Missing the (Bayesian) wood for the trees?

¿Los árboles no dejan ver el bosque Bayesiano?

Jorge Soto-Andrade

University of Chile

Abstract

We intend here to substantiate the claim that the intensive use of trees to tackle Bayesian problems may lead us to "miss the Bayesian wood", particularly if we just focus on the static trees and ignore germane random walks on them. Our main point is that random walks on networks or grids instead, provide a more fruitful and insightful metaphor to address Bayesian problems and fathom the underlying "Bayesian flows". Besides recalling the main tenets of our theoretical background, we discuss below the relation of our claims with related research in this field and give some illustrative classroom examples, arising mainly from our teaching stochastics to non-mathematically inclined first year university students and prospective mathematics teachers.

Keywords: Trees, Bayesian, metaphor, network

Resumen

Nos proponemos aquí fundamentar nuestra afirmación que el uso intensivo de los árboles par abordar problemas Bayesianos nos puede llevar a "no ver el bosque Bayesiano"; particularmente si nos enfocamos solamente en árboles estáticos ignorando los paseos aleatorios relevantes sobre ellos. Nuestro punto principal es que por el contrario los paseos al azar en redes o rejillas, proveen una metáfora más fructífera y perspicaz para enfrentar problemas Bayesianos y discernir los "flujos Bayesianos" subyacentes. Además de recordar los principales principios de nuestro trasfondo teórico, discutimos más abajo la relación de nuestras afirmaciones con investigaciones relacionadas en este campo y damos ejemplos de aula ilustrativos, emergentes, principalmente de nuestra enseñanza de la estocástica a estudiantes universitarios de primer año sin inclinación matemática y futuros profesores de matemáticas.

Keywords: Árboles, bayesiano, metáfora, red.

1. Introduction

This theoretical paper is built around two claims, which we have posited to some extent elsewhere and we intend to further substantiate here. Our first claim is that random walks constitute a royal road to stochastic thinking (Soto-Andrade, 2013, 2015; Soto-Andrade, Díaz-Rojas, & Reyes-Santander, 2018). Our second claim, which is the main focus of this paper, is that we may "miss the Bayesian wood for the trees", because Bayesian problems are better metaphorised as random walks on graphs ike network and grids than on trees (Soto-Andrade et al., 2018).

This paper is structured as follows. In Section 1 below, we recall the main tenets of our theoretical framework, essentially a metaphoric and enactivistic approach to the didactics of mathematics (Lakoff & Núñez, 2000; Proulx & Maheux, 2017; Soto-Andrade, 2018). In Section 2 we recall the role and use of metaphores in the didactics of mathematics, particularly the types of metaphors that most frequently arise in our work with students. In Section 3, we recall our argumentation in favour of our first claim. In Section 4, we apply our metaphorical approach to the study of random walks. In Section 5, we discuss the role of trees in the teaching of stochastics (Batanero & Borovcnik,

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2016; Parzysz, 2011) and relate them to our main concern, to wit Bayesian reasoning, a critical issue in this field indeed (Borovcnik, 2012; Böcherer-Linder et al., 2018; Hoffrage et al., 2015; Zhu & Gigerenzer, 2006). In Section 6 we describe a paradigmatic example of our approach (Rayen's fall) and its experimental background. We end with a discussion, conclusions and open questions in Section 7.

2. Theoretical background: metaphorical approach to didactics of mathematics

Increasing awareness has emerged during the last decades among the mathematics education community that metaphors are not just rhetorical devices but powerful cognitive tools that help us in grasping as well as building new concepts, and also in solving problems in an efficient and friendly way (Diaz-Rojas & Soto-Andrade, 2015; English 1997; Knops et al., 2009; Lakoff & Núñez, 2000; Sfard, 2009; Soto-Andrade 2014, 2018; and many others). In fact, metaphorising (looking at something and seeing something else, metaphorically defined) appears as the often unknowing foundation for human thought (Gibbs 2008). Indeed as suggested by Johnson and Lakoff (2003), our ordinary conceptual system, in terms of which we think and act, is fundamentally metaphorical in nature. Lakoff and Núñez (2000) highlight the intensive use we make of conceptual metaphors that appear—metaphorically—as inference-preserving mappings from a more concrete and transparent 'source domain' into a more abstract and opaque 'target domain', enabling us to fathom the latter in terms of the former.

