Hindawi
Advances in Mathematical Physics
Volume 2018, Article ID 4652516, 9 pages
https://doi.org/10.1155/2018/4652516

Research Article

Willmore-Like Tori in Killing Submersions

Manuel Barros,[1] Óscar J. Garay ,[2] and Álvaro Pámpano[2]

[1]*Departamento de Geometría y Topología, Universidad de Granada, Granada, Spain*
[2]*Department of Mathematics, Faculty of Science and Technology, University of the Basque Country UPV/EHU, Bilbao, Spain*

Correspondence should be addressed to Óscar J. Garay; oscarj.garay@ehu.eus

Received 28 March 2018; Accepted 14 May 2018; Published 12 June 2018

Academic Editor: Remi Léandre

The first variation formula and Euler-Lagrange equations for Willmore-like surfaces in Riemannian 3-spaces with potential are computed and, then, applied to the study of invariant Willmore-like tori with invariant potential in the total space of a Killing submersion. A connection with generalized elastica in the base surface of the Killing submersion is found, which is exploited to analyze Willmore tori in Killing submersions and to construct foliations of Killing submersions made up of Willmore tori with constant mean curvature.

1. Introduction

In 1811 S. Germain pointed out that the free energy controlling the physical system associated with an elastic plate is the *total squared curvature* or *bending energy*. For a surface S in $\mathbb{R}^3$ the bending energy is given by

$$\mathscr{W}(S) = \int_S H^2 dA, \quad (1)$$

where H denotes the mean curvature of S, which, according to a popular model in Biophysics, can be seen also as part of the *elastic energy* of fluid membranes [1]. Notice that, for curves γ isometrically immersed in a Riemannian manifold, H is nothing but the geodesic curvature of γ, κ, and $\mathscr{W}(S)$ boils down to

$$\mathscr{E}(\gamma) = \int_\gamma \kappa^2 ds. \quad (2)$$

Following a classical model of D. Bernoulli, planar curves minimizing the bending energy $\mathscr{E}(\gamma)$ can be thought of as the equilibrium positions of thin *elastic rods* (or *elasticae*). Elastica (and their generalizations) in semi-Riemannian manifolds is a classical subject which has many applications to Physics where they can be used, among other applications, to produce mathematical models of stiff rods, stiff polymers, vortices in fluids, superconductors, membranes, mechanical properties of DNA molecules, and also imaging and visual perception (for more details and references on this subject, see, for example, [2]).

The study of the variational problem associated with $\mathscr{W}(S)$ for surfaces in $\mathbb{R}^3$ goes back to Blaschke's school in the 1920s. Thus, G. Thomsen in [3] obtained the first variation and the Euler-Lagrange equations corresponding to (1). Years later, T. J. Willmore reintroduced the problem in [4] and this is reason why $\mathscr{W}(S)$ is also known as the *Willmore energy* of the surface and its critical points are called *Willmore surfaces*. The Willmore energy is a conformal invariant [5, 6], a property which has been constantly exploited in the vast literature on the subject (see, for example, [2, 7–13]). In the last years, there has been also an intensive investigation of Willmore surfaces in connection with *Willmore problem*, that is, the determination of the minima for the Willmore energy within a given topological class. In particular, T. Willmore proposed in 1965 the following conjecture: for every smooth immersed torus M in $\mathbb{R}^3$, $\mathscr{W}(M) \geq 2\pi^2$ and the equality is obtained at the Clifford torus. The conjecture has been recently proved in [14], although prior to this proof the Willmore conjecture had already been shown for many special cases (see [14–17] and references therein).

A generalization of the Willmore energy is due to B-Y Chen [18]. He extended the Willmore functional to any submanifold M^n of any Riemannian manifold N^m so that the conformal invariance property is preserved. Given a

Riemannian submanifold M^n of N^m the *Chen-Willmore functional* is defined by

$$\mathscr{CW}(M) = \int_M \left(H^2 - \tau_e\right)^{n/2} dV, \quad (3)$$

where H and τ_e denote the *mean curvature* and the *extrinsic scalar curvature* of M, respectively. The energy $\mathscr{CW}(M)$ is conformally invariant and its critical points are known as *Chen-Willmore submanifolds* [18, 19]. In contrast to the case of Willmore surfaces in ambient spaces of constant curvature, there are not many results concerning Chen-Willmore submanifolds in background spaces with nonconstant sectional curvature. Nevertheless, Willmore surfaces and submanifolds have strong connections in Physics with applications to (just to mention a few) the analysis of elastic plates and biological membranes [1, 20] and to bosonic string theories and sigma models (for more details, see [21, 22] and references therein). Several of these applications are based on a beautiful link between Willmore surfaces and elastica which can be established by using a symmetry reduction procedure [12].

In this paper we will study the following generalization of the Willmore energy:

$$\mathscr{W}_\Phi(S) = \int_S \left(H^2 + \Phi_{|_S}\right) dA, \quad (4)$$

where S is a surface of a 3-dimensional Riemannian manifold N, H denotes the mean curvature of S, and $\Phi \in C^\infty(N)$. Here, $\mathscr{W}_\Phi(S)$ will be called *Willmore energy with a potential* Φ and in Section 3 the first variation formula and Euler-Lagrange equation associated with $\mathscr{W}_\Phi$ are computed.

On the other hand, our analysis will be focused mainly on surfaces living in the total space of a *Killing submersion*. A Riemannian submersion $\pi : N \to M$ of a 3-dimensional Riemannian manifold N over a surface M will be called a Killing submersion if its fibers are the trajectories of a complete unit Killing vector field ξ. Killing submersions are determined by two functions on M, its *Gaussian curvature* and the so-called *bundle curvature* (for more details, see [23, 24] and Section 2). A remarkable family of Killing submersions are the homogeneous 3-spaces. With the exception of the hyperbolic 3-space, $\mathbb{H}^3(-1)$, simply connected homogeneous Riemannian 3-manifolds with isometry group of dimension 4 or 6, can be represented by a 2-parameter family $\mathbb{E}(c,\mu)$, where $c, \mu \in \mathbb{R}$. These $\mathbb{E}(c,\mu)$-spaces are 3-manifolds admitting a global unit Killing vector field whose integral curves are the fibers of a certain Riemannian submersion over the simply connected constant Gaussian curvature surface $M(c)$ and, therefore, determine Killing submersions $\pi : \mathbb{E}(c,\mu) \to M(c)$ (for more details, see [25]). A local description of these examples can be given by using the so-called *Bianchi-Cartan-Vranceanu* spaces (see Section 2).

In Section 4, we establish a connection between ξ-invariant Willmore-like tori with ξ-invariant potentials in the total space N of Killing submersions and elastic curves with related potentials in the base surface M. We show also how one can apply this connection to study ξ-invariant Willmore tori in the total space N of a Killing submersion and, in particular, to the case of ξ-invariant Willmore tori in orthonormal frame bundles on Riemannian surfaces. Finally, the previous results are applied in Section 5 to the construction of foliations, of certain Killing submersions, consisting of Willmore tori with constant mean curvature.

