona, por ejemplo, la Proposición 3.49. Además el Lema 3.36 revela, de nuevo haciendo un fuerte uso de la dualidad entre octaedralidad y la w^*-SD2P, que los subconjuntos normantes de la esfera juegan un papel crucial en la octaedralidad de una norma dada.

Bibliography

[Abr] T. A. Abrahamsen, *Linear extensions, almost isometries, and diameter two*, Extr. Math. **30**, 2 (2015), 135–151.

[AHNTT] T. A. Abrahamsen, P. Hájek, O. Nygaard, J. Talponen and S. Troyanski, *Diameter 2 properties and convexity*, Stud. Math. **232**, 3 (2016), 227–242.

[ALL] T. A. Abrahamsen, J. Langemets and V. Lima, *Almost square Banach spaces*, J. Math. Anal. Appl. **434**, no. 2 (2016), 1549–1565.

[ALN] T. A. Abrahamsen, V. Lima and O. Nygaard, *Almost isometric ideals in Banach spaces*, Glasg. Math. J. **56**, 2 (2014), 395–407.

[ALN2] T. A. Abrahansen, V. Lima and O. Nygaard, *Remarks on diameter two properties*, J. Convex Anal. **20**, 2 (2013), 439–452.

[ALNT] T. A. Abrahamsen, V. Lima, O. Nygaard and S. Troyanski, *Diameter two properties, convexity and smoothness*, Milan J. Math **84**, 2 (2016), 231–242.

[ANP] T. A. Abrahamsen, O. Nygaard and M. Põldvere, *New applications of extremely regular functions spaces*, to appear in Pac. J. Math. Preprint available at arXiv.org with reference arXiv:1711.01494.

[AA] Y. A. Abramovic, C. D. Aliprantis, *An Invitation to Operator Theory*, American Mathematical Society, Rhode Island (2002).

[AB] M. D. Acosta and J. Becerra Guerrero, *Weakly open sets in the unit ball of some Banach spaces and the centralizer*, J. Funct. Anal. **259**, 4 (2010), 842–856.

[ABL] M. D. Acosta, J. Becerra Guerrero and G. López-Pérez, *Stability results on diameter two properties,* J. Conv. Anal. **22**, 1 (2015), 1–17.

[ABR] M. D. Acosta, J. Becerra Guerrero, A. Rodríguez Palacios, *Weakly open sets in the unit ball of the projective tensor product of Banach spaces*, J. Math. Anal. Appl. **383**, 2 (2011), 461–473.

[AK] F. Albiac, N.J. Kalton, *Topics in Banach Space Theory*, Springer Inc. (2006).

[AOR] S. Argyros, E. Odell and H. Rosenthal, *On certain convex subsets of c_0,* Lecture Notes in Math. 1332, Functional Analysis, ed. by E. Odell and Rosenthal, Berlin (1988), 80–111.

[AM] A. Avilés and G. Martínez-Cervantes, *Complete metric spaces with property (Z) are length spaces*, J. Math. Anal. Appl. **473**, 1 (2019), 334–344.

[BL] Julio Becerra Guerrero and Ginés López-Pérez, *Relatively weakly open subsets of the unit ball of functions spaces*, J. Math. Anal. Appl. **315**, 2 (2006) 544–554.

[BLPR] J. Becerra Guerrero, G. López-Pérez, A. M. Peralta and A. Rodríguez Palacios, *Relatively weakly open sets in closed balls of Banach spaces, and real JB-triples of finite rank*, Math. Ann. **330**, 1 (2004), 45–58.

[BLRo] J. Becerra Guerrero, G. López-Pérez and A. Rodríguez Palacios, *Relatively weakly open sets in closed balls of C*-algebras*, J. Lond. Math. Soc. **68**, 3 (2003), 753-761.

[BLR1] J. Becerra Guerrero, G. López-Pérez and A. Rueda Zoca, *Big slices versus relatively weakly open subsets in Banach spaces*, J. Math. Anal. App. **428**, 2 (2015), 855–865.

[BLR2] J. Becerra Guerrero, G. López-Pérez and A. Rueda Zoca, *Diametral diameter two properties in Banach spaces*, J. Convex Anal. **25**, 3 (2018), 817–840.

[BLR3] J. Becerra Guerrero, G. López-Pérez and A. Rueda Zoca, *Extreme differences between weakly open subsets and convex combination of slices in Banach spaces*, Adv. Math. **269** (2015), 56–70.