3. Why random walks?

To support our claim that random walks are a royal road to stochastic thinking, we point up first their transversality: they cross boundaries, arising both in the natural and cultural realm, besides providing models for sundry phenomena arising in both of them.

In the first one we find the erratic movement of pollen micro particles discovered by the botanist Robert Brown in 1927 (Powles, 1978), called nowadays "Brownian motion", also observed later in the case of nano-inclusions in metallic alloys, as foreseen by Einstein in 1905 (Preuss, 2002). Already in 60 BC however Lucretius observed dust particles "skirmishing" under sunlight and hypothesised this to be caused by "motions of matter latent and unseen at the bottom" (Powles, 1978). Other examples of random walks in the natural realm are mosquito flights (Pearson, 1905), foraging patterns in human hunter–gatherers (Raichlen et al., 2014) and erratic fluctuations of stock markets (Bachelier, 1900). In fact Pearson (1905) coined the term "random walk" in his query to the journal Nature about the probability distribution of the distance from the origin of a random flying mosquito (a vector of malaria) after a given lapse of time.

In the cultural realm, we find the construction of random hexagrams when consulting Yi Jing, the ancient Chinese oracle (Wilhelm, 1956), that may be seen as a 6-step symmetric random walk on the binary tree, as drawn by Chinese mathematician Shao Yong (Marshall, 2015). Remarkably enough, this random walk, now on an *infinite* binary tree reappears one thousand years later in cosmology as a discrete stochastic model for eternal inflation, which allows for the construction of a multiverse (Harlow et al., 2012; Marcolli, 2017). Another classical example is provided by Saint Francis of Assisi's friars walking across medieval Italy's road network to preach the Gospel, trying to be just instruments of God's will by choosing randomly at each crossroad with the help of the following clever method, devised by Saint Francis himself: At every road junction, he had a friar to whirl nonstop in spite of dizziness and nausea, until he collapsed and fell. Then the whole company would choose the road closest to the direction shown by the friar's head (Anonymous, 1600; Soto-Andrade, 2013).

From a didactic viewpoint, it should be highlighted that random walks are a *visual* embodiment of randomness (we literally *see* randomness in a random walk), that can be easily enacted and simulated, from primary school all the way up to postgraduate school, didactically embodied in what we call "learning sprouts". They can be approached and studied in manifold ways: statistically, metaphorically, probabilistically. They provide "universal models" and metaphors for sundry phenomena.

For example, the classical ruin problem, involving two players (Chung, 1974) can be meaningfully metaphorised as the random walk of a frog on a row of stones whose end stones are crocodiles (absorbing barriers) (Soto-Andrade et al., 2018).

Indeed, random walks facilitate the access of non-mathematically oriented learners to stochastic thinking, enabling them to construct probabilistic notions by themselves, while solving situated concrete problems. In our view they constitute a "learning sprout" for probability and statistics (Soto-Andrade, 2015; Soto-Andrade et al., 2018). Most interesting for us, Bayesian problems can be suitable metaphorized as random walks. See section 4 below for the case of the classical Monty Hall problem.

4. Metaphors for random walks?

Based on our metaphoric approach to the didactics of mathematics (Soto-Andrade, 2018), we encourage students to metaphorise when addressing the study of random walks on graphs. We recall and exemplify below a few helpful metaphors, most frequently used by our learners to explore and figure these random walks, as well as constructing on the way the concept of probability, at various educational levels (Soto-Andrade et al., 2018).

Solomonic (or splitting) metaphor. This metaphor sees the random walker deterministically *splitting* into pieces instead of *walking randomly* according to the given transition probabilities, as King Solomon threatened to do with the disputed baby. For instance, when looking at a frog jumping equally likely right or left on a row of stones in a pond, the Solomonic metaphor sees the frog *splitting* into two halves that go simultaneously right and left, and so on. This 'metaphoric sleight of hand' turns the random walk into a *deterministic* fission process, thus allowing us to reduce probabilistic calculations to deterministic ones, where we just need to keep track of the walker's splitting into pieces: The probability of finding the walker at a given location after n jumps is just the portion of the walker landing there after n splittings. This enables in fact the learners to construct the notion of probability!