2. Killing Submersions

A Riemannian submersion $\pi : N \to M$ of a 3-dimensional Riemannian manifold N over a surface M will be called a *Killing submersion* if its fibers are the trajectories of a complete unit Killing vector field ξ (for more details, see [23, 24]). Fibers of Killing submersions are geodesics in N and form a foliation called the *vertical foliation*. Most of the geometry of a Killing submersion is encoded in a pair of functions: K, τ. Here K represents the *Gaussian curvature* function of the *base surface* M, while τ denotes the so-called *bundle curvature* which is defined as follows. Since ξ is a (vertical) unit Killing vector field, then it is clear that for any vector field, Z, on N, there exists a function τ_Z (which, a priori, depends on the vector field Z) such that $\overline{\nabla}_Z \xi = \tau_Z Z \wedge \xi$, where $\overline{\nabla}$ denotes the Levi-Civita connection of N. Actually, it is not difficult to see that τ_Z does not depend on the vector field Z (see [23] for details) so we get a function $\tau \in C^\infty(N)$, the bundle curvature, satisfying

$$\overline{\nabla}_Z \xi = \tau Z \wedge \xi, \quad (5)$$

where $\wedge$ denotes the vector product. The bundle curvature is obviously constant along the fibers and, consequently, it can be seen as a function on the base surface, $\tau \in C^\infty(M)$. In product spaces $M \times \mathbb{R}$ the projection over the first factor is a Killing submersion, so its bundle curvature is $\tau \equiv 0$. More generally, it is easy to deduce from (5) that $\tau \equiv 0$ in a Killing submersion, if and only if, the horizontal distribution in the *total space*, N, is integrable. It is customary to denote the total space of a Killing submersion by $N(K,\tau)$.

Existence of Killing submersions over a given simply connected surface M for a prescribed *bundle curvature*, $\tau \in C^\infty(M)$, has been proved in [24]. Uniqueness of these submersions, up to isomorphisms, is guaranteed under the assumption that the total space is also simply connected. This fact leads to the classification of Killing submersions on a simply connected surface as quotients of simply connected total spaces under vertical translations. In particular, we can suppose that fibers have finite length since, if necessary, we can always take a suitable quotient under a vertical translation in order to get a circle bundle over M. The following result provides existence of Killing submersions with prescribed bundle curvature over arbitrary Riemannian surfaces

Theorem 1. *Let M be a Riemannian surface with Gaussian curvature K and choose any function $\tau \in C^\infty(M)$. Then there exists a Killing submersion over M with bundle curvature τ, $(N(K,\tau),\pi)$. Moreover, $(N(K,\tau),\pi)$ can be chosen having compact fibers.*

Proof. If M is simply connected, then the result was proved in [24]. Therefore we assume that M is not simply connected and, thus, it has a nontrivial fundamental group $\pi_1(M)$.

Denote by M_o its universal covering and by $p : M_o \to M$ the universal covering map. Now, if g is the metric on M, then we consider M_o endowed with the metric $g_o = p^*(g)$ so that p is a Riemannian covering map. Denote by Γ the group of deck transformations of (M_o, p) which is a subgroup of the group of isometries $Isom(M_o, g_o)$ isomorphic to $\pi_1(M)$.

Consider $\tau_o = \tau \circ p \in C^\infty(M_o)$. Since M_o is simply connected, we know there exists a unique (up to isomorphisms) Killing submersion $\pi_o : N_o \to M_o$ with simply connected total space N_o and bundle curvature τ_o (see [24] for details). Now, if $h \in \Gamma$, then it is obvious that $\tau_o \circ h = \tau_o$. Consequently, once an initial condition has been chosen, there exists a unique isometry $\overline{h} \in Isom(N_o, \overline{g}_o)$ preserving the Killing fiber flow ($\overline{h}_* \xi = \xi$) and satisfying $\pi_o \circ \overline{h} = h \circ \pi_o$. Hence, we have a monomorphism from Γ into $Isom(N_o, \overline{g}_o)$ whose image $\overline{\Gamma}$ is isomorphic to Γ. It is not difficult to see that this determines a properly discontinuous action of $\overline{\Gamma}$ on N_o, defining a Riemannian covering map $\overline{p} : (N_o, \overline{g}_o) \to (N = N_o/\Gamma, \overline{g})$. Finally we define $\pi : N \to M$ by $\pi(\overline{p}(\overline{a})) = p(\pi_o(\overline{a}))$ which satisfies $\pi * (g) = \overline{g}$. It can be checked that it provides a Killing submersion over M with bundle curvature τ. □

Any Killing submersion is locally isometric to one of the following *canonical examples* [24], which include, as we will see later, the so-called *Bianchi-Cartan-Vranceanu spaces* for suitable choices of the functions λ, a, b.

Example 2 (canonical examples). Given an open set $\Omega \subset \mathbb{R}^2$ and $\lambda, a, b \in \mathscr{C}^\infty(\Omega)$ with $\lambda > 0$, the Killing submersion

$$\pi : \left(\Omega \times \mathbb{R}, \mathrm{d}s^2_{\lambda,a,b}\right) \longrightarrow \left(\Omega, \mathrm{d}s^2_\lambda\right), \quad \pi(x, y, z) = (x, y), \tag{6}$$

where

$$\mathrm{d}s^2_{\lambda,a,b} = \lambda^2\left(\mathrm{d}x^2 + \mathrm{d}y^2\right) + \left(\mathrm{d}z - \lambda\left(a\mathrm{d}x + b\mathrm{d}y\right)\right)^2 \tag{7}$$

and

$$\mathrm{d}s^2_\lambda = \lambda^2\left(\mathrm{d}x^2 + \mathrm{d}y^2\right), \tag{8}$$

will be called the canonical example associated with (λ, a, b). Regardless of the values of the functions $a, b \in \mathscr{C}^\infty(\Omega)$, the Riemannian metric given by (7) satisfies that π is a Killing submersion over $(\Omega, \mathrm{d}s^2_\lambda)$ with $\xi = \partial_z$ as unit vertical Killing field. The bundle curvature and the Gaussian curvature are given by

$$2\tau = \frac{1}{\lambda^2}\left((\lambda b)_x - (\lambda a)_y\right), \quad K = -\frac{1}{\lambda^2}\Delta_o(\log\lambda), \tag{9}$$

where Δ_o represents the Laplacian with respect to the standard metric in the plane.