[BLR4] J. Becerra Guerrero, G. López-Pérez and Abraham Rueda Zoca, *Octahedral norms and convex combination of slices in Banach spaces,* J. Func. Anal. **266**, 4 (2014), 2424–2436.

[BLR5] J. Becerra Guerrero, G. López-Pérez and A. Rueda Zoca, *Octahedral norms in spaces of operators*, J. Math. Anal. Appl. **427**, 1 (2015), 171–184.

[BLR6] J. Becerra Guerrero, G. López-Pérez and A. Rueda Zoca, *Octahedrality in Lipschitz-free Banach spaces*, Proc. R. Soc. Edinb. Sect. A Math. **148A**, 3 (2018), 447–460.

[BLR7] J. Becerra Guerrero, G. López-Pérez and A. Rueda Zoca, *Some results on almost square Banach spaces*, J. Math. Anal. Appl. **438**, 2 (2016), 1030–1040.

[BM] J. Becerra Guerrero and M. Martín, *The Daugavet property of C^*-algebras, JB-triples, and of their isometric preduals*, J. Funct. Anal. **224**, 2 (2005), 316-337.

[BR1] J. Becerra Guerrero and A. Rodríguez Palacios, *Banach spaces with the Daugavet property, and the centralizer* , J. Funct. Anal. **254**, 8 (2008), 2294–2302.

[BR2] J. Becerra Guerrero and A. Rodríguez Palacios, *Relatively weakly open sets in closed balls of Banach spaces, and the centralizer*, Math. Z. **262**, 3 (2009), 557–570.

[BeLi] Y. Benyamini and J. Lindestrauss, *Geometric Nonlinear Functional Analysis*, American Mathematical Society (2000).

[Bo-Ro] J. Bourgain and H. P. Rosenthal, *Geometrical implications of certain finite-dimensional decompositions*, Bull. Soc. Math. Belg. Sér. B **32** (1980), 57–82.

[Bou] R. D. Bourgin, *Geometric Aspects of Convex Sets with the Radon-Nikodym Property*, Springer-Verlag Berlin Heidelberg (1983).

[CCGMR] B. Cascales, R. Chiclana, L. García-Lirola, M. Martín and A. Rueda Zoca, *On strongly norm-attaining Lipschitz operators*, to appear in J. Funct Anal. https://doi.org/10.1016/j.jfa.2018.12.006

[CPS] J. M. F. Castillo, P. L. Papini and M. Simões, *Thick coverings for the unit ball of a Banach space*, Houston J. Math. **41**, 1 (2015), 177–186.

[CDW] M. Cúth, M. Doucha and P. Wojtaszczyk, *On the structure of Lipschitz-free spaces*, Proc. Amer. Math. Soc. **144**, 9 (2016), 3833–3846.

[CKK] M. Cúth, O. F. K. Kalenda, P. Kaplický, *Finitely additive measures and complementability of Lipschitz-free spaces*, Isr. J. Math. (2019). https://doi.org/10.1007/s11856-019-1829-y.

[Dau] I. K. Daugavet, *A property of complete continuous operators in the space C*, Usp. Mat. Nauk **18** (5) (1963) 157-158.

[Dev] R. Deville, *A dual characterization of the existence of small combinations of slices*, Bull. Aust. Math. Soc. **37**, 1 (1988), 113-120.

[DGZ] R. Deville, G. Godefroy and V. Zizler. *Smoothness and renormings in Banach spaces.* Pitman Monographs and Surveys in Pure and Applied Math. **64**, 1993. MR1211634 (94d:46012).

[DGH] S. J. Dilworth, M. Girardi and J. Hagler. *Dual Banach spaces which contain isometric copy of L_1*. Bull. Pol. Acad. Sci. Math. **48**, 1 (2000), 1–12.

[Din] S. Dineen, The second dual of a JB^*-triple system. In *Complex Analysis, Functional Analysis and Approximation Theory* (ed. by J. Múgica), 67-69, North-Holland Math. Stud. **125**, North-Holland, Amsterdam-New York, 1986.

[DU] J. Distel, J. J. Uhl, *Vector measures*, American Matematical Society Providence, Rhode Island (1977).