Hydraulic metaphor. This metaphor is a variant of the previous one, where we develop the random walk in space-time and so that the walker appears as a fluid draining down through a tree, or more generally a network, of ducts bifurcating according to the given transition probabilities (Soto-Andrade et al., 2018). This metaphor suggests constructing analogical models made out of a network of ducts and stopcocks to enact jumping frogs or other types of random walks.

Pedestrian metaphor. Learners who dislike calculating with fractions or halving frogs, (in the case of a symmetric random walk) may have the idea of unleashing an army of frogs instead, from the starting stone, and have them split into halves at each stage.

Fittingly, 2^n frogs for a n-step symmetric walk. With the help of this metaphor, students just need to count how many frogs are crouching at each stone after the given number of steps and divide by the total number of frogs, to quantify likelihood. More generally this metaphor looks at a random walk on a graph and sees a company of pedestrians splitting into smaller groups as they progress along a road network. Notice that this provides a natural pedestrian approach to Pascal's triangle.

Borgian (parallel universes) metaphor). This is a variant of the previous ones, where we would see now the frog become double (like acquiring a Doppelgänger), so that we have now two frogs, one jumping right and the other jumping left. Or equivalently, at each jump our Universe splits into two parallel universes, in one of which the frog jumps right and in the other left. After n jumps we will see then an army of 2^n frogs, and we can just count how many frogs are squatting on each stone and divide by the total number of frogs, to quantify likelihood.

The Monty Hall Problem: A metaphoric random walk approach. The main prize – a car – is hidden behind one of 3 doors, while behind the other 2 doors there is a goat. All doors are closed at the beginning. The candidate chooses one. The moderator opens another door and reveals a goat; she then offers to the candidate the option to change his initial choice. Should he change or not? (Borovcnik, 2012).

One friendly way to figure out this problem is to metaphorise it as a simple random walk, as in Figure 1. We leave room for this sort of metaphorisation to emerge among the students. Most of the time it does. To figure out this random walk, a Borgian (or pedestrian) metaphor may arise, which sees 3 walkers choosing a target door (node), of which only one (the red one) hides the prize. When they reach the blue node, halfway to their target, a (Poe-ian) raven points out to each of them that the prize is not hidden at one of the two other nodes and allows them to change their target node if they want. Then the blue arrows indicate the path of the walkers who did not change their target choice and the red arrows indicate the path of their Doppelgängers who did change their minds. We see that only 1 out of the 3 stubborn walkers won the prize, against 2 out of the 3 flexible ones who changed their strategy!

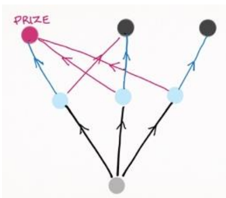

Figure 1. The Monty Hall problem as a random walk.

Random walks occur in some universe, typically a graph in our context, which is often a tree. In the next section we turn to trees and their relation with Bayesian problems.

5. Are we missing the Bayesian wood for the trees?

Trees are ubiquitous nowadays in the teaching of probability (Batanero & Borovcnik, 2016; Batanero & Chernoff, 2018; Dupuis & Rousset-Bert, 1996; Parzysz, 2011). They even appear explicitly in the curricula of several countries, before the introduction of probability, as a way to solve combinatorics problems, as possibility trees or decision trees. As probability trees, they are quite helpful in visualizing and calculating, when tackling problems involving randomness. They have been used to describe and to solve Bayesian problems and their usefulness has been compared with other resources, like contingency tables, nested sets or unit square representations, besides iconic representations (Böcherer-Linder et al., 2018; Borovcnik, 2012; Brasse, 2009; Dupuis & Rousset-Bert, 1996; Parzysz, 2011).