Example 3 (Bianchi-Cartan-Vranceanu spaces). Particularizing the above construction one can get models for all Killing submersions over $\mathbb{R}^2$, $\mathbb{H}^2(c)$ and the punctured sphere $\mathbb{S}^2_*(c)$. Given $c \in \mathbb{R}$, we define $\lambda_c \in \mathscr{C}^\infty(\Omega_c)$ by

$$\lambda_c(x, y) = \left(1 + \frac{c}{4}\left(x^2 + y^2\right)\right)^{-1}, \tag{10}$$

where

$$\Omega_c = \begin{cases} \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < \dfrac{-4}{c}\right\}, & \text{if } c < 0, \\ \mathbb{R}^2, & \text{if } c \geq 0. \end{cases} \tag{11}$$

Then, the metric $\lambda^2_c(\mathrm{d}x^2 + \mathrm{d}y^2)$ in Ω_c has constant Gaussian curvature $K = c$. If, in addition, we choose $a = -\mu y$ and $b = \mu x$ for some real constant μ, then one obtains the metrics of the Bianchi-Cartan-Vranceanu spaces $\mathbb{E}(c, \mu) \equiv \Omega_c \times \mathbb{R}$ ([25, Section 2.3]):

$$\lambda^2_c\left(\mathrm{d}x^2 + \mathrm{d}y^2\right) + \left(\mathrm{d}z + \mu\lambda_c\left(y\mathrm{d}x - x\mathrm{d}y\right)\right)^2. \tag{12}$$

A simple computation, using (9), gives $\mu = \tau$. Thus the Bianchi-Cartan-Vranceanu spaces (BCV spaces) can be seen as the canonical models of Killing submersions with constant bundle curvature and constant Gaussian curvature. Cartan in [26] showed that the examples above cover, in fact, all possible 3-dimensional homogeneous spaces with 4-dimensional isometry group. The BCV family also includes two real space forms, which have 6-dimensional isometry group.

Example 4 (bundle-like metrics). Let $\overline{M}$ be the three-dimensional total space of a principal fiber bundle on a surface, M, and denote by $\pi : \overline{M} \to M$ the natural projection. Consider a principal connection ω and denote by dt^2 the metric on the fiber. Now, given a positive smooth function $f \in C^\infty(M)$ and any Riemannian metric, g, on M, we can define the following *generalized Kaluza-Klein Riemannian metric* on $\overline{M}$:

$$\overline{g} = \pi^*(g) + (f \circ \pi)^2\,\omega^*\left(dt^2\right). \tag{13}$$

In particular, if the function f is constant, then $\overline{g}$ is called a *Kaluza-Klein metric* or a *bundle-like metric*. As it is well known, the following properties are satisfied by the above class of metrics:

(i) The natural action of the structure group, G, on $\overline{M}$ is carried out by isometries of $(\overline{M}, \overline{g})$. Consequently, the fiber flow is associated with a Killing vector field of $(\overline{M}, \overline{g})$.

(ii) The natural projection $\pi : (\overline{M}, \overline{g}) \to (M, g)$ is a Riemannian submersion. Furthermore, its leaves are geodesics in $(\overline{M}, \overline{g})$, if and only if, f is a constant function, i.e., $\overline{g}$ is a bundle-like metric.

(iii) Setting

$$\tilde{g} = \frac{1}{(f \circ \pi^2)}\overline{g} = \pi^*\left(\frac{1}{f^2}g\right) + \omega^*\left(dt^2\right) \tag{14}$$

one sees that generalized Kaluza-Klein metrics are conformal to bundle-like metrics.

Hence, once a suitable conformal change has been made (if necessary), every bundle-like metric provides a Riemannian submersion whose fiber flow is associated with a vertical unit Killing vector field. In other words, we have a Killing submersion. Moreover, we see that homogeneous 3-spaces with 4-dimensional isometry group pose bundle-like metrics and their classes of congruence are completely determined, up to topology, for a pair of constants: the curvature of the base surface c, and the bundle curvature τ.

3. Willmore-Like Energies

Let $(\overline{M}, \overline{g} = \langle , \rangle)$ be a 3-dimensional Riemannian manifold, $\Phi \in C^\infty(\overline{M})$ a smooth function, and S a compact surface with no boundary. In the space of isometric immersions of S in $\overline{M}$, $\mathbf{I}(S, \overline{M})$, we define the following *Willmore-like energy*:

$$\mathscr{W}_\Phi(S) \equiv \mathscr{W}_\Phi(S, \varphi) = \int_S \left(H_\varphi^2 + \Phi_{|_{\varphi(S)}}\right) dA_\varphi, \tag{15}$$

where $\varphi \in \mathbf{I}(S, \overline{M})$, H_φ denotes the mean curvature function of φ and dA_φ is the induced element of area via the immersion φ.

The main purpose of this section is to compute the field equations associated with this kind of functionals. To this end, we consider a *variation* of $\varphi \in \mathbf{I}(S, \overline{M})$. That is, a smooth map $\Psi : S \times (-\delta, \delta) \to \overline{M}$ satisfies the following conditions:

(i) For any $\varepsilon \in (-\delta, \delta)$ the map $\varphi_\varepsilon : S \to \overline{M}$ defined by $\varphi_\varepsilon(m) = \Psi(m, \varepsilon)$ belongs to $\mathbf{I}(S, \overline{M})$.

(ii) $\varphi_0 = \varphi$.

Then, the vector field on Ψ given by

$$V(m, \varepsilon) = \Psi_*\left(\frac{\partial}{\partial \varepsilon}(m, \varepsilon)\right) \tag{16}$$

determines a vector field along φ, $V(m) := V(m, 0)$, which is called the *variational vector field* associated with the variation Ψ. Thus, we can identify the tangent space $T_\varphi(\mathbf{I}(S, \overline{M}))$ with the space of vector fields along φ and, consequently, we have

$$\partial \mathscr{W}_\Phi(S, \varphi)[V] = \frac{\partial}{\partial \varepsilon}\left(\int_S \left(H_{\varphi_\varepsilon}^2 + \Phi\right) dA_{\varphi_\varepsilon}\right)_{|_{\varepsilon=0}}. \tag{17}$$

Now, the following lemma collects some formulae that will be needed in the sequel. They can be obtained using similar computations to those included in [13, 22]. For simplicity, we have eliminated the symbol φ in most of our notation.

Lemma 5. *With the previous notation, the following statements hold:*

(1) *Let $\overrightarrow{H}(m, \varepsilon)$ be the mean curvature vector field of φ_ε at $m \in S$, then*

$$D_\varepsilon \overrightarrow{H}_{|_{\varepsilon=0}} = \frac{1}{2}\left(\Delta V^\perp + \widetilde{A}(V^\perp) + Ric(\mathbf{N}, \mathbf{N})\right) + D_{V^\top}\overrightarrow{H}, \tag{18}$$

where Δ is the Laplacian associated with the connection D in the normal bundle of $\varphi(S)$, $\widetilde{A}$ stands for the Simons operator [13], Ric is the Ricci curvature of $\overline{M}$, and $\mathbf{N}$ denotes the unit normal vector field of φ. Here $()^\top$ and $()^\perp$ denote tangential and normal components, respectively.

(2) *The variation of the area element is given by the following formula:*

$$\frac{\partial}{\partial \varepsilon} dA_{\varphi_{\varepsilon}|_{\varepsilon=0}} = -2\left\langle \overrightarrow{H}, V \right\rangle dA_\varphi + d\theta, \tag{19}$$

where θ is the one-form defined by $\theta(X) = dA_\varphi(V^\top, X)$.