[Dor] L. E. Dor, *Potentials and isometric embeddings in L_1*, Isr. J. Math. **24** (1976), 260–268.

[FHHMPZ] M. Fabian, P. Habala, P. Hájek, V. Montesinos and V. Zizler, *Banach space theory*, Springer Science+Business Media, LLC 2011.

[FG] H. Fetter, B. Gamboa de Buen, *The James Forest*, London Mathematical Society Lecture Notes Series, vol.236, Cambridge University Press, Cambridge (1997).

[Flo] K. Floret, *Natural norms on symmetric tensor products of Banach spaces*, Note Mat. 17 (1997) 153-188.

[Gam] T. W. Gamelin, *Uniform algebras*, Prentice Hall, Inc., Englewood N.J. (1969).

[Gar] L. García Lirola, *Convexity, optimization and geometry of the ball in Banach spaces*, PhD thesis, Universidad de Murcia, 2017. Available at *DigitUM* with reference http://hdl.handle.net/10201/56573.

[GPR] L. García-Lirola, C. Petitjean and A. Rueda Zoca, *On the structure of spaces of vector-valued Lipschitz functions*, Stud. Math. **239**, 3 (2017), 249–271.

[GPrR] L. García-Lirola, A. Procházka and A. Rueda Zoca, *A characterisation of the Daugavet property in spaces of Lipschitz functions*, J. Math. Anal. Appl. **464**, 1 (2018), 473–492.

[GR] L. García-Lirola and A. Rueda Zoca, *Unconditional almost squareness and applications to spaces of Lipschitz functions*, J. Math. Anal. Appl. **451**, 1 (2017), 117–131.

[GGMS] N. Ghoussoub, G. Godefroy, B. Maurey, W. Schachermayer, *Some topological and geometrical structures in Banach spaces*, Mem. Amer. Math. Soc. **387** (1987), 116 p.

[GMS] N. Ghoussoub, B. Maurey and W. Schachermayer, *A counterexample to a problem on points of continuity in Banach spaces*, Proc. Amer. Math. Soc. **99**, 2 (1987), 279–282.

[GMS2] N. Ghoussoub, B. Maurey and W. Schachermayer, *Geometrical implications of certain infinite dimensional decompositions*, Trans. Amer. Math. Soc. **317**, 2 (1990), 541–584.

[Go] A. Godard, *Tree metrics and their Lipschitz-free spaces.* Proc. Amer. Math. Soc. **138**, 12 (2010), 4311-4320.

[God1] G. Godefroy, *A survey on Lipschitz-free Banach spaces*, Commentat. Math. **55**, 2 (2015), 89–118.

[God2] G. Godefroy, *Metric characterization of first Baire class linear forms and octahedral norms,* Studia Math. **95**, 1 (1989), 1-15.

[GK] G. Godefroy and N. J. Kalton, *The ball topology and its applications*, Contemporary Math. **85** (1989), 195–238.

[GM] G. Godefroy and B. Maurey, *Normes lisses et normes anguleuses sur les espaces de Banach separables*, unpublished preprint.

[GMZ] A. J. Guirao, V. Montesinos and V. Zizler, *On preserved and unpreserved extreme points*, in: J. C. Ferrando, M. López-Pellicer (Eds.), Descriptive Topology and Functional Analysis, in: Springer Proc. Math. Stat., vol. 80, Springer International Publishing, 2014, pp. 163-193.

[HL] R. Haller and J. Langemets, *Two remark on diameter 2 propertes*, Proc. Est. Acad. Sci **63**, 1 (2014), 2–7.

[HLLN] R. Haller, J. Langemets, V. Lima and R. Nadel, *Symmetric strong diameter two property*, Mediterr. J. Math. **16** (2019), 35.

[HLP1] R. Haller, J. Langemets, and M. Põldvere, *On duality of diameter 2 properties*, J. Conv. Anal. **22**, 2 (2015), 465–483.

[HLP2] R. Haller, J. Langemets, and M. Põldvere, *Rough norms in spaces of operators*, Math. Nach. **290** (2017), 1678–1688.

[HR] P. Harmand and T. S. S. R. K. Rao, *An intersection property of balls and relations with M-ideals*, Math. Z. **197** 2 (1988), 277–290.

[HWW] P. Harmand, D. Werner and W. Werner, *M-Ideals in Banach spaces and Banach algebras*, Lecture Notes in Math, 1547, Springer-Verlag, Berlin, 1993.