On our side, we claim that trees may lead you to miss the wood, especially the “Bayesian wood”. Indeed, trees as such have no loops, there is just one road leading to Rome (a given node of interest to us) in a tree. On the contrary, a characteristic trait of Bayesian problems, suitably metaphorised, is that we have two or more roads leading to Rome. Then the famous Bayesian question on the “probabilities of the causes” boils down to asking: *among all random travelers getting to Rome, how many came through this road or that one?*

So trees may turn out to be rather awkward as a model for a Bayesian “phase space”: in fact students end up attaching the same label to different nodes, so that they are in fact working on a *quotient space* (a “shadow space”) of the tree, obtained by identifying or merging some nodes. From this viewpoint, more general graphs containing closed circuits, i. e. allowing several paths to connect nodes, typically grids, lattices or networks, constitute a more natural phase space for Bayesian systems than trees. In this sense, relying just on trees, especially probability trees, but also natural frequency trees (Gigerenzer et al., 2007; Hoffrage et al. 2015), tends to hide the Bayesian character of the situation, so that we may “miss the Bayesian wood for the trees”. We illustrate this point through a paradigmatic example (Rayen’s fall) below.

Moreover, in our view, a neglected aspect in this respect is that what is usually most important is not the graphs themselves (either trees or grids), but *random walks* on them (Diaz-Rojas & Soto Andrade, 2015; Soto-Andrade et al., 2018), which provide a crucial dynamical aspect to the visualisation of Bayesian problems.

This meets some of the criticisms in Böcherer-Linder et al. (2018) on the visualisation in terms of trees, who point out as a drawback, that in false positive problems “due to the hierarchical structure of the tree diagram the set of all people tested positive is separated into two distinct parts”. Indeed, if we focus on the corresponding random walk on the associated grid (see Fig. 2 below, for the analogous case of Rayen’s fall) we see two groups of walkers, coming through different gates (“carrier” and “non-carrier”) but *converging* to Rome (“positive test”). Also, the dynamics of the random walk shows clearly why the probability of finding a carrier among those patients with a positive test may be surprisingly low. Böcherer-Linder et al. (2018) also point out that the branching tree visualisation cannot represent discrete and countable objects in a Bayesian context. The pedestrian metaphor for the corresponding random walk on the associated grid however does represent those objects, as the pedestrians enacting the random walk!

Indeed, Bayesian thinking seems to be a difficult endeavour for most learners and users of probability (Brighton & Gigerenzer, 2008; Gigerenzer, 2011, Gigerenzer & Hoffrage, 2007). Kahneman and Tversky (1972, p. 450) even arrived at the conclusion that “man is apparently not a conservative Bayesian: he is not Bayesian at all” and Gould (1992, p. 469) added “our minds are not built (for whatever reason) to work by the rules of probability".

Our theoretical perspective however, suggests that learners may handle probabilistic – particularly Bayesian - situations first by metaphorising them as random walks, on some suitable graph, no necessarily a tree, and then, to figure out the random walk, with a metaphorical sleight of hand, turning the random process into a deterministic one, typically with the help of a hydraulic or a pedestrian metaphor.

In the case of a simple Bayesian problem, involving two contradictory hypotheses (such as being or not a virus carrier) and a single cue (such as a screening test) with two cue values (a positive or negative result), in the terminology of Hoffrage et al. (2015), our approach leads to a 2-step random walk (see example 6.2 below), which in turn may be metaphorised in various way, to be dealt with as a deterministic process.

6. An illustrative example

6.1. Experimental background.

We have tested our metaphoric approach to Bayesian problems with first year humanistically oriented university students at the University of Chile, who intend to major in psychology, sociology, anthropology or law. They have an average total score of 700 points and a maths score of 650 points approximately in our national standardised test (whose national mean is 500 points, with a standard deviation of 110), while a minimum score of 600 points is required to be admitted to the University of Chile. They come mostly from above average schools, where they have however been systematically “taught to the test” in maths: intensive multiple-choice questions drill with no understanding. A small minority of these students is good at learning formulas by heart and applying them following typical routines.