Combining the above two formulae and an argument involving integration by parts, we obtain the first variation formula of the first term in (15):

$$\frac{\partial}{\partial \varepsilon}\left(\int_S \left(H_{\varphi_\varepsilon}^2\right) dA_{\varphi_\varepsilon}\right)_{|_{\varepsilon=0}} = \int_S \Big\langle \Delta \overrightarrow{H} + \left(2H^2 - 2K_S + 2R + \mathrm{Ric}(\mathbf{N}, \mathbf{N})\right)\overrightarrow{H}, V \Big\rangle dA, \tag{20}$$

where K_S denotes the Gaussian curvature of S endowed with the induced metric associated with the initial immersion φ, and R stands for its *extrinsic Gaussian curvature* (that is, the sectional curvature of $\overline{M}$ on the tangent plane of φ).

Now, Lemma 5 can be used once more to compute the first variation formula of the second term in (15):

$$\frac{\partial}{\partial \varepsilon}\left(\int_S \Phi\, dA_{\varphi_\varepsilon}\right)_{|_{\varepsilon=0}} = \int_S \langle \nabla\Phi, V\rangle\, dA - 2\int_S \Phi \left\langle \overrightarrow{H}, V \right\rangle dA - \int_S d\Phi \wedge \theta, \tag{21}$$

where $\nabla\Phi$ denotes the gradient of the potential Φ. Now, it is easy to see that $d\Phi \wedge \theta = V^\top(\Phi)dA$ and then

$$\frac{\partial}{\partial \varepsilon}\left(\int_S \Phi\, dA_{\varphi_\varepsilon}\right)_{|_{\varepsilon=0}} = \int_S \left\langle (\nabla\Phi)^\perp - 2\Phi\overrightarrow{H}, V \right\rangle dA. \tag{22}$$

Finally, we combine (20) and (22) to characterize the extremals of $\mathscr{W}_\Phi(S)$ as the solutions of the following Euler-Lagrange equation:

$$\Delta H + H\left(2H^2 - 2K_S + 2R + \mathrm{Ric}(\mathbf{N}, \mathbf{N}) - 2\Phi\right) + \mathbf{N}(\Phi) = 0. \tag{23}$$

Along this note, solution surfaces of (23), that is extremals of $\mathscr{W}_\Phi(S)$, will be called *Willmore-like surfaces*. Hence, we have

Theorem 6. *Let $(\overline{M}, \overline{g})$ be a 3-dimensional Riemannian manifold, $\Phi \in C^\infty(\overline{M})$ a smooth function, and S a compact surface isometrically immersed in $\overline{M}$. Then S is a Willmore-like surface, if and only if, (23) is satisfied.*

Notice that if the potential Φ is a constant function, say μ (in particular, the case in which $\mathscr{W}_\Phi(S)$ is the Willmore energy in a space of constant curvature μ), then every minimal surface is automatically a Willmore-like surface. However, if the potential is not constant, it follows from (23) that a minimal surface, S, is Willmore-like if and only if $\nabla\Phi$ is tangent to S anywhere.

4. Vertical Willmore Tori in Killing Submersions

Assume we have a Killing submersion $\pi : N(K,\tau) \to M$ of a 3-dimensional Riemannian manifold N over a surface M. Let γ be an immersed curve in (M,g), then $S_\gamma = \pi^{-1}(\gamma)$ is a surface isometrically immersed in N (by the natural inclusion $i = \varphi$) which is invariant under $\mathscr{G} := \{\psi_t : t \in \mathbb{R}\}$, the one-parameter group of isometries associated with the Killing vector field ξ. In fact, any $\mathscr{G}$-invariant surface in N, S, is obtained in this way: $S = \pi^{-1}(\gamma)$, for some curve γ of M. It is usual to call S_γ *the vertical tube (or vertical cylinder) shaped on the curve* γ. Notice that S_γ is embedded if γ is a simple curve, and it is a torus when γ is closed and $\mathscr{G} = \mathbb{S}^1$ is a circle group. If γ is parametrized by its arc-length, then any horizontal lift of γ, $\overline{\gamma}$, is also arc-length parametrized. Now, using as coordinate curves the horizontal lifts of γ and the fibers of the submersion, these vertical tubes S_γ can be parametrized by $\Psi(s,t) = \psi_t(\overline{\gamma}(s))$ and, as a consequence, S_γ are flat. It is also known that the mean curvature of these surfaces H is related to the curvature function of the corresponding cross sections by the formula (for details, see [27]):

$$H = \frac{1}{2}\kappa \circ \pi, \tag{24}$$

κ denoting the geodesic curvature of γ in M. Then, we have

Proposition 7. *Consider a Willmore-like energy $\mathscr{W}_\Phi(S) = \int_S (H^2 + \Phi)dA$ with invariant potential Φ ($\Phi = \phi \circ \pi$) defined on the space of surface immersions in the total space of a Killing submersion $\pi : N(K,\tau) \to M$ with compact fiber, i.e., $\mathscr{G} = \mathbb{S}^1$. If γ is a closed curve in M, then its vertical torus $S_\gamma = \pi^{-1}(\gamma)$ is a Willmore-like surface, if and only if, γ is a extremal of the following elastic-like energy $\mathscr{E}_\phi(\gamma) = \int_\gamma (\kappa^2 + 4\phi)ds$.*

Proof. Let S_γ be a torus over γ. The $\mathbb{S}^1$-action on $N(K,\tau)$ can be naturally extended to $\mathbf{I}(S,N)$ by $\mathscr{W}_\Phi(\varphi) = \mathscr{W}_\Phi(\psi_t \circ \varphi)$, for all $t \in \mathbb{R}$, $\varphi \in \mathbf{I}(S,N)$. On the other hand the space, Σ, of the $\mathbb{S}^1$-invariant immersions, can be identified with

$$\Sigma = \left\{S_\gamma = \pi^{-1}(\gamma) : \gamma \text{ is a closed curve in } M\right\}. \tag{25}$$

In this setting, we can apply the symmetric criticality principle of Palais [28] to reduce symmetry and then $S_\gamma = \pi^{-1}(\gamma)$ is a Willmore torus, if and only if, it is an extremal of $\mathscr{W}_\Phi$ restricted to Σ. Finally, we use (24) to conclude that this happens, precisely, when γ is a critical curve for $\mathscr{E}_\phi$. □

Remark 8. If the potential ϕ is constant, then extremals of the above energy $\mathscr{E}_\phi$ are known as elastic curves (see [29]). Thus, critical curves γ of $\mathscr{E}_\phi$ will be called *elasticae with potential* 4ϕ.

As an application, we obtain the following known characterization of closed elasticae with potential which will be used later

Corollary 9. *A closed unit speed curve $\gamma(s)$ of a Riemannian surface M is an extremal of the elastic-like energy $\mathscr{E}_\phi(\gamma) = \int_\gamma (\kappa^2 + 4\phi)ds$ (κ being its geodesic curvature in M), if and only if,*

$$2\kappa'' + \kappa^3 + 2(K - 2\phi)\kappa + 4\overrightarrow{n}(\phi) = 0, \tag{26}$$

where $()' = d/ds$, $\overrightarrow{n}$ denotes the unit normal of γ in M and K is the Gaussian curvature of M.