[IKR] J. M. Isidro, W. Kaup and A. Rodríguez-Palacios, *On real forms of JB^*-triples*, Manuscr. Math. **86**, 3 (1995), 311-335.

[Iva] Y. Ivakhno, *Big slice property in the spaces of Lipschitz functions*, Visn. Khark. Univ. Ser. Mat. Prylk. Mat. Mekh. **749**, 56 (2006), 109–118.

[Iva2] Y. Ivakhno, *On sets with extremely big slices*, Zh. Mat. Fiz. Anal. Geom. **2**, 1 (2006), 94–103.

[IKW] Y. Ivakhno, V. Kadets and D. Werner, *The Daugavet property for spaces of Lipschitz functions*, Math. Scand. **101**, 2 (2007), 261–279.

[Jam] R. C. James, *A separable somewhat reflexive Banach space with nonseparable dual*, Bull. Am. Math. Soc. **80** (1974), 738–743.

[JZ] K. John and V. Zizler, *On rough norms on Banach spaces*, Commentat. Math. Univ. Carol. **19** (1978), 335–349.

[Joh] J. A. Johnson, *Banach spaces of Lipschitz functions and vector-valued Lipschitz functions*, Trans. Amer. Math. Soc. **148** (1970), 147–169.

[JLPS1] W. B. Johnson, J. Lindenstrauss, D. Preiss and G. Schechtman, *Almost Fréchet differentiability of Lipschitz mappings between infinite-dimensional Banach spaces*, Proc. Lond. Math. Soc. **84** 3 (2002) , 711–746.

[JLPS2] W. B. Johnson, J. Lindenstrauss, D. Preiss and G. Schechtman, *Lipschitz quotients from metric trees and from Banach spaces containing ℓ_1*, J. Funct. Anal. **194**, 2 (2002), 332–346.

[KKW] V. Kadets, N. J. Kalton, D. Werner, *Remarks on rich subspaces of Banach spaces*, Stud. Math. **159**, 2 (2003), 195–206.

[KSW] V. Kadets, V. Shepelska, and D. Werner, *Thickness of the unit sphere, ℓ_1-types, and the Daugavet property*, Houston J. Math. **37**, 3 (2011), 867–878.

[KSSW] V. Kadets, R. V. Shvidkoy, G. G. Sirotkin, and D. Werner. *Banach spaces with the Daugavet property*, Trans. Am. Math. Soc. **352**, 2 (2000), 855–873.

[Kai] S. Kaijser, *A note on dual Banach spaces*, Math. Scand. **41** (1977), 325–330.

[Kal] N. J. Kalton, *Spaces of Lipschitz and Hölder functions and their applications*, Collect. Math. **55**, 2 (2004), 171-217.

[KauP] P. L. Kaufmann, *Products of Lipschitz-free spaces and applications*, Old preprint version (2014), available at arXiv.org with reference 1403.6605

[Kau] P. L. Kaufmann, *Products of Lipschitz-free spaces and applications*, Studia Math. **226** (2015), 213–227.

[Kub] D. Kubiak, *Some geometric properties of Cesàro function space*, J. Convex Anal. **21**, 1 (2014), 189–200.

[Lan] J. Langemets, *Geometrical structure in diameter 2 Banach spaces*, Dissertationes Mathematicae Universitatis Tartuensis 99 (2015), `http://dspace.ut.ee/handle/10062/47446`.

[LLR1] J. Langemets, V. Lima and A. Rueda Zoca, *Almost square and octahedral norms in tensor products of Banach spaces*, RACSAM **111**, 3 (2017), 841–853.

[LLR2] J. Langemets, V. Lima and A. Rueda Zoca, *Octahedral norms in tensor products of Banach spaces*, Q. J. Math. **68**, 4 (2017), 1247–1260.

[LL] J. Langemets and G. López-Pérez, *Bidual octahedral renormings and strong regularity in Banach spaces*, to appear in J. Inst. Math. Jussieu. Available at arXiv.org with reference arXiv:1902.04170.

[LLT] B. Lin, P. Lin and S. Tronyanski, *Characterizations of denting points.* Proc. Am. Math. Soc. **102**, 3 (1988), 526–528.

[LOS] J. Lindenstrauss, G. Olson, Y. Sternfeld. *The Poulsen simplex.* Ann. Inst. Fourier. (Grenoble) **28**, 1 (1978), 91–114.