6.2. Example: Rayen’s fall

We have posed the following Bayesian problem to several cohorts of our humanistically oriented students:

> Rayen lives in the south of Chile and rides her bike along a windy and steep downhill road to school every weekday in winter, when it rains estimatedly 2 days out of 5. On wet road she falls from her bike 1 out of 4 times, on dry road, only 1 out of 9, on the average. If you learned that Rayen fell from her bike today, how likely is that this was a rainy day?

When our students first addressed this Bayesian problem in a test, most of them drew the usual 2-step binary tree to visualise it and their most common error (approx. 2/3 of the class) was to mistake the probability of rain *given* that Rayen fell, with the probability that it rains *and* Rayen falls. So, they got an absurd answer: 1/4 of 2/5 = 1/10 as the estimated probability of rain in case you know that Rayen fell, which is smaller than the probability of rain with no information whatsoever on Rayen’s biking! Curiously enough they made the same mistake in a false positive problem (obtaining that the requested probability is far smaller than the prevalence of the virus!), more often than the typical error reported by Gigerenzer and collaborators (Gigerenzer, 2011; Gigerenzer & Hoffrage, 2007; Hoffrage et al., 2015; Zhu & Gigerenzer, 2006), which is to mix up forward and backward conditional probabilities.

At the next test, after some sessions of significant group work on random walks and Bayesian problems, where some students proposed spontaneously to merge the Fall nodes of the tree, roughly 85 % of students got the right answer, and the remaining 15 % still made the previous mistake. Roughly half of the students who answered correctly drew in fact a grid as in Figure 2 below, most of them as a hydraulic grid rather than as a pedestrian grid, which we had thought to be friendlier for them.

We see that that representing Bayesian problems by probability trees, or better, by natural frequency trees, in the sense of Gigerenzer (Gigerenzer & Hoffrage, 2007; Hoffrage et al., 2015), does not prevent students to make the described mistakes and that Bayesian reasoning (or thinking) becomes more natural and transparent when they explicitly metaphorize the Bayesian “backward question” as a question on concurring paths on a network or grid, metaphorizing the involved random walks in a pedestrian way, and so converging with the friendly natural frequencies approach of Gigerenzer and collaborators (Gigerenzer & Hoffrage, 2007; Hoffrage et al., 2015; Zhu & Gigerenzer, 2006). In this way simple Bayesian problems can be solved even mentally, and more complex ones with just the help of some graph drawing (Hoffrage et al., 2015).

Figure 2 shows renderings of a hydraulic solution and a pedestrian resolution of this Bayesian problem by the students. In fact, students might refer indifferently to 30 days or to 30 pedestrians in this case, noticing that the problem boils down to count how many of the five walkers arriving at destination “Fall” came through the “Rain” gate!

Figure 2. Rayen’s fall: hydraulic and pedestrian grid approach.

From our viewpoint, we foresee that non-mathematically inclined students could be at least as efficient in solving this type of Bayesian problems as mathematically trained students, if they take advantage of such a friendly and intuitive metaphoric random walk on grids approach.

Moreover, our example also opens up the way to realise that a “Bayesian flow” naturally arises in this context, which we obtain when we look at all involved conditional probabilities in the Bayesian situation. See Figure 3 below, drawn by students when trying to visualise the merging of the Fall nodes and the Not Fall nodes of the initial possibility tree. A flow appears here, which is stationary: notice for instance that 1*/4 of 2/5 = 3/5 of 1/6 = 1/10* = probability that it rains *and* Rayen falls = probability that Rayen falls *and* it rains. And so on. So cancelling flows between two adjacent nodes are just the intersection probabilities of those nodes!