Proof. If K is the Gaussian curvature of M, choose any $\tau \in C^\infty(M)$ and a Killing submersion $\pi : N(K,\tau) \to M$ with compact fibers, as guaranteed by Theorem 1. Define $\Phi = \phi \circ \pi$ on $N(K,\tau)$. Now, from Proposition 7, we have that γ is an extremal of the elastic-like energy $\mathscr{E}_\phi(\gamma) = \int_\gamma (\kappa^2 + 4\phi)ds$, if and only if, S_γ is a critical point of $\mathscr{W}_\Phi(S) = \int_S (H^2 + \Phi)dA$. Since S_γ is flat $K_{S_\gamma} = 0$, moreover, a straightforward computation shows that on these vertical flat tori $2R + \mathrm{Ric}(\mathbf{N},\mathbf{N}) = K$, $\mathbf{N}$ being the unit normal along S_γ. Hence (23) gives that S_γ must provide a solution of the following differential equation:

$$\Delta H + H\left(2H^2 + K - 2\phi\right) + \mathbf{N}(\phi) = 0, \tag{27}$$

where H is the mean curvature function of S_γ in $N(K,\tau)$. Finally we combine (24) and (27) to conclude that the curvature of the elastica with 4ϕ-potential must satisfy

$$2\kappa'' + \kappa^3 + 2(K - 2\phi)\kappa + 4\overrightarrow{n}(\phi) = 0, \tag{28}$$

$\overrightarrow{n}$ denoting the unit normal of γ in M. □

On the other hand, as it has been mentioned previously in the introduction, in the context of Killing submersions the Willmore problem can be described as the variational problem associated with the so-called *Chen-Willmore energy*:

$$\mathscr{W}(S) \equiv \mathscr{W}(S,\varphi) = \int_S \left(H_\varphi^2 + R\right)dA_\varphi, \tag{29}$$

where $\varphi \in \mathbf{I}(S, N(K,\tau))$, and R denotes the *extrinsic Gaussian curvature* associated with the immersion φ, that is the sectional curvature of $N(K,\tau)$ restricted to the tangent bundle $d\varphi(TS)$ of the corresponding surface. Recall that extremals of $\mathscr{W}$ are known as *Willmore surfaces*. It should be noticed that, in general, the energy (29) does not coincide with those given in (15), because the potential in (29) is defined on the Grassmannian of two planes, but if $(N(K,\tau),\overline{g})$ has constant sectional curvature, c, then the Willmore energy (29) would correspond to a Willmore-like energy with constant potential $\Phi = c$. Nevertheless, the situation for vertical cylinders shaped on closed curves of the base surface changes and we have the following.

Proposition 10. *Assume that* $\pi : N(K,\tau) \rightarrow M$ *is a Killing submersion with compact fiber, i.e.,* $\mathcal{G} = \mathbb{S}^1$. *Then, a vertical torus over a closed curve* γ *in* M, $S_\gamma = \pi^{-1}(\gamma)$, *is a Willmore surface in* $N(K,\tau)$, *if and only if,* S_γ *is an extremal of the following Willmore-like energy:*

$$\mathscr{W}_{\tau^2}(S) = \int_S \left(H^2 + \tau^2\right) dA. \tag{30}$$

And this happens in turn, if and only if, γ *is an elastica with potential* $4\tau^2$ *in* M.

Proof. The extrinsic Gaussian curvature of vertical cylinders S_γ is related to the bundle curvature by

$$R\left(\overline{X}, \xi, \xi, \overline{X}\right) = \tau^2, \tag{31}$$

where $\overline{X}$ denotes the horizontal lift of the unit tangent vector field $X = \gamma'$. Now, combining (29) and (31) and applying similar arguments to those used in Proposition 7 and the symmetric criticality principle of Palais [28], we draw the conclusion. □

It is well known that compact minimal surfaces in spaces of constant curvature are examples of Willmore surfaces. The Euclidean 3-space does not contain compact minimal surfaces. However, the round three-sphere has plenty of compact minimal surfaces. These Willmore surfaces can be conformally projected by the stereographic projection to the Euclidean space, obtaining so examples of Willmore surfaces in the Euclidean 3-spaces because the Willmore energy is invariant under conformal changes in the surrounding metric. Both the Euclidean 3-space and the round 3-sphere provide examples of total 3-spaces associated with Killing submersions. In the former case $K = \tau = 0$ and π is just the projection over a plane. In the spherical case $K = 4\tau^2$ is a constant and π is the well-known Hopf map from $\mathbb{S}^3$ on $\mathbb{S}^2$. As it has been said previously, besides these spaces (whose group of isometries has dimension 6) the class of homogeneous three-space with 4-dimensional group of isometries provides also nice examples of Killing submersions with both K and τ being constant. In this case vertical minimal tori are automatically Willmore (see [30] and references therein). In contrast, for general Killing submersions we have the following direct consequence of Proposition 10 and (28)

Corollary 11. *Let* $\pi : N(K,\tau) \rightarrow M$ *be a Killing submersion and* $\gamma(s)$ *be a closed geodesic in* M, *then* $S_\gamma = \pi^{-1}(\gamma)$ *is a Willmore torus in* $N(K,\tau)$ *if and only if* $\gamma(s)$ *is a tangent line of* τ^2.

As an application, let us consider, for example, the unit round 2-sphere, $\mathbb{S}^2 \subset \mathbb{R}^3$, which we assume to be centered at the point $p_o = (0,0,1)$. Now, choose the nonnegative function $h \in C^\infty(\mathbb{S}^2)$ obtained as the restriction of the height function, in $\mathbb{R}^3$, to the plane $z = 0$. It is clear that the gradient flow associated with ∇h is made up of the great circles through the origin, which obviously are geodesics. On the other hand, we know that there exists a Killing submersion $N(K = 1, h) \rightarrow \mathbb{S}^2$ with bundle curvature h [24]. This Killing submersion is unique up to isomorphism and in this case, it coincides with the Hopf map $\pi : \mathbb{S}^3 \rightarrow \mathbb{S}^2$ where the 3-sphere is endowed with a suitable metric. Therefore, in the corresponding conformal class, we get a class of minimal Willmore tori having a pair of great circles in common.