[LT] J. Lindestrauss, L. Tzafriri, *Classical Banach spaces*, Springer Verlag Berlin (1977).

[Lop] G. López-Pérez, *The big slice phenomena in M-embedded and L-embedded spaces*, Proc. Am. Math. Soc. **134**, 1 (2005), 273–282.

[LR] G. López-Pérez and A. Rueda Zoca, *Diameter two properties and polyhedrality*, RACSAM **113**, 1 (2019), 131–135.

[LS1] G. López-Pérez and J. Soler Arias, *The convex point of continuity in Banach spaces not containing ℓ_1*, J. Math. Anal. Appl. **378**, 2 (2011), 734–740.

[LS2] G. López-Pérez and J. Soler Arias, *The point of continuity property in Banach spaces not containing ℓ_1*, Isr. J. Math. **191** (2012), 347–361.

[MP] J. Martínez and A. M. Peralta, *Separate weak*-continuity of the triple product in dual real JB^*-triples*, Math. Z. **234**, 4 (2000), 635–646.

[Mil] V. D. Milman, *Geometric theory of Banach spaces. II. Geometry of the unit ball*, Usp. Mat. Nauk 26, 6 (1971), no. 6(162), 73-149 (Russian), Russian Math. Surveys 26 (1971), no. 6, 79-163 (English).

[NW] O. Nygaard, D. Werner, *Slices in the unit ball of a uniform algebra*, Arch. Math. **76**, 6 (2001), 441–444.

[Oja] E. Oja, *Sums of slices in direct sums of Banach spaces*, Proc. Estonian Acad. Sci. **63**, 1 (2014), 8–10.

[PS] A. M. Peralta, L.L. Stach, *Atomic decomposition of real JBW*-triples*, Q. J. Math. **52**, 1 (2001) 79-87.

[Pet] C. Petitjean, *Lipschitz-free spaces and Schur properties*, J. Math. Anal. Appl. **453**, 2 (2017), 894–907.

[Pou] E. T. Poulsen. *A simplex with dense extreme points.* Ann. Inst. Fourier. (Grenoble) **11** (1961), 83–87.

[PR] A. Procházka and A. Rueda Zoca, *A characterisation of octahedrality in Lipschitz-free spaces*, Ann. Inst. Fourier (Grenoble) **62**, 2 (2018), 569–588.

[Rue1] A. Rueda Zoca, *Daugavet property and separability in Banach spaces*, Banach J. Math. Anal. **12**, 1 (2018), 68–84.

[Rue2] A. Rueda Zoca, *Stability results of octahedrality in tensor product spaces and finite representability in* ℓ_1, Q. J. Math. **70**, 1 (2019), 1–5.

[RTV] A. Rueda Zoca, P. Tradacete and I. Villanueva, *Daugavet property in tensor product spaces*, preprint (2019). Available at arXiv.org with reference arXiv:1903.01761.

[Ryan] R. A. Ryan, *Introduction to tensor products of Banach spaces*, Springer Monographs in Mathematics, Springer-Verlag, London, 2002.

[SSW] W. Schachermayer, A. Sersouri and E. Werner, *Moduli of nondentability and the Radon-Nikodým property in Banach spaces*, Isr. J. Math. **65**, 3 (1989), 225–257.

[Shv] R. V. Shvidkoy, *Geometric aspects of the Daugavet property*, J. Funct. Anal. **176**, 2 (2000), 198-212.

[Wea] N. Weaver, *Lipschitz algebras*, World Scientific, 1999.

[D.Wer] D. Werner, *M-ideals and the "basic inequality"*. J. Approximation Theory **76**, 1 (1994), 21–30.

[D.Wer2] D. Werner, *Recent progress on the Daugavet property*, Ir. Math. Soc. Bull. **46** (2001), 77–79.

[E.Wer] E. Werner, *Non dentable solid subsets in Banach lattices failing RNP. Applications to renormings*, Proc. Am. Math. Soc. **107**, 3 (1989), 611–620.

[Wil] A. Wilansky, *Topology for Analysis*, John Wiley (1970).

[Yag] Y. Yagoub-Zidi, *Some isometric properties of subspaces of function spaces*, Mediterr. J. Math. **10**, 4 (2013), 1905–1915.

Index