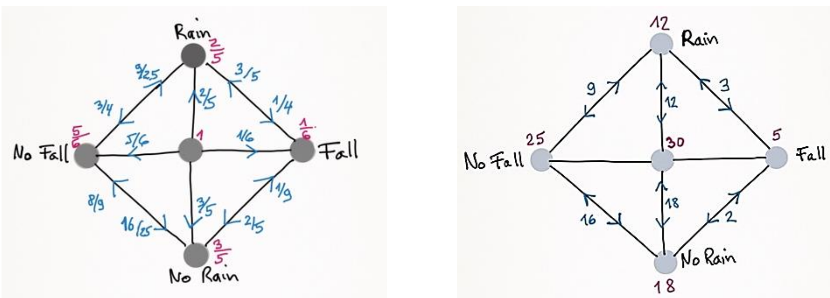

Figure 3. Bayesian flows for Rayen

We claim here that often a network or a grid is a friendlier graph than a tree. Indeed, trying to draw a more symmetrical graph, students realise that the big picture in a Bayesian question is a Bayesian flow on a graph, which can be read and interpreted in sundry ways.

7. Discussion and open questions

We have discussed the intertwining of three main ideas. First, Bayesian problems can be metaphorised in a friendly way as random walks. This allows learners to dispense of Bayes' awesome formula (that many learn by rote in the traditional teaching of probability) and solve concrete Bayesian problems just looking at the germane graph, even mentally. Second, we have seen that the natural stage for Bayesian problems (seen as random walks) is given by networks and grids rather than trees, because of the " several roads leading to the same node" character of those problems. Third, the aforementioned random walks can be studied and solved with the help or hydraulic or pedestrian metaphors, the latter being friendlier for most learners. Here we converge with the natural frequencies rendering of Bayesian problems promoted by Gigegerenzer and collaborators these last decades (Gigerenzer, 2011; Gigerenzer & Hoffrage, 2007; Hoffrage et al., 2015; Weber et al., 2018; Zhu & Gigerenzer, 2006).

Nevertheless, we have observed than among our humanistically-oriented students at the university, there is an unexpected majority which declares to prefer hydraulic metaphors to pedestrian ones, because they find them "conceptually more orderly and transparent", especially when infinite processes are involved. This in spite of the fact that they are not so skilled in manipulating fractions.

We noticed that if - with a more systemic approach - we figure out all involved conditional probabilities in our Bayesian problem, we see a stationary "Bayesian flow" emerge, which could also be fathomed as two cancelling flows going in opposite directions. This could be a sensible metaphor for correlational cases in which a causal relationship is unclear, contrary to the case of Rayen´s fall where we tend to say that Rayen falls *because* of the rain, but not that it rains *because* Rayen falls, although each event "increases" the estimate probability of the other (from 2/5 to 3/5 and from 1/6 to 1/4).

On the other hand, with respect to cognitive rigidity related to Bayesian problems, as reported by Weber et al. (2018), who noticed that a majority of students, when tackling a Bayesian problem couched in natural frequencies, still preferred to switch back to (mostly decimal) probabilities, we observed the following. Our humanistically oriented students, although most of them developed stiff cognitive joints at secondary school, after a few sessions of metaphorising, became more flexible, and were able to seamlessly move from hydraulic to pedestrian metaphors, and backwards. Particularly, they were able to choose autonomously the number of pedestrians they should unleash, when given the relevant data in fraction or decimal form (cf. Engel, 1975). This needs some exercising though, to overcome the weight of the prevailing didactical contract (Brousseau, 1998) at the secondary school, or their previous structural coupling with mathematics (Proulx &. Maheux, 2017; Varela et al., 1991) which does not allow for metaphorizing or the like.

As an open end, we could point out that in a more enactivistic approach (Proulx & Maheux, 2017; Reid & Mgomgelo, 2015; Soto-Andrade, 2018; Varela, 1987; Varela et al., 1991), Bayesian problems, like Rayen's Fall or false positive problems, could be posed just as "situational seeds", with no questions being asked by the teacher, so as to leave room for them to emerge from the learners, who would in fact co-construct the problem (loc. cit.). We may expect that in this case the question about the likelihood of a rainy day each time Rayen falls would not spark as quickly as the urgent question of a patient who got a positive reading in a screening test. This suggests that the context of (mathematically equivalent) Bayesian problems (or flows) plays a determining role in the way that learners may interact or couple with them, although students may easily recognise the equivalence of different Bayesian problems which can be metaphorised by the "same" random walk. These didactic phenomena would deserve further (theoretical and experimental) research.

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