To end this section, we consider frame bundles of surfaces as a particular case of Killing submersions and study Willmore surfaces in this context. Let (M, g) be a Riemannian surface with metric g and Gaussian curvature K. Denote by $\mathbf{F}$M its orthonormal frame bundle. Then, the natural projection, $\pi : \mathbf{F}M \rightarrow M$ gives a principal bundle with structure group $\mathbf{O}(2)$. Certainly the geometry of $\mathbf{F}$M is more closely related to M than that of any other principal bundle. Remember, for example, the popular definitions of the Levi-Civita connection and the curvature operator of (M, g) in terms of the connection form ω and the curvature form Ω of $\mathbf{F}$M. Now, $\mathbf{F}$M can be endowed with bundle-like metrics as follows:

$$\overline{g} = \pi^*(g) + \omega^*\left(dt^2\right), \tag{32}$$

providing a large class of Killing submersions. The corresponding bundle curvature τ can be computed using (31) and [31]:

$$\tau^2 = R\left(\overline{X}, \xi, \xi, \overline{X}\right) = \frac{1}{4}\frac{\left|R_{\mu(\xi)}X\right|^2}{|X|^2\left|\mu(\xi)\right|^2} \circ \pi, \tag{33}$$

where μ is the third O'Neill invariant [32] defined in terms of the connection form $\omega = (\omega_{ij})$ by

$$\mu(\xi)(f) = \sum_{i \neq j}^{2} \omega_{ij}(\xi) f_i \wedge f_j = 2\omega_{12}(\xi) f_1 \wedge f_2, \qquad f = \left(f_1, f_2\right) \in \mathbf{F}M. \tag{34}$$

But $1 = \overline{g}(\xi,\xi) = \langle \omega(\xi), \omega(\xi)\rangle = \sum_{i\neq j}^{2}(\omega_{ij}(\xi))^2 = 2(\omega_{12}(\xi))^2$ and consequently $(\omega_{12}(\xi))^2 = 1/2$. Therefore, $|\mu(\xi)|^2 = 2$. Moreover, we have

$$\begin{aligned} R_{\mu(\xi)} &= 2\omega_{12}(\xi) R_{f_1 f_2}\overline{X} \\ &= \sqrt{2}K\left(\langle X, f_2\rangle f_1 - \langle X, f_1\rangle f_2\right), \end{aligned} \tag{35}$$

and we obtain

$$\left|R_{\mu(\xi)}X\right|^2 = 2K^2\left(\langle X, f_1\rangle^2 + \langle X, f_2\rangle^2\right) = 2K^2, \tag{36}$$

where $|X| = 1$. Hence, from (33) we see that the bundle curvature of the frame bundle Killing submersion is given by

$$\tau^2 = \frac{1}{4}K^2 \circ \pi. \tag{37}$$

Thus, combining Proposition 10 and (37) we have the following.

Corollary 12. *Let γ be a closed curve in a Riemannian surface, M, with Gaussian curvature K. Then $S_\gamma = \pi^{-1}(\gamma)$ is a Willmore torus in the orthonormal frame bundle* $\mathbf{F}$M *if and only if γ is an extremal of the following elastic energy with potential*

$$\mathscr{A}(\gamma) = \int_\gamma \left(\kappa^2 + K^2\right) ds, \tag{38}$$

acting on the space of closed curves in M.

5. Killing Submersions and Willmore Tori Foliations

5.1. Willmore Foliations of Orthonormal Frame Bundles on Surfaces. We first show how one can get minimal Willmore tori families foliating the frame bundle of compact rotation surfaces in $\mathbb{R}^3$.

Proposition 13. *The orthonormal frame bundle of any compact rotation surface in $\mathbb{R}^3$ admits a foliation by minimal Willmore tori.*

Proof. Let S_γ be the rotation torus swept out by revolving around the z-axis a closed curve γ contained in the half plane $z = 0$, $x > 0$. Denote by $R_t(\gamma)$ the meridian obtained by rotating an angle t the profile curve γ. It is obvious that $\pi^{-1}(R_t(\gamma))$ is a minimal torus in $\mathbf{F}S_\gamma$, where $\pi : \mathbf{F}M \to S_\gamma$ denotes the natural projection. Moreover, meridians are not only geodesics, but also they are extremals of the elastic energy $\mathscr{A}$ because $\vec{n}(K) = 0$ (see Corollary 12 and (28)). Consequently, $\{\pi^{-1}(R_t(\gamma)) : t \in \mathbb{R}\}$ constitutes a foliation of $\mathbf{F}S_\gamma$ by minimal Willmore tori. □

Now we want to obtain orthonormal frame bundles of surfaces admitting foliations by nonminimal Willmore tori. To this end, we consider the following construction.

Let $f(s)$ be a smooth function defined on a real interval $I = (a, b)$ and consider the surface $I \times \mathbb{S}^1$ endowed with the warped product metric $ds^2 + f(s)d\theta^2$. The corresponding Riemannian surface is called a *warped product surface* and is simply denoted by $S_f := I \times_f \mathbb{S}^1$. The group $\mathbb{S}^1$ acts trivially on S_f by isometries and the orbits of this action, $\gamma_s(\theta) := \{s\} \times \mathbb{S}^1$, are called fibers of S_f. We first want to determine the warped surfaces for which all fibers are extremals of

$$\mathscr{A}(\gamma) = \int_\gamma \left(\kappa^2 + K^2\right) ds, \tag{39}$$

K being the Gaussian curvature of S_f. According to the results in previous section, they will give rise to orthonormal frame bundles admitting foliations by Willmore tori with constant mean curvature. Then, we have

Proposition 14. *Let $S_f := I \times_f \mathbb{S}^1$ be a warped product surface all whose fibers are critical for the energy functional (39). Then, locally, either f is constant and S_f is a Riemannian product surface, or $f(s)$ is determined by*

$$s + c_2 = \pm\frac{2}{\lambda}\int \frac{df}{\left(-1 + (4/\lambda) f - 2\operatorname{Lam}\left(-(1/\lambda)\, e^{d_1-(2/\lambda)f}\right) - \operatorname{Lam}^2\left(-(1/\lambda)\, e^{d_1-(2/\lambda)f}\right)\right)^{1/2}}, \tag{40}$$

where $c_2, \lambda \in \mathbb{R}^+$ and Lam *denotes the Lambert function [33].*

Proof. The Euler-Lagrange equation for closed extremals of (39) in a surface are obtained using Corollary 9:

$$\begin{aligned} &2\kappa''(t) + \kappa(t)^3 + 2K(t)\kappa(t) - K(t)^2\kappa(t) \\ &\quad + \vec{n}\left(K(t)^2\right) = 0, \end{aligned} \tag{41}$$

where t is the arc-length parameter of the curve, the derivative with respect to t is denoted by $(\,)' = d/dt$, $K(t)$ is the Gaussian curvature of the surface along the curve, and $\vec{n}$ is the normal vector to the curve in the surface.

Assume that every orbit of S_f is critical for (39). In S_f the geodesic curvature of any orbit, $\kappa(s,\theta) = -\dot{f}(s)/f(s)$, is constant on it (where now we are using overdots for d/ds) while the Gaussian curvature is given by $K(s,\theta) = -\ddot{f}(s)/f(s)$. Observe that $\vec{n} \equiv d/ds$ is the normal vector field to the orbits on the surface. Since all orbits are critical, (41) tells us that

$$\begin{aligned} &2f(s)\dot{f}(s)\ddot{f}(s) - \left(\dot{f}(s)\right)^3 + 2f(s)\ddot{f}(s)\dddot{f}(s) \\ &\quad - \dot{f}(s)\left(\ddot{f}(s)\right)^2 = 0, \end{aligned} \tag{42}$$

for $s \in I$. If $f(s)$ is constant, then S_f is locally a Riemannian product which is a flat surface with every orbit being a geodesic. Therefore, orbits of Riemannian products satisfy (41) so that they are critical for (39). If $f(s)$ is not constant, we divide (42) by $f((s))^2$ obtaining

$$\frac{d}{ds}\left(\frac{\left(\dot{f}(s)\right)^2}{f(s)} + \frac{\left(\ddot{f}(s)\right)^2}{f(s)}\right) = 0. \tag{43}$$

and therefore

$$\left(\dot{f}(s)\right)^2 + \left(\ddot{f}(s)\right)^2 = \lambda f(s). \tag{44}$$

for some $\lambda \in \mathbb{R}^+$. For convenience, in the following manipulations we put $y(s) := f(s)$. Then, (44) can be written as

$$\frac{1}{2}\frac{d}{dy}\left(\left(\dot{y}(s)\right)^2\right) = \pm\left(\lambda y(s) - \left(\dot{y}(s)\right)^2\right)^{1/2}. \tag{45}$$

Thus, setting $z(s) = (\dot{y}(s))^2$, $p(y) = dz/dy$, one gets

$$\frac{1}{2}\frac{dz}{dy} = \pm(\lambda y - z)^{1/2},$$
$$y = \frac{1}{\lambda}\left(\frac{1}{4}p^2 + z\right). \quad (46)$$

Differentiating with respect to z the second identity in (46), we have

$$dz = \frac{1}{2}\frac{p^2}{\lambda - p}dp. \quad (47)$$

Integrating this we get

$$z + c_1 = -\frac{\lambda}{2}p - \frac{1}{4}p^2 - \frac{\lambda^2}{2}\log(\lambda - p), \quad (48)$$

with $c_1 \in \mathbb{R}$. Unwinding the changes of variable (48) reduces to

$$c_1 = \mp\lambda\left(\lambda y - \dot{y}^2\right)^{1/2} - \lambda y$$
$$- \frac{\lambda^2}{2}\log\left(\lambda \mp 2\left(\lambda y - \dot{y}^2\right)^{1/2}\right). \quad (49)$$

Manipulating (49), we obtain

$$-\frac{1}{\lambda}e^{d_1-(2/\lambda)y}$$
$$= -\frac{1}{\lambda}\left(\lambda \mp 2\left(\lambda y - \dot{y}^2\right)^{1/2}\right)e^{-(1/\lambda)(\lambda\mp 2(\lambda y - \dot{y}^2)^{1/2})}, \quad (50)$$

for a certain constant $d_1 \in \mathbb{R}$. In other words

$$\mathrm{Lam}\left(-\frac{1}{\lambda}e^{d_1-(2/\lambda)y}\right) = -\frac{1}{\lambda}\left(\lambda \mp 2\left(\lambda y - \dot{y}^2\right)^{1/2}\right), \quad (51)$$

where Lam denotes the Lambert function (see, [33]). Finally, from (51) we have

$$\dot{y}^2 = \frac{\lambda}{4}\left(-1 + \frac{4}{\lambda}y - 2\mathrm{Lam}\left(-\frac{1}{\lambda}e^{d_1-(2/\lambda)y}\right)\right.$$
$$\left. - \mathrm{Lam}^2\left(-\frac{1}{\lambda}e^{d_1-(2/\lambda)y}\right)\right). \quad (52)$$

This concludes the proof. □

5.2. Elasticae with Potential and Willmore Tori in Killing Submersions. For a given Riemannian surface M with Gaussian curvature K, and for a potential $\mu \in C^\infty(M)$, we can consider the following elastic energy:

$$\mathscr{A}_\mu(\gamma) = \int_\gamma \left(\kappa^2 + \mu\right) ds \quad (53)$$

acting on the space of closed curves in M. From Corollary 9 the Euler-Lagrange equation associated with critical curves of this functional is

$$2\kappa'' + \kappa^3 + 2K\kappa - \mu\kappa + \vec{n}(\mu) = 0, \quad (54)$$

κ denoting the geodesic curvature of the extremal curve. In particular, if the potential μ is constant say λ, then the extremals of $\mathscr{A}_\lambda$ are the classical elastic curves (which we will call here λ-elasticae). Now, a natural problem is the following: given an elastica, say $\gamma(s)$, in M, determine those potentials $\mu \in C^\infty(M)$ for which $\gamma(s)$ is an extremal of $\mathscr{A}_\mu$, in other words, a solution of (54). It is clear that these potentials must satisfy the following differential equations along the curve $\gamma(s)$:

$$\vec{n}(\ln(\mu - \lambda)) = \kappa. \quad (55)$$

Therefore, in a neighborhood of $\gamma(s)$ in M, they must have the following form:

$$\mu(s,t) = \exp(\kappa(s)t + \omega(s)) + \lambda, \quad (56)$$

where $\omega(s)$ is an arbitrary function along $\gamma(s)$. Consequently, we have

Theorem 15. *Let $\gamma(s)$ be a λ-elastica with curvature function $\kappa(s)$ lying on a surface M with Gaussian curvature K. For an arbitrary function $\omega(s)$ along $\gamma(s)$ consider*

$$\mu(s,t) = \exp(\kappa(s)t + \omega(s)) + \lambda, \quad (57)$$

defined on a certain neighborhood $t \in (-\epsilon,\epsilon)$ of $\gamma(s)$. Let $\pi : N(K,\tau) \to M$ be a Killing submersion with closed fibers and bundle curvature τ, satisfying $4\tau^2 = \mu(s,t)$ on the chosen neighborhood. Then, $S_\gamma = \pi^{-1}(\gamma)$ is a Willmore torus in $N(K,\tau)$.

Finally, as an illustration we analyze the following problem. In $M = \mathbb{R}^2 - \{(0,0)\}$ consider the family of circles $\{C_t : t > 0\}$ defined by $C_t = \{(x,y); x^2 + y^2 = t^2\}$. It is obvious that this family constitutes the leaves of a foliation in M. Now, we wish to determine those potentials $\mu \in C^\infty(M)$ that make the whole family of circles extremals of the energy (53). Noticed that we are considering M as the once-punctured Euclidean plane and obviously endowed with the Euclidean metric, so $K = 0$ and the curvature of C_t is 1/t. Hence, from (54) we must determine those functions $\mu(s,t)$ which are solutions of the following differential equation:

$$\partial_t\mu = \frac{1}{t}\mu - \frac{1}{t^3}. \quad (58)$$

But, the general solution of (58) is

$$\mu(s,t) = f(s)t + \frac{1}{3t^2}, \quad f \in C^\infty\left(\mathbb{S}^1\right). \quad (59)$$

so that we can prove the following statement.

Corollary 16. *Let $M = \mathbb{R}^2 - \{(0,0)\}$ be the once-punctured Euclidean plane, $f \in C^\infty$ a periodic positive function and define μ by (59). Then, there exists a Killing submersion $\pi : N(0,\tau) \to M$ with bundle curvature given by $4\tau^2 = \mu$ which admits a foliation by Willmore tori with constant mean curvature.*

Proof. Positivity of the function f allows ensuring the existence of $\tau \in C^{\infty}(M)$ satisfying $4\tau^2 = \mu$, where μ is defined in (59). Now, we use Theorem 1 to obtain a Killing submersion, say $\pi : N(0, \tau) \to M$, with compact fibers over M and bundle curvature τ. Since the circles C_t, $t > 0$ are extremals of the energy action (53), we obtain that $S_t = \pi^{-1}(C_t)$ are Willmore tori in N (see Proposition 10). □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This research was partially supported by MINECO-FEDER Grants MTM2014-54804-P and MTM2013-47828-C2-1-P and Gobierno Vasco Grant IT1094-16, Spain. The third author has been supported by Programa Predoctoral de Formación de Personal Investigador No Doctor, GobiernoVasco, 2015.

References

[1] W. Helfrich, "Elastic properties of lipid bilayers: theory and possible experiments," *Zeitschrift für Naturforschung C*, vol. 28, no. 11-12, pp. 693–703, 1973.

[2] O. J. Garay, "Riemannian submanifolds shaped by the bending energy and its allies," in *Proceedings of the Sixteenth International Workshop on Diff. Geom.*, vol. 16, pp. 57–70, 2012.

[3] G. Thomsen, "Uber konforme Geometrie I: Grundlagen der konformen Fächentheorie," *Abhandlungen aus dem Mathematischen Seminar der Universitat Hamburg*, vol. 3, no. 1, pp. 31–56, 1923.

[4] T. J. Willmore, "Mean curvature of immersed surfaces," *Analele ştiinţifice ale Universităţii "Al. I. Cuza" din Iaşi. Matematică (SERIE NOUĂ)*, vol. 14, pp. 99–103, 1968.

[5] W. Blaschke, *Vorlesungen Uber Differentialgeometrie III*, Springer, Berlin, Germany, 1929.

[6] J. H. White, "A global invariant of conformal mappings in space," *Proceedings of the American Mathematical Society*, vol. 38, pp. 162–164, 1973.

[7] M. Barros and B.-Y. Chen, "Stationary 2-type surfaces in a hypersphere," *Journal of the Mathematical Society of Japan*, vol. 39, no. 4, pp. 627–648, 1987.

[8] R. L. Bryant, "A duality theorem for Willmore surfaces," *Journal of Differential Geometry*, vol. 20, no. 1, pp. 23–53, 1984.

[9] I. Castro and F. Urbano, "Willmore surfaces of R4 and the Whitney sphere," *Annals of Global Analysis and Geometry*, vol. 19, no. 2, pp. 153–175, 2001.

[10] D. Ferus and F. Pedit, "S1-equivariant minimal tori in S4 and S1-equivariant Willmore tori in S3," *Mathematische Zeitschrift*, vol. 204, no. 2, pp. 269–282, 1990.

[11] S. Montiel, "Willmore two-spheres in the four-sphere," *Transactions of the American Mathematical Society*, vol. 352, no. 10, pp. 4469–4486, 2000.

[12] U. Pinkall, "Hopf tori in S3," *Inventiones Mathematicae*, vol. 81, no. 2, pp. 379–386, 1985.

[13] J. L. Weiner, "On a problem of Chen, Willmore, et al," *Indiana University Mathematics Journal*, vol. 27, no. 1, pp. 19–35, 1978.

[14] F. C. Marques and A. Neves, "Min-max theory and the Willmore conjecture," *Annals of Mathematics: Second Series*, vol. 179, no. 2, pp. 683–782, 2014.

[15] J. Langer and D. Singer, "Curves in the hyperbolic plane and mean curvature of tori in 3-space," *Bulletin of the London Mathematical Society*, vol. 16, no. 5, pp. 531–534, 1984.

[16] A. Ros, "The Willmore conjecture in the real projective space," *Mathematical Research Letters*, vol. 6, no. 5-6, pp. 487–493, 1999.

[17] P. Topping, "Towards the Willmore conjecture," *Calculus of Variations and Partial Differential Equations*, vol. 11, no. 4, pp. 361–393, 2000.

[18] B.-Y. Chen, "Some conformal invariants of submanifolds and their applications," *Bollettino dell'Unione Matematica Italiana*, vol. 6, pp. 380–385, 1974.

[19] B.-Y. Chen, *Total Mean Curvature and Submanifolds of Finite Type*, vol. 27 of *Series in Pure Mathematics*, World Scientific Publishing Corporation, Singapore, 2nd edition, 1984.

[20] J. C. Nitsche, "Boundary value problems for variational integrals involving surface curvatures," *Quarterly of Applied Mathematics*, vol. 51, no. 2, pp. 363–387, 1993.

[21] M. Barros, "A geometric algorithm to construct new solitons in the O(3) nonlinear sigma model," *Physics Letters B*, vol. 553, no. 3-4, pp. 325–331, 2003.

[22] M. Barros, M. Caballero, and M. Ortega, "Rotational surfaces in L3 and solutions of the nonlinear sigma model," *Communications in Mathematical Physics*, vol. 290, no. 2, pp. 437–477, 2009.

[23] J. M. Espinar and I. S. de Oliveira, "Locally convex surfaces immersed in a Killing submersion," *Bulletin of the Brazilian Mathematical Society*, vol. 44, no. 1, pp. 155–171, 2013.

[24] J. M. Manzano, "On the classification of Killing submersions and their isometries," *Pacific Journal of Mathematics*, vol. 270, no. 2, pp. 367–392, 2014.

[25] B. Daniel, "Isometric immersions into 3-dimensional homogeneous manifolds," *Commentarii Mathematici Helvetici*, vol. 82, no. 1, pp. 87–131, 2007.

[26] É. Cartan, *Leçons sur la Géométrie des Espaces de Riemann*, Gauthier Villars, Paris, 1928.

[27] M. Barros, "Willmore tori in non-standard 3-spheres," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 121, no. 2, pp. 321–324, 1997.

[28] R. S. Palais, "The principle of symmetric criticality," *Communications in Mathematical Physics*, vol. 69, no. 1, pp. 19–30, 1979.

[29] J. Langer and D. A. Singer, "The total squared curvature of closed curves," *Journal of Differential Geometry*, vol. 20, no. 1, pp. 1–22, 1984.

[30] M. Barros, A. Ferrández, and O. J. Garay, "Equivariant Willmore surfaces in conformal homogeneous three spaces," *Journal of Mathematical Analysis and Applications*, vol. 431, no. 1, pp. 342–364, 2015.

[31] A. Gray, "Pseudo-riemannian almost product manifolds and submersions," *Journal of Mathematics and Mechanics on JSTOR*, vol. 16, pp. 715–727, 1967.

[32] B. O'Neill, "The fundamental equations of a submersion," *Michigan Mathematical Journal*, vol. 13, pp. 459–469, 1966.

[33] R. M. Corless, G. H. Gonnet, D. E. Hare, D. J. Jeffrey, and D. E. Knuth, "On the Lambert W function," *Advances in Computational Mathematics*, vol. 5, no. 4, pp. 329–359, 1996.